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QUASI-CORES IN A MONETARY ECONOMY WITH NONCONVEX PREFERENCES

L. S. Shapley and Martin Shubik

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PREFACE

In this Memorandum the authors present some mathematical results concerning a competitive situation arising in economic theory, using the techniques of the theory of games. Game theory is important in its general applicability to a variety of conflict situations—economic, political, and military.

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ABSTRACT

A model of a pure exchange economy is investigated without the usual assumption of convex preference sets for the participating traders. The concept of core, taken from the theory of games, is applied to show that if there are sufficiently many participants, the economy as a whole will possess a solution that is sociologically stable—i.e., that cannot profitably be upset by any coalition of traders.

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QUASI-CORES IN A MONETARY ECONOMY
WITH NONCONVEX PREFERENCES

1. INTRODUCTION

In his study of bilateral monopoly, Edgeworth [8] introduced a solution concept, called the contract curve, which consists of those Pareto-optimal exchanges that are at least as attractive to each monopolist as the initial, no-trade position. He then went on to observe that if the number of entrepreneurs on each side of the market were increased, then the contract curve, or contract set, should diminish in size and converge eventually to the set of competitive allocations, which are those that can be derived from the initial position by direct budgetary optimization on the part of the individual traders under a fixed schedule of prices. This shrinking of the contract set depended on the possibility of recontracting, it being postulated that no proposed allocation of goods would be finally acceptable to the market as a whole if there were any subset of traders who, by exchanging goods only amongst themselves, could all do better.

The recontracting principle corresponds closely to the notion of domination that underlies two important solution concepts in n-person cooperative game theory, to wit, the core and the von Neumann-Morgenstern solution.¹ Indeed, if

¹The core of a game is the set of all undominated outcomes [11, I]. A solution is any set of outcomes, mutually undominating, that collectively dominate all others [21]. The term "core" was introduced by Gillies and Shapley [13, 10] in studying properties of the von Neumann-Morgenstern solutions;

one regards the expanded Edgeworth model as a multi-person game, then the contract curve, as curtailed by recontracting, is precisely the core. The uncurtailed contract curve, on the other hand, is a solution, though not generally the only one. In the original case of bilateral monopoly, the core, the contract curve, and the unique solution all coincide.

A number of refinements and extensions of Edgeworth's convergence theorem have been obtained by Shubik [17], Debreu and Scarf [6, 7], Aumann [2], and Vind [19, 20], who all exploit to some degree the game-theoretic point of view. The preferences of the individual traders are usually assumed to be convex, since, as is well known, without convexity a competitive equilibrium may not exist. The present paper, on the other hand, is concerned especially with markets that possess neither a competitive equilibrium nor a core; indeed, in the presence of either one our chief theorems would be almost trivial to prove.²

The general tenor of our results is that nonconvexity in the preference sets is economically unimportant when the number of participating individuals is large. This may not be a new observation, but here we give it a mathematically precise expression in a fresh context.

More specifically, we show under quite general conditions that even if the true core cannot be assured, because

the core as an independent solution concept was developed by the latter in lectures at Princeton in the fall of 1953.

²In brief, we have: convex preferences - competitive equilibrium - core - strong ϵ -core - weak ϵ -core. The implications are generally not valid in the reverse direction.

of nonconvexity, nevertheless certain quasi-cores will always appear when the number of traders in the market is large enough. These quasi-cores are characterized by the requirement that recontracting traders must show a definite positive profit before being permitted to block a proposed final allocation. A distinction must be made ("weak" vs. "strong" ϵ -cores) depending on whether the ϵ -threshold is applied to the whole set of recontracting traders, or to each one individually. The conditions for existence are somewhat different in the two cases (Theorems 2 and 4).

There is reason to believe that the ϵ -cores, in a double limit process involving both ϵ and the number of traders, would shrink down à la Edgeworth upon the competitive allocations of a certain related economy, obtained by "concavification" of the original utility functions. In the original market these allocations, though feasible, need not be Pareto optimal. The associated prices, on the other hand, may lead under individual optimization only to infeasible allocations—that is, to mismatched supply and demand. The prices will at least be in pseudo-equilibrium, however, in the sense that if the excess demand for any good is positive, then another scheme of individual optimizations can be found that makes it negative, and vice versa. (Nonuniqueness of optimum demand schedules is to be expected when preferences are nonconvex.) As yet we have not found sharp conditions for the convergence of the ϵ -cores, and our mathematical results in this paper deal

only with the existence question.

Recently, new attention has been focused on the economic implications of nonconvex preferences [5, 9, 12, 15]. Unfortunately, convexity is too often assumed in economics merely because tools for handling the opposite assumption are lacking. Not the least important feature of the present investigation, we believe, lies in its contribution to the mathematical technology of nonconcave utility functions. We have already mentioned the device of concavification; we also introduce a condition of "spannability" that helps restore some of the regularity that is lost when concavity is abandoned, and which seems potentially useful beyond the immediate context of cores and competitive equilibria.

Several simple examples are discussed in Secs. 3 and 11 in order to give the reader a feeling for the effects of nonconvex preferences. In an effort to keep clear the main lines of the argument, some of the drier mathematical demonstrations have been placed in appendices.

2. TWO SPECIAL ASSUMPTIONS: MONEY AND EQUAL TASTES

The mathematical results that we present are to a certain extent illustrative, rather than comprehensive, since they rest upon two fairly drastic assumptions, in an otherwise very general setting. This is partly a matter of technical expedience. It is often good policy, especially in the social sciences, to work with simplified models, in order to gain insight and discover techniques for tackling more complex situations. But we should emphasize that

our present assumptions appear to have no direct bearing on the main subject of this investigation, namely, the effects of nonconvex preferences on the existence of cores.

First, we shall assume that in addition to the actual commodities in exchange there is a kind of "money" available, which can be separated out of individual utility functions in the following way:

$$(1) \quad u^i(x_1^i, \dots, x_m^i, \xi^i) = U^i(x_1^i, \dots, x_m^i) + \lambda_i \xi^i.$$

Here x_j^i is the amount of the j^{th} commodity held by the i^{th} individual, ξ^i is his "money" level, and λ_i is a constant.³ We assume that (1) holds to a sufficient degree of approximation for a range of values of ξ^i sufficiently wide to encompass the action of the model. Like the actual commodities, "money" is assumed perfectly divisible and freely transferable; unlike them, is not constrained by an a priori lower bound. This entails no loss of realism, since in practice the relation (1) will begin to fail long before the individual faces bankruptcy, or a hard and fast credit ceiling.

It is worth remarking that this idealized money, though sometimes called "transferable utility," does not entail interpersonal comparisons of utility at the psychological level. Money serves merely to de-couple

³The constant marginal utility for money in (1) is not of itself at issue. We could just as well assume the form $w^i = f_i(U^i + \lambda_i \xi^i)$; with f_i any order-preserving transformation. It would then be merely a matter of relabeling utility levels to go from w^i to u^i .

individual tastes. If a dollar is taken from millionaire X, then his utility is diminished by one dollar's worth of X-utils; if it is given to beggar Y, then the latter's utility is increased by one dollar's worth of Y-utils. There is no psychological comparison of "degrees of happiness." The key phrase "one dollar's worth" has meaning because of the identity (1), which is assumed separately for each individual. Admittedly, it is a poor assumption for a beggar, since it denies any income effect. It might be a good assumption for a millionaire, within a suitably restricted range. In any event, it is not an interpersonal assumption.⁴

⁴It has long been the fashion to formulate elementary models of exchange on a barter basis. Though somewhat removed from everyday experience, such models are easy to think about and work with, and have an air of abstract generality that is appropriate to the fundamental questions that they deal with. Money is, after all, a very complex phenomenon, with many nonelementary ramifications. Build a theory of barter economics first, we are urged, and let "money" appear, if at all, only in the guise of a commodity in exchange.

In game-theoretic models, however, which are usually an order of magnitude more complex than the associated mechanistic models, money paradoxically is often a simplifying factor. The expediency of the "barter-first" approach is therefore called into question. Money, the de-coupler, smooths away many of the difficulties that arise in dealing simultaneously with many independently motivated sources of strategic decision. Both the models and their solutions can be more readily formulated, analyzed, and interpreted if a suitably idealized money is present.

The authors feel that this is not merely a formal theoretical consideration, but that a monetary de-coupler is a very basic feature of human economic behavior in bargaining situations, and that when both goods and information are freely transferable it would be difficult in fact, hence unrealistic in a model, to prohibit the adoption of some such comparator and carrier of value. To treat money as just another commodity, whether this is technically expedient or not, only conceals the unique and remarkable role that it plays in real bargaining processes.

In this view, an elementary model of an exchange

The advantage of the assumption (1), in the present case, is that it permits us to reduce the game model to characteristic-function form (see Sec. 4). This is a powerful analytic and conceptual device first introduced by von Neumann and Morgenstern [21]. Whether our present results could be duplicated without its aid is not yet clear.

The second special assumption is that all individuals in the exchange economy have equal tastes. Formally, this means that their utility functions u^i are interrelated by order-preserving transformations f_{ij} :

$$(2) \quad u^i(x_1, \dots, x_m, \xi) = f_{ij}(u^j(x_1, \dots, x_m, \xi)), \text{ all } i, j.$$

Since we are free to re-calibrate the individual utility scales, this means that all preferences can be expressed by means of a single function u . This kind of assumption is often made in economic modeling.

It will become clear that "equal tastes" does not symmetrize the model to the point of triviality. For one thing, the initial endowments of the traders may be different.

economy could quite properly include an idealized "money" like (1) from the start, and leave to later elaborations the important but less than vital questions of transfer restrictions, income effects, individual spending ceilings, credit costs, etc.

A reader unable to accept this view is, of course, free to regard our results as pertaining only to special economies in which one of the commodities, in plentiful supply, happens to enter the utility functions in the manner (1). What we are trying to suggest, however, is that the hypothesis of a "transferable utility" is not only expedient, in this context, but methodologically respectable as well.

Moreover, the solutions that one gets with nonconvex preferences are often nonsymmetric even when the initial holdings are identical—the "economics" of the situation can drive identical traders to behave differently. This effect will be illustrated in the next section.

The advantage of assuming (2), in the present work, is that it allows us to write down explicitly certain optimizing allocations, beginning at (17) below. This simplifies the mathematical arguments in several places, but similar proofs of similar theorems could be carried out, we believe, without this aid.

The combined effect of our two assumptions is to permit us to postulate a single utility function for goods and money, of the form

$$(3) \quad u(x_1, \dots, s_m, \xi) = U(x_1, \dots, s_m) + \xi,$$

applicable to all traders in the economy.

3. AN EXAMPLE

The following example is intended to give some idea of the role played by money and equal tastes in a market with nonconvex preferences.⁵ The model is highly symmetric having n identical traders and two identical goods, but the competitive and core solutions, when they exist, will require nonidentical behavior on the part of the traders.

The nonconvexity in the preference sets (see Fig. 1) comes from preferred ratios in the consumption process.

⁵This example received brief mention in [7]. Some other simple examples are examined in Sec. 11 below.

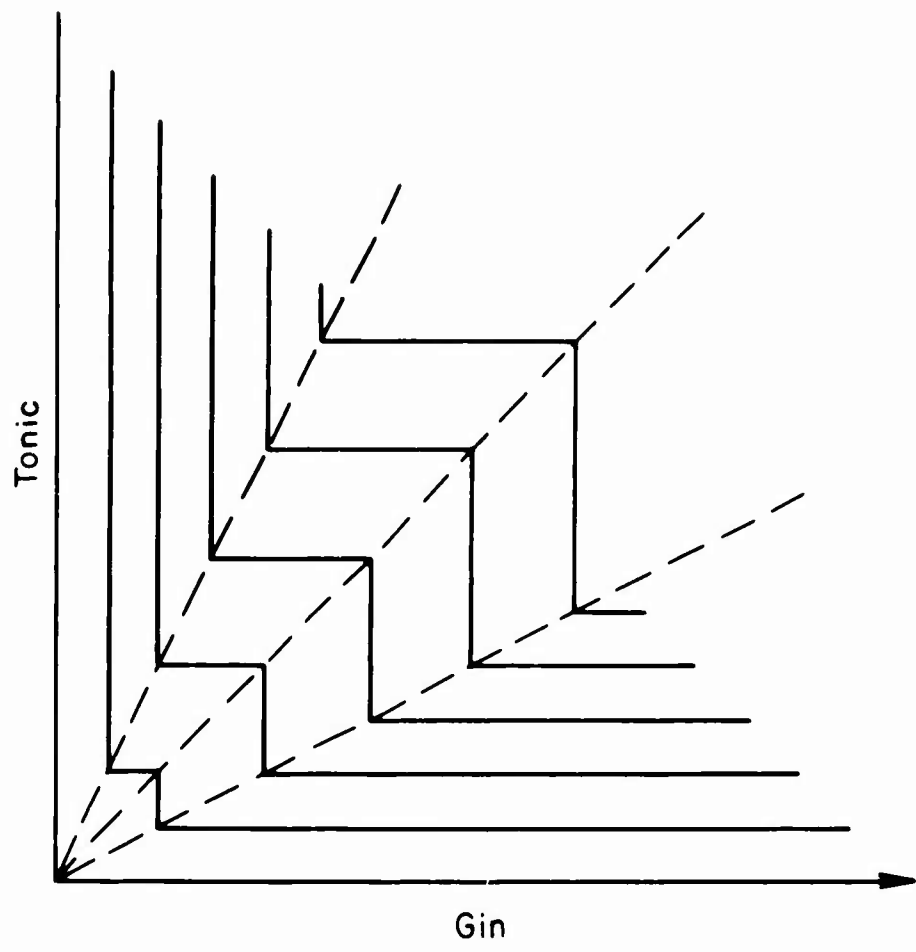


Fig. 1—The commodity space

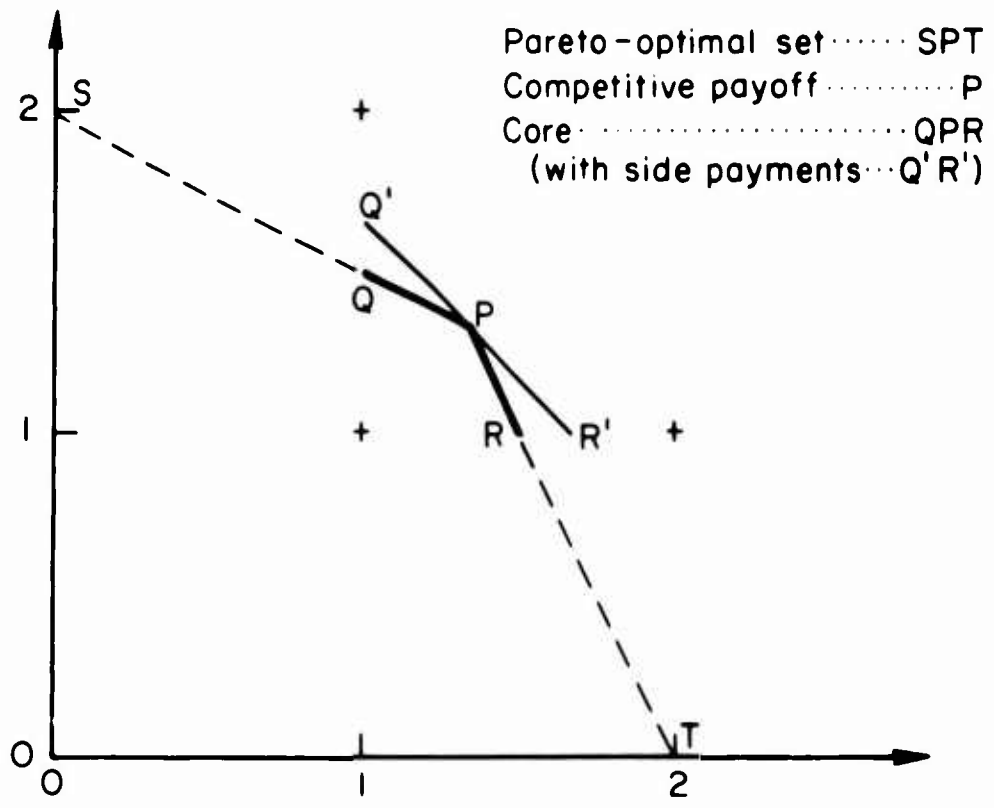


Fig. 2—The utility space

For a numerical utility function, we may take

$$(4) \quad U(x) = \max [\min(2x_1, x_2), \min(x_1, 2x_2)].$$

The traders are assumed to start with one unit of each good apiece.

For example, the goods might be gin and tonic. Each trader is indifferent as between weak drinks (1 to 2) and strong drinks (2 to 1), but he will not take both, and rejects intermediate (or more extreme) concoctions.

In looking for a competitive equilibrium, we first observe that unequal prices will never work, whether or not there is money available, since the traders would all want to buy the cheaper good and sell the other. With equal prices, however, a competitive allocation can sometimes be reached. In fact, if n is even, each man can trade $1/3$ units of one good for an equal amount of the other, and end up with an efficient bundle—either $(4/3, 2/3)$ or $(2/3, 4/3)$. The result is worth $4/3$ to everyone. Of course, it may require cooperative action, or a determined hostess, to decide who is to get which drink!

But if n is odd, there is no series of exchanges (at equal prices) that gives everyone an efficient ratio, and without money there can be no competitive equilibrium. Rather, we have the pseudo-equilibrium situation described in the Introduction, in which the excess demand for either commodity can be construed as either positive or negative, but not zero.

Introducing money (3) changes the picture somewhat. In the even case the competitive prices and final utility levels remain as before, but the allocative possibilities are opened up. Because of the first-degree homogeneity of (4), we can let some of the traders liquidate their holdings for cash, provided that others spend an equal amount to increase their consumption, and all stick to efficient ratios. In the odd case, this same option works to circumvent the previous difficulty in matching supply and demand, and it is easy to find feasible competitive allocations. For example, one man might sell out completely, leaving an even number of traders to divide the goods efficiently. The final payoff, as in the even case, will be exactly $4/3$ to each trader. Thus, we see that it is possible for money to overcome the "ill" effects of nonconvexity.

The homogeneity of (4) was crucial to this result, however. Had we assumed diminishing returns to scale, then money would no longer "save" the odd case. To see this, let us modify the utility function to be the square root of the function in (4). Then the equilibrium prices for each good would have to be (as it turns out) precisely $\sqrt{1/12}$. What is important is not this number, but the fact that each trader would have only two ways to optimize, neither way involving any net money transfer. In short, money would be irrelevant, and there would be no competitive solution.

What of the cores? Returning to the original function (4), and assuming $n = 2$, one can easily determine the Edgeworth contract curve for the no-money case. It proves to be a rather spectacular "curve": four triangular regions arranged in a ring! The core, its image in the utility space, is the bent line QPR shown in Fig. 2. It includes the competitive payoff P, and is included in the set of Pareto-optimal payoffs SPT. The introduction of money is not easily shown in the commodity space because of the added dimension, but in the utility space the effect is simply to move the Pareto set out to the straight line with slope -1 passing through the point P, of which the segment Q'PR' constitutes the core.

For $n > 2$ the recontracting principle takes over with a vengeance: the shrinkage is immediate and total. The cores actually contain no points other than the competitive payoffs. That is, when n is odd and side payments are not permitted, there is no core at all; in all other cases the core is a single point. This drastic curtailment of the core is rather atypical, and may be ascribed to the simple polyhedral form chosen for the indifference map.

These assertions about the cores are not meant to be obvious. The proofs, given in Appendix 1, may be of interest to the reader wishing experience with the techniques of core analysis.⁶

⁶The present example provides no instance of a core without a competitive equilibrium; such cases, however, are easily constructed.

4. GAMES AND CORES

The characteristic function of a game [21, 14] is designed to express the optimum result or results obtainable by each coalition S of players, regardless of the actions of the players outside S . In the transferable-utility case,⁷ it is a real-valued set function $v(S)$, arbitrary except for the condition $v(0) = 0$ and for the property of superadditivity:

$$(5) \quad v(S \cup T) \geq v(S) + v(T), \text{ if } S \cap T = \emptyset.$$

The outcome of the game is conveniently summarized in a payoff vector, α , with components measured in utility units; clearly, we must have

$$(6) \quad \sum_N \alpha_i \leq v(N),$$

where N denotes the set of all players. A payoff vector is called an imputation if it satisfies two further requirements:

$$(7) \quad \sum_N \alpha_i \geq v(N)$$

and

⁷Without transferable utility, the characteristic function is set-valued, and can be defined in two ways: $v(S)$ represents either (a) the set of payoff vectors that S can surely achieve, or (b) the set of payoff vectors that S cannot be prevented from achieving. The two theories that emerge are not generally equivalent, because there is no minimax theorem for coalitions in the absence of transferable utility. (See Aumann and Peleg [4].)

$$(8) \quad x_i \geq v(\{i\}) \text{ for all } i \in N.$$

These correspond respectively to Pareto optimality and individual rationality. Conditions (6), (7), and (8) are always consistent because (5) implies that $v(N) \geq \sum v(\{i\})$.

The more stringent requirement of group rationality:

$$(9) \quad \sum_S x_i \geq v(S) \text{ for all } S \subset N,$$

includes both (7) and (8). The set of payoff vectors thereby delimited, if any, is called the core. Since (6) and (9) may well be inconsistent, however, the core need not exist.

Speaking informally and intuitively, a coreless game ought to be more competitive and harder to stabilize than one with a core. Indeed, in any game, an observed outcome falling outside the core would seem to admit just two interpretations: (a) it is a transient event in some dynamic process, or (b) it is evidence of a social structure among the players that inhibits some coalitions from developing their full potential. On the other hand, an observed outcome in the core tells us nothing about the organization of society; the core is sociologically neutral [16].

The mathematical results in this paper depend on the device of enlarging the core by a small amount. Two related concepts will be needed. If ϵ is a small positive number, we define the strong ϵ -core as the set of payoff

vectors α satisfying

$$(10) \quad \sum_S \alpha_j \geq v(S) - \epsilon, \quad \text{for all } S \subset N.$$

We define the weak ϵ -core as the set of payoff vectors satisfying

$$(11) \quad \sum_S \alpha_i \geq v(S) - s\epsilon, \quad \text{for all } S \subset N,$$

where s denotes the number of elements of S . It is easy to verify the following chain of set-inclusions:

$$\text{weak } \epsilon\text{-core} \supset \text{strong } \epsilon\text{-core} \supset \text{weak } \frac{\epsilon}{n}\text{-core} \supset \dots \supset \text{core},$$

where n is the number of players in the game.

These quasi-cores are not merely technical devices, looking toward an eventual convergence theorem. They provide a way, for example, of taking into account the costs of coalition-forming. Under the weak definition, the costs would depend on the size of the coalition; in the strong case there would be a fixed charge. Alternatively, we might regard the organization costs as negligible, or already included in $v(s)$, but view the ϵ or $s\epsilon$ as a threshold, below which the blocking maneuver implicit in (9)—the actual exercise of "group rationality"—is not considered worth the trouble.

5. THE MARKET MODEL

Let there be m different commodities and t_0 different types of traders, distinguished by the stocks of goods

they hold at the beginning of the trading session. The initial endowment of a player of type "t" will be denoted by a vector

$$a^t = (a_1^t, \dots, a_m^t).$$

If S is a set of players, and if s_t denotes the number of players in S of type "t", then the aggregate initial endowment of S may be written as follows:

$$a(\Delta) = \sum_{t=1}^{t_0} s_t a^t,$$

where Δ , called the profile of S, is an abbreviation for the integer vector (s_1, \dots, s_{t_0}) . The total supply of goods in the game is then $a(\nu)$, where ν is the profile of the set N of all players.

At the conclusion of trading, the players hold bundles x^i , $i \in N$, which must account for the total quantities initially present in the market. Thus we have

$$(12) \quad x^i \in E_+^m, \text{ and } \sum_N x^i = a(\nu),$$

where E_+^m denotes the closed positive orthant of Euclidean m-space. Subject to these constraints, all final allocations are assumed possible. (In particular, the outcome is not assumed to be symmetric as between players of the same type.) In addition, there may be direct transfers of money among the players. Thus, if U is the common utility measure for goods (see (3)), the possible final

payoffs will take the form

$$\alpha_i = U(x^i) - \pi_i, \quad \text{all } i \in N,$$

subject to (12) and $\sum_N \pi_i = 0$.

By symmetry, the characteristic function depends only on the profiles of the coalitions. Since internal money transfers will cancel, we have

$$(13) \quad v_U(\mathcal{A}) = \sup_y \sum_{v=1}^s U(y^v), \quad \text{subject to } \begin{cases} \text{all } y^v \in E_+^m, \\ \sum y^v = a(\mathcal{A}), \end{cases}$$

where s denotes the sum of the s_t . Players outside the coalition do not affect this value, since they can neither force nor be forced into dealings with the coalition members.⁸

To ensure that the "sup" in (13) is finite, we shall assume from the outset that U is bounded above on compact subsets of E_+^m , or, equivalently, that there is a continuous function K such that $U(x) \leq K(x)$ for all $x \in E_+^m$. (Compare condition (15) below.)

Let $\Gamma_U(\mathcal{N})$ designate the game we have just defined. The subscript "U" will serve to distinguish it from an auxiliary game to be introduced in the next section.

⁸Hence, questions regarding the credibility and proper evaluation of threats do not arise in this case, though they play an important role in the general theory of cooperative games.

6. THE COMPETITIVE SOLUTION

An allocation $\{x^{*i}\}$ satisfying (12) is called competitive if there exists a price vector $p = (p_1, \dots, p_m)$ such that, for each individual $i \in N$, the bundle x^{*i} maximizes the expression

$$(14) \quad U(x^i) - p \cdot (x^i - a^i).$$

The numbers p_j are called equilibrium prices. Observe that there are no "budget" conditions $p \cdot (x^i - a^i) = 0$, since our money enters directly into the complete utility function (4). Had we approached this definition via a budgeted money-of-account (taking our present "valuable" money to be the $m+1$ -st commodity), then the usual definition would give us equilibrium prices p'_1, \dots, p'_{m+1} , related to the above by $p_j = p'_j/p'_{m+1}$. Thus, the actual numbers p_j are significant, not just their ratios, as is evident from the form of (14).

Let α_i^* denote the maximum of (14) for a given equilibrium price vector p^* . Then it is a simple matter to show that the vector α^* , which we shall refer to as a competitive imputation, is in the core of the game $\Gamma_U(n)$. Indeed, we have at once

$$\sum_N \alpha_i^* = \sum_N U(x^{*i}) \leq v_U(n),$$

verifying the feasibility requirement (6). To verify the core inequalities (9), we fix $S \subset N$ and $\epsilon > 0$, and, using

(13), find bundles $y^i \in E_+^m$ such that

$$\sum_S U(y^i) \geq v_U(\mathcal{A}) - \epsilon \text{ and } \sum_S y^i = a(\mathcal{A}).$$

Since α^* maximizes, we have

$$\alpha_i^* \geq U(y^i) - p^* \cdot (y^i - a^i) \text{ for each } i \in S.$$

Summing, we obtain

$$\sum_S \alpha_i^* \geq \sum_S U(y^i) \geq v_U(\mathcal{A}) - \epsilon.$$

Since this is valid for arbitrarily small ϵ , (9) follows, completing the proof.

Theorem 1. Every competitive imputation is in the core.

We remark that this result does not depend on equal tastes, since in the proof just given each $U(\cdot)$ could be replaced by $U^i(\cdot)$. We should also emphasize that nothing has been said about the existence of a competitive solution.

7. CONCAVIFICATION OF THE UTILITY FUNCTION

For our theorem on weak ϵ -cones we shall impose almost no conditions on the function U . It need not be concave, continuous, or monotonic, and it may be either bounded or unbounded. We shall require, however, that its asymptotic growth be no more than linear, and that it be bounded from below on all compact subsets of E_+^m . To this end, we assume the existence of a linear function L_0 and a continuous

function K_0 such that the inequalities

$$(15) \quad K_0(x) \leq U(x) \leq L_0(x)$$

hold for all x in the commodity space E_+^m .

Let us now define a function C on E_+^m as follows:

$$(16) \quad C(x) = \sup \sum_{h=1}^{m+1} \lambda_h U(y^h), \text{ subject to } \begin{cases} \lambda_h \geq 0, \sum \lambda_h = 1, \\ y^h \in E_+^m, \sum \lambda_h y^h = x. \end{cases}$$

The finiteness of this "sup" is ensured by (15); indeed, we have $C(x) \leq L_0(x)$. The function C is concave, it majorizes U , and it is the least such function.⁹ We may remark that C is continuous at every interior point x of E_+^m and possesses a linear support there, i.e., a linear function L such that $L \geq C$ and $L(x) = C(x)$. If the "sup" in (16) is actually achieved for all x , we shall say that U is spannable (see Sec. 10).

We intend to use C as an artificial utility function in defining a concave majorant game $\Gamma_C(n)$, identical in every other respect to the game $\Gamma_U(n)$ previously defined. The characteristic function v_C has a simple explicit form. Since players have identical, concave utilities (hence convex preferences), a coalition achieves maximum profit by dividing its total endowment equally among its members, and we have

⁹The use of $m + 1$ spanning points is sufficient to "concavify" any linearly bounded function on E^m . Use of a larger number in (16) would not affect the value of $C(x)$.

$$(17) \quad v_C(\mathcal{A}) = sC\left(\frac{a(\mathcal{A})}{s}\right).$$

We see that $v_C(\mathcal{A})$, like $a(\mathcal{A})$, is homogeneous of degree one:

$$(18) \quad v_C(k\mathcal{A}) = kv_C(\mathcal{A}), \quad k = 0, 1, 2, \dots,$$

indicating constant returns to scale to a uniformly expanding coalition in the artificial game $\Gamma_C(\mathcal{A})$. For the true game $\Gamma_U(\mathcal{A})$, on the other hand, we do not have homogeneity in general, but only the inequality

$$(19) \quad v_U(k\mathcal{A}) \geq kv_U(\mathcal{A}), \quad k = 0, 1, 2, \dots$$

—a consequence of superadditivity (5). Our results on the existence of ϵ -cores will hinge on showing that $v_U(\mathcal{A})$ is nevertheless "almost" homogeneous, in a sense made specific in the lemmas accompanying Theorems 2 and 4.

8. EXISTENCE OF THE WEAK ϵ -CORE

We consider exchange economies based on utility function U satisfying (15).

Theorem 2. For every profile $\mathcal{A} = (n_1, \dots, n_{t_0})$, and for every $\epsilon > 0$, there exists a constant k_0 such that all games $\Gamma_U(k\mathcal{A})$ with $k \geq k_0$ possess weak ϵ -cores.

Lemma on weak ϵ -homogeneity of v_U . For every profile \mathcal{A} and for every $\epsilon > 0$, there exists a constant $k_0 = k_0(\mathcal{A}, \epsilon)$ such that

$$(20) \quad v_C(\Delta) - \epsilon \leq \frac{1}{k} v_U(k\Delta) \leq v_C(\Delta)$$

holds for all $k \geq k_0$.

Proof. Fix Δ and ϵ , and let $x^* = a(\Delta)/s$. Then, by (17),

$$v_C(\Delta) = sC(x^*).$$

Using (16), find a convex representation $x^* = \sum \lambda_h y^h$ such that

$$(21) \quad C(x^*) \leq \sum_{h=1}^{m+1} \lambda_h U(y^h) + \frac{\epsilon}{2s}.$$

Let ℓ_h denote the greatest integer in $k s \lambda_h$, $h = 1, \dots, m+1$.

Then

$$(22) \quad \sum \ell_h y^h \leq k s \sum \lambda_h y^h = k s x^* = a(k\Delta).$$

Thus, in a coalition with profile $k\Delta$, it is possible to assign the bundle y^1 to the first ℓ_1 players, y^2 to the next ℓ_2 players, and so on. If " \leq " holds in (22), there will be goods left over after this allotment, but there will also be at least one player left over, too. Allotting the excess goods equally among the extra players gives them each an allocation y^* that lies within the convex hull of the $\{y^h\}$. Since there are at most m extra players, we can write down an upper bound for the amount that such an allotment of excess might cause to be deducted from the total coalition utility, namely:

$$B = m \cdot \left[\min_{y^*} K_0(y^*) \right], \text{ subject to } y^* \in \text{convex hull of } \{y^h\}.$$

(This is the only use of the function K_0 postulated in (15).) The important fact about this bound is that it is independent of k .

We have thus described a feasible allocation, whose value to the coalition is at least $\sum \lambda_h U(y^h) - B$. Thus

$$v_U(k) \geq \sum \lambda_h U(y^h) - B.$$

Applying (21) and the definitions of λ_h and x^* , we obtain

$$\begin{aligned} v_U(k) &\geq ks \sum \lambda_h U(y^h) + \sum (\lambda_h - ks \lambda_h) U(y^h) - B \\ &\geq ks C(x^*) - \frac{k\epsilon}{2} + \sum (\lambda_h - ks \lambda_h) U(y^h) - B \\ &\geq kv_C - \frac{k\epsilon}{2} - \sum |U(y^h)| - B \\ &= kv_C - \frac{k\epsilon}{2} - \frac{k_0 \epsilon}{2}, \end{aligned}$$

where

$$k_0 = \frac{2}{\epsilon} (\sum |U(y^h)| + B) \geq 0.$$

Then $k \geq k_0$ implies that

$$v_U(k) \geq kv_C - k\epsilon,$$

giving us one side of (20). The other side is a consequence of (18) and the general inequality $v_U \leq v_C$. This completes the proof of the lemma.

Proof of Theorem 2. Let α be the payoff vector associated with a competitive allocation of the concave game $\Gamma_C(n)$. For example, let $x^* = \alpha(n)/n$, and take

$$\alpha_i = C(x^*) - p^* \cdot (x^* - a^i), \quad \text{all } i \in N,$$

where p^* is the gradient of any linear support function $L^* \geq C$ with $L^*(x^*) = C(x^*)$.¹⁰ By Theorem 1, α is in the core of $\Gamma_C(n)$. Moreover, for every k , the k -fold replication of γ is the payoff vector of a competitive allocation of the larger game $\Gamma_C(kn)$, and lies in its core. Denote this k -fold replication (a vector with kn components) by $\alpha^{(k)}$. We shall now construct a nearby imputation $\beta^{(k)}$ of the game $\Gamma_U(kn)$.

Denote the difference $v_C(kn) - v_U(kn)$ by g ; clearly $g \geq 0$. Choose an arbitrary n -vector γ whose components sum to g/k and satisfy

$$0 \leq \gamma_i \leq \alpha_i - v_U(\lambda).$$

where λ denotes the profile of the l -player set $\{i\}$. This is possible because of the two inequalities:

$$g/k \leq v_C(n) - v_U(n) \leq \sum_i (\alpha_i - v_U(\lambda)),$$

and

$$0 \leq v_C(\lambda) - v_U(\lambda) \leq \alpha_i - v_U(\lambda).$$

(The first follows from (18), (19), (7) applied to $\Gamma_C(n)$, and the superadditivity of v_U ; the second is a consequence of (8) applied to $\Gamma_C(n)$.) Let $\gamma^{(k)}$ be the k -fold replication of γ , and define

$$(23) \quad \beta^{(k)} = \alpha^{(k)} - \gamma^{(k)}.$$

¹⁰Regardless of equal tastes, it is a simple matter to find the competitive solution(s) of an exchange economy with money, when utilities are concave.

This is the desired imputation of $\Gamma_U(k\mathcal{N})$. We wish to show that it lies in the weak ϵ -core, in the sense of (11), whenever k is greater than the constant $k_0 = k_0(\mathcal{N}, \epsilon)$ provided by the lemma.

Consider, therefore, an arbitrary subset S of the set of all kn players. Note first that

$$(24) \quad \sum_S \alpha_i^{(k)} \geq v_C(\mathcal{A}) \geq v_U(\mathcal{A}),$$

since $\alpha^{(k)}$ is in the core of $\Gamma_C(k\mathcal{N})$ and $C \geq U$. Also we have

$$(25) \quad \sum_S \alpha_i^{(k)} \leq s \frac{g}{k} = sv_C(\mathcal{N}) - \frac{s}{k} v_U(k\mathcal{N}) \leq s\epsilon,$$

since each $\alpha_i \leq g/k$ and $k \geq k_0(\mathcal{N}, \epsilon)$. Combining (23), (24), and (25) now gives the desired result:

$$(26) \quad \sum_S \beta_i^{(k)} \geq v_U(\mathcal{A}) - s\epsilon.$$

This completes the proof of Theorem 2.

We may observe that $\epsilon \rightarrow 0$ as $\epsilon \rightarrow 0$, so that the weak ϵ -cores can be said in a certain sense to possess a limit point—namely, the imputation α replicated an infinite number of times. Let us state this more precisely. Given any competitive payoff α of the concave game $\Gamma_C(\mathcal{N})$, and any $\delta > 0$, then for all sufficiently small $\epsilon > 0$ and for all $k \geq k_0(\mathcal{N}, \epsilon)$ there is an n -vector at a distance less than δ from α , whose k -fold replication is in the weak ϵ -core of $\Gamma_U(k\mathcal{N})$.

It further appears, in analogy with results of Debreu and Scarf [7], that the above does not hold when α is not a competitive payoff of $\Gamma_C(\nu)$. This would mean that the weak α -cores converge (in the above sense) to exactly the set of competitive payoffs of the concavified game. But we have not carried through a proof of this convergence, and it is possible that some regularity assumptions on U might be required.

9. FURTHER CONDITIONS ON U

The extremely weak condition (15) that we have so far imposed upon the utility function will need reinforcement before a result analogous to Theorem 2 can be obtained for strong α -cores; we shall require "spannability" and a certain amount of differentiability for U . Our primary purpose, of course, is the study of the effects of nonconcavity, not the wholesale abandonment of regularity assumptions of all kinds. Our policy of keeping the hypotheses as general as possible, however, serves to make clear precisely what our results do and do not depend upon.

We shall postulate, then, that U is radially differentiable and spannable. The first assumption means that U is differentiable along all rays in E_+^m emanating from the origin.¹¹ This is considerably weaker (if $m > 1$) than assuming the existence of first partial derivatives

¹¹We shall actually need the existence of radial derivatives only at points where U and C are equal.

$\partial U / \partial x_i$.¹² We do not demand that the radial derivatives be continuous, or, indeed, that U itself be continuous.

The second assumption means (see Sec. 7) that there exists a concave function $C \geq U$ such that for each $x \in E_+^m$, there are $m+1$ (or fewer) points $y^h \in E_+^m$, and weights $\lambda_h \geq 0$, such that

$$\sum \lambda_h = 1, \quad \sum \lambda_h y^h = x, \quad \text{and} \quad \sum \lambda_h U(y^h) = C(x).$$

This clearly implies that U is bounded above by a linear function, as previously assumed (right side of (15)). The matching assumption that U is bounded below by a continuous function (left side of (15)) is not implied, and will in fact not be needed.

10. DISCUSSION OF SPANNABILITY

The notion of spannability may be unfamiliar to many readers, but it appears to be quite fundamental to any investigation of the relaxation of convexity conditions. Hence a short digression is in order, to link this notion to other analytic conditions. Some simple examples will follow in the next section.

Let U be called sublinear if for every linear function L with positive coefficients, the difference $U - L$ has a finite upper bound. For example, logarithmic (Bernoullian) utility is sublinear; also, any bounded utility function is sublinear.

¹²Figure 1 provides an example.

Theorem 3. If U is continuous, sublinear, and strictly increasing, then U is spannable.

This theorem (proved in Appendix 2) is not at all sharp. For example, it is apparent from the proof that "strictly increasing" is hardly necessary; a certain very weak insatiability condition would suffice instead, as follows: For each $x \in E_+^m$ and each $j = 1, 2, \dots, m$, there is a $y \in E_+^m$ differing from x only in the j -th component such that $y_j > x_j$ and $U(y) \geq U(x)$.

Another extension of Theorem 3 results from observing that if U is spannable and L is linear, then $U + L$ is spannable, despite the fact that adding L can destroy both sublinearity and monotonicity (or insatiability).

11. SOME EXAMPLES

Let us now consider a very simple example, having just one commodity and one type of player, in order to show how spannability and differentiability will be crucial to the existence of strong ϵ -cores in the limit. Let $U(x) = [x/2]$, i.e., the greatest integer less than or equal to $x/2$, and let all the initial endowments be 1 unit. Here $C(x) \equiv x/2$, and U is spannable, but not (radially) differentiable. (See Fig. 3a.) For a coalition with s members we have

$$v_U(s) = [s/2] \text{ and } v_C(s) = s/2.$$

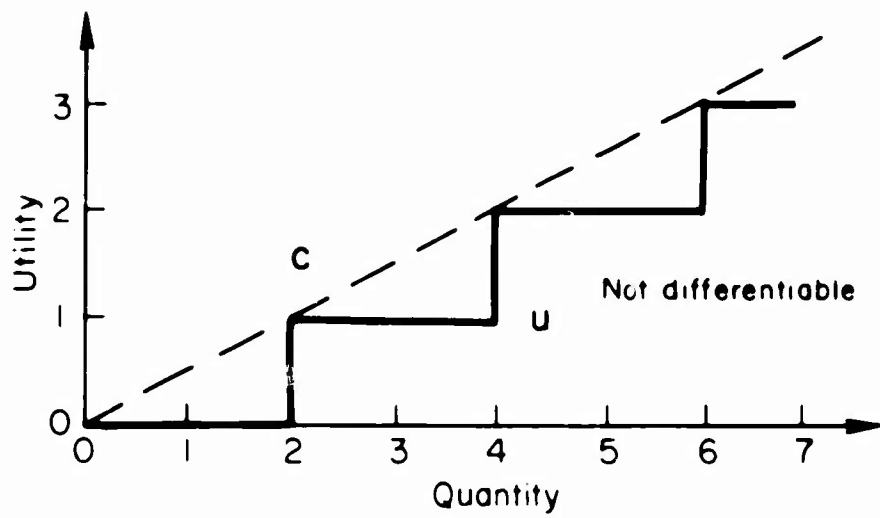


Fig. 3a

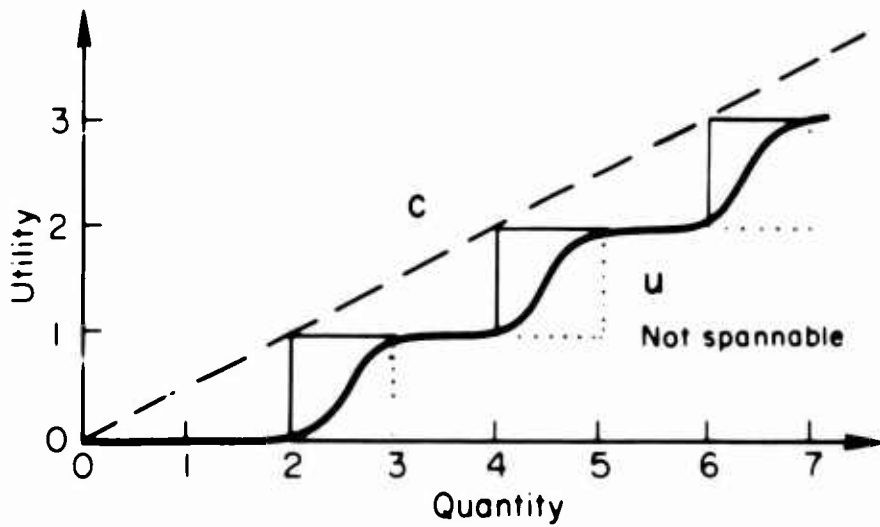


Fig. 3b

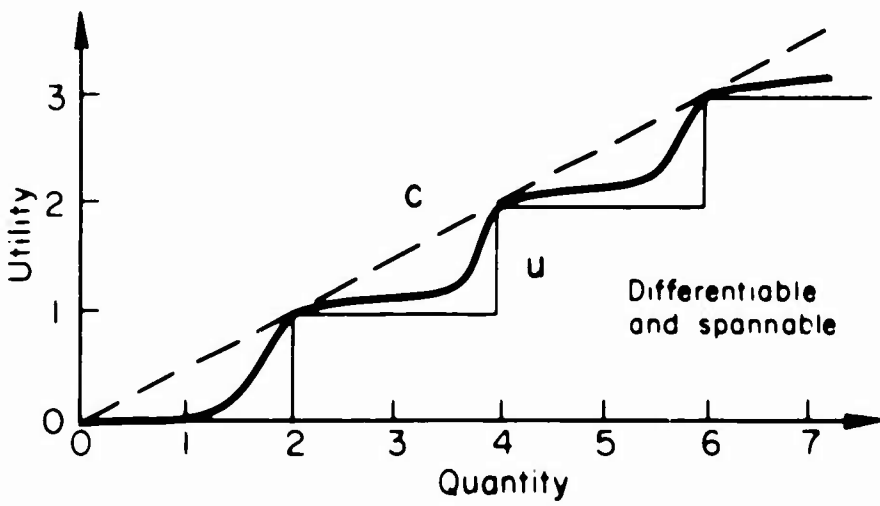


Fig. 3c

If the game happens to have an odd number of players, say $n = 2r + 1$, then in any Pareto-optimal payoff vector, the most-favored player receives at least r/n units. The $2r$ least-favored players therefore receive at most $r - r/n$. This must be compared with the amount r that they can obtain in coalition. The difference, r/n , converges not to 0, but to $1/2$ as $r \rightarrow \infty$. Hence strong ϵ -cores do not exist for large, odd n and small ϵ .

Now let us change U to spoil spannability, but at the same time make U differentiable (Fig. 3b).¹³ One can then verify that

$$(27) \quad v_U(s) = U(s) = \left\lfloor \frac{s-1}{2} \right\rfloor, \quad s = 1, 2, \dots$$

In other words, a coalition can do no better than allot all its goods to one player.¹⁴ If there happens to be an even number of players in this game, say $n = 2r$, then the least-favored set of $n-1$ players will always get $(r-1) - (r-1)/n$ or less, compared with the amount $r-1$ they can obtain in coalition. Again, the difference goes to $1/2$ as $r \rightarrow \infty$, and strong ϵ -cores do not exist for large, even n and small ϵ .

¹³The precise form of the curved parts of U within the small squares is immaterial.

¹⁴Proof: If the coalition allots in integer units, U might as well be the lower step-function in the figure (dotted lines). If fractional shares are used, the result can be no better than discarding one unit and then using the upper step-function. In either case the value (27) is best possible.

Finally, let us restore spannability, as in Fig. 3c, taking care to make U differentiable at the points of contact with C . A coalition with $n = 2r + 1$ members will now be able to make an allotment consisting of $2 + 1/r$ units to r of its members and nothing to the other $r+1$ members, and thereby receive a total utility of $r(1 + 1/2r - O(1/r^2))$. Thus we have

$$v_U(n) = n/2 - O(1/n).$$

This is also valid for even n , trivially. For any n , then, the imputation that gives equal shares to all will assign to each s -player set an amount

$$(s/n)v_U(n) = s/2 - O(1/n).$$

This must be compared with the amount, at most $s/2$, that they can obtain in coalition. In this case the difference does go to zero, and we have strong ϵ -cores in the limit as $n \rightarrow \infty$, for arbitrarily small ϵ .

12. EXISTENCE OF THE STRONG ϵ -CORE

We now consider exchange economies based on a utility function U that is radially differentiable and spannable.

Theorem 4. For every profile κ and for every $\epsilon > 0$, there is a constant k_0 such that every game $\Gamma_U(k\kappa)$ with $k \geq k_0$ has a strong ϵ -core.

Lemma on strong ϵ -homogeneity of v_U . For every profile κ and for every $\epsilon > 0$ there exists a constant k_0 such

that

$$(28) \quad v_C(\Delta) - \frac{1}{k} \geq \frac{1}{k} v_U(k\Delta) \geq v_C(\Delta)$$

holds for all $k \geq k_0$.

Proof. Fix Δ and s , and let $x^* = a(\Delta)/s$. Then

$$(29) \quad v_C(\Delta) = sC(x^*).$$

Using the spannability of U , we can find a convex representation $x^* = \sum \lambda_h y^h$ such that

$$C(x^*) = \sum \lambda_h U(y^h).$$

Given k , we wish to "move" the points y^h slightly, to make the coefficients λ_h integer multiples of $1/ks$. The technical details of this maneuver have been relegated to Appendix 3; the result is a new convex representation $x^* = \sum \mu_h z^h$ for each k , with the property that for each h , $ks\mu_h$ is an integer, and

$$(30) \quad ks|U(z^h) - L^*(z^h)| \rightarrow 0 \text{ as } k \rightarrow \infty,$$

where L^* is any linear function supporting C at x^* . (This result makes essential use of the radial differentiability of U .)

The new representation is a feasible allocation for any coalition with profile $k\Delta$, and we therefore have

$$v_U(k\Delta) \geq \sum ks\mu_h U(z^h).$$

Using (30), we have for sufficiently large k

$$v_U(kA) \geq ks \sum_h \mu_h L^*(z^h) - \epsilon.$$

The right side is equal to $ksL^*(x^*) - \epsilon$, by linearity.

This in turn is equal to $ksC(x^*) - \epsilon$. Applying (29) gives the desired result:

$$v_U(kA) \geq kv_C(A) - \epsilon.$$

The other inequality in (28) is immediate, by (18). This completes the proof of the lemma.

The proof of the theorem itself proceeds exactly like that of Theorem 2 in Sec. 8, but uses the more powerful lemma that we just have established. Note that in order to ensure that the constructed imputation $\beta^{(k)}$ is in the strong ϵ -core, we must use " ϵ/n " for " ϵ " in applying the lemma. The last line of the proof (compare (26) in Sec. 8) will then read as follows:

$$\sum_S \beta_i^{(k)} \geq v_U(A) - \frac{\epsilon C}{kn} \geq v_U(A) - \epsilon.$$

The remarks at the end of Sec. 8, concerning the limiting behavior of weak ϵ -cores, apply equally to the present case.

13. CONCLUSION

The core is a more general economic concept than the competitive allocation, in that it is possible to have markets with a core but no competitive equilibrium, but

not conversely. This can happen whether the absence of a competitive equilibrium is due to nonconvex preferences or to other factors not considered in this paper, such as nonconvex production sets or external production economies. Although no pure price mechanism will clear the market, there may nevertheless exist an imputation of wealth—a core payoff—that is stable against both individual and joint action by the participants. Achieving this imputation may involve taxes, transfer payments, or multiple prices.

There are also cases of interest where the core itself is vacuous. This means, in terms of Edgeworth's mechanism, that there will always be a group wishing to recontract. To resolve the inherent instability of this situation, we must resort to introducing social, cultural, or institutional restraints. The theory of games offers several solution concepts that provide frameworks for the systematic introduction of these factors; among the more promising are the von Neumann-Morgenstern "solutions" [21], the Luce ".-stable pairs" [14], and the Aumann-Maschler "bargaining sets" [3].

Another tool that seems potentially useful in this connection is the quasi-core, as developed in the present paper. It can easily be shown that ϵ -cores (both weak and strong) always exist if ϵ is large enough. The sociological factor involved here can be interpreted as an

organizational cost prerequisite to cooperative action, proportional to the parameter ϵ . Theorems 2 and 4 of this paper indicate that even if ϵ is small, the quasi-cores will exist when the market is large enough. Of course, when ϵ -cores exist for small values of ϵ , it is not unlikely that the core itself exists as well, making the market fully stable against recontracting. But even without a true core, the profit to be gained from recontracting out of an ϵ -core would be small, and a near-stability can be achieved.

By passing to the limit with these quasi-cores, or by direct concavification of the finite model, one can hopefully define a price structure to take the place of the missing competitive equilibrium. These pseudo-equilibria might repay further study. Perhaps (for example) one could find cases where they are the limits of convergent tâtonnement processes [18], when no true equilibrium exists.

Appendix 1

DETERMINATION OF CORES IN THE EXAMPLE OF SECTION 3

Generalizing (4), let the utility function be given by

$$(31) \quad U(x) = \max [\min (ax_1, x_2), \min (x_1, ax_2)],$$

where $a > 1$. Set $A = 2a/(1 + a)$, and note that $a > A > 1$.

Suppose first that money side payments are permitted. Then it is easily verified that any set of s players, $2 \leq s \leq n$, can achieve a combined payoff of sA , and no more. Thus, the payoff vector P that assigns A to each player is undominated, and lies in the core. Any other payoff vector will assign less than $(n-1)A$ to the $n-1$ least-favored players; hence, if $n \geq 3$, that set of players can block it. Thus the core consists of the single point P . When $n = 2$, however, blocking only occurs if one player is assigned less than 1 , or if the two together get less than $2A$. Thus, the core is the line segment joining the points $(1, 2A-1)$ and $(2A-1, 1)$ (Q' and R' in Fig. 2).

Next, let side payments be prohibited, and let $\alpha = (\alpha_1, \dots, \alpha_n)$ be an undominated payoff vector. We may assume that

$$(32) \quad \alpha_1 \leq \alpha_2 \leq \dots \leq \alpha_n.$$

Now, the coalition $\{1, 2\}$ can divide its assets as follows:

$(t, t/a)$ to 1. $(2 - t, 2 - t/a)$ to 2.

and thereby achieve the payoffs $\alpha_1 = t$, $\alpha_2 = 2 - t/a$ for any t between 0 and A (i.e., any point on the line from S to P in Fig. 2). To avoid domination of α by such β , we must have

$$(33) \quad \alpha_1 \geq 2a - a\alpha_2.$$

Combined with (32), this entails

$$(34) \quad \alpha_2 \geq \frac{2a}{a+1} = A.$$

We also have, of course,

$$(35) \quad \alpha_1 \geq 1.$$

Let $\{(x_1^i, x_2^i)\}$ be an allocation that yields α . Let p be the number of indices $i > 1$ such that $x_1^i \leq x_2^i$, and let $q = n - 1 - p$. We note from (31) that $x_1^i \leq x_2^i$ implies that $\alpha_i = \min(ax_1^i, x_2^i)$, and that $x_1^i > x_2^i$ implies that $\alpha_i = \min(x_1^i, ax_2^i)$. Hence

$$p\alpha_2/a + q\alpha_2 \geq \sum_{i=1}^n x_1^i = n - x_1^1.$$

by (32). Hence

$$x_1^1 \leq n - (p/a + q)\alpha_2 = B_1.$$

Similarly

$$x_2^1 \leq n - (p + q/a)\alpha_2 = B_2.$$

Without loss of generality, we can assume that $p \geq q$.

Then $B_1 \geq B_2$, and we have

$$(26) \quad \alpha_1 \leq \min(B_1, aB_2).$$

Case i. Suppose $p = q$. Then $n = 2p + 1$, and

$$(37) \quad \begin{aligned} 1 \leq \alpha_1 \leq B_1 &= (2p + 1) - p\left(\frac{1}{a} + 1\right)\alpha_2 && \text{(by (35) and (36))} \\ &\leq 2p + 1 - p \frac{a + 1}{a} A && \text{(by (34))} \\ &= 1. \end{aligned}$$

Hence there is equality throughout (37), and $\alpha_1 = 1$ and $\alpha_2 = A$. But this violates (33). Hence we cannot have $p = q$.

Case ii. Suppose $p \geq q + 2$. Then

$$\begin{aligned} 1 \leq \alpha_1 \leq aB_2 &= (p + q + 1)a - (pa + q)\alpha_2 && \text{(by (35) and (36))} \\ &\leq a + p(a - aA) + q(a - A) && \text{(by (34))} \\ &\leq a + p(a - aA) + (p - 2)(a - A) = 2A - a \\ &= 1 - \frac{(a - 1)^2}{a + 1} < 1. \end{aligned}$$

Hence we cannot have $p \geq q + 2$. There remains...

Case iii. Suppose $p = q + 1$. Then $n = 2p$, and we have

$$2a - a\alpha_2 \leq \alpha_1 \leq aB_2 = 2pa - (pa + p - 1)\alpha_2,$$

from (33) and (36). Hence

$$(38) \quad (p - 1)(a + 1) \alpha_2 \leq 2(p - 1)a.$$

Now if $p > 1$, this gives $\alpha_2 \leq A$; hence, by (34), $\alpha_2 = A$, and we have equality in (38). But in deriving (38) we made essential use of (33) and most of (32); therefore, equality must prevail in these places as well, and we find that all the α_i are equal to A . In other words, the only possible candidate for the core (if $p > 1$) is the vector $\alpha = (A, A, \dots, A)$. It is easily verified that this vector is in fact a feasible payoff, and is undominated; hence we have a one-point core as claimed. Note that n in this case is even.

If $p = 1$ (i.e., if $n = 2$), then (38) is no restriction, and it is easy to show that the core consists of the bent line from $(1, 2 - \frac{1}{a})$ to (A, A) to $(2 - \frac{1}{a}, 1)$ (QPR in Fig. 2).

Appendix 2

PROOF OF THEOREM 3

Define C as in (16). take x^* interior to E_+^m , and let L be a linear support to C at x^* . Then L is strictly increasing in each x_j , and so is $L/2$. By sublinearity, we can find a function L' parallel to $L/2$ such that $L' \geq U + c$, where c is a preassigned positive constant. Let R denote the region of E_+^m in which $L \leq L'$; clearly R is compact and contains x^* . We now wish to consider convex representations of x^* that "almost" achieve the value $C(x^*)$, in the sense of the "sup" in (16). In order to distinguish between vertices lying within R and those outside, the representations will be written in the following way:

$$(39) \quad x^* = \alpha y + \bar{\alpha} z = \alpha \sum \lambda_h y^h + \bar{\alpha} \sum \mu_k z^k, \quad y^h \in R, \quad z^k \notin R.$$

Here $\bar{\alpha}$ denotes $1 - \alpha$, and is understood to be 0 if there are not points of the second type in the representation, i.e., if z is not well defined. (Note that y is always well defined; this follows from the fact that x is outside the convex set $E_+^m - R$.) Given $\epsilon > 0$, by (16) we can find a representation satisfying

$$\alpha \sum \lambda_h U(y^h) + \bar{\alpha} \sum \mu_k U(z^k) \geq C(x^*) - \epsilon.$$

Hence we have

$$\alpha L(y) + \bar{\alpha} L'(z) = \bar{\alpha} c \leq L(x^*) - c.$$

or, from (39) and the linearity of L ,

$$c \leq \bar{\alpha} [L(z) - L'(z) + c].$$

Since $[L(z) - L'(z) + c] \geq c > 0$ for $z \in E_+^m - R$, we see that

$$\bar{\alpha} = 0 \quad \text{as} \quad \epsilon \rightarrow 0.$$

But since L' and $L/2$ are parallel, the expression in brackets is of the form $L'(z) + c'$, with a new constant c' . Thus, even though $\|z\|$ may be unbounded, we nevertheless have

$$\bar{\alpha} z \rightarrow 0 \quad \text{and} \quad \bar{\alpha} L'(z) \rightarrow 0, \quad \text{as} \quad \epsilon \rightarrow 0.$$

Hence,

$$y = \frac{1}{\alpha} (z^* - \bar{\alpha} z) \rightarrow x^*, \quad \text{as} \quad \epsilon \rightarrow 0,$$

and

$$\begin{aligned} \sum \lambda_h U(y^h) &= \frac{1}{\alpha} [C(x^*) - c - \bar{\alpha} \sum \mu_k U(z^k)] \\ &= \frac{1}{\alpha} [C(x^*) - c - \bar{\alpha} L'(z)] \\ &\rightarrow C(x^*). \end{aligned}$$

Since the y^h are restricted to the compact region R and since U is assumed to be continuous, there exists a limiting representation $x^* = \sum \lambda_h y^h$ with $\sum \lambda_h U(y^h) = C(x^*)$.

Hence C is spanned by U at the arbitrary interior point x^* .

If x^* is not interior to E_+^m , this argument is not directly valid, since a linear support L may not exist. But we can then reduce the dimension of the problem without affecting either the definition of $C(x^*)$ or the hypotheses of continuity, sublinearity, and strict monotonicity. In the reduced problem, x^* will be interior.

Appendix 3

DERIVATION OF (30) IN SECTION 12

Lemma Let L^* be a support to C at x^* . Let there be a convex representation $x^* = \sum \lambda_h y^h$ such that $\sum \lambda_h U(y^h) = C(x^*)$. Assume that U possesses a radial derivative at each y^h . Let s be a fixed integer and k a variable integer. Then there exist convex representations $x^* = \sum \mu_h z^h$, depending on k , such that for each h , $k\mu_h$ is an integer and

$$(30) \quad ks [U(z^h) - L^*(z^h)] \rightarrow 0 \quad \text{as} \quad k \rightarrow \infty.$$

Proof. Assume first that all λ_h are positive. For any k , we can find nonnegative integers i_h with $\sum i_h = ks$ and such that $|ks\lambda_h - i_h| \leq 1$. Now define:

$$\mu_h = i_h/ks \quad \text{and} \quad z^h = \frac{i_h}{\lambda_h} y^h.$$

For k sufficiently large, the μ_h will be positive, and all the statements in the lemma are obviously satisfied, except for (30). To verify the latter, note that L^* is tangent to U at each y^h , and that z^h approaches y^h along the ray Oy^h . (If $y^h = 0$ and there is no ray then (30) is trivial.) Since the derivative of U at y^h exists along that ray, we have

$$\frac{L^*(z^h) - U(z^h)}{z^h - y^h} \rightarrow 0 \quad \text{as} \quad k \rightarrow \infty.$$

(We are concerned, of course, only with values of k for which $z^h \neq y^h$.) But, for large k ,

$$\begin{aligned} \|z^h - y^h\| &= \left| \frac{\lambda_h - \mu_h}{\lambda_h} \right| \|y^h\| \\ &\leq \frac{1/ks}{\lambda_h - 1/ks} \|y^h\| \end{aligned}$$

Hence $ks \|z^h - y^h\|$ goes to zero, and (30) follows

If some of the λ_h are zero, we can set the corresponding $\mu_h = 0$ and $z^h = y^h$, and proceed as above, with h restricted to the indices for which λ_h is positive

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