Office of Naval Research
Contract Nonr-1866 (16) NR-372-012

THE CHARACTERIZATION AND IDENTIFICATION OF SYSTEMS

By

Charles R. Arnold and Kumpati S. Narendra

June 18, 1965

Technical Report No. 471

Cruft Laboratory
Division of Engineering and Applied Physics
Harvard University • Cambridge, Massachusetts

Office of Naval Research
Contract Nonr-1866(16)
NR - 372 - 012

THE CHARACTERIZATION AND IDENTIFICATION OF SYSTEMS

by

Charles R. Arnold and Kumpati S. Narendra

June 18, 1965

The research reported in this document was sponsored in part by support extended to Cruft Laboratory, Harvard University, by the U. S. Army Research Office, the U. S. Air Force Office of Scientific Research, and the U. S. Office of Naval Research under the Joint Services Electronics Program by Contract Nonr-1866(16). Reproduction in whole or in part is permitted for any purpose of the United States Government.

Technical Report No. 471

Cruft Laboratory

Division of Engineering and Applied Physics

Harvard University

Cambridge, Massachusetts

ABSTRACT

This report provides a fairly exhaustive survey of deterministic models for the characterization of systems and of the techniques available for the subsequent identification of some physical system in terms of each of the various classes of models. The term "system" refers, in general, to a nonlinear system and the major emphasis of the report is thus upon nonlinear models and their identification requirements.

Most of the ideas presented in this report are not new. However, many of them are not widely known in the engineering field and it was thus felt desirable to collect, organize, and present them as clearly as possible for the engineer.

THE CHARACTERIZATION AND IDENTIFICATION OF SYSTEMS

by

Charles R. Arnold* and Kumpati S. Narendra
Division of Engineering and Applied Physics
Harvard University, Cambridge, Massachusetts

I. INTRODUCTION

System characterization and system identification are basic problems in system theory and they have received considerable attention in the past. In a recent paper Zadeh [1] places these problems, together with the problem of system classification, at the top of his list of principal problems of system theory. These various problems may be described briefly as follows:

System Characterization The problem of system characterization is concerned primarily with the mathematical representation of a system's input/output relationship. These mathematical representations or models will be in terms of differential equations, state equations, integral equations, etc. Part of the consideration of these models is the requirements to be imposed upon the various related functions such as the impulse response and transfer function so as to express various attributes (e.g., stability, bandwidth, physical realizability) of a physical system.

System Classification The problem of system classification may be stated as follows. Given a family of system classifications $C_1, C_2, C_3, \ldots$, and an unknown physical system S which belongs to one of the C_i's, determine on the basis of observations of the systems input/output, that particular classification C_λ to which the system S belongs.

System Identification Given a class C of system models and some physical system S, the identification problem is to determine that specific model in C which is equivalent to S. The identification is to be accomplished through the observation, often in the presence of noise, of the response of S to various probe functions. Obvious examples are the estimation of impulse responses, transfer functions, and the coefficients of a differential equation.

- - - - - - -

*Staff member, Lincoln Laboratories, M.I.T., Lexington, Massachusetts.

The purpose of this report is to provide a fairly exhaustive survey of deterministic models for the characterization of systems and of the techniques available for the subsequent identification of some physical system in terms of each of the various classes of models. Throughout this report, the term "system" refers, in general, to a nonlinear system. Thus, the major emphasis will be on nonlinear models and their identification requirements. Linear systems are, however, included for continuity. One would not attempt to use a nonlinear model for identification before he has first ruled out the possibility of a linear model. The problem of system classification (e.g., linear or nonlinear) is not explicitly considered in this report since the authors feel that it is subsumed under the general problem of system identification.

Some of the nonlinear models that are described in this report are not very well known in the engineering literature. There is also considerable confusion concerning others. As an example, Wiener's theory means different things to different people. Part of the difficulty is that Wiener's ideas have evolved over the years. In this report three separate stages (or models) are recognized. The first (1942), is referred to as the Volterra expansion; the second (1949), Wiener I; and the third (1958), as Wiener II. On the other hand, Deutsch in his recent monograph [2] called the first (Volterra) model, Wiener's method, and he fails to present the later two. Considerable confusion also exists as to the advantages as well as the limitations of the various models; there is the recurrent problem of the choice of base functions which has not been adequately treated.

In this report an attempt is made to present all the various models for system characterization with their advantages and limitations clearly stated. An attempt is also made to indicate the suitability of using the various models in a particular situation. An entire section is devoted to the considerations which should govern the choice of a model. For the orthogonal expansion models, part of this choice is in the selection of a set of base functions. The ultimate goal is, however, to give the engineer enough details so that he may make an adequate identification of a real system in any particular situation. Toward this goal, the following three questions have to be answered:

1. What models are there available?
2. How does one select a class of models?
3. Having selected a class of models, how does one identify the given system?

The next three sections of this report attempt to answer each of these questions.

No specific applications are given of the ideas expressed in this report. One reason is that there are numerous real applications which use a linear model, yet very few which employ a nonlinear model in the process of identification of an unknown system. One area where some of these ideas may be particularly relevant is that of adaptive control. To date, most of the models have been linear time-invariant and have been made adaptive in order to follow some time-varying linear, nonlinear, or stochastic system. The use of some of the orthogonal nonlinear models that are discussed may lead to real improvements in some adaptive situations with little extra work.

A system S is defined as a physical object which transforms elements, called inputs, from some input set into other elements, called outputs, which lie in some other set called the output set. There is considerable literature on linear multivariable systems, but only a few results on the characterization and identification of multiple input/output nonlinear systems. This report will, however, be limited to systems which have only a single input and a single output connection (see Fig. 1.1).

Figure 1.1 A Single-Input/Single-Output System

If a system responds to two different elements of the input set, say x_1 and x_2, it will also be required to accept their sum $x_1 + x_2$ as an input. Moreover, the scale in which the input elements are measured should not affect their acceptability as inputs. Thus, if x_1 is an input, so must αx_1 for each real number α. This closure of the input set under these two operations (addition and scalar multiplication) are just the requirements that the elements of the input set form a vector space. Henceforth, it is assumed that the input set for the system is a vector space and it will be called the input space. By analogy, if the output set can possibly form an input space for some other system (say in cascade), it will be called the output space.

For general systems, the concepts of controllability and observability have been defined [3]. These concepts are not important in the identification problem in that those portions of any physical system which are uncontrollable, unobservable, or both cannot be identified [4]. Therefore, all models which are considered will be for completely controllable and completely observable systems. If it so happens that a given physical system has, say, an uncontrollable portion, then the response of the uncontrollable portion will appear as noise upon the observations during an identification of the system.

Finally, it is assumed that all systems that are to be identified are asymptotically stable. This implies that the response (output) of the system will essentially go to zero in a finite time after the input has been removed. The ground state is also assumed to be unique and hence, the system will have, essentially, a finite memory. One could hardly hope to identify a system whose output depends in a significant way upon the infinite past history of the input. Only continuous time models are discussed in this report. The corresponding discrete time versions are rather straightforward.

II. THE CHARACTERIZATION OF SYSTEMS

As mentioned in the previous section, the problem of system characterization is concerned primarily with the mathematical representations of a system's input/output relationship. In this section, various mathematical representations (also called models) for linear and nonlinear systems are treated in considerable detail. The transition from one mode of representation to another is also a part of the system characterization problem. However, this segment of the problem is not treated in this report.

A. Linear Time-Invariant Systems

Within the bounds mentioned in the introduction, linear time-invariant systems may be completely characterized by the following models.

1. The Impulse Response – For a given impulse response $h(\ )$, one has an input/output relation in the form

$$y(t) = \int_0^\infty h(\tau)x(t-\tau)d\tau . \qquad (2.1)$$

2. The Transfer Function – Given a transfer function $H(\ )$, one may compute the output for a given input via the relations

$$\begin{aligned} X(s) &= \mathcal{L}[x(t)], \\ Y(s) &= H(s)X(s), \\ y(t) &= \mathcal{L}^{-1}[Y(s)]. \end{aligned} \qquad (2.2)$$

3. The Differential Equation – With the added assumption that the system is "differentiable" and of N^{th} order, one may model it by the constant coefficient linear differential equation

$$\left(\sum_{n=0}^{N} a_n \frac{d^n}{dt^n} \right) y(t) = \left(\sum_{m=0}^{M \leq N} b_m \frac{d^m}{dt^m} \right) x(t) . \qquad (2.3)$$

Alternatively, in state vector form

$$\begin{aligned} G\dot{\underset{\sim}{s}}(t) &= A\underset{\sim}{s}(t) + \underset{\sim}{b}x(t) \\ y(t) &= \langle \underset{\sim}{c}, \underset{\sim}{s}(t) \rangle \end{aligned} \qquad (2.4)$$

where $\underset{\sim}{s}(t)$ is the N-dimensional state vector, G and A are constant N x N matrices, $\underset{\sim}{b}$ and $\underset{\sim}{c}$ are constant N-vectors and $\langle\ ,\ \rangle$ is the inner product. Within the class of completely controllable and completely observable systems, the two models are equivalent. The state vector formulation does have wider applicability and is very much in vogue with present interest in optimal systems. However, the N-th order equation model (2.3) will generally be found preferable in an identification situation.

4. Orthogonal Decompositions – Finally, rather than take the impulse response, one may take its set $\{c_n\}$ of generalized Fourier coefficients as the basic characterization. That is $h(\ )$ is assumed to be expressed in terms of some set $\{\phi_n\}$ of functions orthonormal on $[0, \infty)$ (on $[0, T]$ if the memory span of the system is limited) as

$$h(\tau) = \sum_{n=0}^{\infty} c_n \phi_n(\tau) \qquad (2.5)$$

where

$$c_n = \int_0^{\infty} h(\tau)\, \phi_n(\tau)\, d\tau . \qquad (2.6)$$

Substituting this representation of $h(\ )$ into Eq. (2.1), it is found that the input/output relation can be expressed as

$$y(t) = \sum_{n=0}^{\infty} c_n v_n(t), \tag{2.7}$$

where

$$v_n(t) = \int_0^{\infty} \phi_n(\tau)\, x(t-\tau)\, d\tau . \tag{2.8}$$

Structurally, the model is a parallel bank of orthogonal filters, all driven by the input $x(t)$ and each followed by some gain c_n. The system output is the sum of the various adjusted filter outputs (see Fig. 2.1). Two different systems are thus characterized by two different sets of gains c_n.

In a later section, the choice of base functions $\phi_n(\ )$ will be discussed. The Laguerre set and the orthogonalized exponentials are two sets which have received considerable attention in the engineering literature.

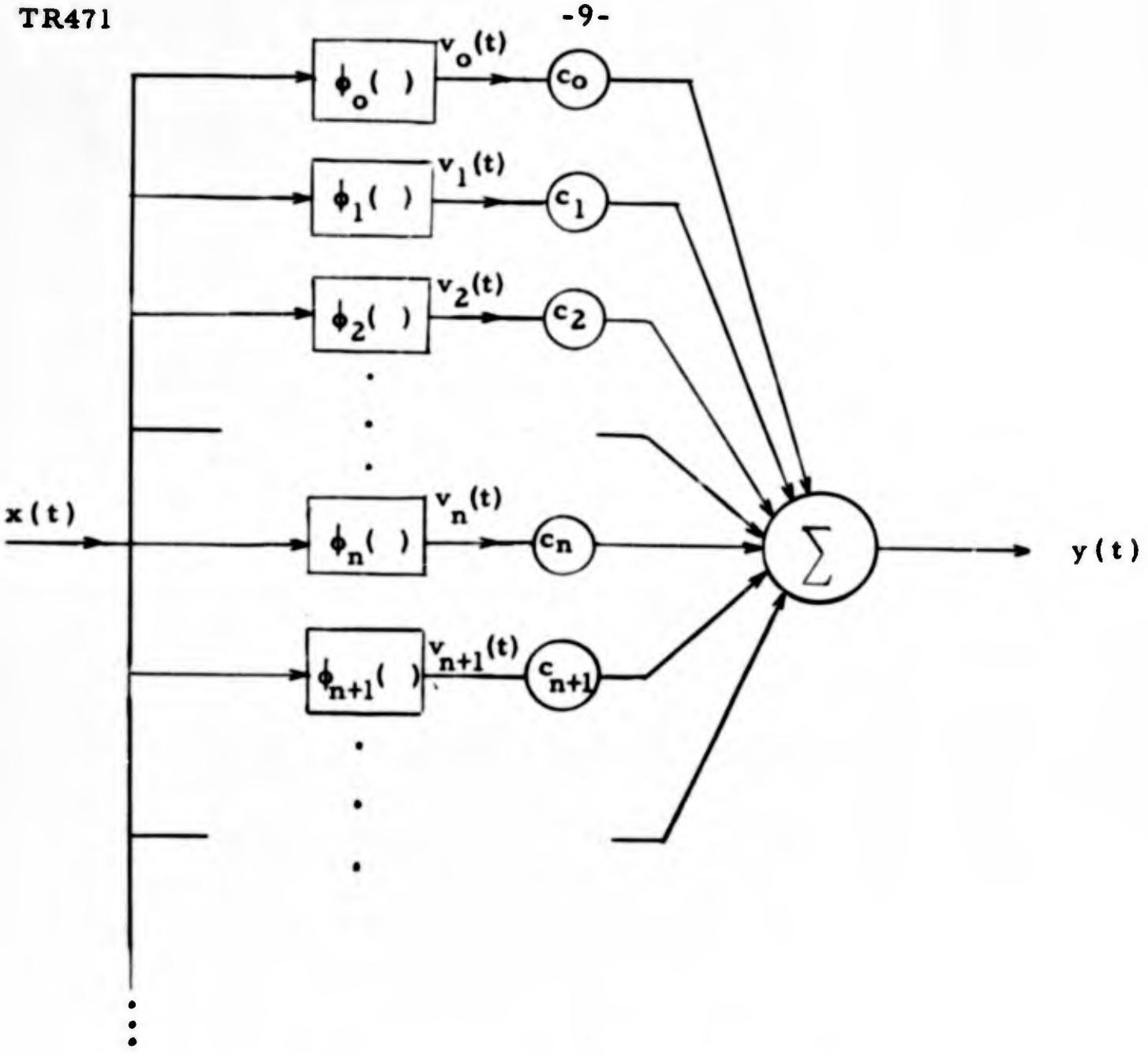

Figure 2.1 An Orthogonal Representation

B. Time-Varying Linear Systems

For time-varying linear systems, let us consider in the same order the generalization of the above models.

1. The Impulse Response [5, 6]. For linear time-varying systems, the impulse response $h(t, \tau)$ is usually defined as the response at time t of a system (initially at rest) to a delta function input at time τ. The corresponding input/output relation for a physical realizable system is given by

$$y(t) = \int_{-\infty}^{t} h(t, \tau)\, x(\tau)\, d\tau \ . \qquad (2.9)$$

However, it is found convenient to introduce an age variable $\xi = t - \tau$ into (2.9). Thus

$$y(t) = \int_{0}^{\infty} h(t, t-\xi)\, x(t-\xi)\, d\xi$$

and hence, define a new* impulse response function, $w(t, \xi) = h(t, t-\xi)$ which is the response at time t due to an impulse applied ξ units before

$$y(t) = \int_{0}^{\infty} w(t, \xi)\, x(t-\xi)\, d\xi \ . \qquad (2.10)$$

2. The Frequency Functions [8]. For a time-variable linear system, the system function (frequency response function) is defined as the Fourier transform

$$H(j\omega, t) = \int_{-\infty}^{\infty} w(t, \xi)\, e^{-j\omega\xi}\, d\xi \ . \qquad (2.11)$$

The corresponding input/output relation is given by

$$y(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(j\omega, t)\, X(j\omega)\, e^{j\omega t}\, d\omega \ , \qquad (2.12)$$

where $X(j\omega)$ is the usual Fourier transform of the input $x(\ )$.

- - - - - - -

*There is still a third impulse response function which may be defined [7].

One may also introduce a twofold transformation of the impulse response called the bifrequency system function defined as

$$\Gamma(j\omega, j\lambda) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} w(t, \xi)\, e^{-j\lambda t - j\omega\xi}\, d\xi dt = \int_{-\infty}^{\infty} H(j\xi, t)\, e^{-j\lambda t}\, dt \quad . \qquad (2.13)$$

With this function, the input/output relation in the frequency domain becomes

$$Y(j\lambda) = \int_{-\infty}^{\infty} \Gamma(j\omega, j\lambda)\, X(j\omega)\, d\omega \; ; \qquad (2.14)$$

the time domain functions being given by the usual Fourier transform pairs.

3. The Differential Equation. The only change in modeling time-varying systems using the differential equation is to make the coefficients time-varying. Thus (2.3) becomes

$$\left(\sum_{n=0}^{N} a_n(t)\, \frac{d^n}{dt^n}\right) y(t) = \left(\sum_{m=0}^{M\leq N} b_m(t)\, \frac{d^m}{dt^m}\right) x(t) \quad . \qquad (2.15)$$

In the state vector form, the coefficient matrices, and vectors now also become time-varying as

$$G(t)\,\dot{\underset{\sim}{s}}(t) = A(t)\,\underset{\sim}{s}(t) + \underset{\sim}{b}(t)\, x(t) \,,$$

$$y(t) = \langle \underset{\sim}{c}(t), \underset{\sim}{s}(t) \rangle \quad . \qquad (2.16)$$

4. Orthogonal Decompositions. Again, as in the time-invariant case, the age dependent impulse response function $w(t, \xi)$ may be decomposed as

$$w(t,\xi) = \sum_{n=0}^{\infty} \theta_n(t)\, \phi_n(\xi) \tag{2.17}$$

where again the set of functions $\phi_n(\ )$ forms an orthonormal set on $[0,\infty)$. Substituting (2.17) into the output relation (2.10), one finds that the output is expressible as

$$y(t) = \sum_{n=0}^{\infty} \theta_n(t)\, v_n(t) \tag{2.18}$$

where again, the $v_n(t)$ are given by (2.8). The resulting model structure is the same as before (Fig. 2.1) except the previous constant gains c_n must now be replaced by the time-varying gains

$$\theta_n(t) = \int_0^{\infty} w(t,\xi)\, \phi_n(\xi)\, d\xi \,. \tag{2.19}$$

C. Nonlinear Time-Invariant Systems

With the consideration of nonlinear systems, one must broaden one's perspective considerably for, by definition, all systems which are not linear are lumped into the nonlinear class. One possible viewpoint is to consider a system mathematically as a nonlinear operator T mapping the input space X into the output space Y. But this is not very fruitful. What is needed is an input/output relation which relates specific input and output functions and which also conveys the evolution of the mapping in time. Symbolically, one wants a relation of the form

$$y(t) = F[x(\tau): -\infty < \tau \leq t] \qquad (2.20)$$

which expresses the value of the system's output at any time t as a function of the past history (i.e., prior to t) of the input. The appropriate mathematical concept is that of a functional which is a mapping from a linear space (the input space X) into the real numbers R (the real value of the system's output at time t). Thus, most of what is discussed in the rest of this section has had its origin in functional analysis.

In retrospect, it should be noted that the general continuous linear functional on the input space X is just (2.10) which specializes in the time-invariant case to (2.1) which was the starting point for linear systems.

One still does not have explicit input/output relations for nonlinear systems. However, just as elementary analysis uses polynomials, Taylor series, orthogonal functions, etc., to study and approximate general functions, functional analysis will yield explicit representation for (2.20) and hence also models for nonlinear systems.

1. The Volterra Series. Consider the system of Fig. 2.2 where $g_1(\ )$ and $g_2(\ )$ are the impulse responses of the linear elements. One may write the system's output as

$$y(t) = \left(\int_0^\infty g_1(\tau_1)\, x(t-\tau_1)\, d\tau_1 \right) \left(\int_0^\infty g_2(\tau_2)\, x(t-\tau_2)\, d\tau_2 \right)$$

$$y(t) = \int_0^\infty \int_0^\infty g_1(\tau_1)\, g_2(\tau_2)\, x(t-\tau_1)\, x(t-\tau_2)\, d\tau_1\, d\tau_2$$

or by symmetry

$$y(t) = \int_0^\infty \int_0^\infty g_1(\tau_2) g_2(\tau_1)\, x(t-\tau_1)\, x(t-\tau_2)\, d\tau_1\, d\tau_2 .$$

If one defines the symmetric kernel

$$h_2(\tau_1, \tau_2) = \tfrac{1}{2}\, g_1(\tau_1)\, g_2(\tau_2) + \tfrac{1}{2}\, g_1(\tau_2)\, g_2(\tau_1) ,$$

the input/output relation may be written in the form

$$y(t) = \int_0^\infty \int_0^\infty h_2(\tau_1, \tau_2)\, x(t-\tau_1)\, x(t-\tau_2)\, d\tau_1\, d\tau_2 . \qquad (2.21)$$

This is a homogeneous functional of second degree in that if the input is changed from $x(\ )$ into $kx(\ )$, the output goes from $y(\ )$ over into $k^2 y(\ )$. Functionals of first degree are just the linear functionals (2.1). The reader can readily see how to synthesize other homogeneous functionals of various degrees. These various homogeneous functionals and their kernels are called Volterra functionals and Volterra kernels, respectively, after Vito Volterra who first introduced them [9] and who also made extensive studies of them [10].

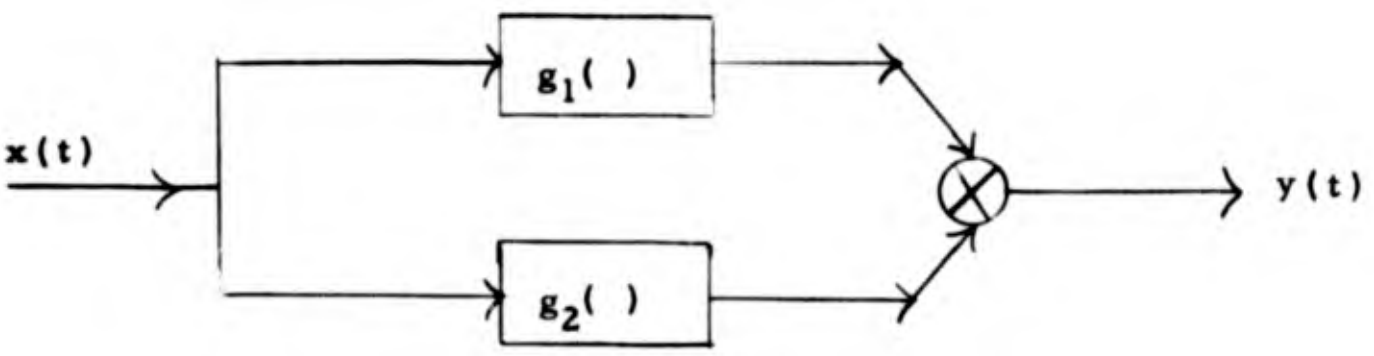

Figure 2.2 A Homogeneous Second Degree Volterra Functional

Volterra also introduced Volterra series

$$y(t) = h_o + \int_0^\infty h_1(\tau_1)\, x(t-\tau_1)\, d\tau_1 + \int_0^\infty \int_0^\infty h_2(\tau_1, \tau_2)\, x(t-\tau_1)\, x(t-\tau_2)\, d\tau_1\, d\tau_2 + \ldots \tag{2.22}$$

which is a functional power series in terms of the various homogeneous functionals. Any well-behaved nonlinear functional (2.20) may be expanded in a Volterra series (2.22) which will have some region of validity. Just as a Taylor series is a representation of a function, a Volterra series is a representation for a functional. However, just as a Taylor series is not very good for approximation purposes over some interval, neither is the Volterra series for any one functional good for approximation over some input space. Yet for functions, one has the Weierstrass approximation theorem which states that any function continuous on a closed interval can be uniformly approximated within any prescribed tolerance (over the interval) by some polynomial. But polynomials result from the truncation of a Taylor series. Similarly, the "polynomials" of functional analysis result from the truncation of Volterra's series. Moreover, the truncated Volterra expansions provide for the generalization of Weierstrass's theorem to continuous functionals as was proved by Frechet [11] in 1910. It is because of this approximation capability that the Volterra functionals are of interest in modeling systems with an eye to future engineering applications. It was Norbert Wiener [12] who first used the Volterra representation in 1942 to treat the response of a nonlinear circuit to noise. Since then there has been an ever increasing rate of application of these ideas to nonlinear systems [13-24].

2. Multidimensional Frequency Transfer Functions. The analysis of linear systems for some inputs is greatly simplified by the introduction of the Fourier or Laplace transform of the impulse response (the transfer function). Analogous results also hold for the nonlinear Volterra functionals but one must use multidimensional transforms of the Volterra kernels. These were also introduced by Wiener [12] in 1942 where their utility in calculating output statistics was demonstrated. Since George [25] has given a fairly complete description of the technique, the general case will not be treated here, but a third-order system will be discussed to illustrate most of the important points.

Thus, consider a system characterized by a third degree Volterra functional

$$y(t) = \int_0^\infty \int_0^\infty \int_0^\infty h_3(\tau_1, \tau_2, \tau_3)\, x(t-\tau_1)\, x(t-\tau_2)\, x(t-\tau_3)\, d\tau_1\, d\tau_2\, d\tau_3 \,. \quad (2.23)$$

Introducing the three-dimensional frequency function

$$H_3(j\omega_1, j\omega_2, j\omega_3) = \int_{-\infty}^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty h(t_1, t_2, t_3) \exp\left\{-j\omega_1 t_1 - j\omega_2 t_2 - j\omega_3 t_3\right\} dt_1\, dt_2\, dt_3 \,. \quad (2.24)$$

One cannot, however, transform Eq. (2.23) directly in that it is not a function of three arguments. As a subterfuge, introduce the functional

$$z(t_1, t_2, t_3) = \int_0^\infty \int_0^\infty \int_0^\infty h_3(\tau_1, \tau_2, \tau_3)\, x(t_1-\tau_1)\, x(t_2-\tau_2)\, x(t_3-\tau_3)\, d\tau_1\, d\tau_2 d\tau_3$$

and then apply the Fourier transformation. One finds that

$$Z(j\omega_1, j\omega_2, j\omega_3) = H_3(j\omega_1, j\omega_2, j\omega_3)\, X(j\omega_1)\, X(j\omega_2)\, X(j\omega_3) \ . \qquad (2.25)$$

To get $y(t)$, one could invert (2.25) to get $z(t_1, t_2, t_3)$ and then set $t = t_1 = t_2 = t_3$. However, this association may be done in the frequency domain. Thus considering

$$y(t) = z(t, t, t) = \left(\frac{1}{2\pi}\right)^3 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} Z(j\omega_1, j\omega_2, j\omega_3) \exp\left\{ j(\omega_1+\omega_2+\omega_3)t \right\} d\omega_1 d\omega_2 d\omega_3 \ ,$$

and substituting $\omega = \omega_1+\omega_2+\omega_3$, one finds that

$$y(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{j\omega t} \left\{ \left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} Z(j\omega_1, j\omega_2, j\omega - j\omega_1 - j\omega_2)\, d\omega_1\, d\omega_2 \right\} d\omega$$

or

$$Y(j\omega) = \left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} Z(j\omega_1, j\omega_2, j\omega - j\omega_1 - j\omega_2)\, d\omega_1\, d\omega_2 \ . \qquad (2.26)$$

Note that from the symmetry of $h_3(\ldots)$ and thus of $Z(\ldots)$, Eq. (2.26) is only one of many possible forms that the frequency relation may take.

In summary, given a system characterized by the frequency function $H_3(j\omega_1, j\omega_2, j\omega_3)$, one may calculate the output for a given input by the relations

$$X(j\omega) = \mathcal{F}[x(t)] = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt \ ,$$

$$Y(j\omega) = \left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H_3(j\omega_1, j\omega_2, j\omega - j\omega_1 - j\omega_2)\, X(j\omega_1)\, X(j\omega_2)\, X(j\omega - j\omega_1 - j\omega_2)\, d\omega_1 d\omega_2, \qquad (2.27)$$

$$y(t) = \mathcal{F}^{-1}[Y(j\omega)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} Y(j\omega)\, e^{j\omega t}\, d\omega \ .$$

This system model is very effective in calculating the response to sinusoidal inputs or the output statistics in the case of white noise excitation.

3. The Differential Equation In the past, the differential equation has been the major vehicle for modeling nonlinear systems. The general implicit expression for the differential equation is usually written as

$$y^{(n)}(t) = f\left(y^{(n-1)}, \ldots, y', y, x\right) \tag{2.28}$$

or in state vector form as

$$\dot{\underset{\sim}{y}} = \underset{\sim}{f}(\underset{\sim}{y}, x) . \tag{2.29}$$

But neither (2.28)nor (2.29) gives a model in which one may identify a given unknown system. To get an explicit class of nonlinear differential models, one must select special cases of (2.28) or (2.29) which contain indeterminate parameters.

For example, on the basis of a priori knowledge, one may want to model a given system in terms of the class of, say, the Riccati equation

$$\dot{y}(t) + ay^2(t) + by(t) = x(t) , \tag{2.30}$$

where the coefficients a and b distinguish one member of the class from another. Other well-known parametric forms of (2.28) are Duffing's equation, van der Pol's equation, Rayleigh's equation, etc. The parametric form of (2.29) may be written as

$$\dot{\underset{\sim}{y}}(t) = \underset{\sim}{g}(\underset{\sim}{y}, x; \underset{\sim}{a}) \tag{2.31}$$

where the functional form of $g(\ldots)$ is known, but the vector $\underset{\sim}{a}$ of parameters characterizes a particular system.

4. Orthogonal Expansions. For anyone who has had to approximate functions by polynomials or to fit analytic expressions to empirical data, the value of orthogonality is well known. To fit a general N-th order polynomial to a set of data in say a least-squares sense requires the solution of an N-th order system of algebraic equations which, moreover, is very poorly conditioned in the mathematical sense for even moderate values of N. Much of the same is true in trying to fit a truncated Volterra expansion to a nonlinear functional, except now the difficulties are compounded even more (c.f. next section) in that the system of equations which must now be solved is a set of Wiener-Hopf integral equations. Through orthogonality, however, one can eliminate the poor conditioning and also the requirement of solving systems of equations. The coefficients in an orthogonal expansion are determined by simple averages. Therefore, there is a need to define a concept of orthogonality for functionals.

Orthogonality is basically an inner product space [26] concept. Two functions from an inner product space (with inner product $< , >$) are orthogonal ($\perp$) when their inner product is zero. For example, for $f, g \in L_2[a, b]$

$$< f, g > = \int_a^b f(t)\, g(t)\, w(t)\, dt = 0 \Rightarrow f \perp g ,$$

or more generally, for any measure space (S, μ)

$$< f, g > = \int_D f \cdot g^* \, d\mu = 0 \Rightarrow f \perp g .$$

Note, that in either case, the inner product contains two essential features: the domain of the functions plus some weighting function or measure. As a domain for functionals, take an ensemble of stationary ergodic processes $X = \{x(t)\}$ with their probability distribution as a measure. Then a well-defined inner product for functionals Φ and Ψ in the space of functions $\mathcal{F}_X$ and X is the ensemble average of their product

$$<\Phi, \Psi>_X = \mathcal{E}_X\{\Phi[x(t)]\ \Psi[x(t)]\} . \tag{2.32}$$

Orthogonality between functionals may now be defined as follows: two functionals Φ and Ψ are orthogonal relative to the ensemble X if

$$<\Phi, \Psi>_X = 0 . \tag{2.33}$$

Moreover, one may say that a set of functionals $\{\Phi_n\}$ forms an orthogonal set over X if

$$<\Phi_m, \Phi_n>_X = \delta_{mn} . \tag{2.34}$$

Finally, given some space $\mathcal{F}_X$ of functionals over X containing some orthonormal set of functionals Φ_n, one may expand an arbitrary member $F[\ ] \in \mathcal{F}_X$ as

$$F[x(t)] = \sum_n A_n \Phi_n[x(t)] \tag{2.35}$$

and, as for functions, the coefficients are determined by the projections

$$A_n = <F, \Phi_n>_X . \tag{2.36}$$

Moreover, because of the assumed ergodicity, the above ensemble averages may be obtained by simple time averages

$$A_n = \overline{F[x(t)]\,\Phi_n[x(t)]}\,.$$

Some specific examples of orthogonal functions may now be considered.

a. Cameron and Martin [27]. In 1947 Cameron and Martin published a paper entitled "The Orthogonal Development of Nonlinear Functionals in Series of Fourier-Hermite Functionals." Basically, their idea was to split the representation for a nonlinear functional into two parts; first, they decomposed the input $x(\ )$ into a generalized Fourier expansion which was then followed by a Hermite expansion of the Fourier coefficients. Engineering-wise, this can be thought of as splitting of a nonlinear system into a linear system with memory followed by a nonlinear no-memory system. For this decomposition to work, however, one can see that one must choose X to be a Gaussian process, for this is the only process which is preserved under linear transformations (i.e., in the memory portion). Moreover, the Gaussianness of the process dictates the choice of the Hermite functions, as will be shown below.

Specifically, Cameron and Martin treated the case of $x(\ )$ from the space $C[0, 1]$ of real functions continuous on $[0, 1]$ with the Wiener measure (now generally known as a Wiener space). They then decomposed $x(\ )$ in terms of any set of functions $\{\phi_n(t)\}$ orthonormal on $[0, 1]$ with the generalized Fourier coefficients given by generalized Stieltjes integrals.

This is equivalent to assuming X to be a white Gaussian process and for the purpose of this report, one must accept the fact that any sample function $x(\ )$ on $[0, 1]$ may be expressed by its generalized Fourier coefficient.*

*For the reader who objects to this decomposition of white noise, we suggest that he pursue the original paper [27].

$$v_n = \int_0^1 \phi_n(\tau)\, x(\tau)\, d\tau \,. \tag{2.37}$$

Since the set of coefficients $\{ v_n \}$ characterizes $x(\ )$ on $[0, 1]$, any functional of x over this same interval should be expressible in the limit as $N \to \infty$ as an ordinary function $f(v_0, v_1, \ldots, v_N)$ of the v_n's. For reasons that will soon become apparent, it is convenient to express this function as a multidimensional Hermite expansion in the coefficients v_0 through v_N. Thus define

$$\Phi_{ij\ldots\ell}[x] = H_i(v_0)\, H_j(v_1) \ldots H_\ell(v_N) \tag{2.38}$$

where $H_k(u)$ denotes the partially normalized Hermite polynomials defined by

$$H_k(u) = (-1)^k\, 2^{-k/2}\, \frac{e^{u^2}}{\sqrt{k!}}\, \frac{d^k}{du^k} \left\{ e^{-u^2} \right\} , \tag{2.39}$$

and satisfying the orthogonality relation

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} H_m(u)\, H_n(u)\, e^{-u^2}\, du = \delta_{mn} \,. \tag{2.40}$$

The orthogonality of the Φ's can now be demonstrated. By definition

$$\langle \Phi_{m_0 m_1 \ldots m_N}, \Phi_{n_0 n_1 \ldots n_N} \rangle = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \Phi_{\underset{\sim}{m}} \Phi_{\underset{\sim}{n}}\, p(v_0, v_1, \ldots, v_N)\, dv_0\, dv_1 \ldots dv_N \tag{2.41}$$

where $p(v_0, v_1, \ldots v_N)$ is the joint probability distribution of the v_n's. But for the assumed white Gaussian input $x(\ )$, the v_n's form an independent jointly Gaussian distributed set of random variables of zero expectation. Also, by the normality of the ϕ_n's, one can show that all the v_n's have the same variance which may be assumed to have been adjusted to equal one half. Thus

$$p(v_0, v_1, \ldots, v_N) = \left(\frac{1}{\sqrt{\pi}}\right)^{N+1} \exp\left\{-\sum_{n=0}^{N} v_n^2\right\} . \qquad (2.42)$$

Substituting this probability density into (2.41) and using the separability of both the exponential function and the Φ's into Hermite polynomials, one may write (2.41) as a product of N+1 separate integrals. The reason for the choice of the Hermite expansion now becomes apparent. The joint Gaussian probability density (2.42) supplies the necessary exponential factor in each of the factor integrals to enable one to take advantage of the orthogonality (2.40) between the individual Hermite polynomials. Thus, one arrives at the orthogonality of the Φ's

$$\langle \Phi_{m_0 m_1 \ldots m_N}, \Phi_{n_0 n_1 \ldots n_N} \rangle = \delta_{m_0 n_0} \delta_{m_1 n_1} \ldots \delta_{m_N n_N} . \qquad (2.43)$$

Returning to the expansion of $f(v_0, v_1, \ldots, v_N)$, by the completeness of the Hermite polynomials, one can write

$$f(v_0, v_1, \ldots, v_N) = \sum_{m_0=0}^{\infty} \sum_{m_1=0}^{\infty} \ldots \sum_{m_N=0}^{\infty} a_{m_0 m_1 \ldots m_N} \Phi_{m_0 m_1 \ldots m_N}$$

where

$$a_{m_0 m_1 \cdots m_N} = \mathcal{E}_X \left\{ f(v_0, v_1, \ldots, V_N)\, \Phi_{m_0 m_1 \cdots m_N} \right\} .$$

Finally, putting all these results together, as $N \to \infty$,

$$f(v_0, v_1, \ldots, v_N) \to F[x(\tau): 0 \leq \tau \leq 1]$$

with

$$a_{m_0 m_1 \cdots m_N} \to A_{m_0 m_1 \cdots m_N} = \langle F, \Phi_{m_0 m_1 \cdots m_N} \rangle_X .$$

That is, the functional $F[\ ]$ may be represented as

$$F[x(\tau): 0 \leq \tau \leq 1] = \lim_{N \to \infty} \sum_{m_0=0}^{\infty} \sum_{m_1=0}^{\infty} \cdots \sum_{m_N=0}^{\infty} A_{m_0 m_1 \cdots m_N} H_{m_0}(v_0) H_{m_1}(v_1) \cdots H_{m_N}(v_N) \tag{2.44}$$

where the v_n's are given by (2.37) and

$$A_{m_0 m_1 \cdots m_N} = \langle F[\ ], H_{m_0} H_{m_1} \cdots H_{m_N} \rangle_X . \tag{2.45}$$

b. Wiener [28]. It was Wiener who saw the implication of Cameron and Martin's paper to nonlinear systems. For the Fourier (or memory) portion of the expansion, he took the Laguerre functions [29] which form a complete orthonormal system on $[0, \infty)$. With these functions, one may represent at any time t, the entire past of an input function $x(\ )$ by the coefficients

$$v_n(t) = \int_0^{\infty} \ell_n(\tau)\, x(t - \tau)\, d\tau , \quad n = 0, 1, \ldots . \tag{2.46}$$

The Laguerre functions are physically realizable as filter responses and, moreover, because of their recursive nature, the $v_n(t)$, may be generated in a ladder network called a Laguerre network [30].

Then conceiving a nonlinear system's output at each instant of time t to be some functional of the system's input (up to that time)

$$y(t) = F[x(\tau): -\infty < \tau \leq t],$$

the Cameron and Martin expansion may be used. Thus,

$$y(t) = \lim_{N\to\infty} \sum_{m_0=0}^{\infty} \sum_{m_1=0}^{\infty} \cdots \sum_{m_N=0}^{\infty} A_{m_0 m_1 \cdots m_N} H_{m_0}(v_0(t)) H_{m_1}(v_1(t)) \cdots H_{m_N}(v_N(t)) \tag{2.47}$$

where now the v_n's are given by (2.46) and where again one must use white Gaussian noise as a test probe to determine the coefficients

$$A_{m_0 m_1 \cdots m_N} = \overline{y(t) H_{m_0}(v_0(t)) H_{m_1}(v_1(t)) \cdots H_{m_N}(v_N(t))} \tag{2.48}$$

by the time averages (for engineering purposes). Figure 2.3 gives a schematic presentation of Wiener's representation of a nonlinear system. The coefficients $A_{m_0 m_1 \cdots m_N}$ (sometimes called Wiener coefficients) distinguish one system from another.

c. Bose [31-33]. One of the difficulties of implementing Wiener's characterization is the large number of multipliers required in the synthesis of the Hermite polynomials. As an alternative to the Hermite polynomials, A. G. Bose proposed the use of a set of "gate functions"

$$\phi_k(v) = \begin{cases} 1 & \text{if } v \text{ is in the k-th subinterval} \\ 0 & \text{otherwise} \end{cases} \tag{2.49}$$

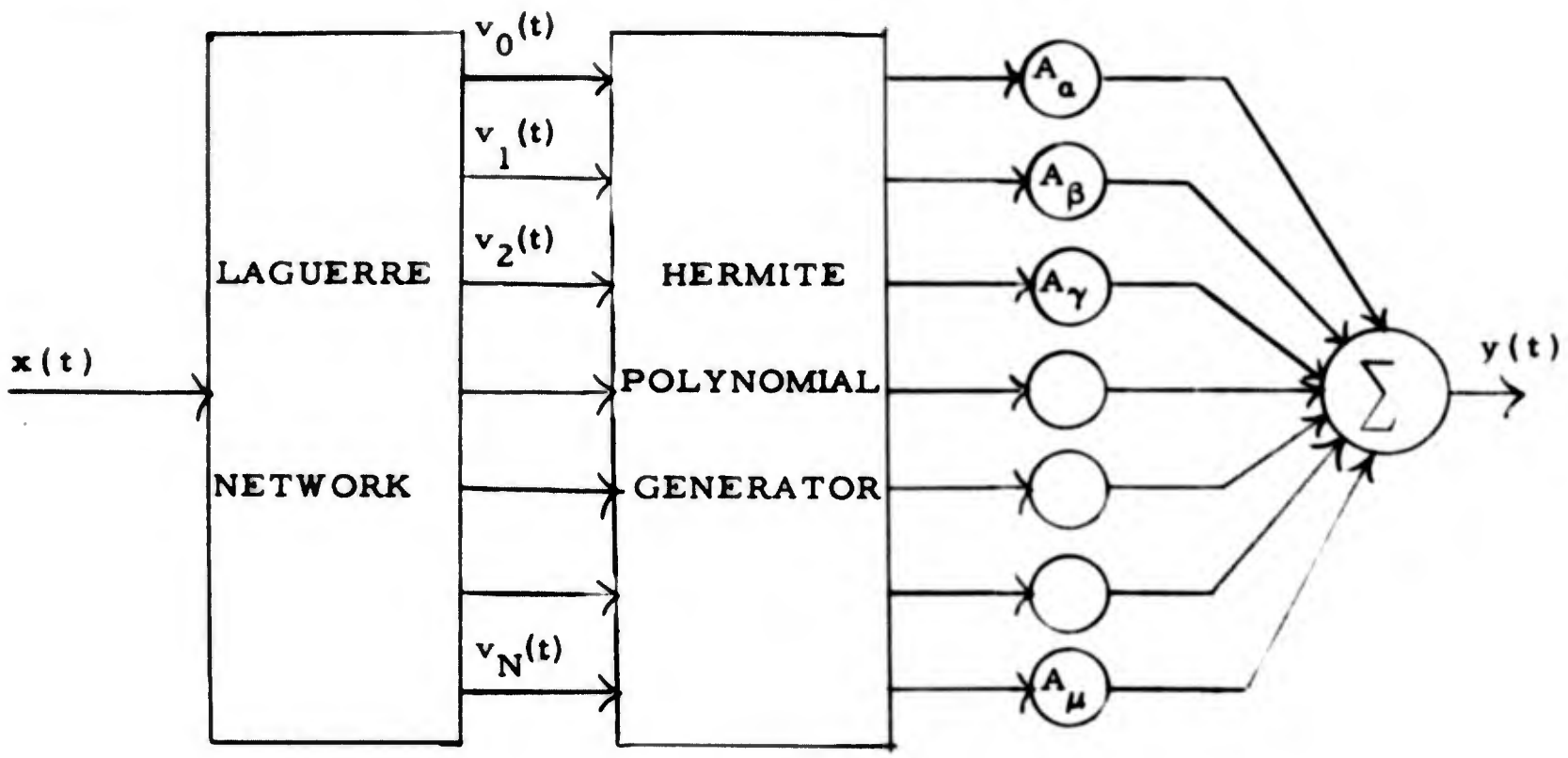

Figure 2.3 The Wiener I Structure

which may be implemented in a more practical fashion as level selection circuits. Bose preserved the Laguerre decomposition of the input. For any bounded input, it is easy to show that the $v_n(t)$ are bounded, say, on $[-V, V]$. Hence, divide $[-V, V]$ into M subintervals (equal or not) with a corresponding set of gate functions. Then analogous to (2.47), one has Bose's representation

$$y(t) = \lim_{\substack{N\to\infty \\ M\to\infty}} \sum_{k_0=1}^{M} \sum_{k_1=1}^{M} \cdots \sum_{k_N=1}^{M} B_{k_0 k_1 \ldots k_N} \phi_{k_0}(v_0(t))\, \phi_{k_1}(v_1(t)) \ldots \phi_{k_N}(v_N(t)), \tag{2.50}$$

where now the coefficients may be determined by the quotient of time averages

$$B_{k_0 k_1 \cdots k_N} = \frac{\overline{y(t)\,\phi_{k_0}(v_0(t))\,\phi_{k_1}(v_1(t))\cdots\phi_{k_N}(v_N(t))}}{\overline{\phi_{k_0}(v_0(t))\,\phi_{k_1}(v_1(t))\cdots\phi_{k_N}(v_N(t))}} \qquad (2.51)$$

for any ergodic process with suitable bandwidth as input. The average in the denominator of (2.51) is required because the gate functions have not been normalized. At first sight, this appears as a real improvement in that one has freed oneself from the requirement of a Gaussian probe function. However, (2.50) is not a true orthogonal functional expansion. As Bose rightly points out, one can think of the Laguerre decomposition (2.46) at any instant as yielding a point $(v_0(t), v_1(t), \ldots, v_N(t))$ in a parameter space characterizing the past history of the input. The multidimensional product of gate functions is just a shorthand for the quantization of this parameter space into cells, and the coefficients $B_{k_0 k_1 \cdots k_N}$ are just the mean value of the output conditioned by the input being represented by a point in the $k_0 k_1 \cdots k_N$-th cell. That is, the coefficients are not derivable by averages which depend upon orthogonal functionals; they are just the measured conditioned means

$$B_{k_0 k_1 \cdots k_N} = \mathcal{E}\left\{y(t) \,|\, x(t) \text{ in } k_0 k_1 \cdots k_N\text{-th cell}\right\} , \qquad (2.52)$$

for a particular parameterization of the past of the input (Bose actually treats the case of a weighted conditional mean).

d. Barrett [34]. Motivated by the use of Hermite polynomials in both the study of Gaussian noise through zero-memory nonlinear devices and the Wiener I expansion, Barrett in 1955 recognized the value of some multivariate Hermite polynomials which had been defined by Grad [35]. In a very heuristic fashion, Barrett arrived at a set of Hermite functionals as the limit of Grad's polynomials as the N-variate argument $\underset{\sim}{x}$ became a continuous function $x(t)$. Then, putting these zero-memory "partially" orthogonal functionals into a functional power series similar to the Volterra series, Barrett arrived at another method for system characterization.

As an introduction to Grad's contribution, consider the set of Hermite polynomials (the usual kind, but different from those defined by (2.39) above) defined by

$$w(x)\, H_n(x) = (-1)^n \frac{d^n}{dx^n} w(x)\,, \tag{2.53}$$

where the weighting function is given by

$$w(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2} x^2\right). \tag{2.54}$$

The first few resulting polynomials are

$$\begin{aligned} H_0(x) &= 1\,, & H_2(x) &= x^2 - 1\,, \\ H_1(x) &= x\,, & H_3(x) &= x^3 - 3x\,. \end{aligned} \tag{2.55}$$

These polynomials also satisfy the orthogonality condition

$$\int_{-\infty}^{\infty} H_m(x)\, H_n(x)\, w(x)\, dx = (n!)\ \delta_{mn}\,. \tag{2.56}$$

In an analogous fashion, for the N-dimensional vector $\underset{\sim}{x}$ (with components x_i), Grad [35] defines the N-variate Hermite polynomials of n-th order by

$$w_N(\underset{\sim}{x})\, \mathcal{H}^{(n)}(\underset{\sim}{x}) = (-1)^n \frac{\partial^n}{\partial x_{i_1} \partial x_{i_2} \cdots \partial x_{i_n}} w_N(\underset{\sim}{x}) , \tag{2.57}$$

where

$$w_N(\underset{\sim}{x}) = \left(\frac{1}{\sqrt{2\pi}}\right)^N \exp\left\{-\frac{1}{2}\sum_{i=1}^{N} x_i^2\right\} . \tag{2.58}$$

The resulting n-th order polynomial is a tensor of order n . The first few polynomials expressed in component form are

$$\mathcal{H}^{(0)} = 1 , \quad \mathcal{H}_i^{(1)} = x_i , \quad \mathcal{H}_{ij}^{(2)} = x_i x_j - \delta_{ij} , \tag{2.59}$$

$$\mathcal{H}_{ijk}^{(3)} = x_i x_j x_k - (x_i \delta_{jk} + x_j \delta_{ki} + x_k \delta_{ij}) .$$

The corresponding orthogonality condition is given by

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \mathcal{H}_{\underset{\sim}{i}_m}^{(m)} \mathcal{H}_{\underset{\sim}{i}_n}^{(n)} w_N(\underset{\sim}{x})\, d\underset{\sim}{x} = \begin{cases} 0 & m \neq n \\ 0 & m = n,\ \underset{\sim}{i}_m \text{ not a permutation of } \underset{\sim}{i}_n \\ n! & m = n,\ \underset{\sim}{i}_m \text{ a permutation of } \underset{\sim}{i}_n \end{cases} \tag{2.60}$$

At this point, it is observed that (2.58) is the joint probability density for N-samples from a white Gaussian process $x(t)$ of unit variance. As $N \to \infty$, one might conceive recovering $x(t)$ from the N-variate vector $\underset{\sim}{x}$ and the N-variate Hermite polynomials yielding the following Hermite functionals

$$\mathcal{H}^{(0)}[x(\ )] = 1 \qquad \mathcal{H}^{(1)}[x(\ )] = x(t)$$

$$\mathcal{H}^{(2)}[x(\ )] = x(t_1)\, x(t_2) - \delta(t_2 - t_1) \tag{2.61}$$

$$\mathcal{H}^{(3)}[x(\ )] = x(t_1)\, x(t_2)\, x(t_3) - x(t_1)\, \delta(t_3 - t_2) + x(t_2)\, \delta(t_1 - t_3) + x(t_3)\, \delta(t_2 - t_1)$$

The orthogonality integral must now be interpreted as a mean value or expectation with the final form given by

$$\mathcal{E}\left\{\mathcal{K}^{(m)}[x(\ );\, t_1, t_2, \ldots, t_m]\,\mathcal{K}^{(n)}[x(\ );\, \tau_1, \tau_2, \ldots, \tau_n]\right\} = \begin{cases} 0 & m \neq n \\ \sum \prod^{n} \delta(t_i - \tau_j) & m = n \end{cases} \tag{2.62}$$

where $\sum \prod^{n}$ stands for the sum over all permutations of the t, τ subscripts in the n-dimensional product of delta functions.

Having arrived at the Hermite functionals, Barrett proceeded to represent the general nonlinear functional (or system) by the functional power series

$$F[x(\tau): -\infty < \tau \leq t] = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} k_n(t; \tau_1, \tau_2, \ldots, \tau_n) \cdot$$

$$\cdot\, \mathcal{K}^{(n)}[x(\ );\, \tau_1, \tau_2, \ldots, \tau_n]\, d\tau_1\, d\tau_2 \cdots d\tau_n \tag{2.63}$$

where by the time-invariance, the kernels could also be written as

$$k_n(t; \tau_1, \tau_2, \ldots, \tau_n) = g_n(t-\tau_1,\ t-\tau_2,\ \ldots,\ t-\tau_n)\,. \tag{2.64}$$

By multiplying both sides of Eq. (2.63) by $\mathcal{K}^{(n)}[x(\ );\, \xi_1, \xi_2, \ldots, \xi_n]$ and taking expectation, one obtains by using the orthogonality condition (2.62)

$$k_n(t; \tau_1, \tau_2, \ldots, \tau_n) = \mathcal{E}\left\{F[x(\ )]\,\mathcal{K}^{(n)}[x(\ );\, \tau_1, \tau_2, \ldots, \tau_n]\right\}\,. \tag{2.65}$$

Thus, finally, Barrett's representation for a system reduces to

$$y(t) = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} k_n(t; \tau_1, \tau_2, \cdots \tau_n) \cdot$$

$$\cdot \mathcal{K}^{(n)}[x(\ ); \tau_1, \tau_2, \ldots, \tau_n] d\tau_1 d\tau_2 \ldots d\tau_n , \qquad (2.66)$$

where the kernels are determined by the averages

$$k_n(t; \tau_1, \tau_2, \ldots, \tau_n) = \mathcal{E}\left\{ y(t) \mathcal{K}^{(n)}[x(\ ); \tau_1, \tau_2, \ldots, \tau_n] \right\} \qquad (2.67)$$

where the excitation $x(\ )$ is a white Gaussian process of unit variance. Barrett has also indicated the procedure to generalize the above characterization to colored Gaussian noise as a probe function.

e. Wiener II [36]. Given any set of elements from an inner product space, one may construct by the Gram-Schmidt procedure [37] a set of mutually orthogonal vectors or functions, as the case may be. All the well-known orthogonal polynomials may be generated from the set of monomial powers of x, i.e.,

$$1, x, x^2, x^3, \ldots, x^n, \ldots, \qquad (2.68)$$

by this procedure with a suitable inner product. As an example, the Hermite polynomials of the previous subsection may be generated by this procedure with the inner product

$$< p_m(x), p_n(x) > = \int_{-\infty}^{\infty} p_m(x)\, p_n(x)\, e^{-x^2/2}\, dx . \qquad (2.69)$$

For the zero degree polynomial, simply take

$$H_0(x) = 1 .$$

For the first degree, take

$$H_1(x) = x + a_1$$

and determine a_1 such that $< H_0 , H_1 > = 0$. Thus,

$$< H_0 , H_1 > = \int_{-\infty}^{\infty} 1 \, (x + a_1) \, e^{-x^2/2} \, dx = a_1 \sqrt{2\pi} = 0 \Rightarrow a_1 = 0 .$$

Hence,

$$H_1(x) = x .$$

For the second degree polynomial, take

$$H_2(x) = x^2 + b_2 x + a_2$$

and a_2 and b_2 are determined by the requirements that

$$< H_0 , H_2 > = \int_{-\infty}^{\infty} 1 \cdot (x^2 + b_2 x + a_2) \, e^{-x^2/2} \, dx = (1 + a_2) \sqrt{2\pi} = 0 \Rightarrow a_2 = -1,$$

$$< H_0 , H_1 > = \int_{-\infty}^{\infty} x \cdot (x^2 + b_2 x + a_2) \, e^{-x^2/2} \, dx = b_2 \sqrt{2\pi} = 0 \Rightarrow b_2 = 0 .$$

Hence, as before

$$H_2(x) = x^2 - 1 .$$

Similarly, to continue the process, one sets

$$H_3(x) = x^3 + c_3 x^2 + b_3 x + a_3$$

and determines a_3 , b_3 , and c_3 by the requirements that

$$< H_0 , H_3 > = 0 , \quad < H_1 , H_3 > = 0 , \quad < H_2 , H_3 > = 0 .$$

These details are included here for two reasons. First, to point out how simple the idea behind the Gram-Schmitt procedure is, and secondly, because one may carry over to functionals just about everything once one has noted that the homogeneous Volterra functionals are to the "polynomial" functionals as the monomial powers of x are to polynomials. One might wonder why this quite natural procedure was not the first to be applied to obtain orthogonal functionals, but it remained for Wiener (in 1958) to carry this through in detail. In fairness to Barrett, we should point out that his earlier results are essentially the same, but his method of derivation was heuristic. What he probably lacked was an adequate definition for the inner product of two functionals. Wiener, on the other hand, had at his command a very adequate integration process based upon a measure (Wiener measure) derived from Brownian motion functions. The interested reader may pursue the rigorous development in Wiener's monograph [36]. For the purposes of this report, however, one can interpret Wiener's differential of a Brownian motion function $dx(t, \alpha)$ as a sample function $x(t)$ from a white Gaussian process of zero means and unit variance. With this interpretation, Wiener's orthogonal G-functionals may now be derived in a way which should be quite acceptable to an engineer, while still preserving the spirit of Wiener's development.

As a starting point, take the set of homogeneous Volterra functionals of a white Gaussian process, namely

$$\mathcal{F}_n[x(\ )] = \int_0^\infty \int_0^\infty \cdots \int_0^\infty h_n(\tau_1, \tau_2, \ldots, \tau_n)\, x(t-\tau_1)\, x(t-\tau_2) \ldots x(t-\tau_n)\, d\tau_1 d\tau_2 \ldots d\tau_n, \tag{2.70}$$

with the expectation operator upon the product of two functionals as in an inner product. Moreover, implicit in its definition, white Gaussian noise is also stationary and ergodic. Thus, one may replace the expectation operator by time averages (denoted by a superposed bar). Proceeding with the Gram-Schmidt orthogonalization process, one may take the zero-order functional to be the constant h_0. Also, independent of the Gram-Schmidt process, all functionals may be normalized to have unity norm square. Thus, the zero degree functional is taken to be

$$G_0[h_0;\ x(\ )] = 1 . \tag{2.71}$$

For the first degree functional, write

$$G_1[\ldots;\ x(\ )] = h_1 + \int_0^\infty h_1(\tau)\, x(t-\tau)\, d\tau$$

where the constant h_1 is determined by the requirement that $< G_1 , G_0 > = 0$. Since

$$< G_1 , G_0 > = \mathcal{E}\left\{ \int_0^\infty h_1(\tau)\, x(t-\tau)\, d\tau \right\} = h_1 + \int_0^\infty h_1(\tau)\, \overline{x(t-\tau)} d\tau = h_1$$

$h_1 = 0$ and to normalize G_1, one requires that

$$< G_1 , G_1 > = \mathcal{E}\left\{ \int_0^\infty h_1(\tau)\, x(t-\tau)\, d\tau \int_0^\infty h_1(\tau)\, x(t-\xi)\, d\xi \right\} = 1 .$$

Since

$$< G_1 , G_1 > = \int_0^\infty \int_0^\infty h_1(\tau)\, h_1(\xi)\, x(t-\tau) x(t-\xi)\, d\tau\, d\xi = \int_0^\infty \int_0^\infty h_1(\tau) h_1(\xi)\, \delta(\tau-\xi)\, d\tau\, d\xi$$

$$= \int_0^\infty h_1(\tau)\, h_1(\tau)\, d\tau = \int_0^\infty h_1^{\,2}(\tau)\, d\tau ,$$

then, the first degree G-functional is

$$G_1 [h_1(\ );\, x(\ )] = \int_0^\infty h_1(\tau)\, x(t-\tau)\, d\tau \qquad (2.72)$$

where the kernel $h_1(\ )$ is subject to the normalizing condition

$$\int_0^\infty h_1^{\,2}(\tau)\, d\tau = 1 . \qquad (2.73)$$

Again, for the second degree functional

$$G_2[\ldots; x(\ )] = h_2 + \int_0^\infty h_2(\tau)\, x(t-\tau) d\tau + \int_0^\infty \int_0^\infty h_2(\tau_1, \tau_2) x(t-\tau_1)\, x(t-\tau_2)\, d\tau_1\, d\tau_2 ,$$

with the constant h_2 and the linear kernel $h_2(\ )$ determined by the requirements that

$$< G_2 , G_0 > = \mathcal{E}\left\{G_2\right\} = h_2 + \int_0^\infty h_2(\tau, \tau)\, d\tau = 0$$

and

$$< G_2 , G_1 > = \mathcal{E}\left\{ h_2 \int_0^\infty h_1(\xi)\, x(t-\xi)\, d\xi + \int_0^\infty \int_0^\infty h_2(\tau) h_1(\xi) x(t-\tau) x(t-\xi)\, d\xi\, d\tau + \right.$$

$$\left. + \int_0^\infty \int_0^\infty \int_0^\infty h_2(\tau_1, \tau_2)\, h_1(\xi)\, x(t-\tau_1)\, x(t-\tau_2)\, x(t-\xi)\, d\tau_1\, d\tau_2\, d\xi \right\} = 0 .$$

The first and third integrals of $< G_2, G_1 >$ are zero, since all odd moments of the Gaussian process vanish, thus requiring

$$< G_2, G_1 > = \int_0^\infty h_2(\tau)\, h_1(\tau)\, d\tau = 0 \quad ,$$

but since the linear kernel $h_1(\ )$ is arbitrary (except for normalizing condition), $h_2(\ )$ must vanish identically for the above equation to hold. Moreover, from the first condition above

$$h_2 = -\int_0^\infty h_2(\tau, \tau)\, d\tau \ .$$

Hence, the second degree G-functional is

$$G_2[h_2(\ ,\ ); x(\ )] = \int_0^\infty \int_0^\infty h_2(\tau_1, \tau_2)\, x(t-\tau_1)\, x(t-\tau_2)\, d\tau_1 d\tau_2 - \int_0^\infty h_2(\tau, \tau) d\tau \qquad (2.74)$$

which is orthogonal to all constants and linear functionals. Before proceeding to normalize $G_2[\ ]$, however, a lemma on the higher moment functions for a Gaussian process will be needed.

Lemma. Let $X_1, X_2, \ldots, X_N$ be a set of correlated Gaussian variates of zero means and covariances $R_{ij} = \mathcal{E}\{X_i X_j\}$. Then

$$\mathcal{E}\{X_1 X_2 \ldots X_N\} = \begin{cases} 0 & N \text{ odd} \\ \sum \prod R_{ij} & N \text{ even} \end{cases} \qquad (2.75)$$

where the notation $\sum \prod$ means the sum over all completely distinct partitions of $X_1 X_2 \ldots X_n$ into pairs. The total number of terms in the summation is $\dfrac{N!}{(\frac{N}{2})!\, 2^{N/2}}$. For proof of this lemma, see[38]. As an example, by the lemma

$$\mathcal{E}\left\{X_1X_2X_3X_4\right\} = R_{12}R_{34} + R_{13}R_{24} + R_{14}R_{23}.$$

In the white noise case

$$\mathcal{E}\left\{x(t-\tau_1)x(t-\tau_2)x(t-\tau_3)x(t-\tau_4)\right\} = \delta(\tau_1-\tau_2)\delta(\tau_3-\tau_4)+\delta(\tau_1-\tau_3)\delta(\tau_2-\tau_4)+\delta(\tau_1-\tau_4)\delta(\tau_2-\tau_3) \tag{2.76}$$

Now, returning to the normalization of $G_2[\ ]$, consider

$$\langle G_2, G_2\rangle = \mathcal{E}\left\{\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty h_2(\tau_1,\tau_2)h_2(\tau_3,\tau_4)x(t-\tau_1)x(t-\tau_2)x(t-\tau_3)x(t-\tau_4)d\tau_1 d\tau_2 d\tau_3 d\tau_4\right.$$

$$\left. - 2\int_0^\infty\int_0^\infty\int_0^\infty h_2(\tau_1,\tau_2)h_2(\tau,\tau)x(t-\tau_1)x(t-\tau_2)d\tau_1 d\tau_2 d\tau + \left(\int_0^\infty h_2(\tau,\tau)d\tau\right)^2\right\}.$$

By the symmetry of the original Volterra kernel $h_2(\,,\,)$, and using (2.76)

$$\langle G_2, G_2\rangle = \left(\int_0^\infty h_2(\tau,\tau)d\tau\right)^2 + 2\int_0^\infty\int_0^\infty h_2^2(\tau_1,\tau_2)d\tau_1 d\tau_2 - 2\left(\int_0^\infty h_2(\tau,\tau)d\tau\right)^2 +$$

$$+ \left(\int_0^\infty h_2(\tau,\tau)\,d\tau\right)^2 .$$

Hence, the condition on the second-order kernel is

$$\langle G_2, G_2\rangle = 2\int_0^\infty\int_0^\infty h_2^{\,2}(\tau_1,\tau_2)\,d\tau_1\,d\tau_2 = 1 . \tag{2.77}$$

If the Gram-Schmidt processes are continued, the third-degree functional is found to be

$$G_3[h_3(\ ,\ ,\ );x(\ )] = \int_0^\infty \int_0^\infty \int_0^\infty h_3(\tau_1,\tau_2,\tau_3)x(t-\tau_1)x(t-\tau_2)x(t-\tau_3)d\tau_1 d\tau_2 d\tau_3 -$$

$$- 3\int_0^\infty \int_0^\infty h_3(\xi,\xi,\tau)\,x(t-\tau)\,d\tau\,d\xi \qquad (2.78)$$

where

$$3!\int_0^\infty \int_0^\infty \int_0^\infty h_3{}^2(\tau_1,\tau_2,\tau_3)\,d\tau_1\,d\tau_2,\,d\tau_3 \;=\; 1\ . \qquad (2.79)$$

The other higher-order G-functionals may be found by continuing the procedure.

Once having the orthogonal Wiener G-functionals, one may represent the general nonlinear system (or functional) by

$$y(t) = F[x(\tau): -\infty < \tau \le t] = \sum_{n=0}^{\infty} G_n[h_n(\tau_1,\tau_2,\ldots,\tau_n);\,x(\ )] \qquad (2.80)$$

where, as before, the characterizing kernels $h_n(\ldots)$ are determined by the average

$$\langle y(t),\, G_\mu[k_\mu(\tau_1,\tau_2,\ldots,\tau_\mu);\,x(\ )]\rangle = \langle G_\mu[h_\mu(\ldots);x(\ )],\ G_\mu[k_\mu(\ldots);x(\ )]\rangle$$

$$= \mu!\int_0^\infty \int_0^\infty \int_0^\infty h_\mu(\tau_1,\tau_2,\ldots,\tau_\mu)\,k_\mu(\tau_1,\tau_2,\ldots,\tau_\mu)\,d\tau_1\,d\tau_2\ldots d\tau_\mu\ . \qquad (2.81)$$

If the identifying kernel is taken as

$$k_\mu(\tau_1;\tau_2,\ldots,\tau_\mu) = \delta(t_1-\tau_1)\,\delta(t_2-\tau_2)\ldots\delta(t_\mu-\tau_\mu)\ ,$$

the system's kernel may be recovered as

$$h_\mu(t_1,t_2,\ldots,t_\mu) = \frac{1}{\mu!}\,\langle y(t)\ ,\ G_\mu\,[k_\mu(\ldots);\,x(\ )]\rangle\ , \qquad (2.82)$$

which represents a higher-order input/output correlation plus other terms (c.f. Section IV).

One may also recover the kernels by assuming that each of the symmetric kernels h_μ and k_μ are decomposable into a sum of products of orthonormal functions, e.g.,

$$h_n(\tau_1, \tau_2, \ldots, \tau_n) = \sum_i \sum_j \cdots \sum_\ell a_{ij\cdots\ell}\, \phi_i(\tau_1)\, \phi_j(\tau_2) \cdots \phi_\ell(\tau_n) \ . \quad (2.83)$$

By substituting these expansions into the integral of Eq. (2.81), using the separability of the kernels, and the orthonormality of the base functions $\phi_i(\ )$, one may determine each of the coefficients $a_{ij\cdots\ell}$ rather than the kernel itself. In fact, if the base functions are chosen to be the Laguerre functions, the Wiener I structure is recovered.

5. The Uryson and Hammerstein Operators. So far, every model for nonlinear systems that has been discussed has been a rather direct extension of some linear model. There are, however, several other classes of integral operators which represent an extension of the linear convolution integral model in a direction different from the Volterra series model. There is considerable mathematical literature [39, 40] relating to these nonlinear integral operators, but only a few references appear in the engineering literature.

a. The Uryson Operator. The nonlinear integral operator

$$\mathcal{U}[x(t)] = \int_\Omega K[t, \tau, x(t)]\, d\tau \quad (2.84)$$

is called an Uryson operator after P. S. Uryson who made the first studies of equations with this operator. There are many results relating to its complete continuity in various spaces, as well as existence and uniqueness

theorems for various equations containing it. For the purposes of the following discussion, it is noted that it may be used as a model for time-varying nonlinear systems in the form

$$y(t) = \int_{\Omega} K[t, \tau, x(\tau)]\, d\tau . \tag{2.85}$$

To obtain a time-invariant model, (2.85) is specialized to the form

$$y(t) = \int_{0}^{\infty} K[\tau, x(t-\tau)]\, d\tau , \tag{2.86}$$

which was first introduced into the engineering literature by Zadeh in a series of three papers [41-43] in 1953. Zadeh designates this class of models as $\mathcal{N}_1$.

b. The Hammerstein Operator. A special case of the Uryson operator is the so-called Hammerstein operator

$$\mathcal{K}[x(t)] = \int_{\Omega} K(\tau, t)\, f[\tau, x(\tau)]\, d\tau , \tag{2.87}$$

named after A. Hammerstein [44]. Again the principal interest lies in (2.87) as a system model, namely

$$y(t) = \int_{\Omega} K(\tau, t)\, f[\tau, x(\tau)]\, d\tau . \tag{2.88}$$

For a nonlinear time-invariant model, specialize (2.88) to

$$y(t) = \int_{0}^{\infty} k(\tau)\, f[x(t-\tau)]\, d\tau \tag{2.89}$$

which will be called the Hammerstein model. Physically, this represents a system composed of a zero-memory nonlinearity f[ ] followed by a linear system with impulse response k( ). See Fig. 2.4. Many partially or completely known real systems are of this structure.

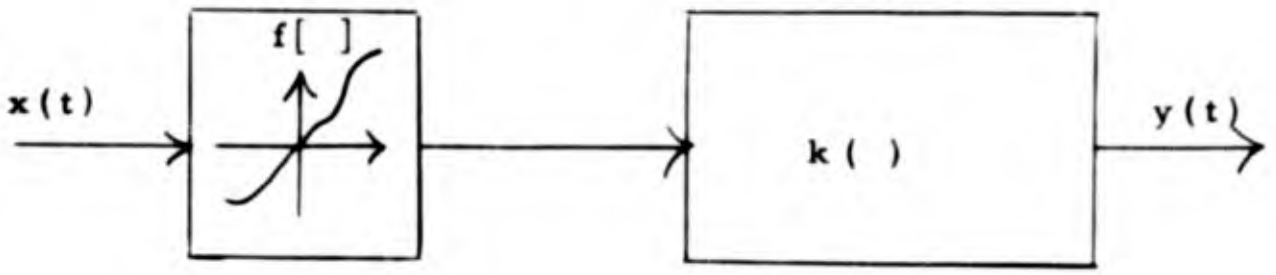

Figure 2.5 A Hammerstein Operator

C. Other Subclasses of the Uryson Model. By approximating the kernel K[,] of (2.86) by a sum of degenerate kernels as is usually done in the theory of linear equations, e.g.,

$$K[\tau, x] = \sum_{n=0}^{N} k_n(\tau)\, f_n[x], \tag{2.90}$$

one arrives at the input/output relation

$$y(t) = \sum_{n=0}^{N} \int_{0}^{\infty} k_n(\tau)\, f_n[x(t-\tau)]\, d\tau \tag{2.91}$$

which is just a parallel bank of Hammerstein models. Zadeh [43] notes that members of this subclass are, in a sense, dense in the Uryson class $\mathcal{N}_1$, in that models from $\mathcal{N}_1$ are either expressible as (2.91) or may be arbitrarily close by a member of (2.91).

Zadeh also notes three subclasses of (2.91). They are the nonlinear tapped delay-line filter with

$$k_n(\tau) = \delta(\tau-\Delta_n), \tag{2.92}$$

the power-series model with

$$f_n[x] = x^n, \tag{2.93}$$

and the piece-wise linear model with

$$f_n[x] = \begin{cases} a_n x + b_n & \text{for } x \in I_n \\ 0 & \text{otherwise} \end{cases}. \tag{2.94}$$

d. Zadeh's Generalization of the Uryson Model Analogous to the Uryson models $\mathcal{N}_1$, Zadeh introduced a hierarchy of system models, $\mathcal{N}_2, \mathcal{N}_3, \ldots$ where the input/output relation of class $\mathcal{N}_2$ is given by

$$y(t) = \int_0^\infty \int_0^\infty K[\tau_1, \tau_2; x(t-\tau_1), x(t-\tau_2)]\, d\tau_1\, d\tau_2 \tag{2.95}$$

and in general, for $\mathcal{N}_n$,

$$y(t) = \int_0^\infty \int_0^\infty K[\tau_1, \tau_2, \ldots, \tau_n; x(t-\tau_1), x(t-\tau_2), \ldots, x(t-\tau_n)]\, d\tau_1\, d\tau_2 \ldots d\tau_n \tag{2.96}$$

As a special subclass of $\mathcal{N}_n$, we note Singleton's [45] tapped delay-line power-series model where

$$y(t) = \sum_i \sum_j \cdots \sum_\ell a_{ij\ldots\ell}[x(t)]^i [x(t-\Delta_1)]^j \ldots [x(t-\Delta_{n-1})]^\ell . \tag{2.97}$$

D. Time-Varying Nonlinear Systems

The various possible time-varying generalizations corresponding to the above nonlinear models will not be enumerated here. From the analogy with the extension of linear time-invariant models to time-varying ones, the details for the nonlinear case are rather obvious. One paper in this direction is by Flake [21]. The mathematical literature on the Uryson and the Hammerstein operators also usually treats the more general case of a time-varying kernel.

III. THE CHOICE OF MODELS AND BASE FUNCTIONS

In Section II, most of the known models for system characterization were enumerated. As a prelude to the system identification problem, which is predicated upon having a system model, an attempt is made in this section to give some guide lines on the choice of a model with an eye toward the future identification of some physical system in terms of this model. The choice of an "adequate" model in any given situation is not an obvious one but depends on a variety of factors, some of which are listed below.

A. General Considerations in the Choice of a System Model

1. Adequacy. The first and prime consideration in choosing a model is that of adequacy. An adequate model is one such that the difference between the responses of the physical system and those predicted from the model meets some criterion for all (or some restricted set) of inputs. For example, a model may be considered adequate if the mean square error (MSE) between the response of the physical system and that obtained from the model is less than some value, say ϵ_o, for all inputs from some class, say all band-limited functions with zero frequency content above ω_o radians/sec. There are many other well-known figures of merit (e.g., the integral square error (ISE), the integral absolute error (IAE), the impulse response area ratio (IRAR), the integrated product of time and absolute error (ITAE), etc. [46]) as well as other restrictions to the class of input functions. No physical system is truly linear, but does become so only after one has established

some adequacy criterion and is subsequently able to identify an adequate linear model. Physical systems which are linear under one criterion may not be linear under a more stringent definition of adequacy.

2. Simplicity. The next important consideration in choosing a model is that of simplicity. One invariably attempts to obtain an adequate fit with a low-order linear time-invariant model. If this is not adequate, one may first want to increase the order of a differential model or to add terms to one's orthogonal expansion before considering the impulse response as a model. If no linear time-invariant model is adequate, one might proceed to the time-varying linear models. The appropriateness of such a model may be determined by considering the stationarity (or lack thereof) of one's estimate for the best linear time-variant model. If no significant variations are found in successive estimates of a linear model, one may proceed directly to the consideration of a nonlinear model. Again, in the nonlinear model one would not take more terms than are required to meet one's adequacy criterion.

3. Identifiability. Another important consideration in the choice of a model is the ease with which a system can be identified. One of the serious shortcomings of the Uryson-Hammerstein operator models is that there are no adequate techniques for their identification. On the other hand, the identification of a system's impulse response in some situations (say, on-line in an adaptive loop) may be much too involved, yet the identification in terms of some orthogonal basis may be quite tractable in that only simple

averages are required. Along these same lines, in the nonlinear case, the identification of mixed Volterra kernels (i.e., a truncated Volterra expansion containing more than one isolated term) will require the solution of a system of integral equations (cf., next section) and thus offers a strong motivation for the various orthogonal functional expansions. In general, one may say that the orthogonal expansion (linear or nonlinear) leads to the easiest identification procedures.

4. The Use of the System. The actual use or input environment of a physical system can yield some very definite constraints upon one's choice of a model. If it is known that the given physical system will be used in a particular situation and will be subjected to a restricted class of inputs, one may possibly want to model only that portion of the physical system which will respond to inputs from the restricted input space. Also, the given physical system may already be in service and one must perform the identification from the existing inputs or with the addition of a small probe signal to the existing inputs. As indicated below, the class of existing inputs and/or additional probe functions will also dictate in some cases the set of base functions to be used in an orthogonal expansion model.

5. The Use of the Model. In addition to the use of the system, the use to be made of the model will in many situations determine the appropriate model. For many control-system applications, the identification of the physical plant in terms of its transfer functions offers the greatest advantage for the subsequent determination of compensation or control

function. Although the impulse response conveys the same information, a good estimate of one of these functions is not readily translated into a good estimate of the other. Therefore, identification should be made directly in terms of the model which one will subsequently use.

6. Implementation. In some applications, it may be desirable to implement the model in real time (or faster) upon a computer (analogue or digital). As examples, one may think of a model reference control loop or a filter to recover signals transmitted through a channel. The choice of a model can greatly affect the complexity of the implementation. For example, consider the implementation of the discrete version of the linear convolution integral, i.e.,

$$y_i = \sum_{j=0}^{J} h_j x_{i-j} , \qquad (3.1)$$

on a digital computer. The calculation for each filtered value y_i will require $(J+1)$ multiplications and J additions plus the overall requirement of $(2J+2)$ memory slots. In the case of large receiving arrays (acoustic, seismic, or radar) where (3.1) must be applied to each sensor input, the amount of computation may be prohibitive. On the other hand, the corresponding differential equation model may lead to a simple second-order difference equation, e.g.,

$$y_n = \alpha_1 y_{n-1} + \alpha_2 y_{n-2} + \beta_0 x_n , \qquad (3.2)$$

which will require only three multiplications and two additions per output value (plus only 5 memory slots) and yet yield an accuracy comparable to the convolution model (3.1) above with $J = 100$.

Again, in the case of orthogonal expansions, one might do well to pick base functions $\phi_i(\ )$ from within the class expressible as confluent hypergeometric functions [47] for two reasons. First, the confluent hypergeometric functions are entire functions and thus possess desirable approximating capability. Secondly, they satisfy recurrence relations which in terms of network synthesis implies that the base functions may be generated by a ladder network [48]. In terms of implementation upon a digital computer, their recursive nature implies that each subsequent base function may be generated from the previous members via a simple difference scheme. Moreover, the evaluation of (2.8) in the orthogonal decomposition of the past of the input $x(\ )$ into the $v_n(t)$'s is also possible by simple recursive schemes in the discrete time case. As a model example, one should mention the Laguerre functions which are expressible as

$$\ell_n(x) = {}_1F_1(n+1;\ 1;\ -x)\ . \tag{3.3}$$

The discrete time values of

$$v_k(t) = \int_0^\infty \ell_k(\tau)\, x(t-\tau) \tag{3.4}$$

may be computed recursively from

$$\begin{aligned} v_o(n) &= Bv_o(n-1) + Ax(n)\ , \\ v_k(n) &= Bv_k(n-1) + Cv_{k-1}(n-1) - v_{k-1}(n)\ . \end{aligned} \tag{3.5}$$

The continuous or analogue case may be implemented by the Laguerre network.

7. Extensibility. Another consideration in choosing a model which may be implemented in the future is that the chosen model can be extended in an easy fashion if the original model proves inadequate. The extension of the linear convolution model (2.1) to a two-term Volterra representation will require major additions to the model's implementation. Also, the identification of mixed Volterra kernels represents a major computation as has already been noted. However, starting from a linear model in the form of an orthogonal expansion, it is a simple matter to add some orthogonal second-order terms. If the base functions for the linear model have been well chosen, they will also be quite adequate in many cases in the extension to a nonlinear model.

8. Gray Boxes and/or a Priori Knowledge. In many physical situations, one has a partial knowledge of the system. As opposed to black boxes, one might call these gray box systems. In modeling gray box systems, one should try to find a model which incorporates as many as possible of the known attributes of the system and yet which has enough freedom to allow the identification of an adequate fit. As an example, a given system may be known to contain a zero-memory nonlinearity followed by some linear dynamics. In this case, the Hammerstein operator may prove a desirable model. In another situation, one might have some knowledge of the physical mechanism by which various elements (or components) of a system function. Here, the differential equation may be the easiest model to generate for the system. It is rather inconceivable how anyone could generate an adequate nonlinear differential equation model in a truly black box situation.

B. The Choice of Base Functions in an Orthogonal Expansion

Having decided upon one of the parametric orthogonal expansion models (i.e., the linear, Cameron and Martin, Wiener I, or Bose), one must then choose a set of base functions $\phi_i(\ )$. Two important considerations dictate the choice of these base functions and in some situations yield conflicting requirements.

1. The Approximation Problem. In the pure identification problem, one is completely free to choose his input probe function. In this situation, the base functions are determined for their approximating capability. In a pure identification situation, one is also free to measure first the spectral transfer characteristic of the given black box under white noise excitation. With this information, one is then in a good position to choose a set of orthogonal base functions $\phi_i(\ )$ for the identification of a linear time-invariant model. In the sense of requiring the fewest number in terms in the expansion, the best set of orthogonal functions is that which has a gain transfer characteristic similar to that measured from the system. For example, if the unknown system has an estimated transfer characteristic similar to that of Fig. 3.1 (e.g., like a low-pass filter), the Laguerre functions will form a very natural basis. The required decomposition may be accomplished with a Laguerre network. The one free parameter of the network, the bandwidth or pole position, should be taken to measure exactly that of the estimated gain characteristics. Then if the system is truly linear, time-invariant, and if it does not have too complicated a phase structure, the identification should be quite satisfactory.

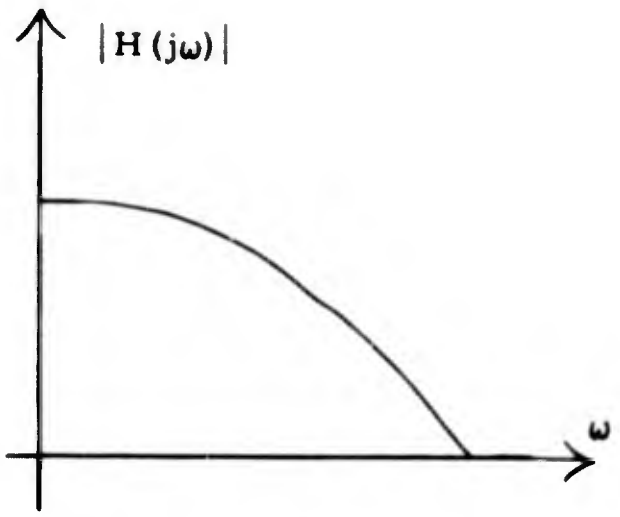

Figure 3.1 Bode Plot

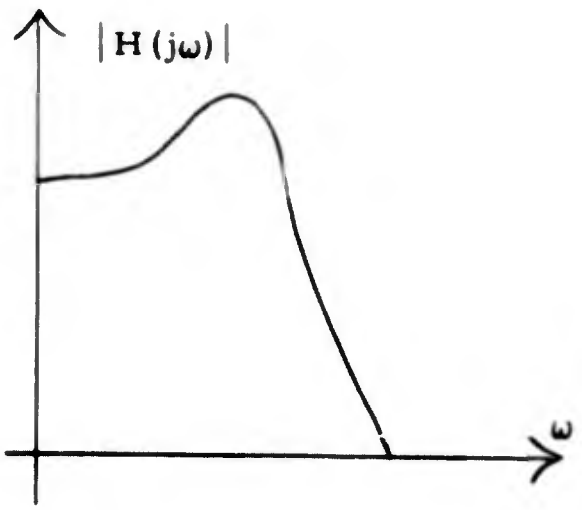

Figure 3.2 Bode Plot

If the estimated transfer characteristic is not similar to a low-pass filter, one will have to find other appropriate base functions or generate (via Kautz's method [49]) one's own starting from a pole-zero pattern deduced from the estimated characteristic. For example, if the estimated gain transfer characteristic of the system is similar to that of Fig. 3.2, the natural choice for $\phi_1(\ )$ is

$$\phi_1(t) = e^{-at} \sin(\omega_o t)$$

with the pole-zero pattern of Fig. 3.3. The next member $\phi_2(\ )$ will have the pole-zero pattern of Fig. 3.4 with

$$\phi_2(\ ) = \mathcal{L}^{-1}\left\{\frac{\omega_o[(s-a)^2+\omega_o^2]}{[(s+a)^2+\omega_o^2]^2}\right\}.$$

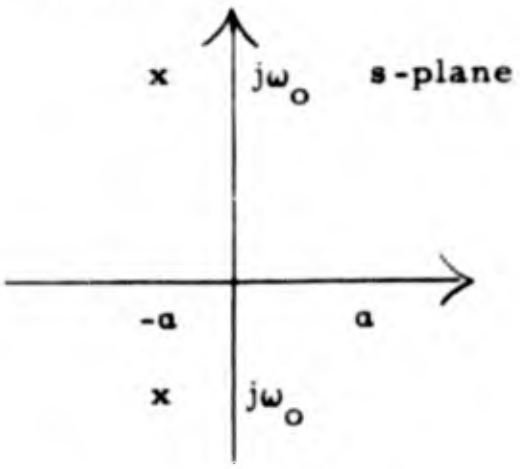

Figure 3.3 Pole Plot

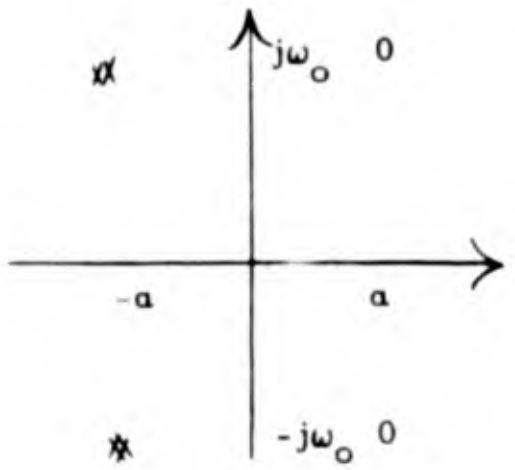

Figure 3.4 Pole-Zero Plot

These and subsequent members will form a subclass of the orthogonalized expotentials[50, 51]. The orthogonalized exponentials have been proposed before, but their appropriateness in a particular situation has been lacking.

On the other hand, say that one is committed to using a Laguerre function, but the estimated gain characteristic is again similar to that of Fig. 3.2. Then to obtain the best approximation with the fewest number of terms, the pole position of the Laguerre network should be chosen to lie on the circle about the origin and through the pair of complex conjugate poles. See Fig. 3.5. For the proof of this simple result, see Head [52].

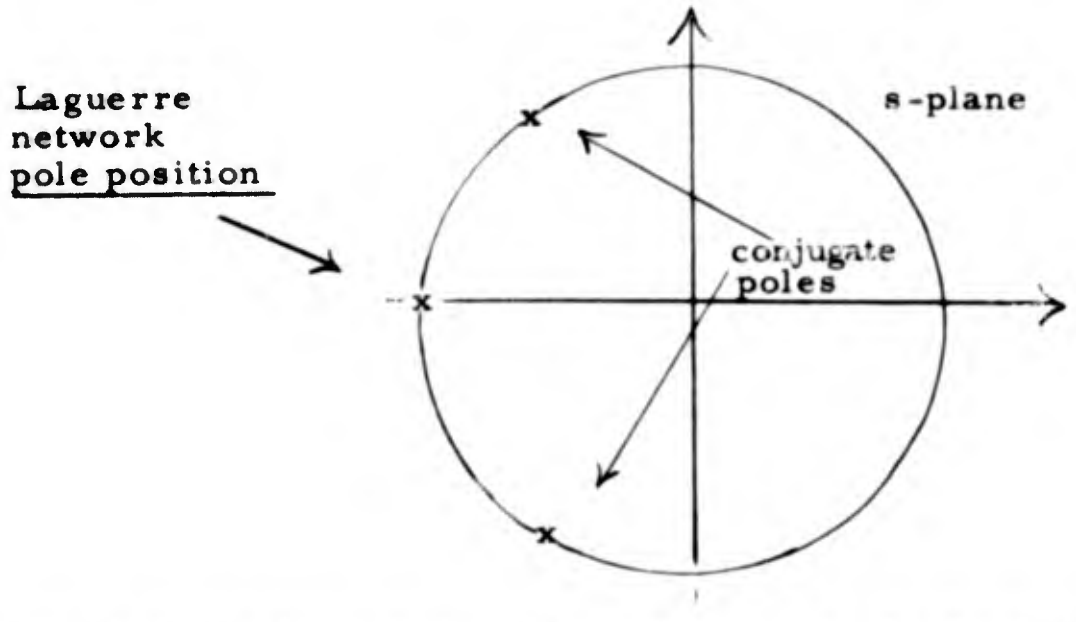

Figure 3.5 Determination of Bandwidth of a Laguerre Network

Head's result is important for two reasons. First, because of the expressed concern in the literature for the choice of the bandwidth of a Laguerre network and secondly, because it illustrates how a given system's pole-zero pattern can determine the free parameters in an orthogonal expansion.

In summary, for a linear time-invariant system, one should choose base functions which yield an adequate approximation in as few terms as is possible and still meet the constraints of practicality. The freedom to estimate a given system transfer gain characteristic first can give one a real handle upon such a choice.

If the linear model is not adequate, then, barring the finding of significant time variations in the system's response, one will have to choose a nonlinear model. The question that naturally arises is whether the base functions already chosen can be utilized. If the unknown system is such that it is responding to a considerable portion (in terms of bandwidth) of the input excitation, then, heuristically, one probably has a good a basis as is possible. The given system may not contain any linear functional; however, one may justify the above basis as yielding the best linear approximation.

On the other hand, the system may have a rather wide-band output and yet be responding to only a narrow-band portion of the input (possible in a nonlinear system). This contingency may be checked by determining the system's describing function [53] or by estimating the input-to-output cross-spectral density. If it so happens that the given system responds to only a narrow frequency band, then possibly one should limit the probe function to the corresponding (in frequency band) narrow-band Gaussian noise. As a result, the base functions are now determined by the considerations below.

2. The Independence Problem. In an in-service identification problem or in other situations where the input function is predetermined (as in the narrow-band case above), the choice of the base functions is determined by the requirement that the resulting coefficient functions, i.e.,

$$v_n(t) = \int_0^\infty \phi_n(\tau)\, x(t-\tau)\, d\tau , \tag{3.6}$$

be uncorrelated in the linear case and independent for the nonlinear model. To see how this requirement arises, consider the case of a linear model with the system represented as

$$y(t) = \sum_{k=0}^{K} C_k\, v_k(t) , \tag{3.7}$$

where it is required to identify the C_k's. One would hope to do this by a simple cross-correlation (as is possible for a white noise probe) between the system's output and one of the $v_k(t)$'s. Thus, consider for a stationary input $x(\ )$

$$\mathcal{E}\left\{v_j(t)\, y(t)\right\} = \mathcal{E}\left\{\int_0^\infty \phi_j(\xi)\, x(t-\xi)\, d\xi \sum_{k=0}^{K} C_k \int_0^\infty \phi_k(\eta)\, x(t-\eta)\, d\eta\right\}$$

$$= \sum_{k=0}^{K} C_k \int_0^\infty \phi_j(\xi) \int_0^\infty \phi_k(\eta)\, \mathcal{E}\left\{x(t-\xi)\, x(t-\eta)\right\} d\eta\, d\xi ,$$

$$\mathcal{E}\left\{v_j(t)\, y(t)\right\} = \sum_{k=0}^{K} C_k \int_0^\infty \phi_j(\xi) \int_0^\infty \phi_k(\eta)\, R_x(\xi-\eta)\, d\eta\, d\xi . \tag{3.8}$$

One is at an impasse unless the base functions $\phi_k(\ )$ are the eigenfunctions of the integral equation

$$\int_0^\infty R_x(\xi-\eta)\,\phi(\eta)\,d\eta = \lambda\phi(\xi). \tag{3.9}$$

If they are, and are properly normalized, then one may write

$$\mathcal{E}\left\{v_j(t)\,y(t)\right\} = \sum_{k=0}^{K} C_k\lambda_k \int_0^\infty \phi_j(\xi)\,\phi_k(\xi)\,d\xi = \sum_{k=0}^{K} C_k\lambda_k\,\delta_{jk} = C_j\lambda_j\,, \tag{3.10}$$

where λ_j is the eigenvalue associated with the $v_j(t)$. Therefore, unless the eigenfunctions of (3.9) are taken as the base functions, the various $v_\nu(t)$ functions will not be uncorrelated and the characterizing coefficients C_k cannot be identified by simple averages.

In the case of a nonlinear model, the $v_n(t)$'s are required to be independent. Thus, the above expansion will also serve our purpose since, for a Gaussian probe $x(\ )$, the $v_n(t)$'s are also Gaussian and by (3.10) uncorrelated, hence, independent.

The above expansion of a random process in terms of the eigenfunctions associated with its covariance kernel is generally known as the Karhunen-Loeve expansion [54]. The Karhunen-Loeve expansion allows one to remove Wiener's whiteness assumption. Moreover, for colored Gaussian inputs, it also dictates a very natural set of base functions.

In summary, in an out-of-service situation (pure identification), the natural probe for a linear model is white noise and for a nonlinear model, white Gaussian noise. In either case, the base functions are chosen for

their approximating ability. In an in-service situation, the input is given and the natural base functions for either a linear or nonlinear model are the eigenfunctions of the input's covariance kernel. For the nonlinear model, however, the input must be Gaussian for all models except Bose's. Finally, in an in-service situation where one may add a test signal to the existing input, the base functions for a linear model may again be chosen for their approximating ability with the natural choice for the test signal being, again, white noise. The identification of in-service systems in terms of nonlinear models with added test signals is an open problem.

IV. THE IDENTIFICATION OF SYSTEMS

The identification problem may be represented schematically as shown in Fig. 4.1. Given a black box, an unknown

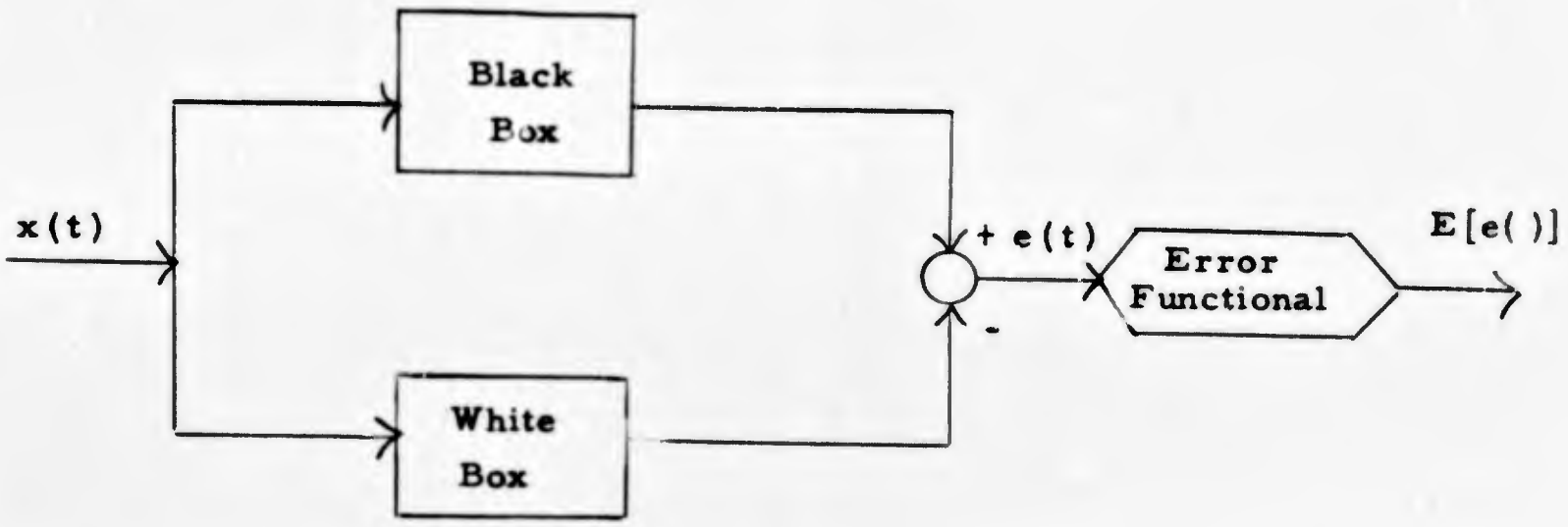

Figure 4.1 The Identification Problem

system with in general noisy output, and a white box representing a class of system

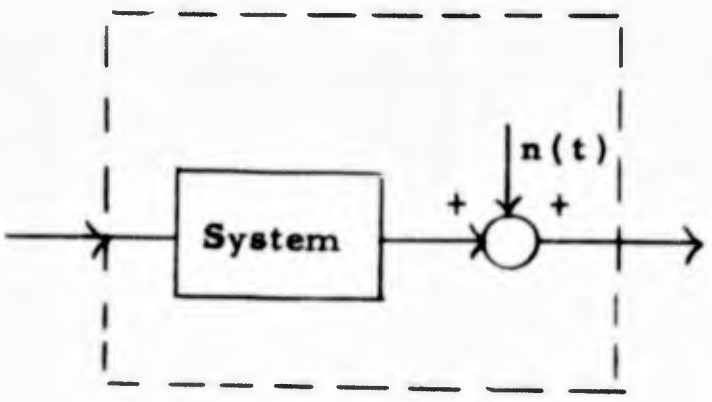

Figure 4.2 The Black Box

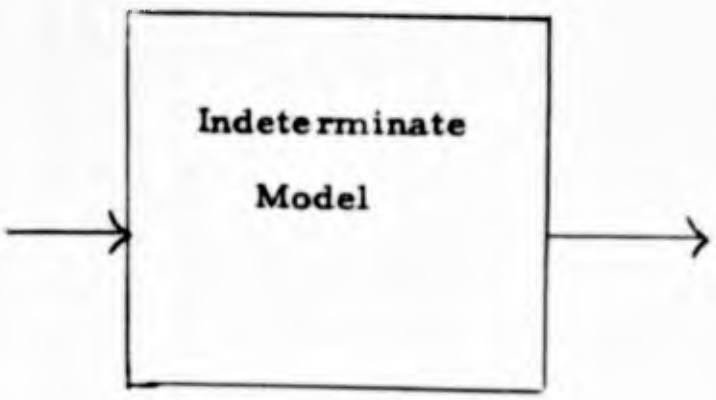

Figure 4.3 The White Box

models, the problem is to specialize the white box such that the resulting model meets some adequacy criterion, the adequacy criterion being in terms of the difference, or some functional thereof, between the outputs of the black and white box for some class of inputs.

The noise upon the system's output is made a part of the black box for several reasons. First, there may be truly additive but unknown noise generated by the system. Secondly, a portion of the system's output may be from some uncontrollable portion and hence will also appear as noise. Finally, the possibility that any particular class of models will yield an absolute identification of a particular system is rather remote. Hence, the additive noise will also allow us to compensate for the inadequacy of any particular model. One last point, if the measuring device also has noise which is unknown, one may just as well lump it in with the system's noise.

The designation of a class of models as a white box is due to Wiener. It is a white box, as opposed to a black box, in that its structure is known but specific functions or parameters are unknown. The process of identification is just the estimation of these various functions or parameters.

The actual process of identification can take many forms. Some are dictated by the application of the system (in-service/out-of-service), others by the identification technique for a given model, and still others by a combination of the two. Some identification procedures will require the implementation of the white box and will allow for its continuous adjustment. Others will not require the implementation of the white box but will require recording, in digital form, the system input and output with subsequent calculations to yield the model.

For each class of models enumerated in Section II, one may attempt the identification of a given unknown system. We obviously cannot treat each class of models in every possible situation. Thus, this section is confined chiefly to the pure identification (out-of-service) problem with as general a probe function as is possible. Many in-service identifications will fit within this scope and, in some other cases, additional techniques are given.

The minimization of mean square error is used as an adequacy criterion because of its mathematical tractability.

A. The Identification of Linear Time-Invariant Systems

1. The Impulse Response. Assume that a given unknown system S can be characterized by an input/output relation in the form of the convolution integral (2.1). Then, to identify S is to determine that specific model within the class represented by (2.1) which is equivalent to S. In the convolution integral (2.1), it is the kernel (or impulse response) $h(\ )$ which makes the model specific. Hence, to identify S is to determine $h(\ )$. This may be accomplished in several ways.

The simplest way to determine $h(\ )$ is via its definition as the response of the system to a single unit impulse. Here, it must be assumed that there is no noise upon the output. If the observations of the system's impulse response are noisy, one can possibly estimate $h(\ )$ by means of numerous observations of the impulse response.

A second method is to excite the system with white random noise of unit spectral density. Then the impulse response h( ) may be determined as the input-to-output cross-correlation function (see Fig. 4.4)

$$h(\tau) = \phi_{xy}(\tau) = \mathcal{E}\left\{x(t)\, y(t+\tau)\right\} . \qquad (4.1)$$

If the system is in-service, one may often be able to add to the existing input x( ) a small test signal s( ). If s( ) is again taken to be white noise, h( ) can be determined as shown in Fig. 4.5

$$h(\tau) = \phi_{sy}(\tau) = \mathcal{E}\left\{s(t)\, y(t+\tau)\right\} . \qquad (4.2)$$

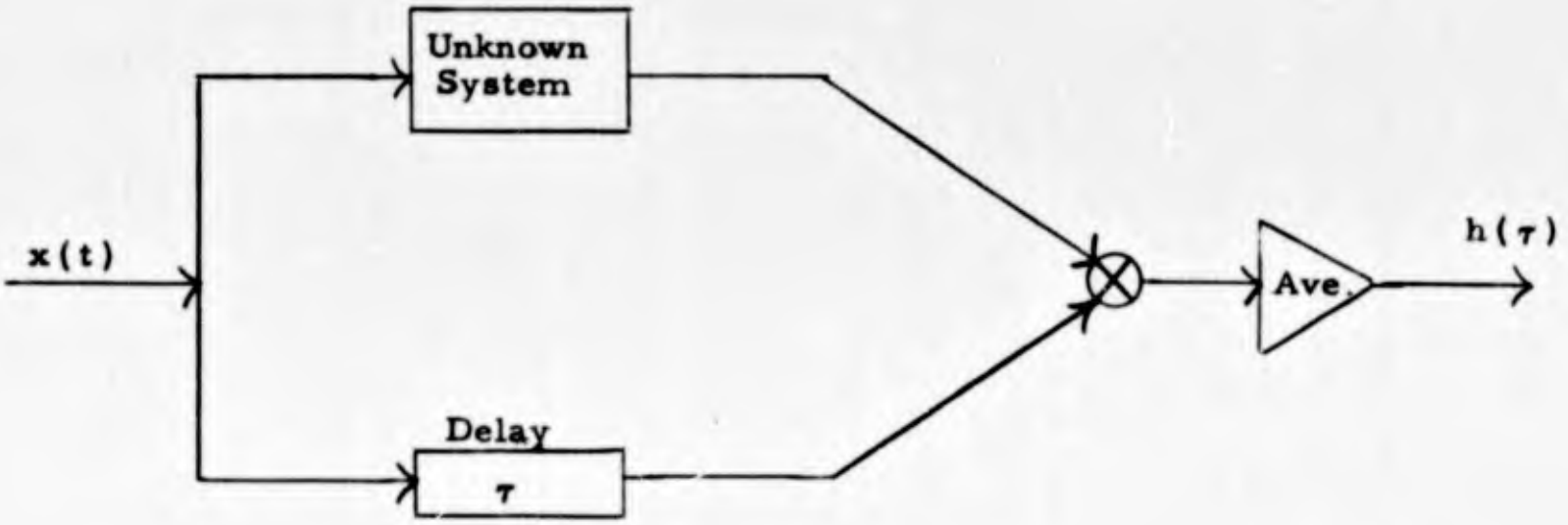

Figure 4.4 Determination of Impulse Response

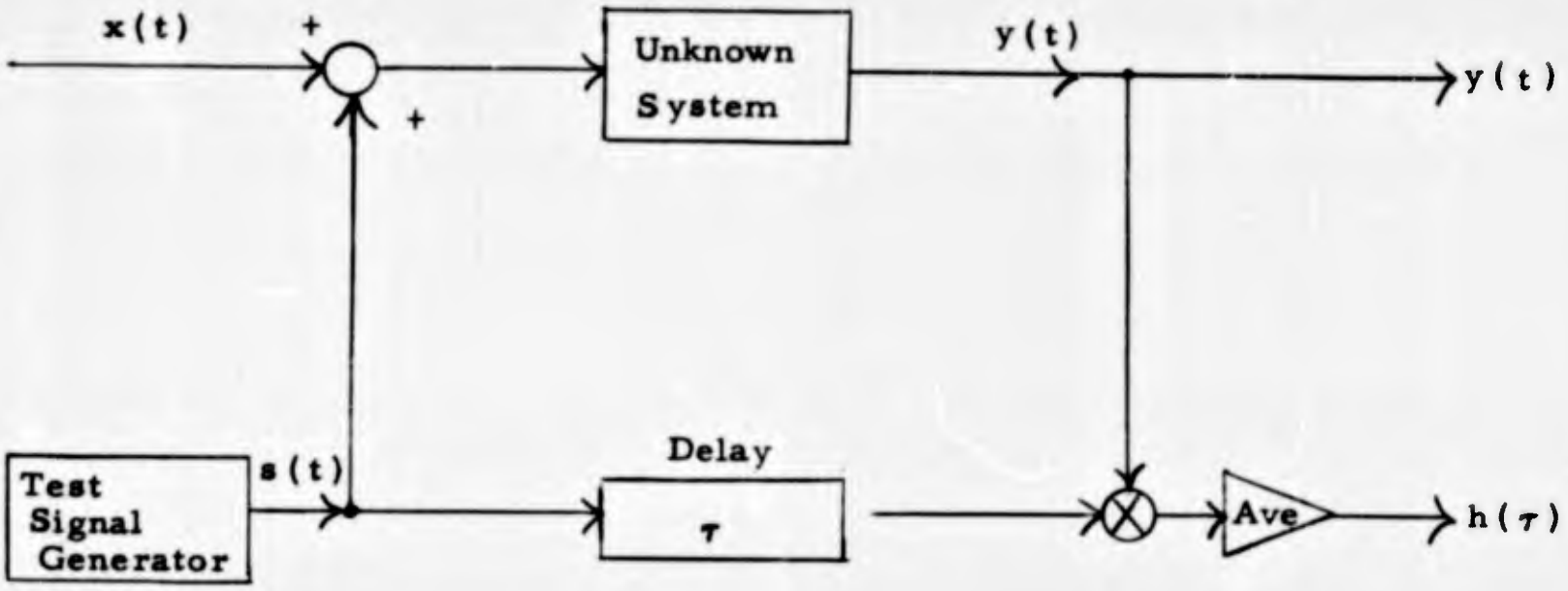

Figure 4.5 In-Service Identification of the Impulse Response

Finally, if the system is in-service and a test signal cannot be added, S can still be identified if the existing input is a sufficiently wide-band stationary input. In this situation, one must determine both the input-to-output cross-correlation $\phi_{xy}(\tau)$ and the auto-correlation function $R_x(\tau)$ of the existing input. Then $h(\ )$ is the solution of the Wiener-Hopf equation

$$\phi_{xy}(\tau) = \int_0^\infty h(\xi)\, R_x(\tau - \xi)\, d\xi \,. \qquad (4.3)$$

For additional details on the estimation of the impulse response, see [55-58].

What if the system is nonlinear? In such a case, the first method may fail to give even an approximation, but any variation of the correlation techniques will always yield the best linear approximation in the mean square sense.

2. The Transfer Function. To identify the transfer function $H(\ )$ of an unknown system, one can again proceed in several ways. The usual procedure is to excite the system with a unit amplitude sinusoid and then measure the amplitude and phase of the output. As one sweeps out the frequency range, $H(\ )$ is determined in the form

$$H(j\omega) = |H(j\omega)|\, e^{j\phi(\omega)}, \tag{4.4}$$

where $|H(j\omega)|$ is the measured output amplitude and $\phi(\omega)$ the measured phase shift. If noise is used as a probe (or it exists as a normal input), one may determine $|H(j\omega)|$ from the relation

$$|H(j\omega)|^2 = \frac{\Phi_{yy}(\omega)}{\Phi_{xx}(\omega)} \tag{4.5}$$

where $\Phi_{xx}(\ )$ and $\Phi_{yy}(\ )$ are the input and output spectra, respectively. If the input noise is white, only the estimate of the spectrum is needed to determine $|H(j\omega)|$. To determine the phase $\phi(\omega)$ or the complete transfer function $H(j\omega)$, the frequency domain version of (4.3), namely

$$\Phi_{xy}(\omega) = H(j\omega)\, \Phi_{xx}(\omega), \tag{4.6}$$

can be used. Thus, one must also estimate the cross-spectral density $\Phi_{xy}(\omega)$. In practice, both the co-spectrum $C_{xy}(\omega)$ and quadrature spectrum $Q_{xy}(\omega)$ must be determined from the even and odd parts of the input-to-output cross-correlation. With these estimates, one has

$$|H(j\omega)| = \frac{\sqrt{C_{xy}^2(\omega) + Q_{xy}^2(\omega)}}{\Phi_{xx}(\omega)}, \qquad \phi(\omega) = \arctan\left\{\frac{Q_{xy}(\omega)}{C_{xy}(\omega)}\right\}. \tag{4.7}$$

Additional details may be found in [59, 60]. If the system is nonlinear, the first method may yield no worthwhile results but the second method will always yield the best mean square linear approximation.

3. The Differential Equation. To identify a time-invariant system in terms of a linear differential equation is to determine the set of coefficients a_n and b_n of Eq. (2.3). The usual technique here is to consider the sampled-data (discrete time) version of (2.3), namely, the difference equation model

$$y(n) + A_1 y(n-1) + \ldots + A_N y(n-N) = B_0 x(n) + B_1 x(n-1) + \ldots + B_M x(n-M) \quad (4.8)$$

with the associated pulse transfer function

$$H(z) = \frac{B_0 + B_1 z^{-1} + B_2 z^{-2} + \ldots + B_M z^{-M}}{1 + A_1 z^{-1} + A_2 z^{-2} + \ldots + A_N z^{-N}} . \quad (4.9)$$

The identification then becomes the determination of the coefficients A_k and B_k from a set of sampled inputs $x(n)$ and outputs $z(n)$ of the system. Usually, the distribution of the $z(n)$'s is not known, so one generally uses a (possibly weighted) least-squares technique to determine the coefficients. One does not, however, define the error in the usual way as a sum of squares of the differences between system and model outputs, e.g.,

$$E = \frac{1}{K+1} \sum_{k=0}^{K} \left\{ z(n-k) - y(n-k) \right\}^2 , \quad (4.10)$$

since the resulting system of equations for the coefficients A_n and B_n is in terms of system-to-model cross-correlation functions. The

minimization of (4.10) by least-squares may, however, be made the basis for an iterative identification procedure wherein the model of the previous stage is used to calculate the required correlation functions. Along these lines, see the note by Steiglitz and McBride [61].

For a tractable least-square problem, one does choose to minimize the term

$$E_K(n) = \frac{1}{K+1} \sum_{k=0}^{K} \left\{ z(n-k) - \hat{z}(n-k) \right\}^2 , \tag{4.11}$$

where the difference Eq. (4.8) is used for $\hat{z}(\ )$, i.e.,

$$\hat{z}(n) = \sum_{i=1}^{M} B_i x(n-i) - \sum_{j=1}^{N} A_j z(n-j) . \tag{4.12}$$

The necessary conditions for minimum of $E_K(n)$, namely,

$$\frac{\partial E_K(n)}{\partial B_i} = 0 \quad \text{and} \quad \frac{\partial E_K(n)}{\partial A_j} = 0 \tag{4.13}$$

will yield a set of linear equations in the unknown coefficients A_n and B_n, and various input and output sample auto- and cross-correlation functions. These may be solved by any one of numerous techniques to yield the solution of the identification problem. See Kalman [62] for additional details.

If the distribution of both input and output are known, one may use maximum likelihood estimates for the coefficients. See Levin [63] and Cox [64].

4. Orthogonal Decompositions. To identify an unknown system in terms of some orthogonal base functions $\phi_n(\ )$, one must determine the set of c_n's such that

$$y(t) = \sum_{n=0}^{N} c_n \int_0^{\infty} \phi_n(\tau)\, x(t-\tau)\, d\tau \ . \tag{4.14}$$

To do this, one must place in turn each of the orthogonal networks in parallel with the unknown system, that is, such that both have the same random input. Then, providing the base functions have been chosen appropriately (cf., Section III), the c_n's may be determined as simple averages

$$c_n = \mathcal{E}\left\{v_n(t)\, y(t)\right\} \ . \tag{4.15}$$

If the base functions are the eigenfunctions in a Karhunen-Loeve expansion, there will be an additional numerical factor in (4.15). In either case, Fig. 4.6 gives the essential details for the identification.

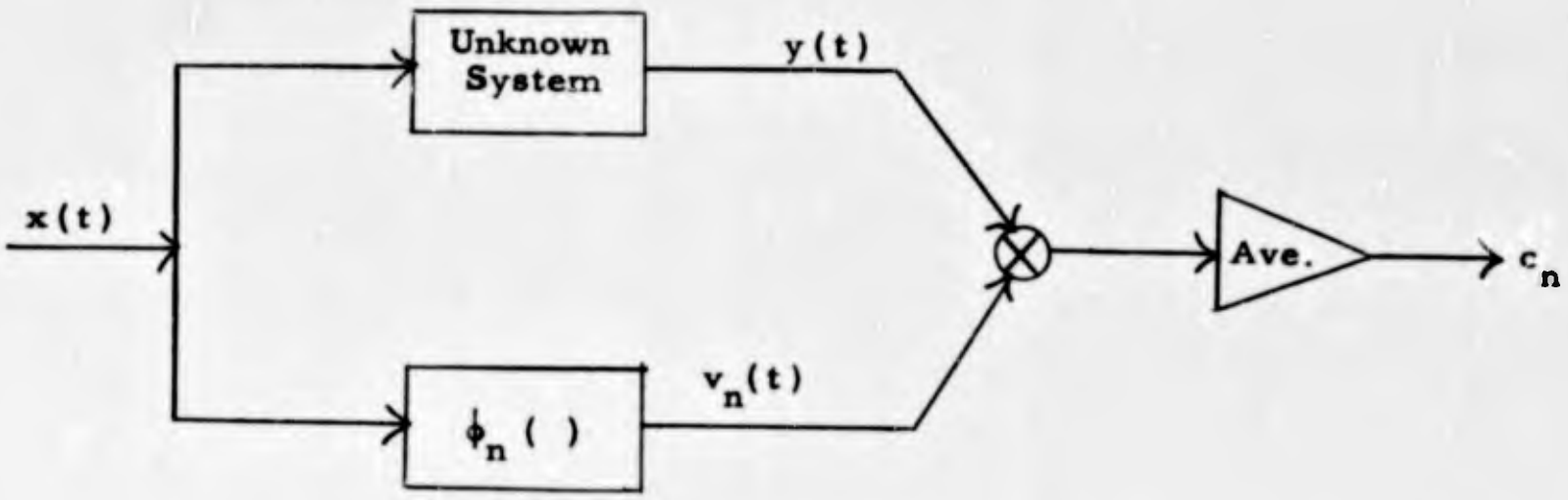

Figure 4.6 Pure Identification of Orthogonal Expansion

If one is an in-service condition but can add a test signal, one should use the configuration of Fig. 4.7 where only the test signal is applied to the orthogonal network.

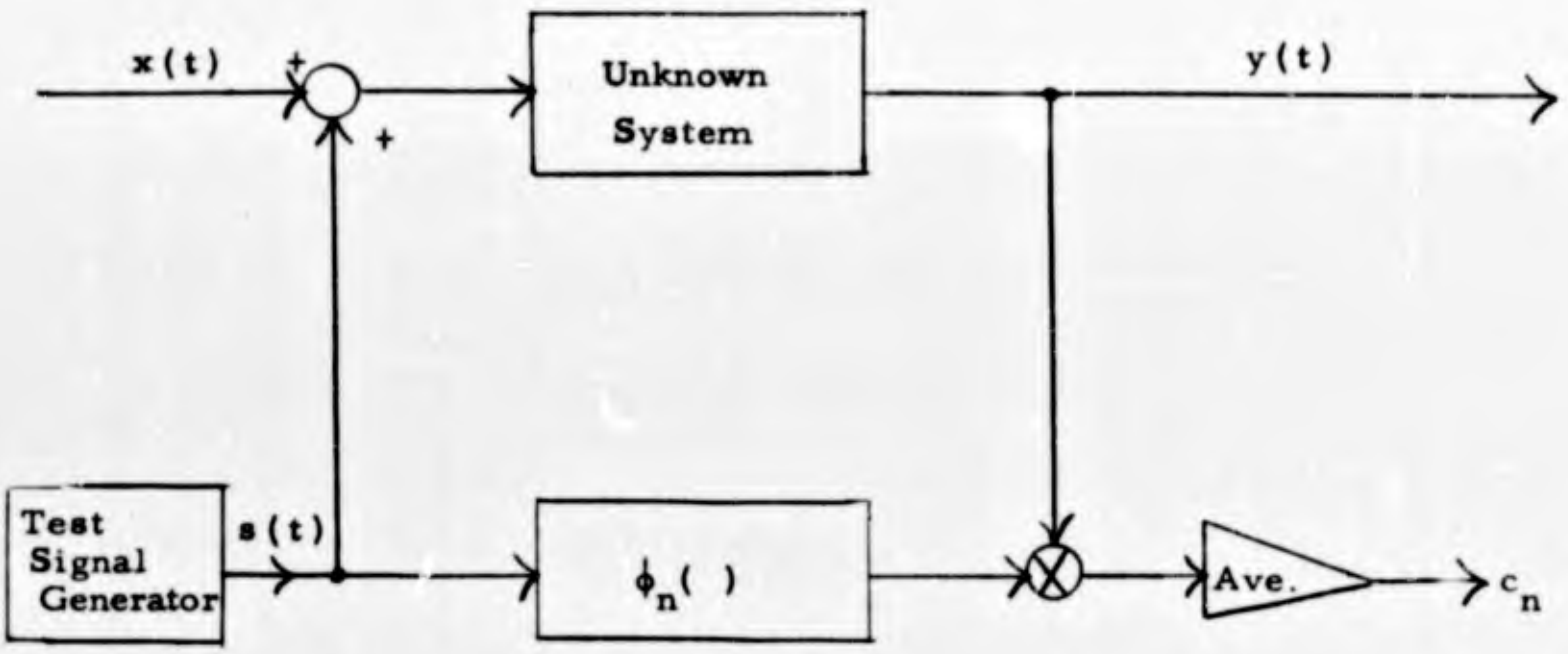

Figure 4.7 In-Service Identification of Orthogonal Expansion

B. The Identification of Time-Varying Linear Systems

In order to identify an unknown system in terms of a time-varying linear model, one must usually assume that the time variations of the system are of such a low frequency that the same technique used for identifying time-invariant models will yield an adequate model before the system changes significantly. With this assumption also goes the implication that the observation noise is of low level. Otherwise, it would require an even slower variation in the actual system to allow additional averaging time to remove the effects of the noise.

We shall not go through all the details, but with the above assumption all the techniques of part A above may be used in a repetitive (possibly overlapping) fashion to follow the time variations of the given system.

One fresh approach has been offered by Kushner [65]. He uses a simple first-order iterative scheme based upon a stochastic version of the steepest descent method to estimate the impulse response of a discrete-time system. For various classes of inputs, Kushner proves convergence of the scheme and he does not assume any stationarity of the system over some period of time.

C. The Identification of Nonlinear Time-Invariant Systems

1. The Volterra Expansion. In order to identify a nonlinear system in terms of a truncated Volterra series expansion (2.22), one must determine the various Volterra kernels, $h_0, h_1(\ ), h_2(\ ,\ ), \ldots, h_N(,\ldots,)$. To present the identification technique, consider the simplest class of models which will indicate all the details, namely, the three-term expansion

$$y(t) = h_0 + \int_0^\infty h_1(\tau)\, x(t-\tau)\, d\tau + \int_0^\infty \int_0^\infty h_2(\xi,\eta)\, x(t-\xi)\, x(t-\eta)\, d\xi\, d\eta . \quad (4.16)$$

If the actual system output is designated by $z(\ )$, one can write the mean square error functional between system and model as

$$E = \mathcal{E}\left\{[z(t)-y(t)]^2\right\} = \overline{[z(t)-y(t)]^2} \quad (4.17)$$

for any stationary ergodic random process as input. Then, substituting the model (4.16) into the error functional (4.17) and employing the usual calculus of variations procedure (i.e., determining the variations δE_0, δE_1, and δE_2 by setting, in turn, $h_0 = h_0 + \epsilon k_1$, $h_1(\tau) + \epsilon k_1(\tau)$, $h_2(\xi, \eta) = h_2(\xi, \eta) + \epsilon k_2(\xi, \eta)$ and then

$$\frac{\partial}{\partial \epsilon} \left\{ \delta E_i \right\} \Big|_{\epsilon=0} = 0 , \qquad i = 0, 1, 2),$$

one obtains the following system of integral equations

$$\overline{z(t)} = h_0 + \int_0^\infty h_1(\xi)\, \overline{x(t-\xi)}\, d\xi + \int_0^\infty \int_0^\infty h_2(\xi, \eta)\, \overline{x(t-\xi)\, x(t-\eta)}\, d\xi\, d\eta \quad ,$$

$$\overline{z(t)\, x(t-\tau)} = h_0 \overline{x(t-\tau)} + \int_0^\infty h_1(\xi) \overline{x(t-\xi) x(t-\eta)}\, d\xi + \int_0^\infty \int_0^\infty h_2(\xi, \eta) \overline{x(t-\xi) x(t-\eta) x(t-\tau)}\, d\xi\, d\eta , \tag{4.18}$$

$$\overline{z(t)\, x(t-\tau) x(t-\lambda)} = h_0\, \overline{x(t-\tau) x(t-\lambda)} + \int_0^\infty h_1(\xi) \overline{x(t-\xi) x(t-\tau) x(t-\lambda)}\, d\xi +$$

$$+ \int_0^\infty \int_0^\infty h_2(\xi, \eta) x(t-\xi) x(t-\eta) x(t-\tau) x(t-\lambda)\, d\xi\, d\eta \quad ,$$

whose solution is the solution of the identification problem. This system of equations is the analogue of the Wiener-Hopf Eq. (4.3) in the identification of a linear system. Their solution, however, is a very formidable problem in the case of a general stochastic input. When the input process is such that the multidimensional correlation functions are not separable, the above integrals do not represent convolution and, hence, spectrum factorization techniques will not work. Katzenelson and Gould [19] offer an iterative procedure of stage-wise

optimization and successive substitution. Hsieh [66] proposes a gradient technique and Eykhoff [67] considers the discrete-time version of (4.16) with the corresponding solution given as the solution of a large order matrix equation in the various measured correlations.

If one is free to choose one's input, the obvious choice will be white Gaussian noise of unit variance. Then the system of integral equations (4.18) becomes

$$\bar{z} = h_0 + \int_0^\infty h_2(\xi, \xi)\, d\xi ,$$

$$\overline{z(t)\, x(t-\tau)} = h_1(\tau) \tag{4.19}$$

$$\overline{z(t)\, x(t-\tau)\, x(t-\lambda)} = \bar{z}\delta(\tau-\lambda) + 2h_2(\tau, \lambda) .$$

Here, the solution is fairly direct; for other higher-order approximations, even this specialized case offers considerable difficulties.

In the intermediate case of colored Gaussian noise, one can use spectrum factorization techniques because of the separability of the various higher-order correlation functions of a Gaussian process.

2. Multidimensional Frequency Functions. For the identification of the multidimensional frequency functions, there is no analogue to the sinusoidal probe technique of linear systems. There is, however, the very natural extension of the noise probe technique. The natural choice is again white Gaussian noise with the subsequent measurement of the various higher-order input-to-output cross-spectral densities. From the system of equations containing these estimates, one solves for the required higher dimensional transfer functions. The details are rather obvious. The analogue of Eqs. (4.19) will be

$$\bar{z} = h_0 + \int_0^\infty H_2(j\omega, -j\omega)\, d\omega$$

$$\Phi_{xz}(\omega) = H_1(j\omega) \qquad (4.20)$$

$$\Phi_{xxz}(\omega_1, \omega_2) = 2\pi \bar{z}\delta(\omega_1 + \omega_2) + 2H_2(j\omega_1, j\omega_2) .$$

The frequency domain identification of nonlinear systems suffers from the same defect as the Volterra expansion models. Without orthogonality, one cannot easily sort out the responses to the various functionals. If one can assume that the unknown system is an isolated homogeneous functional or that it is the sum of one odd and one even order functional, the details become quite simple. For the case of a system composed of only the linear and the quadratic functions, one may see Tick [68] for additional details.

3. The Differential Equation For the identification of an unknown system in terms of nonlinear differential equation, one requires some partial knowledge of the system in order to choose an appropriate parametric form for the equation. Once having chosen a class of differential equations as a white box, one may also be able to determine the parameters from the same a priori knowledge. If not, one is faced with the problem of estimating the remaining unknown parameters. If the unknowns appear linearly, i.e., as simple multiplicative gains upon the various terms as in, say, Duffing's equation

$$\ddot{y}(t) + \omega_0{}^2 y(t) + k y^3(t) = x(t) , \qquad (4.21)$$

then one may use least-squares just as was done for linear equations. For the Duffing's equation, one could write down the error functional E_K over to the last K values and then find the unknown coefficients as the solution to the equations:

$$\frac{\partial E_K}{\partial (\omega_0^2)} = 0 , \quad \frac{\partial E_K}{\partial k} = 0 . \tag{4.22}$$

If the unknown parameters enter the equation in a nonlinear fashion, then least-squares will not work. One can, however, resort to the implementation of the error functional and minimize it by some gradient technique to yield the identification.

Another technique offered by Kumar and Stridhar [69] for the identification of nonlinear equations uses the quasi-linearization method of Bellman and Kalaba. Briefly, their identification technique is to adjoin to the state vector form of the differential model (2.31), the equation $\dot{\underline{a}} = 0$, i.e.,

$$\begin{aligned} \dot{\underset{\sim}{y}} &= \underset{\sim}{g}(y, x; \underset{\sim}{a}) \\ \dot{\underset{\sim}{a}} &= 0 . \end{aligned} \tag{4.23}$$

By defining a new state vector $\underset{\sim}{Z} = \begin{bmatrix} \underset{\sim}{y} \\ \underset{\sim}{a} \end{bmatrix}$, Eqs. (4.23) may be written as

$$\dot{\underset{\sim}{Z}} = F(\underset{\sim}{Z}, x, t) \tag{4.24}$$

Then, for a set of measured values

$$< \underset{\sim}{e}_i , \underset{\sim}{Z}(t_i) > = c_i , \quad i = 1, 2, \ldots, N, \tag{4.25}$$

one has a nonlinear multipoint boundary value problem. By using the quasi-linearization technique of Bellman and Kalaba [70, 71], the nonlinear problem may be solved as a convergent sequence of linear multipoint boundary value problems from which one can determine $\underset{\sim}{Z}(0)$ and, hence, the unknown vector $\underset{\sim}{a}$ of parameters. The uniqueness of solutions and the case of noisy observations have not, however, been treated.

4. The Orthogonal Decompositions. Let us divide the orthogonal functional models of Section II. 4 into two categories. The first, which includes the Cameron and Martin, the Wiener I, and the Bose models, will be called the parametric type in that one must first choose some set of base functions and then the identification amounts to estimating a set of coefficients (parameters). The second category will be called the nonparametric type. It includes the Barrett and Wiener II models which, in essence, are the same.

The pure identification of the parametric nonlinear models has a very close analogy with the identification of the linear orthogonal models. In each case, one must implement the white box and measure the averages between the system's output and the various components of the model's output. These simple averages yield the constants (or gain setting) for the final characterization of the system. Very little more needs to be said for the pure identification problem in that the required averages have been enumerated with the models in Section II. Figure 4.8 gives the required structure for identifying a Wiener I model.

For in-service identification of nonlinear systems analogous to Fig. 4.7 (for linear systems), no simple solution exists. Even if the test signal is statistically independent of the existing input, the very nonlinear nature of the system will mix the inputs in such a fashion that the response to neither can be sorted out. In the case of linear systems, the technique worked because, via superposition, one has only two components to the output which are also statistically independent.

The identification of a system in terms of the nonparametric models will require, in the case of Barrett's model, the generation of the Hermite functionals of the Gaussian input, or in the Wiener approach, the generation of the identifying G-functionals with products of delta functions as kernels. With very little effort, one can see that these are one and the same. In either case, one must then average the system's output with the functional of the input. As one varies the various time delays in the functionals of the input, the average will yield one of the kernels which identify the unknown system. In Fig. 4.9, the structure required for the estimation of the third-order kernel $h_3(\tau_1, \tau_2, \tau_3)$ is given. The analogy here is with that of the determination of the impulse response via input-to-output cross-correlation techniques (compare Figs. 4.4 and 4.9). The structure of Fig. 4.9 is correct even when the Gaussian input is colored.

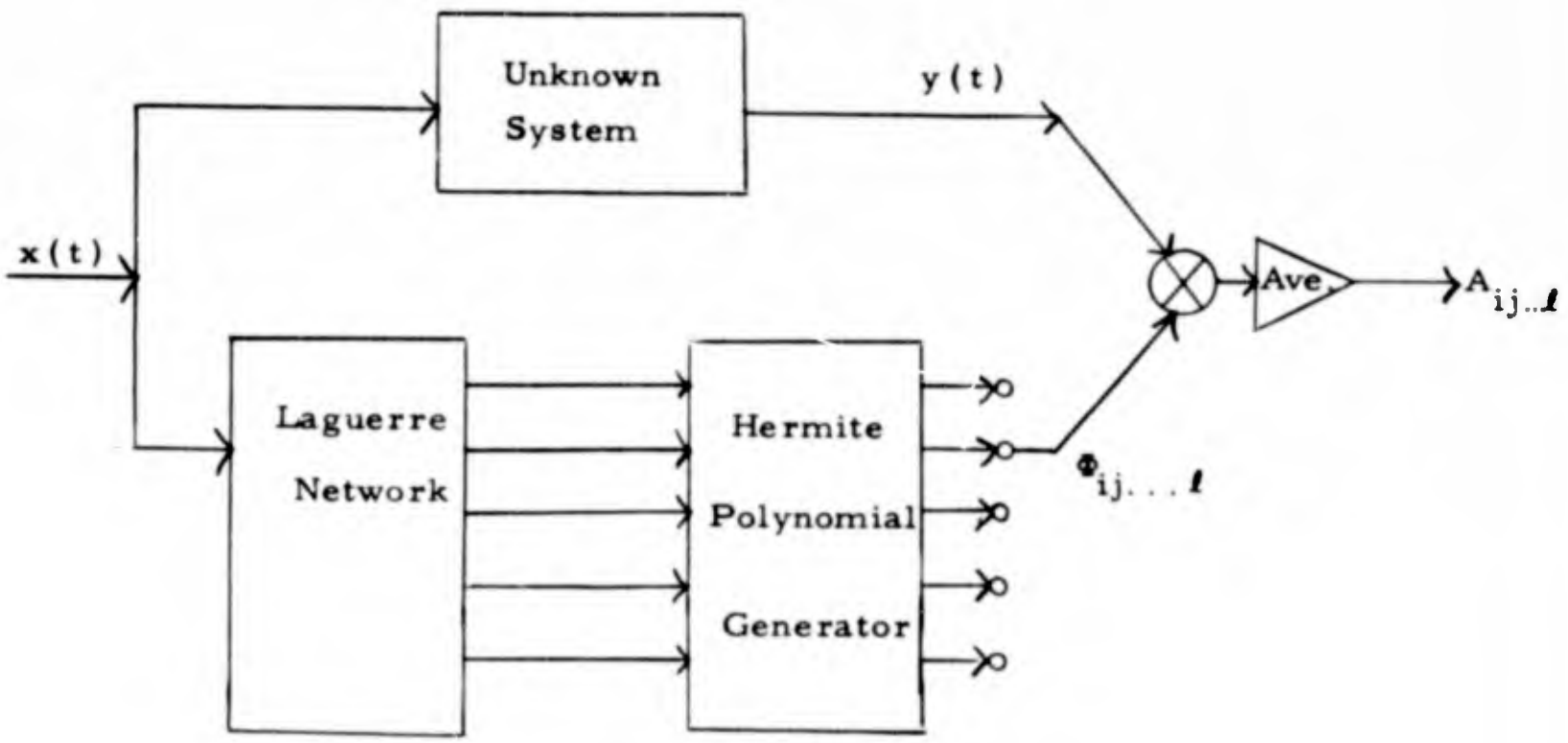

Figure 4.8 Identification as a Wiener I Model

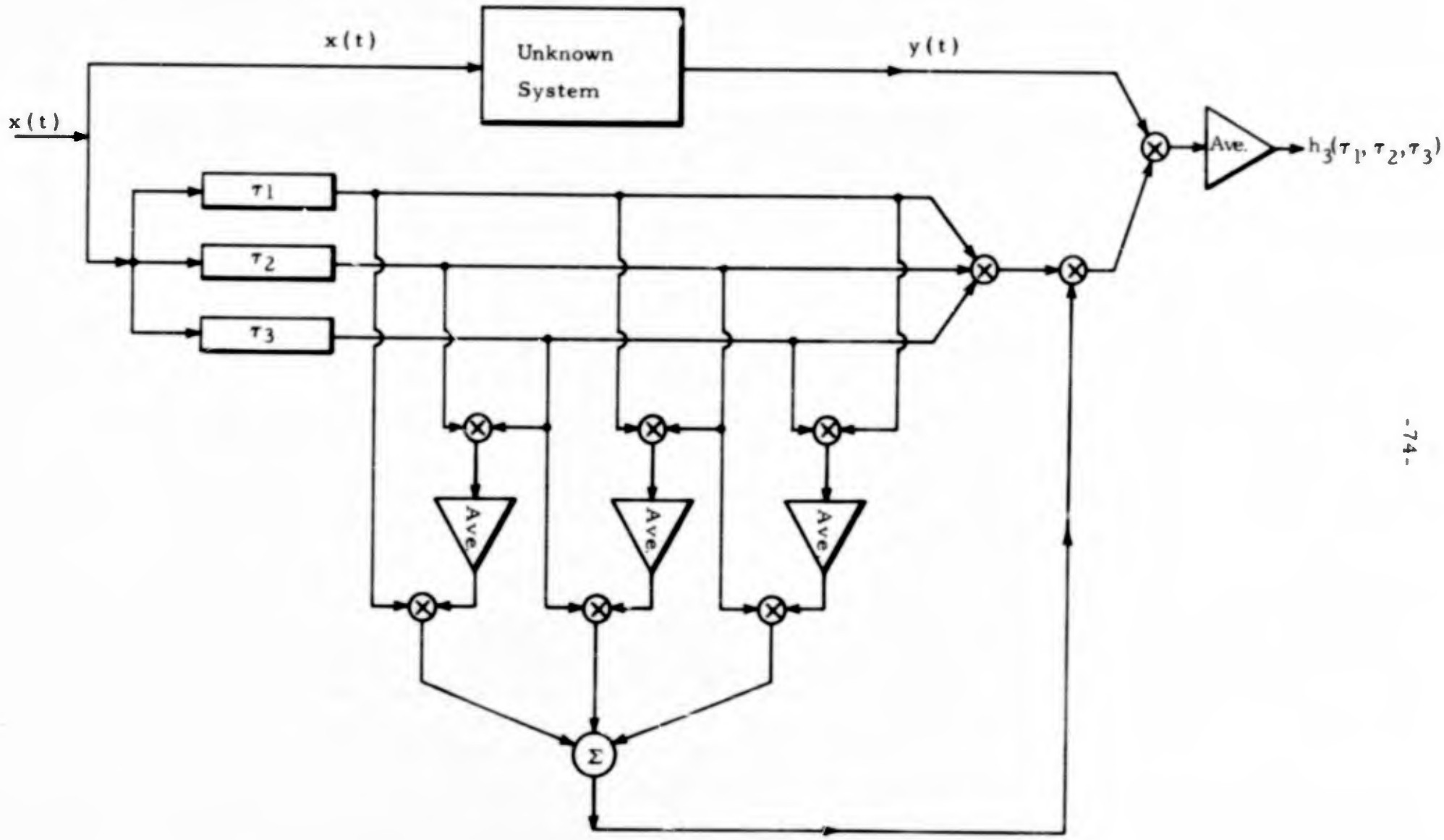

Figure 4.9 Determination of Third Degree Barrett/Wiener II Kernel

Additional investigation is required for the in-service identification of nonlinear systems, but one rather obvious technique is to use a parametric orthogonal model and simple observation of some error functional while the coefficients are adjusted to give a minimum. The orthogonality decouples any effect due to the adjustments made upon the coefficients of the model.

Even if orthogonality does not exist (i.e., for non-Gaussian inputs) but the existing inputs are fairly wide-band, one can still employ the correlation technique of Narendra and McBride [72]. Briefly, the technique is to define the error functional as

$$E_T(t) = \int_{t-T}^{t} e^2(\tau)\, d\tau \qquad (4.26)$$

where $e(\ )$ is the usual error between system and model,

$$e(t) = y(t) - \sum_a A_a \Phi_a \,. \qquad (4.27)$$

Then to employ a gradient technique, one requires that [73]

$$\frac{dA_a}{dt} = -\beta \frac{\partial}{\partial A_a} \left\{ E_T(t) \right\} \,. \qquad (4.28)$$

Since the gains appear linearly, from the above equations, one has

$$\frac{dA_a}{dt} = -2\beta \int_{t-T}^{t} e(\tau)\, \Phi_a [x(\ ); \tau]\, d\tau \,. \qquad (4.29)$$

By formally integrating (4.29), one has

$$A_a(t) = A_a(t_o) - \beta \int_{t_o}^{t} \int_{\tau-T}^{\tau} e(\xi)\, \Phi_a [x(\ ); \xi]\, d\xi\, d\tau \ . \qquad (4.30)$$

Therefore, if one adjusts the gains in a continuous fashion as indicated by (4.30), the error functional will be minimized and the final values $A_a(\infty)$ will yield the required characterization for any of the parametric models.

5. The Uryson-Hammerstein Models. As has already been noted in Section III, one of the chief drawbacks of the Uryson or Hammerstein models is that to date no adequate techniques exist for their identification. The only known approach is one offered by Zadeh [74] in 1956. For models of the Uryson class η_1, Zadeh shows how the unknown kernel function $K[t, x]$ may be determined as the solution to a pair of triple-order Fourier and Hankel transform-inverses of the complete first and second-order describing functions of the system. The scope of the required measurements is considerable, i.e., the measurement of two describing functions for all amplitudes and all frequencies. The triple transform inversion also presents considerable difficulty.

Additional techniques are certainly welcome. The restriction to Hammerstein models may yield very tractable identification procedures.

REFERENCES

1. L. A. Zadeh, "From Circuit Theory to System Theory," Proc. IRE 50, 856-865 (May, 1962).

2. R. Deutsch, Nonlinear Transformations of Random Processes, Prentice-Hall, Inc., New Jersey, 1962.

3. M. Athans and P. Falb, Optimal Control: An Introduction to the Theory and Its Applications, McGraw-Hill Book Co., Inc., New York (in press), Chap. 4.

4. R. E. Kalman, "Mathematical Description of Linear Dynamical Systems," SIAM J. on Control 1, 152-192 (1963).

5. L. A. Zadeh, "The Determination of the Impulsive Response of Variable Networks," J. Appl. Phys. 21, 642-645 (July, 1950).

6. K. S. Miller, "Properties of Impulsive Responses and Green's Functions," IRE Trans. Circuit Theory CT-2, 26-31 (March, 1955).

7. T. Kailath, "Sampling Models for Linear Time-Variant Filters," Tech. Report No. 352, Research Lab. of Electronics, Massachusetts Institute of Technology, Cambridge, Massachusetts, May, 1959.

8. L. A. Zadeh, "Frequency Analysis of Variable Networks," Proc. IRE 38, 291-299 (March, 1950).

9. V. Volterra, "Sopra le Funzioni che Dipendono da Altre Funzioni," R. C. Accad. Naz. Lincei 3, 97-105 (1887).

10. V. Volterra, Theory of Functionals, Dover Publications, New York, 1959.

11. M. Frechet, "Sur les Fonctionnelles Continues," Ann. Ecole Norm. Sup. 27, 193-216 (1910).

12. N. Wiener, "Response of a Non-linear Device to Noise," Report V-16S, Radiation Lab., Massachusetts Institute of Technology, Cambridge, Massachusetts, April, 1942.

13. S. Ikehara, "A Method of Wiener in a Non-linear Circuit," Tech. Report No. 217, Research Lab. of Electronics, Massachusetts Institute of Technology, Cambridge, Massachusetts, December, 1951.

14. R. Deutsch, "On a Method of Wiener for Noise through Nonlinear Devices," IRE Convention Record, part 4, pp. 186-192 (March, 1955).

15. M. B. Brilliant, "Theory of the Analysis of Nonlinear Systems," Tech. Report No. 345, Research Lab. of Electronics, Massachusetts Institute of Technology, Cambridge, Massachusetts, March 3, 1958.

16. R. Mc Fee, "Determining the Response of Nonlinear Systems to Arbitrary Inputs," Trans. AIEE, part II(Appl. and Industry), 80, 189-193 (September, 1961).

17. H. L. Van Trees, Synthesis of Optimum Nonlinear Control Systems, Massachusetts Institute of Technology Press, Cambridge, Massachusetts, 1962.

18. R. H. Flake, "Volterra Series Representation of Nonlinear Systems," Trans. AIEE, part II(Appl. and Industry), 81, 330-335 (January, 1963).

19. J. Katzenelson and L. A. Gould, "The Design of Nonlinear Filters and Control Systems," Information and Control 5, 108-143(June, 1962) and 7, 117-145 (June, 1964).

20. P. Alper and D. C. J. Poortulict, "On the Use of Volterra Series Representation and Higher Order Impulse Responses for Nonlinear Systems," Electronics Lab. Report, Technological University, Delft, Netherlands, June 1963.

21. R. H. Flake, "Volterra Series Representation of Time-Varying Non-linear Systems," Proc. Sec. Int. Conf. IFAC (1963).

22. M.-L. Liou, "Nonlinear-System Analysis and Synthesis," Tech. Report No. 6554-6, Stanford Electronics Lab., Stanford University, Stanford, California, October, 1963.

23. J. B. Berger, "Application of Volterra Series Analysis to an Electro-hydraulic Control Valve," Paper II-3 of 1964 JACC, June, 1964.

24. H. L. Van Trees, "Functional Techniques for the Analysis of the Nonlinear Behavior of Phase-Locked Loops," Proc. IEEE 52, 894-910 (August, 1964).

25. D. A. George, "Continuous Nonlinear Systems," Tech. Report No. 355, Research Lab. of Electronics, Massachusetts Institute of Technology, Cambridge, Massachusetts, July 24, 1959.

26. A. E. Taylor, Introduction to Functional Analysis, John Wiley and Sons, New York, 1961, p. 106.

27. R. H. Cameron and W. T. Martin, "The Orthogonal Development of Non-linear Functions in Series of Fourier-Hermite Functionals," Ann. Math. 48, 385-392 (April, 1947).

28. N. Wiener, "Seminar in Nonlinear Networks," Research Lab. of Electronics, February, 1949 (unpublished).

29. J. W. Head and W. P. Wilson, "Laguerre Functions, Tables and Properties," Proc. IEE C-103, 428-440 (1956).

30. Y. W. Lee, Statistical Theory of Communication, John Wiley and Sons, New York, 1960.

31. A. G. Bose, "A Theory for the Experimental Determination of Optimum Nonlinear Systems," IRE Conv. Record, Part 4, 21-30, March, 1956.

32. A. G. Bose, "A Theory of Nonlinear Systems," Tech. Report No. 309, Research Lab. of Electronics, Massachusetts Institute of Technology, Cambridge, Massachusetts, May 15, 1956.

33. A. G. Bose, "Nonlinear System Characterization and Optimization," Trans. IRE on Circuit Theory CT-6, 30-40 (May 1959).

34. J. F. Barrett, "The Use of Functionals in the Analysis of Non-linear Physical Systems," Statistical Advisory Unit Report No. 1/57, Ministry of Supply, Great Britain. Also, J. Electronics and Control 15, 567-615 (December, 1963).

35. H. Grad, "Note on N-Dimensional Hermite Polynomials," Commun. Pure Appl. Math 2, 325-330 (December, 1949).

36. N. Wiener, Nonlinear Problems in Random Theory, John Wiley and Sons, New York, and Tech. Press, New York, 1958.

37. A. E. Taylor, op. cit., p. 116-117.

38. M. Schetzen, "Average of the Product of Gaussian Variables," Quarterly Progress Report No. 60, Research Lab. of Electronics, Massachusetts Institute of Technology, Cambridge, Massachusetts, January 15, 1961, pp. 137-141.

39. P. M. Anselone (ed.), Nonlinear Integral Equations, University of Wisconsin, Madison, Wisconsin, 1964.

40. M. A. Krasnosel'skii, Topological Methods in the Theory of Nonlinear Integral Equations, Pergamon Press, New York, 1964.

41. L. A. Zadeh, "Optimum Nonlinear Filters," J. Appl. Phys. 24, 396-404 (April, 1953).

42. L. A. Zadeh, "Optimum Nonlinear Filters for Extraction and Detection of Signals," IRE Conv. Record, Part 8, 57-65, March, 1953.

43. L. A. Zadeh, "A Contribution to the Theory of Nonlinear Systems," J. Franklin Inst. 255, 387-408 (May, 1953).

44. A. Hammerstein, "Nichtlineare Integralgleichungen nebst Anwendungen," Acta Math. 54, 117-176 (1930).

45. H. E. Singleton, "Theory of Nonlinear Transducers," Tech. Report No. 160, Research Lab. of Electronics, Massachusetts Institute of Technology, Cambridge, Massachusetts, August 12, 1950.

46. J. E. Gibson, Nonlinear Automatic Control, Mc Graw-Hill Book Co., Inc., New York, 1963, Chap. 11.

47. L. J. Slater, Confluent Hypergeometric Functions, Cambridge University Press, Cambridge, England, 1960.

48. V. O. Mowery, "On Hypergeometric Functions in Iterated Networks," Trans. IEEE on Circuit Theory CT-11, 232-247 (June, 1964).

49. W. H. Kautz, "Transient Synthesis in the Time Domain," Trans. IRE on Circuit Theory CT-1, 29-39 (September, 1954).

50. W. H. Huggins, "Representation and Analysis of Signals, Part I: The Use of Orthogonalized Exponentials," Report No. AFCRC TR-57-357, The Johns Hopkins University, Baltimore, Maryland, September 30, 1957.

51. D. C. Ross, "Orthogonal Exponentials," Proc. Nat. Electronics Conf. 18, 838-849 (October, 1962).

52. J. W. Head, "Approximation to Transients by Means of Laguerre Series," Proc. Cambridge Phil. Soc. 52, 640-651 (October, 1956).

53. J. G. Truxal, Automatic Feedback Control System Synthesis, Mc Graw-Hill Book Co., Inc., New York, 1955, Chap. 10.

54. W. B. Davenport, Jr., and W. L. Root, Random Signals and Noise, Mc Graw Hill Book Co., Inc., New York, pp. 96-101.

55. Y. W. Lee, "Application to Statistical Methods to Communications Problems," Tech. Report No. 181, Research Lab. of Electronics, Massachusetts Institute of Technology, Cambridge, Massachusetts, September, 1950.

56. M. J. Levin, "Optimum Estimation of Impulse Response in the Presence of Noise," Trans. IRE on Circuit Theory CT-7, 50-56 (March, 1960).

57. J. H. Westcott, "The Parameter Estimation Problem," Proc. First Internat. Conf. IFAC, Moscow, 1960, pub. Butterworth, London.

58. J. C. Lindenlaub and G. R. Cooper, "Noise Limitations of System Identification Techniques," Trans. IEEE on Automatic Control AC-8, 43-48 (January, 1963).

59. G. M. Jenkins, "Cross-Spectral Analysis and the Estimation of Linear Open Loop Transfer Functions," Chap. 18 of Time Series Analysis, M. Rosenblatt (ed.), John Wiley and Sons, New York, 1963.

60. G. M. Jenkins, "An Example of the Estimation of a Linear Open Loop Transfer Function," Technometrics 5, 227-245 (May, 1963).

61. K. Steiglitz and L. E. McBride, to be published in Trans. IEEE on Automatic Control.

62. R. E. Kalman, "Design of a Self-Optimizing Control System," Trans. ASME 80, 468-478 (February, 1958).

63. M. J. Levin, "Estimation of a System Pulse Transfer Function in the Presence of Noise," Trans. IEEE on Automatic Control AC-9, 229-235 (July, 1964).

64. H. Cox, "On the Estimation of State Variables and Parameters for Noisy Dynamic Systems," Trans. IEEE on Automatic Control AC-9, 5-12 (January, 1964).

65. H. J. Kushner, "A Simple Iterative Procedure for the Identification of the Unknown Parameters of a Linear Time-Varying Discrete System," J. of Basic Engineering 85, 227-235 (June, 1963).

66. H. C. Hsieh, " The Least-Squares Estimation of Linear and Nonlinear System Weighting Function Matrices," Information and Control 7, 84-115 (March, 1964).

67. P. Eykhoff, "Some Fundamental Aspects of Process-Parameter Estimation," Trans. IEEE on Automatic Control AC-8, 347-357 (October, 1963).

68. L. J. Tick, "The Estimation of Transfer Functions of Quadratic Systems," Technometrics 3, 563-567 (November, 1961).

69. K. S. P. Kumar and Sridhar, "On the Identification of Control Systems by the Quasi-Linearization Method," Trans. IEEE on Automatic Control AC-9, 151-154 (April, 1964).

70. R. E. Bellman, "Functional Equations in the Theory of Dynamic Programming, II. Nonlinear Differential Equations," Proc. Natl. Acad. Sci. 41, 482-485 (July, 1955).

71. R. E. Kalaba, "On Nonlinear Differential Equations, the Maximum Operation, and Monotone Convergence," J. Math. and Mech. 8, 519-574 (July, 1959).

72. K. S. Narendra and L. E. Mc Bride, "Multiparameter Self-Optimizing Systems Using Correlation Techniques," Trans. IEEE on Automatic Control AC-9, 31-38 (January, 1964).

73. H. J. Kelley, "Method of Gradients, Chap. 6 of Optimization Techniques, G. Leitman (ed.), Academic Press, New York, 1962.

74. L. A. Zadeh, "On the Identification Problem," Trans. IRE on Circuit Theory CT-3, 277-281 (December, 1956).

Unclassified

Security Classification

DOCUMENT CONTROL DATA - R&D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author)

Cruft Laboratory
Division of Engineering and Applied Physics
Harvard University, Cambridge, Massachusetts

2a. REPORT SECURITY CLASSIFICATION

Unclassified

2b. GROUP

3. REPORT TITLE

THE CHARACTERIZATION AND IDENTIFICATION OF SYSTEMS

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)

Interim technical report

5. AUTHOR(S) (Last name, first name, initial)

Arnold, Charles R. and Narendra, Kumpati S.

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
18 June 1965	84	74

8a. CONTRACT OR GRANT NO.	9a. ORIGINATOR'S REPORT NUMBER(S)
Nonr-1866(16(	Cruft Laboratory Technical Report No. 471
b. PROJECT NO. NR-372-012	
c.	9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)
d.	

10. AVAILABILITY/LIMITATION NOTICES

Qualified requesters may obtain copies of this report from DDC. All others should apply to the Clearinghouse for Federal Scientific and Technical Information, Sills Building, 5285 Port Royal Road, Springfield, Virginia 22151.

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY
Research supports in part by Division of Engineering and Appl. Phys., Harvard University, Cambridge, Massachusetts	Joint Services Electronics Program Office of Naval Research Washington, D. C.

13. ABSTRACT

This report provides a fairly exhaustive survey of deterministic models for the characterization of systems and of the techniques available for the subsequent identification of some physical system in terms of each of the various classes of models. The term "system" refers, in general, to a nonlinear system and the major emphasis of the report is thus upon nonlinear models and their identification requirements.

Most of the ideas presented in this report are not new. However, many of them are not widely known in the engineering field and it was thus felt desirable to collect, organize, and present them as clearly as possible for the engineer.

DD FORM 1 JAN 64 1473

Unclassified

Security Classification

14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
System Models						
System Characterization						
System Identification						
Nonlinear Systems						

INSTRUCTIONS

1. **ORIGINATING ACTIVITY:** Enter the name and address of the contractor, subcontractor, grantee, Department of Defense activity or other organization *(corporate author)* issuing the report.

2a. **REPORT SECURITY CLASSIFICATION:** Enter the overall security classification of the report. Indicate whether "Restricted Data" is included. Marking is to be in accordance with appropriate security regulations.

2b. **GROUP:** Automatic downgrading is specified in DoD Directive 5200.10 and Armed Forces Industrial Manual. Enter the group number. Also, when applicable, show that optional markings have been used for Group 3 and Group 4 as authorized.

3. **REPORT TITLE:** Enter the complete report title in all capital letters. Titles in all cases should be unclassified. If a meaningful title cannot be selected without classification, show title classification in all capitals in parenthesis immediately following the title.

4. **DESCRIPTIVE NOTES:** If appropriate, enter the type of report, e.g., interim, progress, summary, annual, or final. Give the inclusive dates when a specific reporting period is covered.

5. **AUTHOR(S):** Enter the name(s) of author(s) as shown on or in the report. Enter last name, first name, middle initial. If military, show rank and branch of service. The name of the principal author is an absolute minimum requirement.

6. **REPORT DATE:** Enter the date of the report as day, month, year, or month, year. If more than one date appears on the report, use date of publication.

7a. **TOTAL NUMBER OF PAGES:** The total page count should follow normal pagination procedures, i.e., enter the number of pages containing information.

7b. **NUMBER OF REFERENCES:** Enter the total number of references cited in the report.

8a. **CONTRACT OR GRANT NUMBER:** If appropriate, enter the applicable number of the contract or grant under which the report was written.

8b, 8c, & 8d. **PROJECT NUMBER:** Enter the appropriate military department identification, such as project number, subproject number, system numbers, task number, etc.

9a. **ORIGINATOR'S REPORT NUMBER(S):** Enter the official report number by which the document will be identified and controlled by the originating activity. This number must be unique to this report.

9b. **OTHER REPORT NUMBER(S):** If the report has been assigned any other report numbers *(either by the originator or by the sponsor)*, also enter this number(s).

10. **AVAILABILITY/LIMITATION NOTICES:** Enter any limitations on further dissemination of the report, other than those imposed by security classification, using standard statements such as:

(1) "Qualified requesters may obtain copies of this report from DDC."

(2) "Foreign announcement and dissemination of this report by DDC is not authorized."

(3) "U. S. Government agencies may obtain copies of this report directly from DDC. Other qualified DDC users shall request through _____."

(4) "U. S. military agencies may obtain copies of this report directly from DDC. Other qualified users shall request through _____."

(5) "All distribution of this report is controlled. Qualified DDC users shall request through _____."

If the report has been furnished to the Office of Technical Services, Department of Commerce, for sale to the public, indicate this fact and enter the price, if known.

11. **SUPPLEMENTARY NOTES:** Use for additional explanatory notes.

12. **SPONSORING MILITARY ACTIVITY:** Enter the name of the departmental project office or laboratory sponsoring *(paying for)* the research and development. Include address.

13. **ABSTRACT:** Enter an abstract giving a brief and factual summary of the document indicative of the report, even though it may also appear elsewhere in the body of the technical report. If additional space is required, a continuation sheet shall be attached.

It is highly desirable that the abstract of classified reports be unclassified. Each paragraph of the abstract shall end with an indication of the military security classification of the information in the paragraph, represented as *(TS), (S), (C), or (U)*.

There is no limitation on the length of the abstract. However, the suggested length is from 150 to 225 words.

14. **KEY WORDS:** Key words are technically meaningful terms or short phrases that characterize a report and may be used as index entries for cataloging the report. Key words must be selected so that no security classification is required. Identifiers, such as equipment model designation, trade name, military project code name, geographic location, may be used as key words but will be followed by an indication of technical context. The assignment of links, roles, and weights is optional.

Cruft Laboratory Technical Report No. 471
"The Characterization and Identification of Systems"
By Charles R. Arnold and Kumpati S. Narendra
Contract Nonr-1866(16) / NR-372-012

ERRATA

Page 10, line 5, for "define", read "defining"

Page 10, line 6, insert a comma after "before"

Page 16, line 4* (from the bottom) remove period

Page 16, line 3*, for "One" read "one"

Page 20, line 5, for "space of functions $\mathcal{F}_X$ and", read "space $\mathcal{F}_X$ of functionals on"

Page 24, line 2, remove superfluous dot

Page 29, line 7*, for "unit variance", read "intensity Δt"

Page 31, line 5, for "unit variance", read "unit intensity"

Page 33, line 6*, for "unit variance", read "unit intensity"

Page 35, line 2, for "$x(t-\tau)x(t-\xi)$" read "$\overline{x(t-\tau)x(t-\xi)}$"

Page 37, line 2, add the phrase "of unit intensity"

Page 50, line 4*, for "measure exactly", read "exactly match"

Page 52, Figure 3.3 should indicate a pair of complex conjugate poles

Page 54-55 The development from equation (3.6) through equation (3.10) must be for a finite interval, say [0, T]. On the infinite interval, the only kernel for which physically realizable eigenfunctions exist is $R_x(\tau) = \delta(\tau)$, i. e., for white noise. For colored noise, however, the finite expansion interval [0, T] represents no restriction in that in practice no expansion can be done on $[0, \infty)$.

Page 59, line 2*, for "be means", read "as the mean"

Page 68, line 8*, for "$x(t-\xi)x(t-\eta)x(t-\tau)x(t-\lambda)$" read "$\overline{x(t-\xi)x(t-\eta)x(t-\tau)x(t-\lambda)}$"

Page 69, line 6, for "unit variance", read "unit intensity"

Page 74, Figure 4.9, the second multiplier preceding the final average is incorrect. It should indicate the difference between the preceding multiplier and the sum of the sub-averages.