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MATHEMATICS RESEARCH DECEMBER 1965

D1-82-0501

NUMERICAL INTEGRATION OF AN ORBIT AND ITS CONCOMITANT VARIATIONS BY RECURRENT POWER SERIES

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Mathematical Note No. 445

Mathematics Research Laboratory

BOEING SCIENTIFIC RESEARCH LABORATORIES

December 1965

SUMMARY

Power series expansions (with coefficients obtained by recurrence formulas) are more efficient than other integration procedures for computing concurrently an orbit and the resolvent matrix of its variational equations, in the Restricted Problem of Three Bodies. For the same requirements on accuracy, the series expansions use only about 30 per cent of the computing time of the multi-step procedures, and only 12 to 15 per cent of the computing time of the Runge-Kutta-Nystrom method.

I. INTRODUCTION

Steffensen [1][1)] has set up recurrence formulas to integrate by power series the planar Restricted Problem of Three Bodies; the equations of motion were taken in their Lagrangian form, in the jovicentric synodical coordinate system. The series involved were proved to be convergent as long as the initial point is not lying at one of the singularities. Steffensen's algorithm has been used for the first time, and quite extensively, by Rabe in determining the long period orbits around the Lagrangian equilateral centers of libration, either in the Sun-Jupiter system [2] [3] or in the Earth-Moon case [4]. Fehlberg [5] adapted the method to the Lagrangian equations of the Restricted Problem in the barycentric synodical coordinate system; he also extended it to the problem of a charged particle in the field of a magnetic dipole. In both problems, integration by recurrent power series proved to be faster and more reliable than other integration procedures.

Recently Deprit and Price [6] have brought Steffensen's ideas into play to integrate concurrently the Hamiltonian equations of the Restricted Problem and their related variational equations; in particular, they propose to integrate in one package a periodic orbit, the 4×4-matrix of its fundamental displacements, and its characteristic exponents. It amounts to repeatedly evaluating by recurrence 33 power series; the time step is variable, being adjusted automatically by the program anywhere

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1) Numbers in square brackets refer to References, page 14.

within a preassigned interval. Error control is exerted by 5 first integrals and 6 bilinear identities derived from the symplectic character of the resolvent matrix.

Numerical integration by recurrent power series is not a general purpose algorithm: in each case, the right hand members of the differential system should be manipulated, and the order itself of the system possibly increased, so as to give to the equations the general form considered by Fehlberg. But such an *ad hoc* treatment has precisely the virtue of suiting the numerical integration method to the particular problem under investigation. It often means greater reliability on very long spans of integration, unusually large step-sizes, and an appreciable saving in computing time.

We checked these benefits of the power series in the complete (orbit + variations) integration of the Restricted Problem against more classical methods, like the Runge-Kutta-Nystrom algorithm or the multi-step procedures. To reach an accuracy of at least 9 decimal figures for the characteristic exponents of a Trojan orbit, the power series method proceeds safely by time steps equal to about 1.5 canonical units, whereas a multi-step method with one predictor and two correctors interpolating up to the eighth difference had to progress by integration intervals 4000 times smaller. In these conditions, the accumulation of round-off errors may result in inaccurate numerical approximations to the orbit which in turn reflects back most adversely on the computation of its fundamental displacements.

II. THE RESTRICTED PROBLEM

In the canonical units of mass, length and time, and in reference to the barycentric synodical coordinate system [7], the planar Restricted Problem of Three Bodies is described by the Hamiltonian function

$$H = \tfrac{1}{2}(p_x^2+p_y^2) - (xp_y-yp_x) - (1-\mu)\rho_1^{-1} - \mu\rho_2^{-1}$$

where

$$\rho_1 = |(x+\mu)^2 + y^2|^{\frac{1}{2}},$$
$$\rho_2 = |(x+\mu-1)^2 + y^2|^{\frac{1}{2}}.$$

Let u denote the 4-dimensional vector (x, y, p_x, p_y), H_u the gradient of the Hamiltonian H in the direction of the vector u, H_{uu} the 4×4-matrix which is the Hessian of H; let us also denote by R a 4×4-matrix whose elements are functions of the time, and by J the 4×4-symplectic matrix such that $J^{-1} = -J$. In these notations, our task can be described as the numerical integration of the differential system of order 20

$$\dot{u} = JH_u \tag{1}$$

$$\dot{R} = JH_{uu}R, \tag{2}$$

the initial conditions at $t = 0$ being such that $u(0)$ does not lie either in the phase plane $(x = -\mu, y = 0)$ or $(x = 1-\mu, y = 0)$ of binary collisions, and $R(0)$ is the 4×4-identity matrix I_4. The vector differential equation (1) yields the orbit, while the matrix equation (2) produces the *resolvent* [8] made of four linearly independent

variations belonging to the orbit.

Because the Hamiltonian H is conservative, the equations admit the Jacobi integral

(3) $$H = \text{const.}$$

along the orbit, and the vector integral

(4) $$R^T H_u = \text{const.}$$

for the resolvent of the variational equations. (N.B. We denote by R^T the transpose of the matrix R.)

Because the equations (1) are canonical, the resolvent R is a completely canonical matrix, hence at any time along the orbit,

(5) $$RJR^T = J;$$

this matrix identity reduces to 6 independent bilinear relations between the fundamental variations along the orbit.

To the checks supplied by (3), (4) and (5), Danby suggested to add the vector identity

(6) $$\dot{u}(t) = R(t)\dot{u}(0)$$

which expresses that, when u is a solution of (1), then $\dot{u}$ is a solution of the variational equations (2).

As a check on the accuracy of the numerical integration, the Jacobi integral (3) is rather insensitive, obviously for the reason that it depends

not on the Cartesian components of the velocity, but only on the square of the norm of that vector. On the other hand (5) has proved a severe control of the accuracy. For this reason, we have decided to use

$$\eta = \sup_{0 \leq t \leq T} || R(t) J R^{T}(t) - J ||_{*} \tag{7}$$

as a measure of the absolute error in the numerical integration. (N.B. The norm $||A||_1$ of a matrix A is defined as the sum of the absolute values of its elements.)

We choose as a testing stand the Trojan orbit in the Sun-Jupiter system ($\mu = 0.953875 \times 10^{-3}$) having the initial conditions

$$\begin{aligned} x &= 0.524\ 460\ 984 \\ y &= 0.862\ 013\ 960 \\ p_x &= -0.850\ 551\ 063 \\ p_y &= 0.516\ 376\ 943; \end{aligned}$$

this is an orbit very close to being periodic with period $T = 78.505\ 049\ 481$. Our choice is dictated by the fact that this orbit has been determined to a high accuracy from another source, namely from its representation by d'Alembert series carried up to the fourteenth power of the orbital parameter [9].

II. NUMERICAL PROCEDURES

Among the available integration methods, we choose first the Runge-Kutta-Nystrom algorithm with a local truncation error $0(\underline{h}^5)$, $\underline{h}$ being the step-size. A constant step-size along the entire period T was preferable to a variable one: the former requires only four

intermediate evaluations of the right hand members of (1) and (2) whereas the latter imposes six such evaluations per step if the test for adjusting h is to be performed at each point. The error plot for the Runge-Kutta method is denoted by RK in Figure 2.

The second procedure was the multi-step method defined by the Adams-Bashforth predictor

$$z_n = z_{n-1} + h \sum_{1 \le i \le k+1} b_i^* \dot{z}_{n-i} \tag{8}$$

and the Adams-Moulton corrector

$$z_n = z_{n-1} + h \sum_{0 \le i \le k} b_i \dot{z}_{n-i}. \tag{9}$$

We based our decision to use Adams type formulas mainly on their reliability. We assumed that the predictor-corrector methods for systems of equations behave the same qualitatively as they do for single equations; on this basis, we could make a decision as to the type of methods we should consider [10]; our choice has been confirmed, at least in this case of the Restricted Problem, by the experimental results.

We decided to use the corrector (9) twice per step. The results showed that Adams-Moulton algorithms with only one corrector tended to be unstable, especially as k increased. On the other hand, the accuracy attained by three correctors was only slightly better than with two correctors, for the same k and the same step-size.

The error behaviour for the multi-step method with two corrections, and $k = 5, 6, 7, 8$ successively are summarized in Figure 1. Because the local truncation errors of formulas (8) and (9) are $0(h^{k+2})$, we

would like to use as high a value of $\underline{k}$ as possible and still maintain stability. However, as expected, instability appears at smaller $\underline{h}$ as $\underline{k}$ increases. Nevertheless, since we are interested in keeping the error measure η less than about 10^{-5}, we were led to choose eventually $\underline{k} = 8$. The error plot for $k = 8$ is denoted by PC in Figure 2.

The third procedure was the method by recurrent power series [6]. Here we have to evaluate the coefficients in the 20 power series:

$$x(t+h) = \sum_{n\geq 1} x_n(t)h^{n-1},$$

$$y(t+h) = \sum_{n\geq 1} y_n(t)h^{n-1},$$

$$p_x(t+h) = \sum_{n\geq 1} p_n(t)h^{n-1},$$

$$p_y(t+h) = \sum_{n\geq 1} q_n(t)h^{n-1},$$

for the orbit in its four dimensional phase space, and

$$\delta x^{(i)}(t+h) = \sum_{n\geq 1} k_n^{(i)}(t)h^{n-1},$$

$$\delta y^{(i)}(t+h) = \sum_{n\geq 1} {}^{(i)}_{n}(t)h^{n-1},$$

$$\delta p_x^{(i)}(t+h) = \sum_{n\geq 1} u_n^{(i)}(t)h^{n-1},$$

$$\delta p_y^{(i)}(t+h) = \sum_{n\geq 1} v_n^{(i)}(t)h^{n-1}$$

for each column $(i = 1, 2, 3, 4)$ of the resolvent matrix R. In order to compute these coefficients, it has been found practical to introduce 13 other auxiliary variables to be computed also by power series expansions. The necessary recurrent formulas can be found in [11]; they serve to

compute all coefficients in the above series, starting from the values taken by the unknowns at time t_{-0}. Tests are incorporated in the procedure to decide, with respect to the required accuracy, the number of terms to be used in the series, and the step $\underline{h}$. Then the variables are computed from the series at time $t_0 + h$, and these new values serve to re-initialize the recurrent determination of the power series for a new integration step.

In the problems considered, it turned out that the most practical procedure was to keep the number of terms in the series equal to 16 throughout the entire orbit, and to vary only the time intervals over which the series were employed. To keep the error measure $\eta < 10^{-9}$ over a period, only about 50 points were necessary, which means an unusually large time step of 1·5 unit.

III. COMPARISONS

The computations were performed in double precision on an IBM 7094 from programs written in FORTRAN IV.

In the following figures we have plotted for the various integration methods the error measure η on the fundamental variations versus the computer time used to calculate the Trojan orbit over its period. We mean the actual computation time starting after the data were read in, and ending after the computation was complete, but before any printout occurred; such a time is measured by the machine clock within 120 milliseconds.

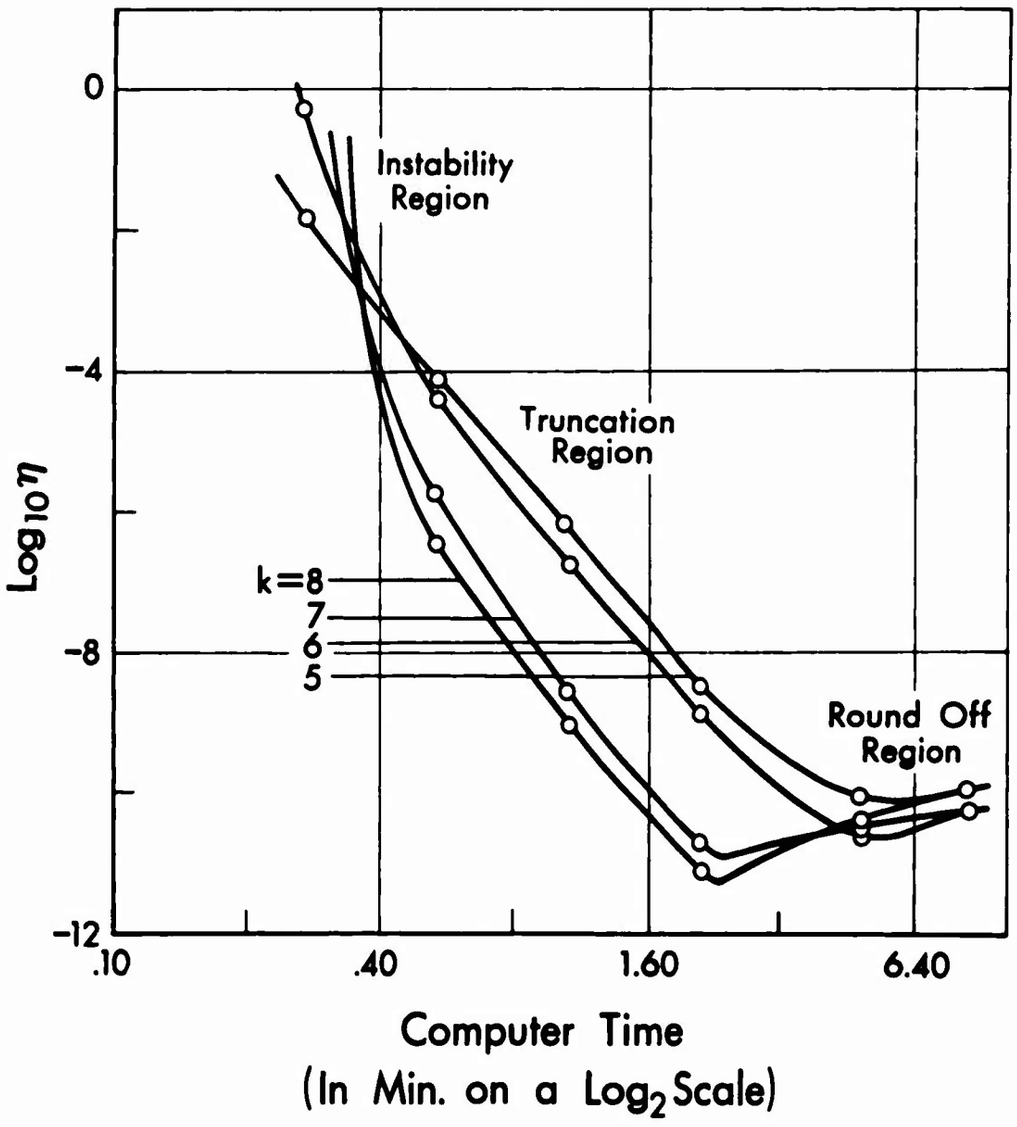
0
-4
-8
-12
$\log_{10}\eta$
Instability
Region
Truncation
Region
k=8
7
6
5
Round Off
Region
.10
.40
1.60
6.40
Computer Time
(In Min. on a $\log_2$ Scale)

Fig. 1

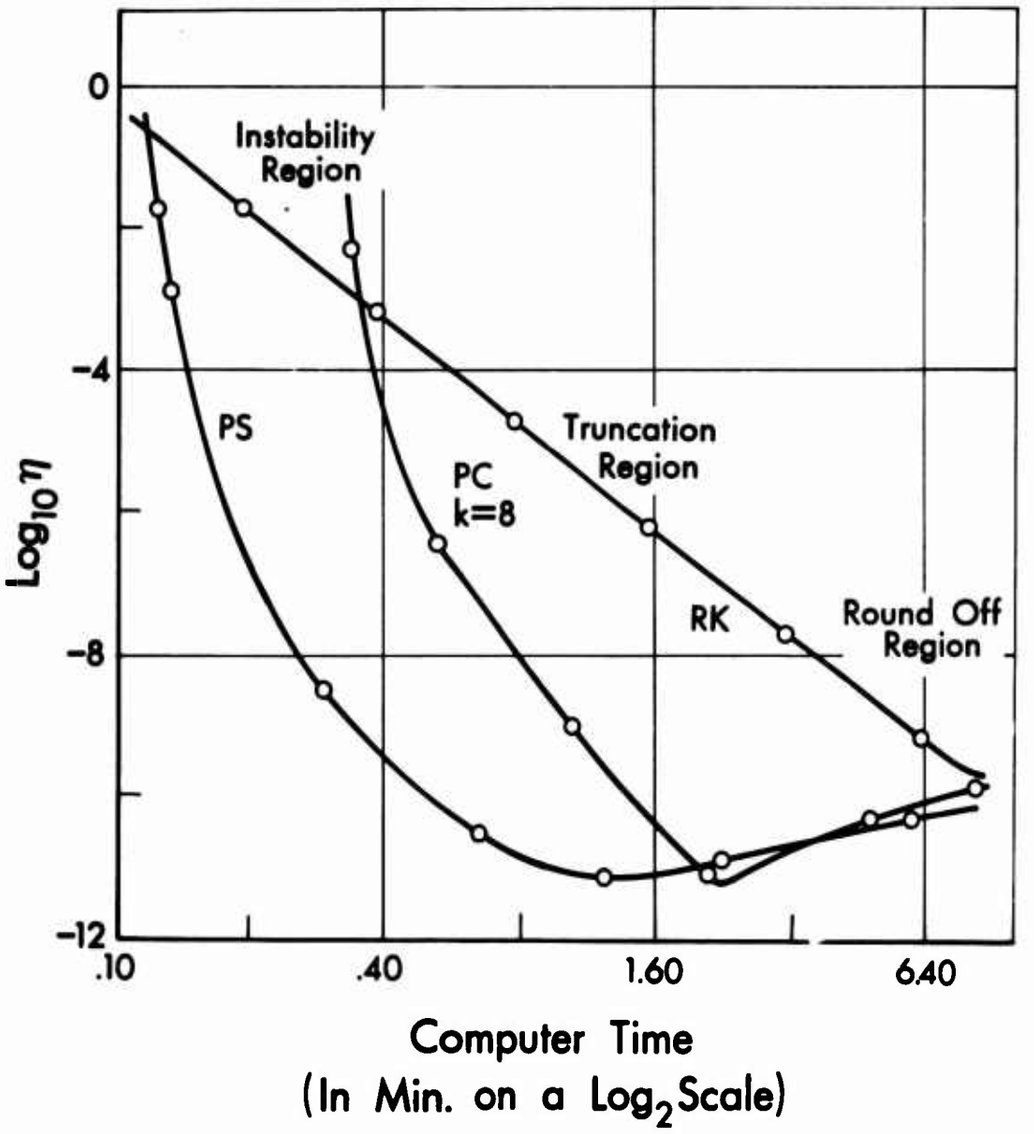
0
-4
-8
-12
Instability Region
PS
PC k=8
Truncation Region
RK
Round Off Region
$\text{Log}_{10}\eta$
.10
.40
1.60
6.40
Computer Time
(In Min. on a Log_2 Scale)

Fig. 2

Such graphs possibly may be criticized on the ground that computer time is too closely dependent on the programmer and his ability to convert a mathematical procedure into an efficient sequence of computer operations. We point out, however, that the Runge-Kutta method is easy enough to program, the predictor-corrector methods are of moderately greater difficulty, whereas the recurrence by power series can require more care. Thus our graphs, if anything, are biased in favor of the Runge-Kutta and the multi-step methods.

The general pattern of the error plots for the particular numerical procedures is as expected. Basically for all methods, there exists a region--the round-off region--where the accumulation of rounding errors is the main contributing factor to the total error; it lies at the right hand side of the figures, because it occurs for too small values of the step $\underline{h}$, hence a significant increase in the amount of computing time. Truncation regions exist at the intermediate values of the computing time; here the main contribution to the total error is the accumulation of the local truncation errors. In the round-off region, the increase in error after the curves passed a minimum is a direct result of the accumulation of rounding errors in the numerical solution of the motion equations (1) rather than the variational equations (2). This trend was indicated by an increasingly inaccurate Jacobi constant (3) in that region.

For all the methods, the time step can be chosen so that η remains less than about 10^{-9} on the entire period; the accuracy can even be increased by a factor of 100 for the methods by recurrent power series and predictor-corrector. In these conditions of maximum accuracy for the last two methods, the program returned at the end of the orbit a Jacobi constant

(3) unchanged with respect to the initial one; the final variational constants (4) were left constant to about 12 significant figures. The orbit came back upon itself with approximately an equal number of significant digits; but the coordinates at the end of the assumed period T differed consistently from the initial ones by 6 figures on the sixteenth place, which indicated that the period T was not determined with sufficient precision.

IV. CONCLUSION

From Figure 2, it comes out quite obviously that the power series is the most economic of the three methods. For an interval of η between 10^{-5} and 10^{-10}, the predictor-corrector requires as much as *three times* the computing time employed by the power series. In the same domain, the Runge-Kutta computing time is greater than that of the power series by a factor *between 5 and 8*.

These conclusions have been confirmed lately by Dr. Roger Broucke at the Jet Propulsion Laboratory, Pasadena, California.

ACKNOWLEDGEMENT

We wish to thank Professor J. M. A. Danby (Yale University) and Dr. J. F. Price (Boeing Scientific Research Laboratories) for helpful suggestions and criticisms.

Mr. Zahar is indebted to The Boeing Company for support during the summer of 1965 in its Scientific Research Laboratories.

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