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Minimizing Functionals on Normed-Linear Spaces

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ERRATA AND ADDITIONS

"Minimizing Functionals on Normed-Linear Spaces"

by

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- Page 3. In (b) of the theorem replace $L_p[0,1]$ by a uniformly convex Banach space.
- Page 4. Line 7 from bottom. Assume E is uniformly convex. By [6] p. 113.
- Page 8. Line 6. for $p > 1$.
- Page 10. Insert after sentence of line 5: and f is uniformly F -differentiable on S if $p > 2$. The inequality $|a|^{2r} + |b|^{2r} - 2|a|^r|b|^r[a,b]/|a||b| \leq |a|^{2r-2}(r^2 + 1)|a - b|^2$ where $|a| > |b|$, and a direct computation show that the F -derivative f' exists and is lipschitz continuous for all $p > 1$.
- Page 11. Line 12. Delete paragraph beginning with "By the theorem of I..."
Replace this with the following: Moreover, the space E_p is uniformly convex. This follows by a theorem of Smulian [12], which states that if the norm in a B -space is uniformly F -differentiable on the unit sphere, then the conjugate space is uniformly convex.
- Page 13. Replace ref [6] by M. Day, Normed Linear Spaces, Academic Press, N. Y., 1962.
- Add:
- [12] V. Smulian. Sur la derivabilité de la normed dans l'espace de Banach, C. R. (Doklady), Acad. Sci. U.R.S.S., Vol. 27, (1940), p. 643-648.

MINIMIZING FUNCTIONALS ON NORMED-LINEAR SPACES

by

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ABSTRACT

This paper extends results of [1], [2], of Goldstein, and [3] of Vainberg concerning steepest descent and related topics. An example is given taken from a simple rendezvous problem in control theory. The problem is one of minimizing a norm on an affine subspace. The problem here is solved in the primal. A solution in the dual is given by Neustadt [4].

I. GENERATION OF MINIMIZING SEQUENCES

Let E be a normed linear space (n. l. space), x_0 an arbitrary point of E and f a functional defined on E . Let S denote the level set $\{x \in E : f(x) \leq f(x_0)\}$ defined at an arbitrary fixed $x_0 \in E$. We denote by $f'(x)$ the Frechet or F -derivative of f at x . We call f *uniformly F -differentiable* on S if f is F -differentiable on S and if $\delta(\epsilon)$ in the definition of the F -derivative is constant on S . The F -derivative of f at x will be denoted by $f'(x)$. If $g \in E^*$ the value of g at x will be denoted by $[g, x]$, and if $h \in E^{**}$ the value of h at $g \in E^*$, by $[h, g]$. Recall that if E and F are n. l. spaces and A is bounded linear operator from E to F , in short $A \in B(E, F)$, and if A is onto then A^{-1} exists and belongs to $B(F, E)$ if and only if for some $m > 0$ and all $x \in E$, $\|Ax\| \geq m\|x\|$; and that $m\|x\| \leq \|Ax\| \leq M\|x\|$ for all x in E implies that $M^{-1}\|y\| \leq \|A^{-1}y\| \leq m^{-1}\|y\|$ for all y in F .

We observe that if E is a reflexive Banach Space, $A \in B(E, E^*)$ and $[Ax, x] \geq m\|x\|^2$ for all $x \in E$, then A is onto and thus has an inverse. For, on the contrary supposition, take $f^0 \notin M = \text{range } A$. Choose g in E^{**} such that $g(f^0) = \text{dist}(f^0, M) > 0$, $\|g\| = 1$ and $g(f) = 0$ for all f in M . Take $\bar{g}$ in E so that $[g, f] = [f, \bar{g}]$ for all f in E^* . Then $0 = [g, Ax] = [Ax, \bar{g}]$ for all x in E .

Thus $[A\bar{g}, \bar{g}] = 0$ while $\|g\| = \|\bar{g}\| = 1$. O. E. D.

Let ϕ denote a bounded map from S to E satisfying the two conditions $[f'(x), \phi(x)] \geq 0$, and given $\epsilon > 0$ there exists $\delta > 0$ such that $[f'(x), \phi(x)] < \delta$ implies $\|f'(x)\| < \epsilon$. Some examples of such mappings are the following:

(1) Let $A \in B(E^*, E)$ such that $[y, Ay] \geq \sigma \|y\|^2$ for all $y \in E^*$ and some $\sigma > 0$. Let $\phi(x) = Af'(x)$ and choose $\delta = \epsilon^2 \sigma$. Then $\|f'(x)\| < \epsilon$.

As a possible candidate for the operator A , suppose f is twice F -differentiable on E . Assume that for some $\mu > 0$, and some x in S the operator $f''(x)$ in $B(E, E^*)$ is onto and is "bounded below", that is, the bilinear functional satisfies $[f''(x)z, z] \geq \mu \|z\|^2$ for all z in E . Then $\|f''(x)z\| \geq \mu \|z\|$ showing that $f''(x)$ has an inverse $[f''(x)]^{-1} = A \in B(E^*, E)$. Since A has a bounded inverse, there exists a number $\sigma > 0$ such that $\|Ay\| \geq \sigma \|y\|$ for all $y \in E^*$. Set $z = Ay$. Then $[f''(x)z, z] = [y, Ay] \geq \mu \sigma^2 \|y\|^2$ showing the candidacy of A .

(2) Suppose E is a reflexive Banach space. By the weak compactness of the unit sphere in E it follows that for some z_0 , $\|z_0\| = 1$, $[f'(x), z_0] = \|f'(x)\|$. Set $\phi(x) = z_0 \|f'(x)\|$. Because $[f'(x), \phi(x)] = \|f'(x)\|^2$, $\phi(x)$ is the analogue of the gradient in Hilbert space. When E is an L_p space the point z_0 is obtained by considerations of equality in Hölder's inequality.

(3) Since $\|f'(x)\| = \sup\{[f'(x), z] : \|z\| = 1\}$, if $0 < \alpha < 1$ a point z_0 exists such $[f'(x), z_0] \geq \alpha \|f'(x)\|$. If for fixed α and all $x \in S$ we can find such z_0 , we may take $\phi(x) = z_0 \|f'(x)\|$.

In what follows let $\Delta(x, \rho) = f(x) - f(x - \rho\phi(x))$ and $g(x, \rho) = \Delta(x, \rho)/\rho[f'(x), \phi(x)]$. Assume E is a normed-linear space and S is the level set of f at x_0 in E . In what follows, assume $0 < \sigma < \frac{1}{2}$.

Theorem. Assume that on S f is uniformly F -differentiable or that the F -derivative f' exists and is uniformly continuous. Set $x_{k+1} = x_k$, when $[f'(x_k), \phi(x_k)] = 0$; otherwise choose* ρ_k so that $\sigma < g(x_k, \rho_k) \leq 1 - \sigma$ when $g(x_k, 1) < \sigma$ or $\rho_k = 1$ when $g(x_k, 1) \geq \sigma$, and set $x_{k+1} = x_k - \rho_k \phi(x_k)$.

(a) If S is bounded or f is bounded below then $\{f'(x_k)\}$ converges to 0 while $\{f(x_k)\}$ converges downward to a limit, L . If S is compact, then every cluster point of $\{x_k\}$ is a zero of f' . In addition, if $\phi(x_k) \rightarrow 0$ and f' has finitely many zeros, $\{x_k\}$ converges.

(b) If S is convex and bounded and f is convex, $L = \inf\{f(x) : x \in S\} = \theta$. If, in addition, E is a reflexive Banach space, then every weak cluster point of $\{x_k\}$ minimizes f on E . If $E = L_p[0, 1]$, then $\{x_k\}$ converges to a unique minimizer of f .

(c) Assume that the Gateaux derivative f'' exists on S and satisfies $\mu \|z\|^2 \leq [f''(x)z, z] \leq M \|z\|^2$ for all $x \in S$, $z \in E$ and some $\mu > 0$. Assume S is convex and E is complete. Then $\{x_k\}$ converges to a unique minimizer of f on E .

*If the Gateaux differential f'' satisfies $f''(x, h, h) \leq \|h\|^2/\rho_0$ for all h in E , x in S and some $\rho_0 > 0$ we can choose ρ_k to satisfy $\delta \leq \rho_k \leq 2\rho_0 - \delta$ with $0 < \delta \leq \rho_0$. The method of steepest descent could also be employed. See [9].

The proof of (a) is given in [1]. The proof there is stated for E , a Hilbert space, but the same proof works when E is taken to be a n. l. space. Two comments might be made, however. S bounded and f' uniformly continuous on S implies that f' is bounded on S . (See e.g. [5], p. 19.) It follows by employing the mean value theorem that f is bounded below on S . The statements f uniformly F -differentiable and the F -derivative f' is uniformly continuous are equivalent. (See [5], p. 45.)

(b) Given $\epsilon > 0$ choose $z' \in E$ such that $f(z') = \theta + \epsilon/2$. Because f' exists at x_k and f is convex, $f(z') \geq f(x_k) + [f'(x_k), z' - x_k]$. Since $\{f'(x_k)\} \rightarrow 0$ and S is bounded, for all k sufficiently large $f(x_k) \leq f(z') + \epsilon/2 = \theta + \epsilon$, showing that $L = \theta$.

If E is reflexive and S is convex, closed, and bounded, then S is weakly compact. Since f is convex, the sets $\{x \in E : f(x) \leq k\}$ are closed, convex, and weakly closed, for all k . Thus f is weakly lower semi-continuous. If z is a weak cluster point of $\{x_k\}$ then for an appropriate subsequence, $\liminf f(x_k) = L \geq f(z)$. Assume $E = L_p[0,1]$ and f is the norm on E . By [6], p. 78, if $\{x_k\}$ converges weakly to z and $f(x_k) \rightarrow z$ then $\{x_k\}$ converges strongly to z . It follows that every weak cluster point of $\{x_k\}$ is a strong cluster point of $\{x_k\}$. Since f' vanishes at every weak cluster point of $\{x_k\}$ and f' vanishes only once by the strict convexity of f , every subsequence of $\{x_k\}$ has the same weak cluster point z , showing that $\{x_k\}$ converges to z .

(c) The hypothesis of (c) imply that f' is Lipschitz continuous and that the set S is bounded. Otherwise S would contain an unbounded sequence, say $\{z_k\}$. By Taylor's theorem if $u \in S$, $f(z_k) \geq f(u) + ||z_k - u||[f'(u) - \mu/2 ||z_k - u||]$, showing that $f(z_k) \geq f(x_0)$, for large k , whence S must be bounded.

We now show that the sequence $\{x_k\}$ is Cauchy. Again by Taylor's theorem if $s > k$, $f(x_s) - f(x_k) \geq [f'(x_k), x_s - x_k] + \mu ||x_s - x_k||^2/2$. Since S is bounded, $||x_s - x_k|| \leq D$ where D is the diameter of S . Thus

$$||x_s - x_k||^2 \leq \frac{2}{\mu} \{f(x_s) - f(x_k) + D ||f'(x_k)||\} \text{ which shows that } \{x_s\} \text{ is}$$

a Cauchy sequence. By the completeness of E $\{x_s\}$ has a limit, say z ,

in E , and $f'(z) = 0$. If z is not unique, then $f'(z_1) = f'(z_2) = 0$, $z_1 \neq z_2$.

Thus $f(z_1) - f(z_2) \geq \frac{\mu}{2} ||z_1 - z_2||^2 \leq f(z_2) - f(z_1)$, a contradiction.

Hence z is unique and is a minimizer of f .

Remarks: Useful remarks may be found in [1], [3] and [9].

II. NEWTONIAN STEPS AND ACCELERATION

Suppose that at the given point x_0 , the function f satisfies the conditions of the first example, namely $\phi(x) = f''_{-1}(x_0)f'(x)$, where

$$f''_{-1}(x_0) = [f''(x_0)]^{-1}. \text{ The corresponding iteration is } x_{n+1} = x_n - \rho_{n-1} f''_{-1}(x_0) f'(x_n).$$

This algorithm, when $\rho_n \equiv 1$ is known as the "modified" Newton's method (see [3], p. 259, or [7], p. 696). In a similar manner if $f''_{-1}(x)$ exists and is uniformly bounded below on S , we may define $\phi(x_n) = f''_{-1}(x_n)f'(x_n)$. We shall do this below. It is clear from what has already been said that ϕ satisfies hypotheses of the above theorem. Our object now is to formulate

an algorithm using $\underline{f}_1''(x_n)f'(x_n) = \phi(x_n)$ which will converge at a superlinear rate.

In the following we set $\Delta(x, \rho) = f(x) - f(x - \rho \underline{f}_1''(x)f'(x))$ and $g(x, \rho) = \Delta(x, \rho) / \rho[\underline{f}_1''(x)f'(x), f'(x)]$.

Theorem. Assume the level set S is a convex subset of a Banach space E . For each x in S assume the F -derivative f'' is continuous on S , $f''(x)$ is onto, $\|f''(x)\| \leq M$, and $[f''(x)z, z] \geq m\|z\|^2$ for some $m > 0$ and all z in E . Set $x_{k+1} = x_k - \rho_{k-1} \underline{f}_1''(x_k)f'(x_k)$, where ρ_k is chosen so that for $\theta < 1/2$, $0 < \theta \leq g(x_k, \rho_k) \leq 1 - \theta$ with $\rho_k = 1$ if possible. Then:

(a) There exists a number N such that if $k > N$ then $\rho_k = 1$.

(b) There is a unique minimizer of f and the sequence $\{x_k\}$ converges to it faster than any geometric progression.

Proof. We have for all x in S that $M\|z\|^2 \geq [f''(x)x, x] \geq m\|z\|^2$ and $m^{-1}\|y\|^2 \geq [y, \underline{f}_1''(x)y] \geq mM^{-2}\|y\|^2$. Thus if $\phi(x) = \underline{f}_1''(x)f'(x)$, then $[f'(x), \phi(x)] \geq mM^{-2}\|f'(x)\|^2$, showing that ϕ satisfies the conditions of the above theorem. Since f'' is bounded on S , f' is Lipschitz continuous, by the mean value theorem. By (c) above $\{x_k\}$ converges to a unique minimizer of f .

Expand $\Delta(x, \rho)$ to two terms in the Taylor series with remainder

$[f''(\xi)h, h]$, where $h = \rho \underline{f}_1''(x)f'(x)$. Set $f''(\xi) = f''(x) + f''(\xi) - f''(x)$.

Then $g(x, \rho) = 1 - \rho/2 - \rho[(f''(\xi) - f''(x))\underline{f}_1''(x)f'(x), \underline{f}_1''(x)f'(x)] / 2[f'(x), \underline{f}_1''(x)f'(x)]$
 $\geq 1 - \rho/2 - \rho\|f''(\xi) - f''(x)\|M^2/2m^3$. Thus $|g(x, \rho) - 1 + \rho/2| \leq \rho\|f''(\xi) - f''(x)\|M^2/2m$. Since $\xi(\rho_k)$ lies between x_k and x_{k+1} $x_0, \xi_0, x_1, \dots$

is a Cauchy sequence; and it, together with its limit z , is a compactum.

Consequently, on this compactum f'' is uniformly continuous, so that

$\{ \|f''(\xi(\rho_k)) - f''(x)\| \}$ converges to 0, showing that the choice $\rho_k = 1$ is eventually feasible.

$$\begin{aligned} \text{To prove (b) we write } x_{k+1} - z &= x_k - z - \rho_{k-1} f''(x_k) f'(x_k) = \\ &= x_k - z - \rho_{k-1} f''(x_k) f''(x_k) (x_k - z) + \rho_{k-1} f''(x_k) [f''(x_k) (x_k - z) - f'(x_k)]. \end{aligned}$$

$$\text{Thus } \|x_{k+1} - z\| = \|x_k - z - \rho_{k-1} f''(x_k) f''(x_k) (x_k - z)\| + \rho_{k-1} \|f''(x_k) [f''(x_k) (x_k - z) - f'(x_k)]\|.$$

$$\begin{aligned} \text{Since } f' \text{ is } F\text{-differentiable at } x_k, \|f'(z) - f'(x_k) - f''(x_k)(z - x_k)\| \\ < \varepsilon \|z - x_k\|. \text{ Thus } \|x_{k+1} - z\| = (1 - \rho_k) \|x_k - z\| + \rho_k m^{-1} \varepsilon \|z - x_k\| \end{aligned}$$

Q.E.D.

Remarks:

(1) Both sides of the inverse of $f''_{-1}(x)$ are used in the proof.

(2) The analogue of the modified Newton process, namely choosing $\phi(x) = f''_{-1}(x_0) f'(x)$ or $f''_{-1}(x_k) f'(x)$ with k fixed also will under the hypothesis of the above theorem generate a sequence converging to a unique minimizer of f . Since $\|x_{k+1} - z\| = \|x_k - z - \rho_{k-1} f''_{-1}(x_0) f''(z) (x_k - z)\| + \rho_{k-1} \| [f''_{-1}(x_0)] \| \varepsilon \|x_k - z\| m^{-1}$ when $\|x_k - z\| < \delta$, the rate of convergence is eventually geometric provided $\|I - \rho_{k-1} f''_{-1}(x_0) f''(z)\| < 1$. Since

$$\|I - \rho_{k-1} f''_{-1}(x_0) f''(z)\| \leq 1 - \rho_k + \rho_k \|f''_{-1}(x_0)\| \|f''(x_0) - f''(z)\|, \text{ if}$$

$\|f''(x_0) - f''(z)\|$ is sufficiently small, $\rho_k \equiv 1$ will generate a sequence converging to z at the rate of geometric progression. A sufficient condition for the global geometric convergence would be $(M/m) < 1/2$, since

$$\|f''(x)\| \leq M \text{ and } \|f''_{-1}(x)\| \leq m^{-1}.$$

(3) Pertinent remarks may be found in [3].

III. EXAMPLE

a) We consider the following problem which arises from a linearized rendezvous problem. See for example [8], [9] and [4]. In [4] this problem is solved in the "dual". We consider here a construction in the "primal". In [8] and [9], we have discussed this problem in the spaces $\mathcal{L}_1$ and $\mathcal{L}_2$; we now discuss the problem in $\mathcal{L}_p$ for $p > 2$. Let $\mathcal{L}_p$ denote the direct sum of n $L_p[0,1]$ spaces. Thus a point $x \in \mathcal{L}_p$ if $x = (x_1, \dots, x_n)$ and $x_i \in L_p[0,1]$; the norm in $\mathcal{L}_p$ will be $\|x\|_p = \left[\int_0^1 |x(t)|^p dt \right]^{1/p}$ where $|x(t)| = \left[\sum_{i=1}^n x_i^2(t) \right]^{1/2}$. Since $\sqrt{n} \max \{x_i(t) : 1 \leq i \leq n\} \geq |x(t)|$, $\|x\|_p$ is well defined. Let $\{u^i : 1 \leq i \leq m\}$ be a linearly independent set in $\mathcal{L}_p$. Set $\frac{1}{p} + \frac{1}{q} = 1$. Since $q < p$, u^i is also in $\mathcal{L}_q$. Given numbers α_i , ($1 \leq i \leq m$) define the affine subspace $M = \{x \in \mathcal{L}_p : [u^i, x] = \alpha_i : 1 \leq i \leq m\}$. We shall consider the problem of minimizing $f(x) = \|x\|_p^p$ on M . The limits $p \rightarrow 1$ and $p \rightarrow \infty$ correspond to the cases of rendezvous with minimum fuel and minimum thrust amplitude respectively. In what follows we shall assume for simplicity that $n = 2$. There are no further difficulties in the general case.

We first observe that if the Gateaux differential (G-differential) of f exists it is given by:

$$\begin{aligned}
f'(x)h &= p \int_0^1 |x(t)|^{p-2} [x_1(t)h_1(t) + x_2(t)h_2(t)] dt \\
&= p \int_0^1 |x(t)|^{p-1} \left[\frac{x_1(t)}{|x(t)|} h_1(t) + \frac{x_2(t)}{|x(t)|} h_2(t) \right] dt \\
&\leq p \|x\|_p^{p/q} [\|h_1\|_p + \|h_2\|_p].
\end{aligned}$$

Here Holder's inequality has been employed on the function $t \rightarrow |x(t)|^{p-1}$ which belongs to $L_q[0,1]$. We have also used $\|\cdot\|_p$ for the norm in $L_p[0,1]$. Thus the G-derivative of f exists.

Observe now that if the second G-differential exists it is given by:

$$\begin{aligned}
[f''(x)h,k] &= p(p-2) \int_0^1 |x(t)|^{p-4} (x_1(t)h_1(t) + x_2(t)h_2(t))(x_1(t)k_1(t) + x_2(t)k_2(t)) dt \\
&\quad + p \int_0^1 |x(t)|^{p-2} (k_1(t)h_1(t) + k_2(t)h_2(t)) dt \\
&= p(p-2) \int_0^1 |x(t)|^{p-2} \left(\frac{x_2(t)}{|x(t)|} h_1(t) + \frac{x_1(t)}{|x(t)|} h_2(t) \right) \left(\frac{x_1(t)}{|x(t)|} k_1(t) + \frac{x_2(t)}{|x(t)|} k_2(t) \right) dt \\
&\quad + p \int_0^1 |x(t)|^{p-2} (k_1(t)h_1(t) + k_2(t)h_2(t)) dt \\
&\leq 2p(p-2) \int_0^1 |x(t)|^{p-2} |h(t)| |k(t)| dt + p \int_0^1 |x(t)|^{p-2} |h(t)| |k(t)| dt.
\end{aligned}$$

If $x \in L_p$, $y \in L_q$ and $z \in L_r$, and $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$, then $\int_0^1 |x(t)y(t)z(t)| dt$

$$\leq \|x\|_p \|y\|_q \|z\|_r. \text{ Since the function } t \rightarrow |x(t)|^{p-2} \text{ belongs to } L_{p'}, \text{ where }$$

$$p' = \frac{p}{p-2}, \text{ and } \frac{1}{p'} + \frac{2}{p} = 1 \text{ it follows that if } u(t) = |x(t)|^{p-2},$$

$$\left[\int_0^1 |u(t)|^{p'} dt \right]^{1/p'} = \|x\|_p^{p-2}, \text{ and } [f''(x)h,k] \leq (2p^2 - 3p) \|x\|_p^{p-2} \|h\|_p \|k\|_p.$$

As before, let S denote the level set of f at x_0 where x_0 will be

subsequently chosen in M . Thus if $x \in S$, $\|x\|_p^{p-2} = [f(x)]^{\frac{p-2}{2}} \leq [f(x_0)]^{\frac{p-2}{2}}$ showing that $[f''(x)h, k]$ is uniformly bounded on S , if h and k are confined to the unit sphere. It follows by Taylor's theorem that f' is F -differentiable on S . By the generalized mean value theorem it further follows that f' is Lipschitz continuous on S .

We now construct x_0 on M . Let $x_0 = \sum_j c_j u^j$. Thus x_0 lies on M if and only if $\sum_j c_j [u^i, u^j] = \alpha_i$. We show that the null space of the matrix $\{[u^i, u^j]\}$ consists only of the 0 element so that c_i is uniquely determined. If for some $c_j \neq 0$, $\sum_j c_j [u^i, u^j] = 0$ then $\sum_i c_i \sum_j [u^i, u^j] = [\sum_i c_i u^i, \sum_j u^j] = 0$, contradicting the linear independence of the set $\{u^i : 1 \leq i \leq n\}$. Let $N = \{x \in \mathcal{E}_p : [u^i, x] = 0, i=1, \dots, m\}$.

We now choose h to maximize $[f'(x), h]$ subject to $\|h\|_p = 1$ and $h \in N$. The maximum is achieved because the sphere meets N in a weakly compact set and the linear function $[f'(x), \cdot]$ is weakly continuous.

The maximization can be accomplished by the method of Euler multipliers

[10]. Let $\phi(h) = \|h\|_p^p - 1$ and $\phi_i(h) = [u^i, h]$. Then a necessary condition that h maximize $f'(x, h)$ subject to $\phi(h) = \phi_i(h) = 0$ is that there exists c_i , $(1 \leq i \leq m+1)$, such that $f'(x)k =$

$$c_1 p \int_0^1 |h(t)|^{p-2} (h_1(t)k_1(t) + h_2(t)k_2(t))dt + \sum_{j=2}^{m+1} c_j [u^{j-1}, k] \text{ for all } k \in \mathcal{E}_p.$$

It follows that

$$p|x(t)|^{p-2}x_i(t) = pc_1|h(t)|^{p-2}h_i(t) + c_2u_i^1(t) + \dots + c_{m+1}u_i^m(t) \quad i=1,2.$$

Let $f_i(t) = (pc_1)^{-1}[p|x(t)|^{p-2}x_i(t) - c_2u_i^1(t) - \dots - c_{m+1}u_i^m(t)]$, and observe that

$$|h(t)|^{2p-2} = f_1^2(t) + f_2^2(t).$$

Therefore:

$$h_i(t) = [(f_1^2(t) + f_2^2(t))^{\frac{1}{2}}]^{\frac{1}{p}} f_i(t) / (f_1^2(t) + f_2^2(t))^{\frac{1}{2}}$$

showing that $h_1 \in L_p$. We now solve the non-linear equations $\phi(h) = 0$, $\phi_i(h) = 0$ ($1 \leq i \leq m$) for $c_2, \dots, c_{m+1}$. If necessary, we replace h by $-h$ to ensure that we have a maximizer rather than a minimizer. The solution is now unique, due to the strict convexity of the sphere in $\mathcal{L}_p$. Because of the uniqueness of the extremal, the h we have constructed must be this extremal.

The subspace N is also an $\mathcal{L}_p$ space. Minimizing $f(x)$ on M is equivalent to minimizing $f(y + x_0)$ on N , with $x = y + x_0$. Clearly the gradient of the function f restricted to N is $h[f'(x), h]$. (See I2, above.) It follows, therefore, if $\phi(x) = h[f'(x), h]$, then ϕ satisfies the conditions required for the theorem of I.

By the theorem of I we may infer that every weak cluster point z of the sequence $\{x^k\}$ minimizes f and $f(x^k)$ converges downward to $f(z)$. Furthermore, since z is unique in the above problem, $\{x^k\} \rightharpoonup z$. We now show, moreover, that $\{x^k\} \rightarrow z$. For each component x^i of x , ($1 \leq i \leq n$) we have that $0 \leq \int_0^1 |x^i(t)|^p dt \leq \int_0^1 |x(t)|^p dt$. Since $\{f(x_k)\}$ converges, the numbers $y_k^i = \int_0^1 |x_k^i(t)|^p dt$ are bounded and the sequence $\{x_k^i\}$ has a weak cluster point. In fact, $\{y_k^i\}$ converges. To prove this, observe that since $\{x_k\} \rightharpoonup z$, $\{x_k^i\} \rightharpoonup z_i$. Take a subsequence $\{x_k^i\}$ such that $\{y_k^i\} \rightarrow y^i$. Since $\{x_k^i\}$ converges both weakly and in L_p norm, $\{x_k^i\}$ converges strongly. Thus $\{x_k\} \rightarrow z'$, say. By continuity $f(x_k) \rightarrow f(z') = f(z)$. By uniqueness, $z = z'$. Therefore, the subsequence $\{x_k^i\} \rightarrow z^i$. Suppose that $\{y_k^i\}$ had another cluster point $\bar{y}^i \neq y^i$.

Take a new subsequence $\{x_k^i\}$ such that $\{y_k^i\} \rightarrow \bar{y}^i$. Again $\{x_k^i\} \rightarrow z^i$, and therefore, $\{y_k^i\} \rightarrow \int_0^T |z^i(\cdot)|^p dt = \bar{y}^i$, a contradiction. It follows, therefore, that $\{x_k\} \rightarrow z$.

(b) The above processes require that at each cycle a non-linear system be solved to determine the gradient. This can be circumvented by imbedding the problem into a Hilbert space. Specifically, assume that the components of u^i are bounded and measurable. Let $\mathcal{L}_2$ denote the direct sum of $L_2[0,1]$ analogously to the above, and define

$M' = \{x \in \mathcal{L}_2 : [u^i, x] = c^i : 1 \leq i \leq m\}$. Let f now be defined on M' . Since f achieves a minimum on M and $M \subset M'$, f also achieves a minimum on M' . Because M is dense in M' , the minima are equal. The gradient of f on M' is merely the restriction of the gradient of f in $\mathcal{L}_2$ to M' and is obtained by orthogonal projection. See [9]. In general, $f'(x)$ does not exist. But if x is bounded and measurable, i.e., $x \in \mathcal{L}_1$, $f'(x) \in \mathcal{L}_2^*$ and $\nabla f(x) \in \mathcal{L}_2$. The set S is bounded in $\mathcal{L}_p$ and this implies S is bounded in $\mathcal{L}_2$, since $\|x\|_2 \leq \|x\|_p$ if $p \geq 2$.

Since f is convex and continuous, S is closed bounded and convex; furthermore, the derivatives of f are densely defined on S . Assume $x_n \in M'$, $x_n \in \mathcal{L}_1$ and $u^i \in \mathcal{L}_1$, $(1 \leq i \leq m)$. Then x_{n+1} is well defined and is also in $\mathcal{L}_1$. To see this, verify that $\nabla f(x_n) \in \mathcal{L}_1$ and the projection of $\nabla f(x_n)$ on the set $\{x \in \mathcal{L}_2 : [u^i, x] = 0, (1 \leq i \leq n)\}$ is also in $\mathcal{L}_1$. We are able to conclude again therefore that there is a unique minimizer z , and that $\{x_k\} \rightarrow z$ and $f(x_k) \searrow f(z)$.

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