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ON THE STRUCTURE AND THE HAUSDORFF DIMENSION OF THE SUPPORT OF A CLASS OF DISTRIBUTION FUNCTIONS INDUCED BY ERGODIC SEQUENCES

TECHNICAL REPORT NO. ESD-TR-65-130

OCTOBER 1965

H. Dym

Prepared for

DEPUTY FOR ENGINEERING AND TECHNOLOGY
ELECTRONIC SYSTEMS DIVISION
AIR FORCE SYSTEMS COMMAND
UNITED STATES AIR FORCE
L. G. Hanscom Field, Bedford, Massachusetts

Project 508G

Prepared by

THE MITRE CORPORATION
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ABSTRACT

In this note we examine the distribution function of the random variable $Z = \sum_{i=1}^{\infty} X_i D^{-i}$ when the random variables $X_1, X_2, \ldots$ form a stationary ergodic sequence and each X_i takes on integer values running from 0 to D-1. We show that the distribution function will be either a step function, or a continuous and purely singular function, or else a linear function of its argument. The form of the distribution function is related to the entropy rate of the random sequence. We discuss this relationship and also the relationship of entropy rate to the concept of Hausdorff dimension.

REVIEW AND APPROVAL

This technical report has been reviewed and is approved.

CHARLES A LAUSTRUP
Lt Col, USAF
Acting Director of Computers
Deputy for Engineering and Technology

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SECTION I

INTRODUCTION

Let $X_1, X_2, \cdots$ be a stationary ergodic sequence of random variables defined on a probability space (Ω, A, P) and mapping into the set $\{0,1,\cdots,D-1\}$. In particular, we may envision Ω as the set of all sequences $\omega = (\omega_1, \omega_2, \cdots)$ with $X_n(\omega) = \omega_n$ and $\omega_n \in \{0,1,\cdots,D-1\}$. To avoid trivialities we assume that $D \geq 2$ and $P(\omega\colon X_1(\omega)=i) > 0$ for $i = 0,1,\cdots,D-1$.

The random variable $Z = \sum_{i=1}^{\infty} X_i D^{-i}$ may be used to transfer the probability structure of the random sequence to the unit interval; each point $\omega \in \Omega$ is mapped by Z into a point $Z(\omega)$ satisfying $0 \leq Z(\omega) \leq 1$. In this note we study the distribution function of Z, $F(a) = P(\omega\colon Z(\omega) < a)$. We show (Theorem 1) that either: F is a step function with k jumps of height $1/k$, or F is continuous and purely singular, or else $F(a) = a$; $F(a) = a$ if and only if the random variables X_i are independent and uniformly distributed. Harris [5] has proved an analogous theorem using different techniques. Our method of proof utilizes Information Theory and illustrates some interesting relationships between the entropy rate, H, of the random sequence and the form of the distribution function. In particular, H is maximal; that is $H = \log D = 1$ (all logarithms will be taken to the base D), if and only if $F(a) = a$. If $0 < H < 1$ then F is purely singular and continuous. If F is a step function then $H = 0$. Presumably there exist ergodic sequences for which $H = 0$ and F is continuous and purely singular though we have not been able to construct any.

In the proof of Theorem 1 we define a set E which is in some sense the support of the distribution function F when it is regarded as a measure. Thus, if F is a step function, E consists of the points of discontinuity of F; if $F(a) = a$ then $E = [0,1]$; if F is continuous and purely singular $\mu(E) = 0$ but $\mu(F(E)) = 1$ (μ is Lebesque measure). We show in Section IV that the Hausdorff dimension of the set E is equal to H. This extends a theorem proved by Kinney [7] for ergodic Markov chains.

SECTION II

SOME INFORMATION THEORETIC PRELIMINARIES

We review briefly some pertinent facts from Information Theory. It will be convenient to introduce the symbol $P(a_1,\cdots,a_n)$ as a shorthand notation for $P(\omega\colon X_1(\omega) = a_1,\cdots,X_n(\omega) = a_n)$ where each $a_i \epsilon \{0,1,\cdots,D-1\}$ for $i = 1,\cdots,n$. Also, as noted earlier, all logarithms will be taken to the base D.

To the process $[P,X_n\colon n \geq 1]$ we associate a set of non-negative numbers.

$$H_n = -\sum_{a_1,\cdots,a_n}^{*} P(a_1,\cdots,a_n) \log P(a_1,\cdots,a_n) \quad .$$

We mean $\sum\limits_{a_1,\cdots,a_n}^{*}$ to signify that the summation is carried out only over those n-tuples which have a positive probability of occurrence; the number of summands is clearly $\leq D^n$. The numbers H_n which are termed (n-fold) entropies satisfy the following simple inequalities:

(a) $H_1 \leq \log D = 1$

$H_1 = 1$ if and only if $P(\omega\colon X_1(\omega) = j) = 1/D$ for $j = 0,\cdots,D-1$

(b) $H_1 \geq H_n - H_{n-1} \geq H_{n+1} - H_n \qquad n = 2,3,\cdots$

$H_1 = H_n - H_{n-1}$ for all $n \geq 2$ if and only if the random variables $X_1,X_2,\cdots$ are independent.

(c) $H_{n+1} - H_n \geq 0$

$H_{n+1} - H_n = 0$ if and only if $P(a_1,\cdots,a_{n+1}) \neq 0$ implies that $P(a_1,\cdots,a_{n+1}) = P(a_1,\cdots,a_n)$

These inequalities may be derived by judicious application of the following inequality which is valid for $x \geq 0$: $\log x \leq \frac{x-1}{\ln D}$; $\log x = \frac{x-1}{\ln D}$ if and only if $x = 1$ where $\ln = \log_e$. For purposes of illustration we sketch the proof of (b).

$$H_{n+1}+H_{n-1}-2\,H_n = \sum^*_{a_1,\cdots,a_{n+1}} P(a_1,\cdots,a_{n+1}) \log \frac{P(a_1,\cdots,a_n)P(a_2,\cdots,a_{n+1})}{P(a_1,\cdots,a_{n+1})P(a_2,\cdots,a_n)}$$

$$\leq \frac{1}{\ln D} \sum^*_{a_1,\cdots,a_{n+1}} P(a_1,\cdots,a_{n+1}) \left[\frac{P(a_1,\cdots,a_n)P(a_2,\cdots,a_{n+1})}{P(a_1,\cdots,a_{n+1})P(a_2,\cdots,a_n)} - 1\right]$$

$$= \frac{1}{\ln D} \left[\sum^*_{a_2,\cdots,a_{n+1}} \frac{P(a_2,\cdots,a_{n+1})}{P(a_2,\cdots,a_n)} \sum^*_{a_1} P(a_1,\cdots,a_n) - 1\right]$$

$$= \frac{1}{\ln D} \left[\sum^*_{a_2,\cdots,a_{n+1}} P(a_2,\cdots,a_{n+1}) - 1\right] = 0$$

In order for equality to hold throughout it is necessary and sufficient that $P(a_1,\cdots,a_{n+1}) \neq 0$ imply

$$P(a_1,\cdots,a_{n+1})\, P(a_2,\cdots,a_n) = P(a_1,\cdots,a_n)\, P(a_2,\cdots,a_{n+1}) \quad (n \geq 1) \quad .$$

This is, however, a necessary and sufficient condition for the random variables $X_1, X_2, \cdots$ to be independent.

From (a), (b) and (c) we deduce that the sequence $\{(H_n - H_{n-1})\}$ is monotone non-increasing, and bounded between 0 and 1. Thus the sequence has a limit which we designate by H:

$$H = \lim_{n \to \infty} (H_n - H_{n-1})$$

Since

$$\lim_{n \to \infty} (H_n - H_{n-1}) = \lim_{n \to \infty} \frac{1}{n} \sum_{j=2}^{n} (H_j - H_{j-1})$$

it follows readily that

$$H = \lim_{n \to \infty} \frac{H_n}{n}$$

Accordingly we term H the entropy rate of the process.

Remarks: 1) The results of this section are valid for sequences which are stationary but not ergodic.

2) Observe that $0 \leq H \leq 1$; $H = 1$ if and only if the random variables X_i are both independent and uniformly distributed.

SECTION III

THE FORM OF THE DISTRIBUTION FUNCTION

It will be convenient to define a shift operator T on the space Ω by the rule $(T\omega)_k = \omega_{k+1}$. For a unilateral stationary sequence of random variables T is a measurable, measure preserving, non-invertible transformation.

Lemma 1

Suppose there exists a point $u \epsilon \Omega$ such that $P(u) = \epsilon > 0$. Then $\epsilon = 1/k$ for some positive integer k, F is a step function with k jumps of height $1/k$ and $H = 0$.

Proof

Designate the characteristic function of the set $\{u\}$, by M_u. Then by the Birkhoff ergodic theorem

$$\lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} M_u (T^j u) = P(u) = \epsilon > 0 .$$

But $M_u (T^j u) = 0$ unless $T^j u = u$. Thus there must exist some smallest positive j, say $j = k$, such that $T^j u = u$. It follows readily that $\epsilon = 1/k$, that the distinct points in Ω with non-zero probability are $u, Tu, \cdots, T^{k-1} u$, and that $P(u) = P(Tu) = \cdots = P(T^{k-1} u) = 1/k$.

For $n \geq k$ $H_n = \log k$ therefore $H = \lim_{n \to \infty} \frac{H_n}{n} = 0$.

Lemma 2

If $P(\omega) = 0$ for every $\omega \epsilon \Omega$ then F is continuous.

Proof

A probability measure is countably additive. Consequently $P(\omega : Z(\omega) \leq a)$ is a right continuous function of a and

$P(\omega : Z(\omega) < a)$ is a left continuous function of a. Thus $P(\omega : Z(\omega) \le a) - P(\omega : Z(\omega) < a) = P(\omega : Z(\omega) = a) = 0$ since there are at most two points in Ω which can be mapped by Z into a and each of these has 0 probability.

Remark 3) F will be strictly monotone increasing if and only if $P(a_1,\cdots,a_n) > 0$ for every finite sequence $a_1,\cdots,a_n$ of elements chosen from $\{0,1,\cdots,D-1\}$.

Lemma 3

The random variables X_i are independent and uniformly distributed if and only if $H = 1$. In this case $F(a) = a$.

Proof

If the random variables X_i are independent and uniformly distributed then $F(g) = g$ at each D-adic rational point, g. The D-adic rational points are dense in the unit interval and, by Lemma 2, F is continuous. Hence $F(a) = a$ for all $0 \le a \le 1$. It was observed in Remark 2 that $H = 1$ if and only if the X_i are independent and uniformly distributed.

Theorem 1

The distribution function F is one of the following 3 types:

(1) a step function with k jumps of height $1/k$ where k is a positive integer,

(2) continuous and purely singular,

(3) $F(a) = a$. $0 \le a \le 1$.

F is of type (3) if and only if $H = 1$. If F is type (1) then $H = 0$. If $H > 0$ F is continuous. (Thus if $0 < H < 1$ then F is type (2).)

Proof

It follows from Lemmas 1 and 2 that either F is type (1), in which case $H = 0$, or F is continuous. If F is continuous and $H = 1$ then, by Lemma 3, $F(a) = a$. It remains to show that if F is continuous and $0 \le H < 1$, then F is purely singular. It suffices to exhibit a set E for which $\mu(E) = 0$ and $\mu(F(E)) = 1$.

Given any $x \in [0,1]$ and any $i \ge 1$ we choose the unique pair of non-negative numbers $\zeta_i = \zeta_i(x)$ and $\eta_i = \eta_i(x)$ such that $\zeta_i(x) + \eta_i(x) = D^{-i}$ and $D^i(x-\eta_i(x))$ and $D^i(x+\zeta_i(x))$ are both non-negative integers.

Consider the set

$$E = \left\{ x : \lim_{i \to \infty} \frac{\log[F(x+\zeta_i) - F(x-\eta_i)]}{\log[\zeta_i + \eta_i]} = H \right\}$$

and note that for each $x \in E$ and each $\epsilon > 0$ there exists a number $N(x,\epsilon)$ such that if $i > N(x,\epsilon)$ then

$$D^{-i(H+\epsilon)} \le F(x+\zeta_i) - F(x-\eta_i) \le D^{-i(H-\epsilon)}$$

If $H < 1$ we can choose $\epsilon > 0$ so that $H + \epsilon < 1$.

Clearly then for each $x \in E$ and $i > N$

$$\frac{F(x+\zeta_i) - F(x-\eta_i)}{\zeta_i + \eta_i} \ge D^{i(1-H-\epsilon)} \to \infty$$

Since F is a bounded monotone increasing function it must have a finite derivative almost everywhere in the sense of Lebesque. (See McShane [9], Thm 34.2, p. 202.) Thus $\mu(E) = 0$.

For $u \epsilon \Omega$ set $[u]_n = \{\omega \epsilon \Omega : \omega_1 = u_1, \cdots, \omega_n = u_n\}$

and let $\Omega^* = \{\omega : \lim_{n \to \infty} \frac{-\log P([\omega]_n)}{n} = H\}$.

By the strong form of McMillan's theorem (Breiman [3] and [4]) $P(\Omega^*) = 1$. The identity

$$F(Z(\omega) + \zeta_i) - F(Z(\omega) - \eta_i) = P([\omega]_i)$$

implies that if $\omega \epsilon \Omega^*$ then $Z(\omega) \epsilon E$. Thus

$$\mu(F(E)) = P(\omega : Z(\omega) \epsilon E) \geq P(\Omega^*) = 1 \quad .$$

McMillan's theorem also implies that if $H > 0$ then $P(\omega) = 0$ for every $\omega \epsilon \Omega$. In this case then, by Lemma 2, F must be continuous.

SECTION IV

ON THE HAUSDORFF DIMENSION OF E

In this section we show that the Hausdorff dimension of the set E constructed in Theorem 1 is equal to H. This is in fact an immediate consequence of a general theorem due to Billingsley [2. Thm 2.4]. We shall give a direct proof, however, establishing in the process Holder conditions on F. The techniques used are similar to those of Kinney [7] and Kinney and Pitcher [8].

Lemma 4

If $x \epsilon E$ then for any $\alpha > 0$

$$\lim_{h \to 0} [F(x+h) - F(x-h)] (2h)^{-(H+\alpha)} = \infty$$

Proof

The proof follows by observing that we can choose a positive $\delta < \alpha$ and i so large that

$$[F(x + \zeta_i + \eta_i) - F(x - \zeta_i - \eta_i)] [2(\zeta_i + \eta_i)]^{-(H+\alpha)}$$

$$\geq 2^{-(H+\alpha)} [F(x + \zeta_i) - F(x - \eta_i)] [\zeta_i + \eta_i]^{-(H+\alpha)}$$

$$\geq 2^{-(H+\alpha)} D^{i(\alpha-\delta)} \to \infty \quad .$$

Lemma 5

For every $\alpha > 0$ there exists a set $A \subset E$ such that $\mu(F(A)) = 1$ and for all $x \epsilon A$

$$\lim_{h \to 0} [F(x+h) - F(x-h)] (2h)^{-H+\alpha} = 0 \quad .$$

Proof

It suffices given arbitrary $\epsilon > 0$ to exhibit a set A with $\mu(F(A)) > 1 - \epsilon$ whose elements satisfy a Holder condition of the above type. Choose a positive $\delta < \alpha$ and consider the set of points, B, for which

$$F(x + \zeta_i) - F(x - \eta_i) > (\zeta_i + \eta_i)^{H-\delta}$$

for infinitely many i. Cover B with a countable collection of intervals, C_λ, each of length $< 3\lambda$ and so chosen that for each $x \in B$ the interval

$$[x - 2\,\eta_i - \zeta_i \;,\; x + \eta_i + 2\,\zeta_i] \subset C_\lambda$$

where i is the smallest integer for which both $\zeta_i + \eta_i < \lambda$ and

$$F(x + \zeta_i) - F(x - \eta_i) > (\zeta_i + \eta_i)^{H-\delta} \quad .$$

Since $\mu(F(B)) = 0$ we can choose λ so small that $\mu(F(C_\lambda)) < \epsilon$.

Set $A = E - C_\lambda$. Clearly $\mu(F(A)) > 1 - \epsilon$. For each $x \epsilon A$ we can choose an integer t so large that $\zeta_t + \eta_t < \lambda$ and for all integers $m \geq t$

$$F(x + \zeta_m) - F(x - \eta_m) < (\zeta_m + \eta_m)^{H-\delta} \quad .$$

This implies that $B \cap [x - \zeta_t - 2\,\eta_t \;,\; x + 2\,\zeta_t + \eta_t] = \emptyset$ and further that there exists an integer $g \geq t$ such that for all $s \geq g$

$$F(x + 2\,\zeta_s + \eta_s) - F(x - \zeta_s - 2\,\eta_s) \leq 3\,(\zeta_s + \eta_s)^{H-\delta} \; .$$

The desired Holder condition for such points x follows easily since $\delta < \alpha$.

Remark 4) Let $\{\alpha_j\}$ be a sequence of positive numbers which decrease monotonically to zero. Let A_j be the set of $x \epsilon E$ for which $F(x+h) - F(x-h) < (2h)^{H-\alpha_j}$ for all sufficiently small positive h. By Lemma 5, $\mu(F(A_j)) = 1$. Clearly the sequence of sets A_j decrease monotonically to a set A with $\mu(F(A)) = 1$ and if $x \epsilon A$ then $F(x+h) - F(x-h)$ $< (2h)^{H-\gamma}$ for any $\gamma > 0$ and all sufficiently small $h > 0$.

We now recall some definitions. The γ dimensional Hausdorff measure of a set $A \subset [0,1]$ is defined by $\Gamma(\gamma; A) = \liminf\limits_{\delta \to 0} \sum\limits_j \mu(I_j)^\gamma$ where the liminf is taken over all countable sets of intervals $\{I_j\}$ such that $\cup I_j \supset A$ and $\mu(I_j) < \delta$. The Hausdorff dimension of the set A, $\beta(A)$, is the number with the property that for any $\epsilon > 0$

$$\Gamma[\beta(A) - \epsilon, A] = \infty \qquad \Gamma[\beta(A) + \epsilon, A] = 0 .$$

We now prove

Theorem 2

If $E^* \subset E$ and $\mu(F(E^*)) > 0$ then $\beta(E^*) = H$.

Proof

Given any fixed $\epsilon > 0$ there exists for each $x \epsilon E^*$ a smallest $i \geq n$ such that

$$F(x + \zeta_i) - F(x - \eta_i) > [\zeta_i + \eta_i]^{H+\epsilon} .$$

We can thus construct a countable set of mutually disjoint intervals $\{I_i^{(n)}\}$ $i = 1,2,\cdots$ each of length $\leq D^{-n}$ and so chosen that

$$\bigcup_i I_i^{(n)} \supset E^* \text{ and } \mu(F(I_i^{(n)})) \geq \mu(I_i)^{H+\epsilon} .$$

Thus

$$1 \geq \mu(F(\bigcup_i I_i^{(n)})) \geq \sum_i (\mu(I_i^{(n)}))^{H+\epsilon} .$$

Hence

$$\Gamma\,(H+\epsilon,\ E^*) \leq 1 \quad \text{and} \quad \beta(E^*) \leq H .$$

To prove the inequality in the other direction fix $\epsilon > 0$ and choose $h_0 > 0$ and a set $A(h_0) \subset E^*$ so that $\mu(F(A(h_0))) \geq \delta > 0$ and for each $x \epsilon A(h_0)$ and every $h \leq h_0$ $\quad F(x+h) - F(x-h) \leq (2h)^{H-\epsilon}$.

Now cover $A(h_0)$ with a set of intervals I_i, each chosen with length $< h_0$. We can assume that each interval I_i contains at least one point, $x_i \epsilon A(h_0)$. Let J_i be the smallest interval symmetric about x_i which contains I_i . Then $\mu(J_i) \leq 2\mu(I_i)$ and correspondingly

$$\Sigma\,(2\mu(I_i))^{H-\epsilon} \geq \Sigma\,(\mu(J_i))^{H-\epsilon} \geq \Sigma\,\mu(F(J_i)) \geq \mu(f(A(h_0))) \geq \delta .$$

Since the chosen covering of $A(h_0)$ was arbitrary, aside from the fact that the length of each interval was constrained to be smaller than h_0 , $\Gamma\,(H-\epsilon,\ A(h_0)) \geq \delta$. Hence $\beta\,(A(h_0)) \geq H$.
But $E^* \supset A(h_0)$ thus $\beta\,(E^*) \geq H$.

Remark 5) The proof as presented is valid for $0 \leq H \leq 1$. The effort expended is, however, only really needed for distribution functions of the second type. For if F is type (1) the set E consists of the k jump points and if F is type (3) $E = [0,1]$. The Hausdorff dimension of a finite or even Countable set of points is 0; whereas the Hausdorff dimension of a bounded set of positive Lebesque measure is 1.

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14. KEY WORDS: Key words are technically meaningful terms or short phrases that characterize a report and may be used as index entries for cataloging the report. Key words must be selected so that no security classification is required. Identifiers, such as equipment model designation, trade name, military project code name, geographic location, may be used as key words but will be followed by an indication of technical context. The assignment of links, rules, and weights is optional.

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