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TRANSLATION

THEORY OF GYROSCOPIC DEVICES
PART II, CHAPTERS 7-11

By

S. S. Rivkin

FOREIGN TECHNOLOGY DIVISION

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U. S. BOARD ON GEOGRAPHIC NAMES TRANSLITERATION SYSTEM

Block	Italic	Transliteration	Block	Italic	Transliteration
А а	*А а*	A, a	Р р	*Р р*	R, r
Б б	*Б б*	B, b	С с	*С с*	S, s
В в	*В в*	V, v	Т т	*Т т*	T, t
Г г	*Г г*	G, g	У у	*У у*	U, u
Д д	*Д д*	D, d	Ф ф	*Ф ф*	F, f
Е е	*Е е*	Ye, ye; E, e*	Х х	*Х х*	Kh, kh
Ж ж	*Ж ж*	Zh, zh	Ц ц	*Ц ц*	Ts, ts
З з	*З з*	Z, z	Ч ч	*Ч ч*	Ch, ch
И и	*И и*	I, i	Ш ш	*Ш ш*	Sh, sh
Й й	*Й й*	Y, y	Щ щ	*Щ щ*	Shch, shch
К к	*К к*	K, k	Ъ ъ	*Ъ ъ*	"
Л л	*Л л*	L, l	Ы ы	*Ы ы*	Y, y
М м	*М м*	M, m	Ь ь	*Ь ь*	'
Н н	*Н н*	N, n	Э э	*Э э*	E, e
О о	*О о*	O, o	Ю ю	*Ю ю*	Yu, yu
П п	*П п*	P, p	Я я	*Я я*	Ya, ya

* ye initially, after vowels, and after ъ, ь; e elsewhere. When written as ё in Russian, transliterate as yё or ё. The use of diacritical marks is preferred, but such marks may be omitted when expediency dictates.

FOLLOWING ARE THE CORRESPONDING RUSSIAN AND ENGLISH

DESIGNATIONS OF THE TRIGONOMETRIC FUNCTIONS

Russian	English
sin	sin
cos	cos
tg	tan
ctg	cot
sec	sec
cosec	csc
sh	sinh
ch	cosh
th	tanh
cth	coth
sch	sech
csch	csch
arc sin	$\sin^{-1}$
arc cos	$\cos^{-1}$
arc tg	$\tan^{-1}$
arc ctg	$\cot^{-1}$
arc sec	$\sec^{-1}$
arc cosec	$\csc^{-1}$
arc sh	$\sinh^{-1}$
arc ch	$\cosh^{-1}$
arc th	$\tanh^{-1}$
arc cth	$\coth^{-1}$
arc sch	sech^{-1}
arc csch	csch^{-1}
rot	curl
lg	log

This book is the second part of a work on the theory of gyroscopic devices (GD's) (the first part was published in 1962). In it the theory of active-type gyroscopic stabilizers, differentiating and integrating gyroscopes, gyroscopic navigational systems, and also certain questions of gyroscopic stabilization are considered.

Basic attention is allotted to determination of dynamic errors of GD's in conditions of random controls and disturbances.

The book is intended for engineering-technical and scientific workers, studying questions of applied gyroscopy, and can serve as a training aid for students and post graduates, studying the theory of GD's.

PREFACE

In this, the second part of a work on the theory of gyroscopic devices (GD's) a theory of active gyroscopic stabilizers, and also of differentiating and integrating gyroscopes is expounded, certain questions of gyroscopic stabilization are considered, and information on the theory of gyroscopic navigational systems is given; among the latter are Foucault gyroscopes, the gyropendulum and the gyrocompass, and also gyroinertial navigational systems.

The theory of the indicated GD's is considered, taking into account the same features which were noted in the preface to Part I of this work, i.e., basic attention is allotted to determination of dynamic errors of GD's in conditions of random controls and disturbances; here there are used probability methods of research. In the analysis of GD's, besides methods, widely employed in applied gyroscopy, there are also called in engineering methods of the theory of automatic control.

The expediency of such an approach found its confirmation in the contemporary tendencies of development of the applied theory of gyroscopes, where in the solution of different problems connected with analysis and synthesis of GD's probability methods of investigation, and also methods of the theory of automatic control obtain ever greater importance.

The author expresses his gratitude to A. A. Sveshnikov, D. R. Merkin, E. I. Sliv, Ya. G. Ostromukhov, and S. S. Matveyev for a number of remarks on the book, and also to scientific editor R. I. Chertkov.

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CHAPTER 7

ACTIVE GYROSTABILIZERS AND THEIR APPLICATION

Designations Appearing in Cyrillic

В = IR = inner ring

в = zh = gyrohousing

г = g = gyroscopic

Г = Gyro = gyroscope

ГВ = GV = gyrovertical

г.м = gm = gyromotor

ГН = DG = directional gyro

Гк = Gh = gyrohousing

ГС = GS = gyrostabilizer

ГТ = GT = gyrotachometer

ГУ = GD = gyro device

г.э = g.e = gyroequator

Д = mot = motor

Д.г = mot.g = motor and gyros

ДМ = TQ = torque device

ДС = SP = signal pickoff

ин = in = inertia

к = c = correcting

К = Fr, R = frame, gimbal ring

М, м = Pend, pend = pendulum

МК = CP = correcting pendulum

н = best = best

Н = OR = outer ring

п = per = periodic

П = P, Pot = potentiometer

п = P = Plat = platform

р = f, cont = frame, control

Р = Dist = distributor

с = vel, s = velocity

с = s = stabilizing

с.д = s.m. = stabilizing motor

СД = SM = stabilizing motor

ст = stat = static

т = fr = friction

ТГ = TG = tachometer generator

у,у = Amp, amp = amplifier

э = e = equator

я = arm = armature

ц = circ = circulation

§ 7.1. Function of Active Gyrostabilizers (GS's). Fundamental Circuits of GS's

1. Function and Types of Active GS's

In the first part of this work [120] directional gyroscopes and vertical gyros were considered, in which astatic gyroscopes with three degrees of freedom are used. Such GD's are devices of readout type (Chapter 1, § 2.4, Par. 1). A common deficiency of readout GD's is their susceptibility to the influence of external disturbances. To increase the accuracy of a GD of this type we carry out "unloading," applying a complex suspension system for the gyroscopic sensor with minimum friction, and also servos, ensuring readings of the sensor with minimum load acting on it. Here, total error of the GD is "composed" of errors of the gyroscopic sensor and errors of the servo. These and other circumstances complicate creation of precision GD's of readout type.

In the last decade in connection with the development of automatic adjustment and control systems it was necessary to increase accuracy of GD's. Furthermore, there arose the necessity of directly stabilizing separate instruments and devices on an object. For solution of these problems there were created GD's, based on the principle of active gyroscopic stabilization.

The essence of the principle of active gyroscopic stabilization consists in compensating external disturbing torques by stabilizing torques artificially created

in the device by a special stabilizing (unloading) motor. Control of the stabilizing motor is realized usually by a signal pickoff (through an amplifier), which fixes small shifts of the gyroscope, caused by external disturbances. With these shifts in a power gyroscope, just as in an astatic one, there appear torques of gyroscopic reactions (gyroscopic torques), which compensate the disturbing torques in the interval of time until the stabilizing motor begins to work.

Such device we call active gyroscopic stabilizers (GS's) or active gyroscopic frames (gyroframes).*

The stabilizing motor "unloads" the GS on only one axis, on that one, for which there is not required application of "delicate" suspensions, as this occurs in GD's of readout type. With respect to the other axis of the suspension the GS is sensitive to external disturbances. Decrease of the shown influences on one axis in active GS's is easier than the same with respect to the two axes of suspension in the readout GD.

It is understandable that a GS does not need a servo for taking readings, since it permits direct mechanical connection with the stabilized object or with the angular motion pickoff of the object, determined by the GS.

These and other circumstances explain the comparatively high accuracy of GD's based on the principle of active gyroscopic stabilization.

In accordance with what has been said, an active GS is what we call an electromechanical gyroscopic device, applied for stabilization of different objects and equipped with a special motor for overcoming the influence on the stabilized object of disturbing torques. At present active GS's are widely applied on ships and aircraft for direct stabilization of separate instruments and devices. Furthermore, certain types of directional gyroscopes, gyroverticals, and combined devices, called directional gyroverticals operate on the principle of active gyroscopic stabilization.

Active GS's are rather complex electromechanical systems. The theory and methods of calculation of GD's, based on the principle of active gyroscopic stabilization, were broadly developed in the works of Soviet scientists B. V. Bulgakov, B. I. Kudrevich, A. Yu. Ishlinskiy, Ya. N. Roytenberg, and others.

In this chapter questions of analysis of power GS's and also application of GS's in certain gyroscopic devices (DG's, GV's and others) are considered. Power GS's can be based not only on astatic gyroscopes with three degrees of freedom, but also

*Subsequently we shall use the first of these terms, and by gyroframe we mean the actual mechanical part of a gyrostabilizer without electrical components of its control networks.

e.g., on floating integrating gyroscopes with two degrees of freedom. Such GS's, finding application in certain gyroinertial navigational systems, are considered in Chapter 11.

Active GS's, depending upon the number of gyroscopes applied in the frame can be one or two gyro. By the number of axes of stabilization power GS's are subdivided into single-axis, double-axis, and triple-axis. A single-axis GS stabilizes the object about one axis; such a stabilizer can be used also, for instance, in a DG of active type. A double-axis GS provides stabilization of the object about two mutually-perpendicular axes, i.e., creates a platform, stabilized with respect to the plane of the horizon; this stabilizer can be used also as a GV of active type. The triple-axis GS stabilizes an object in three mutually-perpendicular axes, i.e., creates a platform stabilized relative to inertial space or to the plane of the horizon and in azimuth; this stabilizer can be used also as a directional gyrovertical, an instrument determining three angles of rotation of an object about its center of gravity.

Depending upon the method of controlling the stabilizing motor we distinguish:

1) active GS's with control for the "angle of precession," using gyroscopes with large angular momentum;

2) active GS's with control not only by the angle of precession, but also by the derivative of this angle, and also by the derivative of the angle of rotation of the GS about the axis of rotation of the frame, using gyroscopes with comparatively small angular momentum.

The latter GS's sometimes are called active stabilizers based on "small" gyroscopes. In this case the weight and dimensions of the stabilizer itself are considerably less than for a stabilizer with control of the angle of precession, but the system of automation for a GS based on "small" gyroscopes is more complex.

Application of active GS's in different gyroscopic devices appeared in the Soviet Union at the end of the 1930's, but the basic principles and schemes of active gyroscopic stabilization were developed in 1924 by S. A. Nozdrovskiy [106, 158]. Analogous schemes appeared abroad considerably later [186, 191].

2. Fundamental Schemes of GS's

The principle active gyroscopic stabilization we shall explain from the example of a single-axis GS with one and two gyros.

a) <u>One-Gyro Stabilizer</u>.

The schematic of a single-axis GS with one gyro is shown in Fig. 7.1. Basic

components of the stabilizer are: gyrohousing Gh with rotor; frame Fr, playing the role of the outer gimbal and rigidly joined to the stabilized objects; signal pickoff SP,* fixed on spin axis Ox of the gyrochamber, called the axis of precession; amplifier Amp, which amplifies current from the signal pickoff and transmits it to the armature circuit of motor SM; stabilizing motor SM, connected by gear transmission to frame Fr and intended for application at torques to the axis of stabilization (axis $O\eta$ of frame rotation), which compensate external disturbing torques acting on the frame; correcting pendulum CP and torque device TQ, which are elements of the correction system of the GS.

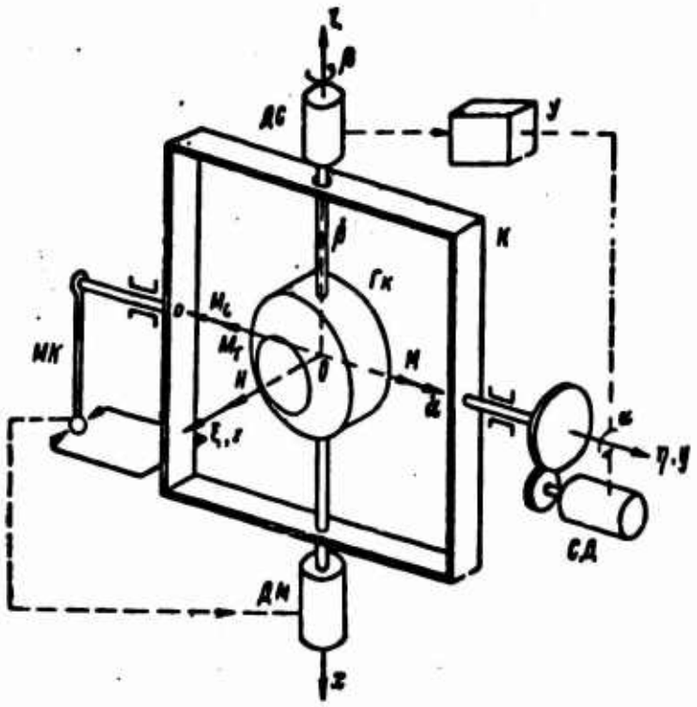

Fig. 7.1. Schematic of a one-gyro GS.

Let us consider briefly the principle of action of a GS. During application of an external disturbing torque M (Fig. 7.1) to the axis of rotation of frame Fr the latter will not turn about its axis of stabilization $O\eta$; due to the gyroscopic resistance of the rotor the frame will preserve its initial, e.g., vertical position, but the gyrohousing Gh will precess, turning about its axis of precession Ox. Due to this motion there appears gyroscopic torque M_g, directed, opposite the external disturbing torque M. This gyroscopic torque in the first instants compensates the disturbing torque, and the frame preserves its initial position. Subsequently with turn of the gyrohousing about is axis of precession Ox a certain angle β signal pickoff SP through amplifier Amp will switch on stabilizing motor; the latter will apply a stablizing torque M_s about the spin axis O_η of the frame, directed opposite external disturbing torque M. Under the influence of torque M_s the gyrochamber will start to precess in the opposite direction and will stop at a certain position, in which stabilizing torque M_s will balance the external disturbing torque M, if the latter does not change. The gyroscope axis will oscillate near this position.

Thus, the magnitude and direction of current intensity in the armature network

*Frequently instead of "signal pickoff" they apply the term "angle pickoff." Subsequently, as in Part I of this work, we shall use the term "signal pickoff."

of the stabilizing motor, and consequently, also the torque, developed by this motor, depend on the angle β of rotation of the housing. In this case we have an active GS with control of the angle of precession. Let us note that current in the motor armature and the signal in the amplifier do not build up instantly, and over the time interval, until stabilizing torque M_s reaches the needed magnitude, stabilization of the frame with the object is realized, as shown earlier, directly by the gyroscope with its precessional motion and accompanying gyroscopic torque M_g, which balances the torque M of external forces. Consequently, in an active GS the gyroscope carries out stabilization only in the first moment; subsequently it provides a stabilizing motor, with the corresponding power of which it is possible to stabilize considerable inertial masses connected with the frame (what is impossible in a GD of readout type).

From the above we understand why we sometimes consider an active GS a tracking system [4] for the external torque applied to the axis of stabilization. Here the gyroscope is a sensor, which "produces" a controlling action proportional to the integral of the disturbing torque applied to the stabilization axis, or, more exactly, the integral of the difference of the disturbing and stabilizing torques.

The stabilizing motor only compensates torques acting on the stabilization axis of the frame. Torques applied along the axis of precession Ox (Fig. 7.1) will cause drift of the frame with respect to its axis rotation, i.e., error of stabilization α. Besides this, there occurs drift of the frame from the assigned position due to rotation of the earth and natural movement of the object, since an active GS, like an astatic gyroscope, does not seek to occupy any definite position with respect to the earth. It follows from this that active GS's can be used, e.g., as directional gyroscopes, but not as gyrocompasses.

To keep the frame in an assigned position with respect to the earth, i.e., for imparting "directional force" (selectivity), we have a correction system, analogous to that applied in a vertical gyro and consisting at a correcting pendulum CP and a torque device TQ. The presence of the pendular correction system permits compensation of the above-indicated drifts of the frame; the principle of action of this sytem is analogous to that considered in Part I, § 2.3, Par. 2a.

b) Two-Gyro Stabilizer.

The one-gyro stabilizer is sensitive to rotations of the object, to which it is attached, about the axis of precession of the gyroscope. Actually, during rotation of frame F (Fig. 7.1), together with an object, about the axis of precession Ox the gyro seeks to preserve a constant position in inertial space, and its rotation about

axis Cx following the object and frame is realized by application by stabilizing motor SM (switched on by signal pickoff SP) of a torque about the axis of the frame, which can lead to oscillations of the frame, i.e., to errors of stabilization.

The frame can be made insensitive to rotation of the object about the axis of precession, i.e., the gyrohousing will turn, following the frame, without switching on the stabilizing motor, if we use a two-gyro stabilizer (Fig. 7.2). In the frame of such a stabilizer are fixed two gyroscopes $Gyro_1$ and $Gyro_2$; shafts of their gyrohousings are joined by a hinged mechanism — an anti-parallelogram A, which permits the gyrohousings to rotate about axes of precession identical angles, but in different directions.* Rotors of the gyroscopes spin in opposite directions, which determines the opposite direction of vectors of angular momentum $\vec{H}$.

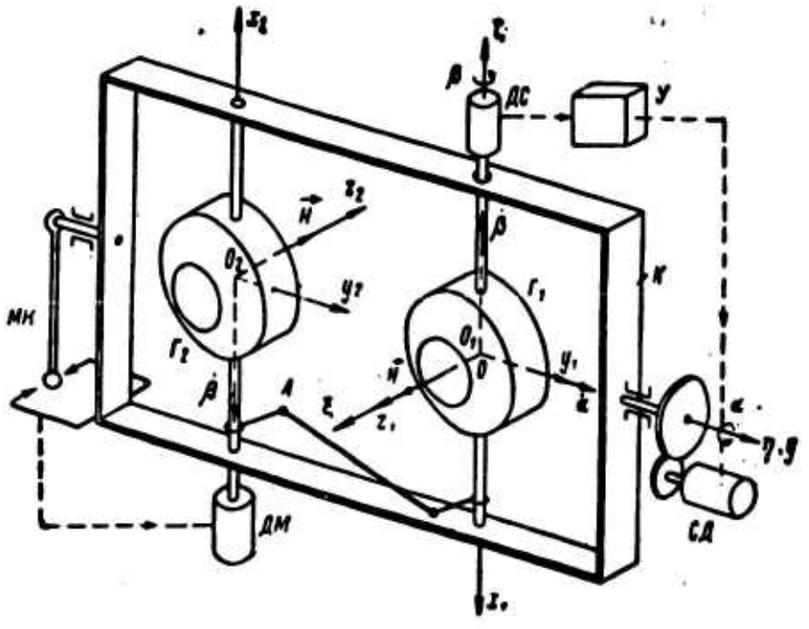

Fig. 7.2. Schematic of a two-gyro GS.

With rotation of the frame about the vertical axis the anti-parallelogram will force the gyrohousing to turn together with the frame, the signal pickup will preserve a neutral position, and the stabilizing motor will not be switched on. With the indicated turn of the gyrohousings there appear gyroscopic torques identical in magnitude, but directed along the axis of stabilization of the frame in opposite directions (due to the different direction of vectors $\vec{H}$ of the gyroscopes), so that they mutually compensate one another. In other respects the arrangement and principle of action of a two-gyro stabilizer do not differ from a one-gyro stabilizer. If to a two-gyro stabilizer (Fig. 7.2) we apply an external disturbing torque about the axis of stabilization, the gyrohousings will precess in different directions (these motions are allowed by the anti-parallelogram), which, through the signal pickoff and amplifier, with switch on the stabilizing motor. Presence of two gyroscopes doubles the angular momentum of the frame.

*In certain schemes of GS's, instead of an anti-parallelogram between axes of precession, there is installed gear transmission. Application of the anti-parallelogram is permissible only in devices, where angles of rotation of the gyrohousings are small.

c) Stabilizer Based on "Small" Gyroscopes

Gyrostabilizers, in which the stabilizing motor is controlled by the angle of precession β, along with high accuracy, also have several deficiencies: relatively great weight and size, considerable time for activating the stabilizer and others. Therefore, stabilizers based on "small" gyroscopes, not having these deficiencies have also found application.

A distinctive feature of such a stabilizer is the fact that the stabilizing motor is controlled not only by the angle of precession β, but also by derivatives $\dot{\beta}$, $\ddot{\beta}$, and also by derivatives $\dot{\alpha}$, $\ddot{\alpha}$, of the angle of rotation of the frame about the axis of stabilization, i.e., a torque of stabilization,

$$M_c = f(\beta, \dot{\beta}, \ddot{\beta}, \ldots, \dot{\alpha}, \ddot{\alpha}, \ldots) \tag{7.1.1}$$

is a function of the shown parameters.

Another distinctive feature of a GS based on "small" gyroscopes consists of the fact that, in it there are applied a stabilizing motor and amplifier with minute time constants. The necessity of smallness of time constants is explained by the following. During use of gyroscopes with a comparatively small angular momentum the initial stabilization of the frame by gyroscopic torque due to precessional movement of gyroscopes is possible only for a very small interval of time, and we need rapid out-in of the stabilizing motor.

In the considered GS with smaller weight and dimensions of the actual stabilizer the system of automation is more complex.

§ 7.2. Dynamic Characteristics of a GS

1. Geometric and Kinematic Parameters

Investigation of a GS starts with composition of differential equations of motion and the corresponding transfer functions. When composing equations we have in mind equations for a single-axis two-gyro stabilizer.

Considering, first, work of a GS on a fixed base, we select as the reference system trihedron $O\xi\eta\zeta$, for which axis $O\eta$ coincides with the axis of rotation of the frame (Fig. 7.2), axis $O\xi$ is horizontal, and axis $O\zeta$ is located in the vertical plane, passing through the axis of the frame; origin O we consider coinciding with the point of suspension O_1 of gyroscope $Gyro_1$.

The position of the spin axis O_1z_1 (Figs. 7.2 and 7.3a) of the rotor of gyroscope $Gyro_1$ relative to its initial direction $O\xi$, or, which is the same, the position of Resal's axes $O_1x_1y_1z_1$, connected with the gyrochamber of gyroscope $Gyro_1$, with respect to axes $O\xi\eta\zeta$, is determined by angles α and β. In Fig. 7.3b the position of

Resal axes $O_2x_2y_2z_2$, connected with the gyrohousing of gyroscope $Gyro_2$, with respect to axes $O\xi\eta\zeta$ is determined also by angles α and β. The initial position (for $\alpha = \beta = 0$) of Resal axes in Fig. 7.3 is designated O_1 (O_2) xyz.

Transformation of axes $O\xi\eta\zeta$ to $O_1x_1y_1z_1$ is recorded in the form of matrix equality

$$[x_1, y_1, z_1] = A_1[\xi, \eta, \zeta]. \tag{7.2.1}$$

where matrix $A_1 = \|a'_{\mu\nu}\|$ ($\mu, \nu = 1, 2, 3$) can be presented in the form of the product of two matrices, corresponding to rotations of axes $O_1x_1y_1z_1$ angles α and β; as a result we have matrix

$$A_1 = \begin{Vmatrix} -\sin\alpha & 0 & -\cos\alpha \\ -\cos\alpha\sin\beta & \cos\beta & \sin\alpha\sin\beta \\ \cos\alpha\cos\beta & \sin\beta & -\sin\alpha\cos\beta \end{Vmatrix} \tag{7.2.2}$$

which can be obtained also from (6.2.2)* by replacement of α by $(90^\circ + \alpha)$ (compare Figs. 7.3a and 1.6).

Since angles α and β in practice are usually rather small, we present matrix A_1 in approximate form:

$$A_1 = \begin{Vmatrix} -\alpha & 0 & -1 \\ -\beta & 1 & 0 \\ 1 & \beta & -\alpha \end{Vmatrix} \tag{7.2.3}$$

Transformation of axes $O\xi\eta\zeta$ to $O_2x_2y_2z_2$ can be recorded in matrix form in the following manner:

$$[x_2, y_2, z_2] = A_2[\xi, \eta, \zeta]. \tag{7.2.4}$$

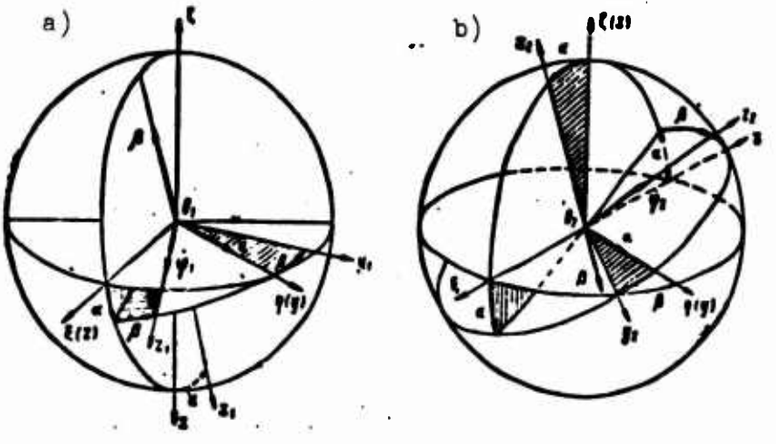

Fig. 7.3. Euler angles, determining the position of gyroscopes.

where for matrix $A_2 = \|a''_{\mu\nu}\|$ ($\mu, \nu = 1, 2, 3$), by analogy with (7.2.2), we obtain:

*Formulas, paragraphs, figures and examples, numbers of which start with 1-6, ertain to Chapters 1-6 of Part I of this work, and subsequently indication of Part I s omitted.

$$A_2 = \begin{Vmatrix} \sin\alpha & 0 & \cos\alpha \\ \cos\alpha\sin\beta & \cos\beta & -\sin\alpha\sin\beta \\ -\cos\alpha\cos\beta & \sin\beta & \sin\alpha\cos\beta \end{Vmatrix} \tag{7.2.5}$$

from which for small α and β we have

$$A_2 = \begin{Vmatrix} \alpha & 0 & 1 \\ \beta & 1 & 0 \\ -1 & \beta & \alpha \end{Vmatrix} \tag{7.2.6}$$

Projections p', q', r' of angular velocity $\vec{\omega}'$ of gyroscope Gyro_1 on Resal axes $O_1x_1y_1z_1$, if trihedron $O\xi\eta\zeta$ is motionless, will be determined by relationships

$$p' = -\dot{\beta};\ q' = \dot{\alpha}\cos\beta;\ r' = \dot{\varphi}_1 + \dot{\alpha}\sin\beta, \tag{7.2.7}$$

where $\dot{\varphi}_1$ — angular velocity of spinning of the rotor of gyroscope Gyro_1.

Likewise for gyroscope Gyro_2 we obtain

$$p'' = -\dot{\beta},\ q'' = \dot{\alpha}\cos\beta,\ \ r'' = \dot{\varphi}_2 + \dot{\alpha}\sin\beta, \tag{7.2.8}$$

where $\dot{\varphi}_2$ — angular velocity of spinning of the rotor of gyroscope Gyro_2.

For calculation of motion of axes $O\xi\eta\zeta$ in inertial space it is necessary in (7.2.7) and (7.2.8) to introduce components of the angular velocity of following $\vec{u}$ of the shown axes (§ 1.2).

2. Differential Equations of Motion of a Two-Gyro GS

a) Composing Equations of Motion of the Gyroframe.

Let us compose equations of motion at the actual gyroframe, supplementing them subsequently by equations, characterizing processes, in electrical components of the GS. A two-gyro gyroframe belongs to multigyroscope systems; the general method of composition of equations of such systems is considered in § 7.6; therefore here we shall limit ourselves to direct composition of equations of the gyroframe, using Lagrange equations (§ 1.5, Par. 2). First we shall consider the case of a fixed base.

Movement of a two-gyro frame is described by a system of four equations, which, by analogy with (1.5.2), can be presented in the following form:

$$\left.\begin{aligned} \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\alpha}}\right) - \frac{\partial T}{\partial \alpha} &= M_\eta \\ \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\beta}}\right) - \frac{\partial T}{\partial \beta} &= -M_\xi \\ \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\varphi}_1}\right) - \frac{\partial T}{\partial \varphi_1} &= M_{z_1} \\ \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\varphi}_2}\right) - \frac{\partial T}{\partial \varphi_2} &= M_{z_2} \end{aligned}\right\} \tag{7.2.9}$$

where T — kinetic energy of system, consisting of a frame and two gyroscopes;

M_η, M_x, M_{z_1}, M_{z_2} — external torques, applied to the frame and to the gyroscope about the indicated axes;*

α, β, φ_1, φ_2 — generalized coordinates (the same as in Par. 1).

Kinetic energy of the system with identical gyroscopes is determined by relationship

$$T = \frac{1}{2}\left\{J_{f\eta}\dot{\alpha}^2 + J_{r.e}\left[(p')^2 + (q')^2\right] + J_{r.e}\left[(p'')^2 + (q'')^2\right] + J(r')^2 + J(r'')^2\right\}, \tag{7.2.10}$$

where $J_{f\eta}$ — moment of inertia of the frame together with stabilized object about its spin axis $O\eta$ (Fig. 7.2) (here there is included the moment of inertia of the armature of the stabilizing motor given with respect to axis $O\eta$, which we will ignore for the time being; see Par. b);

$J_{g.e}$ — equatorial moment of inertia of the gyroscope;

J — moment of inertia of the gyrorotor about its spin axis;

p', q', r', p'' ... — components of instantaneous angular velocity of gyroscopes Gyro_1 and Gyro_2 (Fig. 7.2), determined by relationships (7.2.7) and (7.2.8).

Introducing (7.2.7) and (7.2.8) in (7.2.10), we obtain

$$T = \frac{1}{2}\left[J_{f\eta}\dot{\alpha}^2 + 2J_{r.e}\left(\dot{\beta}^2 + \dot{\alpha}^2\cos^2\beta\right) + J\left(\dot{\varphi}_1 + \dot{\alpha}\sin\beta\right)^2 + J\left(\dot{\varphi}_2 + \dot{\alpha}\sin\beta\right)^2\right]. \tag{7.2.11}$$

Let us note that in expression (7.2.11) the minute component $J_{gh}\dot{\alpha}^2\sin^2\beta$, included in general in the kinetic energy of the gyroscope [see 1.3.61] is rejected, due to smallness of moment of inertia J_{gh} of the gyrohousing about the rotor axis as compared to moments of inertia J of the rotor. In other respects formula (7.2.11) has the same form as relationship (1.3.61) for the kinetic energy of an astatic gyroscope, if in the latter we designate $(J_e + J_{gh.e})$ by $J_{g.e}$ [see (3.2.4)].

Calculating partial derivatives of kinetic energy and substituting them in (7.2.9), we find the system of differential equations of motion of the gyroframe:

$$\left.\begin{aligned}
&\frac{d}{dt}\left[\left(J_{f\eta} + 2J_{r.e}\cos^2\beta\right)\dot{\alpha}\right] + \frac{d}{dt}\left[J\left(\dot{\varphi}_1 + \dot{\alpha}\sin\beta\right)\sin\beta + J\left(\dot{\varphi}_2 + \dot{\alpha}\sin\beta\right)\sin\beta\right] = M_\eta\\
&2J_{r.e}\left(\ddot{\beta} + \dot{\alpha}^2\sin\beta\cos\beta\right) - \left[J\left(\dot{\varphi}_1 + \dot{\alpha}\sin\beta\right) + J\left(\dot{\varphi}_2 + \dot{\alpha}\sin\beta\right)\right]\dot{\alpha}\cos\beta = -M_x\\
&\frac{d}{dt}J\left(\dot{\varphi}_1 + \dot{\alpha}\sin\beta\right) = M_{z_1}\\
&\frac{d}{dt}J\left(\dot{\varphi}_2 + \dot{\alpha}\sin\beta\right) = M_{z_2}
\end{aligned}\right\} \tag{7.2.12}$$

*We note that, $M_x = M_{x_1} + M_{x_2}$ is the resultant external torque acting on axes of precession of the gyrohousings.

For a steady-state of work of the gyroscopes $M_{z_1} = M_{z_2} = 0$ (§ 1.4, Par. 2); therefore, from the third and fourth equations of (7.2.12), by analogy with (1.4.8), we have (considering $\dot{\varphi}_1 = \dot{\varphi}_2$)

$$J(\dot{\varphi}_1 + \dot{\alpha}\sin\beta) = J(\dot{\varphi}_2 + \dot{\alpha}\sin\beta) = \text{const} = H, \tag{7.2.13}$$

where H — angular momentum of the gyroscope, identical for both gyroscopes.

Then, considering (7.2.13), we can characterize motion of the gyroframe by two equations

$$\left.\begin{aligned} \frac{d}{dt}\left[\left(J_{p_1} + 2J_{r.s}\cos^2\beta\right)\dot{\alpha}\right] + 2H\dot{\beta}\cos\beta = M_\eta \\ 2J_{r.s}\left(\ddot{\beta} + \dot{\alpha}^2\sin\beta\cos\beta\right) - 2H\dot{\alpha}\cos\beta = -M_x \end{aligned}\right\} \tag{7.2.14}$$

Usually, during investigation of a GS we use linearized equations of motion, which are obtained from (7.2.14) in the form

$$\left.\begin{aligned} \left(J_{p_1} + 2J_{r.s}\right)\ddot{\alpha} + 2H\dot{\beta} = M_\eta \\ 2J_{r.s}\ddot{\beta} - 2H\dot{\alpha} = -M_x \end{aligned}\right\} \tag{7.2.15}$$

analogous to equations (1.6.3) of motion of an astatic gyroscope.

For a single-gyro frame, as it is easy to see, we have the following equations:

$$\left.\begin{aligned} \left(J_{p_1} + J_{r.s}\right)\ddot{\alpha} + H\dot{\beta} = M_\eta \\ J_{r.s}\ddot{\beta} - H\dot{\alpha} = -M_x \end{aligned}\right\} \tag{7.2.16}$$

Let us compose equations of motion of a gyroframe on a moving base. Considering ourselves limited to derivation of linearized equations, we apply the Kudrevich method (§ 1.6, Par. 2). For this we construct an auxiliary drawing (Fig. 7.4); on axes $O\xi\eta\zeta$ we plot components u_ξ, u_η, u_ζ at the angular velocity of following $\vec{u}$ of these axes, caused by rotation of the earth and spinning of the object (§ 1.2).

Along axis of stabilization $O\eta$ (Fig. 7.4) are located vectors of moments of forces: $J_{f\ \eta}\ddot{\alpha}$ — moment of force of inertia of the frame; $J_{g.e}\ddot{\alpha}$ — moment of force of inertia of gyroscope $Gyro_1$ (or $Gyro_2$); $H(\dot{\beta} + u_\zeta)$, $H(\dot{\beta} - u_\zeta)$ — gyroscopic torques, created by gyroscopes $Gyro_1$ and $Gyro_2$; M_η — torque of external forces applied to the frame. About the axis of precession O_1x_1 of gyroscope $Gyro_1$ there act: $J_{g.e}\ddot{\beta}$ — moment of the force of inertia; $H(\dot{\alpha} + u_\eta)$ — gyroscopic torque; M_{x_1} — external torques applied to the gyroscope. About axis of precession O_2x_2 of gyroscope $Gyro_2$ there act, correspondingly, moments: $J_{g.e}\ddot{\beta}$, $H(\dot{\alpha} + u_\eta)$ and M_{x_2}.

In accordance with D'Alembert's principle the sum of these moments on axis $O\eta$ and O_1x_1 (or O_2x_2) should be equal to zero; therefore, from Fig. 7.4, considering that the gyrohousings are connected by kinematic transmission with a transmission ratio of 1, and, consequently, that moments $J_{g.e}\ddot{\beta}$, $H(\dot{\alpha} + u_\eta)$, M_{x_2}, acting along axis

Ox_2, are transmitted by the anti-parallelogram to axis Ox_1, we have (for φ = const)

$$\left.\begin{aligned} J_{pq}\ddot{\alpha}+J_{r.s}\dot{\alpha}+J_{r.s}\ddot{\alpha}+H\left(\dot{\beta}+u_\zeta\right)+H\left(\dot{\beta}-u_\zeta\right)-M_\eta=0\\ J_{r.s}\ddot{\beta}-H\left(\dot{\alpha}+u_\eta\right)+J_{r.s}\ddot{\beta}-H\left(\dot{\alpha}+u_\eta\right)+M_{x_1}+M_{x_2}=0\end{aligned}\right\} \qquad (7.2.17)$$

or

$$\left.\begin{aligned} (J_{pq}+2J_{r.s})\ddot{\alpha}+2H\dot{\beta}=M_\eta\\ 2J_{r.s}\ddot{\beta}-2H\left(\dot{\alpha}+u_\eta\right)=-\left(M_{x_1}+M_{x_2}\right)=-M_x\end{aligned}\right\} \qquad (7.2.18)$$

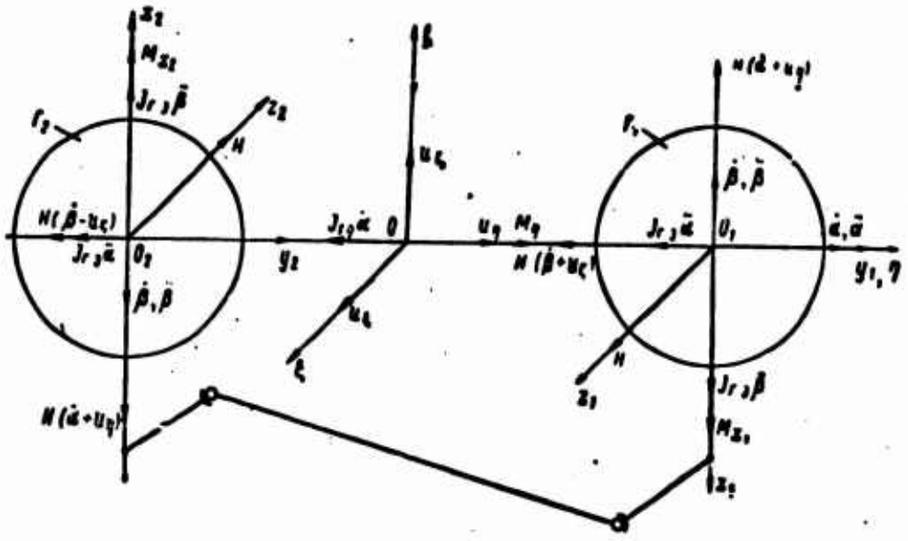

Fig. 7.4. Deriving equations of motion of a two-gyro GS on a moving base.

From the given equations it follows that the behavior of a GS on a moving base is influenced only by component u_η of the angular velocity of following on the spin axis of the frame, for compensation of which it is necessary to apply a correction system (§ 7.1, Par. 2a). According to (7.2.18) a two-gyro stabilizer is insensitive to rotation of the base about vertical $O\zeta$ (with which axes of precession O_1x_1, O_2x_2 coincide) with angular velocity u_ζ, since the gyroscopes, thanks to the anti-parallelogram will turn together with the frame and base, and gyroscopic torques Hu_ζ and $-Hu_\zeta$ appearing here compensate one another.

For a one-gyro frame (Fig. 7.1) we analogously obtain equations

$$\left.\begin{aligned} (J_{pq}+J_{r.s})\ddot{\alpha}+H\left(\dot{\beta}+u_\zeta\right)=M_\eta\\ J_{r.s}\ddot{\beta}-H\left(\dot{\alpha}+u_\eta\right)=-M_x\end{aligned}\right\} \qquad (7.2.19)$$

explaining its distinctive features, shown in § 7.1, Par 2a.

Equations (7.2.19) are composed taking into account components of only the first order of smallness. Equations of motion of the same one-gyro frame when allowing for the most essential components of the second order of smallness we can present in the form [131]

$$\left.\begin{aligned} (J_{pq}+J_{r.s})\ddot{\alpha}+H\left(\dot{\beta}+u_\zeta+u_\zeta\alpha\right)=M_\eta\\ J_{r.s}\ddot{\beta}-H\left(\dot{\alpha}+u_\eta-u_\zeta\beta\right)=-M_x\end{aligned}\right\} \qquad (7.2.20)$$

differing from (7.2.19) by components $Hu_\xi\alpha$ and $Hu_\xi\beta$. In certain cases, for instance during investigation of a one-gyro frame during rolling of a ship, we should take these components into account, i.e., use equations (7.2.20). It is easy to show [131] that in reference to a two-gyro frame analogous components for gyroscopes $Gyro_1$ and $Gyro_2$ (Fig. 7.2) will be mutually compensated and will not enter equations (7.2.18), if during their composition we consider components of the second order of smallness. Therefore, a two-gyro frame will (as compared to a one-gyro) be insensitive not only to rotation of the base around the vertical with angular velocity u_ξ, but also to components $u_\xi\alpha$ and $u_\xi\beta$ of angular velocity u_ξ, which can have a substantial value during rolling of a ship.

b) Expressions of Torques, Acting on a GS.

Torques M_η and M_x, acting on a gyrostabilizer, are composed of control and disturbing torques (§ 2.3, Par. 1). Among control torques are: stabilizing torque M_s about axis $O\eta$, applied by stabilizing motor SM (Fig. 7.2), and correcting torque $M_{c\,x}$, e.g, about axis O_2x_2, applied by torquer device TQ. Among disturbing torques M_1 and M_2 are: friction torques in the suspension axis of the frame and axes of rotation of the gyrohousings; moments of inertia of the motor armature due to oscillations of the frame and object, on which the GS is fixed; moments unbalance torques of the gyroframe and stabilized object, etc. In accordance with what has been said we have

$$\left.\begin{aligned} M_\eta &= M_c + M_1 \\ M_x &= M_{cx} + M_2 \end{aligned}\right\} \tag{7.2.21}$$

where

$$\left.\begin{aligned} M_1 &= M_{tr\eta} + M_{in} + f_1(t) \\ M_2 &= M_{trx} + f_2(t) \end{aligned}\right\} \tag{7.2.22}$$

$M_{fr\ \eta}$, $M_{fr\ x}$ – friction torques in axes of the suspension;

M_{in} – moment of inertia of the motor armature due to oscillations of the frame and object;

$f_1(t)$, $f_2(t)$ – other external disturbing torques.

Stabilizing torque M_s is expressed through the torque of stabilizing motor $M_{s.m}$ on the spin axis of the armature by the relationship

$$M_c = \gamma\eta M_{c.A}, \tag{7.2.23}$$

where γ – gear transmission ratio from the shaft of the stabilizing motor to the axis of stabilization of the frame;

η – transmission factor.

Considering subsequently, for simplicity, (in examining the theory of GS's) that $\eta = 1$, we have

$$M_c = \gamma M_{c.A}. \tag{7.2.24}$$

The expression for $M_{s.m}$ depending upon control parameters and engine characteristics will be given later.

Let us turn to determination of correcting torque M_{cx}. Usually there is used a system of pendular correction (Fig. 7.2), the principle of action of which does not need explanation (see § 2.3, Par. 2a).

For the proportional characteristic of correction (Fig. 2.18a) in the case of a fixed base, when the correcting pendulum maintains a vertical position, correcting torque $M_{c\ x}$, by analogy with (2.3.4), is determined by relationship

$$M_{ux} = -S_1\alpha. \tag{7.2.25}$$

Another possible method of horizontal correction of a GS is carried out [56] means of shifting the center of gravity of G_1 and G_2 (Fig. 7.5) of gyroscopes $Gyro_1$ and $Gyro_2$ relative to their geometric centers O_1 and O_2 a magnitude l in the equatorial plane; gyroscopes here are turned into unique horizontal pendulums.* Upon deflection of the gyroframe an angle of α from its vertical position the weight component $P_{g.m}$ of the gyromotor, equal to $P_{g.m} \sin\alpha \approx P_{g.m}\alpha$, introduces, e.g., to axis of precession O_1x_1 a correcting torque

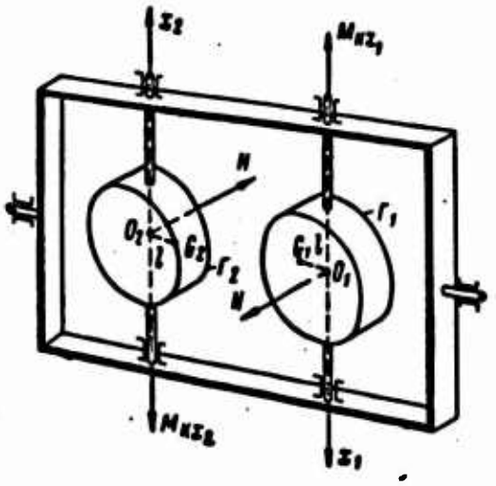

Fig. 7.5. Schematic of horizontal compensation of a GS by shifting the center of gravity of the gyroscopes.

$$M_{ux_1} = -P_{r.u}l\alpha. \tag{7.2.26}$$

Designating $P_{g.m}l = S_2$, we obtain

$$M_{ux_1} = -S_2\alpha, \tag{7.2.27}$$

analogous to (7.2.25).

Thus, both methods of horizontal correction of a GS on a nonoscillatory base are equivalent.

For the case of an oscillatory base, when the correcting pendulum is deflected from the vertical at angles $\chi_1(t)$, as a certain random function of time, correcting torque $M_{c\ x}$, by analogy with (6.2.8), will be

$$M_{ux} = -S_2[\alpha - \chi_1(t)]. \tag{7.2.28}$$

In the second method of correction for the case of an oscillatory base the correcting torque will be determined by formula

$$M_{ux_1} = -\left(P_{r.u}l\alpha + \frac{P_{r.u}}{g}w_\xi l\alpha - \frac{P_{r.u}}{g}w_\xi l\right) \tag{7.2.29}$$

*Such correction is sometimes called mechanical.

or

$$M_{\text{кл}_1} = -S_2\alpha\left(1+\frac{w_\zeta}{g}\right)+S_2\frac{w_\xi}{g}, \tag{7.2.30}$$

w_ξ and w_ζ – acceleration of the center of gravity of the gyroscope along axes $O\xi$ and $O\zeta$ (Fig. 7.2).

One of the forms of disturbances to a GS are friction torques in the suspension axes. In the case of liquid friction and a fixed base friction torques are determined by relationships of type (2.3.72):

$$M_{\text{тр}x} = n_2\dot{\beta};\ M_{\text{тр}\eta} = -n_1\dot{\alpha}. \tag{7.2.31}$$

When the GS is mounted on an oscillating base, e.g., on a ship, when axis $O\eta$ of stabilization of the gyroframe is combined with the longitudinal axis of the ship the liquid friction torque about axis $O\eta$ will be

$$M_{\text{тр}\eta} = -n_1(\dot{\alpha}-\dot{\theta}), \tag{7.2.32}$$

where $\dot{\theta}(t)$ is a random function, characterizing angular velocity of rolling of the ship.

The friction torque $M_{fr\ x}$ along the axis of precession here will not change, i.e., will be determined by the same expression (7.2.31), since yawing of the ship on a course does not affect work of a two-gyro stabilizer.

With dry friction in the axes of the frame suspension, assuming for simplicity that friction torques are symmetric during rotation of the axis of the suspension in various directions, by analogy with (2.3.74) we have

$$M_{\text{тр}x} = K_x \operatorname{sign}\dot{\beta};\ M_{\text{тр}\eta} = -K_y \operatorname{sign}\dot{\alpha}. \tag{7.2.33}$$

Due to certain residual unbalance of a spinning gyroscope [4] the gyrochamber practically always accomplished oscillatory motions with a frequency, equal to the angular velocity of spinning of the rotor, i.e., on the axis of precession there are automatically created oscillations, which translate forces of dry friction into forces of liquid resistance.* Therefore, we can determine friction torque $M_{fr\ x}$ on the axis of precession by formula (7.2.31). This phenomenon on the axis of stabilization usually does not appear because between the gyrohousing of the gyroscope (accomplishing the vibration) and the place of application of the load moment there is a long kinematic line with torsions and gaps, absorbing the oscillation. Therefore, friction

*See also: V. A. Besekerskiy. Application of vibrators to remove nonlinearity in automatic controllers. "Automation and telemechanics," No. 6, Academy of Sciences of USSR, 1947.

torque $M_{fr\ \eta}$ on the axis of stabilization is usually determined by formula (7.2.33). When the GS is set on an oscillating base, e.g., on a ship, for $M_{fr\ \eta}$, by analogy with (2.3.85), we have

$$M_{тр} = -K_1 \operatorname{sign}(\dot{\alpha} - \dot{\theta}). \tag{7.2.34}$$

Let us turn to finding the moment of inertia M_{in} of the armature of the stabilizing motor, caused by oscillations of frame and object. For definitiveness we shall consider the case, when the GS is installed on a ship, experiencing rolling; the axis of stabilization of the frame is parallel to the longitudinal axis of the ship.

We compose equations of motion of the armature of the stabilizing motor in an absolute system of coordinates [132]:

$$J_я \dot{\omega}_я = M_{с.я} - M, \tag{7.2.35}$$

where J_{arm} – moment of inertia of the armature with respect to its spin axis;

ω_{arm} – absolute angular velocity of the armature;

$M_{s.m}$ – torque of the motor;

M – reaction torque of the frame of the stabilizer, applied to the armature of the stabilizing motor.

Let us designate by $\omega'_{arm} = \omega_{arm} - \dot{\theta}$ the relative angular velocity of the motor armature (with respect to the ship; $\dot{\theta}$ – angular velocity of its rolling), and by $\dot{\alpha}' = \dot{\alpha}' - \dot{\theta}$ – the relative angular velocity of the frame; then for gear transmission ratio γ, from the shaft of the stabilizing motor to the shaft of the frame we have

$$\gamma = \frac{\omega'_я}{\dot{\alpha}'} = \frac{\omega_я - \dot{\theta}}{\dot{\alpha} - \dot{\theta}}. \tag{7.2.36}$$

According to (7.2.35) and (7.2.36) we obtain

$$M = -\gamma J_я \ddot{\alpha} + (\gamma - 1) J_я \ddot{\theta} + M_{с.я}. \tag{7.2.37}$$

Then about the axis of the frame we have torque γM, equal to

$$\gamma M = -\gamma^2 J_я \ddot{\alpha} + \gamma(\gamma - 1) J_я \ddot{\theta} + \gamma M_{с.я}. \tag{7.2.38}$$

Since, according to (7.2.24), $\gamma M_{s.m} = M_s$, the remaining components in (7.2.38) are components of the moment M_{in}, i.e.,

$$M_{ин} = -\gamma^2 J_я \ddot{\alpha} + \gamma(\gamma - 1) J_я \ddot{\theta}. \tag{7.2.39}$$

This expression was obtained on the assumption that the number k of axes of transmission (including the axis of the frame and the axis of the stabilizing motor) is odd; in general,

$$M_{ин} = -\gamma^2 J_я \ddot{\alpha} + \gamma[\gamma + (-1)^k] J_я \ddot{\theta}. \tag{7.2.40}$$

Expressions of torques M_η and M_x on the axes of stabilization and precession, according to (7.2.21), (7.2.22), (7.2.24), (7.2.28), (7.2.31), (7.2.32) and (7.2.40), will be:*

$$\left.\begin{aligned} M_\eta &= \gamma M_{e.д} - n_1(\dot\alpha - \dot\theta) - \gamma^2 J_д \ddot\alpha + \\ &+ \gamma[\gamma + (-1)^k] J_д \ddot\theta + f_1(t) \\ M_x &= -S_2[\alpha - \chi_1(t)] + n_2\dot\beta + f_2(t) \end{aligned}\right\} \qquad (7.2.41)$$

Putting (7.2.41) in the equations of motion (7.2.18) of the gyroframe, we obtain:

$$\left.\begin{aligned} I_\eta\ddot\alpha + 2H\dot\beta + n_1\dot\alpha - \gamma M_{e.д} &= \gamma[\gamma + (-1)^k] J_д\ddot\theta + \\ &+ n_1\dot\theta + f_1(t) \\ 2J_{г.э}\ddot\beta - 2H(\dot\alpha + u_\eta) - S_2\alpha + n_2\dot\beta &= -S_2\chi_1(t) - f_2(t) \end{aligned}\right\} \qquad (7.2.42)$$

where

$$I_\eta = J_{p\eta} + 2J_{г.э} + \gamma^2 J_д. \qquad (7.2.43)$$

The above mentioned differential equations of motion of the gyroframe when the stabilizer is set on a oscillating base were first obtained by Ya. N. Roytenberg (see [132, 130]).

Expressions for torques acting on the GS taking into account interference of its individual components can be composed by the method of A. Yu. Ishlinskiy [45, 51] (see § 1.6, Par. 7).

c) Equations, Describing Processes in Electrical Components of a GS.

The equations of motion of a gyroframe obtained above one should supplement by equations, characterizing transients in electrical components of the GS.

In the beginning consider a GS, in which control of the stabilizing motor is by the angle of precession β.

Let us assume that in the GS there is employed a d-c stabilizing motor SM (Fig. 7.2) with independent excitation; the latter develops a torque**

$$M_{e.д} = \Phi i, \qquad (7.2.44)$$

where Φ – magnetic flux of excitation windings;

i – current strength in the motor armature.

Current strength i enters the equation of balance of voltages in the motor

---

*Friction in both axes of the suspension for simplification is assumed liquid, where $n_2\beta$ is the total friction torque in both axes of precession.

**Sometimes the torque of the motor is recorded in the form $M_{s.m} = c_m\Phi i$, where c_m is a constant coefficient for the given motor (see, for instance: M. I. Komisar. Motors of gyroscopic systems, Oborongiz, 1963).

armature

$$l\frac{di}{dt}+ri+c\omega'_{a}=U_{y}, \tag{7.2.45}$$

where l — coefficient of self-inductance of the armature network;

r — resistance of the armature network (including internal resistance of the amplifier);

c — a coefficient, characterizing the dependence of the counter emf induced in the motor armature on the relative angular velocity ω'_{arm} of its rotation;

U_{amp} — voltage fed from amplifier Amp (Fig. 7.2) to the armature of the stabilizing motor.

Substituting in (7.2.45), during work of the GS on a vibrating object (a ship), for ω'_{arm} relationship (7.2.36), we obtain

$$l\frac{di}{dt}+ri+c_{1}\dot{\alpha}=c_{1}\dot{\delta}+U_{y}. \tag{7.2.46}$$

One can present this expression in somewhat different form, if one were to introduce the electromagnetic time constant T_{mot} of the motor armature network:

$$T_{a}=\frac{l}{r}, \tag{7.2.47}$$

$$T_{a}\frac{di}{dt}+i+\frac{c_{1}}{r}\dot{\alpha}=\frac{c_{1}}{r}\dot{\delta}+\frac{1}{r}U_{y} \tag{7.2.48}$$

or, designating

$$\frac{c_{1}}{r}=b,\ \frac{1}{r}=e, \tag{7.2.49}$$

$$T_{a}\frac{di}{dt}+i+b\dot{\alpha}=b\dot{\delta}+eU_{y}. \tag{7.2.50}$$

Voltage U_{amp}, from amplifier Amp (Fig. 7.2), is determined by equation

$$T_{y}\dot{U}_{y}+U_{y}=kU, \tag{7.2.51}$$

where T_{amp} — time constant of the amplifier;

k — amplifier gain;

U — voltage from signal pickoff SP, proceeding to amplifier input.

Voltage U is a function of the angle of precession β of the gyroscopes. In the case of a linear response of the signal pickoff we have*

$$U=-k_{1}\beta, \tag{7.2.52}$$

where k_1 — amplification factor of the considered unit.

Substituting (7.2.52) in (7.2.51) and designating

$$k_{1}k=S_{1}, \tag{7.2.53}$$

*In formula (7.2.52) we take the minus sign, since with a positive disturbing torque the angle of precession of the gyroscope $\beta > 0$ (Fig. 7.2), and for return of the the gyroscope to the intial position stabilizing torque M_s should be directed toward the negative side of axis $O\eta$.

we obtain

$$T_y\dot{U}_y + U_y = -S_1\beta. \tag{7.2.54}$$

Thus, we have the following equations, characterizing transients in electrical components of a GS.*

$$\begin{aligned} &M_{c.x} = \Phi i \text{ (motor's torque)} \\ &T_y\dot{U}_y + U_y = -S_1\beta \text{ (amplifier)} \\ &T_x\frac{di}{dt} + i + b\dot{\alpha} = b\dot{\theta} + cU_y \text{ (motor armature network)} \end{aligned} \tag{7.2.55}$$

Let us compose similar equations for a GS using "small" gyroscopes. In this case control of the motor is realized not only by the angle of precession β, but by a more complicated dependence, including also auxiliary signals. One scheme of a GS using "small" gyroscopes [4] is shown in Fig. 7.6, from which it is clear that to the amplifier there proceed: voltage U_1 from signal pickoff SP, proportional to angle β; voltage U_2 from tachometer TG feedback, proportional to the relative angular velocity of the frame ($\dot{\alpha} - \dot{\theta}$); voltage U_3 from gyrotachometer GT, determining angular velocity $\dot{\theta}$ of rolling of the ship. Consequently, instead of (7.2.52) we have [4]**

$$U = U_1 + U_2 + U_3 = -k_1\beta - k_2(\dot{\alpha} - \dot{\theta}) - k_3\dot{\theta}. \tag{7.2.56}$$

Application in the scheme of (Fig. 7.6) of a tachometer generator and a gyrotachometer permits controlling, besides angle β, also of angular velocity $\dot{\alpha}$, i.e., introduction of a signal from the gyrotachometer gives the possibility of partially ($k_2 \neq k_3$) compensating error of the GS, arising due to the fact that the tachometer generator yields a voltage, proportional, not to absolute angular spin velocity $\dot{\alpha}$ of the frame, but to the relative angular velocity ($\dot{\alpha} - \dot{\theta}$). In general in a GS using "small" gyroscopes the control signal

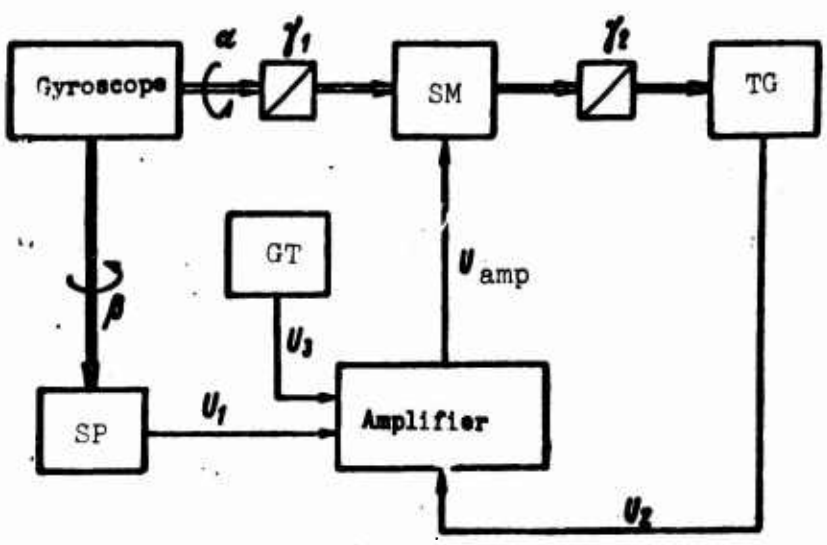

Fig. 7.6. Block diagram of a GS using "small" gyroscopes.

* In general it may also be necessary to consider transients in the torque pickoff TP (Fig. 7.2).

** Coefficients k_2 and k_3 include transmission ratios γ_1 and γ_2 (Fig. 7.6).

" can be formed, for instance, as

$$U = -(k_1\beta + k_2\dot{\beta} + k_3\ddot{\beta} + k_4\dot{\alpha} + k_5\ddot{\alpha}). \tag{7.2.57}$$

In such GS's, without changing parameters of the frame and stabilizing motor, by changing values of coefficients k_1, ... k_5 it is possible to change the nature of motion of the stabilizer. Applying such a law of control, it is possible to increase the speed of operation of the stabilizer, which is equivalent to introduction of a certain additional "fictitious" angular momentum; therefore, in the considered GS's they usually apply gyroscopes with smaller values of H, but the system of automation of these GS's is more complicated.

Considering (7.2.56), instead of (7.2.55) for the GS scheme shown in Fig. 7.6, using "small" gyroscopes, we have:

$$\left.\begin{aligned} &M_{c.A} = \Phi i \\ &T_y\dot{U}_y + U_y = -S_1\dot{\beta} - k_2'(\dot{\alpha} - \dot{\theta}) - k_3'\dot{\theta} \\ &T_A\frac{di}{dt} + i + b\dot{\alpha} = b\dot{\theta} + cU_y \end{aligned}\right\} \tag{7.2.58}$$

where

$$k_2' = kk_2; \quad k_3' = kk_3. \tag{7.2.59}$$

d) <u>System of Differential Equations of Motion of a GS</u>.

The equation of motion of the gyroframe should be supplemented by equations, characterizing transients in electrical components of the GS. Adding to equations (7.2.42) of the gyroframe equations (7.2.55), we obtain a system differential equations of motion for the GS:

$$\left.\begin{aligned} &I_\eta\dot{\alpha} + 2H\dot{\beta} + n_1\dot{\alpha} - Ki = \xi\dot{\theta}(t) + n_1\dot{\theta}(t) + f_1(t) \\ &2J_{r.s}\ddot{\beta} - 2H(\dot{\alpha} + u_\eta) - S_2\alpha + n_2\beta = -S_2\chi_1(t) - f_2(t) \\ &T_y\dot{U}_y + U_y + S_1\beta = 0 \\ &T_A\frac{di}{dt} + i + b\dot{\alpha} - cU_y = b\dot{\theta}(t) \end{aligned}\right\} \tag{7.2.60}$$

where

$$K = \gamma\Phi; \quad \xi = \gamma[\gamma + (-1)^n]J_a. \tag{7.2.61}$$

Analogously, using (7.2.42) and (7.2.58), it is possible to compose equations for a GS using "small" gyroscopes.

System of equations (7.2.60) is composed for a GS, on a ship, experiencing rolling. In the general case of oscillations of the object, $\dot{\theta}$ and $\ddot{\theta}$ in these equations should be replaced by the correspondingly angular velocity ω_η and angular acceleration $\dot{\omega}_\eta$ of the object about axis of stabilization $O\eta$ of the GS (Fig. 7.2).

During investigation of GS on a nonoscillatory base in equations (7.2.60) it is necessary to set $\dot{\theta} = \ddot{\theta} = 0$, and $\chi_1 = 0$, since the correcting pendulum of the GS preserves in these conditions its vertical direction.

Sometimes it is convenient to write the differential equations of motion of a GS in matrix form. Such equations were composed by B. V. Bulgakov [18]. Examples of application of these equations for investigation of GS's were given by Ya. N. Roytenberg [130].

3. Block Diagrams of a GS and Its Transfer Functions

a) Block Diagram of a GS.

In connection with the fact that GS is a rather complicated automatic control system, it is useful to compose its block diagram. To determine transients of the GS it is necessary to find transfer functions with respect to disturbances $f_1(t)$ and $f_2(t)$. Transfer functions can be found from equations of motion (7.2.60) of the GS or by construction of its block diagram and its subsequent transformation. Let us use the second of these methods. Let us note that construction of the block diagram of a GS is very useful also during solution of different problems of analysis and synthesis of active GS's.

As it is known, the block diagrams of automatic control system gives a graphic representation of dependences between Laplace transforms for external disturbances and for variables of the considered system [80a]. Therefore, for construction of the block diagram we use differential equations of the system, recorded in Laplace transforms. Methods of composition and transformation of block diagrams are presented in books on the theory of automatic control (for instance, [107, 113, 80a]).

For the initially considered case of work of a GS on a fixed and nonoscillatory base in equations (7.2.60) one should take $\dot{\theta} = 0$, $\ddot{\theta} = 0$, $\chi_1 = 0$, $u_\eta = 0$; furthermore, we shall consider that the GS does not have correction, i.e., $S_2 = 0$. Then equations (7.2.60) will have form:

$$\left.\begin{aligned} &I_x\ddot{\alpha} + 2H\dot{\beta} + n_1\dot{\alpha} - K\alpha = f_1(t) \\ &2J_{r.}\ddot{\beta} - 2H\dot{\alpha} + n_2\dot{\beta} = -f_2(t) \\ &T_y\dot{U}_y + U_y + S_1\beta = 0 \\ &T_a\frac{di}{dt} + i + b\dot{\alpha} - cU_y = 0 \end{aligned}\right\} \qquad (7.2.62)$$

During investigation of a GS by methods of the theory of automatic control we consider a stabilizer with control by the angle of precession β.

In accordance with what was said above we apply to equations (7.2.62) Laplace transform (1.7.2); then, considering initial conditions null, we obtain:

$$(I_1 s + n_1)\, s\, \alpha(s) + 2Hs\beta(s) - Ki(s) = f_1(s); \quad (T_y s + 1)\, U_y(s) + S_1 \beta(s) = 0;$$
$$(2J_{r.\,b} s + n_2)\, s\, \beta(s) - 2Hs\, \alpha(s) = -f_2(s); \quad (T_\Sigma s + 1)\, i(s) + bs\, \alpha(s) - eU_y(s) = 0. \qquad (7.2.63)$$

Each of these equations one should solve for its output variable:

$$\left.\begin{aligned} \alpha(s) &= \frac{1}{(I_1 s + n_1)s}[-2Hs\beta(s) + Ki(s) + f_1(s)] \\ \beta(s) &= \frac{1}{(2J_{r.\,b} s + n_2)s}[2Hs\,\alpha(s) - f_2(s)] \\ U_y(s) &= -\frac{S_1}{T_y s + 1}\beta(s) \\ i(s) &= \frac{1}{T_\Sigma s + 1}[-bs\,\alpha(s) + eU_y(s)] \end{aligned}\right\} \qquad (7.2.64)$$

Then, each equation can be depicted in the form of some block diagram; the set of them represents the block diagram of the GS, considered as a closed-loop automatic control system. A block diagram of a GS composed thus is shown in Fig. 7.7. Inside each block of the diagram is the expression of its transfer function. Variables (α, β, i, ...) at inputs and outputs of blocks and external disturbances (f_1, f_2) are given in their Laplace transforms, which, for simplicity of recording, are designated by the same letters, as the originals.

The diagram, shown in Fig. 7.7, can be presented somewhat differently, if the closed loop of the biggest number of blocks (1-2-3-4-5-6-7-8) is taken as the main one. Then we obtain a block diagram of the GS (Fig. 7.8), analogous to that given in the book of N. T. Kuzovkov [80a]. The diagram is three-loop; besides the main loop, it contains feedback — 2Hs and direct connection — bs.

We shall show how the structure of the circuit of the GS will change with pendular correction in the case of a motionless base, as before, and an undisturbed state of the correcting pendulum. In accordance with (7.2.60) in (7.2.64) only the second equation will change, taking form

$$\beta(s) = \frac{1}{(2J_{r.\,b} s + n_2)s}[(2Hs + S_2)\alpha(s) - f_2(s)]. \qquad (7.2.65)$$

Consequently, allowing for correction in the block diagram of the GS (Figs. 7.7 and 7.8) only the transfer function of block 2 will change; instead of 2Hs it will have the form ($2Hs + S_2$).

We compose the block diagram of a GS for conditions of a mobile mounting considering disturbances due to vibrations of the object, e.g., a ship. In this case we directly apply equations (7.2.60), which it is necessary in general to supplement by the differential equation of motion of the correcting pendulum. When the axis of stabilization of the frame is directed parallel to the longitudinal axis of the ship

it is possible to use the equation of a pendular inclinometer. Considering for simplicity only components of the first order of smallness, by analogy with (4.6.49), we write the equation of motion of the correcting pendulum of the GS in the form

$$T_{\text{м}}^2\ddot{\chi}_1+2\zeta T_{\text{м}}\dot{\chi}_1+\chi_1=kf_3(t). \tag{7.2.66}$$

where T_{pend} – time constant of the pendulum;

ζ – relative damping factor;

k – 1/g [see (4.3.24)];

$f_e(t)$ – disturbance to the pendulum.

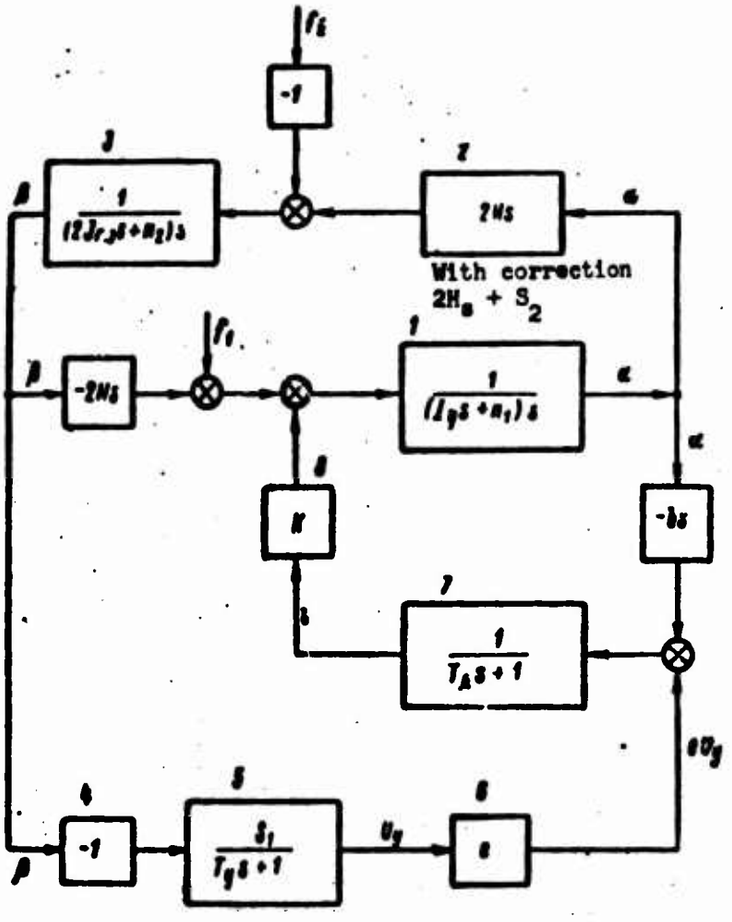

Fig. 7.7. Block diagram of a GS (case of a fixed base).

In the considered case $f_3(t)$ is determined by approximate formula (4.6.41); limiting ourselves to calculation only of rolling of the ship, we have

$$f_3(t)=-z\ddot{\vartheta}(t). \tag{7.2.67}$$

Then instead of (7.2.66) we have

$$T_{\text{м}}^2\ddot{\chi}_1+2\zeta T_{\text{м}}\dot{\chi}_1+\chi_1=-kz\ddot{\vartheta}(t). \tag{7.2.68}$$

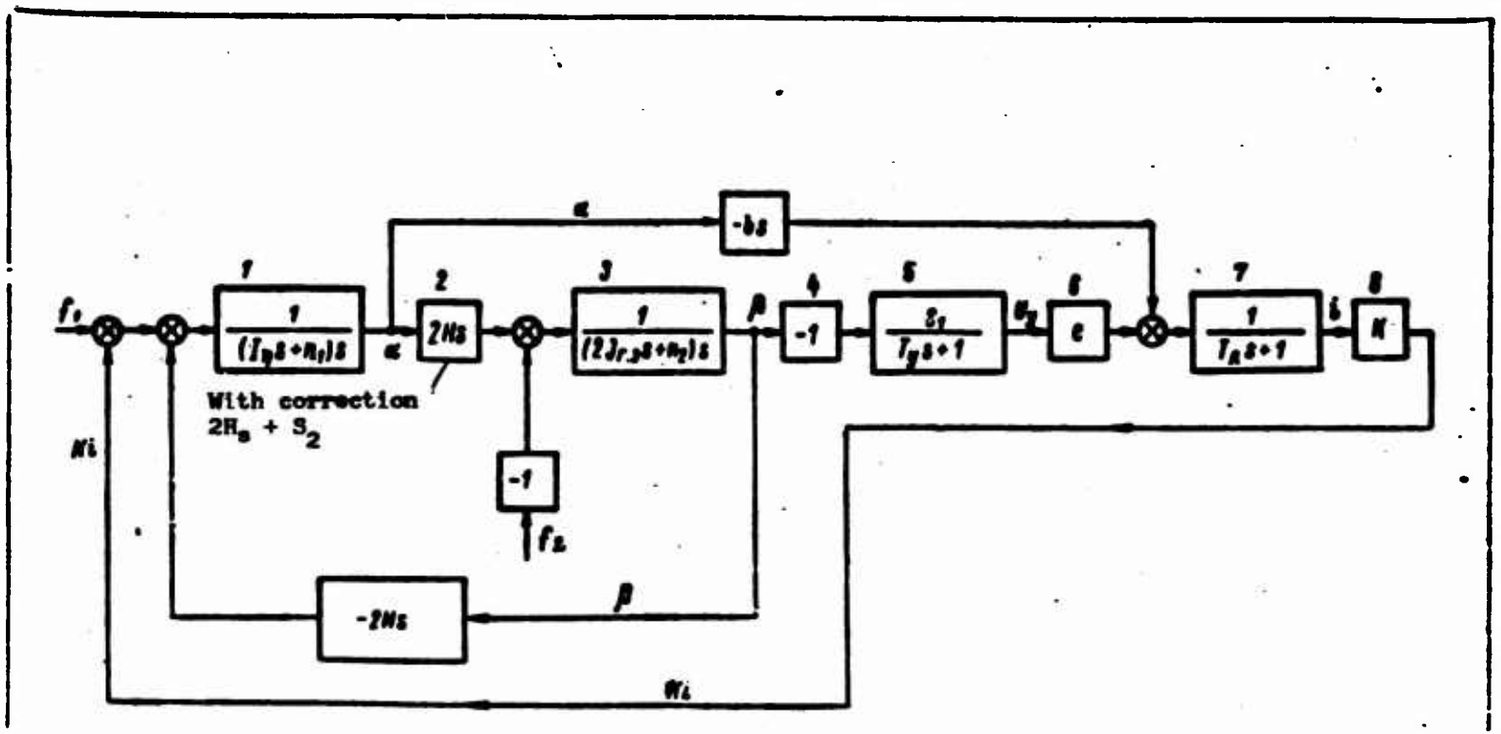

Fig. 7.8. Modified block diagram of the GS (case of a fixed base).

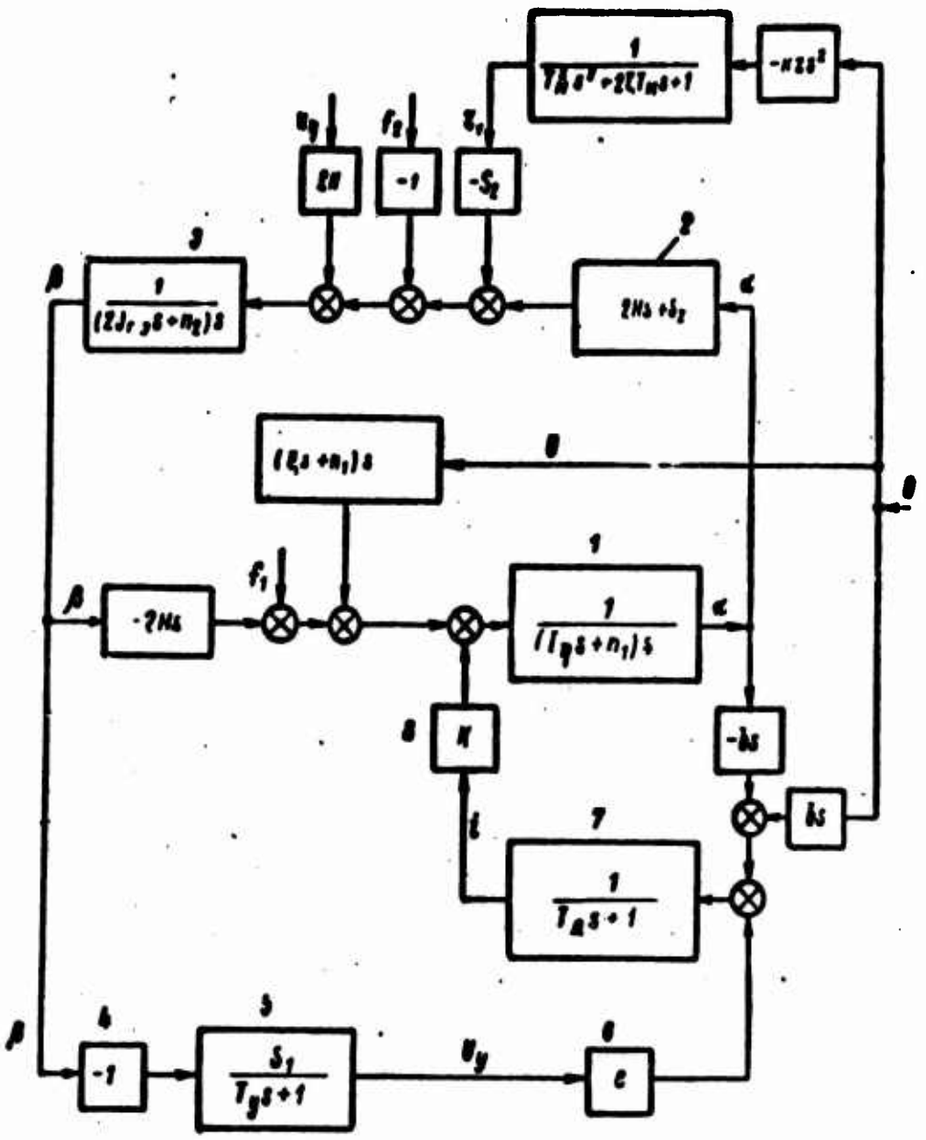

Fig. 7.9. Block diagram of a GS (case of mobile and oscillating base).

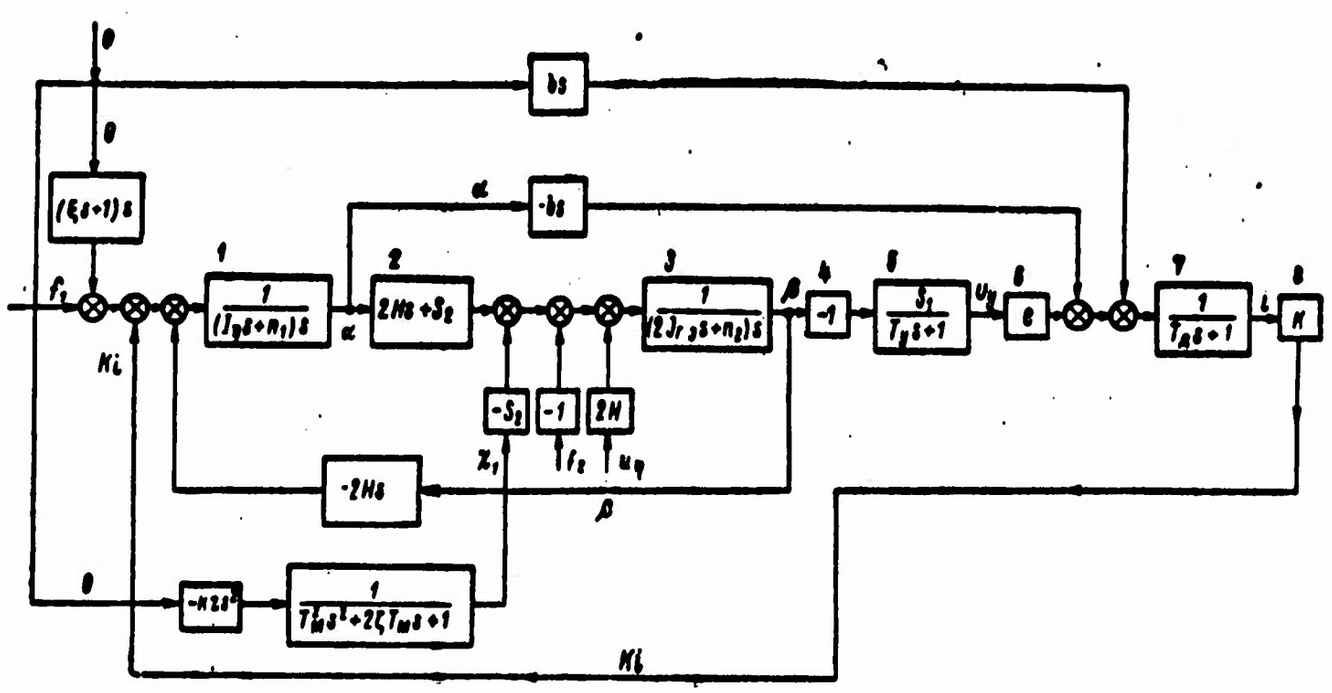

Fig. 7.10. Modified block diagram of a GS (case of mobile and oscillating base).

Introducing (7.2.68) in (7.2.60), we have a complete system of equations of motion of a GS on a rolling ship:

$$\left.\begin{aligned}
&I_{\text{ц}}\ddot{\alpha}+2H\beta+n_1\dot{\alpha}-Ki=\xi\ddot{\theta}(t)+n_1\dot{\theta}(t)+f_1(t)\\
&2J_{\text{р.э}}\ddot{\beta}-2H\left(\dot{\alpha}+u_{\text{ц}}\right)-S_2\alpha+n_2\dot{\beta}=-S_2\chi_1(t)-f_2(t)\\
&T_{\text{ш}}^2\ddot{\chi}_1+2\chi T_{\text{ш}}\dot{\chi}_1+\chi_1=-kz\ddot{\theta}(t)\\
&T_y\dot{U}_y+U_y+S_3\beta=0\\
&T_{\text{я}}\frac{di}{dt}+i+b\dot{\alpha}-eU_y=b\dot{\theta}(t)
\end{aligned}\right\}\qquad(7.2.69)$$

For composition of the block diagram the GS we write equation (7.2.69) in Laplace transforms with zero initial conditions:

$$\left.\begin{aligned}
&\left(I_{\text{ц}}s+n_1\right)s\alpha(s)+2Hs\beta(s)-Ki(s)=\\
&=\left(\xi s+n_1\right)s\theta(s)+f_1(s)\\
&\left(2J_{\text{р.э}}s+n_2\right)s\beta(s)-\left(2Hs+S_2\right)\alpha(s)=2Hu_{\text{ц}}(s)-\\
&-S_2\chi_1(s)-f_2(s)\\
&\left(T_{\text{ш}}^2s^2+2\chi T_{\text{ш}}s+1\right)\chi_1(s)=-kzs^2\theta(s)\\
&\left(T_ys+1\right)U_y(s)+S_3\beta(s)=0\\
&\left(T_{\text{я}}s+1\right)i(s)+bs\alpha(s)-eU_y(s)=bs\theta(s)
\end{aligned}\right\}\qquad(7.2.70)$$

Solving each of these equations for its output variable, we obtain

$$\left.\begin{aligned}
\alpha(s)&=\frac{1}{\left(I_{\text{ц}}s+n_1\right)s}\left[-2Hs\beta(s)+Ki(s)+\right.\\
&\left.+\left(\xi s+n_1\right)s\theta(s)+f_1(s)\right]\\
\beta(s)&=\frac{1}{\left(2J_{\text{р.э}}s+n_2\right)s}\left[\left(2Hs+S_2\right)\alpha(s)+2Hu_{\text{ц}}(s)-\right.\\
&\left.-S_2\chi_1(s)-f_2(s)\right]\\
\chi_1(s)&=-\frac{kz}{T_{\text{ш}}^2s^2+2\chi T_{\text{ш}}s+1}s^2\theta(s)\\
U_y(s)&=-\frac{S_3}{T_ys+1}\beta(s)\\
i(s)&=\frac{1}{T_{\text{я}}s+1}\left[-bs\alpha(s)+eU_y(s)+bs\theta(s)\right]
\end{aligned}\right\}\qquad(7.2.71)$$

From these equations we construct the block diagram of a GS for the case of a mobile and oscillating base (Fig. 7.9). A modified diagram, analogous to that shown in Fig. 7.8, is shown in Fig. 7.10.

b) Transfer Functions of a GS.

For determination of transfer functions of a GS we convert its block diagram by rules known from the theory of automatic control (see, for instance, [107, 113, 80a]). As an example we consider a GS on a motionless and nonoscillating base, a

block diagram of which is shown in Fig. 7.8; here we shall also consider correction of the GS. During transformation of the GS block diagram one should keep in mind that we are mainly interested in change of angle α of GS frame rotation about axis O_η (Fig. 7.2), constituting the error of stabilization of the frame. Therefore, we shall consider variable $\alpha(t)$ the output magnitude and disturbances $f_1(t)$ and $f_2(t)$ the input values and determine transfer functions of the GS by coordinate α with respect to disturbances $f_1(t)$ and $f_2(t)$. Transformation of the GS block diagram, analogous that given below, is carried out in [80a].

In order to determine transfer function $\Phi_{\alpha,f_1}(s) = \Phi_{11}(s)$ of a closed-loop gyrostabilizer system, we transform the part of its block diagram, connected with introduction of disturbance $f_1(t)$, considering $f_2(t) = 0$. Then, using known rules, it is possible to transform the diagram, shown in Fig. 7.8, when $f_2(t) = 0$, to the form presented in Fig. 7.11. In the scheme block 1 is covered by three negative feedbacks (blocks 2, 3 and 4). It is possible to further transform this scheme, replacing 1 and 2 by one block I with transfer function $W_I(s)$ and 3 and 4 also by one block II with transfer function $W_{II}(s)$ as a result we obtain a scheme, which is a closed-loop automatic control system (Fig. 7.12).* Transfer functions $W_I(s)$ and $W_{II}(s)$ are found by known rulers:**

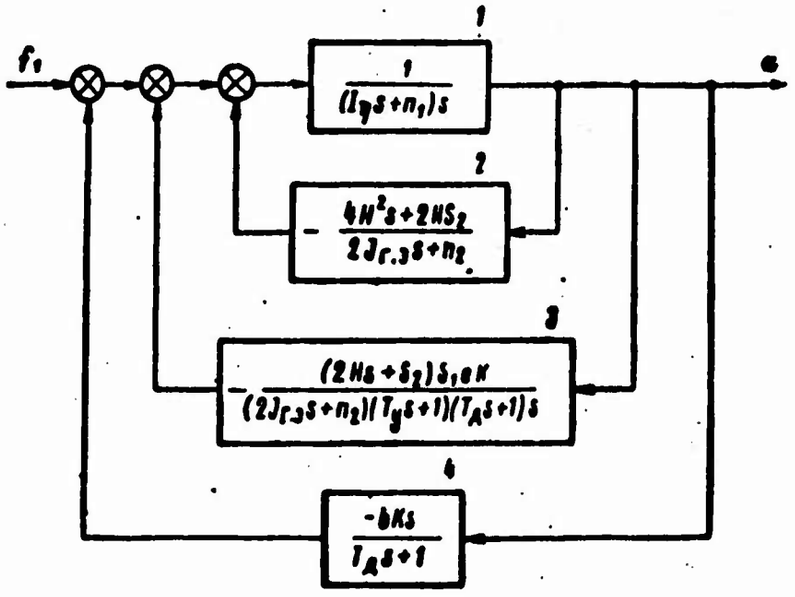

Fig. 7.11. Transformed GS block diagram (for $f_2 = 0$).

$$W_I(s) = \frac{W_1(s)}{1 + W_1(s)\,W_2(s)} = \frac{\dfrac{1}{(I_q s + n_1)s}}{1 + \dfrac{1}{(I_q s + n_1)s} \cdot \dfrac{4H^2 s + 2HS_2}{2J_{r.3} s + n_2}} =$$

$$= \frac{2J_{r.3} s + n_2}{2J_{r.3} I_q s^3 + (I_q n_2 + 2J_{r.3} n_1)s^2 + (n_1 n_2 + 4H^2)s + 2HS_2}; \qquad (7.2.72)$$

$$W_{II}(s) = W_3(s) + W_4(s) = -\left[\frac{(2Hs + S_2)S_1 eK}{(2J_{r.3} s + n_2)(T_y s + 1)(T_A s + 1)s} + \frac{bKs}{T_A s + 1}\right]. \qquad (7.2.73)$$

*Let us note that block I characterizes the gyroframe, and block II characterizing control circuits of the GS.

**The minus sign in (7.2.73) signifies negative feedback in a closed-loop system (see Figs. 7.11 and 7.12).

In the diagram of (Fig. 7.12) we have basic block I, covered by negative feedback in the form of block II; we find transfer function $\Phi_{11}(s)$ by a formula, analogous to (7.2.72):

$$\Phi_{11}(s) = \frac{W_1(s)}{1 + W_1(s)W_{11}(s)}, \tag{7.2.74}$$

or, considering (7.2.72) and (7.2.73), when $T_{mot} = T_{amp} = 0$* we finally will obtain

$$\Phi_{11}(s) = \frac{2J_{r.s}s^2 + n_2 s}{2J_{r.s}I_\Sigma s^4 + [I_\Sigma n_2 + 2J_{r.s}(n_1 + bK)]s^3 + [n_2(n_1 + bK) + 4H^2]s^2 + \overline{+ 2H(S_1 cK + S_2)s + S_1 S_2 cK}}. \tag{7.2.75}$$

During investigation of a GS we may also need the expression for the transfer function of an open-loop GS system (Fig. 7.12) $W_{\alpha, f_1}(s) = W_{II}(s)$. According to the diagram in Fig. 7.12, we have (when $T_{mot} = T_{amp} = 0$):**

$$W_{11}(s) = W_1(s)W_{11}(s) = \frac{2J_{r.s}bKs^2 + n_2 bKs^2 + 2HS_1 cKs + S_1 S_2 cK}{2J_{r.s}I_\Sigma s^4 + (I_\Sigma n_2 + 2J_{r.s}n_1)s^3 + (n_1 n_2 + 4H^2)s^2 + 2HS_2 s}. \tag{7.2.76}$$

Let us turn to determination of transfer function $\Phi_{\alpha, f_2}(s) = \Phi_{12}(s)$ of a closed-loop GS system. For this when $f_1(t) = 0$, considering the diagram in Fig. 7.12, it is possible to present the block diagram of Fig. 7.8 in the form, depicted in Fig. 7.13. The latter diagram can be easily transformed by replacing blocks I and II by one block, whose transfer function as one may see from the preceding, is equal to $\Phi_{11}(s)$. We obtain the diagram, given in Fig. 7.14, which with respect to influence $f_2(t)$ is closed. Then, for transfer function $\Phi_{12}(s)$

$$\Phi_{12}(s) = W_{III}(s)\Phi_{11}(s). \tag{7.2.77}$$

Substituting here for $W_{III}(s)$ the expression of the transfer function of block III (when $T_{mot} = T_{amp} = 0$),*** shown in the diagram of Fig. 7.13, and formula (7.2.75), we obtain:

$$\Phi_{12}(s) = \frac{2Hs + S_2 cK}{2J_{r.s}I_\Sigma s^4 + [I_\Sigma n_2 + 2J_{r.s}(n_1 + bK)]s^3 + [n_2(n_1 + bK) + 4H^2]s^2 + \overline{+ 2H(S_2 cK + S_2)s + S_1 S_2 cK}}. \tag{7.2.78}$$

When necessary we can find the expression for the corresponding transfer function

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*Determination of transfer functions of a GS, taking into account time constants T_{mot} and T_{amp} is given below.

**The minus sign in (7.2.76) is omitted.

***See footnote* on p. 31.

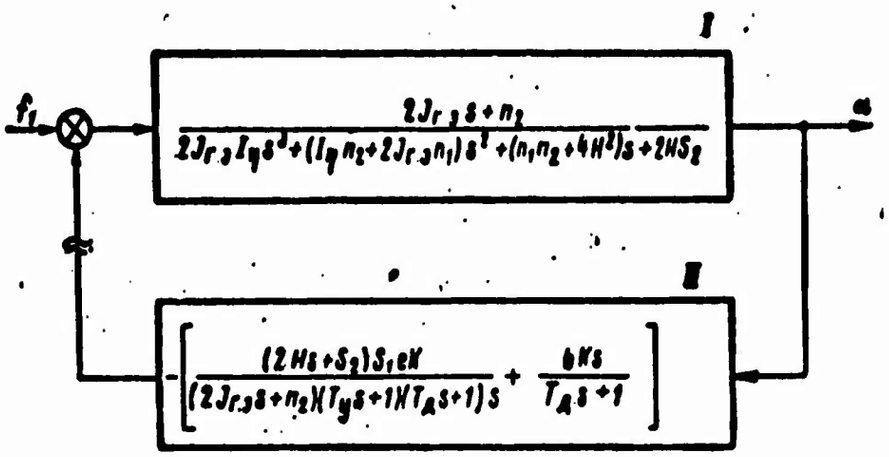

Fig. 7.12. Final form of transformed block diagram of the GS (when $f_2 = 0$).

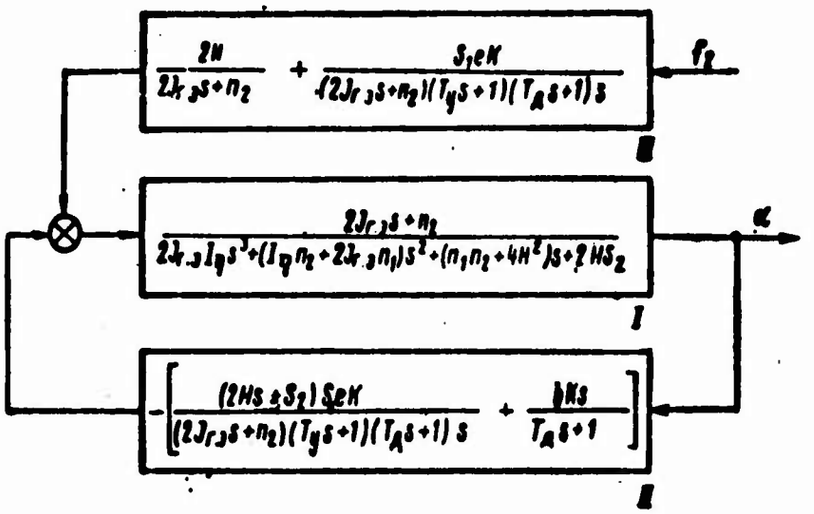

Fig. 7.13. Transformed block diagram of a GS (when $f_1 = 0$).

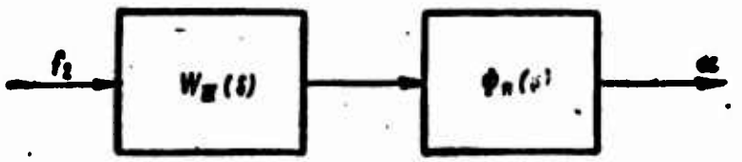

Fig. 7.14. Final form of transformed block diagram of a GS (when $f_1 = 0$).

$W_{12}(s)$ of an open-loop GS system by the same method, as the relationship for $W_{11}(s)$ [see (7.2.76)].

Thus, by transforming the block diagram of the GS there is the possibility of very simply finding expressions of its transfer functions. Furthermore, this method is more graphic than the purely analytic one.

In arriving at expressions for transfer functions time constants T_{mot} and T_{amp} of control networks of the GS were assumed equal to zero. In most cases such an approach, due to smallness of T_{mot} and T_{amp} is fully permissible during investigation of transients of a GS and calculation of its errors. However, during the analysis of certain questions of stability of a GS we may need expressions for transfer functions taking into account the indicated time constants. We obtain the expression for $\Phi_{11}(s)$, if in formula (7.2.74) we directly substitute relationships (7.2.72) and (7.2.73), considering for simplicity that there is no compensation ($S_2 = 0$) (which has no significance, for instance, during investigation of stability of a GS).

Then for $\Phi_{11}(s)$ we have

$$\Phi_{11}(s) = \frac{b_4 s^4 + b_3 s^3 + b_2 s^2 + b_1 s + b_0}{a_6 s^6 + a_5 s^5 + a_4 s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0}, \qquad (7.2.79)$$

where

$$\left.\begin{aligned}
a_0 &= 2HS_1eK \\
a_1 &= n_2(n_1+bK)+4H^2 \\
a_2 &= I_1n_2+2J_{r.s}(n_1+bK)+(n_1n_2+4H^2)(T_A+ \\
&\quad +T_y)+n_2bKT_y \\
a_3 &= 2J_{r.s}I_1+(I_1n_2+2J_{r.s}n_1)(T_A+T_y)+ \\
&\quad +(n_1n_2+4H^2)T_AT_y+2J_{r.s}bKT_y \\
a_4 &= 2J_{r.s}I_1(T_A+T_y)+(I_1n_2+2J_{r.s}n_1)T_AT_y \\
a_5 &= 2J_{r.s}I_1T_AT_y \\
b_0 &= n_2 \\
b_1 &= 2J_{r.s}+n_2(T_A+T_y) \\
b_2 &= 2J_{r.s}(T_A+T_y)+n_2T_AT_y \\
b_3 &= 2J_{r.s}T_AT_y
\end{aligned}\right\} \qquad (7.2.80)$$

If it is necessary to find the expression for transfer function $W_{11}(s)$ of an open-loop GS system taking into account time constants T_{mot} and T_{amp}, then in (7.2.76) instead of $W_I(s)$ and $W_{II}(s)$ one should directly substitute relationships (7.2.72) and (7.2.73); then for $S_2 = 0$ we obtain formula

$$W_{11}(s)=\frac{2J_{r.s}s+n_2}{2J_{r.s}I_1s^3+(I_1n_2+2J_{r.s}n_1)s^2+(n_1n_2+4H^2)s}\times \qquad (7.2.81)$$
$$\times\left[\frac{2HS_1eKs}{(2J_{r.s}s+n_2)(T_ys+1)(T_As+1)s}+\frac{bKs}{T_As+1}\right],$$

which should be simplified for concrete use.

To determine the transfer function $\Phi_{12}(s)$ taking into account time constants T_{mot} and T_{amp} it is possible to use formula (7.2.77), substituting instead of $W_{III}(s)$ the transfer function of the corresponding block (Fig. 7.13), and for $\Phi_{11}(s)$ expression (7.2.79).

Then for $\Phi_{12}(s)$ we obtain

$$\Phi_{12}(s)=\frac{b_3's^3+b_2's^2+b_1's+b_0'}{(a_5s^5+a_4s^4+a_3s^3+a_2s^2+a_1s+a_0)s}, \qquad (7.2.82)$$

where a_0, a_1, ..., a_5 are determined by relationships (7.2.80), and for b_0', ..., b_3' we have

$$b_0'=S_1eK;\quad b_1'=2H;\quad b_2'=2H(T_A+T_y);\quad b_3'=2HT_AT_y. \qquad (7.2.83)$$

During investigation of forced oscillations of a GS, caused by its error in conditions of rolling of the object, for instance a ship, it is necessary to know transfer function of GS in coordinate α with respect to rolling disturbances. Thus, for the considered case of installing a GS on a ship, which experiences rolling, according to (7.2.69) the shown disturbances are connected with angular velocities and accelerations of the rolling $\theta(t)$. Therefore, for calculation of dynamic errors

of the GS in these conditions it is necessary to know transfer function $\Phi_{\alpha,\theta}(s) = \frac{\alpha(s)}{\theta(s)}$ of a closed-loop active GS system.

It is possible to determine $\Phi_{\alpha,\theta}(s)$ by transforming the GS block diagram, shown in Fig. 7.10. This method was stated in detail above; therefore for finding $\Phi_{\alpha,\theta}(s)$ we start directly from equations (7.2.69), considering time constants T_{mot} and T_{amp} so small that their influence on error of the GS can be disregarded; time constant T_{pend} of the correcting pendulum we also consider small (example 4.2). Then, disregarding disturbances ($f_1 = f_2 = 0$) and not considering angular velocity of the reference system of the GS ($u_\eta = 0$), we rewrite system of equations (7.2.69) in the form

$$\left.\begin{aligned} I_y\ddot{\alpha} + 2H\dot{\beta} + n_1\dot{\alpha} - Kl &= \xi\ddot{\theta}(t) + n_1\dot{\theta}(t) \\ 2J_{r.s}\dot{\beta} - 2H\dot{\alpha} - S_2\alpha + n_2\beta &= -S_2\chi_1(t) \\ \chi_1 &= -kz\ddot{\theta}(t) \\ U_y + S_3\beta &= 0 \\ l + b\dot{\alpha} - dU_y &= b\dot{\theta}(t) \end{aligned}\right\} \qquad (7.2.84)$$

After very simple transformations we rewrite equations (7.2.84) in operator form $\left(p = \frac{d}{dt}\right)$:

$$\left.\begin{aligned} (2J_{r.s}p^2 + n_2p)\beta - (2Hp + S_2)\alpha &= S_2kzp^2\theta \\ (2Hp + S_1eK)\beta + [I_yp^2 + (n_1 + bK)p]\alpha &= \\ = [\xi p^2 + (n_1 + bK)p]\theta \end{aligned}\right\} \qquad (7.2.85)$$

To eliminate variable β we find the principal determinant Δ of system (7.2.85) and determinant Δ_α, obtained from Δ by means of replacing the corresponding column by the column of right sides of equations (7.2.85).

We have

$$\begin{aligned} \Delta = \begin{vmatrix} 2J_{r.s}p^2 + n_2p & -(2Hp + S_2) \\ 2Hp + S_1eK & I_yp^2 + (n_1 + bK)p \end{vmatrix} &= 2J_{r.s}I_yp^4 + [I_yn_2 + \\ + 2J_{r.s}(n_1 + bK)]p^3 + [n_2(n_1 + bK) + 4H^2]p^2 &+ \\ + (2HS_1eK + 2HS_2)p + S_1S_2eK. \end{aligned} \qquad (7.2.86)$$

$$\begin{aligned} \Delta_\alpha = \begin{vmatrix} 2J_{r.s}p^2 + n_2p & S_2kzp^2\theta \\ 2Hp + S_1eK & [\xi p^2 + (n_1 + bK)p]\theta \end{vmatrix} &= [2J_{r.s}\xi p^4 + \\ + [2J_{r.s}(n_1 + bK) + n_2\xi - 2HS_2kz]p^3 &+ \\ + [(n_1 + bK)n_2 - S_1S_2eKkz]p^2]\theta. \end{aligned} \qquad (7.2.87)$$

Since, on the basis of Cramer's rule,*

$$\Delta \cdot \alpha = \Delta_\alpha,$$

considering (7.2.86) and (7.2.87), we obtain the following differential equation for

*V. I. Smirnov. Course in higher mathematics, Vol. III, Part 1, State Press for Technical and Theoretical Literature, 1949.

α in operator form:

$$[2J_{r.s}I_y p^4 + [I_y n_2 + 2J_{r.s}(n_1 + bK)]\, p^3 + [n_2(n_1 + bK) + 4H^2]\, p^2 + (2HS_1 eK + 2HS_2)\, p + S_1 S_2 eK]\, z = [2J_{r.s}\xi p^4 + [2J_{r.s}(n_1 + bK) + n_2\xi - 2HS_2 kz]\, p^3 + [n_2(n_1 + bK) - S_1 S_2 eKkz]\, p^2 \quad (7.2.88)$$

Applying to (7.2.88) the Laplace transform, we obtain an expression of the transfer function of the GS, $\Phi_{\alpha,\theta}(s)$, in the form

$$\Phi_{\alpha,\theta}(s) = \frac{2J_{r.s}\xi s^4 + [2J_{r.s}(n_1 + bK) + n_2\xi - 2HS_2 kz]\, s^3 + [n_2(n_1 + bK) - S_1 S_2 eKkz]\, s^2}{2J_{r.s}I_y s^4 + [I_y n_2 + 2J_{r.s}(n_1 + bK)]\, s^3 + [n_2(n_1 + bK) + 4H^2]\, s^2 + (2HS_1 eK + 2HS_2)\, s + S_1 S_2 eK} \quad (7.2.89)$$

The considered transfer function can be obtained by another method. In 1950 B. V. Bulgakov [21] developed a method of investigating automatic control systems with many degrees of freedom, based on use of matrix calculation. This method was applied [21, 130] to analysis of a power GS, particularly for composing its transfer function; here the closed loop of the stabilization system is presented in the form of two blocks (Fig. 7.15): block I consists of the GS frame with its object, the gyroscopes and stabilizing motor; block II is the amplifier. Those who wish to become acquainted with this method are referred to the above mentioned works.

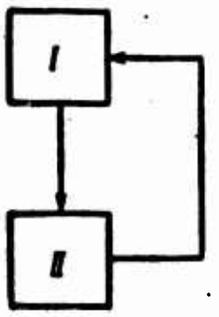

Fig. 7.15. Block diagram of a GS (from B. V. Bulgakov).

§ 7.3. Transient Responses of a GS on a Fixed Base

1. Preliminary Remarks

Determination of transient responses of a GS on a fixed base is a very important question of the theory of active GS's. Here, of interest is analysis of free natural oscillations of a GS, which differ from corresponding motions of the GD, considered in Part I. Most essential is investigation of the stability of the GS, allowing one to establish necessary relationships between basic parameters of a stabilizer. Performance of an active GS is found by analysis of its behavior in the transients, caused by a unit step input. Here, it is necessary to consider, e.g., dry friction torques in axes of the GS suspension, and in practice a nonlinear law of control of the stabilizing motor is also possible, as a consequence of which in the system can be established an independent oscillating mode.

2. Natural Oscillations of a GS

a) Characteristic Equation of a GS.

The starting point of analysis of a GS is determination of its natural oscillations. First, we consider motion of the GS without friction in the axes of the suspension and motion of the GS with liquid friction in the case of a linear law of control of the stabilizing motor. Investigation of motion of a GS allowing for dry friction in the axes of suspension, and also in the case of a nonlinear law of control of the stabilizing motor is given in Par. 4.

For analysis of natural oscillations of a GS it is necessary to have the characteristic equation of the considered dynamic system. The latter is easy to obtain, if we know the expression of the transfer function of a closed-loop GS system. Actually, the expression for the transfer function $\Phi_{11}(s)$ of a closed-loop GS system in coordinate α with respect to disturbance $f_1(t)$ can be written in the form

$$\Phi_{11}(s) = \frac{\alpha(s)}{f_1(s)}, \tag{7.3.1}$$

where $\alpha(s)$ and $f_1(s)$ are Laplace transforms of functions $\alpha(t)$ and $f_1(t)$, respectively.

Presenting $\Phi_{11}(s)$ in the form

$$\Phi_{11}(s) = \frac{M_{11}(s)}{N_{11}(s)}, \tag{7.3.2}$$

where $M_{11}(s)$ and $N_{11}(s)$ are polynomials from s, we obtain the characteristic equation of the closed-loop GS system from equality

$$N_{11}(s) = 0, \tag{7.3.3}$$

which, according to (7.2.74) and (7.2.76), it is also possible to write as follows:

$$1 + W_{11}(s) = 0, \tag{7.3.4}$$

where $W_{11}(s)$ is the transfer function of an open-loop GS system.

Replacings in (7.3.3) by λ, we obtain the characteristic equation of a closed-loop GS system in the form

$$N_{11}(\lambda) = 0. \tag{7.3.5}$$

Consequently, to find the characteristic equation of a GS it is necessary to equate the denominator of transfer function $\Phi_{11}(s)$ of a closed-loop GS system to zero and to replace* s by λ in it.

We shall find the same expression for the characteristic equation of a GS if we use transfer function $\Phi_{12}(s) = \frac{\alpha(s)}{f_2(s)}$, since, for instance, according to (7.2.79) and (7.2.82), from conditions $N_{11}(s) = 0$ and $N_{12}(s) = 0$ we obtain identical characteristic equations.

In accordance with what was presented in § 7.2, Par. 3b, it is possible to compose characteristic equation of a GS for different cases, i.e., taking into account or ignoring time constants of the amplifier and stabilizing motor, etc.

Let us consider, first, the case, when in composing the transfer function of a GS we consider the above-indicated time constants of its control networks; we assume there is no GS correction. Then, according to (7.2.79) and (7.3.5), and also taking into account the fact that during composition of expression (7.2.79) for transfer function $\Phi_{11}(s)$ we removed s from its numerator and denominator, we obtain the characteristic equation of a closed-loop GS system

$$(a_5\lambda^5 + a_4\lambda^4 + a_3\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0)\lambda = 0, \tag{7.3.6}$$

from which it follows that one root of this equation $\lambda_1 = 0$. This means that if the GS, as assumed in the considered case, does not have correction, the position of equilibrium of the stabilizer frame is arbitrary, i.e., the frame maintains any assigned initial value of angle α_0. Therefore, for analysis of free oscillations of a GS near the position of equilibrium it is possible to use characteristic equation

$$a_5\lambda^5 + a_4\lambda^4 + a_3\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 = 0. \tag{7.3.7}$$

Let us find from a concrete example numerical values of coefficients a_0, a_1, ..., a_5.

---

*Replacement of s by λ does not have fundamental significance and usually is carried out [27] in order to not confuse variable of the Laplace transform s with the argument λ of the characteristic equation.

Example 7.1: Determine the values of coefficients a_0, a_1, ..., a_5 of the characteristic equation (7.3.7) of a GS, having the following parameters: angular momentum of gyroscope $H = 22 \cdot 10^4$ g-cm-sec [4]; equatorial moment of inertia of the gyroscope (rotor + gyrohousing) $J_{g.e} = 60$ g-cm-sec^2; moment of inertia of the frame with about axis of stabilization $J_{f\eta} = 3 \cdot 10^4$ g-cm-sec^2; as the stabilizing motor there is used a d-c motor with independent excitation SL-367 [4], for which rated voltage $U_{rat} = 110$ v, net power is 32 watts, armature rotation rate $w'_{arm} = 2500$ rpm, current in armature network $i = 0.9$ amps, torque $M_{s.m} = 1250$ g-cm, resistance of armature $r_{arm} = 1.5$ ohms, moment of inertia of armature $J_{arm} = 0.7$ g-cm-sec^2, electromagnetic time constant of the armature network of the motor $T_{mot} = 0.003$ sec, transmission ratio from the axis of the motor to the stabilization axis of the frame $\gamma = 600$; gain factor of the signal pickoff of the angle of precession $k_1 = 0.1$ v/deg; amplifier gain $k = 10$; time constant of amplifier $T_{amp} = 0.02$ sec [4].

When determining coefficients a_0, a_1, ..., a_5 one can ignore the characteristic equation (7.3.7) of friction on the axes of suspension of the GS, i.e., to take $n_1 = n_2 = 0$.

Solution: 1. According to (7.2.80), when $n_1 = n_2 = 0$, for determination of coefficients a_0, a_1, ..., a_5 we have the following relationships:

$$\left.\begin{aligned} a_0 &= 2HS_1\gamma K; & a_3 &= 2J_{r.s}J_\eta + 4H^2T_AT_y + 2J_{r.s}bKT_y \\ a_1 &= 4H^2; & a_4 &= 2J_{r.s}J_\eta(T_A + T_y) \\ a_2 &= 2J_{r.s}bK + 4H^2(T_A + T_y); & a_5 &= 2J_{r.s}J_\eta T_AT_y \end{aligned}\right\}$$

2. Angular momentum of the GS, equal to angular momentum of the two gyroscopes, will be

$$2H = 2 \cdot 22 \cdot 10^4 = 44 \cdot 10^4 \text{ g-cm-sec.}$$

3. From (7.2.43) we find the moment of inertia of the GS about the axis of stabilization of the frame:

$$J_\eta = J_{p\eta} + 2J_{r.s} + \gamma^2 J_g = 3 \cdot 10^4 + 2 \cdot 60 + 600^2 \cdot 0{,}7 = 2821 \cdot 10^3 \text{ g-cm-sec}^2.$$

4. In accordance with (7.2.53) we determine coefficient S_1:

$$S_1 = k_1 k = 0{,}1 \cdot 57{,}3 \cdot 10 = 57{,}3 \text{ v/radian.}$$

5. By (7.2.44) we calculate magnetic flux Φ of the excitation windings of the stabilizing motor:

$$\Phi = \frac{M_{e.s}}{i} = \frac{1250}{0{,}9} = 1390 \text{ g-cm/amp.}$$

6. For coefficient K, considering (7.2.61), we have:

$$K = \gamma\Phi = 600 \cdot 1390 = 8333 \cdot 10^3 \text{ g-cm/amp.}$$

7. We find coefficient e by formula (7.2.49), understanding in it by r the resistance of the armature network r_{arm}, including internal resistance of the amplifier r_{amp} = 3 ohms:

$$e = \frac{1}{r} = \frac{1}{r_a + r_y} = \frac{1}{1{,}5 + 3} = 0{,}2222 \text{ 1/ohms.}$$

8. We determine the coefficient c of counter emf, included in equation (7.2.45); for quantity c this relationship is obvious:

$$c = \frac{U_a - I_a r_a}{\omega_a'} = \frac{110 - 0{,}5 \cdot 1{,}5}{\frac{6{,}28 \cdot 2500}{60}} = 0{,}4152 \text{ v/sec.}$$

9. By (7.2.49) we find parameter b:

$$b = \frac{\gamma c}{r} = \frac{\gamma c}{r_a + r_y} = \frac{600 \cdot 0{,}4152}{4{,}5} = 55{,}36 \text{ amp-sec.}$$

10. By the formulas given in Par. 1 of the present example we calculate values of coefficients a_0, a_1, ..., a_5:

$$a_0 = 2HS_1eK = 44 \cdot 10^6 \cdot 57{,}3 \cdot 0{,}2222 \cdot 8333 \cdot 10^3 = 4668 \cdot 10^9 \text{ g}^2\text{cm}^2\text{sec};$$

$$a_1 = 4H^2 = (44 \cdot 10^6)^2 = 1936 \cdot 10^9 \text{ g}^2\text{cm}^2\text{sec}^2;$$

$$a_2 = 2J_{r.s}bK + 4H^2(T_A + T_y) = 2 \cdot 60 \cdot 55{,}36 \cdot 8333 \cdot 10^3 + 1936 \cdot 10^6 (0{,}003 + 0{,}02) = 9969 \cdot 10^6 \text{ g}^2\text{cm}^2\text{sec}^3;$$

$$a_3 = 2J_{r.s}I_q + 4H^2T_AT_y + 2J_{r.s}bKT_y = 2 \cdot 60 \cdot 2821 \cdot 10^3 + 1936 \cdot 10^6 \cdot 0{,}003 \cdot 0{,}02 + 2 \cdot 60 \cdot 55{,}36 \cdot 8333 \cdot 10^3 \cdot 0{,}02 = 1562 \cdot 10^9 \text{ g}^2\text{cm}^2\text{sec}^4;$$

$$a_4 = 2J_{r.s}I_q(T_A + T_y) = 2 \cdot 60 \cdot 2821 \cdot 10^3 (0{,}003 + 0{,}02) = 7786 \cdot 10^3 \text{ g}^2\text{cm}^2\text{sec}^5;$$

$$a_5 = 2J_{r.s}I_qT_AT_y = 2 \cdot 60 \cdot 2821 \cdot 10^3 \cdot 0{,}003 \cdot 0{,}02 = 2031 \text{ g}^2\text{cm}^2\text{sec}^6.$$

Let us note that characteristic equation (7.3.7), numerical values of coefficients of which were found above, is used in § 7.3, Par. 3b during the analysis of stability of a GS by the Mikhaylov criterion.

b) Analysis of Natural Oscillations of a GS.

During the analysis of natural oscillations of a GS we shall ignore friction in the axes of the suspension ($n_1 = n_2 = 0$), and also time constants of control networks ($T_{mot} = T_{amp} = 0$). If, in (7.2.80), we set $n_1 = n_2 = 0$, $T_{mot} = T_{amp} = 0$, then, according to (7.3.7), the characteristic equation of the system will have form

$$2J_{r.s}I_q\lambda^3 + 2J_{r.s}bK\lambda^2 + 4H^2\lambda + 2HS_1eK = 0. \tag{7.3.8}$$

Designating

$$\nu^2 = \frac{4H^2}{2J_{r.s}I_q}, \tag{7.3.9}$$

$$n = bK, \quad m = S_1 eK, \tag{7.3.10}$$

we rewrite (7.3.8) in the following form:

$$\lambda^3 + \frac{n}{I_\eta}\lambda^2 + \nu^2\lambda + \nu^2\frac{m}{2H} = 0. \tag{7.3.11}$$

Comparing (7.3.9) with (3.2.8) we see that ν represents frequency of natural notational oscillations of a free gyrostabilizer with the stabilizing motor turned off (K = 0); this follows directly from (7.3.11), since when K = 0, according to (7.3.10), m = n = 0. To find the physical meaning of coefficients n and m it is useful to compose characteristic equation (7.3.11), proceeding from differential equations (7.2.60) of motion of the GS. For the considered case in (7.2.60) one should set $n_1 = n_2 = 0$, $T_{mot} = T_{amp} = 0$, $\dot{\theta} = 0$, $\ddot{\theta} = 0$, $u_\eta = 0$, $f_1 = f_2 = 0$, $S_2 = 0$; then we obtain system of equations

$$\left.\begin{aligned} &I_\eta\ddot{\alpha} + 2H\dot{\beta} - Ki = 0 \\ &2J_{r.s}\ddot{\beta} - 2H\dot{\alpha} = 0 \\ &U_y + S_1\beta = 0 \\ &i + b\dot{\alpha} - eU_y = 0 \end{aligned}\right\} \tag{7.3.12}$$

which we can transform to

$$\left.\begin{aligned} &I_\eta\ddot{\alpha} + 2H\dot{\beta} + bK\dot{\alpha} + S_1eK\beta = 0 \\ &2J_{r.s}\ddot{\beta} - 2H\dot{\alpha} = 0 \end{aligned}\right\} \tag{7.3.13}$$

or, introducing designation (7.3.10),

$$\left.\begin{aligned} &I_\eta\ddot{\alpha} + 2H\dot{\beta} + n\dot{\alpha} + m\beta = 0 \\ &2J_{r.s}\ddot{\beta} - 2H\dot{\alpha} = 0 \end{aligned}\right\} \tag{7.3.14}$$

from which it is easy to show that the characteristic equation of system (7.3.14) is equation (7.3.11).

From (7.3.14) it follows [130, 125] that: 1) coefficient m represents the so-called steepness of the static characteristic of the SM, i.e., the dependence between the torque, applied to the axis of the frame by the stabilizing motor, and the angle of precession β of the gyrohousing; 2) coefficient n represents steepness of the damping torque on the spin axis of the frame, which is caused by the counter emf, which appears during rotation of the armature of the stabilizing motor.

We compose the characteristic equation for a GS taking into account liquid friction in the axes of the suspension, and not taking into account time constants T_{mot} and T_{amp}. Then, according to (7.2.80), we obtain

$$2J_{r.s}I_q\lambda^3+[I_qn_2+2J_{r.s}(n_1+bF)]\lambda^2+$$
$$+[n_2(n_1+bK)+4H^2]\lambda+2HS_1K=0 \qquad (7.3.15)$$

or, considering designations (7.3.9) and (7.3.10),

$$\lambda^3+\left(\frac{n_2}{2J_{r.s}}+\frac{n_1+n}{I_q}\right)\lambda^2+\left[\frac{n_2(n_1+n)}{2J_{r.s}I_q}+\nu^2\right]\lambda+\nu^2\frac{m}{2H}=0; \qquad (7.3.16)$$

it follows from this that the presence of liquid friction changes damping of oscillations of the frame, and also the frequency of natural nutational oscillations of a free GS (m = 0, n = 0), which in this case will be determined by relationship

$$\nu_1^2=\nu^2+\frac{n_1n_2}{2J_{r.s}I_q}. \qquad (7.3.17)$$

Let us consider natural oscillation of a GS in the simplest case, when friction in the suspension axes is not taken into account ($n_1 = n_2 = 0$), and time constants T_{mot} and T_{amp} of control networks are so small that their influence can be ignored. For this we use the characteristic equation of a GS (7.3.11). In investigation of an analogous equation [132] there is analyzed the case, when parameters of the GS are selected so that characteristic equation (7.3.11) has one real and two complex conjugate roots; here, for a stable GS the real parts of roots should be negative:

$$\lambda_1=-\delta+j\mu;\quad \lambda_2=-\delta-j\mu;\quad \lambda_3=-\varepsilon. \qquad (7.3.18)$$

As we know,* such roots of cubic algebraic equation should satisfy the following relationships of coefficients of equation (7.3.11):

$$\left.\begin{aligned}2\delta+\varepsilon&=\frac{n}{I_q}\\ \delta^2+\mu^2+2\varepsilon\delta&=\nu^2\\ \varepsilon(\delta^2+\mu^2)&=\nu^2\frac{m}{2H}\end{aligned}\right\} \qquad (7.3.19)$$

Then, with initial conditions

$$\alpha(0)=\alpha_0,\quad \beta(0)=\beta_0,\quad \dot\alpha(0)=\dot\alpha_0,\quad \dot\beta(0)=\dot\beta_0 \qquad (7.3.20)$$

we can present integrals of differential equations of motion (7.3.14) of the GS, omitting small components (including terms containing $\dot\beta_0$), in the following approximate form [132]:

$$\left.\begin{aligned}\alpha&=\alpha_0-\frac{J_{r.s}}{H}\dot\beta_0e^{-\delta t}+\frac{\dot\alpha_0}{\mu}e^{-\delta t}\sin\mu t+\frac{J_{r.s}}{H}\dot\beta_0e^{-\delta t}\cos\mu t\\ \beta&=\left(\beta_0+\frac{I_q}{2H}\dot\alpha_0\right)e^{-\delta t}+\frac{\delta}{\mu}\beta_0e^{-\delta t}\sin\mu t-\frac{I_q}{2H}\dot\alpha_0e^{-\delta t}\cos\mu t\end{aligned}\right\} \qquad (7.3.21)$$

*See, for instance: V. I. Smirnov. Course in higher mathematics, Vol. 1, State Press for Technical and Theoretical Literature, 1948.

From these expressions it follows that motion of the GS, with values of coefficients selected according to (7.3.18) and (7.3.19), constitutes the superposition of two modes of motion: 1) aperiodic and 2) damped oscillations with frequency μ, which in the case of small damping δ can be close to the frequency ν of nutational oscillations of a free GS (with the stabilizing motor turned off). As can be seen, appearance of the aperiodic component of motion of the GS is caused mainly by initial deflection β_0 of the gyroscopes with respect to the axes of precession. Initial amplitude of damped oscillations of the GS depends basically on the magnitude of initial angular velocity $\dot{\alpha}_0$ of the frame, caused by the external torque instanteously applied to the frame, for instance, due to a shock effect. For determination of the nature of motion of a GS we give a numerical example.

Example 7.2: Find the law of motion of a GS $\alpha(t)$ and $\beta(t)$, if to the GS frame, with parameters given in example 7.1, there is applied an instantaneous torque (shock), importing to the frame initial angular velocity $\dot{\alpha}_0 = 0.2$ 1/sec; the initial angle of displacement of the gyroscope $\beta_0 = 5^\circ$; $\dot{\beta}_0 = 0$; $\alpha_0 = 0$.

Solution: 1. When $\alpha_0 = 0$, according to (7.3.21), we have

$$\left.\begin{aligned} \alpha &= -\frac{J_{r.s}}{H}\varepsilon\beta_0 e^{-\delta t} + \frac{\dot{\alpha}_0}{\mu}e^{-\delta t}\sin\mu t + \frac{J_{r.s}}{H}\varepsilon\beta_0 e^{-\delta t}\cos\mu t \\ \beta &= \left(\beta_0 + \frac{I_\eta}{2H}\dot{\alpha}_0\right)e^{-\delta t} + \frac{\delta}{\mu}\beta_0 e^{-\delta t}\sin\mu t - \frac{I_\eta}{2H}\dot{\alpha}_0 e^{-\delta t}\cos\mu t \end{aligned}\right\}$$

2. By (7.3.9) and (7.3.10) for the data of example 7.1, we calculate ν^2, n, m:

$$\nu^2 = \frac{4H^2}{2J_{r.s}I_\eta} = \frac{1936\cdot10^6}{2\cdot60\cdot3621\cdot10^2} = 5719 \text{ 1/sec}^2;\ \nu = 75{,}62 \text{ 1/sec};$$

$$n = bK = 85{,}36\cdot6333\cdot10^2 = 4613\cdot10^4 \text{ g-cm-sec};$$

$$m = S_0\varepsilon K = 57{,}3\cdot0{,}2222\cdot6333\cdot10^2 = 1061\cdot10^4 \text{ g-cm/radian}.$$

3. We find values of the following quantities:

$$\frac{n}{I_\eta} = \frac{4613\cdot10^4}{3621\cdot10^2} = 163{,}5 \text{ 1/sec};$$

$$\nu^2\frac{m}{2H} = 5719\,\frac{1061\cdot10^4}{44\cdot10^4} = 1379\cdot10^2 \text{ 1/sec}^3.$$

4. Using (7.3.19) and the found numerical values of ν^2, $\frac{n}{I_\eta}$ and $\nu^2\frac{m}{2H}$, we obtain

$$\varepsilon = 127 \text{ 1/sec};\ \delta = 18{,}5 \text{ 1/sec};\ \mu = 27{,}2 \text{ 1/sec}.$$

5. Then on the basis of the formulas given in Par. 1, we have the following laws of change $\alpha(t)$ of the angle of rotation of the frame about the axis of stabilization and $\beta(t)$ of the angle of rotation of the gyroscope about the axis of precession:

$$\left.\begin{aligned}\alpha(t) &= -3{,}022\cdot10^{-3}e^{-127t} + 7{,}39\cdot10^{-3}e^{-18.5t}\sin 27{,}2t + \\ &\quad + 3{,}022\cdot10^{-3}e^{-18.5t}\cos 27{,}2t \\ \beta(t) &= 0{,}2155e^{-127t} + 0{,}4074e^{-18.5t}\sin 27{,}2t - 0{,}1282e^{-18.5t}\cos 27{,}2t\end{aligned}\right\}$$

Graphs of functions $\alpha(t)$ and $\beta(t)$ are shown in Fig. 7.16, from which it follows that natural oscillations of the considered GS attenuate very rapidly, where these oscillations occur with frequency μ, which in the given case, due to considerable damping δ, essentially differs from the frequency ν of nutational oscillations of a free GS.

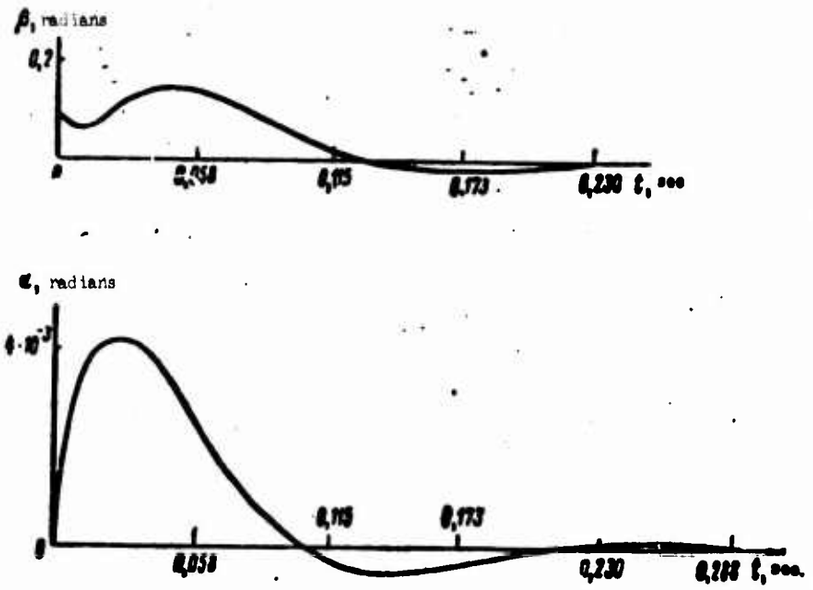

Fig. 7.16. Graphs of functions $\alpha(t)$ and $\beta(t)$.

c) Certain Remarks.

Above we considered natural oscillations of a GS without taking into account the influence on them of liquid friction in the suspension axes and time constants of control networks. Analogously, we analyze oscillations of a GS taking into account these factors. Investigation of free oscillations of a GS under the action, for instance, of an instantaneous external torque or a constant torque can be performed by the same methods of the theory of automatic control as we used during the analysis of corresponding motions of a free gyroscope (§ 3.2, Par. 3).

In the readout GD's, considered in Part 1, during the analysis of their natural oscillations we usually disregard the elastic compliance of components of the gyroscope due to the comparatively small stresses, developed in such GD's. In a GD of active type, in particular in active GS's, these stresses essentially increase, and consequently, elastic deformations of the GS can become considerable. The latter affect natural oscillations, stability and forced oscillations of the GS. An investigation of a GS taking into account elastic compliance of its components is contained in the works of A. Yu. Ishlinskiy [47], G. D. Blyumin, E. I. Sliv [135], and others.

We shall indicate peculiarities of natural oscillations of a GS, taking into account its elastic deformations. Frequency ν of natural nutational oscillations of a free GS (with the stabilizing motor turned off), according to (7.3.9), is determined

by relationship

$$\nu = \frac{2H}{\sqrt{2J_{r.s}I_q}}. \tag{7.3.22}$$

In the work of A. Yu. Ishlinskiy [47] it is shown that the value of the frequency ν of nutation of a GS, found by experimental means, is less than that calculated by the formula (7.3.22), where the difference between these quantities grows with increase of angular momentum; V. I. Kuznetsov perceived the cause of this non-correspondence [47] in elastic compliance of the GS components. In the GS, due to elastic deformations, the gyrohousing may have a certain shift (turn) with respect to the frame about the axis of stabilization. Here, forces of elasticity create a restoring torque, proportional to the amount of deformation. In the shown work there is considered a case, when elastic only the shaft, (axis of stabilization) connecting the stabilized mass with the gyroscopic frame possesses elastic compliance.* In practice, we use formula (7.3.22) with a correction factor (0.5 to 0.7), i.e., we assume [47]

$$\nu = (0{,}5 \div 0{,}7)\frac{2H}{\sqrt{2J_{r.s}I_q}}. \tag{7.3.23}$$

There is considered [135] a case, when elastic only the axis of precession of the GS possesses elastic compliance. On the basis of analysis of natural oscillations of the GS there are obtained expressions for frequencies of its elastic and nutational oscillations. There is also marked a decrease of the frequency of nutational oscillations of a GS, possessing elastic compliance of individual components.

With investigation of natural oscillations of a GS there is connected the problem of selection of optimum damping of the system, i.e., optimum value of steepness n [see (7.3.14)] of the damping torque on the spin axis, at which quenching of natural oscillations of the GS will be most rapid. There exists a rigorous solution [132] of this problem, pertaining to the problem of synthesis of GS's.

Characteristic equation (7.3.8), besides (7.3.11), can also be presented in the form

$$T_{\Sigma.r}^2\lambda^3 + 2\zeta T_{\Sigma.r}\lambda^2 + \lambda + \frac{n}{2H} = 0, \tag{7.3.24}$$

where

$$T_{\Sigma.r}^2 = \frac{2J_{r.s}I_q}{4H^2};\quad 2\zeta T_{\Sigma.r} = \frac{2J_{r.s}bK}{4H^2} = \frac{2J_{r.s}n}{4H^2};$$

$$\zeta = \frac{1}{2H}\sqrt{\frac{J_{r.s}}{2I_q}}\,n. \tag{7.3.25}$$

*For those interested in investigating the influence of elastic deformations of a GS on its natural oscillation we recommend the cited work.

Let us find the physical meaning of parameters in equation (7.3.24). Quantity $T_{mot.g}$ is the electromechanical time constant of the stabilizing motor and gyroscopes, since in I_η [see (7.2.43)] there also enters the reduced moment of inertia $\gamma^2 J_{arm}$ of the motor armature. Quantity ζ, proportional to coefficient n of the damping torque, is the relative coefficient of damping of natural oscillations of the GS. With the stabilizing motor turned off (m = n = 0), instead of (7.3.24), we obtain

$$T_{n.r.}^2 \lambda^2 + 1 = 0, \tag{7.3.26}$$

i.e., the characteristic equation of natural nutational oscillations of a free GS with time constant $T_{mot.g}$, connected, according to (7.3.9) and (7.3.25), with frequency ν of nutational oscillations of the GS by relationship

$$T_{n.r.} = \frac{1}{\nu}, \tag{7.3.27}$$

analogous to (3.2.62) for a free gyroscope.

Thus, in the investigations given earlier we disregarded small electromagnetic time constant T_{mot} of the stabilizing motor, but we considered its electromechanical time constant $T_{mot.g}$.

3. Stability of GS's

a) Preliminary Remarks.

Analysis of stability is an important problem of the study of GS's. Furthermore, guaranteeing stability is one of the first requirements, which parameters of a projected gyrostabilizer should satisfy. Usually stability of a GS is considered with respect to coordinate α, i.e., the angle of rotation of the frame about the axis of stabilization; here, the GS will also be stable with respect to coordinate β, i.e., the angle of rotation of the gyrohousing, about the axes of precession. During investigation of stability of a GS it makes no difference whether we start from transfer function $\Phi_{11}(s)$ [see (7.2.79)] or from transfer function $\Phi_{12}(s)$ [see (7.2.82)]; as it was shown was in Par. 2, in both cases we have identical characteristic equations, which determine stability of the GS.

If we consider a GS without correction, then we arrive at the characteristic equation (7.3.6) of a closed-loop GS system; the position of equilibrium of the frame with respect to the spin axis in this case is arbitrary (Par. 2). In this connection stability of the GS is usually determined, not with respect to coordinate α, but with respect to angular velocity $\dot{\alpha}$ of a rotation of the frame around the axis of stabilization, i.e., in examining stability of a GS they originate from characteristic equation (7.3.7).

Since the GS is a closed-loop of automatic control system, and its work is described by equations of rather high order, for analysis of its stability it is simplest to use methods of automatic control theory, using for this, e.g., the Mikhaylov criterion of stability, the gain-phase criterion of stability, the criterion of stability, based on analysis of logarithmic frequency-response curves, and others. Significantly wide-spread is analysis of stability of automatic control systems by the gain-phase criterion of stability, which permits us to judge the stability of the system in the gain-phase response of an open-loop system [107]. The essential merit of this criterion is that the gain-phase response of an open-loop system can also be obtained experimentally upon supplying the input, for instance, of the open-loop "gyroframe-amplifier" system (Fig. 7.15) a sinusoidal signal of variable frequency.*

In the considered case the expression for the gain-phase response $W_{11}(s)|_{s=j\omega}$ of an open-loop GS system, for instance (7.2.76), i.e., when $T_{mot} = T_{amp} = 0$, is generally rather complicated, and its numerator is significantly more complex than the numerator of the corresponding gain-phase response $\Phi_{11}(s)|_{s=j\omega}$ of a closed-loop GS system, determined by formula (7.2.75). Therefore, for investigation of stability of a GS application of the gain-phase criterion of stability, if characteristic $W_{11}(j\omega)$ has not been determined experimentally, is hampered.

In this case it is simpler to use Mikhaylov the frequency criterion of stability in which we directly consider stability of a closed-loop system.

b) Application of the Mikhaylov Criterion of Stability.

Let us find with help of the Mikhaylov criterion the stability of the GS whose parameters are given in example 7.1, considering time constants T_{mot} and T_{amp} of the control networks. We have characteristic equation (7.3.7) of a closed-loop GS system in the form

$$a_5\lambda^5 + a_4\lambda^4 + a_3\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 = 0.$$

The left part of this equation, which we call the characteristic Mikhaylov polynomial, we designate $D(\lambda)$:

$$D(\lambda) = a_5\lambda^5 + a_4\lambda^4 + a_3\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0. \qquad (7.3.28)$$

Replacing λ by $j\omega$, we have

$$D(j\omega) = a_5(j\omega)^5 + a_4(j\omega)^4 + a_3(j\omega)^3 + a_2(j\omega)^2 + a_1(j\omega) + a_0. \qquad (7.3.29)$$

*Opening of this system is usually produced at the amplifier outlet. To change frequency it is possible to use a selsyn, working as a transformer, the rotor of which is rotated with different angular velocities [136].

With change of ω from $-\infty$ to $+\infty$ the end of vector $D(j\omega)$ describes on complex plane D a curve, which is called the characteristic curve, the hodograph of vector $D(j\omega)$ or the Mikhaylov hodograph.

The Mikhaylov criterion of stability, offered by him in 1938 [100], we usually formulate in the following manner [107]: a system of automatic control will be stable, if with growth of ω from 0 to ∞ vector $D(j\omega)$ turns angle $n\frac{\pi}{2}$, where n is the degree of the equation $D(\lambda) = 0$, which is the same, if the characteristic curve with change of ω from 0 to ∞, beginning with the positive real axis, passes consecutively in the positive direction (i.e., counterclockwise) through n quadrants.

Sometimes the Mikhaylov criterion of stability is applied in another form, which is a corollary of the above criterion. For this will separate in (7.3.29) the real part from the imaginary:

$$D(j\omega) = \varphi(\omega) + j\psi(\omega), \tag{7.3.30}$$

where $\varphi(\omega)$ and $\psi(\omega)$ are the real and imaginary functions of the Mikhaylov hodograph; considering (7.3.29), we have

$$\left.\begin{aligned}\varphi(\omega) &= a_4\omega^4 - a_2\omega^2 + a_0\\ \psi(\omega) &= a_5\omega^5 - a_3\omega^3 + a_1\omega\end{aligned}\right\}. \tag{7.3.31}$$

From (7.3.31) it follows that $\varphi(\omega)$ contains even powers of ω; when $\omega = 0$, $\varphi(\omega)$ crosses the axis of ordinates distance a_0 from the origin; $\psi(\omega)$ contains odd powers of ω; when $\omega = 0$, $\psi(\omega) = 0$. Considering what has been said, we give the formulation of the Mikhaylov criterion of stability in another form [114]: for stability of a linear system of the n-th order it is necessary and sufficient, that all roots of two separately taken equations

$$\varphi(\omega) = 0, \quad \psi(\omega) = 0 \tag{7.3.32}$$

are real and alternating or which is the same, that curves $\varphi(\omega)$ and $\psi(\omega)$, during change of ω from 0 to ∞, cross the axis of abscissas n times in all (including point $\omega = 0$) and that these points of crossing alternate.*

Let us give an example of determination of the stability of a GS.

Example 7.3: Using the Mikhaylov criterion, determine the stability of the GS, whose parameters are given in example 7.1, where we obtained the following numerical values of coefficients $a_0, a_1, \ldots, a_5$ of the characteristic equation of a closed-loop GS system:

$$a_0 = 4668\cdot10^9\ \text{г}^2\text{см}^2\text{сек}^0;\quad a_1 = 1936\cdot10^8\ \text{г}^2\text{см}^2\text{сек}^2;\quad a_2 = 9989\cdot10^6\ \text{г}^2\text{см}^2\text{сек}^3;$$
$$a_3 = 1562\cdot10^6\ \text{г}^2\text{см}^2\text{сек}^4;\quad a_4 = 7786\cdot10^2\ \text{г}^2\text{см}^2\text{сек}^5;\quad a_5 = 2031\ \text{г}^2\text{см}^2\text{сек}^6.$$

*The indicated conditions are valid with simultaneous fulfillment of the following inequalities: $\varphi(0) > 0$, $\psi(0) > 0$; this corresponds to positive direction of movement of the Mikhaylov curve.

Solution: 1. Substituting (7.3.31) in (7.3.32) and considering values of coefficients a_0, a_1, ..., a_5, we have:

$$\left.\begin{aligned}\varphi(\omega) &= 7786\cdot 10^6\omega^4 - 9989\cdot 10^6\omega^2 + 4668\cdot 10^6 = 0\\ \psi(\omega) &= 2031\omega^5 - 1562\cdot 10^6\omega^3 + 1936\cdot 10^6\omega = 0\end{aligned}\right\}$$

2. For stability of the GS roots of polynomials $\varphi(\omega)$ and $\psi(\omega)$ should be real and alternating, where, instead of alternation of roots it is possible to investigate alternating of the squares of the roots. We find roots of polynomial $\psi(\omega)$

$$\omega(2031\omega^4 - 1562\cdot 10^6\omega^2 + 1936\cdot 10^6) = 0.$$

We have

$$\omega_1^2 = 0;$$

$$\omega_{2,3}^2 = \frac{1562\cdot 10^6 \pm \sqrt{(1562\cdot 10^6)^2 - 4\cdot 2031\cdot 1936\cdot 10^6}}{2\cdot 2031};$$

$$\omega_2^2 = 0{,}1256\cdot 10^4\ 1/\text{sec}^2;\ \omega_3^2 = 7{,}565\cdot 10^4\ 1/\text{sec}^2.$$

3. We determine roots of polynomial $\varphi(\omega)$

$$7786\cdot 10^6\omega^4 - 9989\cdot 10^6\omega^2 + 4668\cdot 10^6 = 0.$$

We have

$$\omega_{4,5}^2 = \frac{9989\cdot 10^6 \pm \sqrt{(9989\cdot 10^6)^2 - 4\cdot 7786\cdot 10^6\cdot 4668\cdot 10^6}}{2\cdot 7786\cdot 10^6};$$

$$\omega_4^2 = 0{,}04855\cdot 10^4\ 1/\text{sec}^2;\ \omega_5^2 = 1{,}234\cdot 10^4\ 1/\text{sec}^2.$$

As can be seen from example 7.3, roots of polynomials $\psi(\omega)$ and $\varphi(\omega)$ alternate among themselves and, consequently, the considered GS is stable; curves of functions $\varphi(\omega)$ and $\psi(\omega)$ are shown in Fig. 7.17.* Construction of curves $\varphi(\omega)$ and $\psi(\omega)$ also permits judging of the margin of stability, which here can be characterized, e.g., by percent radio of the narrowest interval between roots to the widest adjacent interval [27].

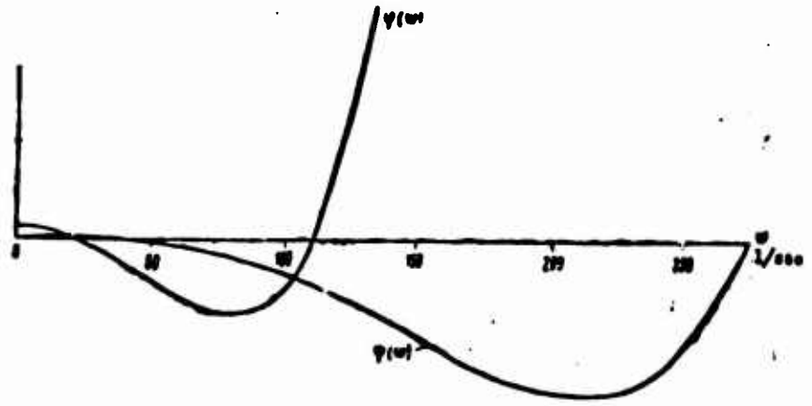

Fig. 7.17. Investigation of stability of a GS by the Mikhaylov criterion.

c) Finding the Basic Relationship Between Parameters of a GS, Ensuring its Stability.

The Mikhaylov criterion, like other criteria, permits us to find the stability of a given GS, for which all parameters are assigned.

In analyzing a GS there

*From the curves in Fig. 7.17 it is clear that $\varphi(0) > 0$ and $\psi(0) > 0$ (see footnote on p. 47).

is usually posed the problem, for assigned basic parameters, to select values of certain other parameters (which can vary in known limits), proceeding from the condition of stability of the system. For solution of such problems we apply special methods of isolating areas of stability, allowing us to select not only values of coefficients, e.g., of the characteristic equation of the GS, but also any of its other parameters, on which these coefficients depend.

For isolating areas of stability and construction of boundaries of stability there exist different methods [107]. For a GS this problem will be considered subsequently for the case when time constants of control networks of the GS can be ignored.

For the basic relationship between parameters of a GS, proceeding from the condition of its stability, we consider the case when the amplifier time constant T_{amp} is negligible, i.e., we have a GS with a "fast-working" amplifier; liquid friction torques in suspension axes we still will ignore ($n_1 = n_2 = 0$). For solution of this problem we use characteristic equation (7.3.7) of a closed-loop GS system, in which coefficients a_0, a_1, ..., a_5 are determined by relationships (7.2.80). If in them we set $n_1 = n_2 = 0$ and the time constant of the amplifier $T_{amp} = 0$, we obtain

$$\left.\begin{aligned} a_0 &= 2HS_1eK; & a_3 &= 2J_{r.s}I_\eta \\ a_1 &= 4H^2; & a_4 &= 2J_{r.s}I_\eta T_\Lambda \\ a_2 &= 2J_{r.s}bK + 4H^2T_\Lambda; & a_5 &= 0 \end{aligned}\right\} \tag{7.3.33}$$

Substituting (7.3.33) in (7.3.7) and considering designations (7.3.10), we have

$$2J_{r.s}I_\eta T_\Lambda\lambda^4 + 2J_{r.s}I_\eta\lambda^3 + (2J_{r.s}n + 4H^2T_\Lambda)\lambda^2 + 4H^2\lambda + 2Hm = 0, \tag{7.3.34}$$

i.e., a characteristic equation of the fourth order.

To determine the relationship between basic parameters, with which the GS is stable we use the Hurwitz criterion. In the case of a characteristic equation of the fourth order necessary and sufficient conditions of stability are the following [114]:

$$\left.\begin{aligned} &a_4 > 0, \quad a_3 > 0, \quad a_2 > 0, \quad a_1 > 0, \quad a_0 > 0; \\ &\Delta_3 = \begin{vmatrix} a_3 & a_1 & 0 \\ a_4 & a_2 & a_0 \\ 0 & a_3 & a_1 \end{vmatrix} = a_1(a_2a_3 - a_4a_1) - a_0a_3^2 > 0 \end{aligned}\right\} \tag{7.3.35}$$

Positiveness of coefficients a_0, a_1, ..., a_4 of characteristic equation (7.3.34) is always realized. Regarding, however, the second condition ($\Delta_3 > 0$), then, substituting in it coefficients a_0, a_1, ..., a_4 from (7.3.33) and considering (7.3.10), we obtain

$$4H^2\left[(2J_{r.s}n + 4H^2T_\Lambda)2J_{r.s}I_\eta - 2J_{r.s}I_\eta T_\Lambda 4H^2\right] - 2Hm\,4J_{r.s}^2I_\eta^2 > 0$$

or

$$2J_{r.s}^2 I_q n H^2 + 4J_{r.s} I_q H^4 T_A - 4J_{r.s} I_q H^4 T_A - J_{r.s}^2 I_q^2 H m > 0,$$

from which we have

$$2Hn - I_q m > 0$$

or

$$\frac{2H}{I_q} > \frac{m}{n}. \qquad (7.3.36)$$

Thus, we obtain the known condition of stability [47] of an active GS, first discovered in 1943 by V. I. Kuznetsov. From (7.3.36) it follows that if we disregard the amplifier time constant T_{amp}, the electromagnetic time constant of the stabilizing motor T_{mot} (of the same order or even less than T_{amp}, see example 7.1) does not influence stability of the GS. As for the electromechanical time constant of the motor, which substantially exceeds its electromagnetic constant, it was considered by us (Par. 2b). Conditions of stability of a GS taking into account time constants T_{mot} and T_{amp} were obtained by A. Yu. Ishlinskiy [47], and also A. N. Roytenberg [132].

The circumstance that electromagnetic time constant T_{mot} of the stabilizing motor (if here time constant T_{amp} of the amplifier is negligible) does not enter into the condition of stability (7.3.36), permits us to obtain this condition by simpler means. For case $T_{mot} = T_{amp} = 0$ we have a third order characteristic equation (7.3.11)

$$\lambda^3 + \frac{n}{I_q}\lambda^2 + \nu^2\lambda + \nu^2\frac{m}{2H} = 0.$$

According to the Hurwitz criterion, for a third order equation conditions of stability have form [114]

$$\left.\begin{aligned} a_3 > 0,\ \ a_2 > 0,\ \ a_1 > 0,\ \ a_0 > 0 \\ \Delta_2 = \begin{vmatrix} a_1 & a_0 \\ a_3 & a_2 \end{vmatrix} = a_2 a_1 - a_3 a_0 > 0 \end{aligned}\right\} \qquad (7.3.37)$$

Positiveness of coefficients $a_0, \ldots, a_3$ of the characteristic equation is always realized. According to the second condition of (7.3.37) we have

$$\frac{n}{I_q}\nu^2 - \nu^2\frac{m}{2H} > 0$$

or

$$\frac{2H}{I_q} > \frac{m}{n},$$

i.e., condition of stability (7.3.36).

Let us find the influence on stability of a CS liquid friction torques in the suspension axes, and we shall limit ourselves to calculation of friction only on the axis of stabilization; then, considering in (7.3.16) $n_2 = 0$, we have a characteristic

equation of the GS in the form

$$\lambda^3 + \frac{n_1 + n}{I_y}\lambda^2 + \nu^2\lambda + \nu^2\frac{m}{2H} = 0. \tag{7.3.38}$$

According to conditions of stability (7.3.37), we obtain

$$\frac{2H}{I_y} > \frac{m}{n_1 + n}, \tag{7.3.39}$$

from which it follows that increase of liquid friction on the spin axis of the frame (axis of stabilization) increases stability of the system; however, increase of coefficient of friction n_1 leads to increase of the external disturbing torque on the axis of the frame during oscillation of the base on which the gyrostabilizer is fixed. Therefore, such a method of increasing the degree of stability of a GS is unacceptable.

We note one more important circumstance. Condition of stability (7.3.36) may also be used for selecting the tentative value of the gear transmission ratio γ from the stabilizing motor to the axis of stabilization of the frame. Considering (7.2.43), and also (7.2.49) and (7.3.10), we rewrite (7.3.36) in the form

$$\frac{\gamma}{J_{p_1} + 2J_{r.s} + \gamma^2 J_a} > \frac{S_1}{2Hc}. \tag{7.3.40}$$

The best value of transmission ratio γ_{best} from the point of view of guaranteeing stability of a GS will be obtained, when the left part of inequality (7.3.40) reaches a maximum; then

$$\gamma_0 = \sqrt{\frac{J_{p_1} + 2J_{r.s}}{J_a}}. \tag{7.3.41}$$

However, such an approach is tentative and during selection of the most advantageous gear ratio one should also consider time constants of the control networks. This problem is considered in detail by A. Yu. Ishlinskiy [47].

b) Isolating Regions of Stability GS.

Selecting basic parameters of the GS, besides the condition of stability (7.3.36) it is useful to isolate regions of stability of the GS; here, for the characteristic third order equation of the gyrostabilizer it is possible to apply the Vyshnegradskiy diagram. The Vyshnegradskiy criterion permits one in simple and graphic form to reveal stability of a third order dynamic system, and also to select parameters of the system, proceeding from the condition of stability.

Let us assume we have for the GS characteristic third order equation (7.3.11), which in general form we write thus:

$$a_3\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 = 0, \tag{7.3.42}$$

where

$$a_0 = v^2 \frac{m}{2H}; \quad a_1 = v^2; \quad a_2 = \frac{n}{l_q}; \quad a_3 = 1. \tag{7.3.43}$$

Let us replace variables, taking

$$u = \lambda \sqrt[3]{\frac{a_3}{a_0}}. \tag{7.3.44}$$

Then characteristic equation (7.3.42) is reduced to so-called normalized form:

$$u^3 + Au^2 + Bu + 1 = 0, \tag{7.3.45}$$

where Vyshnegradskiy parameters A and B are connected with coefficients of characteristic equation (7.3.42) by the following relationships:

$$A = \frac{a_2}{\sqrt[3]{a_0 a_3^2}}; \quad B = \frac{a_1}{\sqrt[3]{a_3 a_0^2}}. \tag{7.3.46}$$

Vyshnegradskiy proved that the necessary and sufficient condition of stability is observance of inequality

$$AB > 1 \ (\text{when } A > 0, \ B > 0), \tag{7.3.47}$$

which we call the Vyshnegradskiy criterion of stability.

Considering (7.3.46) we can rewrite inequality (7.3.47) in the form

$$a_1 a_2 > a_3 a_0, \tag{7.3.48}$$

coinciding with the second condition of (7.3.37).

Substituting in (7.3.48) equality (7.3.43), we obtain the condition of stability of the GS (7.3.36).

Boundaries of stability are determined by equality $a_1 a_2 = a_3 a_0$ or

$$AB = 1 \ (\text{when } A > 0, \ B > 0). \tag{7.3.49}$$

This equality on the plane of parameters A and B (Fig. 7.18)is depicted by a hyperbola, called by the Vyshnegradskiy hyperbola. The region of stability lies above and to the right of the hyperbola; the region between the hyperbola and the coordinate axes, for which AB < 1 (when A > 0, B > 0), is the region of instability.

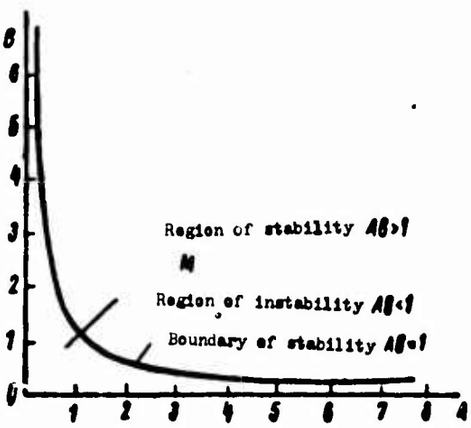

Fig. 7.18. Vyshnegradskiy diagram.

If we know parameters of the GS, by formulas (7.3.43) and (7.3.46) we can determine Vshnegradskiy parameters A and B. Plotting on the diagram (Fig. 7.18) the point with coordinates A, B, it is easy to find whether the GS is stable

or unstable; if the point falls on the hyperbola, then the GS is on the boundary of stability. Since in A and B there enter parameters of the GS, by the Vyshnegradskiy diagram, we very conveniently find the influence of individual parameters on stability of the GS and also select values of parameters ensuring its stability.

Let us consider a numerical example.

Example 7.4: Determine by the Vyshnegradskiy criterion stability of the GS, whose parameters are given in example 7.1; we have: $2H = 44 \cdot 10^4$ g-cm-sec, $I_\eta = 2821 \cdot 10^2$ g-cm-sec^2, $K = 8333 \cdot 10^2$ g-cm-amp, $b = 55.36$ amp-sec, $e = 0.2222$ 1/ohm, $S_1 = 57.3$ v/radian; $\nu^2 = 5719$ 1/sec^2 (example 7.2).

Solution: 1. By formulas (7.3.10) we calculate coefficients n and m:

$$n = bK = 55{,}36 \cdot 8333 \cdot 10^2 = 4613 \cdot 10^4 \text{ g-cm-sec},$$
$$m = S_1 eK = 57{,}3 \cdot 0{,}2222 \cdot 8333 \cdot 10^2 = 1061 \cdot 10^4 \text{ g-cm/radian},$$

2. According to (7.3.43) we calculate a_0, a_1, ..., a_3:

$$a_0 = \nu^2 \frac{m}{2H} = 5719 \frac{1061 \cdot 10^4}{44 \cdot 10^4} = 1379 \cdot 10^2 \text{ 1/sec}^3;$$
$$a_1 = \nu^2 = 5719 \text{ 1/sec}^2;$$
$$a_2 = \frac{n}{I_\eta} = \frac{4613 \cdot 10^4}{2821 \cdot 10^2} = 163{,}5 \text{ 1/sec};$$
$$a_3 = 1.$$

3. We calculate the Vyshnegradskiy parameters A and B by formulas (7.3.46):

$$A = \frac{a_2}{\sqrt[3]{a_0 a_3^2}} = \frac{163{,}5}{\sqrt[3]{1379 \cdot 10^2 \cdot 1}} = 3{,}16;$$
$$B = \frac{a_1}{\sqrt[3]{a_3 a_0^2}} = \frac{5719}{\sqrt[3]{1 \cdot (1379 \cdot 10^2)^2}} = 2{,}14.$$

4. According to the Vyshnegradskiy criterion (7.3.47)

$$AB = 3{,}16 \cdot 2{,}14 = 6{,}76 > 1,$$

i.e., the gyrostabilizer is stable.

Point M with coordinates A and B we plot on the Vyshenegradskiy diagram (Fig. 7.18); it is in the region of stability of the GS.

With help of the same diagram, given the position of point M in the region of stability, i.e., coordinates A and B, it is possible, e.g., for a GS with assigned H and I_η to select parameters m and n. As for selection of the position of point (A, B), we note the following. As it is known [114], depending upon the position of the point on the diagram we obtain different values of the roots of characteristic equation (7.3.42), i.e., a different character of the transient. Vyshnegradskiy

comparatively smaller angular momenta. This is the basic advantage of introduction of derivatives in the law of control of the stabilizing motor.

4. Transient Characteristic of a GS

a) Construction of Transient Response $\alpha(t)$.

During the analysis of an active-type GS of importance is determination of its transient response and basic performance indices during typical disturbance, for which we usually take a unit step input. Behavior of a GS in a transient regime, and namely its natural oscillation, for the case when time constants T_{mot} and T_{amp} of the control networks are not taken into account and the characteristic equation of the closed-loop GS system is cubic, considered in Par. 2.

It is interesting to construct the transient response and determine performance indices of the GS taking into account these time constants of the control networks. Since in this case motion of the GS is described by differential equations of comparatively high order, for investigation of performance it is expedient to use methods, developed in the theory of automatic control, e.g., frequency methods of performance analysis [107, 27]. The merit of these methods is that they are based on the same frequency-response curves, in this case of a GS, which were used by us in analyzing stability of the GS; their other merit is that these methods are essentially graphoanalytical and do not require calculation of roots, which is very important during the analysis of a GS, whose characteristic equation is an equation of the fifth or sixth order.

In the frequency method of analysis of GS performance applied hereafter we used the real frequency-response curve of the GS. Let us write the gain-phase response $\Phi(j\omega)$ of a closed-loop GS system, by analogy with (3.2.40), in the form

$$\Phi(j\omega) = P(\omega) + jQ(\omega), \tag{7.3.60}$$

where $P(\omega)$ and $Q(\omega)$ are the real and imaginary frequency-response curves of the GS.

Consequently, from the known expression for $\Phi(j\omega)$ it is easy determined the real frequency-response curve $P(\omega)$ of the GS. Knowing $P(\omega)$, we can find the transient response $\alpha(t)$ of the GS with a unit step input, which is expressed by the following relationship [107]:

$$\alpha(t) = \frac{2}{\pi}\int_0^\infty \frac{P(\omega)}{\omega}\sin \omega t\, d\omega. \tag{7.3.61}$$

To determine the real frequency-response curve $P(\omega)$ of a GS we consider the case when the GS does not have correction. Let us use expression (7.2.79) of transfer

function $\Phi_{11}(s)$ of a closed-loop GS system on coordinate α with respect to a disturbing torque, applied to the axis of stabilization of the frame,

$$\Phi_{11}(s) = \frac{b_3 s^3 + b_2 s^2 + b_1 s + b_0}{a_5 s^5 + a_4 s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0}, \qquad (7.3.62)$$

where coefficients a_0, ..., a_5, b_0, ..., b_3 are determined through parameters of the GS by relationships (7.2.80).

By analogy with (3.2.37) the gain-phase response $\Phi_{11}(j\omega)$ of a closed-loop GS system can be presented in the form

$$\Phi_{11}(j\omega) = \frac{a(\omega) + jb(\omega)}{c(\omega) + jd(\omega)}, \qquad (7.3.63)$$

where, according to (3.2.38) and (7.3.62), for $a(\omega)$, $b(\omega)$, $c(\omega)$, $d(\omega)$ in the considered case we have relationships

$$\left.\begin{aligned} a(\omega) &= b_0 - b_2\omega^2 \\ b(\omega) &= b_1\omega - b_3\omega^3 \\ c(\omega) &= a_0 - a_2\omega^2 + a_4\omega^4 \\ d(\omega) &= a_1\omega - a_3\omega^3 + a_5\omega^5 \end{aligned}\right\} \qquad (7.3.64)$$

The real frequency-response curve $P(\omega)$ is expressed through $a(\omega)$, $b(\omega)$ ... by formula (3.2.41), i.e.,

$$P_{11}(\omega) = \frac{a(\omega)c(\omega) + b(\omega)d(\omega)}{c^2(\omega) + d^2(\omega)}. \qquad (7.3.65)$$

Subsequently, to determine transient response $\alpha(t)$ of a GS it is necessary in the beginning to construct the real frequency-response curve $P_{11}(\omega)$ of the latter.

Example 7.6: Construct the real frequency-response curve $P_{11}(\omega)$ of the GS whose parameters are given in example 7.1; liquid friction torques in the suspension axes for simplification can be ignored ($n_1 = n_2 = 0$).

Solution: 1. According to example 7.1 for coefficients a_0, ..., a_5 we obtain the following values:

$$\begin{aligned} a_0 &= 1663 \cdot 10^9 \ g^2cm^2sec; & a_3 &= 1562 \cdot 10^8 \ g^2cm^2sec^4; \\ a_1 &= 1936 \cdot 10^8 \ g^2cm^2sec^2; & a_4 &= 7786 \cdot 10^8 \ g^2cm^2sec^5; \\ a_2 &= 9989 \cdot 10^8 \ g^2cm^2sec^3; & a_5 &= 2031 \ g^2cm^2sec^6. \end{aligned}$$

2. According to formula (7.2.80); when $n_2 = 0$ for determination of coefficients b_0, ..., b_3 we have expressions

$$\left.\begin{aligned} b_0 &= 0; & b_2 &= 2J_{r.s}(T_x + T_y) \\ b_1 &= 2J_{r.s}; & b_3 &= 2J_{r.s}T_xT_y \end{aligned}\right\}$$

from which we find the following values of them:

$$\begin{aligned} b_0 &= 0; \\ b_1 &= 2 \cdot 60 = 120 \ g\text{-}cm\text{-}sec^2 \\ b_2 &= 2 \cdot 60\,(0{,}003 + 0{,}02) = 2{,}76 \ g\text{-}cm\text{-}sec^3; \\ b_3 &= 2 \cdot 60 \cdot 0{,}003 \cdot 0{,}02 = 0{,}0072 \ g\text{-}cm\text{-}sec^4. \end{aligned}$$

3. By formulas (7.3.64) and (7.3.65) for different ω we calculate corresponding values of $P_{11}(\omega)$; with the obtained data we constructed a graph of $P_{11}(\omega)$ (Fig. 7.21).

By the shape of curve $P_{11}(\omega)$ with the help of special rules developed in automatic control theory [107], it is possible to judge the character of the transient response $\alpha(t)$. Thus, for instance, the final value of unit step response $\alpha(t)$ in a steady state is equal to the initial value $P_{11}(0)$ of the real frequency-response curve, i.e., static error of the GS will be

$$\alpha(\infty) = \alpha_{st} = P_{11}(0). \tag{7.3.66}$$

Directly from formulas (7.3.64) and (7.3.65) for $\omega = 0$ we have: $a(0) = b_0$, $b(0) = 0$, $c(0) = a_0$, $d(0) = 0$, and, consequently,

$$P_{11}(0) = \frac{b_0}{a_0}. \tag{7.3.67}$$

If we do not take into account friction torque in the axis of precession of the gyroscope ($n_2 = 0$), $b_0 = 0$ [see (7.2.80)], and

$$\alpha_{st} = P_{11}(0) = 0, \tag{7.3.68}$$

i.e., the gyrostabilizer with respect to disturbance along the axis of stabilization, when we disregard friction in the axis of precession, can be considered an astatic system, similar to a readout-type free gyroscope. When we allow for friction in the axis of precession, according to (7.2.80) and (7.3.10),

$$\alpha_{st} = P_{11}(0) = \frac{n_2}{2Hm}, \tag{7.3.69}$$

i.e., the GS is a static system in this case.

Example 7.7: Determine static error of a GS due to liquid friction in its axis of precession during application of a unit step input $f_1 = 1$ kg-cm to the axis of stabilization; the GS has the parameters given in example 7.1; the coefficient of liquid friction in the axis of precession $n_2 = 50$ g-cm-sec.

Solution: By formula (7.3.69) we find

$$\alpha_{st} = \frac{n_2}{2Hm} = \frac{50}{44 \cdot 10^3 \cdot 1061 \cdot 10^4} 10^8 = 1{,}07 \cdot 10^{-6} \text{ radian},$$

i.e., the value of α_{stat} is negligible. Consequently, a GS without correction can be considered an astatic system.

By curve $P_{11}(\omega)$ it is possible to judge other properties of transient $\alpha(t)$; these properties are expounded in detail in courses on the theory of automatic control (see, e.g., [107]).

Using curve $P_{11}(\omega)$ of the real frequency-response of the GS, it is possible to

approximately construct transient response $\alpha(t)$ of a gyrostabilizer with the help of a model trapezoidal frequency-response curves [107]. For this, curve $P_{11}(\omega)$ (Fig. 7.21) is divided into trapezoids, i.e., in the considered case it is replaced by eight trapezoidal frequency characteristics. For each of these characteristics with the help of a special table [154] we construct the corresponding unit step response $\alpha_1(t)$; algebraic summation of ordinates of curves $\alpha_1(t)$ gives curve $\alpha(t)$ (Fig. 7.22) of the transient response of an active GS.

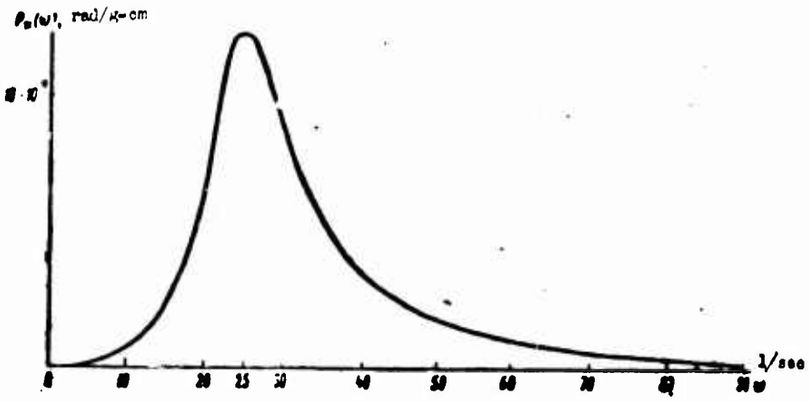

Fig. 7.21. Real frequency-response curve $P_{11}(\omega)$ of a GS.

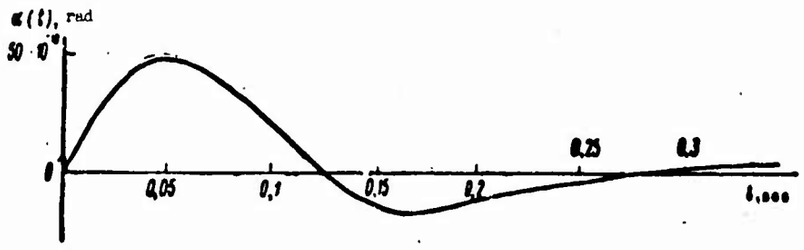

Fig. 7.22. Curve of transient response $\alpha(t)$ of a GS.

From the curve it follows that the transient response of a GS during a unit step input along the axis of stabilization consists of damped oscillations. Duration of the transient response, i.e., control time t_{cont} is approximately 0.3 sec, which is fully acceptable.

Besides the given method for calculations of a GS, in particular for estimating performance and selecting a series of parameters, we construct logarithmic amplitude characteristics of the GS [4]. Such a method is especially convenient during solution of problems of synthesis of GS's, i.e., during selection of its basic parameters.

4) Influence of Dry Friction Torque on the Frame Axis of a GS.

Let us turn to consideration of a more complicated case, considering dry friction torque on the axis of stabilization of the frame; control of the stabilizing motor, also before, we assume to be linear. The presence of dry friction in the axis of stabilization, characterized by a nonlinear law of change of friction torque as a function of the angular velocity $\dot{\alpha}$ of gyroframe rotation, affects the character of the transient response $\alpha(t)$, and under certain conditions it may cause stable periodic oscillations of the GS, i.e., its natural oscillations. Of greatest interest is investigation of natural oscillations of an active-type GS. For solution of this problem it is possible to use the method of approximation, e.g., the method of

harmonic linearization of nonlinearities [114, 115], applied earlier (§ 6.3, Par. 3d).

Being interested in natural oscillations of the GS with respect to angle α of gyroframe rotation, we write the differential equation of motion of the GS in operator form $\left(p = \frac{d}{dt}\right)$, using expression (7.2.79) of the transfer function of a closed-loop GS system:

$$\begin{aligned}(a_5p^5 + a_4p^4 + a_3p^3 + a_2p^2 + a_1p + a_0)\alpha = \\ = (b_3p^3 + b_2p^2 + b_1p + b_0)f_1(t),\end{aligned} \tag{7.3.70}$$

where $f_1(t)$ – disturbance, applied to the axis of stabilization;

$a_0, \ldots, a_5, b_0, \ldots, b_3$ – coefficients, determined by relationships (7.2.80).

As $f_1(t)$ we take the dry friction torque on the axis of stabilization of the frame. Then, taking into account (7.2.34), for the case of a nonoscillating base ($\dot{\theta} = 0$), we have

$$f_1(t) = M_{тр} = -K_y \operatorname{sign} \dot{\alpha}. \tag{7.3.71}$$

Substituting (7.3.71) in (7.3.70), we obtain a nonlinear differential equation for variable α

$$\begin{aligned}(a_5p^5 + a_4p^4 + a_3p^3 + a_2p^2 + a_1p + a_0)\alpha = \\ = -(b_3p^3 + b_2p^2 + b_1p + b_0)K_y \operatorname{sign} \dot{\alpha}.\end{aligned} \tag{7.3.72}$$

For investigation of natural oscillations of a GS we use the method of harmonic linearization of nonlinearities [114, 115]; then we shall seek periodic solution for variable α, by analogy with (6.3.63), in the form

$$\alpha = a_{пер} \sin \omega_{пер} t, \tag{7.3.73}$$

where a_{per} and ω_{per} are amplitude and frequency of the sought periodic solution.

In accordance with the method of harmonic linearization of a nonlinearities the assigned nonlinear function, for instance $F(x, px)$ of variable x, is replaced by an expression of form

$$F(x, px) = qx + \frac{q'}{\omega_{пер}} px, \tag{7.3.74}$$

where q and q' are coefficients of the harmonic linearization.

In the considered case for nonlinear function $K_y \operatorname{sign} \dot{\alpha}$, according to (7.3.74), we have [115, p. 226]

$$K_y \operatorname{sign} p\alpha = \frac{q'}{\omega_{пер}} p\alpha. \tag{7.3.75}$$

Harmonic linearization of nonlinearities of form $K_y \operatorname{sign} \dot{\alpha}$ (Fig. 2.21a), if we consider that friction torque for $\dot{\alpha} = 0$ instantly changes sign, is carried out just as for an ideal relay characteristic of correction (Fig. 2.18e). Then, by analogy with (6.3.64), for the coefficient of harmonic linearization $q'(a_{per})$ we

obtain

$$q'(a_n) = \frac{4K_y}{\pi a_n}. \tag{7.3.76}$$

Substituting (7.3.76) in (7.3.75), we have

$$K_y \operatorname{sign} p\alpha = \frac{4K_y}{\pi a_n \omega_n} p\alpha, \tag{7.3.77}$$

and equation (7.3.72) will take form

$$(a_5 p^5 + a_4 p^4 + a_3 p^3 + a_2 p^2 + a_1 p + a_0)\alpha = -(b_2 p^2 + b_2 p^2 + b_1 p + b_0)\frac{4K_y}{\pi a_n \omega_n} p\alpha$$

or

$$\left[a_5 p^5 + \left(a_4 + b_2 \frac{4K_y}{\pi a_n \omega_n}\right)p^4 + \left(a_3 + b_2 \frac{4K_y}{\pi a_n \omega_n}\right)p^3 + \right.$$
$$\left. + \left(a_2 + b_1 \frac{4K_y}{\pi a_n \omega_n}\right)p^2 + \left(a_1 + b_0 \frac{4K_y}{\pi a_n \omega_n}\right)p + a_0\right]\alpha = 0. \tag{7.3.78}$$

The corresponding characteristic equation will be

$$a_5\lambda^5 + \left(a_4 + b_2 \frac{4K_y}{\pi a_n \omega_n}\right)\lambda^4 + \left(a_3 + b_2 \frac{4K_y}{\pi a_n \omega_n}\right)\lambda^3 +$$
$$+ \left(a_2 + b_1 \frac{4K_y}{\pi a_n \omega_n}\right)\lambda^2 + \left(a_1 + b_0 \frac{4K_y}{\pi a_n \omega_n}\right)\lambda + a_0 = 0. \tag{7.3.79}$$

Replacing λ by $j\omega$, we rewrite (7.3.79) in the following manner:

$$a_5 j\omega^5 + \left(a_4 + b_2 \frac{4K_y}{\pi a_n \omega_n}\right)\omega^4 - \left(a_3 + b_2 \frac{4K_y}{\pi a_n \omega_n}\right)j\omega^3 -$$
$$- \left(a_2 + b_1 \frac{4K_y}{\pi a_n \omega_n}\right)\omega^2 + \left(a_1 + b_0 \frac{4K_y}{\pi a_n \omega_n}\right)j\omega + a_0 = 0. \tag{7.3.80}$$

To find a periodic solution of equation (7.3.72), presenting equation (7.3.80) in the form

$$X(\omega) + jY(\omega) = 0, \tag{7.3.81}$$

it is necessary that by virtue of the Mikhaylov criterion [114] for $\omega = \omega_{per}$ and $a = a_{per}$ conditions $X(\omega) = 0$ and $Y(\omega) = 0$ are simultaneously satisfied (the Mikhaylov curve should pass through the origin of coordinates). This gives two equations*

$$X(\omega_n, a_n) = 0, \quad Y(\omega_n, a_n) = 0, \tag{7.3.82}$$

from which we determined unknown frequency ω_{per} and amplitude a_{per} of the sought periodic solution of the considered equation.

Separating in (7.3.80) the real and imaginary part, we have

$$\left.\begin{aligned} X(\omega) &= \left(a_4 + b_2 \frac{4K_y}{\pi a_n \omega_n}\right)\omega^4 - \left(a_2 + b_1 \frac{4K_y}{\pi a_n \omega_n}\right)\omega^2 + a_0 \\ Y(\omega) &= a_5\omega^5 - \left(a_3 + b_2 \frac{4K_y}{\pi a_n \omega_n}\right)\omega^3 + \left(a_1 + b_0 \frac{4K_y}{\pi a_n \omega_n}\right)\omega \end{aligned}\right| \tag{7.3.83}$$

*Equalities (7.3.82) follow obviously from equation (7.3.81) by force of known properties of complex numbers.

from which, according to (7.3.82), we obtain two equations for determination of frequency ω_{per} and amplitude a_{per}:

$$\left.\begin{aligned} a_4\omega_n^4 + b_3\frac{4K_y}{\pi a_n}\omega_n^3 - a_2\omega_n^2 - b_1\frac{4K_y}{\pi a_n}\omega_n + a_0 = 0 \\ a_5\omega_n^5 - a_3\omega_n^3 - b_2\frac{4K_y}{\pi a_n}\omega_n^2 + a_1\omega_n + b_0\frac{4K_y}{\pi a_n} = 0 \end{aligned}\right\} \qquad (7.3.84)$$

These equations also permit us to determine the dependence of amplitude and frequency of the periodic solution on each parameter of the system with constant values of other parameters.

Natural oscillations correspond to a stable periodic solution. Therefore, determining ω_{per} and a_{per}, it is necessary to check stability of the corresponding periodic solution (7.3.73). For this we use, e.g., condition of stability (6.3.79), in which partial derivatives $\left(\frac{\partial X}{\partial a}\right)_{per}$, $\left(\frac{\partial Y}{\partial \omega}\right)_{per}$ ... are found from equations (7.3.84) by substituting in the latter values of coefficients a_0, ..., a_5, b_0, ..., b_3 from (7.2.80). Solution of this problem, carried out by the same method as is used in § 6.3, Par. 3d, can be carried out by the reader independently.

However, in not all cases does dry friction in the axis of stabilization cause stable periodic motions, i.e., natural oscillation of the GS. Consider a gyrostabilizer, for which time constants T_{mot} and T_{amp} of the control networks are negligible. In this case, according to (7.2.80), for coefficients a_0, ..., a_5 and b_0, ..., b_3 we have the following relationship (liquid friction torques are not considered):

$$\left.\begin{aligned} a_0 = 2Hm;\ a_1 = 4H^2;\ a_2 = 2J_{r.s}n;\ a_3 = 2J_{r.s}I_1; \\ a_4 = 0;\ a_5 = 0; \\ b_0 = 0;\ b_1 = 2J_{r.s};\ b_2 = 0;\ b_3 = 0 \end{aligned}\right\} \qquad (7.3.85)$$

Substituting (7.3.85) in (7.3.84), we obtain equations

$$\left.\begin{aligned} -2J_{r.s}n\omega_n^2 - 2J_{r.s}\frac{4K_y}{\pi a_n}\omega_n + 2Hm = 0 \\ -2J_{r.s}I_1\omega_n^3 + 4H^2\omega_n = 0 \end{aligned}\right\} \qquad (7.3.86)$$

from which we find the frequency

$$\omega_n = \frac{2H}{\sqrt{2J_{r.s}I_1}} \qquad (7.3.87)$$

and amplitude

$$a_n = \frac{4K_y}{\pi}\frac{\sqrt{\frac{2J_{r.s}}{I_1}}}{m - 2H\frac{n}{I_1}}. \qquad (7.3.88)$$

of periodic solution (7.3.73); here ω_{per} coincides with the frequency ν of nutation oscillations of a free GS [see (7.3.9)]. Using condition (6.3.79), it is easy to show that in this case (for $T_{mot} = T_{amp} = 0$) the found periodic solution (7.3.73) is unstable, i.e., natural oscillation of the GS are absent. Here, the presence of dry friction in the axis of the frame will promote damping of natural nutation oscillations of the GS. From the given example it follows that dry friction in the axis of stabilization does not under all conditions lead to natural oscillations of a GS.

c) Influence of a Nonlinear (Relay) Law of Control of the Stabilizing Motor.

In order to analyze the influence of a nonlinear law of control of the stabilizing motor, we assume that the signal pickoff (Fig. 7.2) is a component with an ideal relay characteristic (Fig. 7.23), i.e., voltage taken from it is expressed by nonlinear relationship

$$U = -U_0 f(\beta), \tag{7.3.89}$$

where U_0 — d-c voltage at output of the signal pickoff, and nonlinear function $f(\beta)$, by analogy with (2.3.18), has the form

$$f(\beta) = \begin{cases} +1 & \text{when } \beta > 0 \\ -1 & \text{when } \beta < 0. \end{cases} \tag{7.3.90}$$

Voltage U from the signal pickoff enters the amplifier (Fig. 7.2). The differential equation of the amplifier (7.2.51) upon replacement in it of U by relationship (7.3.89) will be

$$T_y \dot{U}_y + U_y = -kU_0 f(\beta),$$

or, if we disregard the time constant T_{amp} of amplifier,

$$U_y + kU_0 f(\beta) = 0. \tag{7.3.91}$$

If we disregard time constants of the control networks and do not consider friction in the suspension with a nonlinear law of control of the stabilizing motor, equations of the GS are determined by system (7.3.12) on the condition of replacement in it of the third equation by expression (7.3.91); then

$$\left.\begin{aligned} I_y \ddot{\alpha} + 2H\dot{\beta} - Ki &= 0 \\ 2J_{r.s}\ddot{\beta} - 2H\dot{\alpha} &= 0 \\ U_y + kU_0 f(\beta) &= 0 \\ i + b\dot{i} - cU_y &= 0 \end{aligned}\right\} \tag{7.3.92}$$

or

$$\left.\begin{aligned} I_y \ddot{\alpha} + 2H\dot{\beta} + bK\dot{i} + ckKU_0 f(\beta) &= 0 \\ 2J_{r.s}\ddot{\beta} - 2H\dot{i} &= 0 \end{aligned}\right\} \tag{7.3.93}$$

$$\alpha = \alpha_0 - \int_0^t U \cos\varphi \cos K \, d\tau. \tag{7.4.6}$$

Disregarding, with a small value of interval of time t, changes of K and φ, we obtain

$$\alpha = \alpha_0 - Ut \cos\varphi \cos K, \tag{7.4.7}$$

and error of the GS, in this case due to rotation of the earth, will be

$$\alpha^c = -Ut \cos\varphi \cos K, \tag{7.4.8}$$

i.e,, we have the velocity of the error GS.*

With another location of the axis of stabilization of the frame in α^s may include the vehicle velocity. Thus, if the axis of stabilization coincides with $O\xi$, instead of (7.4.5) we have

$$\alpha = \alpha_0 - \int_0^t u_\xi \, d\tau. \tag{7.4.9}$$

According to (1.2.32) $u_\xi = -U \cos\varphi \sin K - \frac{v}{R}$, and, by analogy with (7.4.8), for constant φ, K and v, we have

$$\alpha^c = \left(U \cos\varphi \sin K + \frac{v}{R}\right) t. \tag{7.4.10}$$

To eliminate these velocity errors of a GS, i.e., to maintain the stabilized vehicle in constant position, e.g., with respect to the plane of the horizon, it is necessary to introduce correction.

Above we considered the influence on a GS of induced angular velocity u_η. As for the angular velocity u_ζ of rotation of the system of reference $O\xi\eta\zeta$ around axis $O\zeta$, or the axis of precession O_1x_1 (Fig. 7.2), on the basis of (7.2.18) it follows that a two-gyro stabilizer is insensitive to it. A single-gyro stabilizer, as can be seen from (7.2.19), is sensitive to angular velocity u_ζ, the presence of which causes the stabilizing motor to switch on; this ensures perpendicularity of axis Oz of the gyroscope (Fig. 7.2) to the plane of the frame during its turns about axis $O\zeta$.

Without correction error of the GS will also be influenced by disturbing moments along the axis of precession, for instance, friction torques.

3. Errors of a GS with Correction

To eliminate errors of the GS, considered in Par. 2, and also other position

*The term "velocity error" of a GS, as for a free astatic gyroscope, is conventional, since this error is a consequence of "visible" drift of a GS without a corrector in reference to the earth's coordinate system, i.e., a basic property of a free gyroscope.

errors of the frame there is applied a system of horizontal correction, which is based on use of pendular correction, analogous to that applied in a vertical gyro or to that carried out by means of shifting of the centers of gravity of gyroscopes relative to their geometric centers in the equatorial plane (§ 7.2, Par 2b). However, even with this horizontal correction there can occur (depending upon the characteristic of the applied correction) GS errors both due to rotation of the earth and vehicle motion, and also to disturbing torques about the axis of precession.

To determine these errors we consider the case when in the GS there is applied correction of pendular type. Considering the finding of error in coordinate α, i.e., in the angle of rotation of the frame, we use the second equation of system (7.2.42), rejecting in it inertial term $2J_{s.e}\ddot{\beta}$ (since, by analogy with the preceding, for detecting steady-state motion of a frame it is possible to limit oneself to analysis of its basic precessional motion). Replacing in (7.2.42) the liquid friction torque $n_2\dot{\beta}$ by the dry friction torque $K_x \operatorname{sign}\dot{\beta}$ [see (7.2.33)], we have

$$2H\left(\dot{\alpha}+u_1\right)+S_2\alpha=S_2\chi_1(t)+K_x\operatorname{sign}\dot{\beta}+f_2(t). \tag{7.4.11}$$

Considering designations, analogous to (2.3.5) and (6.2.12), we rewrite (7.4.11) in the form

$$\dot{\alpha}+\varkappa_2\alpha=-u_1+\varkappa_2\chi_1(t)+k_1K_x\operatorname{sign}\dot{\beta}+k_1f_2(t), \tag{7.4.12}$$

where

$$\varkappa_2=\frac{S_2}{2H} \tag{7.4.13}$$

is the specific velocity of horizontal correction, and coefficient k_1 is determined by relationship

$$k_1=\frac{1}{2H}. \tag{7.4.14}$$

From equation (7.4.12) it follows that disturbances from oscillations of the correcting pendulum $\chi_1(t)$ affect the frame through the correction system, in distinction from friction torques $K_x \operatorname{sign}\dot{\beta}$ and other disturbing torques $f_2(t)$ on the axis of precession, which affect motion of the frame directly.

Introducing, by analogy with (6.2.12), the time constant of correction,

$$T=\frac{1}{\varkappa_2}, \tag{7.4.15}$$

we rewrite equation (7.4.12) in the form

$$T\dot{\alpha}+\alpha=-Tu_1+\chi_1(t)+k_1TK_x\operatorname{sign}\dot{\beta}+k_1Tf_2(t). \tag{7.4.16}$$

Let us find the transfer functions of the GS on coordinate α with respect to disturbance $\chi_1(t)$ on the part of the correcting pendulum and with respect to

disturbances $f_2(t)$, directly influencing motion of the frame.

According to (7.4.16) we have

$$\Phi'_{\alpha\alpha}(s) = \frac{\alpha(s)}{\chi_1(s)} = \frac{1}{Ts+1}, \tag{7.4.17}$$

$$\Phi^e_{\alpha\alpha}(s) = \frac{\alpha(s)}{f_2(s)} = \frac{k_1 T}{Ts+1}, \tag{7.4.18}$$

i.e., in this case the frame is considered an aperiodic network.

Let us note that in obtaining expressions (7.4.17) and (7.4.18) we did not consider the amplifier time constant T_{amp} and the electromagnetic time constant of the stabilizing motor T_{mot}, which, as will be shown later, are small as compared to the time constant of correction T. Furthermore, we did not consider inertial terms. Expression (7.4.18) can be directly obtained from general formula (7.2.78), if in it we set $J_{g.e} = 0$, do not take into account liquid torques in the suspension axes ($n_1 = n_2 = 0$), and do not consider work of a stabilizing motor ($K = 0$).

Comparing formulas (7.4.17) and (7.4.18) with expressions (6.2.18), we see that transients and static errors the frame on coordinate α (if we consider motion of the frame only under the influence of correction) will be the same as for a gyro vertical with a linear characteristic of correction (§§ 6.3 and 6.4). Therefore, we only give certain formulas for static errors of the GS.

To determine static errors due to rotation of earth and vehicle motion will assume in (7.4.16) that $\chi_1 = K_x = f_2 = 0$; then

$$T\dot{\alpha} + \alpha = -Tu_\eta, \tag{7.4.19}$$

from which for static error we have

$$\alpha^e_{cr} = -Tu_\eta. \tag{7.4.20}$$

If the axis of the frame is set parallel to the longitudinal axis of the ship, then for the case considered in Par. 2, $u_\eta = U \cos\varphi \cos K$, and

$$\alpha^e_{cr} = -TU \cos\varphi \cos K. \tag{7.4.21}$$

In formula (7.4.20) for static error of a GS due to rotation of earth and vehicle motion we did not consider error $\chi_{1\ stat}$ of the correcting-pendulum, caused by the same factors. Taking into account $\chi_{1\ stat}$ according to (7.4.16), instead of equation (7.4.19), we obtain

$$T\dot{\alpha} + \alpha = -Tu_\eta + \chi_{1cr}, \tag{7.4.22}$$

whence

$$\alpha^e_{cr} = -Tu_\eta + \chi_{1cr}. \tag{7.4.23}$$

Expressions for χ_{1stat} were given in § 4.5.

We shall determine static error of a GS due to dry friction torques in the axis of precession. Considering in (7.4.16) that $u_\eta = 0$, $\chi_1 = 0$, $f_2 = 0$, we have

$$T\dot{\alpha} + \alpha = k_1 T K_x \operatorname{sign} \dot{\beta} \tag{7.4.24}$$

or, considering designation (6.3.19),

$$T\dot{\alpha} + \alpha = \nu_{i2} T \operatorname{sign} \dot{\beta}; \tag{7.4.25}$$

the last equation, by analogy with (6.3.21), we rewrite in form

$$T\dot{\alpha} + \alpha = \pm \nu_{i2} T, \tag{7.4.26}$$

from which we find static error due to friction

$$\alpha_{cr}^{T} = \pm \nu_{i2} T. \tag{7.4.27}$$

Analogously we can determine static errors of the GS due to other disturbances $f_2(t)$ to the axis of precession. Furthermore, with accelerations of the vehicle the correcting-pendulum will have deflections $\chi_1(t)$, which through the system of correction will also cause corresponding static errors of the GS.

Example 7.9: Determine static errors of a GS due to rotation of the earth and friction in the axis of precession. Magnitude of the friction torque $K_x = |M_{fr\ x}| = 5$ g-cm; course of ship $K = 0^\circ$; latitude $\varphi = 0^\circ$. Parameters of the GS are taken from example 7.1. Horizontal correction of the GS has a mixed characteristic (Fig. 7.18b); range of the linear section of correction from $-\Phi$ to $+\Phi$ is 0.5°, i.e., angle $|\Phi| = 0.25^\circ$; maximum value L_x of the modulus of the moment of correction on a constant section of characteristic $L_x = 96$ g-cm.

Solution: 1. Steepness S_2 of the proportional section of the correction characteristic we determine by formula (2.3.11):

$$S_2 = \frac{L_x}{\Phi} = \frac{96 \cdot 57{,}3}{0{,}25} = 22\,000 \text{ g-cm/rad.}$$

2. By formula (7.4.13) we calculate specific velocity of correction ν_2:

$$\nu_2 = \frac{S_2}{2H} = \frac{22\,000}{44 \cdot 10^4} = 0{,}05 \text{ 1/sec.}$$

3. According to (7.4.15) we find the time constant of correction T:

$$T = \frac{1}{\nu_2} = \frac{1}{0{,}05} = 20 \text{ sec.}$$

4. Using (7.4.21), we calculate static error of the GS due to rotation of the earth:

$$\alpha_{cr}^{c} = -TU \cos\varphi \cos K = -20 \cdot 7{,}29 \cdot 10^{-5} \cdot 1 \cdot 1 = -1{,}458 \cdot 10^{-3} \text{ rad.} = -5',$$

5. By formula (6.3.19) we find coefficient η_3:

$$\eta_3 = \frac{K_f}{2H} = \frac{5}{44 \cdot 10^4} = 1{,}14 \cdot 10^{-5} \text{ 1/sec.}$$

6. According to (7.4.27) we find static error of the GS from friction along the axis of precession:

$$\alpha_{fr}^{T} = \pm \eta_3 T = 1{,}14 \cdot 10^{-5} \cdot 20 = \pm 22{,}8 \cdot 10^{-5} \text{ rad.} = \pm 0',8.$$

From example 7.9 it follows that error α_{stat}^{vel} can be substantial; since this error is systematic, in principle it can be compensated (§ 6.4, Par. 2b). The time constant of correction T considerably exceeds time constants T_{mot} and T_{amp} (example 7.1) of the GS control network.

§ 7.5. Dynamic Errors of a GS

1. General Characterization of Dynamic Errors of a GS

Accuracy of an active-type GS is found by means of investigation of its error in dynamics under conditions of continuously varying external disturbances, which normally have a random nature. Therefore, dynamic errors of a GS are random functions of time, for analysis of which it is necessary to apply probability methods. Consequently, calculation of dynamic errors of a GS should be reduced to determination of their probability characteristics.

Of greatest interest is finding of errors of the GS with respect to coordinate α, i.e., in the angle of rotation of the gyroframe about its axis of stabilization. However, for complete characterization of the behavior of a GS in dynamics it is also useful to determine possible values of angle β of rotation of the gyroscope about the axis of precession.

Dynamic errors of a GS basically include the same errors as for astatic gyroscopes of readout type with correction. If we take into account mainly disturbances along the axis of stabilization, then for the case considered earlier of location of it parallel to the longitudinal axis of a ship, subject to rolling, as such disturbances one should consider: 1) moments of inertia of the armature of the stabilizing motor during rolling of it through gear transmission (Fig. 7.2) by the frame during rolling of the ship; 2) friction torques in the axis of stabilization; 3) resisting torques (damping), caused by counter emf induced in the motor armature during rotation of the motor stator together with the deck of the ship during its rolling; 4) torques, transmitted to the frame through the system of horizontal correction and caused by rolling disturbances on the correcting pendulum.

Calculation of dynamic errors of a GS from the above-indicated disturbances we conduct in a linear plan, considering the law of control of the stabilizing motor to be linear, considering liquid friction torques in the suspension axes and taking a linear characteristic of the scheme of horizontal correction of the GS. When necessary we can find dynamic errors of a GS taking into account possible nonlinearities by the same methods as were used in Chapters 5 and 6.

2. Errors of a GS Without a Corrector

Let us first determine errors of GS which has no corrector, in conditions of irregular rolling of the ship; here we shall consider deflections α of the frame with respect to the axis of stabilization, and also angles β of rotation of the gyroscope about the axis of precession. Such errors of a GS we shall find during disturbances,

of indicated above in Par. 1 by figures 1-3.

During calculation of probability curves of errors α and β in dynamics we disregard small time constants T_{mot} and T_{amp} of GS control networks (example 7.1), since they usually very insignificantly affect forced oscillations of the GS. Furthermore, we shall not consider liquid friction torques in the axes of precession ($n_2 = 0$).

To determine probability curves of errors α and β in dynamics we use the general equations of motion (7.2.69) of a GS on an oscillating base, considering in them, in accordance with what was said above, that $n_2 = u_\eta = 0$, $S_2 = 0$, $T_{mot} = T_{amp} = 0$, $\chi_1 = 0$, $f_1 = f_2 = 0$. Then one can present these equations in form

$$\left.\begin{aligned} I_\eta\ddot{\alpha} + 2H\dot{\beta} + n_1\dot{\alpha} - Ki &= \xi\ddot{\theta} + n_1\dot{\theta} \\ 2J_{r.s}\dot{\beta} - 2H\dot{\alpha} &= 0 \\ U_g + S_1\beta &= 0 \\ i + b\dot{\alpha} - cU_g &= b\theta \end{aligned}\right\} \qquad (7.5.1)$$

Designating, by analogy with (7.3.10),

$$n = n_1 + bK, \quad m = S_1cK, \qquad (7.5.2)$$

we rewrite (7.5.1) in form

$$\left.\begin{aligned} I_\eta\ddot{\alpha} + 2H\dot{\beta} + n\dot{\alpha} + m\beta &= \xi\ddot{\theta} + n\dot{\theta} \\ 2J_{r.s}\dot{\beta} - 2H\dot{\alpha} &= 0 \end{aligned}\right\} \qquad (7.5.3)$$

After very simple transformations we present equations (7.5.3) in operational form $\left(p = \frac{d}{dt}\right)$:

$$\left.\begin{aligned} (2J_{r.s}I_\eta p^3 + 2J_{r.s}np^2 + 4H^2p + 2Hm)\alpha &= \\ = (2J_{r.s}\xi p^3 + 2J_{r.s}np^2)\theta & \\ (2J_{r.s}I_\eta p^3 + 2J_{r.s}np^2 + 4H^2p + 2Hm)\beta &= (2H\xi p^2 + 2Hnp)\theta \end{aligned}\right\} \qquad (7.5.4)$$

Applying to (7.5.4) the Laplace transform, we obtain expressions of transfer functions $\Phi_{\alpha,\theta}(s)$, and $\Phi_{\beta,\theta}(s)$ of the GS in form

$$\Phi_{\alpha,\theta}(s) = \frac{\alpha(s)}{\theta(s)} = \frac{2J_{r.s}(\xi s^2 + ns)}{2J_{r.s}I_\eta s^3 + 2J_{r.s}ns^2 + 4H^2s + 2Hm}; \qquad (7.5.5)$$

$$\Phi_{\beta,\theta}(s) = \frac{\beta(s)}{\theta(s)} = \frac{2H(\xi s^2 + ns)}{2J_{r.s}I_\eta s^3 + 2J_{r.s}ns^2 + 4H^2s + 2Hm}. \qquad (7.5.6)$$

Let us note that we can obtain formula (7.5.5) from the general expression (7.2.89) of transfer function $\Phi_{\alpha,\theta}(s)$, if in it we assume for the considered condition $n_1 + bK = n$, $S_2 = 0$, $n_2 = 0$.

By analogy with (7.3.25) we introduce designations

$$T_{\text{д.г}}^2 = \frac{2J_{\text{г.э}}J_{\text{г}}}{4H^2}; \quad 2\zeta T_{\text{д.г}} = \frac{2J_{\text{г.э}}n}{4H^2}; \quad \zeta = \frac{1}{2H}\sqrt{\frac{J_{\text{г.э}}}{2J_{\text{г}}}}\,n. \tag{7.5.7}$$

Then we can present (7.5.5) and (7.5.6) in form

$$\Phi_{\alpha,\theta}(s) = \frac{\frac{T_{\text{д.г}}^2}{J_{\text{г}}}(\xi s^3 + ns^2)}{T_{\text{д.г}}^2 s^3 + 2\zeta T_{\text{д.г}} s^2 + s + \frac{m}{2H}}; \tag{7.5.8}$$

$$\Phi_{\beta,\theta}(s) = \frac{\frac{1}{2H}(\xi s^2 + ns)}{T_{\text{д.г}}^2 s^3 + 2\zeta T_{\text{д.г}} s^2 + s + \frac{m}{2H}}. \tag{7.5.9}$$

We recall (§ 7.3, Par. 2b) that $T_{\text{мот.г}}$ — time constant of gyroscopes and the stabilizing motor, i.e., the reciprocal of the frequency ν [see (7.3.9)] of natural nutational oscillations of a free GS.

Let us find mathematical expectation $\bar{\alpha}$ and $\bar{\beta}$ of errors of the GS. By analogy with (4.6.135) we have

$$\bar{\alpha} = \Phi_{\alpha,\theta}(j\omega)\big|_{\omega=0}\,\bar{\theta}, \tag{7.5.10}$$

$$\bar{\beta} = \Phi_{\beta,\theta}(j\omega)\big|_{\omega=0}\,\bar{\theta}, \tag{7.5.11}$$

where $\Phi_{\alpha,\theta}(j\omega)$ and $\Phi_{\beta,\theta}(j\omega)$ — gain-phase responses of the GS, determined by means of replacement in (7.5.8) and (7.5.9) of s by $j\omega$.

Since $\bar{\theta} = 0$ [see (2.1.34)],

$$\bar{\alpha} = \bar{\beta} = 0. \tag{7.5.12}$$

Let us note that the same result is obtained when $\bar{\theta} \neq 0$, since, according to (7.5.8) and (7.5.9), $\Phi_{\alpha,\theta}(j\omega)|_{\omega=0} = 0$, $\Phi_{\beta,\theta}(j\omega)|_{\omega=0} = 0$, i.e., a GS without a corrector constitutes with respect to rolling disturbance $\theta(t)$ an astatic system.

Let us turn to determination of dispersions D[α], D[β] of random functions α(t), β(t); for this ditch we first find expressions of their spectral densities $S_\alpha(\omega)$ and $S_\beta(\omega)$. By analogy with (7.6.47) we have

$$S_\alpha(\omega) = |\Phi_{\alpha,\theta}(j\omega)|^2 S_\theta(\omega), \tag{7.5.13}$$

$$S_\beta(\omega) = |\Phi_{\beta,\theta}(j\omega)|^2 S_\theta(\omega), \tag{7.5.14}$$

where $S_\theta(\omega)$ — spectral density of angle of roll of the ship.

As an example we give the final expression for $S_\alpha(\omega)$. According to (7.5.8) we find the relationship for $|\Phi_{\alpha,\theta}(j\omega)|^2$:

$$|\Phi_{\alpha,\theta}(j\omega)|^2 = \frac{\frac{T_{\text{д.г}}^4}{J_{\text{г}}^2}(\xi^2\omega^6 + n^2\omega^4)}{T_{\text{д.г}}^4\omega^6 - 2T_{\text{д.г}}^2(1-2\zeta^2)\omega^4 + \left(1 - 4\zeta T_{\text{д.г}}\frac{m}{2H}\right)\omega^2 + \frac{m^2}{4H^2}}. \tag{7.5.15}$$

in which M_5 and Δ_5 are defined in coefficients (7.5.33) of general relationships (7.5.26) and (7.5.27).

For the mean quadratic value σ_β of angle β we have

$$\sigma_\beta = \sqrt{D[\beta]}.$$

Example 7.12: Calculate the mean quadratic σ_β and maximum β_m value of the angle of deflection of gyroscope with respect to the axis of precession for a GS, working in conditions of irregular rolling of a ship. Parameters of the GS and conditions of rolling are the same as in example 7.11.

Solution: 1. By formulas (7.5.34), (7.5.26), (7.5.27), and (7.5.33) we find dispersion D[β]:

$$D[\beta] = \frac{M_5}{2\alpha_0\Delta_5} = 0{,}1408.$$

2. We determine the mean quadratic value σ_β of the angle of deflection of the gyroscope:

$$\sigma_\beta = \sqrt{D[\beta]} = \sqrt{0{,}1408} = 0{,}3752 \text{ radians} = 21.5^\circ$$

3. For z = 3.3 (see example 7.11) we calculate the maximum value β_m:

$$\beta_m = z\sigma_\beta = 3{,}3 \cdot 21{,}5 = 70.95^\circ$$

The obtained value of β_m considerably exceeds quantities encountered in practice. This is connected with the fact that parameters of the GS adopted in the example are not the optimum one, corresponding to a real stabilizer.

An analysis of errors of a GS in conditions of irregular rolling of a ship, carried out by methods, close to those used above, is given in [130].

3. Errors of a GS with a Corrector

To determine dynamic errors of a GS with a corrector, besides the disturbances on the frame of the stabilizer in conditions of irregular rolling of ship considered in Par. 2, it is necessary also to take into account torques, transmitted to the frame through the system of horizontal correction and caused by rolling disturbances to the correcting-pendulum, or torques, caused by displacements of the centers of gravity of gyroscopes during use of other type of horizontal correction for the GS (§ 7.2, Par. 2b). Subsequently, we shall consider a scheme of horizontal correction of a GS using a correcting-pendulum, and we shall limit ourselves to determining error of the GS in coordinate α, i.e., in the angle of rotation of the gyroframe.

To find probability curves for error α in dynamics we use the general equations of motion (7.2.69) of a GS on an oscillating base. With the same assumptions as were shown in Par. 2, but for $S_2 \neq 0$ (the GS is equipped with horizontal correction),

disregarding time constant T_{pend} of the correcting-pendulum, it is possible to present these equations in form

$$\left.\begin{aligned} I_{\eta}\ddot{\alpha}+2H\dot{\beta}+n_1\dot{x}-Ki&=\xi\ddot{\theta}+n_1\dot{\vartheta}\\ 2J_{r.s}\ddot{\beta}-2H\dot{\alpha}-S_2x&=-S_2\chi_1\\ \chi_1&=-kz\ddot{\theta}\\ U_y+S_1\beta&=0\\ i+b\dot{x}-eU_y&=b\dot{\vartheta}\end{aligned}\right\} \qquad (7.5.35)$$

Introducing designations (7.5.2), we obtain

$$\left.\begin{aligned} I_{\eta}\ddot{\alpha}+2H\dot{\beta}+n\dot{x}+m_1\beta&=\xi\ddot{\theta}+n\dot{\theta}\\ 2J_{r.s}\ddot{\beta}-2H\dot{\alpha}-S_2x&=kzS_2\ddot{\theta}\end{aligned}\right\} \qquad (7.5.36)$$

from which it is easy to find the equation for coordinate α, which in operational form we can present as follows:

$$\begin{aligned}&\left[2J_{r.s}I_{\eta}p^4+2J_{r.s}np^3+4H^2p^2+2H(S_2+m)p+mS_2\right]\alpha=\\&=\left[2J_{r.s}\xi p^4+(2J_{r.s}n-2HkzS_2)p^3-kzmS_2p^2\right]\theta.\end{aligned} \qquad (7.5.37)$$

Applying the Laplace transform to (7.5.37), we obtain the expression of the transfer function of the GS, $\Phi_{\alpha,\theta}(s)$ in form

$$\Phi_{\alpha,\theta}(s)=\frac{2J_{r.s}\xi s^4+(2J_{r.s}n-2HkzS_2)s^3-kzmS_2s^2}{2J_{r.s}I_{\eta}s^4+2J_{r.s}ns^3+4H^2s^2+2H(S_2+m)s+mS_2}. \qquad (7.5.38)$$

Let us note that (7.5.38) can be obtained from the more general expression (7.2.89), if in it we set $n_2 = 0$ and consider designations (7.5.2). Transfer function (7.5.38) differs from (7.5.5) by the fact that in it there is considered horizontal correction of the GS. Actually, considering in (7.5.38) $S_2 = 0$, we obtain relationship (7.5.5).

By analogy with (7.5.10), we find that also with correction mathematical expectation $\overline{\alpha} = 0$. This is valid for the given case, when the mean value of the angle of displacement $\overline{\chi}_1$ of the correcting-pendulum in the plane of the frame is taken equal to zero.

Spectral density $S_{\alpha}(\omega)$ of random function $\alpha(t)$ will be determined by expression (7.5.13), if in it as the gain-phase response $\Phi_{\alpha,\theta}(j\omega)$ we use (7.5.38), replacing s by $j\omega$. For dispersion $D[\alpha]$ we have relationship (7.5.19)

$$D[\alpha]=2\int_0^{\infty}S_{\alpha}(\omega)\,d\omega=2\int_0^{\infty}|\Phi_{\alpha,\theta}(j\omega)|^2S_{\vartheta}(\omega)\,d\omega. \qquad (7.5.39)$$

This integral can be calculated by the same method as was applied earlier (Par. 2) in calculating dispersion of random motions of the GS. Letting the reader determine D[α] independently in the form of an exercise, we shall limit ourselves to consideration of random motions of a GS with a corrector, proceeding from precession theory. Eliminating in the second equation of (7.5.36) the inertial term $2J_{g.e}\ddot{\beta}$, we obtain the following differential equation for the angle α_1 of deflection of the frame due to rolling disturbance of the ship on the correcting pendulum:

$$2H\dot{\alpha}_1 + S_z\alpha_1 = -S_z kz\ddot{\theta}, \tag{7.5.40}$$

which permits us in the framework of precession theory to find in the first approximation the influence of random motions of the pendulum on deflections of the frame.

Considering the third equation of system (7.5.35), we rewrite (7.5.40) in form

$$2H\dot{\alpha}_1 + S_z\alpha_1 = S_z\chi_1(t), \tag{7.5.41}$$

where

$$\chi_1(t) = -kz\ddot{\theta}(t) \tag{7.5.42}$$

is a random function, characterizing oscillations of the correcting pendulum in the plane of a rib of a ship during rolling.

Taking into account designations (7.4.13), analogously to (7.4.15), we present (7.5.41) in the following manner:

$$T\dot{\alpha}_1 + \alpha_1 = \chi_1(t), \tag{7.5.43}$$

where T — time constant of horizontal correction of the GS.

Let us note that equation (7.5.43) is analogous to (7.4.16), if in the latter we set $u_n = 0$, $K_x = 0$, $f_2 = 0$, i.e., consider only motion of the gyroframe under the influence of disturbances of the correcting pendulum during rolling.

Let us determine the probability curves of random motions $\alpha_1(t)$ of the GS. Since $\bar{\theta} = 0$, then $\bar{\chi}_1 = 0$; then, according to (7.5.43), upon completion of the transient response,

$$\bar{\alpha}_1 = 0. \tag{7.5.44}$$

To find dispersion $D[\alpha_1]$ it is necessary to know transfer function $\Phi_{\alpha_1,\chi_1}(s)$ in coordinate α_1 with respect to disturbance $\chi_1(t)$. Proceeding from (7.5.43), by analogy with (7.4.17), we have

$$\Phi_{\alpha_1\chi_1}(s) = \frac{1}{Ts+1}, \tag{7.5.45}$$

i.e., in this case the GS is considered an aperiodic network.

The corresponding gain-phase response $\Phi_{\alpha_1,\chi_1}(j\omega)$ will be

$$\Phi_{\alpha_1,\chi_1}(j\omega)=\frac{1}{Tj\omega+1}. \tag{7.5.46}$$

Spectral density $S_{\alpha_1}(\omega)$ of random function $\alpha_1(t)$ will be determined by relationship

$$S_{\alpha_1}(\omega)=|\Phi_{\alpha_1,\chi_1}(j\omega)|^2 S_{\chi_1}(\omega). \tag{7.5.47}$$

For $|\Phi_{\alpha_1,\chi_1}(j\omega)|^2$, according to (7.5.46), we have formula

$$|\Phi_{\alpha_1,\chi_1}(j\omega)|^2=\frac{1}{T^2\omega^2+1}. \tag{7.5.48}$$

Spectral density $S_{\chi_1}(\omega)$, considering (7.5.42) and (2.1.84), we can present in form

$$S_{\chi_1}(\omega)=k^2z^2S_{\eta}(\omega)=k^2z^2\omega^4S_{\eta}(\omega) \tag{7.5.49}$$

or, according to (2.1.55) and (2.1.56),

$$S_{\chi_1}(\omega)=\frac{2A_2\mu_2}{\pi}\frac{k^2z^2b_2^2\omega^4}{\omega^4+2a_2\omega^2+b_2^4}, \tag{7.5.50}$$

where a_2 and b_2^2 are determined by relationships (7.5.17).

Substituting (7.5.48) and (7.5.50) in (7.5.47), we obtain

$$S_{\alpha_1}(\omega)=\frac{1}{T^2\omega^2+1}\frac{2A_2\mu_2}{\pi}\frac{k^2z^2b_2^2\omega^4}{\omega^4+2a_2\omega^2+b_2^4}. \tag{7.5.51}$$

Let us find the expression for dispersion $D[\alpha_1]$:

$$D[\alpha_1]=2\int_0^{\infty}S_{\alpha_1}(\omega)\,d\omega \tag{7.5.52}$$

or, taking into account (7.5.51)

$$D[\alpha_1]=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{T^2\omega^2+1}\frac{B_2\omega^4}{\omega^4+2a_2\omega^2+b_2^4}d\omega, \tag{7.5.53}$$

where

$$B_2=4A_2\mu_2k^2z^2b_2^2, \tag{7.5.54}$$

b_2^2 is replaced by b_2, and, according to (7.5.17),

$$b_2^2=\mu_2^2+\lambda_2^2. \tag{7.5.55}$$

Determination of $D[\alpha_1]$ can be reduced to calculation of a tabular integral of form (4.6.71). For this we present (7.5.53) in the following manner:

$$D[\alpha_1]=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{B_2\omega^4}{|(Tj\omega+1)(\omega^2-2\mu_2j\omega-b_2^2)|^2}d\omega. \tag{7.5.56}$$

Integral (7.5.56) can be reduced to form (4.6.71), if in (7.5.56) we consider

the numerator (multiplied by j) equal to $G_n(\omega)$, and the term in the denominator of this expression under the sign of the modulus, equal to $H_n(\omega)$. Then, in accordance with (4.6.72) and (7.5.56), we have for coefficients a_0 ... and b_0 ... the following formulas:

$$\left.\begin{aligned} a_0 &= Tj; & b_0 &= B_0 j \\ a_1 &= 1 + 2\mu_2 T; & b_1 &= 0 \\ a_2 &= -\left(2\mu_2 + b_2^2 T\right) j; & b_2 &= 0 \\ a_3 &= -b_2^2; \end{aligned}\right\} \tag{7.5.57}$$

Considering the known expression of integral I_n [153] in coefficients a_0, ..., a_3 and b_0, ..., b_2, we obtain a formula for $D[\alpha_1]$ in form

$$D[\alpha_1] = \frac{-a_2 b_0 + a_0 b_1 - \frac{a_0 a_1 b_2}{a_3}}{2a_0(a_0 a_3 - a_1 a_2)}. \tag{7.5.58}$$

Substituting (7.5.57) in (7.5.58) and taking into account (7.5.54) and (7.5.55), we finally obtain

$$D[\alpha_1] = \frac{A_2 \mu^2 x^2 \left(\mu_2^2 + \lambda_2^2\right)\left[2\mu_2 + \left(\mu_2^2 + \lambda_2^2\right) T\right]}{T\left[\lambda_2^2 T^2 + \left(1 + \mu_2 T\right)^2\right]}. \tag{7.5.59}$$

For the mean quadratic value of error of the GS we have

$$\sigma_{\alpha_1} = \sqrt{D[\alpha_1]}. \tag{7.5.60}$$

Example 7.13: Calculate the mean quadratic value σ_{α_1} of error of a GS with horizontal correction, resulting from disturbances from rolling of the ship on the correcting-pendulum, for the initial data of examples 7.1, 7.2 and 7.10; coefficient $k = 1/g = 0.102$ sec^2/m, $z = -1$ m; the time constant of correction $T = 20$ sec. (example 7.9). Determine also maximum error α_{1m}, assuming that the probability Q_z that the current angle α_1 will exceed the limit α_{1m} is no greater than 0.1%.

Solution: 1. By formula (7.5.59) we find dispersion $D[\alpha_1]$:

$$D[\alpha_1] = 0{,}1731 \cdot 10^{-6}.$$

2. According to (7.5.60), we determine the mean quadratic error σ_{α_1} of the GS:

$$\sigma_{\alpha_1} = \sqrt{D[\alpha_1]} = \sqrt{0{,}1731 \cdot 10^{-6}} = 0{,}4161 \cdot 10^{-3} \text{ radians} = 1.4'.$$

3. For $Q_z = 0.1\%$ by Table 1 (§ 4.6, Par. 10) we find $z = 3.3$; by analogy with (4.6.87), we calculate the maximum value of GS error ($\bar{\alpha}_1 = 0$):

$$\alpha_{1m} = z\sigma_{\alpha_1} = 3{,}3 \cdot 1{,}4 = 4.6'$$

From example 7.13 it follows that error α_1 of the GS, caused by disturbances of the correcting-pendulum during rolling, is comparatively small. This circumstance

is connected with the great angular momentum of gyroscopes, usually applied in active GS's, which causes a sizeable time constant T for correction.

4. Certain Remarks

Above we considered dynamic errors of a GS under the influence of random disturbances. Sometimes it is necessary to find errors of a GS under disturbances, fairly close to sinusoidal. In this case they can be determined by methods developed in automatic control theory; here, they used expressions of transfer functions of the GS with respect to disturbances, which were obtained earlier. Thus, for instance, during determination of error α of a GS without a corrector arising from rolling of a ship, varying by sine wave law $\theta = \theta_0 \sin \omega t$, we obtain, on the basis of known relationships of automatic control theory [107],

$$\alpha(t) = C_0\theta(t) + C_1\dot{\theta}(t) + \frac{C_2}{2!}\ddot{\theta}(t) + \ldots + \frac{C_{m-1}}{(m-1)!}\theta^{(m-1)}(t), \tag{7.5.61}$$

where C_0, C_1, ..., C_{m-1} are the so-called coefficients of errors, which, in the considered case of finding errors of a GS from rolling, are expressed through derivatives of the transfer function $\Phi_{\alpha,\theta}(s)$ of the GS by relationships of form [107]

$$C_i = \left[\frac{d^i\Phi_{\alpha,\theta}(s)}{ds^i}\right]_{s=0}. \tag{7.5.62}$$

where $\Phi_{\alpha,\theta}(s)$ is determined by formula (7.5.5).

When we only know limits of change of the external disturbances, but it is impossible to make any assumptions about their actual law of change, we are limited to calculating the maximum (accumulated) deflection of the GS. Such problems are considered by Ya. N. Roytenberg [132] and N. T. Kuzovkov [80a]; solution of these problems is based on the method of calculating maximum possible deflection (accumulated deflection) occurring with the most unfavorable law of change of external disturbances limited in modulos, offered by B. V. Bulgakov [19]. Those wishing to become acquainted with methods of calculation of deflection of a GS for external disturbances limited in modulus are directed to the indicated works.

An analysis of errors of a GS which possesses elastic compliance of elements during nonfulfillment of the condition of equal rigidity (§ 5.7, Par. 4), in conditions of vibrations of the base was made by E. I. Sliv [135].

§ 7.6. Application of Active-Type GS's

1. General Remarks

Active GS's, the theory of which was expounded in the preceding sections, are widely applied in different GD's on ships and aircraft. They are used for direct stabilization of separate instruments and devices. Certain types of directional gyroscopes, vertical gyros and combined devices, directional gyrohorizons (directional gyroverticals) operate on the principle of active gyroscopic stabilization. In these devices there are applied single, double, and triple-axis active GS's and are correspondingly one, two or three gyro-frames with one or two gyros each.

Single-axis GS's can be used as directional gyroscopes or indicators of one of the two angles of deflection of a vehicle with respect to the plane of the horizon, and also for direct stabilization relative to one axis of instruments and devices connected to them. Double-axis GS's can be used as a gyro vertical or for direct stabilization of a certain platform with respect to the plane of the horizon. Triple-axis GS's can be applied for determination of three angles of rotation of a vehicle about its center of gravity, i.e., as a directional gyrohorizons and also for direct stabilization of a plane with respect to the plane of the horizon and in heading.

As examples of GS's we shall briefly consider directional gyroscopes, gyroverticals and directional gyroverticals of active type.

2. Directional Gyro of Active-Type

a) Schematic of the Instrument.

A directional gyroscope of active type has the same function as the readout DG (§ 5.1). As was shown earlier, in an active-type DG there is used a single-axis GS with one or two gyros. During analysis of a DG of readout type it was found that for decrease of errors it is advisable to set the DG on a base stabilized with respect to the plane of the horizon; this is also usually done in DG's of active type. However, it is possible to indicate a series, e.g., of aircraft devices [111], in which the DG is placed on a mounting, rigidly joined with the aircraft.

In Fig. 7.25 is a schematic of a DG consisting of a one-gyro active GS, analogous to that shown in Fig. 7.1. If such a DG is fixed on a mounting, rigidly joined with the vehicle (ship, aircraft), then as the signal pickoff SP there usually is used a physical pendulum with a pickoff (contact attachment, potentiometer, inductive pickoff), which through amplifier Amp switches on stabilizing motor SM. With installation of a DG on a base stabilized with respect to the plane of the horizon,

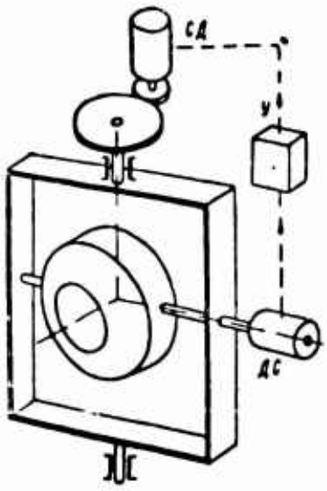

Fig. 7.25. Schematic of a DG (one-gyro GS).

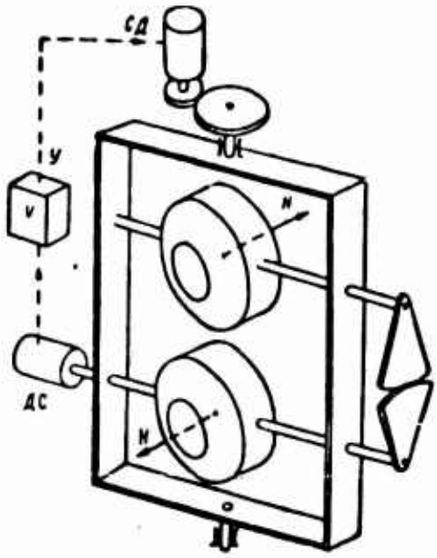

Fig. 7.26. Schematic of a DG (two-gyro GS).

application of a physical pendulum becomes unnecessary and the signal pickoff can be a sensor, fixing deflection of the axis of the gyroscope from the perpendicular to the plane of the frame. It is necessary to note that an active-type DG can also be equipped with a system of azimuth correction, similar to that applied in DG of readout type (Fig. 5.1).

In Fig. 7.26 is a schematic of a DG, consisting of a two-gyro active GS, analogous to that shown in Fig. 7.2. Instead of an anti-parallelogram for such a DG axes of precession of the gyroscopes are connected by gear quadrants. The two-gyro DG, as compared to the single-gyro, possesses the advantage of a two-gyro active GS, shown earlier.

b) Certain Problems of Theory.

The theory of a single-axis active GS, applied in a DG, was considered in detail in the preceding sections of the present chapter. A number of propositions of the theory of readout DG's (Chapter 5) are valid also for active DG's; therefore, here we shall only consider certain problems of the theory of active DG's.

When the DG is located directly on a rolling ship and during work of horizontal correction, the axis of the gyroscope acquires accumulated azimuth deflection (§ 5.7, Par. 6), the magnitude of which may be considerable (example 5.20). If, however, a DG of readout type or one based on a one-gyro active GS is set on a base, stabilized with respect to the plane of horizon, the shown effect of azimuth drift of the axis of the gyroscope does not completely disappear; here, gyro drift will be caused by errors of stabilization of the base. It is interesting to find the magnitude of azimuth drift of a DG, caused by random errors of stabilization of the base.

Let us assume that a one-gyro active DG (Fig. 7.25) is fixed on a plane, assigned by a gyrovertical and stabilized with respect to the plane of the horizon. In the initial moment axis Oz of the gyroscope (Fig. 7.27) forms angle α_0 with the spin axis of the outer gimbal of the platform suspension, coinciding with the longitudinal

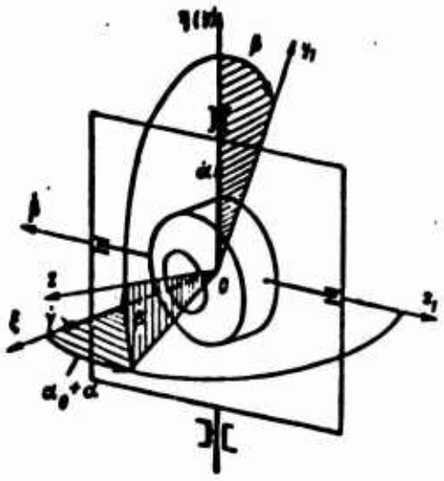

Fig. 7.27. Determining azimuth drift of a DG.

axis Oξ of the ship. Consequently, we shall be interested in azimuth drift of axis Oz of gyroscope relative to its initial direction, determined by angle α_0, which is not small.

During solution of the problem at hand we shall limit ourselves to consideration of the particular case when the gyrovertical, and consequently also the stabilized plane have error in the plane of the ship's ribs, characterized by random function $\gamma(t)$ with mathematical expectation $\overline{\gamma} = 0$. Correlation function $K_\gamma(\tau)$ of random function $\gamma(t)$ is determined by relationship

$$K_\gamma(\tau) = \sigma_\gamma^2 e^{-\mu|\tau|}\left(\cos\lambda\tau + \frac{\mu}{\lambda}\sin\lambda|\tau|\right). \qquad (7.6.1)$$

where σ_γ^2 — dispersion of error of the gyrovertical (plane);

μ — coefficient of irregularity of error;

λ — predominant frequency of change of error, taken subsequently as equal to the frequency of natural rolling of the ship.

To show the accumulated azimuth drift of the axis of the DG, caused by errors of platform stabilization, the influence of which appears during work of the unloading system of the DG, we shall limit ourselves to investigation of the basic precessional motion, using, by analogy with (5.7.129), shortened equations

$$Hq_1 = M_{x_1}; \quad -Hp_1 = M_{y_1}. \qquad (7.6.2)$$

When determining p_1 and q_1 one should consider the induced angular velocity $\dot{\gamma}$ of oscillations of the platform, stabilized from the gyrovertical, and also components $\dot{\alpha}$ and $\dot{\beta}$ of the angular velocity of motion of the gyroscope with respect to this plane; according to Fig. 7.27 we have

$$\left.\begin{aligned} p_1 &= -\dot{\beta} - \dot{\gamma}\sin(\alpha_0 + \alpha) \\ q_1 &= \dot{\alpha}\cos\beta - \dot{\gamma}\cos(\alpha_0 + \alpha)\sin\beta \end{aligned}\right\} \qquad (7.6.3)$$

Considering angles α and β small, we obtain*

$$\left.\begin{aligned} p_1 &= -\dot{\beta} - \dot{\gamma}\sin\alpha_0 \\ q_1 &= \dot{\alpha} - \dot{\gamma}\beta\cos\alpha_0 \end{aligned}\right\} \qquad (7.6.4)$$

*By analogy with the solution of the corresponding problem for a readout DG (§ 5.7, Par. 6) during determination of α for a DG of power type we shall limit ourselves to investigation of equations of the first approximation, which are obtained on the assumption that $\alpha_0 \gg \alpha$.

Introduce (7.6.4) in (7.6.2); then

$$\left.\begin{aligned} H\left(\dot{\alpha}-\dot{\gamma}\beta\cos\alpha_0\right) &= M_{x_1} \\ -H\left(-\dot{\beta}-\dot{\gamma}\sin\alpha_0\right) &= M_{y_1} \end{aligned}\right\} \tag{7.6.5}$$

As also in examining the analogous question for a DG of readout type (§ 5.7, par. 6), we take $M_{x_1} = 0$, and in M_{y_1} we consider only stabilizing torque M_s, applied by the stabilizing motor. Then for M_s, considering that in an active DG, fixed on a stabilized platform, without a correcting, pendulum, for a linear law of control of the stabilizing motor we obtain

$$M_{y_1} = M_s = -S\beta. \tag{7.6.6}$$

Introducing (7.6.6) in (7.6.5) and considering $M_{x_1} = 0$, we have:

$$\left.\begin{aligned} \dot{\alpha} &= \dot{\gamma}\beta\cos\alpha_0 \\ \dot{\beta}+\dot{\gamma}\sin\alpha_0 &= -\frac{S}{H}\beta \end{aligned}\right\} \tag{7.6.7}$$

Considering (2.3.5) and (2.3.7), we rewrite (7.6.7) in form

$$\left.\begin{aligned} \dot{\alpha} &= \dot{\gamma}\beta\cos\alpha_0 \\ T\dot{\beta}+\beta &= -T\dot{\gamma}\sin\alpha_0 \end{aligned}\right\} \tag{7.6.8}$$

where T is the time constant of correction.

Let us present the error of platform stabilization, i.e., disturbance $\gamma(t)$, in the form of a spectral expansion; by analogy with (4.6.149), we have

$$\gamma(t) = \int_{-\infty}^{\infty} e^{j\omega t}\, d\Phi_\gamma(\omega). \tag{7.6.9}$$

Introducing (7.6.9) in the second equation of (7.6.8), we obtain upon completion of the transient response a steady-state solution of this equation in the form

$$\beta(t) = \int_{-\infty}^{\infty} W(j\omega)\, e^{j\omega t}\, d\Phi_\gamma(\omega), \tag{7.6.10}$$

where $W(j\omega)$ is the gain-phase response of a DG with a system of active unloading.

According to the second equation of (7.6.8), for transfer function W(s) we have

$$W(s) = \frac{\beta(s)}{\gamma(s)} = -\frac{Ts}{Ts+1}\sin\alpha_0, \tag{7.6.11}$$

from which

$$W(j\omega) = -\frac{Tj\omega}{Tj\omega+1}\sin\alpha_0; \tag{7.6.12}$$

substituting (7.6.12) in (7.6.10), we obtain

$$\beta(t) = -\int_{-\infty}^{\infty} \frac{Tj\omega}{Tj\omega+1}\sin\alpha_0 e^{j\omega t}\, d\Phi_\gamma(\omega). \tag{7.6.13}$$

Let us consider now the first equation of system (6.6.8),

$$\dot{\alpha} = \dot{\gamma}\beta\cos\alpha_0. \tag{7.6.14}$$

We are interested in "system" drift of the DG in azimuth, the rate of which $\bar{\dot{\alpha}}$ is the mathematical expectation of random function $\dot{\alpha}(t)$. Applying to (7.6.14) the operation of mathematical expectation, we obtain

$$\bar{\dot{\alpha}} = \overline{\dot{\gamma}\beta}\cos\alpha_0. \tag{7.6.15}$$

Let us present $\dot{\gamma}(t)$ in the form of a spectral expansion:*

$$\dot{\gamma}(t) = \int_{-\infty}^{\infty} e^{j\omega t}\, d\Phi_{\dot{\gamma}}(\omega). \tag{7.6.16}$$

Let us find $\overline{\dot{\gamma}\beta}$; considering (7.6.13) and (7.6.16), we have

$$\overline{\dot{\gamma}(t)\beta(t)} = -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-j\omega t}\, d\Phi_{\dot{\gamma}}^{*}(\omega)\,\frac{Tj\omega_1}{Tj\omega_1+1}\sin\alpha_0 e^{j\omega_1 t}\, d\Phi_{\gamma}(\omega_1), \tag{7.6.17}$$

where magnitude $d\Phi_{\dot{\gamma}}^{*}(\omega)$ is complex-conjugate with $d\Phi_{\dot{\gamma}}(\omega)$.

But, by (4.6.150)

$$\overline{d\Phi_{\dot{\gamma}}^{*}(\omega)\, d\Phi_{\gamma}(\omega_1)} = S_{\dot{\gamma}\gamma}(\omega)\,\delta(\omega-\omega_1)\, d\omega\, d\omega_1, \tag{7.6.18}$$

where $S_{\dot{\gamma}\gamma}(\omega)$ – mutual spectral density of random functions $\dot{\gamma}(t)$ and $\gamma(t)$;

$\delta(\omega - \omega_1)$ – delta-function.

Then, instead of (7.6.17), considering also expressions of type (4.6.151), we obtain

$$\overline{\dot{\gamma}(t)\beta(t)} = -\int_{-\infty}^{\infty}\frac{Tj\omega}{Tj\omega+1}\sin\alpha_0 S_{\dot{\gamma}\gamma}(\omega)\, d\omega. \tag{7.6.19}$$

To determine $S_{\dot{\gamma}\gamma}(\omega)$ we first find the cross-correlation function $K_{\dot{\gamma}\gamma}(\tau)$. By analogy with (2.1.86), considering (2.1.67), we have**

$$K_{\dot{\gamma}\gamma}(\tau) = \sigma_{\gamma}^2\frac{\mu^2+\lambda^2}{\lambda}e^{-\mu|\tau|}\sin\lambda\tau. \tag{7.6.20}$$

---

*Let us note that $\dot{\gamma}(t)$, instead of by (7.6.16), can be presented in the form $\dot{\gamma}(t) = \int_{-\infty}^{\infty} e^{j\omega t}\, j\omega\, d\Phi_{\gamma}(\omega)$; then, instead of (7.6.19), we obtain $\overline{\dot{\gamma}(t)\beta(t)} = \int_{-\infty}^{\infty}\frac{T\omega^2}{Tj\omega+1}\sin\alpha_0 S_{\gamma}(\omega)\, d\omega$ and then formula (7.6.24), which is found from expansion (7.6.16). Application of formula (7.6.16) has as its purpose demonstration of how the mutual spectral density $S_{\dot{\gamma}\gamma}(\omega)$ is determined.

**In writing formula (7.6.20) we considered, for instance, that $K_{\dot{x}x}(-\tau) = K_{\dot{x}x}(-\tau)$ [153], and the relationship for $K_{\dot{x}\dot{x}}(\tau)$ [with correction in (2.1.87)] has the form $K_{\dot{x}\dot{x}}(\tau) = -A\frac{b^2}{\lambda}e^{-\mu|\tau|}\sin\lambda\tau$, where $A = \sigma_{\gamma}^2$.

Substituting (7.6.20) in (2.1.42), after transformations we obtain an expression for $S_{\dot{\gamma}\dot{\gamma}}(\omega)$ in the form

$$S_{\dot{\gamma}\dot{\gamma}}(\omega) = \frac{\sigma_\gamma^2(\mu^2+\lambda^2)}{2\pi\lambda}\left[\frac{\omega+\lambda}{\mu^2+(\omega+\lambda)^2} - \frac{\omega-\lambda}{\mu^2+(\omega-\lambda)^2}\right]. \tag{7.6.21}$$

Introducing (7.6.21) in (7.6.19), we find

$$\overline{\dot{\gamma}(t)\beta(t)} = -\int_{-\infty}^{\infty} \frac{\sigma_\gamma^2(\mu^2+\lambda^2)}{2\pi\lambda} \cdot \frac{T/\omega}{T/\omega+1}\sin\alpha_0 \times$$
$$\times\left[\frac{\omega+\lambda}{\mu^2+(\omega+\lambda)^2} - \frac{\omega-\lambda}{\mu^2+(\omega-\lambda)^2}\right]d\omega. \tag{7.6.22}$$

Using for calculation of the obtained integral methods of the theory of residues, we have

$$\overline{\dot{\gamma}(t)\beta(t)} = \sigma_\gamma^2(\mu^2+\lambda^2)\sin\alpha_0\frac{T}{(1+\mu T)^2+\lambda^2T^2}. \tag{7.6.23}$$

Substituting (7.6.23) in (7.6.15), we finally obtain

$$\bar{\dot{\alpha}} = \frac{1}{2}\sigma_\gamma^2(\mu^2+\lambda^2)\frac{T}{(1+\mu T)^2+\lambda^2T^2}\sin 2\alpha_0. \tag{7.6.24}$$

Let us note that formula (7.6.24) has generally the same form as the expression for "system" drift of a readout DG during rolling of a ship, determined by relationship (5.7.149). The basic difference is that in the considered case the $\bar{\dot{\alpha}}$ is proportion to the dispersion $D[\gamma] = \sigma_\gamma^2$ of error of stabilization of the GV platform, on which is the DG fixed, whereas for a DG, placed directly on a rolling ship, $\bar{\dot{\alpha}}$ is proportional to dispersion $D[\theta]$ of the angle of roll of the ship. In both cases the maximum drift rate of the DG axis in azimuth occurs at $\alpha_0 = 45^\circ$.

Example 7.14: Calculate the mathematical expectation $\bar{\dot{\alpha}}$ of azimuth drift of a DG of active type with a linear law of control of the stabilizing motor, fixed on a gyrovertical platform, stabilized with respect to the plane of the horizon. In the initial moment the axis of the gyroscope forms an angle $\alpha_0 = 45^\circ$ with the spin axis of the outer gimbal of the platform suspension, coinciding with the longitudinal axis of the ship; the mean quadratic value of error of the gyrovertical $\sigma_\gamma = 10'$; the coefficient of irregularity of error $\mu = 0.1$ 1/sec; predominant frequency of change of error $\lambda = 0.42$ 1/sec; the time constant of correction $T = 0.1$ sec.

Solution: By formula (7.6.24) we find $\bar{\dot{\alpha}}$:

$$\bar{\dot{\alpha}} = \frac{1}{2}\sigma_\gamma^2(\mu^2+\lambda^2)\frac{T}{(1+\mu T)^2+\lambda^2T^2}\sin 2\alpha_0 = 0{,}5\cdot 0{,}8462\cdot 10^{-5}(0{,}01+0{,}1764)\times$$
$$\times\frac{0{,}1}{(1+0{,}1\cdot 0{,}1)^2+0{,}1764\cdot 0{,}01}\cdot 1 = 0{,}07718\cdot 10^{-6}\ \text{rdn/sec}$$
$$= 0{,}016 \text{ angular min/min}$$

From example 7.14 it follows that $\bar{\dot{\alpha}}$ is minute and considerably less than the value of $\bar{\dot{\alpha}}$, which occurs for a DG, fixed directly on a rolling ship (example 5.20). Consequently, from the considered point of view positioning of the DG, on a base stabilized with respect to the plane of the horizon, is very advisable.

There arises the question of the possibility of total elimination of DG drift $\bar{\dot{\alpha}}$ caused by oscillations of the base. It has been shown [104] that this can be carried out by application in the DG of a two-gyro stabilizer. Actually, in reference to a two-gyro GS for components of angular velocities of the first and second gyros. instead of (7.6.4), we obtain

$$\left.\begin{aligned} p_1' &= -\dot{\beta} - \dot{\gamma}\sin\alpha_0; \quad q_1' = \dot{\alpha} - \dot{\gamma}\beta\cos\alpha_0 \\ p_1'' &= -\dot{\beta} + \dot{\gamma}\sin\alpha_0; \quad q_1'' = \dot{\alpha} + \dot{\gamma}\beta\cos\alpha_0 \end{aligned}\right\} \tag{7.6.25}$$

For $M_{y_1} = M_s$, by analogy with (7.6.6), we have

$$M_{y_1} = M_e = -S\beta. \tag{7.6.26}$$

Substituting (7.6.25) and (7.6.26) in (7.6.2) and considering that both gyroscopes are connected by a coupling, we find the equations of motion of a two-gyro DG $\left(T = \frac{2H}{S}\right)$

$$\left.\begin{aligned} 2H\dot{\alpha} &= 0 \\ T\dot{\beta} + \beta &= 0 \end{aligned}\right\} \tag{7.6.27}$$

differing from equations (7.6.8) of a one-gyro DG.

From the first equation of (7.6.27) it follows that $\bar{\dot{\alpha}} = 0$ q.e.d.

3. Gyroverticals of Active Type

a) Schematic of the Instrument.

A GV of active type in principle is designed for the same purposes, as a GV of readout type (§ 6.1, Par. 1). As shown in Par. 1 of this section, an active GV is based on use of a double-axis GS.

Depending upon the number of applied gyroscopes we distinguish two-gyros and four-gyro GV's. In Fig. 7.28 is a schematic of a two-gyro active GV [17,165]. Stabilized platform P, consisting of the inner gimbal of a Cardan suspension, is fixed in the outer gimbal ring R, whose spin axis is directed parallels, for instance, to the longitudinal axis of the vehicle. On platform P there are located two one-gyro frames with gyroscopes Gyro_1 and Gyro_2. On the axes of precession of the gyros there are fixed signal pickoffs SP_1 and SP_2, which through amplifiers Amp_1 and Amp_2 control, correspondingly, stabilizing motors SM_1 and SM_2, connected by mechanical transmissions with axes of the gimbals of the GV suspension. Consequently, disturbing

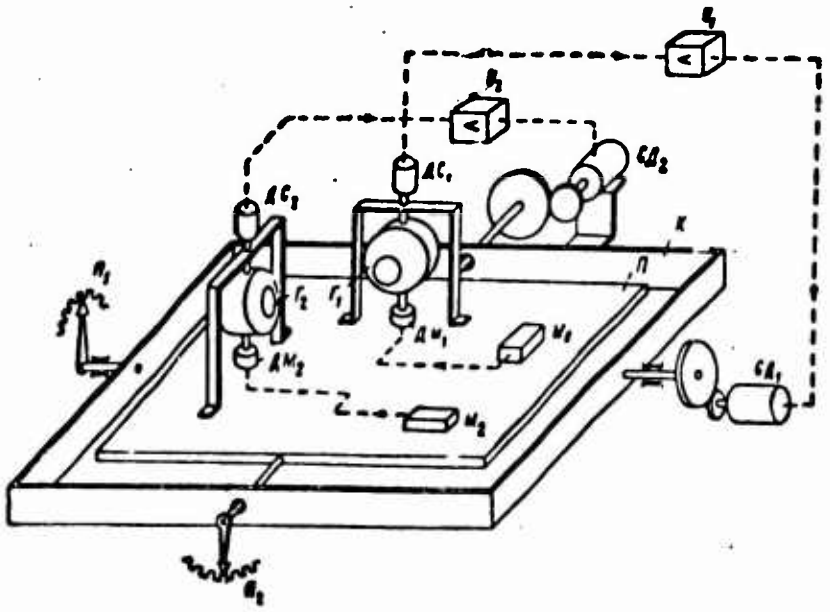

Fig. 7.28. Schematic of a two-gyro active GV.

torques to the shown axes are compensated by torques, applied by the stabilizing motors. To keep platform P in a horizontal position there is applied a system of pendulum correction; here pendulums $Pend_1$ and $Pend_2$, fixing deflections of the platform from the horizontal plane, control torquers TQ_1 and TQ_2, fixed on the axes of precession of gyroscopes $Gyro_1$ and $Gyro_2$; the principle of action of correction we assume is known. Angles of deflections of the vehicle relative to the horizontal plane are removed in the form of corresponding voltages from potentiometers Pot_1 and Pot_2.

Since the considered GV is a combination of two one-gyro activated frames, then with the horizontal location of axes of the gyroscopes shown in Fig. 7.28 work of the system of active-type stabilization will be influenced by rotation of the vehicle about the vertical axis, for instance, turn of the aircraft. This deficiency can be partially eliminated by setting in the two-gyro active GV axes of gyroscopes in vertical position [17, Fig. 468]; however, even in this case during deflection of the axes of the gyroscopes from vertical position the influence e.g., turn of the aircraft on work of the GV will appear. This deficiency of a two-gyro vertical with active stabilization is removed in a four-gyro active GV.

In Fig. 7.29 is a schematic of a four-gyro active GV, developed and investigated in detail by B. V. Bulgakov and Ya. N. Roytenberg [22, 20]. The arrangement of this GV differs from the two-gyro GV (Fig. 7.28) by the fact that on stabilized platform P there are placed two gyroframes, each of which consists of two gyros $Gyro_1$ and $Gyro_2$, and $Gyro_3$ and $Gyro_4$, and axes of precession of the gyroscopes are connected, correspondingly, anti-parallelograms (couplings) A_1 and A_2. On the axes of precession of gyroscopes $Gyro_2$ and $Gyro_4$ there are fixed signal pickoffs SP_1 and Sp_2, which through amplifiers Amp_1 and Amp_2 control stabilizing motors SM_1 and SM_2. The system of correction for platform P with respect to the horizontal plane consists of pendula $Pend_1$ and $Pend_2$ and torquers TQ_1 and TQ_2 controlled by them, fixed on the axes of precession of gyroscopes $Gyro_1$ and $Gyro_3$. Angles of deflections of the vehicle with respect to the horizontal plane are removed in the form of corresponding voltages

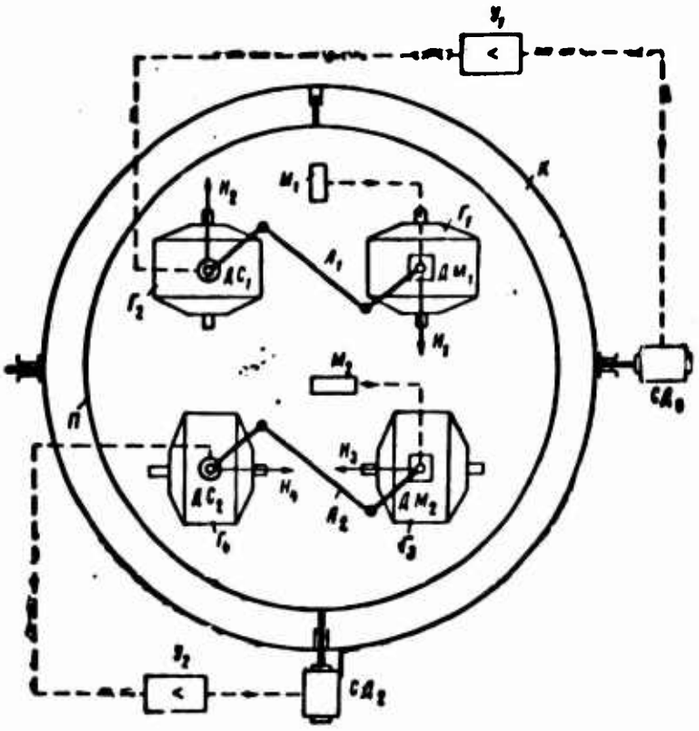

Fig. 7.29. Schematic of a four-gyro active GV.

from potentiometers or other pick-offs, located on the axes of the Cardan suspension of the GV (these are not shown in Fig. 7.29).

The four-gyro active vertical has essential merits in comparison with the two-gyro vertical. In it in principle one can compensate ballistic deviations, caused by vehicle accelerations. The four-gyro vertical possesses the same known advantages during work on an oscillating base.

The given schematics do not exhaust possible variants of active GV's. Of the four-gyro GV's one should, in particular, indicate the so-called multigyrovertical [127, 135, 169]. The fundamental distinction of this GV is that to give it selectivity with respect to the vertical in it, instead of a correction system consisting of pendula and torque devices (Fig. 7.29), there is carried displacement of the center of gravity of the whole system (Cardan suspension and gyroscopes) with respect to the point of suspension and, besides, there is introduced elastic linkage of the gyroscopes with the platform. Thus, the considered GV is a spherical pendulum, i.e., belongs to gyro-pendulum verticals (Chapter 11).

Let us turn to consideration of certain problems of the theory of active GV's. Here, for generality we shall consider the four-gyrovertical whose schematic is shown in Fig. 7.29.

b) Elements of the Theory of a Four-Gyro Active GV.

Equations of motion. To define the position of the GV we select as system of reference of axes $O\xi\eta\zeta$ (Fig. 1.6), which we assume are connected with the path of the vehicle, e.g., of a ship. Resal's axes Ox_1y_1z we connect with the platform of the GV, and point E may be called the vertex of the platform. The position of axes Ox_1y_1z, i.e., of the platform, with respect to axes $O\xi\eta\zeta$ is determined by angles α and β, where α is the angle of turn of external Cardan ring R (Fig. 7.29), and β is the angle of rotation of platform P. Transformations of axes Ox_1y_1z to $O\xi\eta\zeta$

is characterized in the case of small α and β by matrix (1.1.31). With the housings of gyroscopes $Gyro_1$-$Gyro_4$ (Fig. 7.30) we connect, respectively, axes $O_i x_i y_i z_i$ ($i = 1, 2, 3, 4$). Position of gyroscopes $Gyro_1$ and $Gyro_2$ with respect to the platform, i.e., of axes Ox_1y_1z (Fig. 7.30), is determined by the angle δ of rotation of these gyroscopes about their axes of precession O_1z_1 and O_2z_2. Likewise the position of gyroscope $Gyro_3$ and $Gyro_4$ with respect to the platform is determined by the angle γ of rotation of them about axes of precession O_3z_3 and O_4z_4. Angles α, β, δ, γ one should take as generalized coordinates.

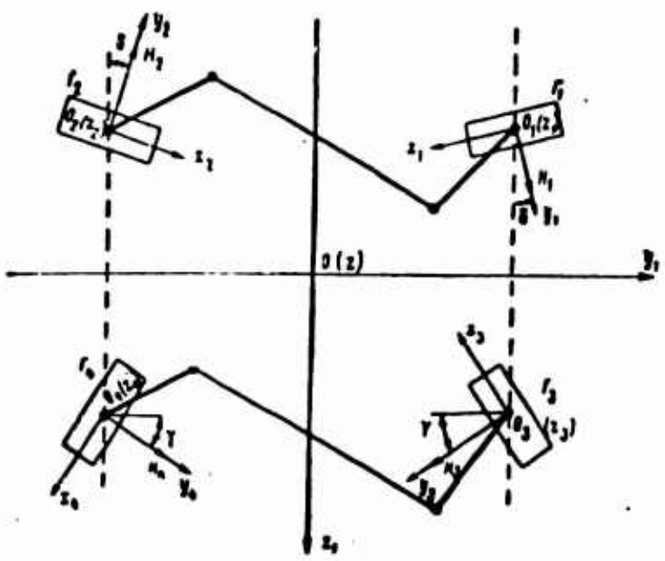

Fig. 7.30. Composing equations for a four-gyro active GV.

Motion of the considered system will be described by four equations: two of them characterize motion of the platform with respect to axes of the Cardan suspension, and the other two characterize motion of gyroscopes $Gyro_1$ and $Gyro_2$, $Gyro_3$ and $Gyro_4$ about their axes of precession with respect to the platform of the GV.

Equations of motion of the GV can be presented in the form of conditions of equilibrium between applied torques, gyrotorques and moments of forces of inertia (as this was done for a four-gyro vertical [127]):

$$\left.\begin{aligned} I_{y_1}\ddot{\alpha} &= M_{y_1} + M_{r\,y_1} \\ I_{x_1}\ddot{\beta} &= -M_{x_1} - M_{r\,x_1} \\ J_{r.e_1}\ddot{\delta} &= M_{z_1} + M^{I}_{r\,z_1} \\ J_{r.e_3}\ddot{\gamma} &= M_{z_3} + M^{II}_{r\,z_3} \end{aligned}\right\} \qquad (7.1.28)$$

where $I_{y_1} = I_\eta$ and I_{x_1} – moments of inertia of the platform with all connected components with respect to axes Oy_1 and Ox_1, respectively;

$J_{g.e_1}$ – moment of inertia of gyroscopes $Gyro_1$ and $Gyro_2$ with respect to axes O_1z_1 and O_2z_2;

$J_{g.e_3}$ – moment of inertia of gyroscopes $Gyro_3$ and $Gyro_4$ with respect to axes O_3z_3 and O_4z_4;

M_{x_1}, M_{y_1}, M_{z_1}, M_{z_3} – projections of torques, applied to systems on corresponding axes;

$M^{I}_{r\,z_1}$, $M^{II}_{r\,z_3}$ – projections on axis O_1z_1 and O_3z_3 of gyro torque of gyro pair I, $Gyro_1$ and $Gyro_2$ and, accordingly of pair II, $Gyro_3$ and $Gyro_4$.

Let us give expression of torques in equations (7.6.28). As torques $M_{y_1} \approx M_\eta$ and M_{x_1} about the suspension axes we consider torques M_{s_1} and M_{s_2}, applied by stabilizing motors, and disturbing torques $f_1(t)$ and $f_2(t)$; we have

$$\left.\begin{aligned} M_{y_1} &\approx M_\eta = M_{c_1} + f_1(t) \\ M_{x_1} &= M_{c_2} + f_2(t) \end{aligned}\right\} \qquad (7.6.29)$$

Considering the law of control of the stabilizing motor to be linear, and disregarding time constants of control networks, we have for M_{s_1} and M_{s_2} the following expressions:

$$M_{c_1} = -S_1\delta; \quad M_{c_2} = -S_1\gamma, \qquad (7.6.30)$$

where S_1 – steepness of the static characteristic of the SM.

Substituting (7.6.30) in (7.6.29), we obtain

$$\left.\begin{aligned} M_{y_1} &= -S_1\delta + f_1(t) \\ M_{x_1} &= -S_1\gamma + f_2(t) \end{aligned}\right\} \qquad (7.6.31)$$

On axes of precession of the gyroscopes we take into account only correcting torques $M_{c\,z_1}$ and $M_{c\,z_2}$, applied by torque devices, i.e.,

$$M_{z_1} = M_{k\,z_1}; \quad M_{z_2} = M_{k\,z_2}. \qquad (7.6.32)$$

Considering deflections of correcting pendulums during change of velocity and heading of the ship, by analogy with (2.3.66), we have

$$M_{z_1} = M_{k\,z_1} = S_2\left(\alpha + \frac{v\omega_k}{g}\right); \quad M_{z_2} = M_{k\,z_2} = -S_2\left(\beta - \frac{\dot{v}}{g}\right), \qquad (7.6.33)$$

where S_2 – steepness of the characteristic of correction;

v, $\dot{v}$ – velocity and acceleration of motion of the ship;

ω_{circ} – angular velocity of circulation of the ship.

Signs in formulas (7.6.33) are selected in such a manner that for $\dot{v} = 0$, $\omega_{circ} = 0$ the GV platform, under the influence of the work of correction returns to the vertical.

For gyro torque $\vec{M}_g$ of the system, by analogy with (1.3.34), we have

$$\vec{M}_r = \sum_{i=1}^{4} \vec{H}_i \times \vec{\Omega}_i, \qquad (7.6.34)$$

where $\vec{H}_1$ – natural angular momentum of the i-th gyroscope;

$\vec{\Omega}_1$ – its absolute angular velocity.

Then

$$\left.\begin{aligned} M_{rx_1} &= \sum_{i=1}^{4} (H_{iy_1}\Omega_{iz} - H_{iz}\Omega_{iy_1}) \\ M_{ry_1} &= \sum_{i=1}^{4} (H_{iz}\Omega_{ix_1} - H_{ix_1}\Omega_{iz}) \end{aligned}\right\} \qquad (7.6.35)$$

In the case of small δ and γ, according to Fig. 7.30 on the condition $H_1 = H$ ($1 = 1, 2, 3, 4$), for projections of $\vec{H}_1$ on axes Ox_1y_1z we obtain

$$\left.\begin{aligned} H_{1x_1} &= H;\ H_{1y_1} = H\delta;\ H_{1z} = 0 \\ H_{2x_1} &= -H;\ H_{2y_1} = H\delta;\ H_{2z} = 0 \\ H_{3x_1} &= H\gamma;\ H_{3y_1} = -H;\ H_{3z} = 0 \\ H_{4x_1} &= H\gamma;\ H_{4y_1} = H;\ H_{4z} = 0 \end{aligned}\right\} \qquad (7.6.36)$$

Let us find projections of absolute angular velocities of gyroscopes on axes Ox_1y_1z, considering matrix (1.1.31):

$$\left.\begin{aligned} \Omega_{1x_1} &= -\dot{\beta} + u_\xi - u_\zeta\alpha;\ \Omega_{2x_1} = -\dot{\beta} + u_\xi - u_\zeta\alpha \\ \Omega_{1y_1} &= \dot{\alpha} + u_\eta - u_\zeta\beta;\ \Omega_{2y_1} = \dot{\alpha} + u_\eta - u_\zeta\beta \\ \Omega_{1z} &= \dot{\delta} + u_\xi\alpha + u_\eta\beta + u_\zeta;\quad \Omega_{2z} = -\dot{\delta} + u_\xi\alpha + u_\eta\beta + u_\zeta \\ \Omega_{3x_1} &= -\dot{\beta} + u_\xi - u_\zeta\alpha;\ \Omega_{4x_1} = -\dot{\beta} + u_\xi - u_\zeta\alpha \\ \Omega_{3y_1} &= \dot{\alpha} + u_\eta - u_\zeta\beta;\ \Omega_{4y_1} = \dot{\alpha} + u_\eta - u_\zeta\beta \\ \Omega_{3z} &= \dot{\gamma} + u_\xi\alpha + u_\eta\beta + u_\zeta;\ \Omega_{4z} = -\dot{\gamma} + u_\xi\alpha + u_\eta\beta + u_\zeta \end{aligned}\right\} \qquad (7.6.37)$$

where u_ξ, u_η, u_ζ — projections on axes $O\xi\eta\zeta$ of induced angular velocity $\vec{u}$ of the reference system, caused by rotation of earth and motion of the ship.

Substituting (7.6.36), (7.6.37), and (7.6.35) and removing terms of the second and higher orders of smallness relative to the coordinates and their derivatives, we obtain

$$\left.\begin{aligned} M_{gx_1} &= 2H\beta u_\zeta - 2H\dot{\gamma} \\ M_{gy_1} &= -2H\gamma u_\zeta - 2H\dot{\beta} \end{aligned}\right\} \qquad (7.6.38)$$

Let us find the projection $M^{I}_{g\,z_1}$ of the gyro torque of gyroscopes $Gyro_1$ and $Gyro_2$, and also projection $M^{II}_{g\,z_3}$ of the gyro torque of gyroscopes $Gyro_3$ and $Gyro_4$. Since axes of precession of these gyroscopes are connected by anti-parallelograms,

$$\left.\begin{aligned} M^{I}_{gz_1} &= M_{gz_{1_1}} - M_{gz_{2_1}} \\ M^{II}_{gz_2} &= M_{gz_{3_2}} - M_{gz_{4_2}} \end{aligned}\right\} \qquad (7.6.39)$$

Determining the shown gyrotorques by method mentioned above, we have

$$\left.\begin{aligned} M^{I}_{gz_1} &= 2H\left(\dot{\alpha} + u_\eta - u_\zeta\beta\right) \\ M^{II}_{gz_2} &= -2H\left(\dot{\beta} - u_\zeta + u_\zeta\alpha\right) \end{aligned}\right\} \qquad (7.6.40)$$

We substitute (7.6.31), (7.6.33), (7.6.38), and (7.6.40) in (7.6.28); we have

$$\left.\begin{aligned} I_{y_1}\ddot{\alpha} &= -S_1\delta + f_1(t) - 2H\dot{\gamma}u_\zeta - 2H\dot{\delta} \\ I_{x_1}\ddot{\beta} &= S_1\gamma - f_2(t) - 2H\dot{\alpha}u_\zeta + 2H\dot{\gamma} \\ J_{z_1}\ddot{\delta} &= S_2\left(\alpha + \frac{\omega_{\text{orb}}}{g}\right) + 2H\left(\dot{\alpha} + u_\eta - u_\zeta\beta\right) \\ J_{z_1}\ddot{\gamma} &= -S_2\left(\beta - \frac{\dot{\vartheta}}{g}\right) - 2H\left(\dot{\beta} - u_\xi + u_\zeta\alpha\right) \end{aligned}\right\} \qquad (7.6.41)$$

If we do not consider small nutation oscillations of the GV, in equations (7.6.41) it is possible to reject the inertial terms; then

$$\left.\begin{aligned} \dot{\delta} + \frac{S_1}{2H}\delta + u_\zeta\gamma &= \frac{1}{2H}f_1(t) \\ \dot{\gamma} + \frac{S_1}{2H}\gamma - u_\zeta\delta &= \frac{1}{2H}f_2(t) \\ \dot{\alpha} + \frac{S_2}{2H}\alpha - u_\zeta\beta &= -u_\eta - \frac{S_2}{2H}\cdot\frac{\omega_{\text{orb}}}{g} \\ \dot{\beta} + \frac{S_2}{2H}\beta + u_\zeta\alpha &= u_\xi + \frac{S_2}{2H}\cdot\frac{\dot{\vartheta}}{g} \end{aligned}\right\} \qquad (7.6.42)$$

Introducing the specific rates of correction $\varkappa_1$ and $\varkappa_2$ [see (2.3.5)] and coefficient k_1 [see (7.4.14)],

$$\varkappa_1 = \frac{S_1}{2H}, \quad \varkappa_2 = \frac{S_2}{2H}, \quad k_1 = \frac{1}{2H}, \qquad (7.6.43)$$

we rewrite (7.6.42) in form

$$\left.\begin{aligned} \dot{\delta} + \varkappa_1\delta + u_\zeta\gamma &= k_1 f_1(t) \\ \dot{\gamma} + \varkappa_1\gamma - u_\zeta\delta &= k_1 f_2(t) \\ \dot{\alpha} + \varkappa_2\alpha - u_\zeta\beta &= -u_\eta - \varkappa_2\frac{\omega_{\text{orb}}}{g} \\ \dot{\beta} + \varkappa_2\beta + u_\zeta\alpha &= u_\xi + \varkappa_2\frac{\dot{\vartheta}}{g} \end{aligned}\right\} \qquad (7.6.44)$$

For the considered case, when axes $O\xi\eta\zeta$ of the reference system are connected with the path of the ship, quantities u_ξ, u_η, u_ζ are determined by relationships (1.2.32) and (1.2.33). Replacing in the latter $K = -\omega_{\text{circ}}$ and considering $|\omega_{\text{circ}}| \gg |U\sin\varphi + \frac{v_E}{R}\tan\varphi|$, we have

$$\left.\begin{aligned} u_\xi &= -U\cos\varphi\sin K - \frac{v}{R} \\ u_\eta &= U\cos\varphi\cos K \\ u_\zeta &= 0 \end{aligned}\right\} \qquad (7.6.45)$$

In formulas (7.6.44) instead of $\varkappa_1$ and $\varkappa_2$ it is possible to introduce the corresponding time constants of correction [see (2.3.7)]

$$T_1 = \frac{1}{\varkappa_1}; \quad T_2 = \frac{1}{\varkappa_2}. \tag{7.6.46}$$

Then, also considering (7.6.45) (we omit subscript "circ"), we rewrite equations (7.6.44) in form

$$\left.\begin{aligned}
&T_1\dot{\delta} + \delta + T_1\omega\gamma = k_1 T_1 f_1(t)\\
&T_1\dot{\gamma} + \gamma - T_1\omega\delta = k_1 T_1 f_2(t)\\
&T_2\dot{\alpha} + \alpha - T_2\omega\beta = -T_2 U\cos\varphi\cos K - \frac{\omega_0}{g}\\
&T_2\dot{\beta} + \beta + T_2\omega\alpha = -T_2\left(U\cos\varphi\sin K + \frac{v}{R}\right) + \frac{\dot{v}}{g}
\end{aligned}\right\} \tag{7.6.47}$$

from which it follows that the first two equations, characterizing motion of gyroscopes with respect to the platform, can be integrated independently of the other two equations, which determine motion of the stabilized GV platform.

System of shortened equations (7.6.47) can also be composed very simply by the method, developed by D. R. Merkin (§ 1.6, Par. 5); application of this method to composition of shortened equations of motion of a four-gyrovertical shown in his book [90, p. 234]. It is necessary to heed the note [90, p. 235] about the fact that obtaining analogous equations of motion of a four-gyrovertical from complete equations (7.6.41) becomes valid, if in the latter we consider dissipative forces (for instance, liquid friction torques), thanks to which the system becomes stable and its nutational oscillations are damped, and considerably faster than the basic precessional motion. After we shift from the system of complete equations (7.6.41) allowing for dissipative forces to shorten equations (7.8.47), in the latter, during study of the basic precessional motion dissipative forces can be ignored.

Mode of setting the GV and its static errors due to rotation of the earth and vehicle motion. Let us consider the case of motion of ship with constant speed and heading; then, considering in the last two equations of system (7.6.47) $\dot{v} = 0$, $\omega = 0$, we obtain equations

$$\left.\begin{aligned}
&T_2\dot{\alpha} + \alpha = -T_2 U\cos\varphi\cos K\\
&T_2\dot{\beta} + \beta = -T_2\left(U\cos\varphi\sin K + \frac{v}{R}\right)
\end{aligned}\right\} \tag{7.6.48}$$

which coincide, if we consider (7.6.45), with equations (6.4.5) of a readout GV which has a linear characteristic of correction, or with equation (7.4.19) of an active GS with pendular corrector.

Integrating (7.6.48) for φ = const and K = const, we obtain

$$\left.\begin{aligned} \alpha &= \alpha^{e}_{cr} + (\alpha_0 - \alpha^{e}_{cr}) e^{-\frac{t}{T_1}} \\ \beta &= \beta^{e}_{cr} + (\beta_0 - \beta^{e}_{cr}) e^{-\frac{t}{T_1}} \end{aligned}\right| \qquad (7.6.49)$$

where

$$\alpha^{e}_{cr} = -T_2 U \cos\varphi \cos K; \quad \beta^{e}_{cr} = -T_2\left(U\cos\varphi \sin K + \frac{v}{R}\right). \qquad (7.6.50)$$

Quantities α^{vel}_{stat} and β^{vel}_{stat}, by analogy with (6.4.7) or with (7.4.21), represent static errors of an active GV from rotation of the earth and motion of the ship, normally called velocity errors. Calculation of these errors by formula (7.6.50) is given in example 6.4. From (7.6.49) it follows that motion of vertex E (Fig. 1.1) of the stabilized platform of the GV toward the vertical is accomplished by aperiodic law. Upon completion of the transient response, according to (7.6.49),

$$\alpha = \alpha^{e}_{cr}; \quad \beta = \beta^{e}_{cr}, \qquad (7.6.51)$$

i.e., vertex E of the platform will be deflected from the vertical the shown magnitudes and the modulus of this deflection is equal to $\sqrt{\left(\alpha^{vel}_{stat}\right)^2 + \left(\beta^{vel}_{stat}\right)^2}$. Compensation of velocity errors of a GV is possible by means of eliminating from readings the latter quantities α^{vel}_{stat} and β^{vel}_{stat}, determined by formulas (7.6.50), by a suitable computer.

Using the first two equations of system (7.6.47), one can determine the regime of installation of the gyroscopes with respect to the platform. When $\omega = 0$, $f_1(t) = f_2(t) = 0$ we have

$$T_1\dot{\delta} + \delta = 0, \quad T_1\dot{\gamma} + \gamma = 0, \qquad (7.6.52)$$

from which

$$\delta = \delta_0 e^{-\frac{t}{T_1}}, \quad \gamma = \gamma_0 e^{-\frac{t}{T_1}}, \qquad (7.6.53)$$

i.e., we obtain the same aperiodic law of motion of the gyroscopes relative to their axes of precession.

Ballistic errors of a GV and their compensation. Above we considered velocity errors of a GV without accelerations of the ship ($\dot{v} = 0$, $\omega = 0$). During change of course and velocity of a ship, besides velocity errors, the instrument also has so-called ballistic errors (§ 6.4, Par. 3). We shall use detailed investigations of ballistic errors available in the literature [20, 22, 132, 127] in a brief account here of the question. In the creation of GV schemes one of important problems is ensuring in the instrument conditions of compensation of ballistic errors, i.e., its undisturbability by vehicle accelerations. Without realization of this condition it is impossible to create a precision GV.

We shall first give the general relationship for ballistic errors of the considered GV. For this we use the last two equations of system (7.6.44); considering (7.6.45) and dropping subscript "circ," we have

$$\left.\begin{aligned}\dot{\alpha}+x_3\alpha-\omega\beta&=-U\cos\varphi\cos K-x_2\frac{\delta\omega}{g}\\ \dot{\beta}+x_3\beta+\omega\alpha&=-U\cos\varphi\sin K-\frac{v}{R}+x_3\frac{\delta v}{g}\end{aligned}\right\}\qquad(7.6.54)$$

Following henceforth the calculations of Ya. N. Roytenberg [132], taking into account (7.6.50), and (7.6.46), and introducing designations (for $v \neq \text{const}$ and $K \neq \text{const}$ we replace α_{stat}^{vel}, β_{stat}^{vel} by α^{vel}, β^{vel})

$$\alpha_1=\alpha-\alpha^c,\quad \beta_1=\beta-\beta^c,\qquad(7.6.55)$$

we rewrite equation (7.6.54) in form

$$\left.\begin{aligned}\dot{\alpha}_1+x_3\alpha_1-\omega\beta_1&=-\mu x_2\frac{\delta\omega}{g}\\ \dot{\beta}_1+x_3\beta_1+\omega\alpha_1&=\mu x_2\frac{\delta v}{g}\end{aligned}\right\}\qquad(7.6.56)$$

where

$$\mu=1+\frac{v^2}{x_2^2};\qquad(7.6.57)$$

$$\nu=\sqrt{\frac{g}{R}}.\qquad(7.6.58)$$

Quantity ν, according to (4.5.16), represents the frequency of oscillations of a simple pendulum, whose length is equal to the radius of the earth R.

By analogy with (6.3.4) we introduce complex deflection z_1 of the vertex of the GV platform:

$$z_1=\alpha_1+j\beta_1.\qquad(7.6.59)$$

Multiplying the second equation of (7.6.56) by $j=\sqrt{-1}$, adding it to the first, and considering (7.6.59), we obtain

$$\dot{z}_1+(x_3+j\omega)z_1=j\mu x_2 y(t),\qquad(7.6.60)$$

where

$$y(t)=\frac{\dot{v}+j\delta\omega}{g}.\qquad(7.6.61)$$

Integrating (7.6.60), we have

$$z_1=\left[z_1(0)+j\mu x_2\int_0^t e^{x_3t+j\int_0^t\omega\,dt}\,y(t)\right]e^{-x_3t-j\int_0^t\omega\,dt}.\qquad(7.6.62)$$

In the expression obtained the first component characterizes motion of the GV, caused by initial deflections; this motion attenuates by aperiodic law. We write the

second component:

$$z_1^{\prime} = j\mu x_3 e^{-\omega_3 - j\int_0^t \omega dt} \int_0^t e^{\omega_3 + j\int_0^t \omega dt} y(t)\,dt. \tag{7.6.63}$$

It characterizes error of the GV due to accelerations of the ship, i.e., ballistic error. Such ballistic errors of a GV are investigated in great detail by B. V. Bulgakov [20].

Let us turn to finding the possibility of compensation of ballistic errors of a four-gyro vertical. We shall limit ourselves to a presentation of the method of compensation proposed by B. V. Bulgakov and Ya. N. Roytenberg [22].

The essence of this method consists in connecting a correcting pendulum Pend_1 (Fig. 7.29) to torquer TQ_2, which is connected to gyroscopes Gyro_3 and Gyro_4, and correcting pendulum Pend_2 to torquer TQ_1, connected with gyroscopes Gyro_1 and Gyro_2. Here, the first two equations of system (7.6.44) do not change. The second two equations will change, since in (7.6.33) $M_{c\,z_1}$ and $M_{c\,z_2}$ will change places due to switching of correction; taking into account (7.6.45) and (7.6.43) we give these equations form

$$\left.\begin{aligned} \dot{\alpha} - (x_3 + \omega)\beta &= -U\cos\varphi\cos K - x_2\frac{\dot{v}}{g} \\ \dot{\beta} + (x_3 + \omega)\alpha &= -U\cos\varphi\sin K - \frac{v}{R} - x_2\frac{v\omega}{g} \end{aligned}\right| \tag{7.6.64}$$

Introducing

$$z = \alpha + j\beta, \tag{7.6.65}$$

we replace equations (7.6.64) by one equation,

$$\dot{z} + j(x_3 + \omega)z = -U\cos\varphi e^{jK} - \frac{jv}{R} - \frac{x_2(\dot{v} + j\omega v)}{g}. \tag{7.6.66}$$

We designate

$$\zeta = z - z^{\varepsilon}, \tag{7.6.67}$$

where

$$z^{\varepsilon} = \frac{jU\cos\varphi e^{jK} - \frac{v}{R}}{x_3} = T_2\left(jU\cos\varphi e^{jK} - \frac{v}{R}\right) \tag{7.6.68}$$

is velocity error of the GV.

Then, taking such into account (7.6.67) and (7.6.58), we rewrite (7.6.66) as follows:

$$\dot{\zeta} + j(x_3 + \omega)\zeta = -\frac{x_2}{l}\left(1 - \frac{v^2}{x_3^2}\right)(\dot{v} + j\omega v). \tag{7.6.69}$$

From this equation it follows that if the specific velocity of correction $\varkappa_2 = \frac{S_2}{2H}$ is selected in such a manner that $\varkappa_2 = \pm\nu$, equation (7.6.69) will become homogeneous, i.e., the GV will be undisturbed by translational accelerations of the ship and, consequently, does not have ballistic errors. Thus, the considered GV, which in its construction belongs to astatic systems, manifests pendular properties, i.e., behaves as a gyroscopic pendulum with a period of precession (for $\omega = 0$)*

$$T = \frac{2\pi}{\nu} = 84{,}4 \text{ min}$$

Indeed, e.g., in the case of a fixed base ($U = \omega = 0$, $v = 0$) equations (7.6.64) have form

$$\dot{\alpha} - \varkappa_2\beta = 0; \quad \dot{\beta} + \varkappa_2\alpha = 0. \tag{7.6.70}$$

Replacing, in accordance with the above-indicated principle of compensation of ballistic errors, $\varkappa_2$ by ν, instead of (7.6.70), we obtain

$$\dot{\alpha} - \nu\beta = 0, \quad \dot{\beta} + \nu\alpha = 0$$

or

$$\ddot{\alpha} + \nu^2\alpha = 0, \quad \ddot{\beta} + \nu^2\beta = 0, \tag{7.6.71}$$

from which we have

$$\left.\begin{aligned} \alpha &= C_1\cos\nu t + C_2\sin\nu t \\ \beta &= -C_1\sin\nu t + C_2\cos\nu t \end{aligned}\right\} \tag{7.6.72}$$

where integration constant C_1 and C_2 are determined from the initial conditions.

From (7.6.72) one can see that the GV accomplishes periodic oscillations with frequency ν; the period of these oscillations, as was indicated above, is $T = \frac{2\pi}{\nu}$.

Consequently, in the considered GV, by selecting the corresponding value of the parameter of correction it is possible in principle to ensure the condition of undisturbability of M. Schuler, i.e., compensate ballistic error of the instrument.

The shown method of compensation of ballistic errors is one of the varieties of the method of integral correction (see Chapter 11). It is possible also to provide damping of the shown precessional motion of a GV. For this [22] it is necessary that voltage fed to each of the torque devices (Fig. 7.29) is a linear combination of signals from both correcting pendulums.

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*Let us note that when $\omega = \text{const}$ the GV accomplishes precessional oscillations with period $T \approx \frac{2\pi}{\omega}$, since $\omega \gg \nu$. However, here, the GV does not lose the property of undisturbability, because when $\varkappa_2 = \pm\nu$ equation (7.6.69) becomes homogeneous.

c) Certain Remarks.

Above, we considered only individual questions of the theory of an active-type GV. More detailed investigations can be carried out with help of the general methods of analysis of active GS's, given in the preceding sections of this chapter. We shall indicate several works on the theory of active GV's. In the book of N. T. Kuzovkov [80b] are complete differential equations of a double-axis gyrostabilized platform (active gyrovertical) composed with help of the method of A. Yu. Ishlinskiy (§ 1.6, Par. 7), corresponding transfer functions and block diagrams of the system. Articles of the same author are devoted to investigation of a gyrostabilized platform with large angles of displacement [77] and to the motion of a gyrostabilized platform, in a Cardan suspension [78]. There are offered [158] several GV schemes, built on the principle of active gyrostabilization.

4. Triple-Axis GS

In contemporary gyroscope technology a considerable place is occupied by so-called triple-axis active GS's. They are applied in gyroscopic device of combined type, intended for determination of three angles of rotation of a vehicle around its center of gravity, i.e., the represented combination of a directional gyroscope (gyroazimuth) and a gyrovertical (gyrohorizon), and, therefore, are called directional gyrohorizons. Furthermore, triple-axis GS's are used for direct stabilization of a certain platform with respect to the horizontal and in azimuth.

In Fig. 7.31 is a schematic of an active directional gyrohorizon [108, Fig. III. 18; 165; 47, p. 271]. The instrument is a combination of three one-gyro activated frames, basic elements of which are gyroscopes $Gyro_1$, $Gyro_2$, $Gyro_3$, fixed on platform P, stabilized relative to the horizontal and azimuth. The latter is suspended in a Cardan suspension, consisting of outer OR and inner IR rings. On the axes of precession of gyroscopes $Gyro_1$, $Gyro_2$, and $Gyro_3$ are fixed signal pickoffs SP_1, SP_2, and SP_3, which through amplifiers Amp_1, Amp_2, and Amp_3, control stabilizing motors SM_1, SM_2, and SM_3, respectively, which ensures realization in the instrument of the principle of active gyroscopic stabilization. In the considered scheme during rotation of the vehicle, e.g., the aircraft, around the vertical axis together with the latter moves the Cardan suspension, where platform P preserves a constant position in azimuth. Therefore, each of the signal pickoffs SP_1 and SP_2 should simultaneously control both stabilizing motors SM_1 and SM_2. For this, signals from pickoffs SP_1 and SP_2 enter amplifiers Amp_1 and Amp_2 through special distributor Dist, distributing these signals depending upon the angle of rotation of the aircraft.

The system of correction of platform P with respect to the horizontal plane consists of a two-component pendulum Pend (for instance, a liquid electrolytic level, Fig. 4.6) and torquers TQ_1 and TQ_2 controlled by it, fixed on the axes of precession of gyroscopes $Gyro_1$ and $Gyro_2$. Angles of roll γ, pitch ϑ and yaw ψ of an aircraft are taken in the form of corresponding voltages from potentiometers Pot_1, Pot_2, and Pot_3, respectively.

The considered active directional gyrohorizon possesses a series of basic merits as compared to readout DG's and GV's. One of them is that, in readings of angles of yaw and rotations of the aircraft there are no gimbal and turning errors.

The theory of triple-axis active GS's can be considered by the same methods as were applied above during the analysis of double-axis stabilizers. Let us note that for composition of differential equations of motion of such complex gyroscopic devices, as a triple-axis GS, there is frequently applied the second method of Lagrange (§ 1.5). A. Yu. Ishlinskiy showed that composition of equations of motion of a complex gyroscopic devices can be comparatively simply carried out by means of successive application of the theorem of angular momentum to the whole mechanical system of the corresponding device and to its separate components. Application of this method to composition of equations of motion of a triple-axis GS is given in his article [4]. Certain questions of the dynamics of a platform stabilized in space are considered in [20]. During development of the theory of triple-axis active GS's of importance is the question of cross connections between channels of stabilization. To account for this circumstance it is necessary to start from nonlinear differential equations of motion of the GS, in which we take into account components (for instance, gyroscope torques), caused by these cross connections. For integration of such equations we usually apply the method of successive approximations, where corresponding solutions of the equations are presented in the form of a series expansion of power's of the small parameters in the equations, i.e.,

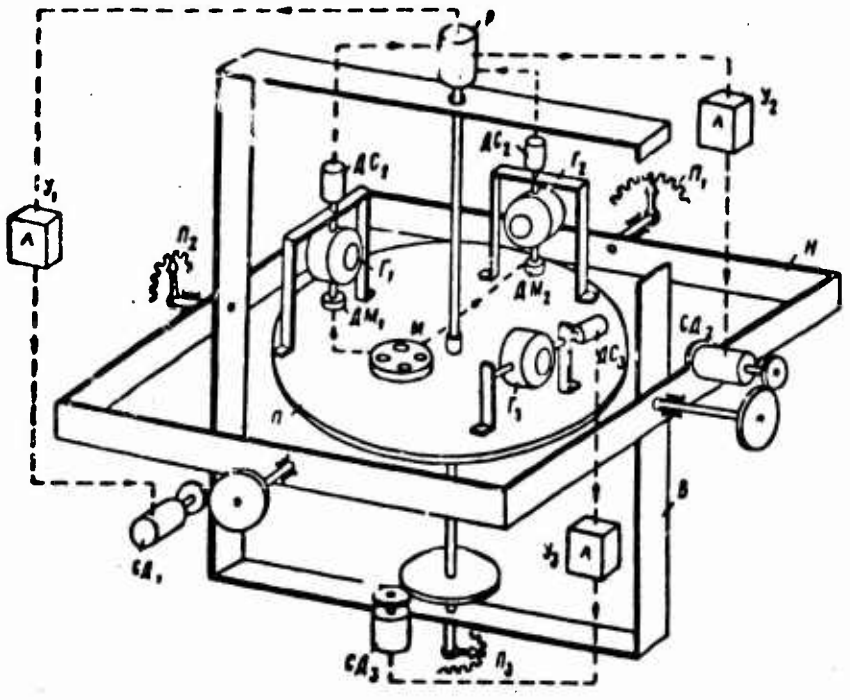

Fig. 7.31. Schematic of a three-gyro GS (an active-type directional gyrohorizon).

we use the method, applied by us during the analysis of the motion of a physical pendulum on an oscillating base (§ 4.6, Par 12d). When accounting for cross connections between channels of stabilization there is found systematic drift* of the GS platform relatively to its assigned position.

*D. S. Pel'por and N. P. Sumarokov. Motion of a triple-axis gyroscopic stabilizer on an oscillating base. "News of higher educational institutions. Instrument-making," issue V, No. 2, 1962; B. I. Nazarov. On error of gyrostabilizers. "News of AS of USSR," Dept. of Tech. Sci., Industrial Cybernetics, No. 2, 1963.

C H A P T E R 8

DIFFERENTIATING GYROSCOPES

Designations Appearing in Cyrillic

в = dis = disturbing

в = f = forced

в = rest = restoring

в = turn = turning

В = AS = actuating screw

ВГ = VG = vibrating gyro

ВК = IR = inner ring

Г = g = gyroscope

Г = GR = gyrorotor

Г = N = nut

ГВ = GV = gyrovertical

ГК = Osc = oscillator

ГН = DG = directional gyro

град = deg = degrees

ГС = GS = gyrostabilizer

г.т = gt = gyrotachometer

ГТ = GT = gyrotachometer

ГТА = GTA = gyrotachoaccelerometer

ГУ = GD = gyroscopic device

г.э = g.e = gyroequator

Д = d = damping, damped

д = dyn = dynamic

Д = mot = motor

Д = D = damper

Д = Mot = motor

ДГ = D.fG = differentiating gyro

ДМ = TQ = Torquer

ДС = SP = signal pickoff

ДУС = RTP = rate-of-turn pickoff

з = del = delay

И = I = indicator

ин = in = inertial

К = C = carriage

К = C = coil

К = R = ring

кр = crit = critical

н = r = rise

НК = OR = outer ring

опт = opt = optimum

Ос = B = base

п = par = parameters

П = Pl = plate

П = Pot = potentiometer

Пл = Pl = plate

пр = sp = spring

Пр = Sp = spring

р = adj = adjustment

р = res = resonance

Р = L = lever

рад = rad = radious

С = Rod = rod (tine of tuning fork)

сек = sec = second

ст = stat = static

т = fr = friction

Т = TR = torsion rod

У,у = Amp, amp = amplifier

ФД = PD = phase detector

э = e = equator

Э = EM = electromagnet

я = arm = armature

§ 8.1. Function and Types of Differentiating Gyroscopes (Dif G's)

In different systems of automatic adjustment and control gyroscopic devices intended for determining angular velocities and angular accelerations of a vehicle are widely applied. Such devices belong to the class of differentiating gyroscopes (Dif G's). Differentiating (§ 2.4, Par 1) is what we call a gyroscope, usually having two degrees of freedom, which differentiates the input signal.* In solving a number of problems of automatic control of a vehicle (ship, aircraft), for instance, for realization of stabilization about its center of gravity, besides angular deflections of the vehicle it is necessary to determine angular velocities and, in certain cases, angular accelerations of deflections. Thus, for instance, for proper work of the roll damper of a ship besides the angle of roll θ it is necessary to know angular velocity $\dot{\theta}$ of change of this angle. Analogously, input data of an autopilot, carrying out automatic stabilization of an aircraft relative to its center of gravity, are angles ψ, ϑ, γ (Fig. 2.4) of deflections of the aircraft, and also angular velocities $\dot{\psi}$, $\dot{\vartheta}$, $\dot{\gamma}$ and in certain cases angular accelerations $\ddot{\psi}$, $\ddot{\vartheta}$, $\ddot{\gamma}$. On the same aircraft the Dif G can be used as a turn indicator, i.e., as an instrument, indicating to the pilot the direction of angular velocity of turning of the aircraft relative to the assigned direction of straight-line movement.

Dif G's, depending upon parameters determined by them are divided into gyroscopes intended for measurement of angular velocities of a vehicle on gyroscopes intended for simultaneous measurement of angular velocities and angular accelerations of a vehicle.** Differentiating gyroscopes which determine angular velocities of a vehicle are called gyrotachometers, rate-of-turn indicators, precession or rate gyroscopes and rate-of-turn pickoffs (RTP); subsequently we shall apply the term gyrotachometer (GT). Differentiating gyroscopes, measuring angular velocities and

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*A differentiating gyroscope can be defined also as a gyroscope, reacting to gyroscope torques, caused by rotation of a vehicle.

**Usually instruments of this type measure the sum of angular velocity and angular acceleration of the vehicle.

angular accelerations of a vehicle, we call gyrotachoaccelerometers, indicators of angular velocity and acceleration, damping or accelerating – rate gyroscopes; henceforth, we shall use the term gyrotachoaccelerometor (GTA). We shall consider GT's in most detail, since they are a very widespread type of gyroscopic device; we shall describe GTA's more briefly, as their application is less widespread.

There exist gyroscopic devices, which determine the sum of the angular deflection of a vehicle and its angular velocity. Such gyroscopes, a variant of the GT, are called integrodifferentiating gyroscopes; they will be considered in Chapter 9.

§ 8.2. Gyroscopic Tachometers (GT's)

1. Function and Types of GT's

The gyrotachometer is a gyroscopic device used to determine angular velocity of the base (vehicle) on which it is mounted. One of the distinctive features of a GT is that it measures absolute angular velocity of the vehicle, i.e., the derivative of the angle of rotation of it in inertial space. Consequently, fixed on a base which is motionless with respect to the earth, a GT in principle is able to measure the component of induced angular velocity of sidereal rotation of the earth; however, creation of a GT, possessing threshold sensitivity necessary for this, is a very complicated technical problem. GT's applied in practice are cruder instruments and can only determine relative angular velocity of a vehicle, many times larger than the angular velocity of rotation of the earth.

We distinguish GT's, based on use of astatic gyroscopes with two and with three degrees of freedom; most widely applied are GT's of the first type. Usually a GT has one gyroscope. It is possible to create a GT with two gyroscopes like a two-gyro active GS (§ 7.1, Par. 2b).

Besides GT's based on use of astatic gyroscopes, there exist GT's based on devices, containing vibrating components; such instruments have acquired the name of gyrotachometers of vibration type or vibratory gyros (VG's) in view of the possibility of applying them not only as differentiating gyroscopes. Brief information on VG's will be given in § 8.4 of this chapter.

Gyrotachometers normally applied are of two types: 1) direct-reading GT's for which the measured angular velocity of the vehicle is fixed in the form of the angle of rotation of the instrument frame (gimbal ring) relative to its initial position; 2) GT's "with zero-drive" of the gyroscope axis (frame of the instrument), in which measurement of angular velocity of the vehicle is carried out by means of compensating the gyroscopic torque (caused by rotation of the vehicle) by a certain artificially

created torque, which ensures retention of the gyroscope axis near the zero position.

2. Schematic of a GT. Basic Relationships

The schematic of a GT is shown in Fig. 8.1. The basic element of the instrument is an astatic rate gyroscope; the gyrorotor is fixed in gimbal ring R, which is sometimes called the frame of the instrument; rotation of the ring is limited by spring Sp, necessary for creation of a restoring torque. Quenching of natural oscillations of gyroscope is carried out by damper D. Readings of the instrument, proportional to the angle of rotation of the frame about axis $O\eta$ (Oy), are taken in the form of voltage from potentiometer Pot. Axes $O\xi\eta\zeta$ are connected with the vehicle (ship, aircraft); axis Ox_1yz, combined in their initial position (when $\beta = 0$) with $O\xi\zeta$, are connected with the frame of instrument and in this case are Resal's axes (§ 1.1). Axis $O\xi$ (Ox) for the position of the gyroscope axis in Fig. 8.1 is the measuring, input axis, or the axis of sensitivity, since in this case the GT is intended for determination of component ω_ζ of the angular velocity $\vec{\omega}$ of rotation of the vehicle; axis $O\eta$ (Oy) is called the output axis, since the angle β of rotation of the frame about this axis is proportional, as will be shown below, to the measured angular velocity ω_ζ.

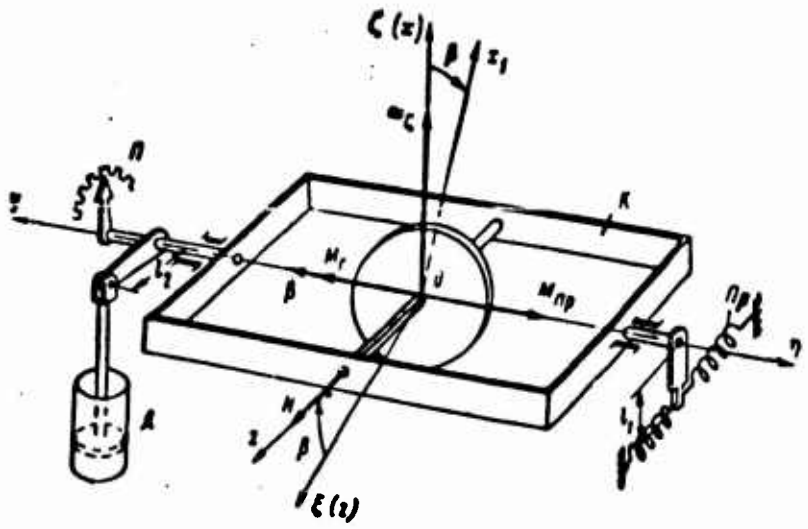

Fig. 8.1. Schematic of a GT.

Let us derive the basic relationship for a GT, establishing the functional connection between input variable $\omega_\zeta(t)$ and the output variable $\beta(t)$. During rotation of the vehicle about axis $O\zeta$ with angular velocity ω_ζ, axis Oz of the gyroscope will change its position with the same velocity ω_ζ, which it is possible to consider the angular velocity of forced precession. Due to this motion of axis Oz of the gyroscope there appears gyroscopic torque $\vec{M}_g$, directed along axis $O\eta$ (Oy), which, according to Foucault's rule, drives the gyro axis to coincide with axis $O\eta$. Gyroscopic torque $\vec{M}_g$, by analogy with (1.3.33), is determined by relationship

$$\vec{M}_r = \vec{H} \times \vec{\omega}_\zeta. \qquad (8.2.1)$$

where $\vec{H}$ – natural angular momentum of the gyroscope;

$\vec{\zeta}0$ – unit vector of axis 0ζ.

The magnitude of the gyroscopic torque is

$$M_r = H\omega_\zeta \sin\left(\vec{H}, \vec{\zeta}^0\right) = H\omega_\zeta \sin(90^\circ - \beta) = H\omega_\zeta \cos\beta. \qquad (8.2.2)$$

If angle β of rotation of the frame is limited to a sufficiently small value, measured normally be a quantity of several degrees, then (8.2.2) can be approximately written in form

$$M_r = H\omega_\zeta. \qquad (8.2.3)$$

Thus, for small angles β gyroscopic torque M_g is proportional to the component of angular velocity ω_ζ measured by the instrument, i.e., M_g can be considered the input variable.

The spring counteracts rotation of the instrument frame, introducing torque M_{sp}, which seeks to return the frame to its initial position, i.e., in this case is a restoring (position) torque. This torque is determined by relationship (2.3.32), i.e.,

$$M_{sp} = -c\beta, \qquad (8.2.4)$$

where c – stiffness coefficient of the spring, equal to the torque created by the spring upon deflection of the gyroscope angle β of one radian.

To determine quantity c we introduce coefficient c_1 of linear spring force of the spring; linear strain of the spring upon rotation of the frame angle β will be $l_1 \sin\beta \approx l_1\beta$, where l_1 – arm of the lever (Fig. 8.1); then the force developed by the spring will be expressed by relationship $c_1 l_1 \beta$; consequently, torque of the spring will be

$$M_{sp} = -c_1 l_1^2 \beta.$$

Comparing this expression with (8.2.4), we have

$$c = c_1 l_1^2. \qquad (8.2.5)$$

In deriving the basic relationship characterizing work of a GT we shall not take into account the equatorial component of angular momentum of the gyroscope, inertia of the frame, the damping torque, and also different disturbing torque, and

we shall use one of the shortened equations (1.6.30) of a gyroscope in D'Alembert's form. Considering the second of these equations in projections on axis Oy (Fig. 8.1) and taking for gyroscopic torque expression (8.2.3) and, as the external torque, the torque M_{sp}, created by the spring [see (8.2.4)], we obtain

$$M_r + M_{sp} = 0 \tag{8.2.6}$$

or

$$H\omega_\zeta - c\beta = 0,$$

from which we find basic relationship

$$\beta = \frac{H}{c}\omega_\zeta, \tag{8.2.7}$$

characterizing work of the GT.*

From (8.2.7) it follows that angle β of rotation of the frame is proportional to the measured angular velocity ω_ζ of the vehicle.

This formula explains why the considered gyroscope is called differentiating. Indeed, if by α we designate the angle of rotation of the vehicle about axis $O\zeta$ with angular velocity $\omega_\zeta = \dot{\alpha}$, then we can write (8.2.7) in form

$$\beta = \frac{H}{c}\dot{\alpha}$$

or in operational form $\left(p = \frac{d}{dt}\right)$

$$\beta = \frac{H}{c}p\alpha, \tag{8.2.8}$$

from which it follows that the output variable – angle β – is proportional to the first derivative of the input variable – the angle α of rotation of the gyroscope about measuring axis $O\zeta$, i.e., the GT is a differentiator.

Another approach is possible in principle, when as the input signal we take the angular velocity ω_ζ. Here, if we do not consider the series factors mentioned above, we can consider the GT as an amplifying network with gain factor $\frac{H}{c}$ [see (8.2.7)], i.e., the functional dependence between input variable ω_ζ and output variable β in the first approximation has the form of a simple linear dependence.

Above we said that the restoring torque is created in the GT by a spring. A deficiency of the application of the spring is the instability of readings of the

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*Detailed explanation of physical processes occurring in a GT is given in the work of S. O. Fridlender and M. S. Kozlov [65].

instrument, e.g., during change of the temperature regime. However, springs, due to simplicity of construction, have had very wide application in GT's.

Sometimes, instead of a spring for creating the restoring torque in a GT, we apply torsion rods or electrical springs. As a torsion rod we use a steel rod of round or rectangular section; one end of the torsion rod is rigidly fastened to the GT frame, and other is fastened to the instrument housing. The torsion rod works on torsion and creates here a restoring torque. Besides this, the torsion plays the role of one of semiaxes of the GT frame. When we use a torsion rod the angles of rotation of the frame are minute, that which decreases the sensitivity of the instrument. Torsion rods are applied in certain types of floating differentiating gyroscopes (Par. 7).

To show the advantages of an electrical spring we shall consider the question of the peculiarities of taking from potentiometer Pot (Fig. 8.2) of a gyrotachometer an output signal in the form of voltage U, proportional to angle β of rotation of the GT frame [171]. Let us designate by U_0 the voltage of the supply source, which corresponds to a certain angle β_0 of rotation of the cursor of the potentiometer. The maximum voltage removed from the potentiometer, relative to its median point, will be $U_1 = \frac{U_0}{2}$; with turn of the cursor a unit angle the voltage from the potentiometer will change by magnitude $\frac{U_0}{2\beta_0}$. With turn of the GT frame angle β from the potentiometer there will be taken voltage

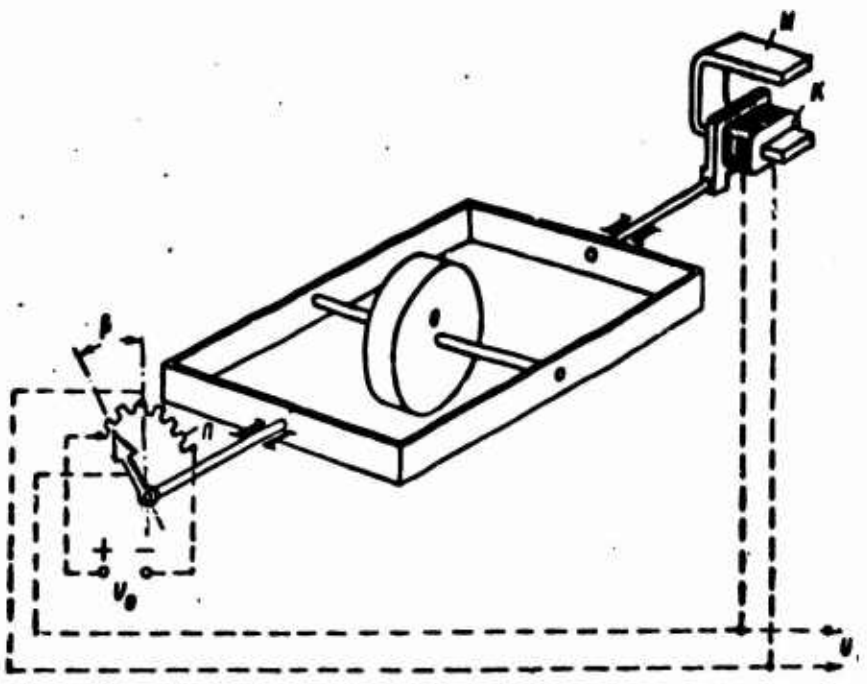

Fig. 8.2. Schematic of a GT with an electrical spring.

$$U = \frac{U_0}{2\beta_0}\beta. \qquad (8.2.9)$$

Substituting here (8.2.7), we obtain

$$U = \frac{U_0}{2\beta_0} \cdot \frac{H}{c}\omega_c. \qquad (8.2.10)$$

Thus, voltage taken from the potentiometer is proportional to the angular velocity ω_ζ of the vehicle, measured by the instrument. From (8.2.10) it follows that fluctuation of feed voltage U_0 leads to a corresponding change of output voltage U, and consequently, to error in measurement by the instrument of angular velocity ω_ζ.

When using in the GT an electrical spring accuracy of the GT does not depend on fluctuation of feed voltage U_0; this is one of its essential advantages. In the schematic of a GT shown in Fig. 8.2 the electrical spring consists of coil C connected to frame GT and permanent magnet M; to coil C there is fed voltage U, taken from the GT potentiometer. With turn of the instrument frame coil C will shift in the field of permanent magnet M; interaction of this field with coil current creates a restoring torque M_{sp}, determined by relationship

$$M_{sp} = -mU, \tag{8.2.11}$$

where m – constant proportionality factor.

Substituting (8.2.3) and (8.2.11) in (8.2.6), we find

$$U = \frac{H}{m}\omega_\zeta, \tag{8.2.12}$$

i.e., voltage U, taken from the potentiometer, will be proportional to the measured angular velocity ω_ζ of the vehicle and does not depend, as in the case of a mechanical-spring, on the feed voltage U_0 [see (8.2.10)].

Another merit of the electrical spring is that with it there disappears the necessity of having a special damper, since it itself creates a damping torque. Furthermore, application of an electrical spring permits easy tuning of the instrument by means of introduction of an adjusting resistance in its circuit.

Let us consider the problem of quenching natural oscillations of the GT frame. Usually for this we apply a pneumatic damper (D in Fig. 8.1), which introduces a damping torque M_d, determined by relationship (2.3.33), i.e.,

$$M_д = -b\dot{\beta}, \tag{8.2.13}$$

where b – damping factor, equal to the torque, created by damper, when angular velocity $\dot{\beta}$ is 1 radian/sec.

Let us find the expression for coefficient b [17]. Velocity of the piston with respect to the cylinder of the damper will be equal to $l_2\dot{\beta}$, where l_2 – a lever

arm (Fig. 8.1); then for the force developed by the damper we have relationship $al_2\dot{\beta}$, where a – specific force of damping, i.e., the force developed with movement of the piston with a velocity of 1 cm/sec; consequently, the damping torque will be $M_d = -al_2^2\dot{\beta}$. Comparing this expression with (8.2.13), we find for coefficient b the following relationship:

$$b = al_2^2. \tag{8.2.14}$$

Pneumatic dampers applied in GT's have these essential deficiencies: they lower sensitivity of the instrument due to increase of frictional resistance; they hamper balancing of the mobile part of the instrument; the damping factor when a GT is set, for instance, on an aircraft varies with change of altitude.

In GT's there are also applied liquid dampers, for instance in the floating differentiating gyroscope (Par. 7).

Let us consider the schematic of the so-called <u>electrical damper</u> (Fig. 8.3). On the GT frame there is fastened a copper plate Pl, which during turn of the frame shifts in the gap between two cores of electromagnet EM of the damper. In the plate there are induced Foucault currents; interaction of them with the magnetic field of the electromagnet creates a damping torque. With increase of the velocity $\dot{\beta}$ of rotation of the GT frame speed the rate of crossing by the plate of magnetic lines of force and the current induced in it are increased, and, consequently, also the damping torque.

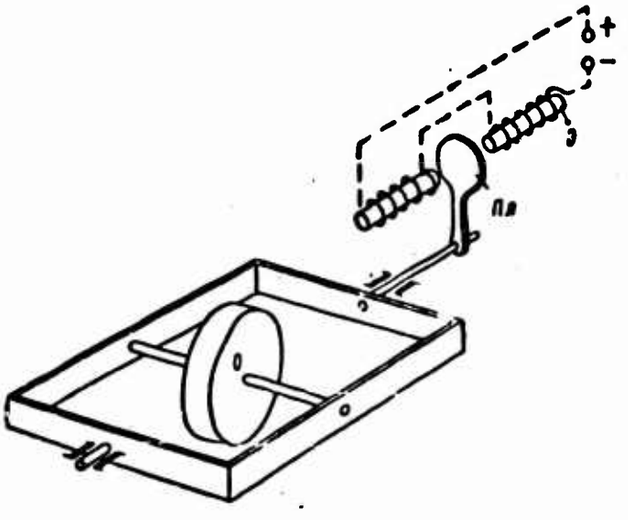

Fig. 8.3. Schematic of a GT with electrical damper.

The electrical damper has a number of advantages: in it there is no dry friction; the damping factor does not change when the GT is installed on an aircraft with change of altitude, etc. However, the electrical damper is more complicated than the pneumatic one; sometimes it has considerable size, and, therefore, it is applied less frequently in GT's.

3. Dynamic Characteristics of a GT

a) <u>Geometric and Kinematic Parameters</u>

Analysis of a GT starts with composition of the differential equations of motion

and the corresponding transfer functions. As was shown in Par. 2, axes $O\xi\eta\zeta$ (Fig. 8.1) are assumed connected with the vehicle; axes Oxyz are rigidly fastened to the instrument frame; in the initial position (when $\beta = 0$) axes Oxyz coincide with $O\xi\eta\zeta$. With turn of the GT frame about $O\eta$ (Oy) axes Oxyz will occupy position Ox_1yz; these axes in this case are Resal's axes.

By analogy with (1.1.27), transformation of axes $O\xi\eta\zeta$ to Ox_1yz we write in the form of the following matrix equality:

$$[x_1, y, z] = A_1[\xi, \eta, \zeta], \tag{8.2.15}$$

where matrix $A_1 = \|a'_{\mu\nu}\|$ $(\mu, \nu = 1, 2, 3)$ has the form

$$A_1 = \begin{Vmatrix} -\sin\beta & 0 & \cos\beta \\ 0 & -1 & 0 \\ \cos\beta & 0 & \sin\beta \end{Vmatrix}. \tag{8.2.16}$$

In case of small β we have approximately,

$$A_1 = \begin{Vmatrix} -\beta & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & \beta \end{Vmatrix} \tag{8.2.17}$$

Angular velocity of axes $O\xi\eta\zeta$, connected constantly with the vehicle, we shall designate by $\vec{\omega}$; then

$$\vec{\omega} = \omega_\xi \vec{i}^0 + \omega_\eta \vec{j}^0 + \omega_\zeta \vec{k}^0. \tag{8.2.18}$$

With the location in Fig. 8.1 the gyrotachometer determines component ω_ζ of angular velocity. With another location of the instrument we can measure component ω_ξ or ω_η. Thus, the GT determines components of angular velocity of a vehicle about the axes connected with it; here one should consider that ω_ξ, ω_η, ω_ζ are expressed through the angular velocities, interesting us, of rotation of the object by rather complex dependences. Actually, for a ship for axes $O\xi\eta\zeta$ (Fig. 8.1) one should take ship axes Oxyz (Fig. 2.2); comparing these figures, we have

$$\omega_\xi = \omega_y; \quad \omega_\eta = \omega_x; \quad \omega_\zeta = -\omega_z. \tag{8.2.19}$$

Substituting (2.1.17) here, we obtain

$$\omega_\xi = \dot{\psi}\cos\theta + \dot{\varphi}\cos\psi\sin\theta; \quad \omega_\eta = \dot{\theta} - \dot{\varphi}\sin\psi;$$

$$\omega_\zeta = -\dot{\varphi}\cos\psi\cos\theta + \dot{\psi}\sin\theta. \tag{8.2.20}$$

Thus, with the location in Fig. 8.1, the gyrotachometor will measure angular velocity ω_ζ, which characterizes the projection on axis $O\zeta$ of angular velocity $\dot{\varphi}$ of yawing of the ship and angular velocity $\dot{\psi}$ of its trim. If the roll angles are sufficiently small,

$$\omega_\zeta \approx -\dot{\varphi}, \tag{8.2.21}$$

i.e., the considered GT (Fig. 8.1) will measure the angular velocity of yawing of the ship. Here, two other GT's according to (8.2.20), will determine angular velocities $\dot{\psi}$ heaving and $\dot{\theta}$ rolling of ship, since

$$\omega_\xi \approx \dot{\psi}, \quad \omega_\eta \approx \dot{\theta}. \tag{8.2.22}$$

When necessary the GT can, e.g., in "pure" form determine angular velocity $\dot{\varphi}$ of yawing of a ship, but for this it is necessary to set it on a platform stabilized with respect to the horizontal.

During composition of the differential equations of motion of a GT by Lagrange or Euler equations it is necessary to have expressions of angular velocities. Projections of the absolute angular velocity of Resal's axes Ox_1yz

$$\vec{\omega}_1 = \vec{\omega} + \dot{\beta}\vec{\gamma}^0 \tag{8.2.23}$$

on the same axes, if we consider (8.2.16), are determined by relationships

$$\left.\begin{aligned} p_1 &= -\omega_\xi \sin\beta + \omega_\zeta \cos\beta \\ q_1 &= -\omega_\eta + \dot{\beta} \\ r_1 &= \omega_\xi \cos\beta + \omega_\zeta \sin\beta \end{aligned}\right\} \tag{8.2.24}$$

Components on the same axes of absolute angular velocity of the rotor, $\vec{\omega}'$,

$$\vec{\omega}' = \vec{\omega}_1 + \dot{\varphi}\vec{z}^0 \tag{8.2.25}$$

(where $\dot{\varphi}$ – angular velocity of rotation of the rotor) will be

$$\left.\begin{aligned} p &= -\omega_\xi \sin\beta + \omega_\zeta \cos\beta \\ q &= -\omega_\eta + \dot{\beta} \\ r &= \omega_\xi \cos\beta + \omega_\zeta \sin\beta + \dot{\varphi} \end{aligned}\right\} \tag{8.2.26}$$

b) Composition of Differential Equations of Motion of a GT

For composition of differential equations of motion of a GT we use the Kudrevich method (§ 1.6, Par. 2). First, we consider the case when the vehicle, e.g., a ship, accomplishes oscillatory motion about its center of gravity; here the mean value (mathematical expectation) of angles of deflections of it from the equilibrium position are equal to zero. Composition of equations of motion of a GT, fixed on an aircraft, turning with banking, is given later.

Let us construct an auxiliary drawing (Fig. 8.4). On axes $O\xi\eta\zeta$ are plotted components ω_ξ, ω_η, ω_ζ of angular velocity $\vec{\omega}$ of these axes. Along the axis Oy of rotation of the GT frame are located vectors of moments of forces: M_{in} – moment of forces of inertia of the frame; M_{sp} – torque, created by the spring; M_d – torque of damping; M_{GD} – gyroscopic torque; M_{dis} – disturbing torque.

In accordance with D'Alembert's principle the sum of these moments on axis Oy should be equal to zero:

$$M_{in} + M_{sp} + M_d + M_{GD} + M_{dis} = 0. \qquad (8.2.27)$$

We shall find expressions for the moments in (8.2.27). Moment of forces of inertia of the frame, caused by rotation of the vehicle about axis $O\eta$ with angular acceleration $\dot{\omega}_\eta$ and turn of the frame relative to the vehicle with angular acceleration $\ddot{\beta}$, is determined by relationship

$$M_{in} = -J_{g.e}\left(\dot{\omega}_\eta - \ddot{\beta}\right). \qquad (8.2.28)$$

where $J_{g.e}$ – sum of equatorial moment of inertia of the rotor, frame (housing) of the gyroscope and mobile parts of the damper and spring.

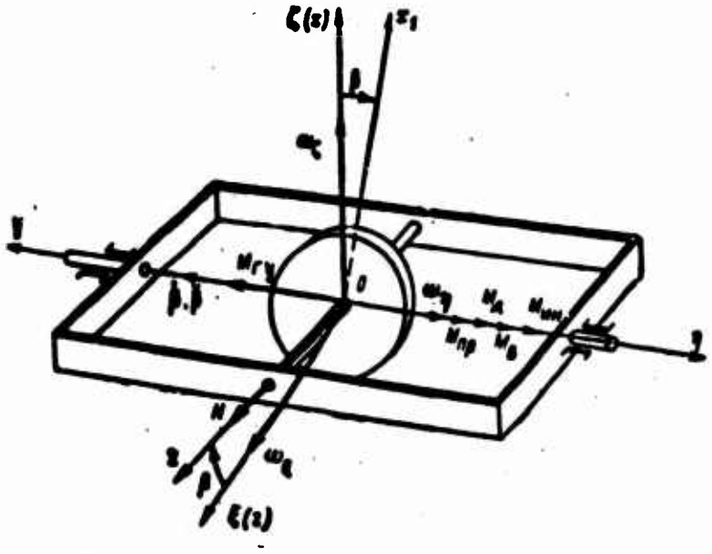

Fig. 8.4. Deriving equations of motion of a GT.

Torque, introduced by the spring, is determined by formula (8.2.4)

$$M_{sp} = -c\beta. \qquad (8.2.29)$$

For the torque of damping we found relationship (8.2.13)

$$M_d = -b\dot{\beta}. \qquad (8.2.30)$$

Let us define the expression of gyroscopic torque; by analogy

with (1.3.34), we have

$$\bar{M}_r = \bar{H} \times \bar{\omega}, \tag{8.2.31}$$

from which the projection of $\bar{M}_g$ on axis Oy will be

$$M_{ry} = -(H_\xi \omega_\xi - H_\xi \omega_\xi) = H(\omega_\xi \cos\beta - \omega_\xi \sin\beta). \tag{8.2.32}$$

To disturbing torque M_{dis} belong friction torques in the suspension axis of the frame, unbalance torques, etc. Subsequently, we assume that

$$M_\beta = M_{fr} + f(t), \tag{8.2.33}$$

where M_{fr} – friction torque in the frame axis;

$f(t)$ – other disturbing torques.

In the case of liquid friction, by analogy with (2.3.72), we have

$$M_r = -n_y \dot{\beta}. \tag{8.2.34}$$

With dry friction in the frame suspension axis, considering for simplicity friction torques to be symmetric during rotation of the suspension axis in various directions, by analogy with (2.3.84), for $M^0_{fr\ y} = 0$ we obtain

$$M_r = -K_y \operatorname{sign} \dot{\beta}. \tag{8.2.35}$$

Substituting (8.2.28)-(8.2.30), (8.2.32) and (8.2.33) in (8.2.27), we have*

$$J_{r.\beta}(\ddot{\beta} - \dot{\omega}_\eta) + b\dot{\beta} + c\beta = H(\omega_\xi \cos\beta - \omega_\xi \sin\beta) + M_r + f(t). \tag{8.2.36}$$

---

*Let us note that the exact equation of motion of a GT can be obtained by using Euler's equations (1.4.1), in form

$$J_{r.\beta}(\ddot{\beta} - \dot{\omega}_\eta) + b\dot{\beta} + c\beta = H(\omega_\xi \cos\beta - \omega_\xi \sin\beta) - (J_{rx} - J_\beta)\left(\frac{\omega_\xi^2 - \omega_\xi^2}{2} \sin 2\beta + \omega_\xi \omega_\xi \cos 2\beta\right) + M_r + f(t),$$

where J_{gx} – moment of inertia of rotor and frame with respect to axis Ox_1 (Fig. 8.4);

J_B – moment of inertia of the frame with respect to axis Oz.

This equation differs from (8.2.36) by the presence of component

$$(J_{rx} - J_\beta)\left(\frac{\omega_\xi^2 - \omega_\xi^2}{2} \sin 2\beta + \omega_\xi \omega_\xi \cos 2\beta\right),$$

which in modulus is essentially less than other components; therefore, we usually investigate GT's on the basis of equation (8.2.36).

In this equation inertial term $J_{g.e}(\ddot{\beta} - \dot{\omega}_{\eta})$ in principle cannot be dropped, since the GT has only one determining coordinate β [90], and, therefore, with any angular momentum of the gyrscope it is impossible to use the shortened equation of (§ 1.6, Par. 8). In equation (8.2.36) is inertial moment $J_{g.e}\omega_{\eta}$ due to angular acceleration of object with respect to an axis $O\eta$. This moment introduces distortion in the GT readings. Usually the magnitude of the corresponding so-called inertial error of the GT is small; however we can, in general, compensate this error, if in the instrument we establish a flywheel, connected to the GT frame by a gear transmission. The mathematical foundation for the condition of compensation of inertial error of a GT is given in the book of D. S. Pel'por et al. [17].

In GT constructions usually applied in practice the limiting value of angle β of deflection of the frame from its zero position if limited to several degrees. Therefore, in equation (8.2.36) it is fully permissible to consider $\cos\beta \approx 1$, $\sin\beta \approx \beta$ and, consequently, we can present it in form

$$J_{r.s}\ddot{\beta} + b\dot{\beta} + (c + H\omega_{\xi})\beta = H\omega_{\zeta} + J_{r.s}\dot{\omega}_{\eta} + M_{\tau} + f(t). \qquad (8.2.37)$$

If $\dot{\omega}_{\eta} = 0$, $\omega_{\xi} = 0$, $M_{fr} = f(t) = 0$, when $\omega_{\zeta} = \text{const}$ for a steady (static) position of the frame, determined by angle β_{stat}, considering in (8.2.37) $\dot{\beta} = \ddot{\beta} = 0$, we have relationship

$$\beta_{cr} = \frac{H}{c}\omega_{\zeta}, \qquad (8.2.38)$$

coinciding with (8.2.7).

If, with the above-indicated conditions we also have $\omega_{\xi} = \text{const}$, instead of (8.2.38) in static conditions we have

$$\beta_{cr} = \frac{H}{c + H\omega_{\xi}}\omega_{\zeta} = \frac{1}{1 + \frac{H}{c}\omega_{\xi}} \cdot \frac{H}{c}\omega_{\zeta}, \qquad (8.2.39)$$

i.e., the GT, besides reacting to angular velocity ω_{ζ} (for measurement of which it is intended), will react also to component ω_{ξ} of angular velocity of the vehicle along axis $O\xi$, which distorts readings of the instrument. To decrease the influence of ω_{ξ} the coefficient c of spring force of the spring we select usually sufficiently large, so that for all possible values of $\frac{H}{c}\omega_{\xi} \ll 1.0$. Of the possibility of total compensation of this GT error we shall say more later (Par. 7).

Let us transform equation (8.2.37), considering $\omega_{\xi} = 0$:

$$J_{r.s}\ddot{\beta} + b\dot{\beta} + c\beta = H\omega_{\zeta} + J_{r.s}\dot{\omega}_{\eta} + M_{\tau} + f(t). \qquad (8.2.40)$$

or

$$\ddot{\beta}+\frac{b}{J_{r.e}}\dot{\beta}+\frac{c}{J_{r.e}}\beta=\frac{H}{J_{r.e}}\omega_{\zeta}+\dot{\omega}_{\eta}+\frac{1}{J_{r.e}}[M_{\tau}+f(t)]. \qquad (8.2.41)$$

We designate

$$n^2=\frac{c}{J_{r.e}}, \qquad (8.2.42)$$

from which we obtain the formula for the frequency n of natural sustained oscillations of the GT:

$$n=\sqrt{\frac{c}{J_{r.e}}}. \qquad (8.2.43)$$

Then for the period of these oscillations of the GT we have

$$T_{r.r}=\frac{2\pi}{n}=2\pi\sqrt{\frac{J_{r.e}}{c}}. \qquad (8.2.44)$$

Let us transform ratio $\frac{b}{J_{g.e}}$ as follows:

$$\frac{b}{J_{r.e}}=\frac{b}{J_{r.e}}\cdot\frac{\sqrt{\frac{c}{J_{r.e}}}}{\sqrt{\frac{c}{J_{r.e}}}}=2\zeta n, \qquad (8.2.45)$$

where

$$\zeta=\frac{b}{2\sqrt{J_{r.e}c}}. \qquad (8.2.46)$$

As shown in § 4.3, Par. 2, quantity ζ constitutes the relative damping factor.

We introduce designations

$$k_1=\frac{H}{J_{r.e}};\quad \rho_1=\frac{1}{J_{r.e}}. \qquad (8.2.47)$$

Considering (8.2.42), (8.2.45) and (8.2.47), we rewrite equation (8.2.41) in the form

$$\ddot{\beta}+2\zeta n\dot{\beta}+n^2\beta=k_1\omega_{\zeta}+\dot{\omega}_{\eta}+\rho_1[M_{\tau}+f(t)]. \qquad (8.2.48)$$

If in (8.2.48) we set $\dot{\omega}_{\eta}$ = 0, M_{fr} = f(t) = 0, then we obtain an equation, analogous to the equation (4.3.19) of oscillations of a physical pendulum.

Let us present (8.2.48) in another form; for this, by analogy with (4.3.23), we

introduce time constant T of the gyrotachometer,

$$T=\frac{1}{n}. \tag{8.2.49}$$

We divide (8.2.48) by n^2 and designate

$$\left.\begin{aligned} k&=\frac{k_1}{n^2}=\frac{H}{J_{r.e}}\cdot\frac{J_{r.e}}{c}=\frac{H}{c}\\ \rho&=\frac{\rho_1}{n^2}=\frac{1}{J_{r.e}}\cdot\frac{J_{r.e}}{c}=\frac{1}{c}\end{aligned}\right\} \tag{8.2.50}$$

Considering (8.2.49) and (8.2.50), we rewrite equation (8.2.48) in form

$$T^2\ddot{\beta}+2\zeta T\dot{\beta}+\beta=k\omega_\zeta+T^2\dot{\omega}_\eta+\rho\left[M_\tau+f(t)\right]. \tag{8.2.51}$$

analogous under the assumptions shown earlier to the equation (4.3.25) of motion of a physical pendulum.

When composing differential equations of motion of a GT by the Kudrevich method we assumed that in the instrument there is used an astatic rate gyroscope. In connection with this we note that during composition of equations of the GT it is possible conditionally to consider it a gyroscope with three degrees of freedom, taking as the outer gimbal ring the housing of the instrument, rigidly joined with the vehicle. In this case it is possible to use equations (1.4.2) of motion of a free gyroscope. Being interested in motion of the frame about axis Oy, one should turn to the second of the shown equations; we replace in it J_e by $J_{g.e}$ and reject small component $J_e pr_1$, caused by centripetal acceleration; then we have

$$J_{r.e}\frac{dq}{dt}-Hp_1=M_{y_1}=M_y. \tag{8.2.52}$$

Ccnsidering (8.2.24) and (8.2.26), we obtain

$$J_{r.e}\left(\ddot{\beta}-\dot{\omega}_\eta\right)-H\left(-\omega_\xi\sin\beta+\omega_\zeta\cos\beta\right)=M_y. \tag{8.2.53}$$

For moment M_{amp} we have

$$M_y=M_{sp}+M_d+M_{dis}. \tag{8.2.54}$$

Substituting for M_{sp}, M_d, M_{dis} their expressions (8.2.29), (8.2.30), (8.2.33) and the resulting relationship in (8.2.53), we arrive at equation (8.2.36) found earlier.

Finally, we compose the equation of motion of a GT for the case when it is fixed on an aircraft, accomplishing a regular turn with constant angle of transverse bank γ. Here, as it is known, angle γ is connected with the angular velocity of turn ω_{turn} and with the radius of turn r_{turn} by the following relationship:

$$\operatorname{tg}\gamma = \frac{\omega_{turn}^2 r_{turn}}{g}. \qquad (8.2.55)$$

We assume that axes $O\xi\eta\zeta$ (Fig. 8.5) are rigidly joined with the aircraft, and $O\eta$ coincides with its longitudinal axis, and $O\xi$ is directed along the right wing; the initial position of these axes when $\gamma = 0$ we designate by $O\xi_0\eta_0\zeta_0$. Together with the aircraft the GT will also bank, and it will no longer measure the angular velocity of turning ω_{turn}, but its projection on the normal axis $O\zeta$ of the aircraft. Actually, the angular velocity of turning ω_{turn} gives on axes $O\xi$ and $O\zeta$ the components

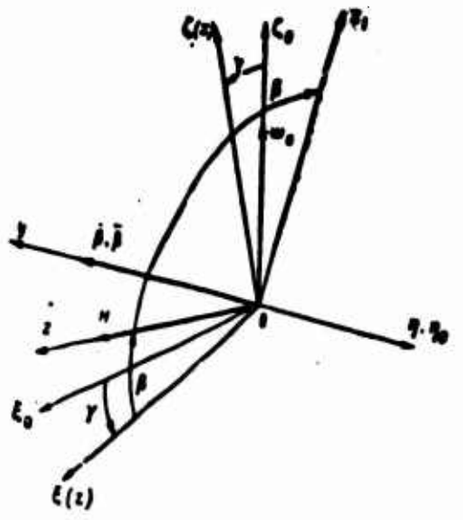

Fig. 8.5. Readings of a GT during turning of an aircraft.

$$\omega_\xi = -\omega_0 \sin\gamma; \quad \omega_\zeta = \omega_0 \cos\gamma. \qquad (8.2.56)$$

Substituting (8.2.56) in (8.2.32), we obtain the following expression for gyroscopic torque:

$$M_{r\beta} = H\omega_0(\cos\beta\cos\gamma + \sin\beta\sin\gamma) = H\omega_0\cos(\beta-\gamma). \qquad (8,2.57)$$

If β is small as compared to γ, we have, approximately

$$M_{r\beta} = H\omega_0\cos\gamma. \qquad (8.2.58)$$

which confirms what we indicated above.

Let us compose for the considered case the equation of motion of the GT. For this it suffices to introduce (8.2.56) in equation (8.2.36); for simplicity in the latter we take $M_{fr} = f(t) = 0$; considering also that when $\gamma = \text{const}$, $\dot{\omega}_\eta = 0$, instead of (8.2.36) we obtain

$$J_{r.o}\ddot{\beta} + b\dot{\beta} + c\beta = H\omega_0\cos(\beta-\gamma). \qquad (8.2.59)$$

For the position of static equilibrium we have

$$\beta_{cr} = \frac{H}{c}\omega_0\cos(\beta_{cr}-\gamma) \qquad (8.2.60)$$

or for a small value of β_{stat}

$$\beta_{cr} = \frac{H}{c}\omega_{\eta}(\cos\gamma + \beta_{cr}\sin\gamma),$$

from which

$$\beta_{cr} = \frac{1}{1 - \frac{H}{c}\omega_{\eta}\sin\gamma} \cdot \frac{H}{c}\omega_{\eta}\cos\gamma. \qquad (8.2.61)$$

Usually coefficient c is selected sufficiently large so that component $\frac{H}{c}\omega_{turn}$ sin γ can be disregarded as compared to unity; then, approximately, we have

$$\beta_{cr} = \frac{H}{c}\omega_{\eta}\cos\gamma. \qquad (8.2.62)$$

i.e., the GT, instead of angular velocity of turn ω_{turn}, will fix its component on the measuring axis $O\zeta$ (Fig. 8.5) of the instrument. Here, if the angle of roll γ of the aircraft is determined by a GV, then from the reading on the GT, characterized by formula (8.2.62), it is possible to separate the value ω_{turn} interesting us, using, for instance, the corresponding computer.

c) Transfer Functions and Frequency-Response Curves of a GT.

Let us determine transfer function W(s) of a GT with respect to the useful input signal. For this in equation (8.2.48) we put disturbances $\dot{\omega}_{\eta} = 0$, $M_{fr} = f(t) = 0$; then

$$\ddot{\beta} + 2\zeta n\dot{\beta} + n^2\beta = k_1\omega_{\xi}. \qquad (8.2.63)$$

We designate by α the angle of rotation of the vehicle about axis $O\zeta$ (Fig. 8.1); then angular velocity is

$$\omega_{\xi} = \dot{\alpha}. \qquad (8.2.64)$$

Considering (8.2.64), we rewrite (8.2.63) in form

$$\ddot{\beta} + 2\zeta n\dot{\beta} + n^2\beta = k_1\dot{\alpha}. \qquad (8.2.65)$$

Let us present (8.2.65) in operational form $\left(p = \frac{d}{dt}\right)$:

$$(p^2 + 2\zeta np + n^2)\beta = k_1 p\alpha. \qquad (8.2.66)$$

Applying to (8.2.66) the Laplace transform, we obtain the following expression for transfer function GT:

$$W(s) = \frac{\beta(s)}{\alpha(s)} = \frac{k_1 s}{s^2 + 2\zeta ns + n^2}. \qquad (8.2.67)$$

We can present this formula in different form, if we divide its numerator and denominator by n^2 and consider relationships (8.2.49) and (8.2.50); then we obtain

$$W(s) = \frac{ks}{T^2s^2 + 2\xi Ts + 1}. \qquad (8.2.68)$$

From (8.2.67) and (8.2.68) it follows that the GT with respect to the angle of rotation of the vehicle about the measuring axis can be considered an oscillating differentiator [114], since to the instrument input there proceeds the derivative of the angle of rotation of the vehicle. In (8.2.68) T – time constant of the GT, and k – its transmission factor. If T is very small, instead of (8.2.68), we obtain, approximately,

$$W(s) = ks, \qquad (8.2.69)$$

i.e., the "ideal" GT is a differentiator.

For determination of the transfer function of a GT another approach is possible, in which as the input signal we take the angular velocity ω_ζ of the vehicle. Then, according to (8.2.63), transfer function of the GT with respect to influence ω_ζ will be determined by relationship

$$W(s) = \frac{\beta(s)}{\omega_\zeta(s)} = \frac{k_1}{s^2 + 2\xi ns + n^2} \qquad (8.2.70)$$

or, considering (8.2.49) and (8.2.50),

$$W(s) = \frac{k}{T^2s^2 + 2\xi Ts + 1}. \qquad (8.2.71)$$

Comparing (8.2.70) and (8.2.71) in accordance with (4.3.21) and (4.3.26), we see that the transfer function of the GT has the same form as the transfer function of a physical pendulum, i.e., the GT with respect to angular velocity measured by it can be considered an oscillating unit. Such an approach to the study of the dynamics of a GT is the most expedient, since here there is the possibility of using results obtained during the study of a physical pendulum (Chapter 4).

Example 8.1. Determine the frequency n of natural undamped oscillations GT, the period of these oscillations T_{gt}, time constant T, transmission coefficient k, steady-state (static) value β_{stat} of the angle of displacement of the instrument frame for the following initial data: angular momentum of the gyroscope H = 2093 g-cm-sec (example 3.2), equatorial moment of inertia of gyroscope together with

frame $J_{g.e} = 1.01$ g-cm-sec^2, stiffness coefficient of the spring c = 1200 g-cm/rad, constant value of measured angular velocity $\omega_\zeta = 2$ deg/sec.

Solution: 1. By formula (8.2.43) we find frequency n of natural oscillations of the GT:

$$n = \sqrt{\frac{c}{J_{r.s}}} = \sqrt{\frac{1200}{1,01}} = 34,47 \text{ 1/sec.}$$

2. According to (8.2.44) we calculate period $T_{g.t}$ of oscillations of the GT:

$$T_{r.t} = \frac{2\pi}{n} = \frac{2 \cdot 3,14}{34,47} = 0,1823 \text{ sec.}$$

3. Using (8.2.49), we determine time constant T:

$$T = \frac{1}{n} = \frac{1}{34,47} = 0,02901 \text{ sec.}$$

4. By formula (8.2.50) we find transmission factor k;

$$k = \frac{H}{c} = \frac{2093}{1200} = 1,744 \text{ sec.}$$

5. According to (8.2.38), we calculate the magnitude β_{stat} of static deflection of the GT frame:

$$\beta_{cr} = \frac{H}{c}\omega_\zeta = k\omega_\zeta = 1,744 \cdot 2 = 3.488^\circ.$$

To determine the gain-phase response W(jω) in (8.2.71) we replace s by jω; then

$$W(j\omega) = \frac{k}{1 - T^2\omega^2 + 2\zeta T j\omega}. \qquad (8.2.72)$$

If for W(s) we use formula (8.2.70), then for W(jω) will also have

$$W(j\omega) = \frac{k_1}{n^2 - \omega^2 + 2\zeta n j\omega}. \qquad (8.2.73)$$

Let us define amplitude A(ω), phase φ(ω), real P(ω) and imaginary Q(ω) frequency-response curves of the GT.

By the same method as for a physical pendulum (§ 4.3, Par. 3), we obtain formulas (4.3.34) and (4.3.35):

$$A(\omega)=\frac{k}{\sqrt{(1-T^2\omega^2)^2+4\zeta^2T^2\omega^2}};\quad \varphi(\omega)=-\operatorname{arc\,tg}\frac{2\zeta T\omega}{1-T^2\omega^2}; \tag{8.2.74}$$

$$P(\omega)=\frac{k(1-T^2\omega^2)}{(1-T^2\omega^2)^2+4\zeta^2T^2\omega^2};\ Q(\omega)=\frac{-2k\zeta T\omega}{(1-T^2\omega^2)^2+4\zeta^2T^2\omega^2}. \tag{8.2.75}$$

The gain-phase response $W(j\omega)$ of the GT, i.e., the curve, described by end of vector $W(j\omega)$ on the complex plane during change of ω from 0 to ∞, has the same form as for a physical pendulum (Fig. 4.14). From the graph it follows that during action on the GT of angular velocity $\omega_\zeta(t)$, varying by periodic law, steady-state oscillation of the GT will also be periodic oscillations of the same frequency as $\omega_\zeta(t)$. Modulus $W(j\omega)$ reaches a maximum at a certain frequency $\omega < n$, where n is the frequency of natural oscillations of the GT.

Knowing the gain-phase response $W(j\omega)$ of the GT, by formula (2.3.105) we can find the (weighting) unit impulse function of the GT

$$k_1(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}W(j\omega)e^{j\omega t}d\omega, \tag{8.2.76}$$

which characterizes the transient response during application to the GT of a unit impulse function, or delta-function $\delta(t-t_0)$.

During investigation of errors of a GT in dynamics it is also necessary to know the transfer function of the GT with respect to disturbances. To determine this transfer function, which we shall subsequently designate by Y(s), we use equation (8.2.51) of the GT, considering $f(t) = 0$; then

$$T^2\ddot{\beta}+2\zeta T\dot{\beta}+\beta=k\omega_\zeta+T^2\dot{\omega}_\eta+\varepsilon M_r. \tag{8.2.77}$$

For axes $O\xi\eta\zeta$ (Fig. 8.4) we take ship axes Oxyz (Fig. 2.2): the axis of sensitivity of the GT coincides with axis $O\zeta$, and in this case the GT is designed for measurement of angular velocity $\dot{\varphi}$ of yawing of the ship; indeed, according to (8.2.21), $\omega_\zeta \approx -\varphi(t)$. Then the disturbing angular acceleration $\dot{\omega}_\eta$, considering (8.2.22), we find equal to angular acceleration $\ddot{\theta}$ of rolling of the ship, i.e., $\dot{\omega}_\eta = \ddot{\theta}$. Let us assume that in the axis of suspension of the frame there is liquid friction, determined by formula (8.2.34),

$$M_r=-n_1\dot{\beta}. \tag{8.2.78}$$

Considering what has been said and formula (8.2.78), we rewrite (8.2.77) in the following manner:

$$T^2\ddot{\beta} + 2\zeta T\dot{\beta} + \beta = k\omega_\zeta + T^2\ddot{\theta} - \rho n_1 \dot{\beta}. \tag{8.2.79}$$

Designating

$$\zeta_1 = \zeta + \frac{\rho n_1}{2T}, \tag{8.2.80}$$

we have

$$T^2\ddot{\beta} + 2\zeta_1 T\dot{\beta} + \beta = k\omega_\zeta + T^2\ddot{\theta}. \tag{8.2.81}$$

Presence in the right part of this equation of disturbance $T^2\ddot{\theta}$ causes error ε of the GT reading. Since $\omega_\zeta(t)$ is the useful signal, then for error ε we obtain the following differential equation:

$$T^2\ddot{\varepsilon} + 2\zeta_1 T\dot{\varepsilon} + \varepsilon = T^2\ddot{\theta}. \tag{8.2.82}$$

We rewrite it in operational form:

$$(T^2p^2 + 2\zeta_1 Tp + 1)\varepsilon = T^2p^2\theta. \tag{8.2.83}$$

Applying the Laplace transform to (8.2.83), we obtain the expression for the transfer function Y(s) of a GT with respect to disturbance from rolling

$$Y(s) = \frac{\varepsilon(s)}{\theta(s)} = \frac{T^2s^2}{T^2s^2 + 2\zeta_1 Ts + 1}. \tag{8.2.84}$$

4. Transient Responses of a GT

a) Behavior of GT in a transient response.

One of the important stages of analysis of a GT is determination of its behavior in a transient response, and also investigation of its errors in steady state, i.e., analysis of its static accuracy. Behavior of GT in transient response with some typical disturbance, for which we usually take a unit step input (§ 2.3, Par. 3d), is best characterized by the so-called performance indices. Since motion of the GT is described by a differential second order equation (8.2.51), analogous to the equation (4.3.25) of motion of a physical pendulum, the performance indices of the GT are determined just as performance indices of the physical pendulum (§ 4.4). Therefore, we note only specific peculiarities for a GT.

The first question which arises here concerns stability of the GT. The stability of a GT, which can be considered an oscillating circuit [Par. 3 and (8.2.71)], is most simply judged by its gain-phase response $W(j\omega)$, determined by formula (8.2.72). The hodograph of $W(j\omega)$ has a form, analogous to that shown in Fig. 4.14, from which it follows that the GT is stable for any positive coefficients of the corresponding characteristic equation. More complicated is the question of stability of the GT in the presence of angular velocity ω_ξ of the vehicle with respect to axis $O\xi$, not coinciding with the measuring axis $O\zeta$ (Fig. 8.4) of the instrument, when its motion is described by differential equation (8.2.37). Here in general, there can occur parametric resonance, as also for a physical pendulum (§ 4.6, Par. 12c).

Subsequently we shall be interested in change of the angle β of rotation of the frame of the GT in a transient response, caused by a typical disturbance in the form of abrupt change of the input value by one and with zero initial conditions, i.e., during a unit step input [1] [see (2.3.107)]. Function $\beta_h(t)$, which determines change of β at the GT output under the shown conditions, is the unit step response and is expressed by general relationship (2.3.108).

Let us use equation (8.2.51) of motion of the GT frame, which, in the absence of disturbances $\dot{\omega}_\eta = 0$, $M_{fr} = f(t) = 0$, will be written in form

$$T^2\ddot{\beta} + 2\xi T\dot{\beta} + \beta = k\omega_\zeta. \qquad (8.2.85)$$

In the considered case, when motion of the GT is described by a differential second order equation, the transient, i.e., transient function of the GT, and its performance indices can be determined analytically. Having the goal of finding the unit step response $\beta_h(t)$ of the GT at application to it of a unit step input $\omega_\zeta(t) = [1]$ with zero initial conditions, we rewrite (8.2.85) in form

$$T^2\ddot{\beta} + 2\xi T\dot{\beta} + \beta = k[1]. \qquad (8.2.86)$$

The particular solution of this equation will be

$$\beta_\nu = k[1]; \qquad (8.2.87)$$

it characterizes readings of the GT in a steady-state conditions, i.e., static deflection $\beta(\infty)$ of the GT frame; consequently [see (8.2.50)],

$$\beta(\infty) = k[1] = \frac{H}{c}[1]. \qquad (8.2.88)$$

Then, we find the general solution of homogeneous equation

$$T^2\ddot{\beta} + 2\zeta T\dot{\beta} + \beta = 0 \qquad (8.2.89)$$

for zero initial conditions. We have the corresponding characteristic equation

$$T^2\lambda^2 + 2\zeta T\lambda + 1 = 0, \qquad (8.2.90)$$

roots of which will be

$$\lambda_{1,2} = -\frac{1}{T}\left(\zeta \pm \sqrt{\zeta^2 - 1}\right). \qquad (8.2.91)$$

Parameters of the GT are selected in such a way that

$$\zeta^2 - 1 < 0; \qquad (8.2.92)$$

then roots λ_1 and λ_2 will be complex:

$$\lambda_1 = -\nu + jn_d; \quad \lambda_2 = -\nu - jn_d, \qquad (8.2.93)$$

where

$$\nu = \frac{\zeta}{T}; \quad n_d = \frac{\sqrt{1-\zeta^2}}{T}; \qquad (8.2.94)$$

ν – coefficient, characterizing the rate of damping of oscillations of the GT:

n_d – frequency of natural damped oscillations of the GT.

Considering (8.2.49), we rewrite (8.2.94) in form

$$\nu = n\zeta; \quad n_d = n\sqrt{1-\zeta^2}; \qquad (8.2.95)$$

when $\zeta = 0$,

$$\nu = 0 \text{ and } n_d = n.$$

The general solution of homogeneous equation (8.2.90) will be

$$\beta = e^{-\frac{\zeta}{T}t}\left(C_1 \cos\frac{\sqrt{1-\zeta^2}}{T}t + C_2 \sin\frac{\sqrt{1-\zeta^2}}{T}t\right). \qquad (8.2.96)$$

where constants C_1 and C_2 are determined by the initial conditions.

Then, considering (8.2.87) and (8.2.96), we obtain the solution of equation (8.2.86), i.e., unit step response $\beta_h(t)$ with unit step input, considering initial

condi ons zero [when $t = 0$, $p_h(0) = 0$, $\dot{p}_h(0) = 0$], in form

$$\beta_h(t) = k\left[1 - e^{-\frac{\zeta}{T}t}\left(\cos\frac{\sqrt{1-\zeta^2}}{T}t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\frac{\sqrt{1-\zeta^2}}{T}t\right)\right]. \quad (8.2.97)$$

Example 8.2. Calculate and construct unit step response $\beta_h(t)$ of a GT for parameters of the instrument, given in example 8.1; relative damping factor of oscillations of the GT frame $\zeta = 0.5$.

Solution. By formula (8.2.97) for different values of time t we determine angles of deflection of the GT frame, and then construct the graph of unit step response $\beta_h(t)$, shown in Fig. 8.6.

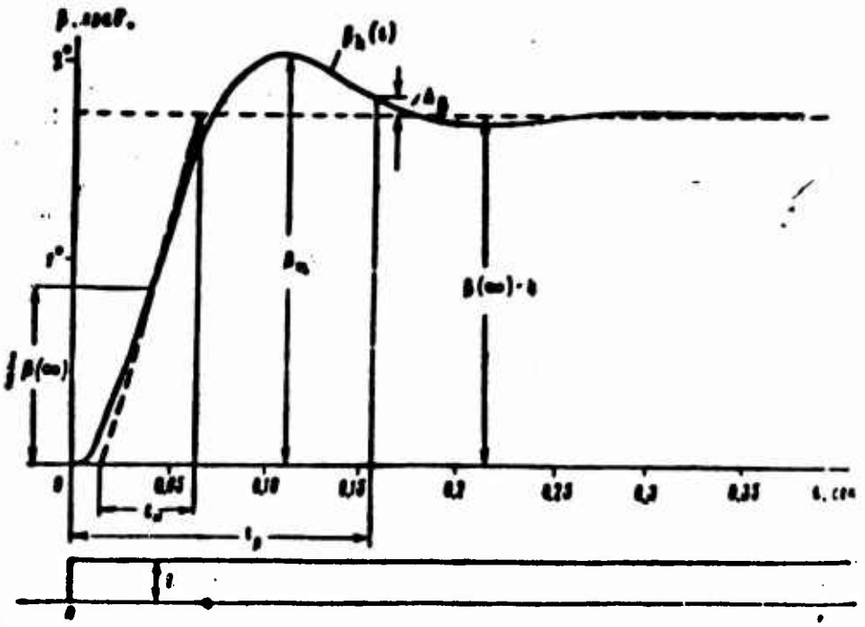

Fig. 8.6. Transient response of a GT for a unit step input. Performance indices.

b) Definition of Performance Indices of a GT.

Having the graph $\beta_h(t)$ of the unit step response, it is easy to determine the so-called performance indices of the considered dynamic system, characterizing behavior of the GT in the transient response.

Among performance indices, according to what was shown in § 4.4, Par. 2b, are: 1) maximum deflection β_m of the GT frame (Fig. 8.6) or magnitude of overshoot $\frac{\beta_m - \beta(\infty)}{\beta(\infty)}$ 100% where $\beta(\infty)$ – static deflection; 2) static deflection $\beta(\infty)$; 3) time the transient response (or time of adjustment) t_{adj}, i.e., the least of the values for time (counted from the moment of application of the disturbance), after which there occurs inequality $|\beta(t) - \beta(\infty)| \leq \Delta_\beta$, where Δ_β – a given small constant;

4) number of oscillations of deflection $\beta(t)$ during the time of transient response t_{adj}. Let us note that with respect to the GT quantity t_{adj} is usually called the time of bringing the frame of the instrument to equilibrium position.

Example 8.3. Using the graph of unit step response $\beta_h(t)$ (Fig. 8.6), determine performance indices, characterizing behavior of the GT in a transient response; take Δ_β equal to 5% $\beta(\infty)$.

Solution. 1. On the graph in Fig. 8.6 we find the magnitude of the maximum deflection β_m of the GT frame from its equilibrium position:

$$\beta_m = 2.05^\circ.$$

2. We find the magnitude of static deflection $\beta(\infty)$:

$$\beta(\infty) = 1.74^\circ.$$

3. We determine the magnitude of overshooting:

$$\frac{\beta_m - \beta(\infty)}{\beta(\infty)} 100\% = \frac{2.05 - 1.74}{1.74} 100\% = 17.8\%.$$

4. We determine the value of Δ_β:

$$\Delta_\beta = 5\% \beta(\infty) = 0.05 \cdot 1.74 = 0.087^\circ.$$

5. On the graph we find the time of adjustment (setting time):

$$t_{adj} = 0.156 \text{ sec}.$$

Sometimes the performance indices include the so-called rise time t_r, which is determined in the following manner [80b]: to curve $\beta_h(t)$ (Fig. 8.6) there is passed a tangent at a point whose ordinate is $\frac{1}{2}\beta(\infty)$; the segment of the tangent included between points of its crossing with the axis of abscissas and with horizontal line $\beta(\infty)$, is projected on the axis of abscissas; the magnitude of this projection is equal to rise time t_r. For unit step response $\beta_h(t)$, shown in Fig. 8.6, rise time comprises $t_r = 0.05$ sec. Rise time is an important performance index, since it characterized the speed of operation of the system.

The above-indicated performance indices were found directly from curve $\beta_h(t)$ of the unit step response of the GT. In the considered case, when motion of the GT frame is described by a differential second order equation, construction of curve $\beta_h(t)$ and determination by it of performance indices present no difficulty. For more

complicated systems in the theory of automatic control there have been developed special methods of determining performance indices without constructing the curve of the unit step response; here, we use the frequency responses of the system (see, for instance, [27, 107]). A number of useful empirical formulas for calculating performance indices by the form of the amplitude frequency $A(\omega)$ characteristic of the system [see (8.2.74)], obtained by means of static treatment of results of investigation of many linear systems on an analog computer, are given in [80b].

c) Certain Remarks.

Formula (8.2.97) for the unit step response $\beta_h(t)$ of a GT is valid when the instrument is influenced by the angular velocity of the vehicle in the form of a unit step function. Sometimes, during presentation of the theory of the gyrotachometer (see, e.g., [57]) they consider the characteristic of the transient response of a GT with constant (but not equal to unity) value of angular velocity of the vehicle. Corresponding formulas can be obtained from those given earlier, if in them instead of unit step function [1] we introduce the magnitude of angular velocity ω_ζ = const. Then, according to (8.2.88), we obtain an expression for static deflection of the GT frame:

$$\beta_{cr} = \Delta\omega_\zeta = \frac{H}{c}\omega_\zeta. \qquad (8.2.98)$$

analogous to (8.2.38).

In the same manner we obtain the expression for the GT transient response during influence on it of angular velocity ω_ζ = const by multiplication of the right side of (8.2.97) by ω_ζ; then considering (8.2.98), we have

$$\beta(t) = \beta_{cr}\left[1 - e^{-\frac{\zeta}{T}t}\left(\cos\frac{\sqrt{1-\zeta^2}}{T}t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\frac{\sqrt{1-\zeta^2}}{T}t\right)\right]. \qquad (8.2.99)$$

For the same parameters ζ and T of the gyrotachometer function $\beta(t)$ will have the same form as transient function $\beta_h(t)$, shown in Fig. 8.6.

In certain particular cases it may be of interest to obtain the transient response of a GT for a more complicated form of disturbance, for instance, in the form of half sine wave, triangluar, or square impulses, etc. For these inputs there have been constructed [86] transient responses of a system which is described by a second order equation (accelerometer).

Above we considered transients responses of a GT for a unit step input and with zero initial conditions. If it is necessary to reveal the influence on the transient

response of a GT of initial conditions, we find the solution of equation (8.2.85), given angle $\beta(0)$ and angular velocity $\dot{\beta}(0)$ of deflection of the GT frame at the time of beginning of motion ($t = 0$). The final expression for $\beta(t)$ in this case is found without special labor.

To construct the graph of the unit step response $\beta_h(t)$ of the GT according to formula (8.2.97) it is necessary to know the basic parameters of the instrument: transmission factor k, time constant T and relative damping factor ζ. For a manufactured instrument these parameters are usually known or can be easily determined experimentally. However, during designing of an instrument there arises the very important problem of determining optimum values of these parameters with given requirements on the transient response, i.e., on performance indices and dynamic accuracy of the instrument.

Presentation of the method of selection of optimum values of basic parameters of a GT, as of other types of GD's, is beyond the scope of this book. Therefore, we shall limit ourselve to only certain remarks. Problems of the method of selection of parameters of a GT are considered in a number of works [57, 108, 174, 165]. Certain general considerations about calculation of basic parameters of recording instruments were shown by A. N. Krylov [67].

During selection of parameters of a GT one must first of all substantiate magnitude $T_{g.t}$ of the period of its natural undamped oscillations, and then by formula (8.2.44) determine the value of the frequency n of oscillations. Selecting period $T_{g.t}$, one should consider the indications of A. N. Krylov [67], who subdivided recording instruments by their assignment into three types: 1) instruments which should not respond to the action of disturbing forces, such as, e.g., bank indicators, seismographs, and so forth; 2) instruments, which are intended for recording namely "disturbing" force, such as: indicators, galvanometers, and so forth; 3) instruments, intended for recording a power impulse, as, e.g., a ballistic galvanometer.

The GT belongs to the second type of instrument, since it is intended for fixing angular velocity $\omega_\zeta(t)$ of a vehicle. For this type of instrument [67]: "1) it is necessary first of all, even if approximately, to give to self an account of the nature of action of disturbing forces: are they periodic or not, if periodic, what is the period, if not periodic, how do they grow; 2) if the instrument is intended for recording force and, thus, it is desirable so that it, as it were, tracked it statically, the period of free oscillations of the instrument should be small as compared to the period of the force or the duration of its build-up."

In accordance with the first recommendation of A. N. Krylov it is necessary to know the nature of change of angular velocity $\omega_\zeta(t)$. In certain cases $\omega_\zeta(t)$ is a periodic function; then we should know its period, equal to the period of oscillations of the vehicle about axis $O\zeta$ (Fig. 8.4). Since the GT should fix (measure) the "external influence," i.e., angular velocity of the vehicle ω_ζ, it is desirable that the GT frame tracks ω_ζ statically, i.e., that the angle β_{stat} deflection of the frame from its zero position be proportional to angular velocity $\omega_\zeta(t)$ [see (8.2.98)]. For this, in accordance with the second of the shown recommendations, period $T_{g.t}$ of natural oscillations of the GT frame should be small as compared to the period of oscillations of the vehicle about axis $O\zeta$, or, which is the same, frequency n of natural oscillations of the frame should be great as compared to the frequency of oscillations of the vehicle. Usually in a GT frequency n is selected 5-10 times larger than the frequency of vehicle oscillations.

However, in practice, $\omega_\zeta(t)$ most frequently is not an assigned periodic function, but a random function of time. Therefore, it is necessary to know the nature of the curve of spectral density of random function $\omega_\zeta(t)$ and to take it into account during selection of the frequency n of natural oscillations of the GT.

Thus, from the above-indicated considerations we select the frequency n of natural oscillations of the GT, and consequently also its time constant $T = 1/n$. Further, according to (8.2.43), knowing for the selected type of gyroscope moment of inertia $J_{g.e}$, one can determine the required value of the coefficient c of spring force of the spring.

Depending upon the required reading scale of the instrument, considering formula (8.2.38) and given, for instance, maximum value ω_{ζ_m} of the measured angular velocity and the accompanying maximum deflection β_{stat_m} of the GT frame, we find the angular mementum H of the gyroscope. For the selected type of gyroscope and known value of the axial moment of inertia J for it we determine the necessary angular velocity of spin of the gyrorotor $\Omega = \frac{H}{J}$. The transmission factor k of the GT is determined by formula (8.2.50), i.e., $k = \frac{H}{c}$.

We shall give certain considerations in the matter of selection of the optimum value of damping factor ζ. The question of selecting the optimum value of ζ of an oscillatory systems of GT type, described by a differential second order equation, is considered in detail in the literature [57, 108, 174]. In a number of works [57, 108] there is developed a method of selecting the optimum value of ζ, which originates from a certain permissible error Δ_β (Fig. 8.6) of instrument readings and a minimum value of the time of setting the GT frame in equilibrium position; also taken into

account [108] is instrument error in dynamics.

For selection of ζ in the first approximation it is possible to originate from the following considerations. From the point of view of decreasing the time of transient response (time of setting) parameter ζ should increase, but this leads to growth of dynamic errors of the GT. By analogy with what was presented in § 4.3, Par. 2, the damping [see (8.2.46)] which ensures the fastest asymptotic approximation of the GT frame to equilibrium position, is called critical; to it there corresponds a critical value of the attenuation factor

$$b_{кр} = 2\sqrt{J_{г.р}c}, \tag{8.2.100}$$

at which the GT loses its oscillation properties, i.e., motion of the GT frame to the position of equilibrium will be aperiodic. As optimum value of the attenuation factor we usually take [174]

$$b = 0{,}707 b_{кр} = 1{,}414\sqrt{J_{г.р}c}. \tag{8.2.101}$$

Substituting (8.2.101) in (8.2.46), for the optimum value ζ_{opt} we obtain

$$\zeta_{опт} = 0{,}707. \tag{8.2.102}$$

Above we spoke of the influence of ζ on the transient function [see (8.2.97)] of a GT. The question of the influence of ζ on error of a GT in dynamics is considered in Par. 6.

During the analysis of the transient of a GT we did not consider friction in axis of rotation of the instrument frame. Since time of setting the frame in equilibrium position (time of the transient response) usually is small in comparison with the period of oscillations of the vehicle, with accuracy sufficient for practical purposes it is possible to disregard the influence of friction in transient conditions, and consider only its influence on the magnitude of static deflection of the GT frame, i.e., find the magnitude of distortion in the steady-state value of instrument readings; this distortion characterizes, corresponding, instrument error of the GT.

5. Instrument Error of a GT

One of basic requirements presented to a GT, as to other types of GD, is to provide required accuracy of readings.

Let us stop to determine instrument errors of a GT (§ 2.4, Par. 2). In a GT there is used an astatic gyroscope, analogous to that applied in the DG and GV.

Instrument errors of the astatic gyroscope utilized in a DG were investigated in detail in § 5.5 and 5.7; therefore, here, in examining instrument errors of GT we shall limit ourselves to a few remarks.

Among instrument errors of GT is is possible to list:

1) error caused by friction torques, on the axis of suspension of the frame of the instrument;

2) errors caused by variations of parameters of the GT;

3) errors caused by static and dynamic unbalance of the gyro unit;

4) errors of reading pickoff of the GT, errors from imperfection of current leads, etc.

Let us consider errors caused by friction torques on the axis of suspension of the frame. This torque includes: the friction torque in the axis of the frame; dry friction torque in the damper on the axis of the frame; friction torques in the readout pickoff of the GT (for instance, potentiometer) and in places of connection of springs brought to the same axis. Friction torques cause distortion of readings of the instrument due to appearance of a region of stagnation, damping of natural oscillations of the GT even in the case of absence or off-position of the damper, errors in dynamics.

Let us find the influence of friction on static error of the GT. The problem of accounting for friction during calculations of errors of a GT in dynamics is expounded in Par. 6.

To determine the region of stagnation of a GT due to friction we use equation (8.2.51), which when $\omega_\zeta = 0$, $\dot{\omega}_\eta = 0$, $f(t) = 0$ will have form

$$T^2\ddot{\beta} + 2\varkappa T\dot{\beta} + \beta = \rho M_r. \tag{8.2.103}$$

Taking into account (8.2.35) and (8.2.50), we have

$$T^2\ddot{\beta} + 2\varkappa T\dot{\beta} + \beta = -\frac{K_y}{c}\operatorname{sign}\dot{\beta}, \tag{8.2.104}$$

from which, by one of the methods of the nonlinear theory of oscillations, one can determine transient $\beta(t)$ taking into account dry friction in the axis of suspension of the GT frame. According to (2.3.75), we rewrite (8.2.104) in the form

$$T^2\ddot{\beta} + 2\varkappa T\dot{\beta} + \beta = \mp\frac{K_y}{c}, \tag{8.2.105}$$

from which static error of the GT due to friction will be

$$\varepsilon_{cr}^{r} = \beta_{cr}^{r} = \mp \frac{K_y}{c}. \qquad (8.2.106)$$

Quantity ε_{stat}^{fr} determines the region of stagnation of the GT due to friction; inside this region the instrument will not give readings; consequently, ε_{stat}^{fr} characterizes the threshold sensitivity of the GT to angular velocity of the vehicle,* expressed in the angle of turn of the GT frame.

The minimum angular velocity $\omega_{\zeta_{min}}$ measured by the GT is determined by the magnitude of the least gyroscopic torque $M_g = H\omega_{\zeta_{min}}$ [see (8.2.3)], which overcomes friction in the axis of the frame, i.e., $\omega_{\zeta_{min}}$ is found from condition

$$H\omega_{\zeta_{min}} > |M_y| \qquad (8.2.107)$$

or

$$\omega_{\zeta_{min}} > \frac{K_y}{H}.$$

The value of $\omega_{\zeta_{min}}$ determines the threshold of sensitivity of the instrument to angular velocity of the vehicle. To increase the sensitivity of a GT, i.e., decrease the value of $\omega_{\zeta_{min}}$, one should decrease the magnitude of friction torque K_y and increase angular momentum H of the gyroscope.

Example 8.4. Determine the region of stagnation of a GT due to friction, and also the minimum angular velocity $\omega_{\zeta_{min}}$ at which the GT reacts; parameters of the GT are shown in example 8.1; magnitude of friction torque on the axis of rotation of the GT frame, is taken equal to $|M_{fr}| = K_y = 3$ g-cm.

Solution. 1. By formula (8.2.106) we determine the magnitude of the dead zone of the GT frame due to friction:

$$\varepsilon_{cr}^{r} = \mp \frac{K_y}{c} = \mp \frac{3}{1200} = \mp 0{,}0025 \text{ rad} = \mp 8.6:$$

2. According to (8.2.107) we calculate the least angular velocity $\omega_{\zeta_{min}}$, to which the GT reacts:

$$\omega_{\zeta_{min}} > \frac{K_y}{H} = \frac{3}{2093} = 1{,}433 \cdot 10^{-3} \text{ 1/sec} = 0{,}0821 \text{ deg/sec}.$$

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*Besides what is shown, the threshold of sensitivity of a GT is determined by the threshold of sensitivity of the signal pickoff.

From example 8.4 it follows that the considered GT, of course, cannot sense the angular velocity of sidereal rotation of the earth $U = 7.29 \cdot 10^{-5}$ 1/sec.

Another form of instrument errors of a GT is caused by variations of parameters of the instrument. Considering (8.2.5), we rewrite (8.2.7) in the form

$$\beta = \frac{H}{c_1 l_1} \omega_c. \qquad (8.2.108)$$

where c_1 — coefficient of linear spring force of the spring;

l_1 — arm of a lever (Fig. 8.1).

Changes of H, c_1, l_1 relative to their design values will cause change of angle β, i.e., error of instrument readings. Let us designate true (design) values of H, c_1, l_1 by H_0, c_{10}, l_{10}; to them there corresponds angle β_0 of rotation of the GT frame, equal to

$$\beta_0 = \frac{H_0}{c_{10} l_{10}} \omega_c. \qquad (8.2.109)$$

We assume that in a manufactured instrument the shown parameters differ from their design values by δH, δc_1, δl_1 and are, correspondingly,

$$H = H_0 + \delta H; \quad c_1 = c_{10} + \delta c_1; \quad l_1 = l_{10} + \delta l_1; \qquad (8.2.110)$$

here angle β will differ from the design value of β by

$$\varepsilon^{par} = \delta\beta = \beta - \beta_0. \qquad (8.2.111)$$

which represents error of the GT from change of the shown parameters of the instrument.

Usually changes of these parameters, in this case δH, δc_1, δl_1, are small as compared to their design values H_0, c_{10}, l_{10}. To determine $\varepsilon^{par} = \delta\beta$ we expand the expression for β in a Taylor series in the vicinity of design values H_0, c_{10}, l_{10}, we reject small components, starting from the second order of smallness, and we replace differentials by finite increments, for which we take δH, δc_1, δl_1. Then, in accordance with (8.2.108), we have

$$\varepsilon^{par} = \delta\beta = \frac{\partial\beta_0}{\partial H_0} \delta H + \frac{\partial\beta_0}{\partial c_{10}} \delta c_1 + \frac{\partial\beta_0}{\partial l_{10}} \delta l_1. \qquad (8.2.112)$$

For partial derivatives we have

$$\frac{\partial \beta_0}{\partial H_0} = \frac{1}{c_{10} l_{10}^2} \omega_\zeta; \quad \frac{\partial \beta_0}{\partial c_{10}} = -\frac{H_0}{c_{10}^2 l_{10}^2} \omega_\zeta; \quad \frac{\partial \beta_0}{\partial l_{10}} = -\frac{2H_0}{c_{10} l_{10}^3} \omega_\zeta. \tag{8.2.113}$$

Substituting (8.2.113) in (8.2.112), we find

$$\varepsilon^{par} = \beta_0 \left[\frac{\delta H}{H_0} - \frac{\delta c_1}{c_{10}} - \frac{2\delta l_1}{l_{10}} \right]. \tag{8.2.114}$$

Thus, to determine ε^{par} it is necessary to know changes δH, δc_1, δl_1 of parameters of the GT, which one should establish by experiment proceeding from manufacturing tolerances of the instrument.

The other forms of instrument errors of a GT, e.g., due to static and dynamic unbalances of the gyro unit, etc., can be determined by methods applied during calculation of corresponding instrument errors of a free astatic gyroscope (§ 5.5).

6. Dynamic Errors of a GT

a) Characteristic of Dynamic Errors of a GT.

Most fully characterizing errors of a GT are their dynamic errors, arising in real conditions of use of the instrument during continuous change of the measured parameter (angular velocity of the vehicle) and with variable external disturbances. Among dynamic errors of a GT are:

1) dynamic errors appearing during change of angular velocity of the vehicle by harmonic law;

2) random dynamic errors appearing during change of angular velocity of the vehicle by random law;

3) error from friction in the spin axis of the instrument frame;

4) errors caused by components of angular velocity of the vehicle along axes which do not coinciding with the measuring axis of the instrument.

Let us turn to consideration of the shown errors of a GT.

b) Dynamics of a GT Under Harmonic Inputs.

Let us consider the case when the input signal, i.e., the angular velocity $\omega_\zeta(t)$ of the vehicle, measured by the instrument, varies by harmonic law. Here, disturbances are not considered, since there are subsequently considered as random functions of time.

First, we shall consider the general case, when the measured angular velocity $\omega_\zeta(t)$ is any definite function of time. Let us find the angle of deflection $\beta(t)$ of the instrument frame appearing here. Let us use differential equation (8.2.48) for the GT frame; considering in it disturbances $\dot{\omega}_\eta = 0$, $M_{fr} = 0$, $f(t) = 0$, we

rewrite (8.2.48) as follows:

$$\ddot{\beta} + 2\zeta n_д \dot{\beta} + n^2 \beta = k_1 \omega_\zeta(t). \qquad (8.2.115)$$

The integral of this equation can be presented in the form

$$\beta(t) = k_1 e^{-\zeta n t}(C_1 \cos n_д t + C_2 \sin n_д t) +$$

$$+ \frac{k_1}{n_д} e^{-\zeta n t} \int_0^t e^{\zeta n \tau} \omega_\zeta(\tau) \sin n_д (t - \tau) d\tau, \qquad (8.2.116)$$

where C_1 and C_2 are constants, determined by initial conditions,

$$n_д^2 = n^2 - \zeta^2 n^2 = n^2(1 - \zeta^2). \qquad (8.2.117)$$

Quantity n_d, for which we earlier obtained formula (8.2.94), is analogous to (8.2.117) (since $1/T = n$; T – time constant of the GT), constitutes the frequency of natural damped oscillations of the instrument frame.

The first component of formula (8.2.116), analogous to expression (8.2.96), characterizes atural damped oscillations of the GT. A certain time interval after switching on the instrument, when its natural oscillations are damped, it will accomplish oscillations, determined, according to (8.2.116), by formula

$$\beta(t) = \frac{k_1}{n_д} e^{-\zeta n t} \int_0^t e^{\zeta n \tau} \omega_\zeta(\tau) \sin n_д (t - \tau) d\tau. \qquad (8.2.118)$$

By the operating principle of the GT deflection β of the instrument frame should in some scale characterize the angular velocity of the vehicle $\omega_\zeta(t)$, measured by it. In order to find this, we shall use a transformation of formula (8.2.118); calculating this integral in parts, where $\omega_\zeta(0) = 0$, we obtain

$$\beta(t) = \frac{k_1}{n_д^2} \omega_\zeta(t) - \frac{k_1}{n_д^2} e^{-\zeta n t} \int_0^t \frac{d}{d\tau}\left[e^{\zeta n \tau} \omega_\zeta(\tau)\right] \cos n_д (t - \tau) d\tau. \qquad (8.2.119)$$

Let us consider the first component, designating it by β_1; considering (8.2.117) and (8.2.50), we have

$$\beta_1 = \frac{k_1}{n_д^2} \omega_\zeta(t) = \frac{k_1}{n^2(1 - \zeta^2)} \omega_\zeta(t) = \frac{k}{1 - \zeta^2} \omega_\zeta(t). \qquad (8.2.120)$$

Comparing (8.2.120) with (8.2.98), we see that β_1 for $\zeta = 0$ constitutes static deflection of the GT frame, i.e., deflection of it under the condition that the

measured angular velocity of the vehicle, $\omega_\zeta(t)$, acts on the instrument "statically." The difference between β_1 and β_{stat} is

$$\Delta\beta = \beta_1 - \beta_{cr} = \frac{k}{1-\zeta^2}\omega_\zeta - k\omega_\zeta = \frac{\zeta^2 k}{1-\zeta^2}\omega_\zeta. \qquad (8.2.121)$$

The second component of formula (8.2.119) is that correction, which it is necessary to add to "static" deflection in order to obtain "dynamic" deflection β. The magnitude of this correction is

$$\beta_\Delta = -\frac{k_1}{n_1^2}e^{-\zeta n t}\int_0^t \frac{d}{d\tau}\left[e^{\zeta n \tau}\omega_\zeta(\tau)\right]\cos n_1(t-\tau)\,d\tau \qquad (8.2.122)$$

A. N. Krylov [67] called this dynamic correction. In this case it is the dynamic error of the GT during measurement by it of angular velocity of a vehicle, varying by the law $\omega_\zeta(t)$. It has been shown [67] that deflection of a system is close to static when the period of natural oscillations of it is small as compared to the period of change of the input. Here, in accordance with what was said above, dynamic correction will be small. In the case of a GT, as shown in Par. 4, this condition is satisfied, i.e., the period of natural oscillations of a GT is small as compared to the period of change of the input $\omega_\zeta(t)$. For known parameters of the GT and with assigned character of $\omega_\zeta(t)$ dynamic error of the GT can be calculated by formula (8.2.122).

The expression for deflections of a GT with an assigned law of change of angular velocity $\omega_\zeta(t)$, given earlier, were obtained by means of direct integration of the differential equations of motion of the GT. This problem can be solved, if we know the frequency-response curve of the GT. Actually, knowing its gain-phase response $W(j\omega)$, by formula (8.2.76) we can find the impulse response $k_\beta(t)$ of the GT, and then, using an expression, analogous to (3.2.66), we can find transient response $\beta(t)$, caused by the angular velocity $\omega_\zeta(t)$ of rotation of the vehicle:

$$\beta(t) = \int_0^t \omega_\zeta(t-\tau)\,k_\beta(\tau)\,d\tau. \qquad (8.2.123)$$

Most simply investigated is the dynamics of a GT in that particular case, quite often encountered, when $\omega_\zeta(t)$ varies by harmonic law. Let us assume that

$$\omega_\zeta(t) = a_\zeta \sin \omega t, \qquad (8.2.124)$$

where a_ζ – peak value of angular velocity;

ω – angular frequency of its change.

Let us substitute (8.2.124) in (8.2.115); then equation GT will obtain form

$$\ddot{\beta} + 2\xi n\dot{\beta} + n^2\beta = k_1 a_\zeta \sin \omega t. \qquad (8.2.125)$$

Considering natural oscillation of the GT to be damped, we shall limit ourselves to consideration of forced oscillations $\beta_\text{в}(t)$ of the instrument frame. From automatic control theory we know (see, for instance, [107]) that during application to a dynamic system of a harmonic input its forced oscillations, in this case of the GT frame, will be determined by relationship

$$\beta_\text{в}(t) = a_\zeta A(\omega) \sin[\omega t + \varphi(\omega)]. \qquad (8.2.126)$$

where $A(\omega)$ and $\varphi(\omega)$ – gain and phase frequency-response curves of the GT, determined by formulas (8.2.74).

From (8.2.126) it follows that forced oscillations of the GT frame, caused by harmonic input $\omega_\zeta(t)$, are also a harmonic function of time, distinguished from the input in amplitude and phase, but having the same angular frequency ω, as the input.

Let us divide both parts of equation (8.2.125) by n^2; considering (8.2.49) and (8.2.50), we obtain

$$T^2\ddot{\beta} + 2\xi T\dot{\beta} + \beta = ka_\zeta \sin \omega t. \qquad (8.2.127)$$

For the amplitude response $A(\omega)$ and phase response $\varphi(\omega)$ of the GT we have relationships (8.2.74), i.e.,

$$A(\omega) = \frac{k}{\sqrt{(1-T^2\omega^2)^2 + 4\xi^2T^2\omega^2}}; \quad \operatorname{tg}\varphi(\omega) = -\frac{2\xi T\omega}{1-T^2\omega^2}. \qquad (8.2.128)$$

According to (8.2.126) amplitude of forced oscillations of the GT will be

$$\beta_\text{в}^0 = a_\zeta A(\omega) = \frac{a_\zeta k}{\sqrt{(1-T^2\omega^2)^2 + 4\xi^2T^2\omega^2}}. \qquad (8.2.129)$$

If angular velocity of the vehicle influenced the GT statically, i.e., had constant value a_ζ, the corresponding static deflection of the GT frame, according to (8.2.127), would be

$$\beta_{\text{ст}} = a_\zeta k. \qquad (8.2.130)$$

Ratio $\frac{p_f^0}{p_{stat}}$, designated by λ, is

$$\lambda = \frac{p_f^0}{p_{st}} = \frac{1}{\sqrt{(1-T^2\omega^2)^2 + 4\zeta^2 T^2 \omega^2}}. \qquad (8.2.131)$$

Quantity λ is called the coefficient of dynamicity [67].

We designate

$$\xi = T\omega; \qquad (8.2.132)$$

considering (8.2.49), we have

$$\xi = \frac{\omega}{n}, \qquad (8.2.133)$$

i.e., ξ is the ratio of the frequency ω of change of angular velocity of the vehicle to the frequency n of natural oscillations of the GT. Introducing (8.2.132) in (8.2.131), we obtain

$$\lambda = \frac{1}{\sqrt{(1-\xi^2)^2 + 4\zeta^2\xi^2}}. \qquad (8.2.134)$$

An investigation of the dependence of coefficient λ on ζ and ξ was performed by A. N. Krylov [67]. This question is expounded in detail in courses of theoretical mechanics (see, for instance, [85]). Therefore, we shall only give certain brief information.

From (8.2.131) it follows that for decrease of dynamic amplitude distortions of GT readings the coefficient of dynamicity λ in the range of frequencies in which the frequency ω of oscillations of the vehicle is located, should be sufficiently close to one. To show the dependence $\lambda(\xi, \zeta)$ we construct curves (Fig. 8.7) of coefficient $\lambda(\xi)$ for different values of ζ. From these curves it follows that when $\xi \ll 1$ coefficient of dynamicity λ differs little from one. This explains the recommendation of A. N. Krylow [67] that we select a relationship between frequencies of natural oscillations of the system and the input so that coefficient ξ is considerably less than one.

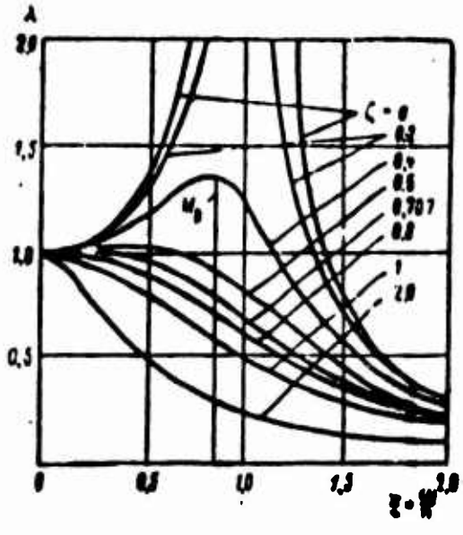

Fig. 8.7. Curves of $\lambda(\xi, \zeta)$.

The curves of Fig. 8.7 permit us to establish the optimum value of the damping factor ζ. Actually, for values of ζ lying in the range $0 < \zeta < 0.707$, curve $\lambda(\xi)$ has resonance peak M_{res}, the magnitude of which is determined by known relationship (see, e.g., [80b; 85])

$$M_p = \frac{1}{2\zeta\sqrt{1-\zeta^2}}. \qquad (8.2.135)$$

For values of $\zeta > 0.707$ we have $M_{res} < 1$, i.e., curves $\lambda(\xi)$ do not have a resonance peak. It is necessary to note that the maximum value of coefficient λ, and, consequently, also of the amplitude of forces oscillations of the GT will not be at $\xi = \frac{\omega}{n} = 1$ (point of resonance for $\zeta = 0$), but at $\xi = \sqrt{1-2\zeta^2}$ [85], i.e., at frequency $\omega = n\sqrt{1-2\zeta^2}$. Using formula (8.2.135), it is easy to construct the curve of the dependence of relative damping factor ζ from resonance peak M_{res} (Fig. 8.8). From the curves in Fig. 8.7 and Fig. 8.8 it becomes understandable why, during selection of parameters of a GT for the optimum value of relative damping factor we usually take ζ = = 0.707. In this case, for a considerable range of values of ξ (Fig. 8.7) coefficient λ differs little from one, and the resonance peak is practically absent (Fig. 8.8).

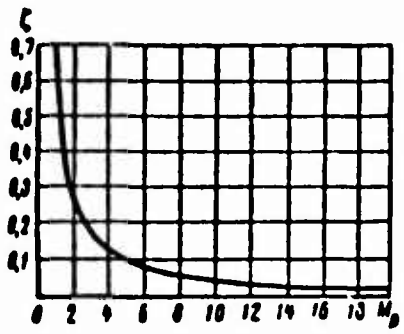

Fig. 8.8. Curve of the dependence of coefficient ζ on the magnitude of M_{res}.

For dynamic error of a GT for $\omega_\zeta(t)$, which follows a harmonic law, we have

$$\varepsilon_\lambda = \omega_\zeta(t) - \frac{1}{k}\beta_\omega(t) \qquad (8.2.136)$$

or, considering (8.2.126) and (8.2.124),

$$\varepsilon_\lambda = a_\zeta \sin\omega t - \frac{a_\zeta}{k} A(\omega)\sin[\omega t + \varphi(\omega)]. \qquad (8.2.137)$$

For convenience of calculation [114] error ε_{dyn} is characterized by two magnitudes: error in amplitude

$$\varepsilon_\lambda^0 = \frac{a_\zeta - \frac{a_\zeta}{k}A(\omega)}{a_\zeta} 100\% = \left[1 - \frac{A(\omega)}{k}\right]100\% = \left[1 - \frac{1}{\sqrt{(1-T^2\omega^2)^2 + 4\zeta^2T^2\omega^2}}\right]100\% \qquad (8.2.138)$$

and error in phase

$$\Delta\varphi = \varphi(\omega). \qquad (8.2.139)$$

Example 8.5. Determine characteristic of forced oscillations of a GT having the parameters given in example 8.1, during input in it of angular velocity of the vehicle, varying by harmonic law $\omega_\zeta = a_\zeta \sin \omega t$, where $a_\zeta = 3^\circ/\text{sec}$, $\omega = 2$ 1/sec; the GT's relative damping factor $\zeta = 0.707$.

Solution. 1. By formula (8.2.133) we find the value of coefficient ξ:

$$\xi = \frac{\omega}{n} = \frac{2}{34,47} = 0,05802.$$

2. According to (8.2.134) we determine the magnitude of the coefficient of dynamicity λ:

$$\lambda = \frac{1}{\sqrt{(1-\xi^2)^2 + 4\zeta^2\xi^2}} = \frac{1}{\sqrt{(1-0,05802^2)^2 + 4\cdot 0,707^2\cdot 0,05802^2}} = 1,0002.$$

3. By formula (8.2.129), considering (8.2.132), we calculate the value of the amplitude β_f^0 of forced oscillations of the GT frame:

$$\beta_f^0 = \frac{a_\zeta \lambda}{\sqrt{(1-\xi^2)^2 + 4\zeta^2\xi^2}} = a_\zeta\lambda = 3\cdot 1,744\cdot 1,0002 = 5.233^\circ.$$

4. Using (8.2.128) and (8.2.132), we find phase shift angle $\varphi(\omega)$:

$$\operatorname{tg}\varphi(\omega) = -\frac{2\zeta T\omega}{1-T^2\omega^2} = -\frac{2\zeta\xi}{1-\xi^2} = -\frac{2\cdot 0,707\cdot 0,05802}{1-0,05802^2} = -0,08232$$

5. According to (8.2.130) we calculate the magnitude of static deflection β_{stat} of the GT frame under the condition that angular velocity of the vehicle acts "statically," i.e., has a constant value:

$$\beta_{GT} = a_\zeta k = 3\cdot 1,744 = 5.232^\circ.$$

6. By formula (8.2.138) we determine dynamic error of the GT in amplitude:

$$\varepsilon_A^0 = \left[1 - \frac{1}{\sqrt{(1-T^2\omega^2)^2 + 4\zeta^2T^2\omega^2}}\right]100\% = \left[1 - \frac{1}{\sqrt{(1-\xi^2)^2 + 4\zeta^2\xi^2}}\right]100\%;$$

$$|\varepsilon_A^0| = \left|\left[1 - \frac{1}{\sqrt{(1-0,05802^2)^2 + 4\cdot 0,707^2\cdot 0,05802^2}}\right]100\%\right| = 0.02\%.$$

c) Dynamics of a GT During Random Inputs.

Characteristics of the input signal. Angular velocity $\omega_\zeta(t)$, most frequently measured by the instrument, and also disturbances, caused by friction in the axis of rotation of the instrument frame and components of angular velocity of the vehicle along axes, not coinciding with the measuring axis of the instrument, are random functions of time. Here, readings $\beta(t)$ of the GT and also its errors $\varepsilon(t)$ in dynamics are also random functions of time, determined by the corresponding probability characteristics. First, we shall consider the reaction of the instrument to a random input $\omega_\zeta(t)$; disturbances we consider absent.

Let us turn to the schematic of a GT (Fig. 8.1), fixed on a ship (Par. 3a). The instrument with the shown location of the axes will measure component $\omega_\zeta(t)$ of angular velocity of the ship, which, according to (8.2.21), with small roll angles is equal to angular velocity of yawing, i.e.,

$$\omega_\zeta(t) = -\dot{\varphi}(t). \tag{8.2.140}$$

Considering in equation (8.2.51) disturbances $\dot{\omega}_{\eta} = 0$, $M_{fr} = 0$, $f(t) = 0$ and considering (8.2.140), we have

$$T^2\ddot{\beta} + 2\chi T\dot{\beta} + \beta = -k\dot{\varphi}(t). \tag{8.2.141}$$

Thus, to the GT input there proceeds $\dot{\varphi}(t)$, which is a normal stationary random function of time (§ 2.1, Par. 3a). For analysis during this motion of the GT frame it is necessary first of all to know the probability characteristics of randum function $\dot{\varphi}(t)$.

Correlation function $K_\varphi(\tau)$ of the angle of yawing of the ship, according to (2.1.52), if drop indices in the letter designations, can be presented by relationship

$$K_\varphi(\tau) = Ae^{-\mu|\tau|}\left(\cos\lambda\tau + \frac{\mu}{\lambda}\sin\lambda|\tau|\right). \tag{8.2.142}$$

where $A = D[\varphi]$ – dispersion of angles of yawing;

μ – coefficient of irregularity of angles of yawing;

λ – predominant frequency of yawing from irregular swell.

Correlation function $K_{\dot{\varphi}}(\tau)$ or random function $\dot{\varphi}(t)$, by analogy with (2.1.69, will be

$$K_{\dot{\varphi}}(\tau) = A(\mu^2 + \lambda^2)e^{-\mu|\tau|}\left(\cos\lambda\tau - \frac{\mu}{\lambda}\sin\lambda|\tau|\right). \tag{8.2.143}$$

For the spectral density $S_\varphi(\omega)$ of random function $\varphi(t)$, according to (2.1.55), we have

$$S_\varphi(\omega) = \frac{2A\mu}{\pi} \cdot \frac{b^2}{\omega^4 + 2a\omega^2 + b^4}, \qquad (8.2.144)$$

where, by analogy with (2.1.56),

$$a = \mu^2 - \lambda^2; \quad b^2 = \mu^2 + \lambda^2. \qquad (8.2.145)$$

The spectral density $S_{\dot\varphi}(\omega)$ of random function $\dot\varphi(t)$, if we consider (2.1.81), will be determined by relationship

$$S_{\dot\varphi}(\omega) = \omega^2 S_\varphi(\omega) \qquad (8.2.146)$$

or, introducing (8.2.144), we have

$$S_{\dot\varphi}(\omega) = \frac{2A\mu}{\pi} \cdot \frac{b^2\omega^2}{\omega^4 + 2a\omega^2 + b^4}. \qquad (8.2.147)$$

Example 8.6. Construct the curve of spectral density $S_{\dot\varphi}(\omega)$ of the angular velocity $\dot\varphi(t)$ of yawing of a ship, measured by a GT, for data given in example 4.8:

$$A = D[\varphi] = 0{,}6695 \cdot 10^{-3}, \quad \mu = 0{,}03 \text{ 1/sec}, \quad \lambda = 0{,}21 \text{ 1/sec}.$$

Solution. By formula (8.2.147), considering (8.2.145), we calculate ordinates of $S_{\dot\varphi}(\omega)$ for different values of ω. The graph of curve $S_{\dot\varphi}(\omega)$ is shown in Fig. 8.9.

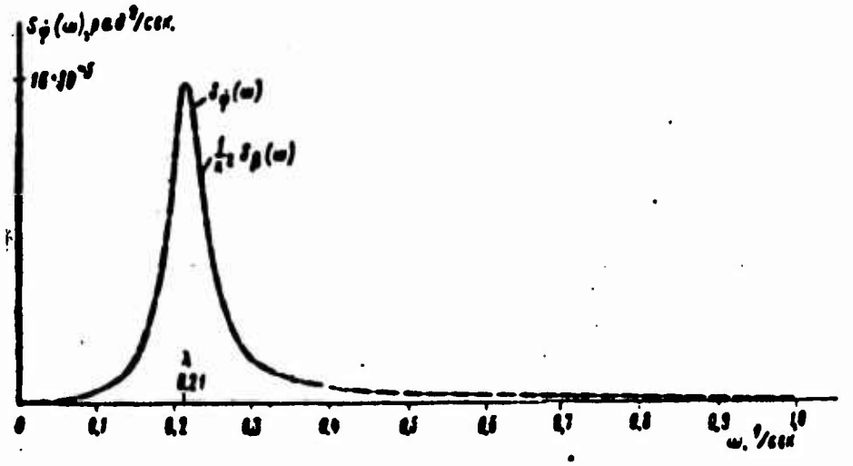

Fig. 8.9. Curves of spectral densities $S_{\dot\varphi}(\omega)$ and $S_\mu(\omega)$.

Dispersion of angular velocity of yawing $D[\dot\varphi]$, by analogy with (2.1.72), will be

$$D[\dot\varphi] = (\mu^2 + \lambda^2) D[\gamma]; \qquad (8.2.148)$$

for the mean quadratic value of $\sigma_{\dot\varphi}$ we have

$$\sigma_{\dot{\varphi}} = \sqrt{\mu^2 + \lambda^2}\,\sigma_{\varphi}. \qquad (8.2.149)$$

Example 8.7. Calculate the dispersion and mean quadratic value of angular velocity of yawing of a ship for the conditions of example 8.6.

Solution. 1. By formula (8.2.148) we calculate the value of dispersion of angular velocity of yawing of the ship $D[\dot{\varphi}]$:

$$D[\dot{\varphi}] = (\mu^2 + \lambda^2)\,D[\varphi] = (0{,}03^2 + 0{,}21^2)\,0{,}6695 \cdot 10^{-3} = 0{,}3013 \cdot 10^{-1}\ 1/\text{sec}^2.$$

2. We determine the mean quadratic value of the angle of yawing σ_{φ}:

$$\sigma_{\varphi} = \sqrt{D[\varphi]} = \sqrt{0{,}6695 \cdot 10^{-3}} = 0{,}02587 \text{ rad} = 1^{\circ}29'.$$

3. According to (8.2.149) we find the mean quadratic value of the angular velocity of yawing $\sigma_{\dot{\varphi}}$:

$$\sigma_{\dot{\varphi}} = \sqrt{\mu^2 + \lambda^2}\,\sigma_{\varphi} = \sqrt{0{,}03^2 + 0{,}21^2} \cdot 0{,}02587 = 0{,}005187\ 1/\text{sec} =$$

$$= 0{,}3144 \text{ deg/sec}.$$

Characteristics of the output signal. Let us turn to determination of probability characteristics of random function $\beta(t)$ at the GT output. In accordance with (8.2.141), $\beta(t)$ also is a normal stationary random function of time. Mathematical expectation $\bar{\beta}$ upon completion of the transient response, if $\overline{\dot{\varphi}(t)} = 0$ [see (8.1.34)] will also be equal to zero:

$$\bar{\beta} = 0. \qquad (8.2.150)$$

Let us determine the spectral density $S_{\beta}(\omega)$ of random function $\beta(t)$. By analogy with (4.6.47) we have

$$S_{\beta}(\omega) = |W(j\omega)|^2 S_{\dot{\varphi}}(\omega), \qquad (8.2.151)$$

where $W(j\omega)$ – gain-phase response of the GT, determined by relationship (8.2.72).

Substituting (8.2.72) and (8.2.147) in (8.2.151), we obtain

$$S_{\beta}(\omega) = \frac{k^2}{(1 - T^2\omega^2)^2 + 4\xi^2 T^2 \omega^2} \cdot \frac{2A\mu}{\pi} \cdot \frac{b^2\omega^2}{\omega^4 + 2a\omega^2 + b^4}. \qquad (8.2.152)$$

Example 8.8. Construct the curve of the spectral density $S_{\beta}(\omega)$ of random

function $\beta(t)$ of GT oscillations under conditions of irregular yawing of a ship for the initial data of examples 8.1 and 8.6; relative damping factor $\zeta = 0.707$.

Solution. By formula (8.2.152), considering (8.2.145), we calculate ordinates of $S_\beta(\omega)$ for different values of ω. The graph of curve $S_\beta(\omega)$ is shown in Fig. 8.10.

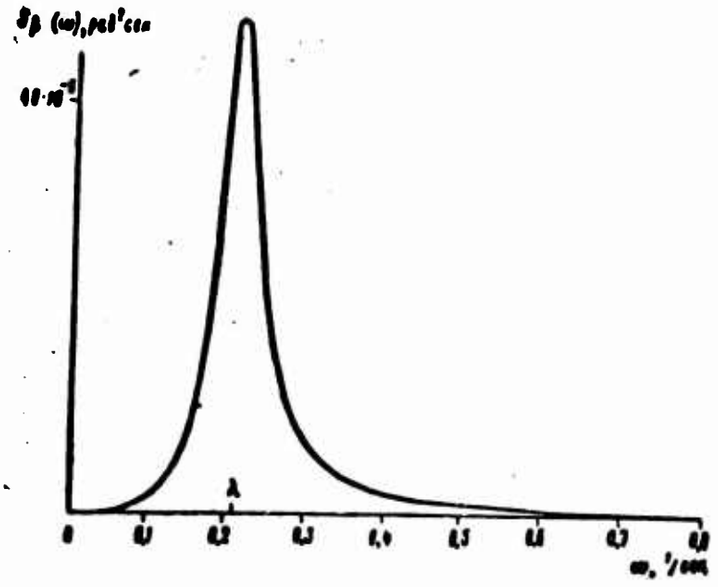

Fig. 8.10. Curve of spectral density $S_\beta(\omega)$.

From Fig. 8.10 it is clear that spectrum $S_\beta(\omega)$ of angles of deflection of the GT frame is similar to the spectrum $S_{\dot\varphi}(\omega)$ of angular velocity of yawing of the ship (Fig. 8.9). In order to find how closely these curves coincide we find the output spectrum of the GT in the scale of angular velocity of the vehicle, i.e., considering (8.2.7) and (8.2.50), we construct in Fig. 8.9 curve $\frac{1}{k^2} S_\beta(\omega)$ (dotted line). It is clear that curves $S_{\dot\varphi}(\omega)$ and $\frac{1}{k^2} S_\beta(\omega)$ practically coincide. This is explained by the fact that [see (8.2.147) and (8.2.152)] the square of the modulus of the transfer function of the GT, $|W(j\omega)|^2 = \frac{k^2}{(1-T^2\omega^2)^2 + 4\zeta^2T^2\omega^2}$, in the region of substantial values of frequencies of the spectrum $S_{\dot\varphi}(\omega)$ of the input signal is practically constant and the coefficient of dynamicity,

$$\lambda = \frac{1}{\sqrt{(1-T^2\omega^2)^2 + 4\zeta^2T^2\omega^2}} = \frac{1}{\sqrt{(1-\quad)^2 + 4\zeta^2\quad}}$$

differs little from one. Consequently, the GT reproduces the random accidental input signal well.

Let us determine dispersion $D[\beta]$ of the angle of rotation of the frame of the instrument. We use a formula of type (2.1.41); considering (8.2.152), we obtain

$$D[\beta] = \int_{-\infty}^{\infty} \frac{k^2}{(1-T^2\omega^2)^2 + 4\zeta^2T^2\omega^2} \cdot \frac{2A_1 A_2}{\pi} \cdot \frac{b^2\omega^2}{\omega^4 + 2a\omega^2 + b^4}\, d\omega. \qquad (8.2.153)$$

We designate

$$B = 4k^2A_1A_2b^2; \qquad (8.2.154)$$

then we rewrite (8.2.153) in form

$$D[\beta] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{(1-T^2\omega^2)^2 + 4\zeta^2T^2\omega^2} \cdot \frac{B\omega^2}{\omega^4 + 2a\omega^2 + b^4}\,d\omega. \qquad (8.2.155)$$

Determination of D[β] can be reduced to calculation of a tabular integral of form (4.6.71). For this we present (8.2.155) as follows:

$$D[\beta] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{B\omega^2}{|(1-T^2\omega^2 + 2\zeta Tj\omega)(\omega^2 - 2\mu j\omega - b^2)|^2}\,d\omega. \qquad (8.2.156)$$

Integral (8.2.156) can be reduced to form (4.6.71), considering in (8.2.156) the numerator (multiplied by j) equal to $G_n(\omega)$ and the form in the denominator of this expression under the sign of the modulus, equal to $H_n(\omega)$. Then, in accordance with (4.6.72) and (8.2.156), we obtain for coefficients a_0 ... and b_0 ... the following formulas:

$$\left.\begin{aligned} a_0 &= -T^2 & b_0 &= 0\\ a_1 &= 2T(\zeta + T\mu)j; & b_1 &= 0\\ a_2 &= 1 + 4\zeta T\mu + T^2b^2 & b_2 &= Bj\\ a_3 &= -2(\zeta Tb^2 + \mu)j; & b_3 &= 0\\ a_4 &= -b^2 \end{aligned}\right\} \qquad (8.2.157)$$

Considering the known expression of integral I_n [153] in terms of coefficients a_0 ... a_4, b_0 ... b_3, we obtain a formula for D[β] in form

$$D[\beta] = \frac{b_0(-a_1a_4 + a_2a_3) - a_0a_3b_1 + a_0a_1b_2 + \frac{a_0b_3}{a_4}(a_0a_3 - a_1a_2)}{2a_0(a_0a_3^2 + a_1^2a_4 - a_1a_2a_3)}. \qquad (8.2.158)$$

Substituting (8.2.157) in (8.2.158) and taking into account (8.2.154), we obtain

$$D[\beta] = k^2\frac{b^2\left(1 + \frac{T\mu}{\zeta}\right)}{(1-T^2b^2)^2 + 4[T^2\zeta^2b^2 + T\mu(\zeta + T\mu) + T^2\mu b^2]}D[\varphi]. \qquad (8.2.159)$$

For the mean quadratic value σ_β of the angle of deflection of the GT frame we have

$$\sigma_\beta = \sqrt{D[\beta]}. \qquad (8.2.160)$$

Example 8.9. Calculate the mean quadratic value σ_β of deflection of the frame of the GT during measurement by it of the angular velocity of yawing of a ship for the initial data of examples 8.1 and 8.6; determine also limiting value β_m of the

angle of rotation of the frame, assuming that the probability Q_z that the current angle β exceeds the limit β_m should not be more than 0.1%.

Solution. 1. By formula (8.2.159) we find $D[\beta]$:

$$D[\beta] = 9{,}153 \cdot 10^{-5}.$$

2. According to (8.2.160) we calculate σ_β:

$$\sigma_\beta = \sqrt{D[\beta]} = \sqrt{9{,}153 \cdot 10^{-5}} = 0{,}9567 \cdot 10^{-2} \text{ rad} = 33'.$$

3. For $Q_z = 0.1\%$ by Table 1. (§ 4.6, Par. 10) we determine $z = 3.3$; then, by analogy with (4.6.87), we calculate β_m (when $\bar{\beta} = 0$)

$$\beta_m = z\sigma_\beta = 3{,}3 \cdot 0{,}9567 \cdot 10^{-2} = 1°49'.$$

Dynamic error of reproduction of the input signal. Let us find the dynamic error of a GT during measurement by it of the angular velocity of a vehicle, representing a random function of time.

We determine dynamic error $\varepsilon(t)$ of the GT, as usual in statistical dynamics [153], by the difference between the useful input signal $\omega_\zeta(t)$ and $\frac{1}{k}\beta(t)$ at the output:

$$\varepsilon(t) = \omega_\zeta(t) - \frac{1}{k}\beta(t), \tag{8.2.161}$$

where to get the general formula angular velocity $\dot{\varphi}(t)$ of yawing of the ship is designated by $\omega_\zeta(t)$.

For mathematical expectation $\overline{\varepsilon(t)}$ we have

$$\overline{\varepsilon(t)} = \overline{\omega_\zeta(t)} - \frac{1}{k}\overline{\beta(t)}$$

or, considering (8.2.150) and the fact that $\overline{\omega_\zeta(t)} = 0$,

$$\overline{\varepsilon(t)} = 0. \tag{8.2.162}$$

Let us determine dispersion $D[\varepsilon]$. For this we present input $\omega_\zeta(t)$ in the form of a spectral expansion, analogous to (4.6.149):

$$\omega_\zeta(t) = \int_{-\infty}^{\infty} e^{i\omega t} d\Phi_{\omega_\zeta}(\omega). \tag{8.2.163}$$

Substituting (8.2.163) in equation (8.2.85), we obtain upon completion of the

transient a steady-state solution of this equation in a form, analogous to (4.6.153):

$$\beta(t) = \int_{-\infty}^{\infty} W(j\omega) e^{j\omega t} d\Phi_{\omega_\zeta}(\omega). \qquad (8.2.164)$$

Substituting (8.2.163) and (8.2.164) in (8.2.161), we obtain

$$\varepsilon(t) = \int_{-\infty}^{\infty} \left[1 - \frac{1}{k} W(j\omega)\right] e^{j\omega t} d\Phi_{\omega_\zeta}(\omega). \qquad (8.2.165)$$

Comparing (8.2.165) with (8.2.164), we note that dynamic error $\varepsilon(t)$ can be considered the output quantity of a system with transfer function

$$W_\varepsilon(s) = 1 - \frac{1}{k} W(s), \qquad (8.2.166)$$

to whose input there proceeds $\omega_\zeta(t)$. Let us note that $W_\varepsilon(s)$ is the transfer function of error [153]

$$W_\varepsilon(s) = \frac{\varepsilon(s)}{\omega_\zeta(s)}. \qquad (8.2.167)$$

Considering what has been said, we can easily find the spectral density $S_\varepsilon(\omega)$ of error $\varepsilon(t)$. By analogy with (8.2.151), we have

$$S_\varepsilon(\omega) = |W_\varepsilon(j\omega)|^2 S_{\omega_\zeta}(\omega), \qquad (8.2.168)$$

where for $W_\varepsilon(j\omega)$, according to (8.2.166), we obtain

$$W_\varepsilon(j\omega) = 1 - \frac{1}{k} W(j\omega) \qquad (8.2.169)$$

or, taking into account (8.2.72),

$$W_\varepsilon(j\omega) = 1 - \frac{1}{k} \frac{k}{1 - T^2\omega^2 + 2\xi T j\omega} = \frac{2\xi T j\omega - T^2\omega^2}{1 - T^2\omega^2 + 2\xi T j\omega}. \qquad (8.2.170)$$

Substituting relationship (8.2.170) in (8.2.168), and for $S_{\omega_\zeta}(\omega) = S^*_\varphi(\omega)$ expression (8.2.147), we obtain

$$S_\varepsilon(\omega) = \frac{T^4\omega^4 + 4\xi^2 T^2\omega^2}{(1 - T^2\omega^2)^2 + 4\xi^2 T^2\omega^2} \cdot \frac{2A_\varphi}{\pi} \cdot \frac{b^2\omega^2}{\omega^4 + 2a\omega^2 + b^4}. \qquad (8.2.171)$$

Example 8.10. Construct the graph of the spectral density $S_\varepsilon(\omega)$ of dynamic error $\varepsilon(t)$ of a GT, which measures the angular velocity of yawing of a ship $\omega_\zeta(t) = -\dot{\varphi}(t)$, for the initial data of example 8.8.

Solution. By formula (8.2.171), considering (8.2.145), we calculate ordinates

of $S_\varepsilon(\omega)$ for different values of ω. The graph of curve $S_\varepsilon(\omega)$ is shown in Fig. 8.11.

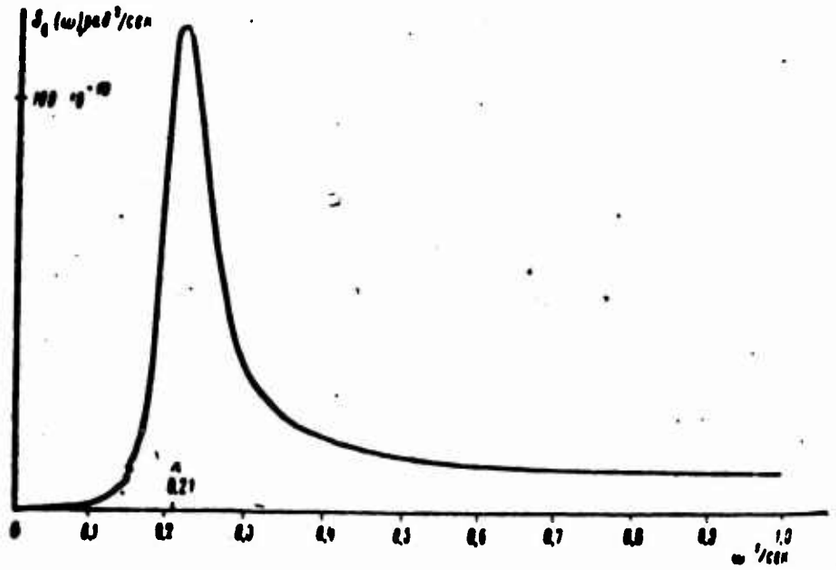

Fig. 8.11. Curve of spectral density $S_\varepsilon(\omega)$.

We determine dispersion D[ε]:

$$D[\varepsilon] = \int_{-\infty}^{\infty} S_\varepsilon(\omega)\, d\omega \tag{8.2.172}$$

or, considering (8.2.171), we obtain

$$D[\varepsilon] = \int_{-\infty}^{\infty} \frac{T^4\omega^4 + 4\zeta^2 T^2\omega^2}{(1-T^2\omega^2)^2 + 4\zeta^2 T^2\omega^2} \cdot \frac{4A_\mu}{\pi} \cdot \frac{b^2\omega^2}{\omega^4 + 2a\omega^2 + b^4}\, d\omega. \tag{8.2.173}$$

We designate

$$B_1 = 4T^4 A_\mu b^2; \quad B_2 = 4\zeta^2 T^2 A_\mu b^2. \tag{8.2.174}$$

Then (8.2.173) we rewrite in form

$$D[\varepsilon] = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{(1-T^2\omega^2)^2 + 4\zeta^2 T^2\omega^2} \cdot \frac{B_1\omega^2 + B_2\omega^4}{\omega^4 + 2a\omega^2 + b^4}\, d\omega. \tag{8.2.175}$$

Determination of D[ε] can be reduced to calculation of a tabular integral of form (4.6.71). For this we present (8.2.173), by analogy with (8.2.156), in the following form:

$$D[\varepsilon] = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{B_1\omega^2 + B_2\omega^4}{|(1-T^2\omega^2 + 2\zeta T j\omega)(\omega^2 - 2\mu j\omega - b^2)|^2}\, d\omega. \tag{8.2.176}$$

Integral (8.2.176) can be reduced to form (4.6.71), considering in (8.2.176) the numerator (multiplied by j) equal to $G_n(\omega)$, and the term in the denominator of

this expression under the sign of the modulus, equal to $H_n(\omega)$. Then, in accordance with (4.6.72) and (8.2.176), we obtain for coefficients a_0 ... and b_0.... the following formulas:

$$\left.\begin{array}{ll} a_0 = -T^2; & b_0 = B_1 j \\ a_1 = 2T(\zeta + T\mu) j; & b_1 = B_2 j \\ a_2 = 1 + 4\zeta T\mu + T^2 b^2; & b_2 = 0 \\ a_3 = -2(\zeta T b^2 + \mu) j; & b_3 = 0 \\ a_4 = -b^2 & \end{array}\right\} \qquad (8.2.177)$$

Considering the known expression of integral I_n [153] in coefficients a_0 ... a_4 and b_0 ... b_3, we have a formula for D[ε] in form

$$D[\varepsilon] = \frac{b_0(-a_1a_4 + a_2a_3) - a_0a_3b_1 + a_0a_1b_3 + \frac{a_0b_3}{a_4}(a_0a_3 - a_1a_2)}{2a_0\left(a_0a_3^2 + a_1^2a_4 - a_1a_2a_3\right)}. \qquad (8.2.178)$$

Substituting (8.2.177) in (8.2.178), taking into account (8.2.174) and considering $A = D[\varphi] = \frac{D[\dot{\varphi}]}{b^2} = \frac{D[^{\omega}\zeta]}{b^2}$, we obtain formula

$$D[\varepsilon] = \frac{T^2\zeta^2\left(b^2 + \frac{\mu}{T\zeta}\right) + T\left(\frac{\mu}{\zeta} + 4T\mu^2 + 4\zeta\mu T^2b^2 + T^3b^4\right)}{(1 - T^2b^2)^2 + 4\left[T^2\zeta^2b^2 + T\mu(\zeta + T\mu) + T^3\zeta\mu b^2\right]} D[\omega_\zeta]. \qquad (8.2.179)$$

which expresses dispersion of dynamic error of the GT depending upon characteristics $D[\omega_\zeta]$, b and μ of the input, and also parameters of the gyrotachometor T and ζ.

Considering that $D[\omega_\zeta] = D[\dot{\varphi}] = b^2D[\varphi]$, instead of (8.2.179) we obtain

$$D[\varepsilon] = b^2\frac{T^2\zeta^2\left(b^2 + \frac{\mu}{T\zeta}\right) + T\left(\frac{\mu}{\zeta} + 4T\mu^2 + 4\zeta\mu T^2b^2 + T^3b^4\right)}{(1 - T^2b^2)^2 + 4\left[T^2\zeta^2b^2 + T\mu(\zeta + T\mu) + T^3\zeta\mu b^2\right]} D[\varphi]. \qquad (8.2.180)$$

For the mean quadratic value σ_ε of dynamic error of the GT we have

$$\sigma_\varepsilon = \sqrt{D[\varepsilon]}. \qquad (8.2.181)$$

Example 8.11. Calculate the mean quadratic value σ_ε of dynamic error of a GT during measurement by it of the angular velocity of yawing of a ship, which is a random function of time, for the initial data of examples 8.1 and 8.6; determine also the limiting value ε_m of GT error, assuming that the probability Q_z that the current value ε of error exceeds limit ε_m should not be more than 0.1%.

Solution. 1. By formula (8.2.180) we find D[ε]:

$$D[\varepsilon] = 0{,}5615 \cdot 10^{-7} \text{ 1/sec}^2.$$

2. According to (8.2.181) we calculate σ_ε:

$$\sigma_\varepsilon = \sqrt{D[\varepsilon]} = \sqrt{0{,}5615 \cdot 10^{-7}} = 0{,}237 \cdot 10^{-3} \text{ 1/sec} = 0.0136^\circ/\text{sec}.$$

3. The same method as in example 8.9 we determine ε_m (for $\bar{\varepsilon} = 0$):

$$\varepsilon_m = z\sigma_\varepsilon = 3{,}3 \cdot 0{,}0136 = 0{,}0449 \text{ °/sec}.$$

From example 8.11 it follows that dynamic error of the GT during measurement of angular velocity of the vehicle, consisting of a random function of time, is comparatively small.

Dynamic error ε(t) upon a random signal entering the GT, considered above, is a result both of amplitude and phase distortions of instrument readings. During solution of certain problems it is of interest to reveal for the same conditions the mean value of phase shifts of output quantity β(t) with respect to input $\omega_\zeta(t)$.

The probability characteristic of these phase shifts, or, more exactly, the time of delay of the considered dynamic system, is a cross-correlation function of input and output signals [3]. In this case we have

$$K_{\omega_\zeta \beta}(\tau) = M[\omega_\zeta(t)\beta(t+\tau)]. \qquad (8.2.182)$$

If the GT ideally reproduces the input signal, i.e., when, according to (8.2.7) and (8.2.50), $\beta(t) = k\omega_\zeta(t)$, the cross-correlation function will be

$$K_{\omega_\zeta \beta}(\tau) = kM[\omega_\zeta(t)\omega_\zeta(t+\tau)] = kK_{\omega_\zeta}(\tau), \qquad (8.2.183)$$

in other words, will be proportional to correlation function $K_{\omega_\zeta}(\tau)$ of the input signal. As it is known [141], $K_{\omega_\zeta}(\tau)$ has a maximum at $\tau = 0$; consequently, the same will take place for cross-correlation function $K_{\omega_\zeta\beta}(\tau)$, i.e., in the case of an ideal GT phase shifts or time delays will be absent. For a real GT the cross-correlation function, the approximate form of which is shown in Fig. 8.12, has a maximum, not at $\tau = 0$, but with some time shift $\tau = t_{del}$, which cnaracterizes the magnitude of the average delay time of the system. Thus, determination of phase shifts or time delays of the GT during measurement of a random input signal is reduced to finding the expression for the cross-correlation function $K_{\omega_\zeta\beta}(\tau)$ of

input and output signals. We can determine delay time t_{del} by investigation of function $K_{\omega\zeta,\beta}(\tau)$ for a maximum or directly on the graph of this function.

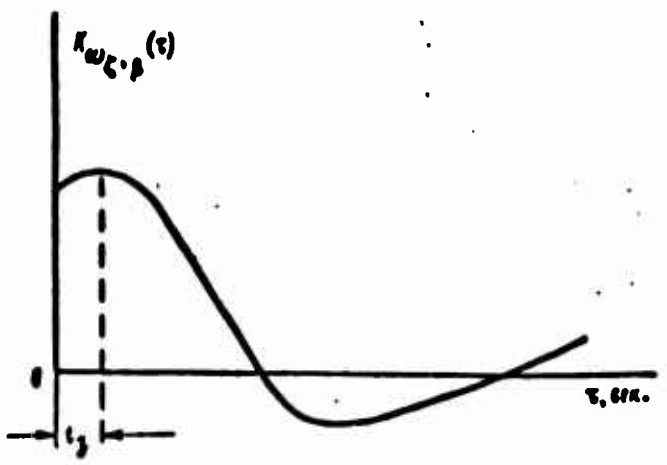

Fig. 8.12. Determining delay of a GT.

We shall give certain indications about the method of determining $K_{\omega\zeta,\beta}(\tau)$. Introducing in formula (8.2.182) for angle $\beta(t)$ its expression (8.2.123) through impulse response $k_\beta(t)$, it is easy to convert $K_{\omega\zeta,\beta}(\tau)$ to form [3]

$$K_{\omega\zeta,\beta}(\tau)=\int_0^\infty k_\beta(l)K_{\omega\zeta}(l-\tau)\,dl, \qquad (8.2.184)$$

The correlation function $K_{\omega\zeta}(\tau)$ of the input signal is assumed known [see (8.2.143)]. We give the derivation of the expression for the impulse response $k_\beta(t)$ of the GT. For $k_\beta(t)$ we had a general expression (8.2.76), i.e.,

$$k_\beta(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}W(j\omega)e^{j\omega t}\,d\omega, \qquad (8.2.185)$$

where for the GT the gain-phase response $W(j\omega)$ is determined by formula (8.2.72).

Considering (3.2.53) and (3.2.54), we present $W(j\omega)$ in form

$$W(j\omega)=\frac{k}{T^2(\lambda_2-\lambda_1)}\left[\frac{1}{\omega-\lambda_1}-\frac{1}{\omega-\lambda_2}\right], \qquad (8.2.186)$$

where λ_1 and λ_2 – roots of equation $1 - T^2\omega^2 + 2\zeta Tj\omega = 0$ [see (8.2.72)], determined by relationships $\lambda_{1,2}=\frac{\zeta}{T}j\pm\frac{\sqrt{1-\zeta^2}}{T}$.

Substituting (8.2.186) in (8.2.185), we obtain an integral,

$$k_\beta(t)=\frac{1}{2\pi T^2}\cdot\int_{-\infty}^{\infty}\frac{k}{\lambda_2-\lambda_1}\left[\frac{1}{\omega-\lambda_1}-\frac{1}{\omega-\lambda_2}\right]e^{j\omega t}d\omega \qquad (8.2.187)$$

of type (3.2.55), whose solution, in light of (3.2.56), will be

$$k_\beta(t)=\frac{k}{T^2(\lambda_2-\lambda_1)}j\left[e^{j\lambda_1 t}-e^{j\lambda_2 t}\right] \qquad (8.2.188)$$

or, taking into account expressions for λ_1 and λ_2,

$$k_\beta(t)=\frac{ke^{-\frac{\zeta}{T}t}}{T\sqrt{1-\zeta^2}}\sin\frac{\sqrt{1-\zeta^2}}{T}t. \qquad (8.2.189)$$

Substituting (8.2.189) and (8.2.143) in (8.2.184), we can find the expression

for cross-correlation function $K_{\omega\zeta,\beta}(\tau)$.

Dynamic error due to disturbances. Let us turn to a presentation of the method of determining dynamic errors of the GT, caused by friction in the axis of rotation of the frame, and also the influence of components of angular velocity of the vehicle along axes, not coinciding with the measuring axis of the instrument (Par. a). We shall consider the same gyrotachometer, intended for measurement of angular velocity of yawing $\dot{\varphi}$ of a ship. In this case, as shown in Par. 3 of this section, in equations (8.2.77) disturbing angular acceleration $\dot{\omega}_\eta$ is equal to angular acceleration $\ddot{\theta}$ of rolling of the ship. Considering what has been said, and also friction torques in the axis of rotation of the frame, determined by relationship (8.2.78),* we obtained the equation of motion of the GT frame (8.2.81), and also equation (8.2.82) of error of the GT, caused by these disturbances. The transfer function of the GT Y(s) with respect to the disturbance of rolling is determined by relationship (8.2.84). Consequently, the spectral density $S_{\varepsilon_1}(\omega)$ of dynamic error $\varepsilon_1(t)$ from rolling disturbance will be

$$S_{\varepsilon_1}(\omega) = |Y(j\omega)|^2 S_\theta(\omega). \tag{8.2.190}$$

where the gain-phase response $Y(j\omega)$ is found by replacement in (8.2.84) of s by $j\omega$, i.e.,

$$Y(j\omega) = \frac{-T^2\omega^2}{1 - T^2\omega^2 + 2\zeta_1 T j\omega}; \tag{8.2.191}$$

coefficient ζ_1 is determined by relationship (8.2.80).

The spectral density $S_\theta(\omega)$ of the angle of rolling of the ship in (8.2.190), according to (2.1.58), has form

$$S_\theta(\omega) = \frac{2A_2\mu_2}{\pi} \cdot \frac{b_2^2}{\omega^4 + 2a_2\omega^2 + b_2^4}. \tag{8.2.192}$$

Substituting (8.2.191) and (8.2.192) in (8.2.190), we have expression

$$S_{\varepsilon_1}(\omega) = \frac{T^4\omega^4}{(1 - T^2\omega^2)^2 + 4\zeta_1^2 T^2\omega^2} \cdot \frac{2A_2\mu_2}{\pi} \cdot \frac{b_2^2}{\omega^4 + 2a_2\omega^2 + b_2^4}, \tag{8.2.193}$$

analogous to the formula (8.2.171) for the spectral density $S_\varepsilon(\omega)$ of dynamic error $\varepsilon(t)$ of the GT during reproduction by it of the useful random input signal.

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*Besides this, in the axis of rotation of the frame there also appear random friction torques. In order to allow for the influence of these torques on GT error, it is necessary to know their probability characteristics.

For dispersion $D[\varepsilon_1]$ of error $\varepsilon_1(t)$ we have

$$D[\varepsilon_1] = \int_{-\infty}^{\infty} S_{\varepsilon_1}(\omega)\, d\omega \qquad (8.2.194)$$

or, considering (8.2.193), we obtain integral* .

$$D[\varepsilon_1] = \int_{-\infty}^{\infty} \frac{1}{k^2} \cdot \frac{T^4\omega^4}{(1-T^2\omega^2)^2 + 4\zeta_1^2 T^2\omega^2} \cdot \frac{2A_{\xi'\xi_2}}{\pi} \cdot \frac{b_2^2}{\omega^4 + 2a_2\omega^2 + b_2^4}\, d\omega, \qquad (8.2.195)$$

which is calculated by the same method as (8.2.173). As a result, we obtain formula

$$D[\varepsilon_1] = \frac{1}{k^2} \cdot \frac{T^4\left(b_2^2 + \frac{\mu_1}{T\zeta_1}\right)b_2^2}{(1-T^2b_2^2)^2 + 4\left[T^2\zeta_1^2b_2^2 + T\mu_2(\zeta_1 + T\mu_2) + T^3\zeta_1\mu_2b_2^2\right]} D[\theta]. \qquad (8.2.196)$$

For the mean quadratic value σ_{ε_1} of dynamic error of the GT we have

$$\sigma_{\varepsilon_1} = \sqrt{D[\varepsilon_1]}. \qquad (8.2.197)$$

Example 8.12. Calculate the mean quadratic value σ_{ε_1} of dynamic error $\varepsilon_1(t)$ of a GT, caused by friction in the axis of rotation of the frame of the instrument and the angular velocity of translational motion of rolling of the ship relative to this axis, for initial data of examples 8.1 and 7.10; determine also limiting value ε_{1m} of GT error, assuming that the probability Q_z that the current value ε_1 of error exceeds limit ε_{1m} should not be more than 0.1%; the coefficient of liquid friction in the axis of rotation of the GT frame $n_1 = 3$ g-cm-sec.

Solution. 1. By formula (8.2.50) we find coefficient ρ:

$$\rho = \frac{1}{c} = \frac{1}{1200} = 0{,}8333 \cdot 10^{-3}\ 1/\text{g}\cdot\text{cm}$$

2. According to (8.2.80) we determine ζ_1:

$$\zeta_1 = \zeta + \frac{\rho n_1}{2T} = 0{,}707 + \frac{0{,}8333 \cdot 10^{-3} \cdot 3}{2 \cdot 0{,}02901} = 0{,}7501.$$

3. Using relationship (8.2.196), we calculate $D[\varepsilon_1]$:

$$D[\varepsilon_1] = 0{,}3153 \cdot 10^{-8}\ 1/\text{sec}^2$$

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*According to (8.2.82) error $\varepsilon \equiv \bar{\varepsilon}_1$ is expressed in an angular measure. To find the corresponding error $\varepsilon_1(t)$ in the angular velocity measured by the GT, similar to error $\varepsilon(t)$ [see (8.2.161)], in formula (8.2.195), according to (8.2.7) and (8.2.50), introduce coefficient $\frac{1}{k^2}$.

4. By formula (8.2.197) we determine σ_{ε_1}:

$$\sigma_{\varepsilon_1} = \sqrt{D[\varepsilon_1]} = \sqrt{0{,}3153 \cdot 10^{-8}} = 0{,}5516 \cdot 10^{-4} \text{ 1/sec} = 0{,}0032 \text{ °/sec}$$

5. By the same method as in example 8.9, we calculate the limiting value of dynamic error of the GT (when $\bar{\varepsilon}_1 = 0$):

$$\varepsilon_{1m} = 3\sigma_{\varepsilon_1} = 3{,}3 \cdot 0{,}0032 = 0{,}0106 \text{ °/sec}$$

From example 8.12 it follows that dynamic error $\varepsilon_1(t)$ of the GT is less than $\varepsilon(t)$ (see example 8.11).*

During estimation of error of the GT we started from a differential equation, e.g., (8.2.40), with constant coefficients. It was obtained from a more general equation (8.2.37), where we also considered component ω_ξ of angular velocity of the vehicle along axis $O\xi$ (Fig. 8.4). Considering disturbances $\dot{\omega}_\eta = 0$, $M_{fr} = f(t) = 0$, since their influence was considered earlier, we rewrite (8.2.37) in form

$$J_{\beta\beta}\ddot{\beta} + b\dot{\beta} + [c + H\omega_\zeta(t)]\beta = H\omega_\xi(t). \qquad (8.2.198)$$

where the measured angular velocity $\omega_\zeta(t)$ and disturbing angular velocity $\omega_\xi(t)$ are usually random functions of time. We divide both sides of equation (8.2.198) by magnitude c and consider relationship (8.2.43), (8.2.45), (8.2.50); then, instead of (8.2.198), we obtain

$$T^2\ddot{\beta} + 2\zeta T\dot{\beta} + [1 + k\omega_\zeta(t)]\beta = k\omega_\xi(t). \qquad (8.2.199)$$

We obtained a differential equation of type (4.6.100), characterizing random motions of a physical pendulum during irregular rolling of a ship, taking into account vertical oscillations of the point of suspension of the pendulum. If dispersion $k^2D[\omega_\xi(t)]$ is small as compared to unity, for integration of (8.2.199) we can apply the same method of successive approximations as was used for solution of equation (4.6.100). For problems, encountered in practice, the assumption $k^2D[\omega_\xi(t)] \ll 1$, is usually satisfied. Those interested in a possible method of solution of equation (8.2.199) are referred to § 4.6, Par. 12e. Allowance for

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*In reality error $\varepsilon_1(t)$, due to disturbances, can be larger than error $\varepsilon(t)$, since among the disturbances we did not consider random friction torques, torques of residual unbalance of the frame, torques of loading of the frame by current feeds, etc.

disturbing angular velocity $\omega_\xi(t)$ leads to appearance of systematic deflection* of the GT frame, as for a physical pendulum (§ 4.6, Par 12e).

Let us note that during use of an equation in which we consider a component of the second order of smallness [for instance, (8.2.198)], it should be definitized. For this, using Euler's equations (1.4.1), we obtain the exact equation of motion of the GT:**

$$J_{r.a}\left(\ddot{\beta}-\dot{\omega}_\eta\right)+b\dot{\beta}+c\beta=H\left(\omega_\zeta\cos\beta-\omega_\xi\sin\beta\right)-$$
$$-\left(J_{rx}-J_a\right)\left(\frac{\omega_\zeta^2-\omega_\xi^2}{2}\sin 2\beta+\omega_\xi\omega_\zeta\cos 2\beta\right)+M_r+f(t).$$

Considering $\dot{\omega}_\eta = 0$, $M_{fr} = f(t) = 0$ and limiting ourselves to calculation of components to the second order of smallness, we obtain

$$J_{r.a}\ddot{\beta}+b\dot{\beta}+\left[c+H\omega_\xi(t)\right]\beta=H\omega_\zeta(t)-\left(J_{rx}-J_a\right)\omega_\xi(t)\,\omega_\zeta(t).$$

This equation differs from (8.2.198) by component $(J_{gx}-J_f)\,\omega_\xi(t)\;\omega_\zeta(t)$, which with a correlational connection of random functions $\omega_\xi(t)$ and $\omega_\zeta(t)$ also causes systematic deflection of the GT frame.

7. Certain Additional Problems of GT Theory

Above we gave the principles of GT theory, based on use of an astatic rate gyroscope. We shall briefly give certain additional problems of theory.

a) Appraisal of a GT with Two Degrees of Freedom.

A GT based on use of an astatic rate gyroscope, along with sufficient simplicity of structure and reading accuracy acceptable for practice, nevertheless possesses certain fundamental deficiencies.

Deflections of tne GT frame in general are nonlinearly connected with the angular velocity of the vehicle, measured by it. Therefore, we seek to make the angular region of deflections of the frame sufficiently small, and this affects accuracy of the instrument. Practically, complete removel of this deficiency is attained in a GT with forced return to zero (see point d).

A GT, besides the component angular velocity of the vehicle along the measuring axis, also reacts to other components of angular velocity, i.e., a GT with two degrees of freedom does not have a clearly expressed axis of sensitivity. To decrease the

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*A. P. Korzhov. On systematic errors in certain gyroscopic instruments. "Problems of applied gyroscopy," Issue 1, Scientific and technical society of instrument-making industry, Sudpromqiz, 1958.

**See footnote on p. 125.

influence, e.g., of angular velocity ω_ξ [see (8.2.199)] we seek to increase the coefficient c of spring force of the spring, which leads to decrease of the transmission factor k of the GT. However, this decreases sensitivity of the instrument. Decrease of the influence on a GT of the disturbance of components of angular velocity of the vehicle along axes not coinciding with the measuring axis of the instrument is carried out in a GT zero drive. This deficiency is best removed by use in the GT of two gyroscopes, connected by an anti-parallelogram, as in a two-gyro active-type gyrostabilizer (§ 7.1, Par 2b). However, here, the complexity and dimensions of the instrument increase.

Accuracy of GT readings is substantially influenced by change of its parameters, primarily of the whole kinetic torque H of the gyroscope and the coefficient c of spring force of spring. Therefore, it is very important to keep these parameters constant.

Friction and load on the axis of rotation of the frame affect the region of stagnation and sensitivity of the instrument. Therefore, decrease of these disturbing torques should be considered an important task in designing such instruments.

b) Gyrotachometer with Pendulum.

Usually in a GT, to create a restoring torque, balancing the gyroscopic torque (caused by angular velocity of the vehicle), a spring is applied, introducing torque $M_{sp} = -c\beta$ [see (8.2.4)]. For this purpose, instead of a spring, we can use a pendulum, rigidly joined to the gyro housing (frame) of the GT. Such a device was realized in one type of GT, the so-called Shilovskiy orthoscope [152].* Indeed, if on the GT frame we set a load of weight P at distance a from its axis rotation, then during turn of the frame angle β, thanks to the presence of the pendulum, there will be created torque $M = -Pa \sin\beta \approx -Pa\beta = -c\beta$, analogous to the torque of a spring. A peculiarity of this instrument is that when during installation, e.g., on an aircraft which accomplishes a regular turn with transverse bank γ, it can indicate this angle, i.e., serve as a bank indicator; however, this can be carried out only at a definite speed of flight of the aircraft [152]. It is necessary to note that due to a series of deficiencies such instruments have not been widely used.

c) On a Gyrotachometer with Floatation Suspension.

One of the essential deficiencies of the above considered GT's is the presence of a considerable friction torque and load on the axis of rotation of the frame,

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*[Translation Editor's Note: This scientist is also referred to in English as P. P. Schilowsky.]

caused by application of ball bearings, of a potentiometer as angle-data pickoff of the frame of the instrument, etc.

It is possible to decrease friction torques in the axis of rotation of the GT frame by application of a liquid suspension, in which the so-called unloading of supports is attained by the lift of the liquid. Simultaneously, the liquid serves to produce the required damping; therefore, in a GT with liquid suspension we lose the necessity to have a special damper. One of the important merits of devices with a liquid suspension is their high vibration and shock resistance and strength.

At present the most advance structural variant of a GT is the so-called floated differentiating gyroscope [149]. This instrument is very similar to the floated integrating gyroscope, which will be considered in detail in the next chapter. The basic difference of the floated differentiating gyroscope from the integrating one consists of the fact that in the former there is a special element, which creates the restoring torque required in the GT; as this element they usually use [149] an elastic torsion rod (Par. 2), one end of which is rigidly fixed to the float, and the other to the case of the instrument. Floated differentiating gyroscopes possess considerably greater sensitivity and accuracy than ordinary GT's.

As for the theory of such instruments, it in essence does not differ at all from GT theory. Detailed description of constructions of floated differentiating gyroscopes, and also necessary theoretical bases are given in the book of G. A. Slomyanskiy and Yu. N. Pryadilov [149].

Peculiarities of a liquid suspension in connection with its use in a floated integrating gyroscope are considered in Chapter 9.

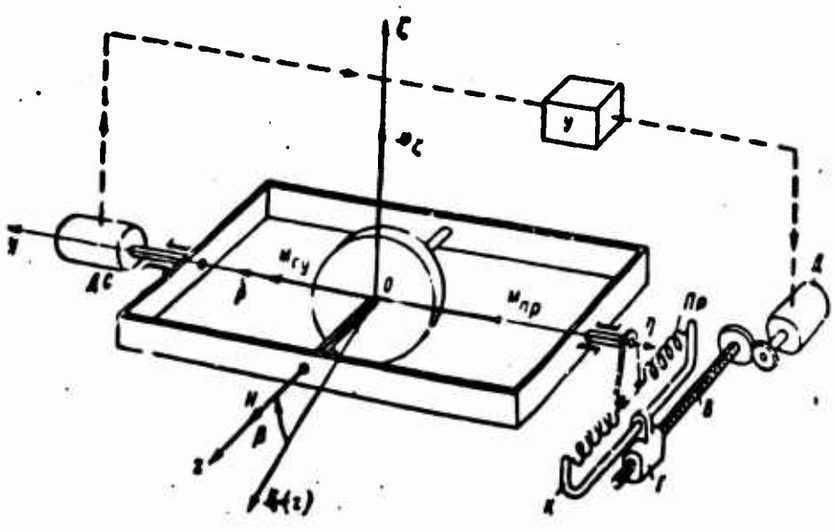

Fig. 8.13. Schematic of a GT with forced return to zero (with spring).

d) Gyrotachometer with Forced Return to zero.

As said earlier, a number of deficiencies of a GT based on use of an astatic rate gyroscope, are removed by means of forced return of the gyroscope to zero [111]. A schematic of the instrument is shown in Fig. 8.13.

The principle of work of a GT with zero drive is the following. With component

ω_ζ of angular velocity of the vehicle (along axis $O\zeta$) there appears gyroscopic torque $M_{g\,y}$, seeking, according to Foucault's rule, to combine the axis of the gyroscope with axis $O\zeta$. Torque $M_{g\,y}$ is balanced by torque M_{sp}, introduced by spring Sp. With turn of the frame angle β the signal pickoff SP, sitting on its spin axis, will switch on through amplifier Amp motor Mot. The motor through actuating screw AS and nut N will start to shift carriage C to which spring Sp is attached; the GT frame will be returned by the spring to zero position, since only in this case will the signal pickoff turn off the motor. It is obvious that linear shift of carriage C and nut N relative to their zero position will be proportional to the measured angular velocity ω_ζ.

Advantages of this scheme in comparison with the previously considered GT scheme without zero drive of the gyroscope are the following. Since the frame is always near the zero position, readings of the instrument can practically be considered linear with respect to the angular velocity ω_ζ measured by it. Due to this same circumstance the GT will be insensitive to the component of angular velocity along axis $O\xi$, i.e., a GT zero drive has only one axis of sensitivity $O\zeta$. The motor in the instrument permits compensating the influence of friction torque and other disturbing and load torques on the axis of rotation of the frame.

Creation of a GT with forced return to zero by a more simple scheme is possible [149]. This scheme differs from the above by the fact that it does not have a spring and other connected elements. The restoring torque here is created directly by the motor, i.e., an electrical device with application of proportional feedback. As can be seen from the scheme in Fig. 8.14, with turn of the instrument frame the signal pickoff SP turns on motor Mot through amplifier Amp, which returns to frame to the zero position. Theoretical research of such a scheme on the example of a floated differentiating gyroscope with feedback is given in [149].

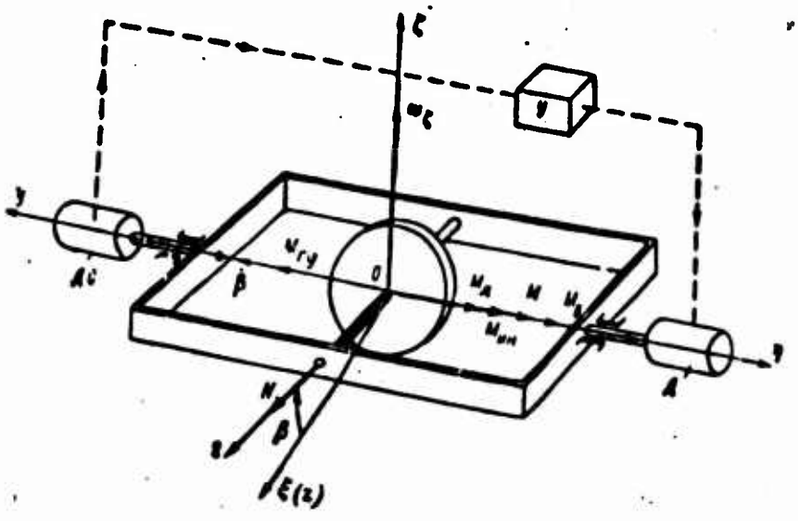

Fig. 8.14. Schematic of a GT with forced return to zero (with proportional feedback).

We shall limit ourselves to composition of the system of differential equations of the given GT scheme. In this case we have a servo, analogous to the servo in an

active gyroscopic stabilizer (§ 7.2, Par. 2). Therefore, here, it is possible in principle to use the same method of composition of equations of motion of the GT, as was applied during composition of differential equations of the active GT.

Let us compose the system of differential equations of motion of a GT with feedback; we assume that the GT is equiped with a damper, which is not shown in Fig. 8.14. In the considered case along the spin axis Oy of the GT frame there are located vectors of torques: M_{in} – moment of inertia of the frame and motor armature; M – torque, introduced by the motor; M_d – damping torque; $M_{g\ y}$ – gyroscopic torque; M_{dis} – disturbing torque. By analogy with (8.2.27) we have

$$M_{in} + M + M_d + M_g + M_{ry} = 0. \tag{8.2.200}$$

The moment of inertia of the GT frame is determined by relationship (8.2.28); moment of inertia of the motor armature, according to (7.2.39), considering that the transmission ratio $\gamma = 1$, will be $-J_{arm}\ddot{\beta}$; then

$$M_{in} = -J_{r.s}\left(\ddot{\beta} - \dot{\omega}_\zeta\right) - J_a\ddot{\beta}, \tag{8.2.201}$$

where J_{arm} – moment of inertia of the motor armature with respect to its axis of rotation.

Designating

$$J'_{r.s} = J_{r.s} + J_a, \tag{8.2.202}$$

we rewrite (8.2.201) in form

$$M_{in} = -J'_{r.s}\ddot{\beta} + J_{r.s}\dot{\omega}_\zeta. \tag{8.2.203}$$

Torque M_{dis}, according to (8.2.33) and (8.2.34) (for the case of liquid friction), will be

$$M_g = -n_1\dot{\beta} + f(t). \tag{8.2.204}$$

For the gyroscopic torque, considering that in a GT with zero drive angle β will be minute, in accordance with (8.2.32) we have, approximately,

$$M_{ry} = H\omega_\zeta. \tag{8.2.205}$$

Damping torque M_d, by analogy with (8.2.13), will be

$$M_d = -b\dot{\beta}. \tag{8.2.206}$$

Substituting (8.2.203)-(8.2.206) in (8.2.200), we obtain

$$J'_{r.s}\ddot{\beta}+b\dot{\beta}=H\omega_{\zeta}+M+J_{r.s}\dot{\omega}_{i}-n_{1}\beta+f(t). \tag{8.2.207}$$

This equation should be supplemented by equations, characterizing transients in electrical components of the GT.

Torque, introduced by the motor, by analogy with (7.2.44), we write in form*

$$M=-\Phi i, \tag{8.2.208}$$

where Φ – magnetic flux of excitation windings;

i – current in motor armature.

Current i enters into the equation of balance of voltages in the motor armature network, according to (7.2.45), in the following manner:

$$l\frac{di}{dt}+ri+c'\omega'_{s}=U_{y}, \tag{8.2.209}$$

where l – coefficient of self-inductance of the armature network;

r – resistance of the armature network (including internal resistance of the amplifier);

c′ – coefficient, characterizing dependence of the counter emf induced in the armature on relative angular velocity ω_{arm} of its rotation;

U_{amp} – voltage fed from amplifier to motor armature.

Since $\omega'_{arm}=-\dot{\beta}$ (Fig. 8.4), instead of (8.2.209) we obtain

$$l\frac{di}{dt}+ri-c'\dot{\beta}=U_{y}. \tag{8.2.210}$$

Introducing electromagnetic time constant T_{mot} of the motor armature network and other designations

$$T_{\Lambda}=\frac{l}{r},\quad \frac{c'}{r}=b_{1},\quad \frac{1}{r}=e, \tag{8.2.211}$$

we rewrite (8.2.210) in the form

$$T_{\Lambda}\frac{di}{dt}+i-b_{1}\dot{\beta}=eU_{y}. \tag{8.2.212}$$

Voltage from the amplifier U_{amp}, according to (7.2.51), is determined by equation

$$T_{y}\dot{U}_{y}+U_{y}=k'U, \tag{8.2.213}$$

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*See footnote on p. 20.

where T_{amp} – time constant of the amplifier;

U – voltage from signal pickoff, proceeding to amplifier input;

k' – amplifier gain.

Voltage U is a function of angle β of rotation of the frame:

$$U = k_1'\beta, \tag{8.2.214}$$

where k_1' – transmission coefficient of the considered unit.

Substituting (8.2.214) in (8.2.213) and designating

$$k_1'k' = S, \tag{8.2.215}$$

we obtain

$$T_y\dot{U}_y + U_y = S\dot{\beta}. \tag{8.2.216}$$

Thus, we have the following system of equations:

$$\left.\begin{aligned} &J'_{r.s}\ddot{\beta} + b\dot{\beta} = H\omega_\zeta + M + J_{r.s}\dot{\omega}_\zeta - n_1\ddot{\beta} + f(t) && \text{(GT frame)}\\ &M = -\Phi i && \text{(motor)}\\ &T_y\dot{U}_y + U_y = S\dot{\beta} && \text{(amplifier)}\\ &T_a\frac{di}{dt} + i - b_u\dot{\beta} = cU_y && \text{(motor armature network)} \end{aligned}\right\} \tag{8.2.217}$$

On the basis of this system of equations we can find the transient, performance indices and dynamic errors of the GT, using for this the same methods, as were applied during investigation of an active GS.

Readings of the considered GT are taken in the form of current strength i at the amplifier output. We shall show that it is proportional to the angular velocity ω_ζ measured by the instrument; disturbances we consider absent, i.e., $\dot{\omega}_\eta = 0$, $n_1 = 0$, $f(t) = 0$; then system of equations (8.2.217) can be written in form

$$\left.\begin{aligned} J'_{r.s}\ddot{\beta} + b\dot{\beta} &= H\omega_\zeta - \Phi i\\ T_y\dot{U}_y + U_y &= S\dot{\beta}\\ T_a\frac{di}{dt} + i - b_u\dot{\beta} &= cU_y \end{aligned}\right\} \tag{8.2.218}$$

From the first equation of this system it follows that in steady (static) position

$$i_{ст} = \frac{H}{\Phi}\omega_\zeta, \qquad (8.2.219)$$

i.e., current strength i is proportional to the magnitude of angular velocity ω_ζ.

Let us solve system of equations (8.2.218) for i; considering time constants T_{amp} and T_{mot} negligible, we have

$$J'_{г.э}\frac{d^2i}{dt^2} + (b + \Psi b_1)\frac{di}{dt} + eS\Psi i = eSH\omega_\zeta + Hb_1\dot{\omega}_\zeta, \qquad (8.2.220)$$

i.e., an equation, analogous in structure to (8.2.40). Thus, the considered system has properties of a tachometer. It should be noted that the counter emf in the motor armature affects damping of the system, since in coefficient with $\frac{di}{dt}$ there enters quantity b_1 [see (8.2.211)]. However, in the right side of equation (8.2.220) there appears disturbance $Hb_1\dot{\omega}_\zeta$, which will introduce distortion in the readings of the instrument. Therefore, coefficient c' of counter emf, induced in the motor armature, should be as small as possible; then $b_1 = \frac{c'}{r}$ will also be small and instead of (8.2.220) we will have

$$J'_{г.э}\frac{d^2i}{dt^2} + b\frac{di}{dt} + eS\Phi i = eSH\omega_\zeta, \qquad (8.2.221)$$

from which for a steady state we obtain formula (8.2.219) found earlier.

e) Gyrotachometer with Three Degrees of Freedom

As tachometers we can also use gyroscopes with three degrees of freedom. One of the simplest schemes of a GT [111], based on use of a three-degree-of-freedom gyroscope, is shown in Fig. 8.15. With the internal ring of the Cardan suspension is connected a sighting device; with the help of this device we observe the vehicle, angular velocity ω_ζ of displacement of which should be determined. In order to keep the sighting device directed towards the vehicle, to torquer TQ we should apply from the control system a certain torque M, which ensures precessional motion of the external gimbal ring together with the sighting device after motion of the vehicle. Since, here, the angular velocity of precession $\omega_p = \frac{M}{H}$ should be equal to ω_ζ, the magnitude of applied torque M will be proportional to the measured angular

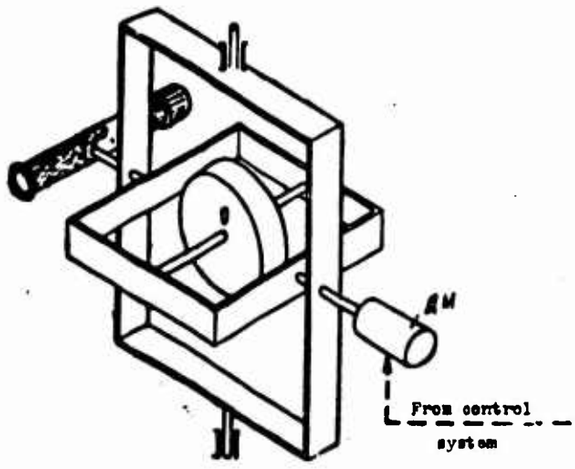

Fig. 8.15. A GT base on an astatic gyroscope with three degrees of freedom.

velocity of movement of the vehicle: $M = H\omega_\zeta$.

We know a fundamentally different method of use of a three-degree-of-freedom gyroscope as a tachometer. To the gyroscope there is applied a certain torque, proportional to the angle of deflection of its axis with respect to the axis, connected with the vehicle, angular velocity of which it to be measured. In a steady state of work the axis of the gyroscope will lag behind the axis connected with the vehicle, and this angle is a measure of the angular velocity of the latter.

One scheme of a GT, realizing this method, was considered by B. I. Kudrevich [71]. In Fig. 8.16 there is shown magnet (or electromagnet) 1, which can spin about axis nn, connected with the vehicle, whose angular velocity ω is to be measured; vector $\vec{\omega}$ is directed along axis Ox, perpendicular to the plane of the figure. Rotor 2, made in the form of an incomplete sphere, rotates about axis Oz; axes Oxyz are connected with the rotor; O is the point of suspension of the rotor; angular mementum of the gyroscope is H. We designate by M the torque created by the rotating magnetic field; β is the angle of deflection of axis Oz of the gyroscope with respect to direction nn.

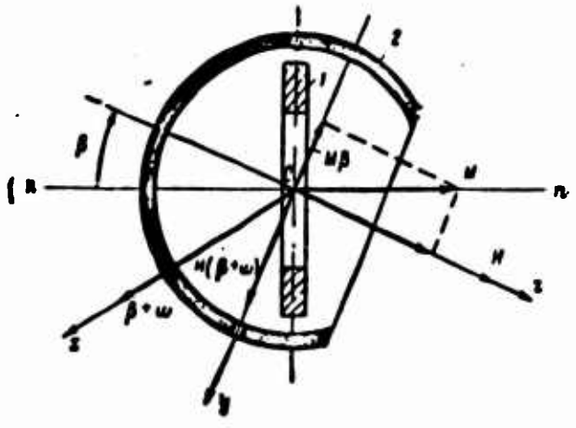

Fig. 8.16. The theory of a "lagging" gyroscope.

We shall show that a steady state angle β is proportional to the measured angular velocity ω. First, we shall consider the case when $\omega = 0$, and determine the law of motion of the gyroscope. Let us expand vector $\vec{M}$ in components along axes Oz and Oy. The component of this torque $-M \sin \beta \approx M\beta$ along axis Oy causes precessional motion of the gyroscope about axis Ox with angular velocity

$$\dot{\beta} = -\frac{M\beta}{H}.$$

Designating $\frac{M}{H} = \varkappa$, we have

$$\dot{\beta} + \varkappa\beta = 0. \qquad (8.2.222)$$

If, for $t = 0$, $\beta(0) = \beta_0$,

$$\beta = \beta_0 e^{-\varkappa t}, \qquad (8.2.223)$$

i.e., axis Oz of the gyroscope will seek to coincide with axis nn by aperiodic law.

Consequently, motion of the gyroscope has the same nature as motion of a gyrovertical with a linear characteristic of correction (§ 6.3, Par. 2b).

If axis nn rotates in space about axis Ox with angular velocity ω, instead of equation (8.2.222) we obtain the following:

$$(\dot{\beta} + \omega) + k\beta = 0. \qquad (8.2.224)$$

from which for a steady state with ω = const we have

$$\beta_{ст} = -\frac{1}{k}\omega. \qquad (8.2.225)$$

Consequently, the considered instrument is a gyrotachometer, i.e., with movement of axis nn after the vehicle axis Oz of the gyroscope will be deflected (lag) from axis nn by angle β, proportional in a steady state to angular velocity ω of the vehicle. In connection with this such gyroscopes are sometimes called "lagging."

Analogous schemes of gyrotachometers are also given in [30, 31].

§ 8.3. Gyrotachoaccelerometer (GTA)

1. Function of the Instrument

In § 8.1, it was shown that in practice there are applied differentiating gyroscopes, which serve to determine the sum of angular velocity and angular acceleration of a vehicle. Such gyroscopes are usually called gyrotachoaccelerometers (GTA) or damping gyroscopes. In GTA's there is used an astatic gyroscope with three degrees of freedom, fixed in a Cardan suspension, where angles of rotations of the inner and outer gimbal rings are bounded by springs.

2. Schematic of a GTA

The schematic of the GTA and its description are given in a number of books and articles (for instance, [17, 57, 112]), which we shall use during our account of this problem. Rotor GR (Fig. 8.17) is fixed in a Cardan suspension, consisting of inner IR and outer OR rings; turns of rings are limited by springs Sp_1 and Sp_2. On the axis of the inner ring IR is located lever L_1, connected with spring Sp_1, which is fastened in the housing of the instrument. On the same axis is fixed lever L_2, united with the pull rod of damper D. Outer ring OR has an elastic connection with the housing of the instrument through spring Sp_2. Readings of the instrument, characterizing the sum of angular velocity and angular acceleration of rotation of the vehicle about axis $O\zeta$, proportional to the angle of rotation of the outer ring about $O\zeta$, are taken in the form of voltage from potentiometer Pot. It should be noted that

in a real construction of the instrument [112] Sp_2 spring is selected so rigid that the maximum possible angle of rotation of the outer ring OR does not exceed several tenths of a degree; angle of rotation of the inner ring IR does not exceed several degrees.

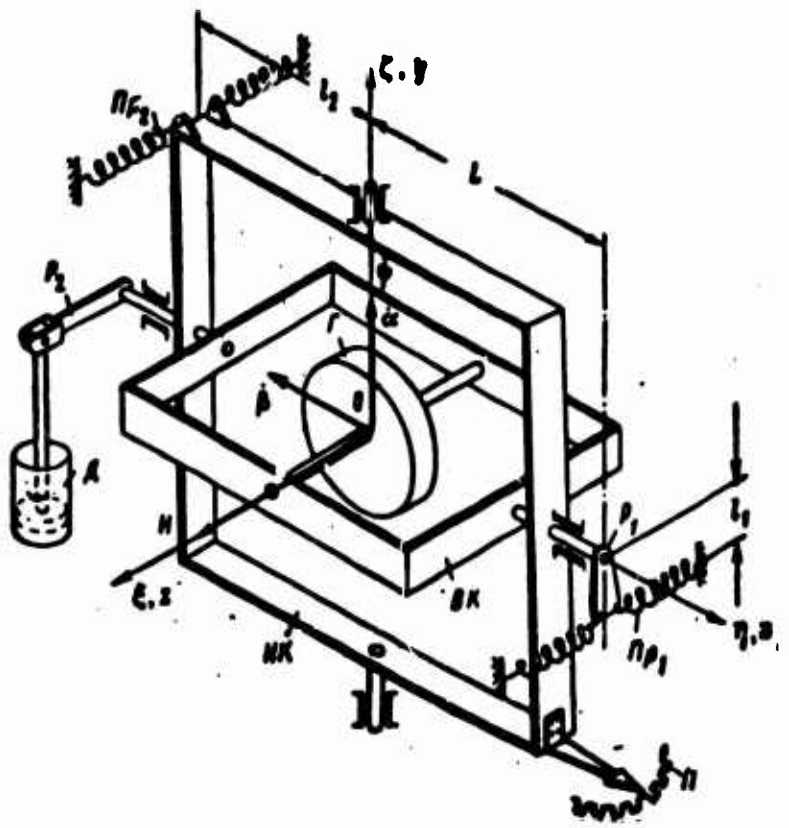

Fig. 8.17. Schematic of a GTA.

3. Basic Relationships for a GTA

For concreteness we shall consider axes $O\xi\eta\zeta$ connected with the aircraft; axes Oxyz, combined in initial position (when $\alpha = \beta = 0$) with axes $O\xi\eta\zeta$, are connected with the inner gimbal ring (Resal's axes). Let us assume that the GTA is designed for measurement of angular velocity $\dot{\psi}$ and angular acceleration $\ddot{\psi}$ of the aircraft about axis $O\zeta$. In accordance with this, spin axis Oy of the outer ring is combined with axis $O\zeta$, and spin axis Ox of the inner ring is connected with the longitudinal axis of the aircraft $O\eta$. Thus, axis $O\zeta$ is the input or measuring axis. Let us designate by α of the angle of rotation of the outer ring with respect to the instrument housing, β – the angle of rotation of the inner ring relative to the outer. Readings of the instrument are taken in the form of the angle α of rotation of the outer ring, which, as will be shown later, is connected with measured parameters $\dot{\psi}$ and $\ddot{\psi}$ of oscillatory motion of the aircraft by the following relationship:

$$\alpha = A\dot{\psi} + B\ddot{\psi}, \tag{8.3.1}$$

where A and B – certain constant coefficients, depending on parameters of the instrument.

Consequently, spin axis Oy of the outer ring, coinciding with $O\zeta$, is the output axis.

Let us derive basic relationship for the GTA, i.e., expression (8.3.1), which establishes the functional connection between input quantities $\dot{\psi}$, $\ddot{\psi}$ and the outlet quantity α. First, we shall consider the case when $\dot{\psi}$ = const. Here, the GTA will

behave as a gyrotachometer. Actually, with great spring force of spring Sp_2 one may assume that in the first approximation, due to smallness of $\dot{\alpha}$ as compared with $\dot{\psi}$, the outer and inner rings will turn about axis $O\zeta$ together with the housing of the instrument and aircraft with angular velocity $\dot{\psi}$, where under the action of gyroscopic torque $M_{g_1} \sim H\dot{\psi}$, directed on axis Oy, the inner ring, according to (8.2.7), will turn an angle β, equal to

$$\beta = \frac{H}{c_1}\dot{\psi};$$

coefficient c_1, considering (8.2.5) and Fig. 8.17, we determine by formula $c_1 = c_1' l_1^2$, and then

$$\beta = \frac{H}{c_1' l_1^2}\dot{\psi}. \qquad (8.3.2)$$

where c_1' – coefficient of linear spring force of spring Sp_1;

l_1 – length of lever L_1.

In this case spring Sp_1 develops force

$$F_1 = c_1 l_1 \beta,$$

and its torque M_1 with respect to axis $O\zeta$ will be

$$M_1 = c_1 l_1 \beta L. \qquad (8.3.3)$$

Torque M_1 will cause turn of the outer ring with respect to the housing of the instrument a certain angle α_1, and also compression of spring Sp_2, which will create with respect to axis $O\zeta$ torque

$$M_2 = -c_2 l_2^2 \alpha_1. \qquad (8.3.4)$$

where c_2 – coefficient of linear spring force of spring Sp_2.

Torques M_1 and M_2 are balanced, i.e.,

$$M_1 + M_2 = 0;$$

substituting (8.3.3) and (8.3.4), we obtain for angle α_1 the following expression:

$$\alpha_1 = \frac{c_1 l_1 L}{c_2 l_2^2}\beta = \frac{HL}{l_1 l_2 c_2}\dot{\psi}. \qquad (8.3.5)$$

coinciding with (8.3.1) if we designate

$$A = \frac{HL}{l_1 l_2 c_2} \qquad (8.3.6)$$

and consider that in this case $\dot{\psi}$ = const and, consequently, $\ddot{\psi}$ = 0.

Let us turn to analysis of the general case, when the aircraft has angular acceleration $\ddot{\psi}$. If $\ddot{\psi} > 0$, the inner ring will turn about axis $O\gamma$ with angular velocity $\dot{\beta}$, determined from (8.3.2):

$$\dot{\beta} = \frac{H}{c_1 l_1^2}\ddot{\psi}. \tag{8.3.7}$$

Here, there appears a gyroscopic torque M_{g_2}, directed along axis $O\zeta$ and equal to

$$M_{r_2} \approx -H\dot{\beta} = -\frac{H^2}{c_1 l_1^2}\ddot{\psi}. \tag{8.3.8}$$

Torque M_{g_2} causes turn of the outer ring and additional compression of spring Sp_2, which creates torque M_2':

$$M_2' = c_2 l_2^2 \alpha_2. \tag{8.3.9}$$

Torques M_{g_2} and M_2' are balanced, i.e.,

$$M_{r_2} + M_2' = 0;$$

substituting (8.3.8) and (8.3.9), we obtain for angle α_2 the following expression:

$$\alpha_2 = \frac{H^2}{c_1 c_2 l_1^2 l_2^2}\ddot{\psi}, \tag{8.3.10}$$

where $\frac{H^2}{c_1 c_2 l_1^2 l_2^2}$, according to (8.3.1), we designate by B:

$$B = \frac{H^2}{c_1 c_2 l_1^2 l_2^2}. \tag{8.3.11}$$

Thus, total angle of rotation α of the outer ring will be

$$\alpha = \alpha_1 + \alpha_2 = \frac{HL}{c_2 l_1 l_2^2}\dot{\psi} + \frac{H^2}{c_1 c_2 l_1^2 l_2^2}\ddot{\psi} = A\dot{\psi} + B\ddot{\psi}. \tag{8.3.12}$$

Consequently, the considered instrument indeed gives the sum of angular velocity and angular acceleration of rotation of the vehicle.

4. Equation of Motion of a GTA

We give without derivation the equation of motion of a GTA in coordinate α [112]:

$$H^2\ddot{\alpha} + al_2^2\left(c_1 L^2 + c_2 l_2^2\right)\dot{\alpha} + c_1 c_2 l_1^2 l_2^2 \alpha = Hc_1 l_1 L\dot{\psi} + H^2\ddot{\psi}, \tag{8.3.13}$$

where a – constant included in the expression of the damping coefficient (8.2.14).

From equation (8.3.13) it follows that angle α, if we disregard small "dynamic correction" (§ 8.2, Par. 6b), is determined by a relationship, coinciding with (8.3.12). Natural oscillations of the outer gimbal ring under the influence of the damping torque attenuate. Frequency n of natural undamped oscillations, according to (8.3.13), is determined by expression

$$n = \frac{l_1 l_2 \sqrt{c_1 c_2}}{H}. \tag{8.3.14}$$

For the location of the GTA with respect to the axes of the aircraft in Fig. 8.17 its readings will also be influenced by oscillations of the aircraft relative to its longitudinal axis, i.e., the angular velocity and angular acceleration of roll. To eliminate this influence we set the instrument in such a manner that the spin axis Ox of the inner frame is inclined to the longitudinal axis of the aircraft a certain angle ϑ_0, determined by relationship [109]

$$\vartheta_0 = \operatorname{arc\,tg} \frac{h}{L}. \tag{8.3.15}$$

Let us find the transfer function of the GTA. According to (8.3.13), we have

$$W(s) = \frac{\alpha(s)}{\psi(s)} = \frac{H^2 s^2 + H c_1 l_1 L s}{H^2 s^2 + a l_3^2 \left(c_1 L^2 + c_2 l_2^2\right) s + c_1 c_2 l_1^2 l_2^2}, \tag{8.3.16}$$

i.e., from the dynamic point of view the GTA can be considered an oscillating circuit with the introduction of derivatives.

As for instrument errors of a GTA, they are basically analogous to instrument errors of a GT. Therefore, we shall be limited to giving only formulas for minimum angular velocity $\dot{\psi}_{min}$ and minimum angular acceleration $\ddot{\psi}_{min}$, characterizing the threshold of sensitivity of the GTA; we have [17]

$$\dot{\psi}_{min} > \frac{K_x}{H}, \quad \ddot{\psi}_{min} > c_1 l_1 \frac{K_y}{H^2}, \tag{8.3.17}$$

where K_x and K_y – correspondingly, moduli friction torques in the spin axes of the inner and outer gimbal rings.

Detailed descriptions of the scheme, constructions of GTA's and its basic characteristics are given in the literature [112, 17].

§ 8.4. Vibratory Gyroscopes (VG's)

1. Function and Types of VG's

One of the essential deficiencies of all GD's considered till now, based on use of a gyroscope in a Cardan suspension, is the presence of friction in the

suspension axes, which lowers accuracy of readings of these devices.

The effort, by all possible ways, to decrease friction in the suspension axes led to creation of gyroscopes with air suspension (§ 6.3, Par. 2a), and also gyroscopes with floating suspension.

Along with this there was noted the fundamental possibility of creating a GD of the type of a vibratory gyroscopes (VG), in which the gyroscope, in general, does not have a Cardan suspension; there also may be no fast-rotating rotor; instruments of this last type can be related to gyroscopes only be convention. By a vibratory gyroscope, in general, we understand an instrument, containing vibrating components and reacting to the gyrotorque, caused by rotation of the vehicle. The VG in principle of action is a gyroscopic tachometer, since the input quantity is the gyroscopic torque from rotation of the vehicle, and the output is a function of angular velocity of the vehicle.

We distinguish VG's of rotor type and rod type. The former contain a revolving rotor; for VG's of the latter type the sensor is several vibrating masses, e.g., rods, similar to tines of a tuning fork.

2. Schemes of VG's of Rotor Type. Basic Relationships

One of the possible schemes of a VG of rotor type is presented in Fig. 8.18a, where the figures designate; 1) symmetric rotor; 2) its spin axis; 3) elastic rods.

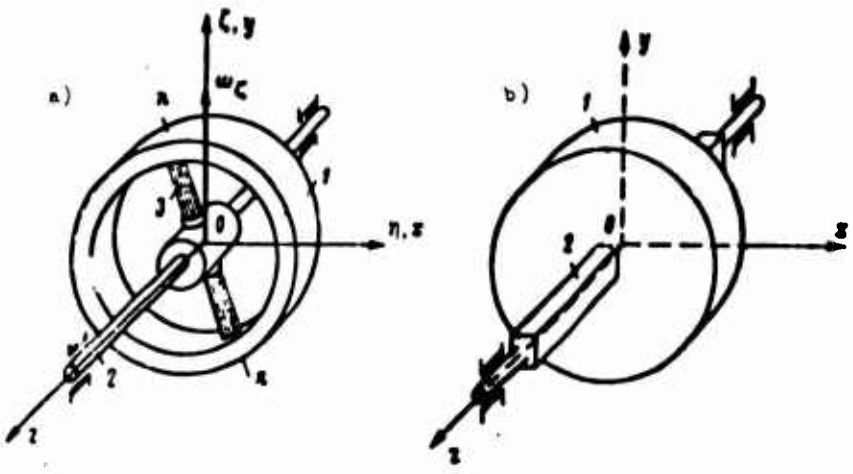

Fig. 8.18. Schematics of VG's of Rotor type.

A peculiarity of the considered VG is that its rotor has the possibility of additionally rotating about axis nn, perpendicular to axis Oz, thanks to the presence of its elastic connection to axis Oz, carried out by elastic rods; axis nn is called the vibration axis. In Fig. 8.18b there is given a somewhat different scheme; here rotor 1 is fixed on its spin axis 2, in the form of a flat spring.

Let us consider the essence of the phenomena occurring in a VG. Let us assume that the VG rotor rotates about axis Oz with angular velocity Ω; angle of spin of the rotor $\varphi = \Omega t$; its angular momentum $H = J\Omega$. Let us assume that the base of the VG, i.e., the vehicle on which it is fixed rotates with angular velocity ω_ζ, directed

along the axis 0ζ (Fig. 8.19). Here, there appears gyroscopic torque

$$M_r \approx H\omega_\zeta. \qquad (8.4.1)$$

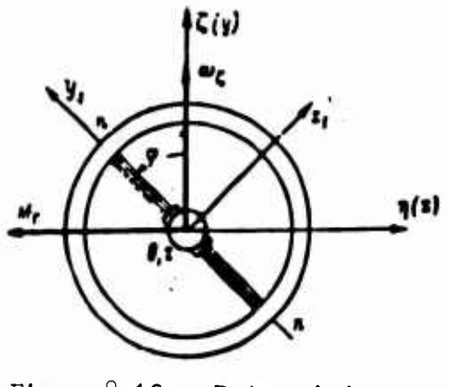

Fig. 8.19. Determining $M_{g\,x_1}$ and $M_{g\,y_1}$.

Components of this torque on axes $0y_1$ and $0x_1$ will be

$$M_{ry_1} = H\omega_\zeta \sin\varphi; \quad M_{rx_1} = -H\omega_\zeta \cos\varphi. \qquad (8.4.2)$$

Components $M_{g\,y_1}$, directed along the axis of vibration nn, causes vibratory oscillations of the rotor about this axis, since during turn of the rotor about axis Ox 90° from the position shown in Fig. 8.19, torque $M_{g\,y_1}$ changes sign. As will be shown below, from the amplitude of these oscillations it is possible to judge the magnitude of angular velocity of the vehicle. Component $M_{g\,x_1}$ of gyroscopic torgue about axis $0x_1$ does not affect motion of the gyroscope; it only causes a certain dynamic pressure of the rotor on its supports.

We compose the differential equation of motion of the rotor about the axis of vibration nn ($0y_1$). Here, we consider that vibration of the rotor about axis $0x_1$ is absent by virtue of the corresponding rigidity of the construction. Let us use Euler's equations (1.3.27). As generalized coordinates we take the angle φ of spin of the rotor (Fig. 8.20) and the angle α of rotation of the rotor about the axis of vibration $0y_1$; axes connected with rotor after these turns will occupy position $0x_1y_1z$. Axes $0\xi\eta\zeta$ connected with the vehicle are rotated about axis 0ζ with angular velocity ω_ζ, which should also be determined by the VG. The second and third equations of (1.3.7), replacing M_y by M_{y_1} and considering that $J_x = J_y = J_e$, $J_z = J$, we rewrite in form

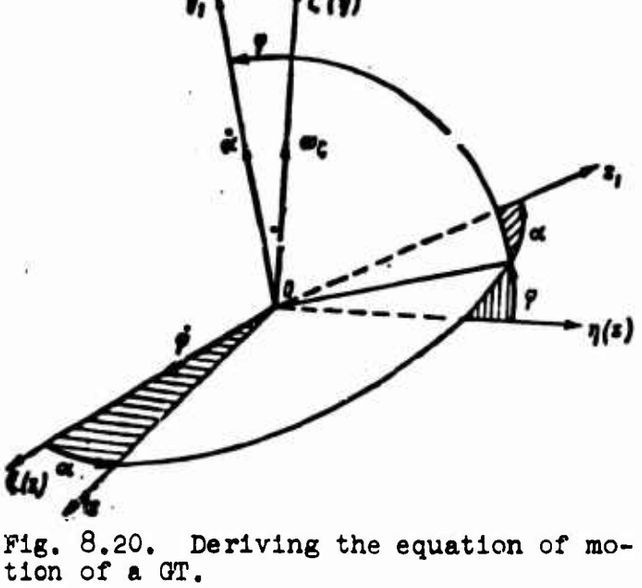

Fig. 8.20. Deriving the equation of motion of a GT.

$$\left.\begin{aligned} J_e\left(\frac{dq}{dt} + rp\right) - Jrp &= M_{y_1} \\ J\frac{dr}{dt} &= M_z \end{aligned}\right\} \qquad (8.4.3)$$

Projections of absolute angular velocity of axes $0x_1y_1z$ on these same axes, according to Fig. 8.20, will be

$$\left.\begin{aligned} p &= \omega_c \sin\varphi\cos\alpha - \dot{\varphi}\sin\alpha \\ q &= \omega_c \cos\varphi + \dot{\alpha} \\ r &= \omega_c \sin\varphi\sin\alpha + \dot{\varphi}\cos\alpha \end{aligned}\right\} \tag{8.4.4}$$

We substitute (8.4.4) in (8.4.3):

$$\left.\begin{aligned} &J_0\left[\ddot{\alpha} + \dot{\omega}_c\cos\varphi - \omega_c\dot{\varphi}\sin\varphi + \right. \\ &\left. + \left(\omega_c\sin\varphi\sin\alpha + \dot{\varphi}\cos\alpha\right)\left(\omega_c\sin\varphi\cos\alpha - \dot{\varphi}\sin\alpha\right)\right] - \\ &- J\left(\omega_c\sin\varphi\sin\alpha + \dot{\varphi}\cos\alpha\right)\left(\omega_c\sin\varphi\cos\alpha - \dot{\varphi}\sin\alpha\right) = M_{y_1} \\ &\frac{d}{dt}J\left(\omega_c\sin\varphi\sin\alpha + \dot{\varphi}\cos\alpha\right) = M_z. \end{aligned}\right\} \tag{8.4.5}$$

Since resisting torques with respect to axis Oz of the gyroscope are balanced by an active torque, the resultant torque $M_z = 0$. Then, from the second equation of system (8.4.5) we obtain

$$J\left(\dot{\varphi}\cos\alpha + \omega_c\sin\varphi\sin\alpha\right) = \text{const} = H = J\Omega. \tag{8.4.6}$$

Angle α of rotation of the rotor about the axis of vibration is sufficiently small; therefore, according to (8.4.5) and (8.4.6), with an accuracy of small components of the first order we obtain the following equation of motion of the rotor with respect to the axis of vibration:

$$J_0\ddot{\alpha} + (J - J_0)\,\Omega^2\alpha = M_{y_1} + H\omega_c\sin\varphi - J_0\dot{\omega}_c\cos\varphi. \tag{8.4.7}$$

Since $\varphi = \Omega t$,

$$J_0\ddot{\alpha} + (J - J_0)\,\Omega^2\alpha = M_{y_1} + H\omega_c\sin\Omega t - J_0\dot{\omega}_c\cos\Omega t. \tag{8.4.8}$$

As the torque M_{y_1} of external forces, applied to the gyroscope with respect to axis Oy_1 (Fig. 8.19 and 8.20), we consider the restoring torque of forces of elastic strains of rods and damping torques; then, by analogy with expressions (8.2.4) and (8.2.13), we have

$$M_{y_1} = -c_1\alpha - b\dot{\alpha}. \tag{8.4.9}$$

Substituting (8.4.9) in (8.4.8), we obtain

$$J_0\ddot{\alpha} + b\dot{\alpha} + \left[c_1 + (J - J_0)\,\Omega^2\right]\alpha = H\omega_c\sin\Omega t - J_0\dot{\omega}_c\cos\Omega t. \tag{8.4.10}$$

In the right part of the equation the second member is small as compared to the

first. Therefore, the differential equation of small oscillations of the rotor about the axis of vibration, introducing designation

$$c = c_1 + (J - J_0)\Omega^2, \tag{8.4.11}$$

we write finally in form

$$J_0\ddot{\alpha} + b\dot{\alpha} + c\alpha = H\omega_\zeta(t)\sin\Omega t, \tag{8.4.12}$$

very similar to equation (8.2.41) of motion of a GT frame.

Let us present (8.4.12) in the following form:

$$\ddot{\alpha} + \frac{b}{J_0}\dot{\alpha} + \frac{c}{J_0}\alpha = \frac{H}{J_0}\omega_\zeta(t)\sin\Omega t. \tag{8.4.13}$$

By analogy with (8.2.43), (8.2.45) – (8.2.47), we introduce designations

$$n = \sqrt{\frac{c}{J_0}};\quad \frac{b}{J_0} = 2\zeta n;\quad \zeta = \frac{b}{2\sqrt{J_0 c}};\quad k_1 = \frac{H}{J_0}. \tag{8.4.14}$$

Then we have

$$\ddot{\alpha} + 2\zeta n\dot{\alpha} + n^2\alpha = k_1\omega_\zeta(t)\sin\Omega t, \tag{8.4.15}$$

where n – frequency of natural undamped vibratory oscillations of the rotor;

ζ – relative attenuation factor.

If we designate by $T = \frac{1}{n}$ the time constant of the VG, (8.4.15) can be rewritten in form

$$T^2\ddot{\alpha} + 2\zeta T\dot{\alpha} + \alpha = k\omega_\zeta(t)\sin\Omega t, \tag{8.4.16}$$

where

$$k = \frac{k_1}{n^2} = \frac{H}{J_0}\cdot\frac{J_0}{c} = \frac{H}{c}. \tag{8.4.17}$$

Since natural oscillation of a VG rotor rapidly attenuate thanks to the presence of damping, we determine its forced oscillations, which are characterized by a particular solution of equation (8.4.16).

The form of this particular solution is determined by the right side of equation (8.4.16), i.e., input $k\omega_\zeta(t)\sin\Omega t$. We shall limit our consideration to the simplest case, when the vehicle rotates about axis 0ζ (Fig. 8.20) with constant angular velocity, i.e., $\omega_\zeta(t) = \omega_\zeta^0 = \text{const}$. Then we rewrite equation (8.4.16) in form

$$T^2\ddot{\alpha} + 2\zeta T\dot{\alpha} + \alpha = k\omega_\zeta^0 \sin \Omega t. \qquad (8.4.18)$$

The transfer function of the VG is determined by the same relationship (8.2.68), as the transfer function of a GT. Then the amplitude and phase frequency-response curves of the VG will be expressed by formulas (8.2.74), if in them we set $\omega = \Omega$ i.e.,

$$A(\Omega) = \frac{k}{\sqrt{(1 - T^2\Omega^2)^2 + 4\zeta^2 T^2\Omega^2}}; \quad \varphi(\Omega) = -\operatorname{arc\,tg}\frac{2\zeta T\Omega}{1 - T^2\Omega^2}. \qquad (8.4.19)$$

Forced oscillations of the VG rotor, by analogy with (8.2.126), are determined by relationship

$$\alpha_0(t) = \omega_\zeta^0 A(\Omega) \sin[\Omega t + \varphi(\Omega)], \qquad (8.4.20)$$

where $A(\Omega)$ and $\varphi(\Omega)$ during constant angular velocity $\Omega = \text{const}$ of rotor spin will also be constant [see (8.4.19)].

From formula (8.4.20) it follows that when $\omega_\zeta^0 = \text{const}$ the frequency of forced oscillations of the rotor is equal to the angular velocity Ω of its spinning; the amplitude of forced oscillations is proportional to the measured angular velocity ω_ζ^0 of the vehicle. Thus, the VG indeed executes the function of a GT. Increase of the amplitude of vibrations of the rotor can be attained by increasing of the gain-frequency response $A(\Omega)$ of a VG, for which, according to (8.4.19), one should increase coefficient $k = \frac{H}{c}$ [see (8.4.17)] and select parameters of system so that $1 - T^2\Omega^2 = 0$, i.e., the VG works in conditions of resonance.* Thus, measuring in one way or another angle α of rotation of the rotor about the axis of vibration, one can determine the angular velocity of rotation of the vehicle.

It should be noted that a VG of the considered type spinning of the rotor, in general, is not obligatory. In principle an analogous phenomenon can be obtained, if, for instance, axis 2 of the rotor (Fig. 8.18b), carried out in the form of a flat spring, is rigidly secured at both ends and instead of rotation we impart to the rotor torsional oscillations relative to this axis. Such a VG does not contain rotating parts, and therefore possesses known advantages as compared to the VG considered earlier. Basic relationships for a VG based on torsional oscillations

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*Quantity c, according to (8.4.11), depends on coefficient c_1 of rigidity of the elastic rods of the VG, and also on $(J - J_e)\Omega^2$, which it is possible to vary by selecting the angular velocity Ω of spinning of the rotor and the relationship between its moments of inertia J and J_e.

of the rotor can be obtained from the above-mentioned formulas, if in them we assume that the angle φ of natural torsional oscillations of the rotor obeys harmonic law $\varphi = \varphi_0 \sin \omega_0 t$, where φ_0 -- amplitude, and ω_0 – frequency of these torsional oscillations.

Vibratory gyroscopes of rotor type, apparently due to technical difficulties of realization, have not been widely applied in practice. There is information [36] on developments abroad of a VG of another type, called a gyrotron; the schematic and basic relationships for this type of VG are given below.

3. Schematic of a VG of Rod Type (Gyrotrom). Basic Relationships

One scheme of a VG of rod type, finding practical application, is based on use of the properties of vibrating masses – rods, comprising tines of a vibrator – a tuning fork; this instrument is known by the name of a Sperry gyrotron [36]. The sensor of the instrument is a vibrator (Fig. 8.21a), consisting of rods Rod_1 and Rod_2, elastic torsion rod TR, connecting the rods with the base of the vibrator B, and plate Pl rigidly fastened to the torsion rod.

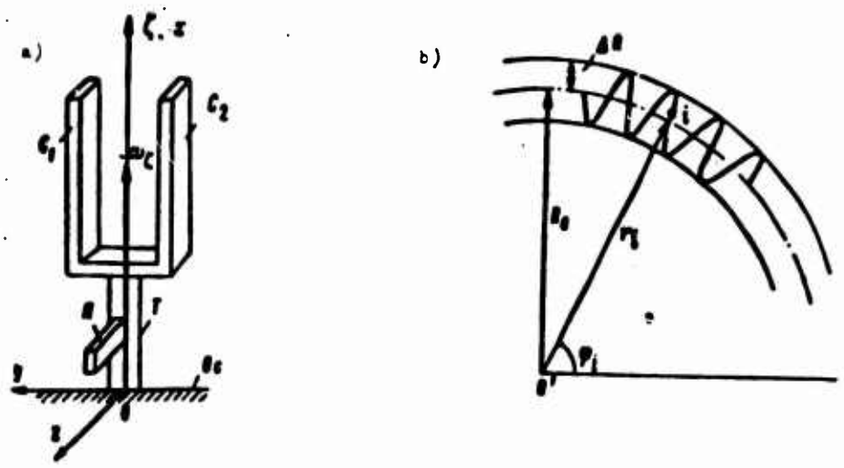

Fig. 8.21. On the principle of operation of a gyrotron.

The principle of action of such a sensor is as follows. By special electromagnets the rod-tines of the tuning fork are set into oscillatory motion. If the base of the instrument, i.e., the vehicle, does not rotate about axis $O\zeta$, oscillations of the rods are in plane $O\zeta(x)_y$ with constant amplitude and frequency depending on parameters of the system. If the vehicle together base B of the vibrator turns about axis $O\zeta$ with angular velocity ω_ζ, there appear torsional oscillations of the vibrator, and, consequently, also of the elastic torsion rod TR and plate Pl about axis $O\zeta(x)$; as will be shown later, the amplitude of these oscillations is proportional to angular velocity ω_ζ.

When the vibrating tines of the tuning fork approach one another, the moment of inertia of the fork about axis $O\zeta$, decreases, and the angular velocity of torsional oscillations about axis increases. If the tines move apart, the moment of inertia

of the fork increases, and its angular velocity decreases. Thus, the vibrating tuning fork constitutes a system with a moment of inertia variable with respect to axis $O\zeta(x)$.

To determine the moment of inertia of the tuning fork we consider motion of the i-th particle (Fig. 8.21b), of one of the tines. Since the frequency of oscillation of the tines exceeds the angular velocity of rotation of the base of the instrument by many times, the path of this particle is a very compressed sine curve. Let R_0 denote the average distance from the axis of the tuning fork to each of the tines; ΔR – the amplitude of vibrations of the tines; r_i – the distance of the i-th particle from the tuning fork axis. For r_i we have the evident relationship

$$r_i = R_0 + \Delta R \sin \omega_0 t, \tag{8.4.21}$$

where ω_0 – frequency of natural vibrations of the tuning fork tines.

The moment of inertia of the i-th particle of mass m_i about axis $O\zeta$ will be

$$I_i = m_i r_i^2 = m_i (R_0 \mp \Delta R \sin \omega_0 t)^2 = m_i R_0^2 \left(1 \mp \frac{\Delta R}{R_0} \sin \omega_0 t\right)^2.$$

Since $\frac{\Delta R}{R_0} \ll 1$, we have, approximately,

$$I_i = m_i R_0^2 \left(1 + 2 \frac{\Delta R}{R_0} \sin \omega_0 t\right). \tag{8.4.22}$$

Summing moments of inertia of all particles of the system, we find its moment of inertia

$$I = I_0 \left(1 + 2 \frac{\Delta R}{R_0} \sin \omega_0 t\right), \tag{8.4.23}$$

where

$$I_0 = m R_0^2 \quad . \tag{8.4.24}$$

– mean value of the moment of inertia;

$$m = \sum_{i=1}^{n} m_i \tag{8.4.25}$$

– mass of the whole system.

Let us find moment M of inertia of the tuning fork, appearing during rotation of the base of the vibrating tuning fork with angular velocity ω_ζ. During this motion projections ω_{r_i} and ω_{φ_i} of acceleration $\vec{\omega}_i$ of the i-th particle in the polar system of coordinates (r, φ) are determined by the following relationships [84]:

$$\left.\begin{aligned} w_{r_i} &= \ddot{r}_i - r_i\dot{\varphi}^2 \\ w_{\varphi_i} &= \frac{1}{r_i}\cdot\frac{d}{dt}\left(r_i^2\dot{\varphi}\right) = r_i\ddot{\varphi} + 2\dot{\varphi}\dot{r}_i \end{aligned}\right\} \tag{8.4.26}$$

Since $\dot{\varphi} = \omega_\zeta$,

$$\left.\begin{aligned} w_{r_i} &= \ddot{r}_i - r_i\omega_\zeta^2 \\ w_{\varphi_i} &= r_i\dot{\omega}_\zeta + 2\omega_\zeta\dot{r}_i \end{aligned}\right\} \tag{8.4.27}$$

Force of inertia $\vec{F}_i$, corresponding to acceleration $\vec{w}_i$, will be

$$\vec{F}_i = -m_i\vec{w}_i = -m_i\left(w_{r_i}\vec{r}_i^0 + w_{\varphi_i}\vec{\tau}_i^0\right). \tag{8.4.28}$$

where $\vec{r}_i^0$ – unit vector of direction r_i;

$\vec{\tau}_i^0$ – unit vector of direction, perpendicular to r_i.

The moment of this force with respect to the axis of the tuning fork will be

$$\vec{M}_i = r_i\vec{r}_i^0 \times \vec{F}_i = -r_i\vec{r}_i^0 \times m_i\left(w_{r_i}\vec{r}_i^0 + w_{\varphi_i}\vec{\tau}_i^0\right).$$

from which for the modulus of the moment we have

$$M_i = -r_i m_i w_{\varphi_i} = -r_i m_i\left(r_i\dot{\omega}_\zeta + 2\omega_\zeta\dot{r}_i\right). \tag{8.4.29}$$

For $\dot{r}_i$, according to (8.4.21), we have

$$\dot{r}_i = \Delta R\omega_0\cos\omega_0 t. \tag{8.4.30}$$

Angular velocity ω_ζ of rotation of the vehicle about axis 0ζ we consider to vary by harmonic law

$$\omega_\zeta = a_\zeta\cos\omega t, \tag{8.4.31}$$

where a_ζ – peak of angular velocity;

ω – angular frequency of change of angular velocity.

We have

$$\dot{\omega}_\zeta = -a_\zeta\omega\sin\omega t. \tag{8.4.32}$$

Substituting (8.4.21), (8.4.30) – (8.4.32) in (8.4.29), we obtain

$$M_i = -m_i\left[-(R_0+\Delta R\sin\omega_0 t)^2 a_\zeta\omega\sin\omega t + 2(R_0+\Delta R\sin\omega_0 t)a_\zeta\cos\omega t\cdot\Delta R\omega_0\cos\omega_0 t\right] =$$

$$= -m_i R_0 a_\zeta\omega_0\left\{-R_0\left(1+\frac{\Delta R}{R_0}\sin\omega_0 t\right)^2\frac{\omega}{\omega_0}\sin\omega t + 2\Delta R\cos\omega_0 t\cos\omega t - 2\Delta R\frac{\Delta R}{R_0}\sin\omega_0 t\cos\omega_0 t\cos\omega t\right\}. \quad (8.4.33)$$

As was shown earlier, $\frac{\Delta R}{R_0} \ll 1$; parameters of the tuning fork are selected in such a way that $\frac{\omega}{\omega_0} \ll 1$; therefore, instead of (8.4.33), we have, approximately,

$$M_i = -2m_i R_0\Delta R a_\zeta\omega_0\cos\omega_0 t\cos\omega t. \quad (8.4.34)$$

Summing all particles of the system, we obtain

$$M = -2mR_0\Delta R a_\zeta\omega_0\cos\omega_0 t\cos\omega t, \quad (8.4.35)$$

where m – mass of the tuning, determined by formula (8.4.25).

This is an approximate expression of the inertial moment, appearing during rotation of the base of a vibrating tuning fork with angular velocity ω_ζ. From (8.4.35) it follows that M constitutes moment of Coriolis forces of inertia, analogous to the torque of gyroscopic reaction; the magnitude of this torque is proportional to angular velocity $\omega_\zeta = a_\zeta\cos\omega t$ [see (8.4.31)].

Let us consider the schematic of the Sperry gyrotron [36] (Fig. 8.22). Basic components of the instrument are: rods [tines] Rod_1 and Rod_2, elastic torsion rod TR, base B of the vibrator, plate Pl; we discussed the function and interaction of these components earlier (Fig. 8.21a). Rod_1 and Rod_2 of the tuning fork impart oscillations with a frequency of about 2000 cps with the help of electromagnets EM_1 and EM_2, which are fed from oscillator Osc through amplifier Amp_1. Let us note that with such a high frequency of natural oscillations of the tines the influence on the instrument of various disturbances considerably decreases.

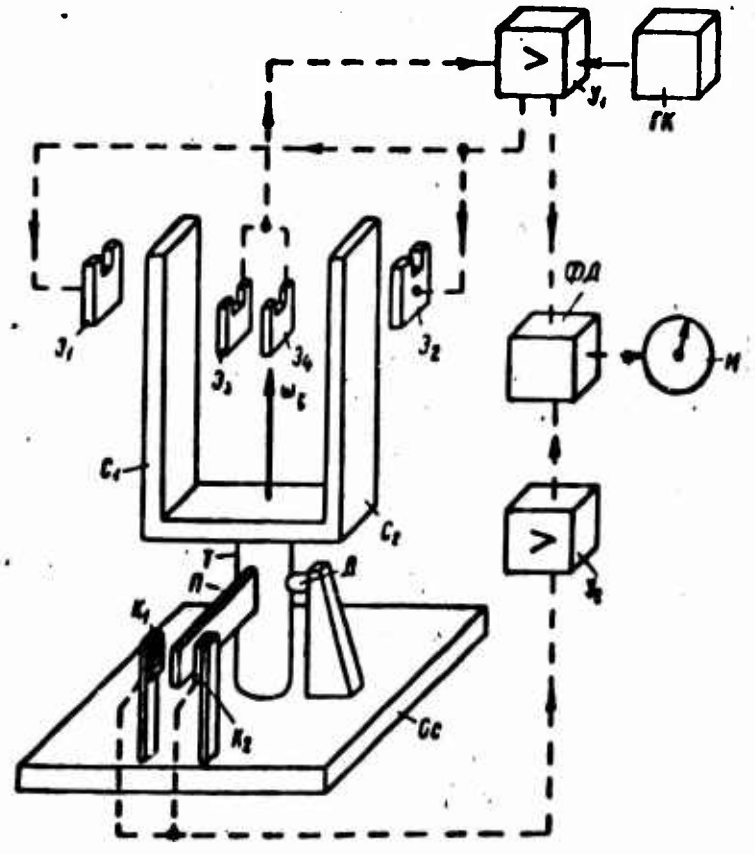

Fig. 8.22. Schematic of a gyrotron.

During rotation of the base B of the instrument with angular velocity ω_ζ, as shown earlier, there appears torque, causing oscillations of the torsion rod TR; here, the natural frequency of torsional oscillations of the tuning fork about its axis is equal to the natural frequency of vibration of the tines. To realize vibration of the tines with natural frequency, besided electromagnets EM_1 and EM_2 there also are electromagnets EM_3 and EM_4; the latter pickoff natural oscillation of the tuning fork tines, and corresponding electrical signals proceed to amplifier Amp_1, and from it to electromagnets EM_1, and EM_2, which sustain natural oscillation of the tuning fork tines. For quenching natural torsional oscillations of the tuning fork about its axis there is a damper D.

During forced oscillations of the torsion rod TR and plate Pl in the field of coils C_1 and C_2 of the signal pickoff there is taken the corresponding electrical signal, which proceeds through amplifier Amp_2 to phase detector PD, to which there is also introduced the signal from electromagnets EM_3 and EM_4 through amplifier Amp_1. The presence of such an electrical circuit permits us to realize equality of frequencies of torsional oscillations of the tuning fork and vibration of its tines. At the output of phase detector PD there is obtained a d-c signal, whose polarity depends on the direction of rotation of the vehicle, and whose magnitude is proportional to the angular velocity ω_ζ of rotation of the latter. Readings of the instrument are fixed on indicator I.

The theory of the gyrotron is the subject of a number of works (see, e.g., [192, 146]), which we use subsequently.

Let us compose the differential equation of torsional oscillations of the vibrator about axis $O\zeta$ (Fig. 8.21a). We designate by α the angle of rotation of the vibrator about axis $O\zeta$. The equation of these torsional oscillations we record in the form

$$I_0\ddot{\alpha} = M_\zeta, \tag{8.4.36}$$

where I_0 – moment of inertia of the vibrator about axis $O\zeta$;

M_ζ – total external torque, acting on the vibrator about the same axis.

In equation (8.4.36) the moment of inertia of the vibrator is assumed constant and equal to the mean value of the moment of inertia, determined by formula (8.4.24). Investigation of the gyrotron, taking into account the variable value of the moment of inertia of the vibrator [see (8.4.23)] is given in article [192].

Torque M_ζ we shall present in the form

$$M_{\zeta} = -M + M_{\text{e}} + M_{\text{д}}, \tag{8.4.37}$$

where M – the moment of Coriolis forces of inertia, appearing in the vibrator during rotation of the vehicle [see (8.4.35)];

M_{rest} – restoring torque of forces of elastic strains of the torsion rod;

M_d – torque of damping.

By analogy with (8.2.4) and (8.2.13) we have

$$M_{\text{e}} = -c\alpha; \quad M_{\text{д}} = -b\dot{\alpha}. \tag{8.4.38}$$

Expression (8.4.35) we change to form

$$M = -mR_0\Delta R a_{\zeta}\omega_0 \left[\cos\left(\omega_0 + \omega\right)t + \cos\left(\omega_0 - \omega\right)t\right]. \tag{8.4.39}$$

Substituting (8.4.39) and (8.4.38) in (8.4.37) and the resulting expression in (8.4.36), we have

$$I_0\ddot{\alpha} + b\dot{\alpha} + c\alpha = mR_0\Delta R a_{\zeta}\omega_0 \left[\cos\left(\omega_0 + \omega\right)t + \cos\left(\omega_0 - \omega\right)t\right]. \tag{8.4.40}$$

By analogy with (8.2.43), (8.2.45)-(8.2.47) we introduce designations

$$n = \sqrt{\frac{c}{I_0}}; \quad \frac{b}{I_0} = 2\zeta n; \quad \zeta = \frac{b}{2\sqrt{I_0 c}}; \quad k_1 = \frac{mR_0\Delta R\omega_0}{I_0}. \tag{8.4.41}$$

Then

$$\ddot{\alpha} + 2\zeta n\dot{\alpha} + n^2\alpha = k_1 a_{\zeta}\left[\cos\left(\omega_0 + \omega\right)t + \cos\left(\omega_0 - \omega\right)t\right], \tag{8.4.42}$$

where n – frequency of natural undamped torsional oscillations of the tuning fork;

ζ – relative attenuation factor.

If we designate by $T = \frac{1}{n}$ the time constant of the gyrotron, (8.4.42) can be presented in form

$$T^2\ddot{\alpha} + 2\zeta T\dot{\alpha} + \alpha = ka_{\zeta}\left(\cos\Omega_1 t + \cos\Omega_2 t\right). \tag{8.4.43}$$

where

$$k = \frac{k_1}{n^2} = \frac{mR_0\Delta R\omega_0}{I_0} \cdot \frac{I_0}{c} = \frac{mR_0\Delta R\omega_0}{c}; \tag{8.4.44}$$

$$\Omega_1 = \omega_0 + \omega; \quad \Omega_2 = \omega_0 - \omega. \tag{8.4.45}$$

As was shown earlier, the frequency of vibration of the tines and of torsional

oscillations of the tuning fork are equal, i.e.,

$$\omega_0 = n. \tag{8.4.46}$$

Since natural torsional oscillations of the vibrator rapidly attenuate due to damping, we determine its forced oscillations, which are characterized by a particular solution of equation (8.4.43).

The transfer function of the gyrotron is determined by the same relationship (8.2.68) as the transfer function of a GT. Then the amplitude and phase frequency-response curves of the gyrotron will be expressed by formulas (8.2.74) if in them we set $\omega = \Omega_1$ (1 = 1, 2), i.e.,

$$A(\Omega_i) = \frac{k}{\sqrt{(1 - T^2\Omega_i^2)^2 + 4\zeta^2 T^2 \Omega_i^2}};$$
$$\varphi(\Omega_i) = -\operatorname{arc\,tg}\frac{2\zeta T\Omega_i}{1 - T^2\Omega_i^2} \quad (i = 1, 2). \tag{8.4.47}$$

Forced torsional oscillations of the tuning fork about its axis, by analogy with (8.4.20), are determined by relationship

$$a_0(t) = a_c \sum_{i=1}^{2} A(\Omega_i) \cos[\Omega_i t + \varphi(\Omega_i)]. \tag{8.4.48}$$

This formula can be simplified, if we transform the expressions for $A(\Omega_1)$ and $\varphi(\Omega_1)$, considering that $\frac{\omega}{\omega_0} \ll 1$.

Substituting in (8.4.47) relationships (8.4.45), (8.4.46) and $T = \frac{1}{n}$, we have [146]

$$\left.\begin{aligned} A(\Omega_1) &= \frac{k}{\sqrt{\left[1 - \frac{(\omega_0 + \omega)^2}{\omega_0^2}\right]^2 + 4\zeta^2 \frac{(\omega_0 + \omega)^2}{\omega_0^2}}} \approx \frac{k}{\frac{2}{\omega_0}\sqrt{\omega^2 + \zeta^2\omega_0^2}} \\ A(\Omega_2) &= \frac{k}{\sqrt{\left[1 - \frac{(\omega_0 - \omega)^2}{\omega_0^2}\right]^2 + 4\zeta^2 \frac{(\omega_0 - \omega)^2}{\omega_0^2}}} \approx \frac{k}{\frac{2}{\omega_0}\sqrt{\omega^2 + \zeta^2\omega_0^2}} \\ \tan\varphi(\Omega_1) &= -\frac{2\zeta\frac{\omega_0 + \omega}{\omega_0}}{1 - \frac{(\omega_0 + \omega)^2}{\omega_0^2}} \approx \frac{\zeta\omega_0}{\omega} \end{aligned}\right\} \tag{8.4.49}$$

$$\left.\begin{aligned}
&\operatorname{tg}\varphi(\Omega_1)=-\frac{2\zeta\frac{\omega_0-\omega}{\omega_0}}{1-\frac{(\omega_0-\omega)^2}{\omega_0^2}}\approx-\frac{\zeta\omega_0}{\omega}\\
&\varphi(\Omega_1)=\varphi_1=\operatorname{arc\,tg}\frac{\zeta\omega_0}{\omega}\qquad \varphi(\Omega_2)=\varphi_2=-\operatorname{arc\,tg}\frac{\zeta\omega_0}{\omega}=-\varphi(\Omega_1)=-\varphi_1
\end{aligned}\right\}\ (8.4.49)\ (\text{contd})$$

Considering (8.4.49), we rewrite (8.4.48) in form

$$a_\zeta(t)=\frac{h a_\zeta}{\frac{2}{\omega_0}\sqrt{\omega^2+\zeta^2\omega_0^2}}\left[\cos\left[(\omega_0+\omega)t+\varphi_1\right]+\cos\left[(\omega_0-\omega)t-\varphi_1\right]\right] \tag{8.4.50}$$

or

$$a_\zeta(t)=\frac{h\omega_0 a_\zeta}{\sqrt{\omega^2+\zeta^2\omega_0^2}}\cos\omega_0 t\cos(\omega t+\varphi_1) \tag{8.4.51}$$

and, taking into account (8.4.44), we finally obtain

$$a_\zeta(t)=\frac{mR_0\Delta R\omega_0^2}{\sqrt{\omega^2+\zeta^2\omega_0^2}}a_\zeta\cos\omega_0 t\cos(\omega t+\varphi_1). \tag{8.4.52}$$

From formula (8.4.52) it follows that the amplitude of forced torsional oscillations of the vibrator is directly proportional to the angular velocity of rotation of the vehicle. Consequently, the gyrotron permits measuring the angular velocity of a vehicle, directed along its axis of symmetry $O\zeta$ (Fig. 8.21a), which is therefore the axis of sensitivity of the instrument. According to formula (8.4.51), sensitivity of the instrument to the angular velocity of rotation of the vehicle depends on coefficient k, expressed through parameters of the system by relationship (8.4.44); from this point of view coefficient k should be as large as possible.

Using terms applied in radio engineering (see, e.g., [34, 89]), it is possible to say that the gyrotron modulates the input signal, which represents the angular velocity of the vehicle $\omega_\zeta = a_\zeta \cos \omega t$ [see (8.4.31)]. Modulation, as is known, is the process of changing one or several parameters of a signal under the influence of another signal. In the considered case the modulated signal $\omega_\zeta(t)$ is the harmonic oscillations with constant amplitude a_ζ; these oscillations are called the <u>carrier.</u> The signal, influencing one or another parameter of the carrier oscillations, is called the <u>modulating signal.</u> In this case the modulating signal is the vibratory oscillations of tines of the tuning fork. Here we have <u>amplitude modulation</u>, since

the amplitude of the carrier oscillations is proportional to the modulating signal. The graph of oscillations modulated in the gyrotron is shown in Fig. 8.23. From this point of view parameters in formula (8.4.51) can be called: a_ζ – amplitude of unmodulated carrier oscillations; ω – frequency and φ_1 – phase of carrier oscillations; ω_0 – frequency of modulating signal; $\frac{k\omega_0}{\sqrt{\omega^2+\zeta^2\omega_0^2}}$ – coefficient of modulation.

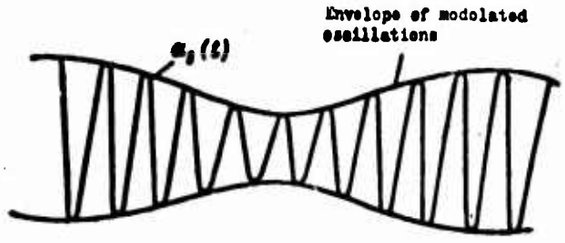

Fig. 8.23. Graph of modulated oscillations.

4. Appraisal of VG's

We shall briefly appraise VG's (see, e.g., [43]), considering one of the most useful schemes applied practically, in a gyrotron:

1) the instrument has one axis of sensitivity and does not react to components of angular velocity of the vehicle on the other axes;

2) there is no Cardan suspension, thanks to which we practically reduce to zero dry friction; the presence of constant vibration lowers the threshold of sensitivity;

3) the high frequency of natural oscillations makes the gyrotron practically insensitive to various disturbances;

4) the output signal is taken in the form of modulated oscillations, which gives the possibility of performing different operations with it: amplification, conversion, integration, differentiation, etc;

5) readings of the instrument are linear in a wide range;

6) there is the possibility of substantially decreasing dimensions of the instrument;

7) the gyrotron does not contain any fragile components, which makes it considerably more durable than gyroscopes, especially under strong mechanical influences (blows, vibration, etc.);

8) the absence of friction parts considerably increases the service life of the instrument.

For creation of a high-quality instrument it is necessary to overcome a number of technical difficulties, connected with the following:

1) guaranteeing high accuracy of manufacture of vibrating components;

2) realization of special care in balancing the vibrator;

3) preservation of constancy of temperature;

4) guaranteeing constancy of elastic properties of components of the vibrator;

5) measurement of comparatively small angles of twisting of the elastic torsion rod, etc.

It should be indicated that possibilities of application of VG's are very diverse. Most simply, the instrument is used as a meter of angular velocity of a vehicle. The VG can find application also in systems of active gyroscopic stabilization. In the case of application of high-precision VG's they can be used also in systems of inertial navigation (Chapter 11).

C H A P T E R 9

INTEGRATING GYROSCOPES

Designations Appearing in Cyrillic

в = for = forced

в = spin = spin axis

вых = o = output

г = g = gyroscopic

Г = GR = gyrorotor

ГВ = GV = gyrovertical

ГН = DG = directional gyro

Гс = Gh = gyrohousing

ГС = GS = gyrostabilizer

ГТ = GT = gyrotachometer

ГУ = GD = gyroscopic device

г.э = g.e = gyroequator

д = d = damping

д = dyn = dynamics

Д = D = liquid suspension

Д = PO = pickoff

д.м = tq = torquer

ДМ = TQ = torquer

ДС = SP = signal pickoff

ИГ = IG = integrating gyro

ИДГ = IDG = integro-differentiating gyro

К = Fr, R = frame, gimbal ring

М = Pend = pendulum

отр = adj = adjustment

п = f = float

п = par = parameter

П = P = platform

П = Pot = potentiometers

п.г = g.f = gyrofloat

ПГ = GF = gyrofloat

ПДГ = FDG = floated differentiating gyro

ПИГ = HIG = hermetic integrating gyro

Пр = Sp = spring

Пщ = Br = bearing

рад = rad = radians

сек = sec = second

с.д = s.m = stabilizing motor

СМ = SM = stabilizing motor

т = fr = friction

у = set = setting

у = Amp = amplifier

Ш = Gr = gear

Ц.Т. = c.g. = center of gravity

э = ass = assigned

§ 9.1. Function and Types of Integrating Gyroscopes (IG's)

In systems of automatic adjustment and control there are applied GD's intended for integration of certain parameters, characterizing motion of the controlled system. Such devices are integrating gyroscopes (IG's). Integrating (§ 2.4, Par. 1) is what we call a gyroscope which carries out integration of the input signal.

Let us note that integration is a natural property of a gyroscope. Indeed from the law of precession (§ 1.3, Par. 6) it follows that the three-degree-of-freedom astatic gyroscope is essentially an integrator, since its angle of precession is equal to the integral with respect to time of the external torque applied to it. In other words, an astatic gyroscope, if we do not consider

nutation oscillations of its axis, can be considered an integrating network (with transmission factor $k = \frac{1}{H}$, where H — angular momentum of the gyroscope) with respect to an input disturbance on the so-called cross-axis (§ 3.2, Par. 3f). The transient response at the output of this network is proportional to the integral in time of the mentioned disturbance.

Depending upon determination of parameters of the object IG's are divided into gyroscopes intended for measurement of: 1) angles of turn of the object, obtained in the instrument by means of integration of angular velocity of the object; 2) the sum of angles of turn of an object and its angular velocities; 3) linear velocity of an object, obtained in the instrument by means of integration of linear acceleration of the c.g. of the object.

The integrating gyroscope which determines the angle of rotation of an object by means of integration of the components of its angular velocity is called a rate-integrating gyro. In such IG's, by analogy with the GT (§ 8.2, Par. 2), we normally use a two-degree-of-freedom astatic gyroscope.

In distinction from the GT, for the IG there is no component, designed for creation of a restoring torque, and the device for quenching natural oscillations of the GT frame is replaced in the IG by a very powerful damper. The most advanced instrument of this type is the hermetic integrating gyroscope (HIG). The gyroscopic rate-integrator, measuring angles of rotation of an object can, in principle, be used in the same cases when, for a given purpose, there is applied a three-degree-of-freedom astatic gyroscope (e.g., as a DG or GV).

The integrating gyroscope, determining sum of the angle of rotation of an object and its angular velocity, is called an integro-differentiating gyroscope (IDG). This variety of IG also is based on use of a two-degree-of-freedom astatic gyroscope and differs from the GT only by the fact that between the instrument frame and the damper there is introduced an elastic connection, e.g., in the form of a spring. Such an instrument in the circuit of an autopilot permits us to directly find, e.g., the sum of the angle and rate of yaw of an aircraft, i.e., to immediately obtain the law of control of its course.

An integrating gyroscope which determines linear velocity of an object by means of integration of linear acceleration of its c.g. is called a gyroscopic integrator of linear accelerations. In an IG of this type, in distinction from the preceding, they usually apply a three-degree-of-freedom gyroscope with its

c.g. offset with respect to the point of suspension, due to which the instrument becomes sensitive to linear accelerations of the object. Such IG's are used for determination of linear velocity of an object [58, 161, 195].

In this chapter we present in most detail rate-integrating gyro, which have recently had wide application, especially in inertial navigational systems (Chapter 11); more briefly, we shall consider gyroscopic integrators of linear accelerations.

§ 9.2. General Characterization and Schematics of Rate-Integrating Gyro

1. Function and Types of GI's

The rate-integrating gyro is a GD, designed for integration of the angular velocity of the base (object), on which it is established, i.e., for determination of the angle of rotation of the object. One distinctive feature of this IG is that it, similarly to the GI (§ 8.2, Par. 1), reacts to the absolute angular velocity of the object, thanks to which it determines the angle of rotation of the latter in inertial space.

There exist different types of IG. The most wide spread is the IG with a liquid damper, and also the floated IG, in which the hydrostatic suspension simultaneously serves as the integrating damper. There also applied integro-differentiating gyroscopes. Let us give a general characterization and schematics of these types of IG's.

2. Schematic of an IG with Liquid Damper. Basic Relationships

The schematic of an IG with a liquid damper is shown in Fig. 9.1. The basic component of the instrument is a two-degree-of-freedom astatic gyroscope; the gyrorotor is fixed in a gimbal ring R (frame of instrument). For integration of the input signal we have the integrating liquid damper D, sometimes called a dashpot damper [30]. Readings of the instrument, proportional to the angle β of frame rotation about axis $O\eta$ (Oy), are taken in the form of voltage from potentiometer Pot.

In some IG schemes on axis $O\eta$ there is set a torquer TQ (controller [149]), which is used, e.g., to compensate disturbing torques, constant in magnitude, acting on this axis. Another function of the torquer during use of it in a floated

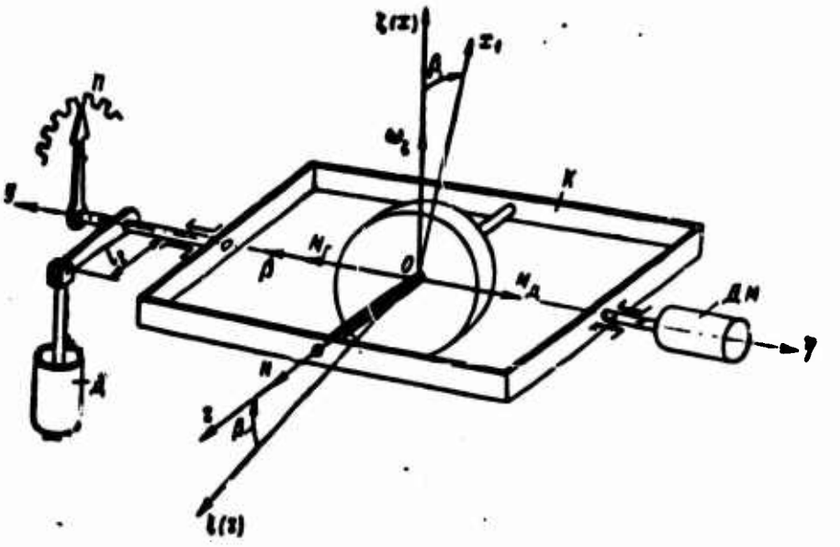

Fig. 9.1. Schematic of an IG with a liquid damper.

IG will be shown later.

Axes $O\xi\eta\zeta$ (Fig. 9.1) we shall consider connected with the vehicle (ship, aircraft); axes Ox_1yz, combined in the initial position (where $\beta = 0$) with $O\xi\eta\zeta$, are connected with the frame of the instrument and are Resal's axes (§ 1.1). Axis $O\zeta$ for the location of the gyroscope axis in Fig. 9.1 is the measuring, input, or sensitivity axis, since in this case the IG reacts to component $\omega\zeta$ of angle velocity $\vec{\omega}$ of rotation of the vehicle, and thanks to presence of integrating damper D gives at the output angle α of rotation of the vehicle about axis $O\zeta$. Axis $O\eta$ (Oy) is the output axis, since angle β of rotation of the frame about it is proportional, as we shall show later, to the integral with respect to time of angular velocity $\omega\zeta$, i.e., to angle α of rotation of the vehicle about axis $O\zeta$.

Comparing this scheme of an IG with the scheme of a GT (Fig. 8.1), we see that for the IG there is no spring Sp, provided in the GT for creation of restoring torque M_{sp}, and the pneumatic damper, used in a GT only for quenching natural oscillations of the instrument frame, is replaced in the IG by a rather powerful liquid damper, whose main assignment is integration of the input signal. Let us note that the liquid damper, in distinction from the pneumatic one, almost does not have dry friction; its characteristics are more stable, and it creates considerably greater damping torque.

Let us find the basic relationship for the considered IG, establishing the functional connection between the input $\omega_\zeta(t)$ and the output $\beta(t)$. With turn of the vehicle about axis $O\zeta$ with angular velocity ω_ζ, similarly to how it is in a GT

(§ 8.2, Par. 2), there appears gyroscopic torque $\vec{M}_g$, directed along axis $O\eta$ (Oy) and seeking to combine the axis of the gyroscope with axis $O\zeta$. The magnitude of this torque is determined by expression (8.2.2) or for small β, by relationship (8.2.3). Consequently, gyroscopic torque M_g, as well as angular velocity ω_ζ, is the input quantity.

During turn of frame IG angle β the piston of the liquid damper shifts relative to its cylinder. It is assumed that the liquid used in the damper has great density, and the coefficient of viscosity does not depend on the speed of movement. Consequently, rotation of the IG frame is prevented by torque M_d, introduced by the damper and determined by expression (8.2.13).

When deriving the basic relationship, characterizing work of an IG, we do not take into account of the equatorial component of angular momentum of the gyroscope, inertia of the frame, and also various disturbing torques, and we use one of the abbreviated equations (1.6.30) of a gyroscope in D'Alembert form. Considering the second of the shown equations in projections on axis Oy (Fig. 9.1) and taking as gyroscopic torque expression (8.2.3), and as the external torque the torque M_d, introduced by the damper [see (8.2.13)], we obtain

$$M_r + M_d = 0 \tag{9.2.1}$$

or

$$H\omega_\zeta - b\dot{\beta} = 0,$$

from which

$$\dot{\beta} = \frac{H}{b}\omega_\zeta. \tag{9.2.2}$$

i.e., the velocity $\dot{\beta}$ of rotation of the instrument frame is proportional to the angular velocity ω_ζ of rotation of the vehicle about the measuring axis $O\zeta$.

Integrating (9.2.2) and considering that in the initial moment when $t = 0$, $\beta(0) = 0$, we have

$$\beta = \frac{H}{b}\int_0^t \omega_\zeta(\tau)\,d\tau, \tag{9.2.3}$$

i.e., the angle β of rotation of the IG frame is proportional to the integral with respect to time of angular velocity ω_ζ of the vehicle.

If by α we designate the angle of vehicle rotation about axis $O\zeta$ with angular velocity $\omega_\zeta = \dot{\alpha}$, we can write (9.2.3) in form

$$\beta = \frac{H}{b}\int_0^t \dot{\alpha}(\tau)\,d\tau. \qquad (9.2.4)$$

from which, with initial conditions t = 0, $\alpha(0) = 0$, we have

$$\beta = \frac{H}{b}\alpha. \qquad (9.2.5)$$

Thus, the angle β of rotation of the IG frame is proportional to angle α of vehicle rotation about the measuring axis $O\zeta$ in inertial space. If axis $O\zeta$ is direct along the vertical, then for a aircraft angle α will be the angle of yawing ψ. Consequently, in this case the IG determines angles of yawing and turns of the aircraft in azimuth, i.e., executes the same function as a DG with a three-degree-of-freedom astatic gyroscope (Chapter 5). With help of similar, but differently poisitioned IG's, we can also determine pitch ϑ and bank angles γ of the aircraft.

Expressions (9.2.2) and (9.2.5) are valid only for small values of β, which for the assigned parameters H and b of the instrument occur, when angles α of vehicle rotation are small. In another case we will be required to use an IG together with a servo drive, working the IG frame into zero position, similarly to a GT with zero drive of the frame (§ 8.2, Par. 7).

It is necessary to recall that the considered type of IG is a rather crude instrument. In fact, the minimum angular velocity $\omega_{\zeta_{min}}$, which the IG can sense, as in the case of a GT, is determined by formula (8.2.107), i.e., depends for a given angular momentum H of the gyroscope on the magnitude $K_{set} = |M_{fr}|$ of the given moment of friction on the axis of rotation of the instrument frame. In this connection accuracy of an IG with the normally applied suspension of a frame on ball bearings turns out to be low; such instruments are not widely used, since they possess larger errors than a three-degree-of-freedom astatic gyroscopes. Therefore, more detailed information on the theory of the rate-integrating gyroscope will be given later in the example of an IG of float type.

3. Schematic of an Integro-Differentiating Gyroscope. Basic Relationships

The schematic of an IDG differs from the IG scheme shown in Fig. 9.1 by the fact that between the gimbal ring (frame of the instrument) and damper D there is an elastic connection. The latter can be realized, e.g., in the form of a cylindrical spring, one end of which is connected to a lever (fastened to the frame), and

other to the piston of the damper.

The basic relationship for an IDG, ignoring inertia of its individual parts, we can derive as follows [109]. We designate by η deformation of the spring due to movement of the piston; deformation of the same spring due to turn of the frame β angle will be βl_2 (l_2 — length of the lever, Fig. 9.1); total linear deformation of the spring will be $(\beta l_2 - \eta)$. Here, the force developed by the spring will be $(\beta l_2 - \eta)c_1$, where c_1 — coefficient of linear spring force of the spring. On the other hand, the force developed by the damper is equal to magnitude $a\dot{\eta}$, where a — specific force of damping [see (8.2.14)]. If we disregard the mass of the damper piston, from the condition of equality of forces developed by the damper and spring, we have

$$a\dot{\eta} = (\beta l_2 - \eta)c_1. \tag{9.2.6}$$

In the considered scheme of an instrument the gyroscopic torque H_{ω_ζ} during rotation of the vehicle with angular velocity ω_ζ is balanced by torque of the spring, equal to $(\beta l_2 - \eta)c_1 l_2$, i.e.,

$$(\beta l_2 - \eta)c_1 l_2 = H\omega_\zeta. \tag{9.2.7}$$

Excluding η from (9.2.6) and (9.2.7), we have

$$\beta = \frac{H}{c_1 l_2}\left(\frac{\omega_\zeta}{a} + \frac{\dot{\omega}_\zeta}{c_1}\right). \tag{9.2.8}$$

If, considering, e.g., yawing of an aircraft, we designate $\omega_\zeta = \psi$, we can write (9.2.8) in form

$$\beta = \frac{H}{c_1 l_2}\left(\frac{\dot{\psi}}{a} + \frac{\ddot{\psi}}{c_1}\right)$$

from which, for initial conditions $t = 0$, $\beta(0) = 0$, $\psi(0) = 0$, $\dot{\psi}(0) = 0$, we obtain

$$\beta = \frac{H}{c_1 l_2}\left(\frac{\psi}{a} + \frac{\dot{\psi}}{c_1}\right). \tag{9.2.9}$$

i.e., the angle β of rotation of the instrument frame is proportional to the sum of angle ψ and angular velocity $\dot{\psi}$ of yawing of the aircraft.

In deriving relationship (9.2.9) we did not allow for inertia of separate components of the IDG, or disturbances. Allowance for these factors leads to dynamic errors, which can be determined by the same methods as for corresponding error of a GT (§ 8.2, Par. 6).

4. Schematic of a Hermetic Integrating Gyroscope

On GD's, during use of them in inertial navigational systems, in systems of automation stabilization of objects, etc., we place high demands with respect to accuracy of readings during work in conditions of large overloads and vibration, shock and other disturbances. Therefore, development in recent years of precision GD's has proceeded to lowering the friction torques in supports of the gyroscope suspension, guaranteeing of vibration and shock strength and resistance, etc. Satisfaction of these requirements was the creation of gyroscopes with floating suspension, which acquired the name of floated gyros. In these gyroscopes due to the buoyancy of the liquid friction torques in the suspension substantially decrease. Liquid (hydrostatic) suspension ensures damping of oscillations of the sensor, and also integration of the input signal; on this feature of a liquid suspension are based the integrating properties of the floated gyro. Simultaneously, the liquid suspension ensures high performance of the instrument in conditions of vibration, shock and other disturbances.*

Among the different types of floated gyros the most widely applied is the hermetic integrating gyroscopes (HIG). The schematic of an HIG is shown in Fig. 9.2. Gyrorotor GR fixed in frame Fr, which is a float of cylindrical shape; axis

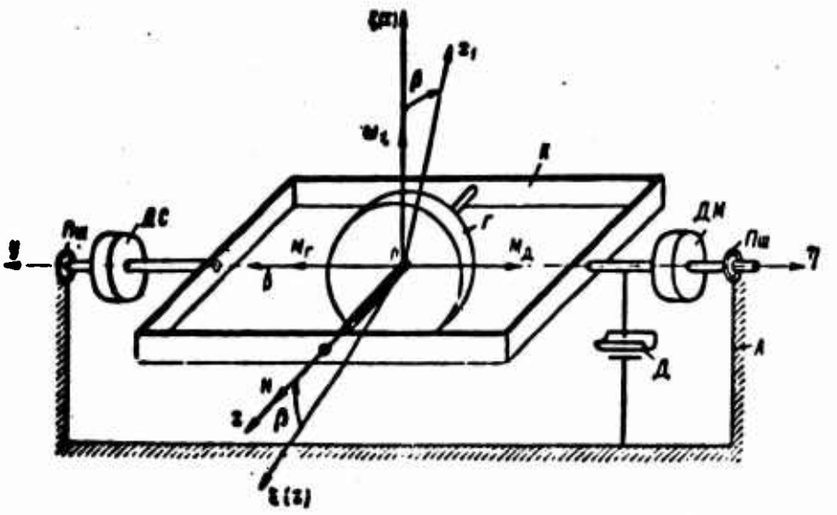

Fig. 9.2. Schematic of an HIG.

*It should be noted that the idea of a floated gyro with two degrees of freedom was first proposed in the Soviet Union by L. I. Tkachev. In 1945 he developed a construction of a floated IG [149], for which he was issued an author's certificate. In the United States the first developments of floated gyros were conducted at the Massachusetts Institute of Technology under the leadership of C. Draper [33].

Oy (Oy) of rotation of the float is fixed in bearings Br, located in housing A of the instrument, which also has a cylindrical shape. The gap between the float and the housing, and also all free space inside the housing is filled by liquid with a high specific weight. This system forms a liquid suspension, designated in Fig. 9.2 by letter D. Buoyancy of the liquid should be equal to the weight of the gyro unit; bearings Br of the suspension turn out to be practically completely free from load; they are made in the form of jewel pivots (like these applied in clockworks) with a minimum moment of friction. Liquid in gap between cylindrical surfaces of the float and instrument housing provides damping, the torque of which is strictly proportional to the first power of angle velocity $\dot{\beta}$ of rotation of the float [see (8.2.13)]. Forces of viscous friction in the damper are used in the IG for integration with respect to time of the input signal – the angular velocity of the vehicle. Application of a liquid suspension partially protects the axis of suspension (axis of rotation of the float) from influence on it of vibration, blows, etc.

In the instrument there is provided automatic control of temperature, which is necessary to maintain: 1) constancy of specific weight of the liquid, which should be close to the average specific weight of the floated gyrounit; 2) constancy of viscosity of the liquid, and consequently, also of the damping factor; 3) constancy of the position of the e.g. of the float and of the center of pressure of the liquid with respect to the axis of rotation of the gyro unit.

Instrument readings, proportional to angle β of rotation of the float about axis $O\eta$ (Oy), are taken in the form of voltage from an inductive signal pickoff SP, called a microsyn [149, 181, 32]. On other side of the axis there is fixed a microsyn – torquer TQ. The latter serves to apply a torque to the axis of the float, necessary for bringing the float (axis of the gyroscope) to zero position, to compensate magnitude of disturbing torques constant in magnitude (from unbalance of the gyrounit, due to the presence of leads, etc.) and for allowance for induced rotation of the reference system.

In a floated IG, as in an IG with a liquid damper (Fig. 9.1), axis $O\zeta$ is the measuring, or input axis, since the HIG reacts to component ω_ζ of angular velocity of the vehicle about this axis; axis $O\eta$ (Oy) is the output axis.

The principle of action of the HIG is analogous to the principle of action of an IG with a liquid damper (Par. 1). For an HIG relationships (9.2.2), (9.2.3),

and (9.2.5) are valid, i.e., the angular velicity $\dot{\beta}$ of rotation of the float is proportional to the measured angular velocity ω_ζ, and angle β is (proportional to the angle of rotation of the vehicle about axis $O\zeta$ in inertial space). Since torques of friction and other disturbances to the axis of rotation of the float are negligible, the HIG is sensitive to minute angular velocities of the vehicle, which ensures high accuracy of its readings.

From what has been said it follows that the HIG permits one to determine the angle of rotation of a vehicle relative to one of the axes connected with it. Therefore, it can be applied as a DG, and with combination of two instruments — as a GV. However, installation of a precision IG, rigidly fastened to a vehicle, is not always expedient. Most acceptable is use of an HIG with a servo drive; here, there is obtained a one-axis active stabilizer; in the case of application of two instruments we have a two-axis active stabilizer, or a GV of active type, and in the case of three HIG's — a triple-axis stabilizer. This explains the wide use of HIG's in inertial navigational systems.

One should not completely reject the possibility of rigid fastening of an HIG on a vehicle and application of it without a servo drive. As shows C. Draper [180], often for control of vehicles on the latter they rigidly set three rate gyros, which determine angular deflections of the vehicle from its initial orientation. As for inertial navigational systems of high accuracy, it is considered [180] that in this case the HIG should be set on a stabilized base, and should by used together with servo drives.

Considering the possibility of using an HIG in contemporary precision GD's, below we consider the theory of it. Here, for establishment of basic properties of the HIG we shall show its peculiarities during work without a servo drive. Theory of an HIG with servo drive for the case of its application in an inertial navigational system will be considered in Chapter 11.

§ 9.3. Hermetic Integrating Gyroscopes and Their Application

1. Cutaway of an HIG and Certain of its Features

A simplified cutaway [149] of an HIG is shown in Fig. 9.3. Gyroscope 1 is fixed in hermetic float 2, whose axis is set in bearings 3, located in the hermetic-case 4 of the instrument. Gap 5 between the case and the float, filled with liquid

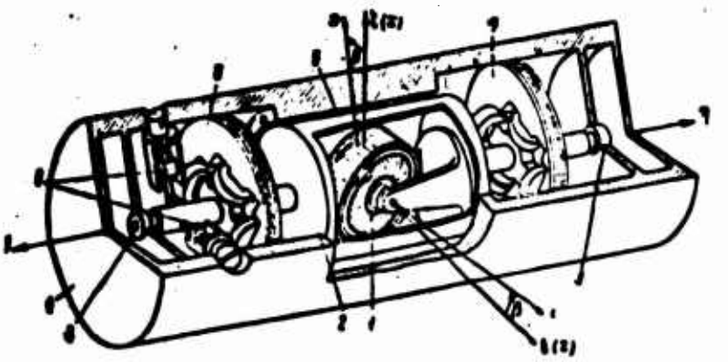

Fig. 9.3. Cutaway of an HIG.

in cylindrical section AB of the floated gyroassembly, will form together with the mentioned components an integrating damper. The magnitude of the gap is of the order of several tenths of a millimeter, which for a large coefficient of viscosity of the liquid causes a considerable coefficient of damping. The necessity of having a high coefficient of damping is determined by the effort to obtain the smallest possible time constant for the HIG, which, as we shall show later, decreases dynamic instrument errors. On the axis of rotation of the float there are fixed signal pickoff 6 and torquer 7. The function of all the components of the instrument were discussed during the description of its schematic (Fig. 9.2).

The floated gyrounit should be carefully balanced by means of combination of its c.g., with the center of pressure and reduction to zero of its buoyancy. Removal of slope of the float and reduction of its buoyancy to zero are attained by bobs, fixed on axis y on both sides of the floated gyrounit (not shown in Fig. 9.3). Balancing of the float with respect to axis x is produced by shifting four nuts 8, which are twisted on screws, fixed on the shaft of the float (in Fig. 9.3 we see only two screws with nuts).

A more detailed description of the HIG, its separate elements and constructions the reader can find in the corresponding literature (e.g., [181, 149]). Certain peculiarities of the HIG, theory of its hydrostatic suspension, and also results of experimental investigations are considered in the report of C. Draper, and others [33]. In these works there are also numerical values of parameters of HIG's (produced by American firms), which will be used later in examples.

Depending upon the magnitude of angular momentum H of the gyroscope HIG's are divided into two groups: 1) comparatively low precision instruments with small H, intended for measurement of large angular velocities; 2) precision instruments with large H, applied in active stabilization system, where angular velocities of

the stabilized platform (on which the HIG is fixed) are small. The difference in values of H in these cases is determined by the strength of the bearings of the float during action on them of gyroscopic torque, caused by rotation of the vehicle.

For the considered instruments angle β of rotation of the float we seek to have sufficiently small or even close to zero by using an HIG with a servo drive. Here, there are ensured: 1) constancy of the coefficient of viscosity of the damping liquid; 2) linearity of readings of the instrument and insensitivity of it to components of angular velocity of the vehicle on axes, not coinciding with the measuring axis; 3) large range of measured angular velocity; 4) small magnitudes of bearing reactions during rotation of the instrument with considerable angular velocities.

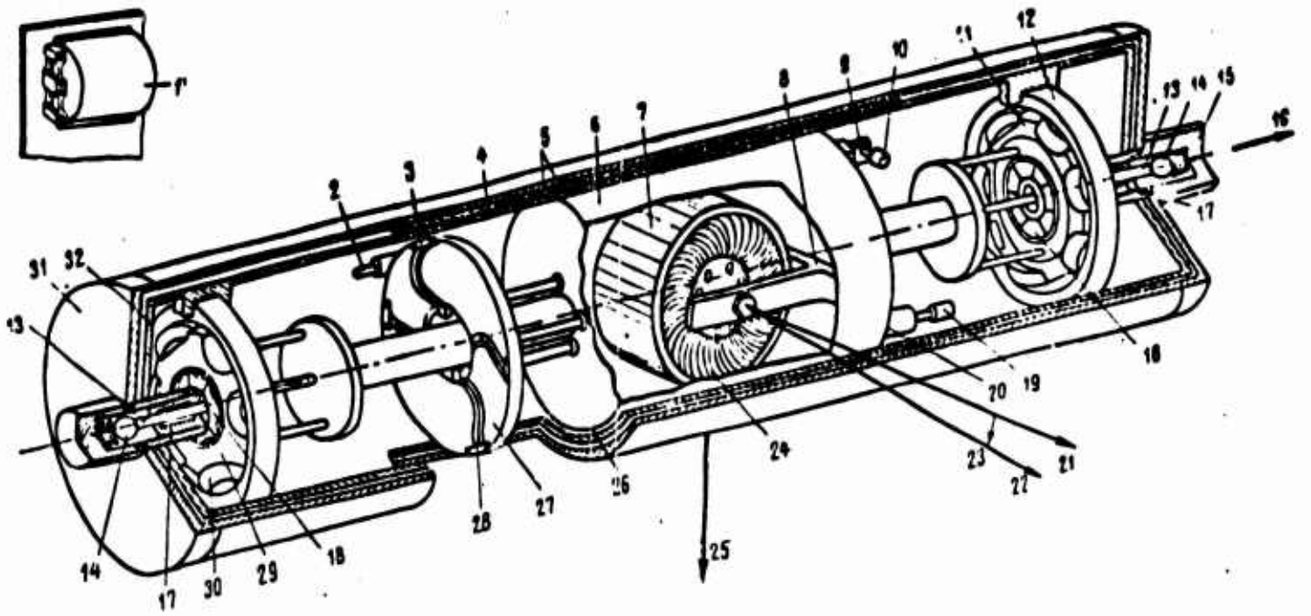

Fig. 9.4. Detailed cutaway of a floated IG. 1) linkage of assembled gyroscope; 2) adjustment of plate with flex-leads; 3) flex-lead terminals; 4) winding of electric heater; 5) temperature pickoff; 6) float; 7) gyrorotor; 8) frame of Cardan suspension; 9) unbalance compensator (lever-type); 10) adjusting screw of unbalance compensator; 11) rotor of torquer and magnetic suspension unit; 12) stator of torquer; 13) support end of the float shaft (journal); 14) ball, pressed by a spring; 15) mounting (installing) flange; 16) output axis; 17) bearing of float suspension; 18) stator of magnetic suspension unit; 19) adjusting screw of compensator of unequal rigidity; 20) compensator of unequal rigidity; 21) initial position of axis of angular momentum; 22) current position of axis of angular momentum; 23) angle of rotation of the float; 24) stator of gyromotor; 25) input axis; 26) gap of damper; 27) unit (plate) of flex-leads; 28) flex-leads for feeding the gyromotor; 29) rotor of signal pickoff and magnetic suspension unit; 30) stator of signals pickoff; 31) temperature shield; 32) instrument case.

Usually, the HIG is an instrumant of precision type. According to some data [149], drift of an HIG can comprise 0.1 angular minutes/minute and even less. One of the peculiarities of the HIG is that it has high sensitivity and a large range

of measurement of angular velocities.

We shall give (Fig. 9.4) a more detailed cutaway of an HIG recently published [180] (designations of separate elements of the instrument are shown in thethe figure caption). Distinctive features of this diagram are the presence of compensators of unequal rigidity (due to difference in the coefficients of elasticity of the structure of the float in directions of the input axis and the axis of sensitivity), and also of the magnetic suspension, ensuring centering of the axis of the float.

Let us note on the technique of production, assembly and adjustement of floated gyros of precision-type we present special requirements [149, 181].

For tests of floated gyros there have been developed special equipment and methods of experimental investigations [149, 181, 148, 147].

2. Dynamic Characteristics of the HIG

a) Geometric and Kinematic Parameters

Analysis of HIG's starts with composition of differential equations and the corresponding transfer functions. From comparison of Figs. 9.1 and 8.1 it follows that axis $O\xi\eta\zeta$ and Ox_1yz in the investigation of an HIG and a GT are selected identically. Therefore, geometric and kinematic parameters of the HIG and GT are also identical (§ 8.2, Par. 3a). Transformation of axes $O\xi\eta\zeta$ to Ox_1yz is expressed by matrix equality (8.2.15), where matrix A_1 of this transformation comes from relationship, (8.2.16) or (8.2.17). Components ω_ξ, ω_η, ω_ζ of angular velocity $\vec{\omega}$ of axes $O\xi\eta\zeta$, connected, for instance, with a ship, are determined by formulas (8.2.20) for (8.2.21) and (8.2.22), i.e., an HIG whose axis of sensitivity coincides with axis $O\zeta$, will react to the angular velocity $\omega_\zeta \approx -\dot{\psi}$ of yawing of the ship.

b) Composition of Differential Equations of an HIG

In an HIG, like a GT, we used an astatic rate gyro. Presence of like components in both instruments permits us during composition of differential equations of motion of the HIG to directly use differential equations of the GT, obtained in § 8.2, Par. 3b; here, one should consider specific differences between the considered instruments.

As the starting point we shall use differential equation (8.2.36) of motion of the GT frame. In this equation: 1) equatorial moment of inertia $J_{g.e}$ of the gyroscope, representing the sum of the equatorial moment of inertia of the rotor, the frame (housing) of the gyroscope and moving parts, one should replace by moment of inertia $J_{g.f}$ of the gyrofloat about its spin axis; 2) due to the absence in an HIG of an element, introducing restoring torque in the instrument, magnitude c one should take equal to zero; 3) in the right part it is necessary to introduce torque M_{tq}, applied by the torquer (Fig. 9.1) about the spin axis of the float. In accordance with this, using (8.2.36) we find the differential equation of the HIG in form*

$$J_{g.f}\left(\ddot{\beta} - \dot{\omega}_\eta\right) + b\dot{\beta} = H\left(\omega_\zeta \cos\beta - \omega_\xi \sin\beta\right) + M_{fr} + M_{tq} + f(t). \tag{9.3.1}$$

where $J_{g.f}$ — moment of inertia of gyrofloat about its spin axis;

β, $\dot{\beta}$, $\ddot{\beta}$ — angle, angular velocity and angular acceleration of rotation of the gyrofloat about axis $O\eta$ (Fig. 9.2);

b — damping factor of integrating damper;

ω_η — component of angular acceleration of the vehicle on axis $O\eta$ (disturbing angular acceleration);

$\dot{\omega}_\zeta$ — component of angular velocity of the vehicle on the axis of sensitivity of the instrument;

ω_ξ — component of angular velocity of the vehicle on axis $O\xi$ (disturbing angular velocity);

H — angular momentum of the gyroscope;

M_{tq} — torque applied by the torquer;

M_{fr} — friction torque in the float axis;

f(t) — other disturbing torques.

---

*We can write the exact differential equation of motion of an HIG, considering the corresponding equation for a GT, given in the footnote to p.122 as follows:

$$J_{n.f}\left(\ddot{\beta} - \dot{\omega}_\eta\right) + b\dot{\beta} = H\left(\omega_\zeta \cos\beta - \omega_\xi \sin\beta\right) - \\ - \left(J_{fx} - J_{spin}\right)\left(\frac{\omega_\zeta^2 - \omega_\xi^2}{2}\sin 2\beta + \omega_\xi\omega_\zeta \cos 2\beta\right) + M_{fr} + M_{tq} + f(t),$$

where J_{fx} — moment of inertia of the gyrofloat with respect to axis Ox (Fig. 9.2);

J_{spin} — moment of inertia of float (without rotor) with respect to axis Oz.

Torque M_{fr} constitutes the torque of residual friction in bearings of the gyrofloat; for the case of liquid friction M_{fr} is determined by relationship (8.2.34); during calculation of dry friction, it is determined by (8.2.35).

Let us give the expression of torque M_{tq}. For microsyns [149] applied in a floated IG as torquers, the magnitude of torque M_{tq} is proportional to current intensity i, of the control signal fed to the torquer, i.e.,

$$M_{x.u} = -k_1 i. \tag{9.3.2}$$

where k_1 — transmission factor of the torquer.

Let us find the expression for damping factor b of the integrating damper. For this we designate (Fig. 9.5):

l — length of damper (length of housing of gyrofloat), cm;

r — radius of float 1, cm;

δ — gap between float and case of instrument 2, cm;

η — absolute viscosity of liquid in the gap of the integrating damper, poises;

S — area of gap, cm^2;

v — linear velocity of a point on the surface of the float, cm/sec.

During determination of damping factor b we consider only the damping torque of the cylindrical surface of the float, not taking into account the damping torque of its end surfaces. For the magnitude of the damping torque of the cylindrical surface of the float we have expression

$$M_x = \frac{rSv\eta}{\delta}. \tag{9.3.3}$$

Since $v = r\dot{\beta}$ ($\dot{\beta}$ — angular velocity of spinning of the gyrofloat about its axis), and $S = 2\pi rl$, instead of (9.3.3) we obtain

$$M_x = \frac{2\pi r^3 l\eta}{\delta}\dot{\beta}. \tag{9.3.4}$$

Comparing (9.3.4) with (8.2.13), we find an expression for the magnitude of the damping factor:

$$b = \frac{2\pi r^3 l}{\delta}\eta. \tag{9.3.5}$$

If η is expressed in poises, to find factor b in g-cm-seconds, instead (9.3.5), we have

$$b = \frac{2\pi r^3 l}{981\delta}\eta. \tag{9.3.6}$$

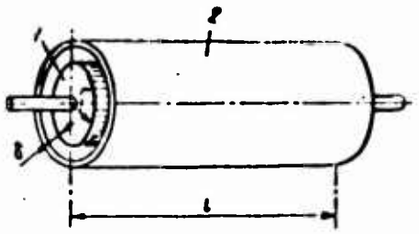

Fig. 9.5. Determination of the damping factor.

Magnitude b sometimes is called the specific damping torque, since, according to (8.2.13), b is numerically equal to the torque created by the damper when the angular velocity of spinning of the float is one radian per second. For a hermetic integrating gyroscope of type 10^4 No. 79 of the Massachusetts Institute of Technology [181, 149] design values of magnitudes b, δ, η are the following: δ = 0.0251 cm; η = 65 poises; b = 20.93 g-cm-sec.

Disturbing torques f(t) in equation (9.3.1) are caused by: incomplete balance of the gyrofloat; flex-lead torques; harmful torques, created by signal pickoffs and torquers, etc.

Before passing to further transformation of equation (9.3.1), we note the following. Often during the presentation of HIG theory [181, 149] as the output magnitude they consider, not to angle β of float rotation, but the corresponding voltage U_o, received from the signal pickoff and connected with β by the linear dependency

$$U_{вых} = k_2\beta, \tag{9.3.7}$$

where k_2 — transmission factor of the signal pickoff.

Introducing (9.3.7) in (9.3.1), we obtain the equation of the HIG for variable U_o. However, subsequently in presentation of HIG theory, as of the theory of other types of GD, we shall take as the variable angle β of rotation of the gyrofloat.

As has been shown, the limiting value of angle β of rotation of the float is small. If we also take into account the fact that the HIG is usually used with a servo system, in equation (9.3.1) it is possible to make substitutions cos β ≈ 1, sin β ≈ β; then

$$J_{в.г}\ddot{\beta} + b\dot{\beta} = H(\omega_z - \omega_\xi\beta) + J_{в.г}\dot{\omega}_\eta + M_{д.м} + M_т + f(t). \tag{9.3.8}$$

Since in the above-indicated cases $\omega_\zeta \gg \omega_\xi\beta$, instead of (9.3.8) we usually use the following equation:

$$J_{в.г}\ddot{\beta} + b\dot{\beta} = H\omega_z + J_{в.г}\dot{\omega}_\eta + M_{д.м} + M_т + f(t). \tag{9.3.9}$$

Consider a steady regime of work of an HIG, by which one should understand such a regime in which $\dot{\beta}$ = const, $\ddot{\beta}$ = 0. If ω_η = 0, M_{tq} = f(t) = 0, for a steady

regime we have expression

$$\beta = \frac{H}{b}\omega_\zeta,$$

coinciding with (9.2.2), i.e., the angular velocity of rotation of the gyrofloat is proportional to the angular velocity ω_ζ of rotation of the vehicle about the measuring axis $O\zeta$ (Fig. 9.2). Considering ω_ζ = const (since by the condition for a steady-state regime β = const), for zero initial conditions we obtain

$$\beta = \frac{H}{b}\omega_\zeta t. \tag{9.3.10}$$

In the considered case $\omega_\zeta t$ – angle α of rotation of the vehicle about axis $O\zeta$; therefore, instead of (9.3.10) we have expression

$$\beta = \frac{H}{b}\alpha,$$

coinciding with (9.2.5). In other words, angle β of rotation of the gyrofloat in a steady-state operating conditions of the instrument is proportional to angle α of vehicle rotation. As will be shown later, if the time constant of the HIG is sufficiently small, the above-mentioned expression will also be valid when $\omega_\zeta \neq$ $\neq$ const. Here, instead of (9.3.10) we obtain formula

$$\beta = \frac{H}{b}\int_0^t \omega_\zeta(\tau)\,d\tau$$

[see (9.2.3)]. Thus, in general, if the time constant of the HIG is sufficiently small, angle β of rotation of the floated gyrounit is proportional to the time integral of angular velocity ω_ζ of the vehicle.

We make certain transformations of equation (9.3.0); dividing both parts of it by b, we obtain

$$\frac{J_{n.r}}{b}\ddot{\beta} + \dot{\beta} = \frac{H}{b}\omega_\zeta + \frac{J_{n.r}}{b}\dot{\omega}_\eta + \frac{1}{b}[M_{д.м} + M_т + f(t)]. \tag{9.3.11}$$

We designate

$$T = \frac{J_{n.r}}{b}; \tag{9.3.12}$$

$$k = \frac{H}{b}; \quad \rho = \frac{1}{b}. \tag{9.3.13}$$

Considering (9.3.12) and (9.3.13), we rewrite (9.3.11) in form

$$T\ddot{\beta} + \dot{\beta} = k\omega_\zeta + T\dot{\omega}_\zeta + \rho\left[M_{д.м} + M_т + f(t)\right]. \qquad (9.3.14)$$

The physical meaning of the designations introduced above is the following: T — time constant of the HIG, characterizing its transient response; k — dimensionless transmission factor of the HIG, characterizing the dependence of $\dot{\beta}$ on ω_ζ in steady-state operating conditions for the instrument; ρ — transmission factor of the HIG with respect to torques applied to the spin axis of the float, characterizing the dependence of $\dot{\beta}$ on the shown torques in steady-state conditions.

Let us assume that the torquer is turned off and that there are no disturbances [$M_{tq} = M_{fr} = f(t) = 0$, $\dot{\omega}_\eta = 0$]; then we write equation (9.3.14) in form

$$T\ddot{\beta} + \dot{\beta} = k\omega_\zeta. \qquad (9.3.15)$$

from which, for small time constant T, even when $\omega_\zeta \neq$ const, with sufficient accuracy we have relationship

$$\dot{\beta} = k\omega_\zeta. \qquad (9.3.16)$$

which, according to (9.3.13), is another form of expression (9.2.2).

c) Transfer Functions and Frequency-Response Curves of an HIG

Let us determine the transfer function W(s) of an HIG with respect to the useful input signal, i.e., to angular velocity $\omega_\zeta(t)$. For this we use equation (9.3.15), presenting it in operational form $\left(p = \frac{d}{dt}\right)$:

$$(Tp + 1)p\beta = k\omega_\zeta. \qquad (9.3.17)$$

Applying to (9.3.17) the Laplace transform, we obtain the following expression for the transfer function of the HIG:

$$W(s) = \frac{\beta(s)}{\omega_\zeta(s)} = \frac{k}{(Ts + 1)s}. \qquad (9.3.18)$$

From formula (9.3.18) it follows that for a sufficiently small time constant T the HIG can be considered an integrating network, whose transfer function for $T = 0$, according to (9.3.18), we write in form

$$W(s) \doteq \frac{k}{s}. \qquad (9.3.19)$$

Taking T into account it is possible to consider the HIG a series connection of integrating and aperiodic [see (5.6.68)] networks, i.e.,

$$W(s) = W'(s)\,W''(s). \tag{9.3.20}$$

where

$$W'(s) = \frac{k}{s}; \quad W''(s) = \frac{1}{Ts+1}. \tag{9.3.21}$$

Properties of an integrating network were considered in the example of an astatic gyroscope with three degrees of freedom (§ 3.2, Par. 3b); properties of an aperiodic network were studied in the example of the same astatic gyroscope, but with a system of proportional correction (§ 5.6, Par. 3d). During influence on the HIG of input angular velocity $\omega_\zeta(t)$, e.g., in the form of a unit step function [1] (2.3.107), its transient response $\beta(t)$ is the superposition of the motion of gyrofloat by aperiodic law (aperiodic network) and motion of the gyrounit proportional to the integral of the input. Here, the smaller the time constant T, the faster the shown aperiodic motion and the more the HIG approximates an ideal integrating network.

Example 9.1: Determine time constant T and transmission factor k of an HIG of type 10^4 No. 79 of the Massachusetts Institute of Technology and also angular velocity $\dot{\beta}$ of rotation of the gyrofloat for the following initial data [181, 149]: moment of inertia of the gyrofloat with respect to its spin axis $J_{g.f} = 0.036$ g-cm-sec^2; damping factor of the integrating damper b = 20.39 g-cm-sec; angular momentum of gyroscope H = 10.19 g-cm-sec; constant value of measured angular velocity of object $\omega_\zeta = 0.5$ deg/sec.

Solution 1: By formula (9.3.12) we find time constant T:

$$T = \frac{J_{g.f}}{b} = \frac{0{,}036}{20{,}39} = 0{,}001766\ \text{sec}.$$

2. According to (9.3.13) we determine transmission factor k:

$$k = \frac{H}{b} = \frac{10{,}19}{20{,}39} = 0{,}4998 \approx 0{,}5.$$

3. Using (9.3.16), we find angular velocity $\dot{\beta}$ of rotation of the gyrofloat in steady-state conditions:

$$\dot{\beta} = k\omega_\zeta = 0{,}5 \cdot 0{,}5 = 0{,}25\ \text{deg/sec}.$$

From example 9.1 it follows that the time constant of the HIG is minute (experimentally measured value of it comprises [181, 149] T = 0.0027 sec); this indicates fast flow of the transient response, characterizing aperiodic motion of the gyrofloat.

Let us find the frequency characteristics of the HIG, considering it first as an ideal integrating network. Then for the gain-phase response of the HIG we have formula (3.2.88)

$$W(j\omega) = \frac{k}{j\omega}. \tag{9.3.22}$$

Amplitude $A(\omega)$, phase $\varphi(\omega)$, real $P(\omega)$ and imaginary $Q(\omega)$ frequency characteristics of the HIG, by analogy with (3.2.89) and (3.2.90), will be determined by relationships

$$A(\omega) = \frac{k}{\omega}; \quad \varphi(\omega) = -\frac{\pi}{2}; \tag{9.3.23}$$

$$P(\omega) = 0; \quad Q(\omega) = -\frac{k}{\omega}. \tag{9.3.24}$$

The gain-phase characteristic $W(j\omega)$ of an ideal floated IG will be determined by the same curve as for an integrating network (Fig. 3.8).

Let us now determine the frequency-response curve of a real HIG, taking into account its time constant T. Comparing (9.3.15) with (3.2.36), for $a(\omega)$ ... $d(\omega)$ in accordance with (3.2.38), we obtain

$$a(\omega) = k; \quad b(\omega) = 0; \quad c(\omega) = -T\omega^2; \quad d(\omega) = \omega. \tag{9.3.25}$$

Substituting (9.3.25) in (3.2.37), we find the gain-phase response of the HIG:

$$W(j\omega) = \frac{k}{-T\omega^2 + j\omega}. \tag{9.3.26}$$

Introducing (9.2.25) in (3.2.41), we obtain an expression for frequency-response curves $A(\omega)$, $\varphi(\omega)$, $P(\omega)$, $Q(\omega)$:

$$\left.\begin{aligned} A(\omega) &= \frac{k}{\omega\sqrt{1+T^2\omega^2}}; \quad \varphi(\omega) = \operatorname{arc\,tg}\frac{1}{T\omega} \\ P(\omega) &= -\frac{kT}{1+T^2\omega^2}; \quad Q(\omega) = -\frac{k}{\omega}\cdot\frac{1}{1+T^2\omega^2} \end{aligned}\right\} \tag{9.3.27}$$

If in (9.3.27) we set T = 0, we obtain formulas (9.3.23) and (9.2.24) for frequency-response curves of an ideal HIG.

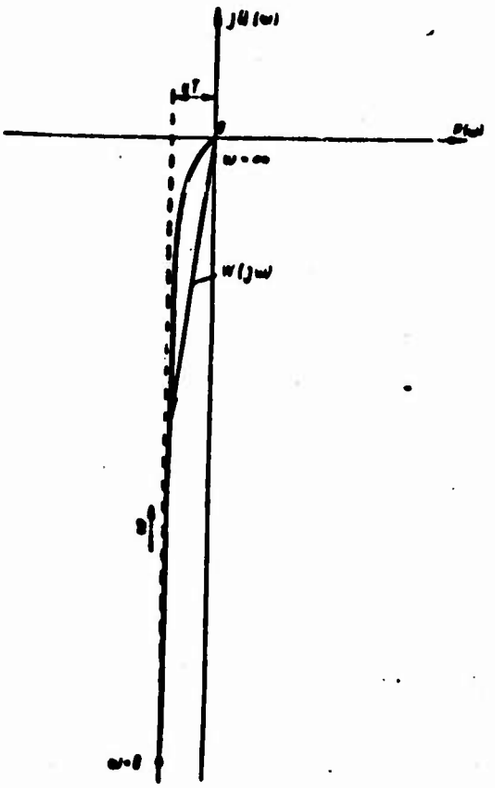

Fig. 9.6. Gain-phase characteristic of an HIG.

Example 9.2: Calculate and construct the graph of the gain-phase characteristics $W(j\omega)$ of a HIG for the initial data of example 9.1 and with change of ω from 0 to ∞.

Solution: Since, according to (3.2.40) $W(j\omega) = P(\omega) + jQ(\omega)$, by formula, (9.2.37) we calculate points of curve $W(j\omega)$ by their coordinates $P(\omega)$ and $Q(\omega)$. The graph of the gain-phase characteristic $W(j\omega)$ of the HIG is constructed in Fig. 9.6.

From Fig. 9.6 it follows that the graph of the gain-phase characteristic $W(j\omega)$ of the HIG practically has the same form as curve $W(j\omega)$ for an integrating network (Fig. 3.8).

Knowning the gain-phase response $W(j\omega)$ of the HIG, by formula (2.3.105) we can find impulse (weighting) transfer function of the HIG,

$$k_g(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} W(j\omega)e^{j\omega t}d\omega, \qquad (9.3.28)$$

which characterizes the transient response during application to the HIG of a unit impulse function, or delta function $\delta(t - t_0)$.

During investigation of HIG errors in dynamics it is necessary to know also the transfer function of the HIG with respect to disturbances. To determine this transfer function, which we henceforth shall designate by $Y(s)$, we use equation (9.2.14), considering in it $M_{tq} = f(t) = 0$; then we have

$$T\ddot{\beta} + \dot{\beta} = k\omega_\zeta + T\dot{\omega}_\eta + \rho M_{fr}, \qquad (9.3.29)$$

i.e., as the disturbances we consider component ω_η of angular acceleration of the vehicle along axis $O\eta$ (Fig. 9.2) and the residual friction torque M_{fr} in the bearings of the float.

For axes $O\xi\eta\zeta$ (Fig. 9.2) we take ship axes Oxyz (Fig. 2.2). The axis of sensitivity of the HIG coincides with axis $O\zeta$, and it will react to angular velocity $\dot{\varphi}$ of yawing of the ship; in fact, according to (8.2.21), $\omega_\zeta \approx -\dot{\varphi}(f)$.

Then, the disturbing angular acceleration $\dot{\omega}_\eta$, if we consider (8.2.22), is equal to angular acceleration $\ddot{\theta}$, of rolling of the ship, i.e., $\dot{\omega}_\eta = \ddot{\theta}$. Let us assume that in the axis of suspension of the gyrofloat there is liquid friction, determined by formula (9.2.34). Considering what has been said and (8.2.34), we rewrite (9.3.29) as follows:

$$T\ddot{\beta} + \dot{\beta} = k\omega_\zeta + T\ddot{\theta} - \rho n_1 \beta. \tag{9.3.30}$$

Designating

$$T_1 = \frac{T}{1+\rho n_1}, \quad k_1 = \frac{k}{1+\rho n_1}, \tag{9.3.31}$$

we have

$$T_1\ddot{\beta} + \dot{\beta} = k_1\omega_\zeta + T_1\ddot{\theta}. \tag{9.3.32}$$

Presence in the right part of this equation of disturbance $T_1\ddot{\theta}$ causes error ε_1 in the HIG reading. Since $\omega_\zeta(t)$ is the useful signal, then for error ε_1 we obtain the following differential equation:

$$T_1\ddot{\varepsilon}_1 + \dot{\varepsilon}_1 = T_1\ddot{\theta}. \tag{9.3.33}$$

We rewrite it in operational form:

$$(T_1 p + 1)p\varepsilon_1 = T_1 p^2\theta. \tag{9.3.34}$$

Applying the Laplace transform to (9.3.34), we obtain an expression of transfer function Y(s) of the HIG with respect to the disturbance of rolling:

$$Y(s) = \frac{\varepsilon_1(s)}{\theta(s)} = \frac{T_1 s}{T_1 s + 1}. \tag{9.3.35}$$

3. Transient Responses of the HIG

Let us consider behavior of an HIG in a transient response for a certain typical disturbance, for which we take a unit step input (§ 2.3, Par. 3d). The first question appearing here concerns stability of the HIG. H is simplest of all to judge stability of an HIG by its gain-phase response $W(j\omega)$ (Fig. 9.6), from which it follows that the HIG is stable for any positive coefficients of the corresponding characteristic equation.

Subsequently we shall be interested in change of the angle rotation of the gyrofloat in the transient, caused by a typical influence in the form of an abrupt

change of the input value per unit and for zero initial conditions, i.e., for a unit step input [1] [see (2.3.107)]. Function $\beta_h(t)$, which characterizes change of β at the HIG output for the shown conditions, is a transient function and is determined by relationship (2.3.108).

Planning to find function $\beta_h(t)$ of the HIG for application to it of unit step input $\omega_\zeta(t) = [1]$ with zero initial conditions, we use equation (9.3.15), which we rewrite in form

$$T\ddot{\beta} + \dot{\beta} = k[1]. \tag{9.3.36}$$

We designate the angular velocity $\dot{\beta}$ of rotation of the gyrofloat by Ω:

$$\dot{\beta} = \Omega. \tag{9.3.37}$$

Then

$$T\dot{\Omega} + \Omega = k[1]. \tag{9.3.38}$$

From (9.3.38) it follows that if as the output we consider $\Omega = \dot{\beta}$, with respect to Ω the HIG is an aperiodic network with transfer function

$$W(s) = \frac{\Omega(s)}{\omega_\zeta(s)} = \frac{k}{Ts+1} \tag{9.3.39}$$

and gain-phase response

$$W(j\omega) = \frac{k}{Tj\omega+1}. \tag{9.3.40}$$

The frequency-response curve of the HIG, as for an aperiodic unit, will be determined by relationships (5.6.71) and (5.6.72). The gain-phase characteristic for this case is presented in Fig. 5.16. The curve begins (for $\omega = 0$) on the real axis a distance k from the origin of coordinates; with increase of ω the modulus of $W(j\omega)$ decreases; when $\omega = \infty$ the phase shifts -90°. During application to the HIG of a periodic input $\omega_\zeta(t)$ angular velocity $\Omega(t) = \dot{\beta}(t)$ of rotation of the float will change by a periodic law of the same frequency as $\omega_\zeta(t)$; with a frequency of oscillations close to zero, $\Omega(t)$ should not have a phase shift; with a frequency near infinity, $\Omega(t)$ will lag in phase an angle approaching 90°.

Using (9.3.38), one can determine the transient function $\Omega_h(t)$ of angular velocity $\Omega = \dot{\beta}$. A particular solution of this equation will be

$$\Omega_r = k[1]; \tag{9.3.41}$$

it characterizes angular velocity of rotation of the float in steady-state conditions, i.e., [see (9.3.13)]

$$\Omega_{cr} = \dot{\beta}_{cr} = \dot{\beta}(\infty) = k[1] = \frac{H}{b}[1]. \tag{9.3.42}$$

what coincides with (9.2.2) for $\omega_\zeta(t) = [1]$.

According to (9.3.38), the transient function $\Omega_h(t) = \dot{\beta}_h(t)$ for a unit step input and for zero initial conditions [when $t = 0$, $\Omega_h(0) = 0.1$], will be

$$\Omega_h(t) = k\left(1 - e^{-\frac{t}{T}}\right). \tag{9.3.43}$$

Example 9.3: Calculate and construct transient function $\Omega_h(t)$ of the angular velocity of rotation of the gyrofloat for parameters of the instrument, given in example 9.1.

Solution 1: By formula (9.3.43) for different values of t we determine values of Ω_h, and then construct graph $\Omega_h(t)$ (Fig. 9.7).

2. Using relationship (6.3.10), we determine the setting time t_{set} of angular velocity $\Omega(t) = \dot{\beta}(t)$ with accuracy of 99%:

$$t_y = 4{,}6T = 4{,}6 \cdot 0{,}001766 = 0{,}008124 \text{ sec}.$$

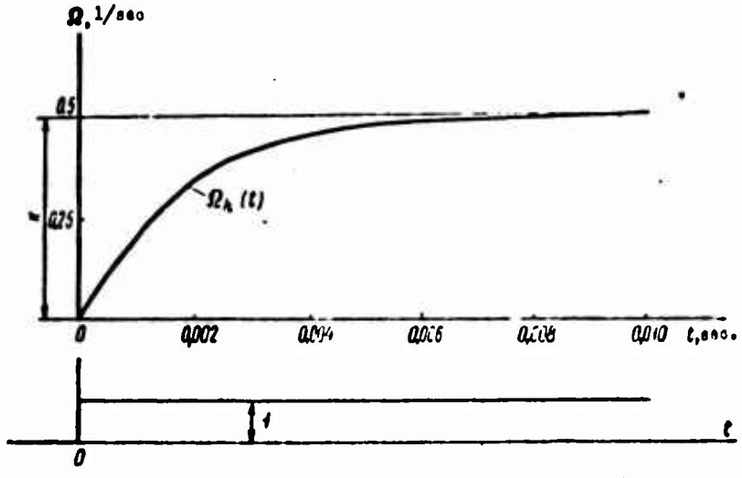

Fig. 9.7. Unit step response $\Omega_h(t)$ for a unit step input.

From example 9.3 and Fig. 9.7 it follows that the unit step response $\Omega_h(t)$ changes by aperiodic law; setting time t_{set}, proportional to time constant T of the HIG, is minute. On conclusion of the transient response the set value of angular velocity $\Omega(t) = \dot{\beta}(t)$ of rotation of the float is proportional to the input.

To determine the unit step response $\beta_h(t)$ we use equation (9.3.36). Integrating it once for during zero initial conditions, we obtain

$$T\dot{\beta} + \beta = k[1]t. \tag{9.3.44}$$

Since T is small, the unit step response $\beta_h'(t)$ of an ideal HIG will be, approximately

$$\beta_h'(t) = k[1]t, \tag{9.3.45}$$

i.e., the same as for an ideal integrating network (Fig. 9.8), and consequently, the angle β of rotation of the float for an input in the form of a unit step input [1] will be equal to the integral of this influence, and it increases proportionally to time t.

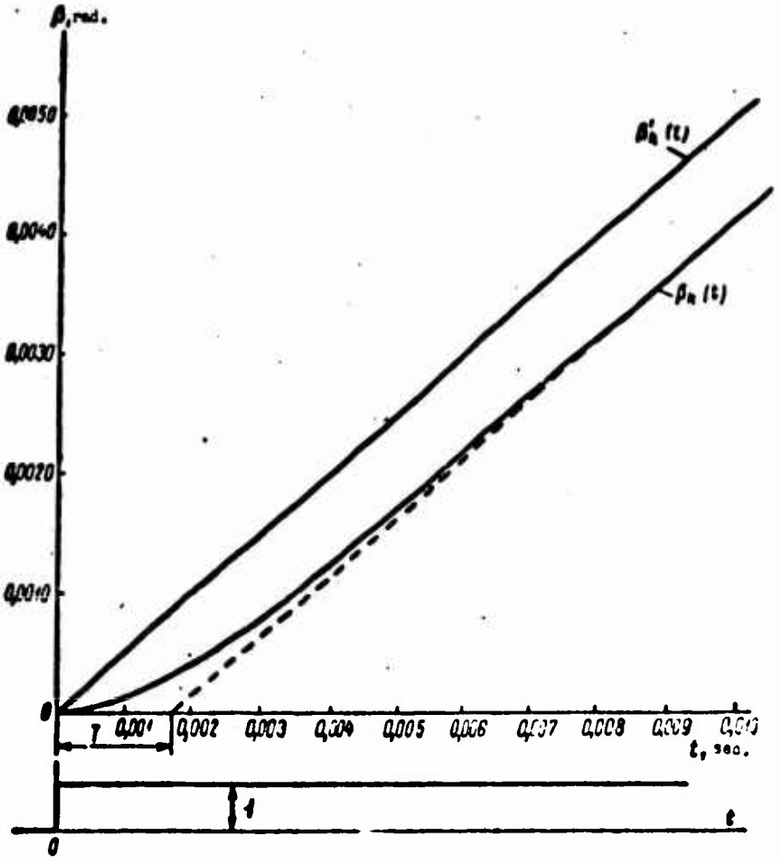

Fig. 9.8. Unit step responses $\beta_h'(t)$ of an ideal integrating network and $\beta_h(t)$ of an HIG.

Let us find the unit step response $\beta_h(t)$ of the HIG taking into account its time constant T. For this we find the physical meaning of [1] t, in the right part of equation (9.3.44). Since in the considered case [1] is the unit value of angular

velocity ω_ζ of rotation of the vehicle about the measuring axis $O\zeta$ (Fig. 9.2) of the HIG, [1] t is the current value of angle α of rotation of the vehicle about axis $O\zeta$ per unit angular velocity of its rotation.

Designating

$$[1]t = \alpha(t), \tag{9.3.46}$$

we rewrite (9.3.44) in form

$$T\dot{\beta} + \beta = k\alpha(t), \tag{9.3.47}$$

from which, for zero initial conditions [t = 0, $\alpha(0) = \beta(0) = 0$], we have

$$\beta_h(t) = \frac{k}{T} e^{-\frac{t}{T}} \int_0^t \alpha(\tau) e^{\frac{\tau}{T}} d\tau. \tag{9.3.48}$$

Integrating the right side (9.3.48) in parts, we obtain

$$\beta_h(t) = k\left[\alpha(t) - \int_0^t \dot{\alpha}(\tau) e^{-\frac{t-\tau}{T}} d\tau\right]. \tag{9.3.49}$$

Consequently, angle β of rotation of the float is proportional to angle α of rotation of the vehicle; $k\int_0^t \dot{\alpha}(\tau) e^{-\frac{t-\tau}{T}} d\tau$ characterizes methodical instrument error in dynamics. For its decrease it is important that time constant T be sufficiently small and the value of $\dot{\alpha}$ not be too large. In the considered case $\dot{\alpha} = \omega_\zeta = [1]$; therefore, performing integration in (9.3.49) and considering (9.3.46), we obtain the expression for the unit step response $\beta_h(t)$ of the HIG taking into account time constant T in the form

$$\beta_h(t) = k[1]\left\{t - T\left(1 - e^{-\frac{t}{T}}\right)\right\}. \tag{9.3.50}$$

For a minute T formula (9.3.50) becomes expression (9.3.45), characterizing the unit step response $\beta_h'(t)$ of an ideal integrating network.

Example 9.4: Calculate and construct the unit step response $\beta_h(t)$ of an HIG for parameters of the instrument, given in example 9.1.

Solution: By formula (9.3.50) for different values of time t we determine angle β of float rotation, and then construct the graph of unit step response $\beta_h(t)$ (Fig. 9.8).

From comparison of $\beta_h(t)$ and $\beta_h'(t)$ (Fig. 9.8) it follows that thanks to the smallness of time constant T the HIG differs little from an ideal integrating network.* In this connection performance indices of the HIG are the same as for integrating networks [107].

Formula (9.3.50) for the unit step response $\beta_h(t)$ of a HIG is valid during influence on the instrument of angular velocity of the vehicle in the form of a unit step function. It is possible to consider the characteristic of the transient response of an HIG for a constant (but not equal to unity) value of angular velocity of the vehicle. Then the expression for the transient response of the gyrofloat during influence on the instrument of angular velocity ω_ζ = const is obtained by multiplication of the right side of (9.3.50) by ω_ζ:

$$\beta(t)=k\omega_\zeta\left[t-T\left(1-e^{-\frac{t}{T}}\right)\right]. \tag{9.3.51}$$

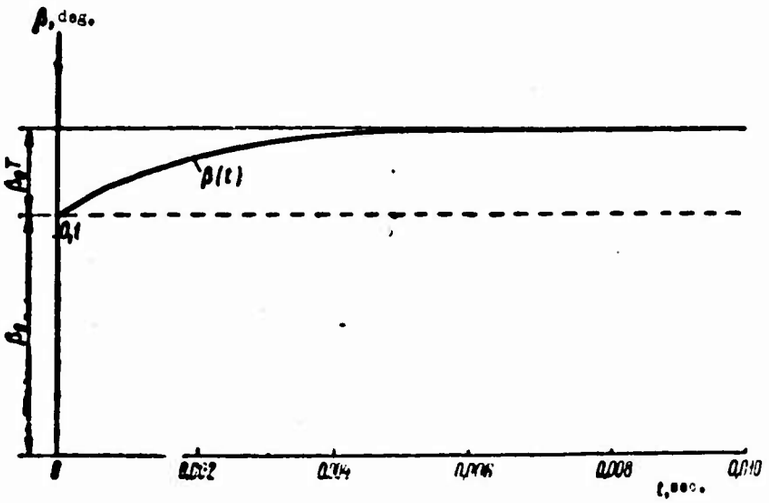

Fig. 9.9. Transient response of an HIG for non-zero initial conditions.

Function $\beta(t)$ has the same form as the unit step response $\beta_h(t)$ shown in Fig. 9.8.

Above we considered the transients response of an HIG for zero initial conditions. It is interesting to find the influence on a transient response of an HIG

*Let us note that curve $\beta_h(t)$ has an asymptote, whose equation is obtained, if in (9.3.50) we set $t \to \infty$; then $\beta_h(t) = k\,[1]\,(t - T)$. This asymptote is shown in Fig. 9.8 by the dotted line; it cuts off on the axis of abscissas a segment equal to time constant T. This circumstance can be used when determining T by the curve of the transient response, obtained experimentally [149].

of non-zero initial conditions. As an example let us consider free motion ($\omega_\zeta = 0$) of the gyrofloat, if in initial moment where t = 0 it had deflection $\beta(0) = \beta_0$ and initial angular velocity $\dot{\beta}(0) = \dot{\beta}_0$. Then, instead of (9.3.15), we have equation

$$T\ddot{\beta} + \dot{\beta} = 0, \tag{9.3.52}$$

which we shall integrate for the above-indicated initial conditions; we obtain

$$\beta(t) = \beta_0 + \dot{\beta}_0 T\left(1 - e^{-\frac{t}{T}}\right). \tag{9.3.53}$$

Transient response $\beta(t)$, determined by relationship (9.3.53), is shown in Fig. 9.9.

Example 9.5: Determine the steady-state value of the angle β of rotation of the gyrofloat whose parameters are given in example 9.1, if in the initial moment t = 0 the angle of deflection of axis of the gyroscope $\beta_0 = 0.1^\circ$, and to the float there is imparted an initial angular velocity $\dot{\beta}_0 = 20$ deg/sec.

Solution: By formula (9.3.53) for steady-state conditions upon completion of the transient response we have

$$\beta = \beta_0 + \dot{\beta}_0 T = 0{,}1 + 20 \cdot 0{,}001766 = 0.135^\circ.$$

From example 9.5 it follows that the biggest error is introduced by initial deflection β_0 of the gyroscope axis; influence of initial angular velocity $\dot{\beta}_0$ for a small time constant T is immaterial.

4. Instrument Errors of the HIG

We present HIG's, applied in precision GD's, very high requirements with respect to accuracy of their readings. Therefore, the subsequent account is connected with questions of accuracy of these instruments.

First, we shall consider instrument error of a HIG, caused by imperfection of its construction, deviations of characteristics of its components from their rated values, imprecisions of manufacture, assembly, etc. Instrument errors, like other ones, can be systematic and random (§ 2.4, Par. 2). Systematic errors in principle can be compensated by means of application, e.g., of corresponding correcting torques (with the help of a torquer), magnitudes of which are selected and introduced on the basis of experimental research of the instrument under

conditions, close to real. As for random errors, in principle it is impossible to compensate them. Therefore, during investigation, design and manufacture of such gyroscopes most important is lowering of the random components of errors to practically acceptable magnitudes.

In connection with the specific peculiarities of HIG's (caused mainly by application of a hydrostatic suspension) during theoretical calculations of its instrument errors there arise considerable difficulties. Therefore, for revealing this kind of error we usually use experimental methods.

Let us give the characteristic of disturbing torques and of HIG instrument errors caused by them. During use in real conditions an HIG may be influenced by the following disturbing torques:

1) moments of the force of gravity and inertia during shifting of the float e.g;

2) torques of hydrostatic pressure, appearing with unbalance of the gyrofloat and during change of the specific weight of the liquid;

3) torques of residual friction in suspension bearings and also torques of viscous friction due to arbitrary motion of the liquid;

4) torques, caused by elastic strains of the gyrofloat;

5) torques, caused by variation of parameters of the HIG, e.g., of the angular momentum of the gyroscope, the damping factor of the liquid, its temperature, etc.;

6) torques, caused by flex-leads and also by signal pickoffs and torquers, etc.;

These and other disturbances lead to the appearance of HIG instrument errors, to which it is possible to relate the following:

1) error due to unbalance of the gyrofloat;

2) errors caused by elastic strains of components of the gyrofloat;

3) errors caused by torques of different forces of friction;

4) errors caused by variations of HIG parameters;

5) errors of the signal pickoff and the torquer; errors due to imperfections of flex-leads, etc.

Unable to consider in detail all the above-indicated errors, we limited ourselve to certain brief remarks.

Friction torques, as in the case of a GT (§ 8.2, Par. 5), limit angular velocity of the vehicle, received by the instrument. In connection with this the minimum vehicular angular velocity $\omega_{\zeta min}$ to which the HIG reacts is determined by

formula (8.2.107)

$$\omega_{\zeta_{min}} > \frac{K_{set}}{H},$$

where K_{set} – magnitude of the friction torque on the axis of rotation of the gyrofloat;

H – angular momentum of the gyroscope.

Let us find the influence of friction torques on operation of an HIG. We use equation (9.3.14), considering in it $\omega_\eta = 0$, $M_{tq} = f(t) = 0$; then

$$T\ddot{\beta} + \dot{\beta} = k\omega_\zeta + \varepsilon M_r. \tag{9.3.54}$$

Taking into account for the residual dry friction torque in the bearings expression (8.2.35) and formula (9.3.13), we have

$$T\ddot{\beta} + \dot{\beta} = k\omega_\zeta - \frac{K_{set}}{b}\operatorname{sign}\dot{\beta}. \tag{9.3.55}$$

According to (2.3.75), we rewrite (9.3.55) in form

$$T\ddot{\beta} + \dot{\beta} = k\omega_\zeta \mp \frac{K_{set}}{b}. \tag{9.3.56}$$

If we disregard time constant T, for $\omega_\zeta > \omega_{\zeta_{min}}$ in the first approximation the angle β of rotation of the float will be determined by the relationship* [for $\beta(0) = 0$]

$$\beta = k\int_0^t \omega_\zeta(\tau)\,d\tau \mp \frac{1}{b}K_{set}t. \tag{9.3.57}$$

The first component of this formula is the useful output [see (9.2.2)]; the second component is error ε^{fr} of the HIG due to friction

$$\varepsilon^{r} = \frac{1}{b}K_{set}t. \tag{9.3.58}$$

To decrease ε^{fr} it is desirable to have a magnitude of K_{set} of friction torque on the axis of rotation of the float as small as possible and as large as possible a coefficient of damping b.

Example 9.6: Determine error ε^{fr} of an HIG due to friction in the axis of rotation of the float for an instrument with the parameters given in example 9.1; the magnitude of the residual friction torque in the bearings K_{set} = 0.1 g-cm; the

*In formula (9.3.57) t is the interval of time during which ω_ζ does not change sign.

time, during which angular velocity ω_ζ of rotation of the vehicle does not change sign, comprises t = 5 sec.

Solution: By formula (9.3.58) find error ε^{fr}:

$$\varepsilon^{fr} = \frac{1}{b} K_\gamma t = \frac{1}{20{,}39} 0{,}1 \cdot 5 = 0{,}02452 \text{ rad} = 1.405^\circ$$

From example 9.6 it follows that error ε^{fr} is substantial; therefore, quantity K_{set} should be smaller than assumed in the given example.

It is necessary to note that in distinction from an HIG, for which error ε^{fr} grows in time, for a GT analogous error when K_{set} = const remains constant [see (8.2.106)].

Friction torques during oscillations of the vehicle also cause forced oscillations of the float, which usually are very insignificant.

Let us consider instrument errors of an HIG, caused by changes of its basic parameters; among these parameters are: angular momentum of the gyroscope; damping factor; temperature of the liquid, with change of which in turn the specific gravity of the liquid, its viscosity, the location of the c.g. of the gyrofloat and the center of pressure with respect to the axis of rotation of the float change; electrical parameters of the signal pickoff and the torquer, etc.

As an example consider instrument error of the HIG due to change of angular momentum H of the gyroscope and of damping factor b of the liquid with respect to their design values. Let us assume that angular velocity of the vehicle ω_ζ = = const and in steady-state conditions upon completion of the transient response angle β of rotation of the float is determined by relationship (9.3.10)

$$\beta = \frac{H}{b} \omega_\zeta t.$$

Change of quantities H and b with respect to their design values will cause change of angle β, i.e., error of the instrument. Let us designate true (design) values of H and b by H_0 and b_0; to them there corresponds angle β_0 of rotation of the float equal to

$$\beta_0 = \frac{H_0}{b_0} \omega_\zeta t. \qquad (9.3.59)$$

Assume that in the prepared instrument these parameters differ from their design values by δH and δb, and comprise, respectively,

$$H = H_0 + \delta H; \quad b = b_0 + \delta b, \qquad (9.3.60)$$

where angle β differs from the design value β_0 by

$$\varepsilon^{\beta} = \delta\beta = \beta - \beta_0. \tag{9.3.61}$$

which represents error of the HIG due to changes of the shown parameters of the instrument.

By the same method we used during determination of analogous instrument error of a GT (§ 8.2, Par. 5) for ε^{par} in the given case, by analogy with (8.2.112), we find relationship

$$\varepsilon^{\beta} = \delta\beta = \frac{\partial\beta_0}{\partial H_0}\delta H + \frac{\partial\beta_0}{\partial b_0}\delta b. \tag{9.3.62}$$

For partial derivatives, according to (9.3.59), we have

$$\frac{\partial\beta_0}{\partial H_0} = \frac{1}{b_0}\omega_\zeta l; \quad \frac{\partial\beta_0}{\partial b_0} = -\frac{H_0}{b_0^2}\omega_\zeta l. \tag{9.3.63}$$

Substituting (9.3.63) in (9.3.62), we find

$$\varepsilon^{\beta} = \beta_0\left(\frac{\delta H}{H_0} - \frac{\delta b}{b_0}\right). \tag{9.4.64}$$

Thus, for determination of ε^{par} it is necessary to know the change δH, δb of parameters of the HIG, which should be established experimentally.

Other forms of instrument error of an HIG can be determined by the same methods as were expounded in § 5.5 in reference to calculation of instrument errors of an astatic gyroscope with three degrees of freedom.

5. Dynamic Errors of an HIG

Among dynamic errors of an HIG, characterizing its accuracy in real operating conditions during continuous change of the measured parameter and varying external disturbances, it is possible to list the same errors as belong to GT's and were considered in § 8.2, Par. 6a. We turn to a brief analysis of these errors.

a) Dynamics of an HIG During Harmonic Disturbances

Here we consider the case when the input signal changes by harmonic law. Here, disturbances are not considered, since they are considered subsequently as random functions of time.

We use equation (9.3.15) of motion of the gyrofloat, in which the input variable is the angular velocity $\omega_\zeta(t)$ of rotation of the vehicle about the measuring

axis 0ζ (Fig. 9.2), or equation (9.3.47), in which the input variable represents the angle α(t) of rotation of the vehicle about the same axis 0ζ. Solution of the latter equation is determined by expression (9.3.49), characterizing forced motion of the gyrofloat. This expression, replacing $\dot{\alpha}(t)$ by $\omega_\zeta(t)$, we can write in form

$$\beta_0(t) = kz - k\int_0^t \omega_\zeta(\tau) e^{-\frac{t-\tau}{T}} d\tau. \qquad (9.3.65)$$

The first composed of this formula characterizes the angle $\beta_1 = k\alpha$ of rotation of the float, proportional to angle α of vehicle rotation, i.e., β_1 is the useful output variable. The second component is the methodical error ε_{dyn} of the instrument in dynamics:

$$\varepsilon_д = -k\int_0^t \omega_\zeta(\tau) e^{-\frac{t-\tau}{T}} d\tau. \qquad (9.3.66)$$

This error depends on the character of angular velocity $\omega_\zeta(t)$ of the vehicle and on time constant T of the HIG. In the considered case ε_{dyn} is the dynamic error of the HIG during measurement by it of the angle of rotation of the vehicle α(t). In the terminology of A. N. Krylov [67] ε_{dyn} is the dynamic correction (§ 8.2, Par. 6b). To decrease dynamic error it is necessary that time constant T be small and that angular velocity $\omega_\zeta(t)$ of the vehicle not be too great.

It is most simple to investigate the dynamics of an HIG in that particular, but frequently encountered case, when α(t) varies by harmonic law

$$\alpha(t) = a_\zeta \sin \omega t, \qquad (9.3.67)$$

where a_ζ – peak value of the angle of rotation of the vehicle;

ω – frequency of its change.

Substituting (9.3.67) in (9.3.47), we obtain

$$T\dot{\beta} + \beta = ka_\zeta \sin \omega t. \qquad (9.3.68)$$

Considering the initial conditions null, we limit our consideration to forced oscillations $\beta_{for}(t)$ of the gyrofloat.

By analogy with (8.2.126) $\beta_{for}(t)$ will be determined by relationship

$$\beta_0(t) = a_\zeta A(\omega) \sin[\omega t + \varphi(\omega)], \qquad (9.3.69)$$

where A(ω) and φ(ω) are amplitude and phase frequency characteristics of the HIG.

From (9.3.69) it follows that forced oscillations of the HIG, caused by a harmonic disturbance $\alpha(t)$, are also a harmonic function of time, differing from the disturbance in amplitude and phase, but having the same frequency ω, as the same disturbance.

Since, according to (9.3.47), the HIG with respect to input variable $\alpha(t)$ can be considered an aperiodic network, for it the frequency characteristics $A(\omega)$ and $\varphi(\omega)$, by analogy with (5.6.71), will be determined by relationships

$$A(\omega)=\frac{k}{\sqrt{1+T^2\omega^2}};\quad \varphi(\omega)=-\operatorname{arc\,tg} T\omega. \tag{9.3.70}$$

According to (9.3.69) the amplitude of forced oscillations of the HIG will be

$$\beta_0^0=a_\zeta A(\omega)=\frac{a_\zeta k}{\sqrt{1+T^2\omega^2}}. \tag{9.3.71}$$

If the input variable $\alpha(t)$ acts on the HIG statically, i.e., has constant value a_ζ, the corresponding angle of float rotation, according to (9.2.5) and (9.3.13), is

$$\beta=a_\zeta k. \tag{9.3.72}$$

Ratio $\frac{\beta_{for}^0}{\beta}$, designated λ, is

$$\lambda=\frac{\beta_0^0}{\beta}=\frac{1}{\sqrt{1+T^2\omega^2}}. \tag{9.3.73}$$

Variable λ is in the considered case the coefficient of dynamicity (§ 8.2, Par. 6b). From (9.3.73) it follows that to decrease dynamic distortions of HIG readings in amplitude the coefficient of dynamicity λ in the range of frequencies in which the frequency ω of oscillations of the vehicle might be, should be sufficiently close to unity. Therefore, time constant T of the HIG should be as small as possible.

Example 9.7: Calculate and construct the graph of the coefficient of dyanmicity λ, for the HIG whose parameters are given in example 9.1.

Solution: By formula (9.3.73) for different ω we determine the value of coefficient λ, and then construct the graph of $\lambda(\omega)$ (Fig. 9.10).

From Fig. 9.10 it follows that for a substantial region of frequencies ω coefficient λ practically does not differ from unity.

For dynamic error of the HIG during change of $\alpha(t)$ by harmonic law we have

$$\varepsilon_\lambda=\alpha(t)-\frac{1}{k}\beta_0(t) \tag{9.3.74}$$

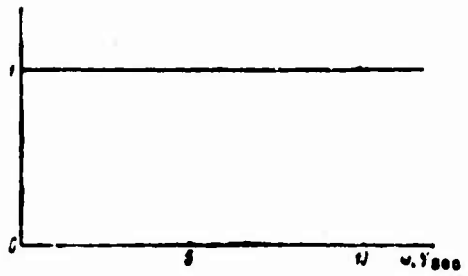

Fig. 9.10. Graph of $\lambda(\omega)$.

or, considering (9.3.67) and (9.3.69),

$$\varepsilon_{\text{д}} = a_{\zeta} \sin \omega t - \frac{a_{\zeta}}{k} A(\omega) \sin [\omega t + \varphi(\omega)]. \qquad (9.3.75)$$

By analogy with (8.2.138) and (8.2.139), we have the following expressions for errors in amplitude and phase:

$$\varepsilon_{\text{д}}^{1} = \frac{a_{\zeta} - \frac{a_{\zeta}}{k} A(\omega)}{a_{\zeta}} = \left[1 - \frac{A(\omega)}{k}\right] 100\% = \left[1 - \frac{1}{\sqrt{1 + T^2\omega^2}}\right] 100\%; \qquad (9.3.76)$$

$$\Delta\varphi = \varphi(\omega). \qquad (9.3.77)$$

Example 9.8: For the HIG with the parameters given in example 9.1, determine the characteristics of forced oscillations of the gyrofloat under the action on the instrument of oscillations of the vehicle, varying by harmonic law

$$\varepsilon(t) = a_{\zeta} \sin \omega t,$$

where $a_{\zeta} = 3^{\circ}$, $\omega = 2 \cdot 1/\text{sec}$.

Solution 1: By formula (9.3.73) we determine the coefficient of dynamicity λ:

$$\lambda = \frac{1}{\sqrt{1 + T^2\omega^2}} = \frac{1}{\sqrt{1 + 0{,}001766^2 \cdot 2^2}} = 0.999937$$

2. According to (9.3.71), considering (9.3.73), we calculate the value of amplitude β_{for}^{0} of forced oscillations of the gyrofloat:

$$\beta_{\text{в}}^{0} = \frac{a_{\zeta} k}{\sqrt{1 + T^2\omega^2}} = a_{\zeta} k \lambda = 3 \cdot 0{,}5 \cdot 0{,}9999937 = 1.499991^{\circ}.$$

3. Using (9.3.70), we find the angle of phase shift $\varphi(\omega)$

$$\operatorname{tg} \varphi(\omega) = -T\omega = -0{,}001766 \cdot 2 = -0{,}003532; \quad \varphi(\omega) = -0^{\circ}12',$$

4. According to (9.3.72) we determined angle β of rotation of the float without dynamic distortion:

$$\beta = a_{\zeta} k = 3 \cdot 0{,}5 = 1.5^{\circ}.$$

5. By formula (9.3.76) we find dynamic error of the HIG in amplitude:

$$\varepsilon_{\text{д}}^{a} = \left[1 - \frac{1}{\sqrt{1 + T^2\omega^2}}\right] 100\% = \left[1 - \frac{1}{\sqrt{1 + 0{,}001766^2 \cdot 2^2}}\right] 100\% = 0.00063\%.$$

From example 9.8 it follows that due to the smallness of time constant T

dynamic distortions of HIG readings during change of the input variable by harmonic law are minute.

b) Dynamics of an HIG During Random Actions

Characteristics of the input signal. As we shown [see (9.3.15)], the input variable for an HIG is angular velocity $\omega_\zeta(t)$ of the vehicle. If we use the modified equation (9.3.47) of the HIG, as the input variable it is possible to consider angle $\alpha(t)$ of rotation of the vehicle. Most frequently $\omega_\zeta(t)$, $\alpha(t)$, and also disturbances caused by residual friction in the axis of rotation of the gyrofloat and components of angular velocity of the vehicle along axes, not coinciding with the measuring axis of instrument, are random functions of time. Here, both HIG readings $\beta(t)$ and also its error $\varepsilon(t)$ in dynamics are also random functions of time, determined by the corresponding probability characteristics.

First, we shall consider reaction of the instrument on to random input $\omega_\zeta(t)$ or $\alpha(t)$; disturbances we consider absent. The Hig with the location of axes shown in Fig. 9.1, and installed on a ship will measure the angle of yawing of the latter (§ 9.3, Par. 2a). Considering (8.2.140), we have

$$\omega_\zeta(t) = -\dot{\varphi}(t), \quad \alpha(t) = -\varphi(t), \tag{9.3.78}$$

where $\varphi(t)$ – the angle of yawing of the ship.

Substituting (9.3.78) in equation (9.3.47), we obtain

$$T\dot{\beta} + \beta = -k\varphi(t). \tag{9.3.79}$$

As it is known, $\varphi(t)$ is a normal stationary random function of time with a mathematical expectation equal to zero (§ 2.1, Par. 3a). Its correlation function $K_\varphi(\tau)$ is determined by formula (8.2.142), and spectral density $S_\varphi(\omega)$ is found by relationship (8.2.144).

Example 9.9: Construct the curve of the spectral density $S_\varphi(\omega)$ of the angle of yawing of a ship for the initial data (given in example 4.8): $A = D[\varphi] = 0.6695 \cdot 10^{-3}$; $\mu = 0.03$ 1/sec; $\lambda = 0.21$ 1/sec.

Solution: By formula (8.2.144), considering (8.2.145), we calculate ordinates of $S_\varphi(\omega)$ for different values of ω. The curve $S_\varphi(\omega)$ is shown in Fig. 9.11 (solid line).

Characteristics of the output signal. Let us turn to determination of probability characteristics of random function $\beta(t)$ at the HIG output. In

accordance with (9.3.47), $\beta(t)$ is also a normal stationary random function of time The mathematical expectation $\overline{\beta}$ upon completion of the transient response, considering that $\overline{\varphi(t)} = 0$ [see (2.1.34)], will also be equal to zero:

$$\overline{\beta} = 0. \tag{9.3.80}$$

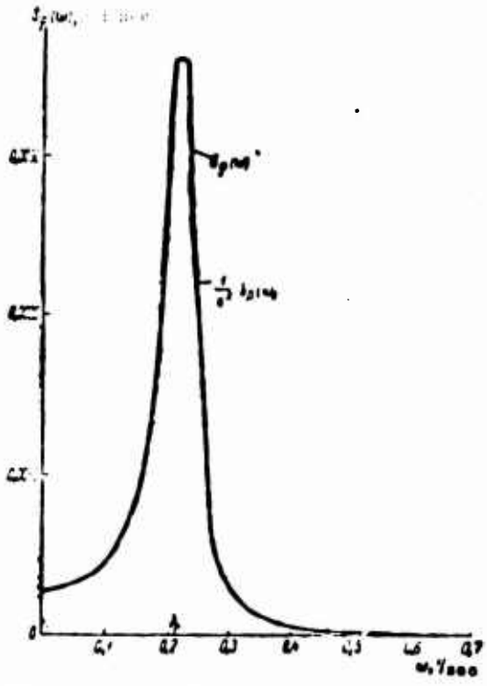

Fig. 9.11. Curves of spectral densities $S_\varphi(\omega)$ and $S_\beta(\omega)$.

We determine the spectral density $S_\beta(\omega)$ of random function $\beta(t)$. By analogy with (4.6.47), we have

$$S_\beta(\omega) = |W(j\omega)|^2 S_\varphi(\omega), \tag{9.3.81}$$

where $W(j\omega)$ – gain-phase response of the HIG determined, according to equation (9.3.47), by relationship

$$W(j\omega) = \frac{k}{1 + Tj\omega}. \tag{9.3.82}$$

Substituting (9.3.82) and (8.2.144) in (9.3.81), we obtain

$$S_\beta(\omega) = \frac{k^2}{1 + T^2\omega^2} \cdot \frac{2A\mu}{\pi} \cdot \frac{b^2}{\omega^4 + 2a\omega^2 + b^4}. \tag{9.3.83}$$

Example 9.10: Construct the graph of spectral density $S_\beta(\omega)$ of random function $\beta(t)$ of the angle of rotation of the gyrofloat in conditions of irregular yawing of a ship for the initial data of examples 9.1 and 9.9.

Solution: By formula (9.3.83), considering (8.2.145), we calculate ordinates of $S_\beta(\omega)$ for different values of ω. The graph of curve $S_\beta(\omega)$ is shown in Fig. 9.12.

From Fig. 9.12 it is clear that the spectrum $S_\beta(\omega)$ of angles of turns of the gyrofloat is similar to spectrum $S_\varphi(\omega)$ of angles of yawing of the ship (Fig. 9.11). In order to find how closely these curves coincide, we find the output spectrum of the HIG in the scale of the angle of rotation of the ship. For this using (9.2.5) (for $\alpha = \varphi$) and (9.3.13), we construct in Fig. 9.11 curve $\frac{1}{k^2} S_\beta(\omega)$ (shown in the figure in the form of a dotted line). It is clear that curves $S_\varphi(\omega)$ and $\frac{1}{k^2} S_\beta(\omega)$ practically coincide. This is explained by the fact that [see (8.2.144) and (9.3.83)] the square of the modulus of the transfer function of the

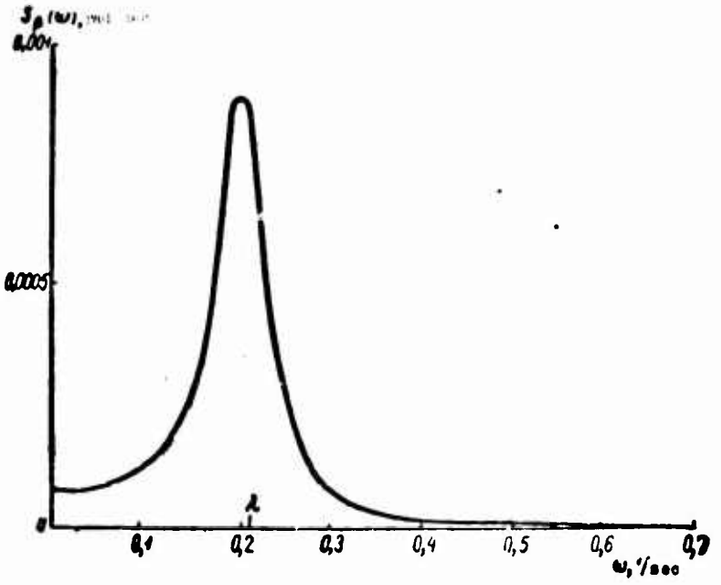

Fig. 9.12. Curve of spectral density $S_\beta(\omega)$.

HIG $|W(j\omega)|^2 = \dfrac{k^2}{1 + T^2\omega^2}$ in the region of essential values of frequencies of spectrum $S_\varphi(\omega)$ of the input signal is practically constant, and the coefficient of dynamicity $\lambda = \dfrac{1}{\sqrt{1 + T^2\omega^2}}$ differs little from unity, and consequently, the HIG reproduces a random input signal well.

We determine dispersion [3] of the angle of rotation of the gyrofloat. We use a formula of type (2.1.41); considering (9.3.83), we obtain

$$D[\beta] = \int_{-\infty}^{\infty} \frac{k^2}{1 + T^2\omega^2} \cdot \frac{2A_1}{\pi} \cdot \frac{b^2}{\omega^4 + 2a\omega^2 + b^4}\, d\omega. \qquad (9.3.84)$$

We designate

$$B = 4k^2 A_1 b^2 \qquad (9.3.85)$$

and rewrite (9.3.84) in the form of integral

$$D[\beta] = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{1 + T^2\omega^2} \cdot \frac{B}{\omega^4 + 2a\omega^2 + b^4}\, d\omega, \qquad (9.3.86)$$

analogous (6.5.21). Then for D[β], considering (6.5.27) and comparing (9.3.85) with (6.5.22), we obtain finally (A = D [φ]):

$$D[\beta] = k^2 \frac{1 + 2\mu T}{\lambda^2 T^2 + (1 + \mu T)^2} D[\varphi]. \qquad (9.3.87)$$

For the mean quadratic value σ_β of the angle of rotation of the gyrofloat we have

$$\sigma_\beta = \sqrt{D[\beta]}. \qquad (9.3.88)$$

Example 9.11: Calculate the mean quadratic value σ_β of the angle of rotation of the gyrofloat during measurement by it of the angle of yawing of a ship for initial data of examples 9.1 and 9.9; determine the limiting value β_m of angle β, assuming that the probability Q_z that the current angle β will exceed limit β_m,

should not be more than 0.1%.

Solution 1: By formula (9.3.87) we find $D[\beta]$;

$$D[\beta] = 0{,}0001672.$$

2. According to (9.3.88) we calculate σ_β

$$\sigma_\beta = \sqrt{D[\beta]} = \sqrt{0{,}0001672} = 0{,}01293 \text{ radians} = 0^\circ 44'.$$

3. For $Q_z = 0.1\%$ by Table 1 (§ 4.6, Par. 10) we determine $z = 3.3$; by analogy with (4.6.87), we calculate β_m (for $\overline{\beta} = 0$):

$$\beta_m = z\sigma_\beta = 3{,}3 \cdot 0{,}01293 = 0{,}04267 \text{ radians} = 2^\circ 27'.$$

Dynamic error of reproduction of the input signal. Let us find the dynamic error of an HIG during measurement by it of the angle of rotation of the vehicle representing a random function of time. Dynamic error $\varepsilon(t)$ of the HIG, by analogy with the corresponding error of a GT (§ 8.2, Par. 6b), is determined by the difference between useful input signal $\alpha(t)$ and quantity $\frac{1}{k}\beta(t)$ at the output [see (8.2.161)]:

$$\varepsilon(t) = \alpha(t) - \frac{1}{k}\beta(t). \tag{9.3.89}$$

where for production to find the complete formula the angle $\varphi(t)$ of yawing of the ship is designated $\alpha(t)$.

By analogy with (8.2.162), for the mathematical expectation of error $\varepsilon(t)$ we have

$$\overline{\varepsilon(t)} = 0. \tag{9.3.90}$$

Let us find dispersion $D[\varepsilon]$. We first find spectral density $S_\varepsilon(\omega)$ of error $\varepsilon(t)$. By the same method by which we obtained the corresponding expression $S_\varepsilon(\omega)$ for a gyrotachometer [see (8.2.168)], we find

$$S_\varepsilon(\omega) = |W_\varepsilon(j\omega)|^2 S_\alpha(\omega). \tag{9.3.91}$$

where the transfer function of error $W_\varepsilon(s)$ is determined by general expression (8.2.169).

Substituting formula (9.3.82) in (8.2.169), we have

$$W_\varepsilon(j\omega) = 1 - \frac{1}{k}W(j\omega) = 1 - \frac{1}{k}\cdot\frac{k}{1+Tj\omega} = \frac{Tj\omega}{1+Tj\omega}. \tag{9.3.92}$$

Introducing relationship (9.3.92) in (9.3.91), and formula (8.2.144) for $S_\alpha(\omega) = S_\varphi(\omega)$, we obtain

$$S_\varepsilon(\omega) = \frac{T^2\omega^2}{1+T^2\omega^2} \cdot \frac{2A\mu}{\pi} \cdot \frac{b^2}{\omega^4 + 2a\omega^2 + b^4}. \tag{9.3.93}$$

For dispersion $D[\varepsilon]$ we have

$$D[\varepsilon] = \int_{-\infty}^{\infty} S_\varepsilon(\omega)\, d\omega \tag{9.3.94}$$

or, considering (9.3.93),

$$D[\varepsilon] = \int_{-\infty}^{\infty} \frac{T^2\omega^2}{1+T^2\omega^2} \cdot \frac{2A\mu}{\pi} \cdot \frac{b^2}{\omega^4 + 2a\omega^2 + b^4}\, d\omega. \tag{9.3.95}$$

We designate

$$B_1 = 4T^2 A\mu b^2 \tag{9.3.96}$$

and rewrite (9.3.95) in the form of integral

$$D[\varepsilon] = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{1+T^2\omega^2} \cdot \frac{B_1\omega^2}{\omega^4 + 2a\omega^2 + b^4}\, d\omega, \tag{9.3.97}$$

analogous to expression (6.5.36). Then for $D[\varepsilon]$, considering (6.5.40) and comparing (9.3.96) with (6.5.37), we finally obtain ($A = D[\varphi]$)

$$D[\varepsilon] = \frac{T^2(\mu^2+\lambda^2)}{\lambda^2T^2 + (1+\mu T)^2} D[\varphi]. \tag{9.3.98}$$

For the mean quadratic value σ_ε of dynamic error of the HIG we have

$$\sigma_\varepsilon = \sqrt{D[\varepsilon]}. \tag{9.3.99}$$

Example 9.12: Calculate the mean quadratic value σ_ε of dynamic error of an HIG during measurement by it of the angle of yawing of a ship, which is a random function of time, for the initial data of examples 9.1 and 9.9. Determine also the limiting value ε_m of HIG error, considering that the probability Q_z that the current value ε of error will exceed limit ε_m, should not be more than 0.1%.

Solution 1: By formula (9.3.98) we find $D[\varepsilon]$:

$$D[\varepsilon] = 0{,}09395 \cdot 10^{-9}.$$

2. According to (9.3.99) we calculate σ_ε

$$\sigma_\varepsilon = \sqrt{D[\varepsilon]} = \sqrt{0{,}09395 \cdot 10^{-9}} = 0{,}9693 \cdot 10^{-5} \text{ radians} = 1.9''.$$

3. By the same method as in example 9.11, we determine ε_m (for $\overline{\varepsilon} = 0$):

$$\varepsilon_m = r\sigma_\varepsilon = 3{,}3 \cdot 0{,}9693 \cdot 10^{-5} = 3{,}199 \cdot 10^{-5} \text{ radians} = 6.5''.$$

From example 9.12 it follows that dynamic error of the HIG during measurement by it of the angle of rotation of a vehicle, which is a function of time, is minute.

During solution of certain problems in estimating accuracy of an HIG we may need to find for the same conditions the mean value of phase shifts of the output variable $\beta(t)$ with respect to input variable $\alpha(t)$. The probability characteristic of these phase shifts, or, more exactly, the lag time of the HIG is cross-correlation function $K_{\alpha,\ \beta}(\tau)$ of the input and output signals.

This characteristic is determined by the same method as the corresponding cross-correlation function $K_{\omega_\zeta,\beta}(\tau)$ [see (8.2.182)] during investigation of the same question in reference to a GT (§ 8.2, Par. 6b).

Dynamic error due to disturbances. Let us find a method of determining dynamic errors of an HIG, caused by residual friction torques M_{fr} in bearings of the float* and by the influence of component $\dot{\omega}_\eta$ of angular acceleration of the vehicle about axis $O\eta$ of rotation of the gyrofloat (Fig. 9.2). In the case of installation of the HIG on a ship for determination of the angle of yawing the differential equation of the gyrofloat, taking into account the shown disturbances, appearing during rolling, is determined by relationship (9.3.32). We also obtained equation (9.3.33) of HIG error caused by these disturbances. The transfer function of the HIG Y(s) with respect to disturbances of rolling is determined by relationship (9.3.35). Consequently, the spectral density $S_{\varepsilon_1}(\omega)$ of dynamic error $\varepsilon_1(t)$ from disturbance of rolling will be

$$S_{\varepsilon_1}(\omega) = |Y(j\omega)|^2 S_\vartheta(\omega), \tag{9.3.100}$$

where the gain-phase response $Y(j\omega)$ is found by replacement in (9.3.35) of s by $j\omega$:

$$Y(j\omega) = \frac{T_1 j\omega}{T_2 j\omega + 1}. \tag{9.3.101}$$

*See footnote on p. 161.

Substituting in (9.3.100) formula (9.3.101) and relationship (9.2.192) for the the spectral density $S_\theta(\omega)$ of the angle of rolling of the ship, we have

$$S_{\varepsilon_1}(\omega) = \frac{T_1^2\omega^2}{1+T_1^2\omega^2}\cdot\frac{2A_2\mu_2}{\pi}\cdot\frac{b_2^2}{\omega^4+2a_2\omega^2+b_2^4}. \qquad (9.3.102)$$

For dispersion $D[\varepsilon_1]$ of error $\varepsilon_1(t)$ we have

$$D[\varepsilon_1] = \int_{-\infty}^{\infty} S_{\varepsilon_1}(\omega)\,d\omega \qquad (9.3.103)$$

or, considering (9.3.102), we obtain integral

$$D[\varepsilon_1] = \int_{-\infty}^{\infty}\frac{T_1^2\omega^2}{1+T_1^2\omega^2}\cdot\frac{2A_2\mu_2}{\pi}\cdot\frac{b_2^2}{\omega^4+2a_2\omega^2+b_2^4}\,d\omega, \qquad (9.3.104)$$

which is calculated by the method, repeatedly applied in the preceding chapters. As a result we obtain the following final formula ($A_2 = D[\theta]$):

$$D[\varepsilon_1] = \frac{T_1^2b_2^2}{\lambda_2^2T_1^2+(1+\mu_2T_1)^2}D[\theta]. \qquad (9.3.105)$$

For the mean quadratic value σ_{ε_1} of dynamic error of the HIG we have

$$\sigma_{\varepsilon_1} = \sqrt{D[\varepsilon_1]}. \qquad (9.3.106)$$

Example 9.13: Calculate the mean quadratic value σ_{ε_1} of dynamic error $\varepsilon_1(t)$ of an HIG, caused by residual friction in the axis of rotation of the gyrofloat and angular accelerated of induced motion of rolling of the ship about this axis, for the initial data of examples 9.1 and 7.10. Determine also the limiting value ε_{1m}, of the HIG error assuming that the probability Q_z that the current value ε_1 of error will exceed limit ε_{1m}, should not be more than 0.1%; the coefficient of liquid friction in the axis of rotation of the float $n_1 = 0.3$ g-cm-sec.

Solution 1: By formula (9.3.13) we find coefficient ρ:

$$\rho = \frac{1}{b} = \frac{1}{20,39} = 0,04904\ \frac{1}{\text{g-cm-sec}}.$$

2. According to (9.3.31) we determine T_1:

$$T_1 = \frac{T}{1+\rho n_1} = \frac{0,001766}{1+0,04904\cdot0,3} = 0,001740\ \text{sec}.$$

3. Using relationship (9.3.105), we calculate $D[\varepsilon_1]$:

$$D[\varepsilon_1] = 2,0516\cdot10^{-6}.$$

4. By formula (9.3.106) we determine σ_{ε_1}:

$$\sigma_{\varepsilon_1} = \sqrt{D[\varepsilon_1]} = \sqrt{2{,}0516 \cdot 10^{-8}} = 1{,}4323 \cdot 10^{-4} \text{ radians} = 0.5'.$$

5. By the same method as in example 9.11, we calculate the limiting value ε_{1m} of dynamic error of the HIG (for $\bar{\varepsilon}_1 = 0$):

$$\varepsilon_{1m} = 3\sigma_{\varepsilon_1} = 3{,}3 \cdot 0{,}5 \approx 1.65'.$$

From example 9.13 it follows that dynamic error $\varepsilon_1(t)$ of the HIG is greater than error $\varepsilon(t)$ (see, e.g., 9.12).

When estimating dynamic errors of the HIG we started from a differential equation (9.3.9) with constant coefficients. It was obtained from the more general equation (9.3.8), which contains variable component ω_ξ of angular velocity of the vehicle along axis $O\xi$ (Fig. 9.1). Considering disturbances $\dot{\omega}_\eta = 0$, $M_{fr} = f(t) = 0$, since the influence of them was considered earlier, we rewrite equation (9.3.8) with the torquer turned off ($M_{tq} = 0$) in form

$$J_{a.r}\ddot{\beta} + b\dot{\beta} = H[\omega_\zeta(t) - \omega_\xi(t)\beta], \tag{9.3.107}$$

where the input angular velocity $\omega_\zeta(t)$ and the disturbing angular velocity $\omega_\xi(t)$ are normally random functions of time. We divide both parts of equation (9.3.107) by b and consider designations (9.3.12) and (9.2.13); then, instead of (9.2.107), we obtain

$$T\ddot{\beta} + \dot{\beta} = k[\omega_\zeta(t) - \omega_\xi(t)\beta]. \tag{9.3.108}$$

We have differential equation with variable coefficients, close in form to the equation (4.6.100) of oscillations of a physical pendulum during irregular rolling of a ship. Since, usually, angles β of turn of the gyrofloat are very small, dispersion $D[\omega_\xi(t)\beta] \ll D[\omega_\zeta(t)]$ and for integration of (9.3.108) we can apply the same method of successive approximations as was used during investigation of equation (4.6.100). The solution of this problem for conditions when $\omega_\zeta(t)$ and $\omega_\xi(t)$ are periodic functions of time is found in [103.1]. In these works it is shown that with the accepted conditions the gyrofloat, besides oscillatory motions, has systematic drift, proportional to time. It follows from this that in an HIG it is very important to eliminate the disturbing influence of component $\omega_\xi(t)$ of angular velocity of the vehicle. For this, according to (9.3.108), angles of rotation of float should be close to zero, which is usually attained in practice by means of application of an HIG with servo drive.

If we use equation (9.3.107), in which we consider a component of the second order of smallness, then we should definitize it. For this, using Euler's equations (1.4.1), we obtain the exact differential equation of motion of the HIG:*

$$J_{\text{в.г}}\left(\ddot{\beta}-\dot{\omega}_{\eta}\right)+b\dot{\beta}=H\left(\omega_{\xi}\cos\beta-\omega_{\zeta}\sin\beta\right)-$$
$$-\left(J_{\text{пр}}-J_{\text{э}}\right)\left(\frac{\omega_{\zeta}^{2}-\omega_{\xi}^{2}}{2}\sin 2\beta+\omega_{\xi}\omega_{\zeta}\cos 2\beta\right)+M_{\text{д.в}}+M_{\text{т}}+f(t).$$

Considering $\dot{\omega}_{\eta}=0$, $M_{tq}=M_{ir}=f(t)=0$ and limiting ourselves to calculation of components of the second order of smallness, we obtain

$$J_{\text{в.г}}\ddot{\beta}+b\dot{\beta}=H\left[\omega_{\zeta}(t)-\omega_{\xi}(t)\beta\right]-\left(J_{\text{пр}}-J_{\text{э}}\right)\omega_{\xi}(t)\,\omega_{\zeta}(t).$$

Above we considered certain basic questions of the theory of just HIG's. The theory of HIG's in case of application with a servo drive, for instance, for stabilization of a platform, is expounded in Chapter 11.

6. Application of HIG's

a) General Remarks

Two-degree-of-freedom floated gyros have essential advantages over astatic gyroscopes in a Cardan suspension, usually applied. Basic advantages of two-degree-of-freedom floated gyros are: 1) high accuracy; 2) small susceptibility to vibration, shock and other disturbances; 3) possibility of use for solving a broad class of problems, usually entrusted to GD's. In this connection two-degree-of-freedom floated gyroscopes have in recent years begun to be applied in the following types of precision GD's:

1) in gyrotachometers for determination of components of angular velocity of a vehicle;

2) in rate-integrating gyroscopes for determination of angles of turn of a vehicle about its axes.

3) in gyroscopic integrators of linear acceleration for determination of components of linear velocity of a vehicle;

4) as sensers systems of active gyroscopic stabilization for direct

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*See footnote on p. 207.

stabilization for direct stabilization of individual instruments and devices, and also in DG's, GV's, azimuth-horizon gyros of active type and inertial navigational systems.*

The gyrotachometer, in which they use a two-degree-of-freedom floated gyro, differs from the HIG by the fact that it has a special component, usually in the form of an elastic torsion rod, which creates a restoring torque. Another difference of this gyrotachometer, called by a floated differentiating gyroscope (FDG) is that it requires a smaller damping torque, and consequently, lower viscosity of the liquid than an HIG, since in an FDG this torque serves only to quench natural oscillations of the gyrofloat. Schemes of such FDG's and specifications are given, e.g., in the works of G.A. Slomyanskiy [149] and C. Draper [181]. There exist FDG's for which restoring torque is created by an electrical device with application of proportional electrical feedback. Such an FDG constitutes a GT with forced return to zero and it is usually called [149] a feedback FDG. The principle of action of such a GT was presented in § 8.2, Par. 7 (Fig. 8.14). Theory of FDG's, not differing in principle from theories of a usual gyrotachometer (§8.2), is presented in detail in the two works mentioned.

Application of a two-degree-of-freedom floated gyro as an IG and the theory of this device were considered earlier.

The floated gyro is also used in the gyroscopic integrator of linear accelerations of a vehicle; here the instrument reacts to the moment of forces of inertia during accelerations of a vehicle due to displacement of the c.g., of the gyrofloat with respect to its axis of rotation. The principle of action of such a gyroscopic integrator is considered in the next section.

Let us turn to analysis of the application of a two-degree-of-freedom floated gyro (HIG) as the sensor of a system of active gyroscopic stabilization.

b) Application of an HIG in a System of Active Gyroscopic Stabilization

Preliminary remark. The HIG is basically applied in systems of active gyroscopic stabilization. This instrument permits us here in principle to solve the same problems as during use of an active GS based on astatic gyroscopes with three

*Possibilities of using of two-degree-of-freedom floated gyros are not to those enumerated.

degrees of freedom (§ 7.6, Par. 1). Indeed, application of an HIG in systems of active gyroscopic stabilization allows us: 1) to realize devices, ensuring direct stabilization of individual components and instruments; 2) to create DG's, GV's and azimuth-horizon gyros of active type; 3) to realize stabilization with respect to three axes of platforms for use of them in inertial navigational systems, etc.

In all these cases the HIG is applied with a servo drive. Therefore, we shall consider the principle of construction of a scheme of joint work of an HIG and a servo drive; the theory of such systems is presented in Chapter 11. First, let us note that application of an HIG with a servo drive allows us to obtain a one-axis active GS, i.e., lets us stabilize a platform relative to one axis. To produce a two-axis GS, i.e., a platform, stabilized relative to two axes, it is necessary to have two HIG's with the corresponding servo drives. Finally, for creation of a triple-axis GS, providing a platform stabilized relative to three axes, there are applied three HIG's with servo drives.

It is necessary to note that a GS using floated IG's as compared to an active GS using a free astatic gyroscopes has one peculiarity. In the latter external disturbing torques are compensated not only by the torque of the stabilizing motor, but also gyroscopic torques, created by the free gyroscopes. In a GS using floated IG's stabilization is carried out only by the torque of the stabilizing motor. In this relation the considered type of GS, in general, more closely approximates a readout-type GS (see § 10.1). However, since in the GS using floated IG's the latter are connected with the stabilizer mechanically, and not through servo systems (as in readout-type GS's), we will arbitrarily list GS's using floated IG's with active GS's.

Stabilization of a single-axis platform relative to inertial space. Let us consider first, how an HIG is used in a one-axis active GS for stabilization of some platform relative to one axis. In order to relate this question to the earlier analysed case of application of an HIG for determination of the angle of yawing of a ship in heading, we shall consider stabilization of a one-axis platform in bearing. The axis of stabilization of such a platform is directed along the vertical, and we obtain a scheme of an active GS, analogous to the scheme of a DG of active type (§7.6, Par. 2a), but using an HIG.

In Fig. 9.13 we give the schematic of a one-axis active GS and HIG. On platform P, stabilized in azimuth, there is fixed an HIG, whose schematic was shown in

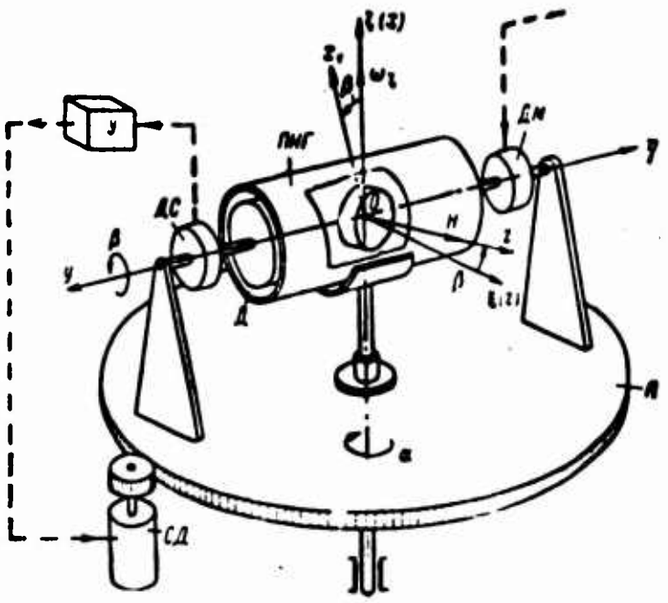

Fig. 9.13. Schematic of a one-axis GS with an HIG (axis of stabilization is vertical).

Fig. 9.2 (see also Fig. 9.3). The basic element of the system is a gyrofloat rotating about axis $O\eta$ (Oy) relative to the instrument housing, which is rigidly connected with platform P. Between the cylindrical surfaces of the gyrounit and the housing of the HIG is liquid. This system will form a float suspension, also fulfilling the role of integrating damper D. Angle β of rotation of the float is fixed by signal pickoff SP, which through amplifier Amp controls stabilizing motor SM, connected through gear transmission with platform P. Motor SM serves for direct stabilization of platform P in azimuth.

On the other end of the axis $O\eta$ (Oy) of rotation of the float is fixed torquer TQ, which applies to the HIG torques for initial drive of the gyroscope axis to zero position ($\beta = 0$), and also for realization (if necessary) of an assigned law of motion of the platform in azimuth.

Subsequently by azimuth stabilization of platform P we shall mean retention of it in a constant position with respect to inertial space or to another coordinate system, moving in an assigned manner relative to the latter.

Let us first consider azimuth stabilization of platform P relative to inertial space, upon realization of which the platform will not accomplish turns about axis $O\zeta$. Axis $O\zeta$ is the sensitive, or input axis (§ 9.2, Par. 4) of the floated IG, and axis $O\eta$ (Oy) is the output axis. The principle of work of the system consists in the given case of the following.

We assume that the base or object on which the GS is fixed turns about axis $O\zeta$ (Fig. 9.13) angle α (relative to a direction fixed in inertial space) with velocity $\omega_\zeta = \alpha$. Together with the base platform P will start to rotate. According to the known principle of action of the integrating gyroscope (§ 9.2, Par. 2) here there appears a gyroscopic torque, directed along axis $O\eta$ (Oy) and causing rotation of the gyrofloat about this axis angle β, which is proportional to the integral in time of ω_ζ [see (9.2.3)] or angle α [see (9.2.5)]. Voltage

proportional to angle β, is transmitted by signal pickoff SP through amplifier Amp to stabilizing motor SM. The latter starts to turn platform P about axis $O\zeta$ with angular velocity of adjustment $\omega_{adj} = -\omega_\zeta$ and turns the platform angle α in a direction opposite its initial rotation together with the base. Here the gyrofloat will return to the initial position, at which angle $\beta = 0$, and the servo drive will be turned off. Actually, the servo drive works continuously and thanks to this platform P preserves constant azimuth orientation relative to inertial space or, more exactly, will be deflected at small angle, determined by the threshold of sensitivity of the IG and servo drive. Here, angle β will remain sufficiently small. Such operating conditions of the considered system we call conditions of geometric stabilization.

A process analogous to the considered one will also occur in the case of a fixed base of the GS or object, but with action on the platform of a certain disturbing torque with respect to axis $O\zeta$. Under the influence of this torque the platform will start to rotate about axis $O\zeta$ with a certain angular velocity ω_ζ and the floated HIG, similarly to the preceding case, will turn on the servo drive. The latter will apply to axis $O\zeta$ a torque, equal in magnitude, but directed directed opposite the disturbing one, and will turn the platform with angular velocity $\omega_{adj} = -\omega_\zeta$ in a direction the opposite of the direction of its rotation under the influence of the disturbing torque.

Thus, application of an HIG together with a servo drive effects stabilization, i.e., constant orientation of the platform relative to inertial space. Here, the gyroscope axis will be near the zero position ($\beta = 0$), which ensures the most favorable conditions of work of the HIG. Let us note that in the considered scheme the floated gyro does not carry any load in stabilization of the platform, but is only the sensor of the active stabilization system.

Application of an HIG with servo drive permits ensuring of high accuracy of devices of gyroscopic stabilization. Thus, according to C. Draper [180], it is not of any special complexity to ensure accuracy of stabilization of 1 angular minute. One may assume that creation of devices of active gyroscopic stabilization with a two-degree-of-freedom integrating gyroscope with servo drive is more promising for guaranteeing high accuracy of stabilization than gyrostabilizers with a three-degree-of-freedom astatic gyroscope. However, very great technical difficulties, appearing during creation of hermetic integrating gyroscopes, still

limit their wide application.

Stabilization of a single-axis platform with respect to coordinates system, moving in an assigned manner with respect to inertial space. Let us consider how we realize another form of platform stabilization, i.e., stabilization in some coordinate system, accomplishing assigned motion relative to inertial space. We assume that the coordinate system rotates about the vertical $O\zeta$ (Fig. 9.13) with assigned angular velocity $\omega_{ass}(t)$ relative to inertial space. It is required to carry out constant azimuth orientation of platform P in the assigned coordinate system, i.e, the platform should rotate together with this coordinate system about axis $O\zeta$ with angular velocity $\omega_{ass}(t)$ relative to inertial space.

The operating principle of the system consists in this case of the following. For rotation of the platform with angular velocity $\omega_{ass}(t)$ to the torquer TQ there should be sent a control signal, proportional to $\omega_{ass}(t)$. The torquer will apply to spin axis $O\eta$ (Oy) of the gyrofloat torque M_{tq}. The magnitude of this torque, according to (9.3.2), is proportional to current intensity i of the control signal, i.e., in the considered case to angular velocity $\omega_{ass}(t)$. Under the action of torque M_{tq} the gyrofloat will turn about axis $O\eta$ (Oy) a certain angle β. Voltage, proportional to angle β, is transmitted by signal pickoff SP through amplifier Amp to stabilizing motor SM. The latter, trying to form the axis of the gyroscope to its initial position ($\beta = 0$), will start to turn the platform about axis $O\zeta$ with angular velocity ω_{adj} in a direction, coinciding with rotation of the considered coordinate system with angular velocity $\omega_{ass}(t)$. Rotation of platform P together with the HIG with angular velocity ω_{adj} will cause a gyroscopic torque M_g, directed along axis $O\eta(Oy)$ in a direction opposite torque M_{tq}. When ω_{adj} becomes equal in magnitude to ω_{ass} ($\omega_{adj} = \omega_{ass}$), the gyroscopic torque will balance torque M_{tq}, the float will come to a position, close to zero ($\beta \approx 0$), and stabilized platform P will turn continuously together with the HIG around axis $O\zeta$ with the assigned angular velocity $\omega_{ass}(t)$ relative to inertial space.

In the considered case we have the following relationships. By analogy with (8.2.3), for the gyroscopic torque M_g caused by rotation of the platform we have

$$M_r = H\omega_{отр} = H\omega_{\text{п}}. \tag{9.3.109}$$

Torque M_{tq}, applied by the torquer, is determined by formula (9.3.2). So that the gyrofloat does not spin about its axis, by analogy with (9.2.1), this

$$u_\zeta = U\sin\varphi + \frac{v_E}{R}\operatorname{tg}\varphi.$$

Then, according to (9.3.111), current i introduced to torquer TQ (Fig. 9.13) by the control signal, should be proportional to angular velocity u_ζ i.e.,

$$i = \frac{H}{k_1}u_\zeta = \frac{H}{k_1}\left(U\sin\varphi + \frac{v_E}{R}\operatorname{tg}\varphi\right). \qquad (9.3.113)$$

With realization of the shown correction platform P will preserve with respect to the meridional plane an assigned initial angle α_0, and relative to inertial space it will turn at angle α, determined, by virtue of (9.3.112) and (9.3.113), by following relationship:

$$\alpha = \alpha_0 + \int_0^t \left(U\sin\varphi + \frac{v_E}{R}\operatorname{tg}\varphi\right)d\tau. \qquad (9.3.114)$$

Examples of application of HIG's is devices of active gyroscopic stabilization. Let us consider the scheme of a single-axis GS with an HIG, in which the platform should be stabilized in horizontal position relative to one axis. Such a scheme is shown in Fig. 9.14. On stabilized platform P is fixed the HIG, whose schematic

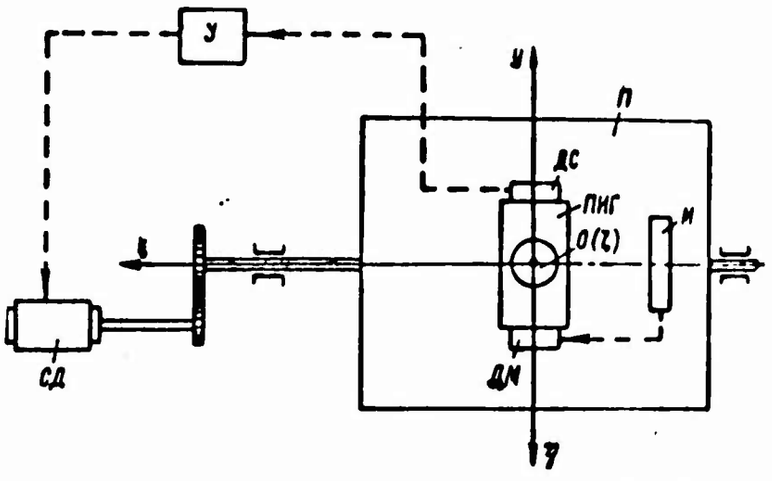

Fig. 9.14. Schematic of a single-axis GS with an HIG (axis of stabilization is horizontal).

was shown in Fig. 9.2. The HIG is located here so that the gyroscope axis in its initial position (when $\beta = 0$) is combined with vertical $O\zeta$, which is perpendicular to the plane of the drawing; input axis $O\xi$ is horizontal and is directed along the spin axis of platform P; output axis $O\eta$ (Oy) is also horizontal. By analogy with what we considered earlier, during rotation of platform P about axis $O\xi$ the HIG through signal pickoff SP and amplifier Amp will turn on stabilizing motor SM,

which will turn the platform in the opposite direction; as a result the platform will maintain constant position relative to inertial space. In this case the system works in conditions of geometric stabilization.

Let us consider how combination of platform P with the horizontal plane is ensured. For simplicity assume that axis $O\xi$ of the platform is directed along the meridional line, about which the plane of the horizon revolves with angular velocity U cos φ [see (1.2.21)]. For realization of turn of platform P about axis $O\xi$ with angular velocity U cos φ following the plane of the horizon it is necessary to introduce to the torquer TQ a control signal, current i of which should be proportional to angular velocity U cos φ. Then the scheme shown in Fig. 9.14 will operate in a regime of spatial integration and the platform will turn angle $\int U \cos \varphi \, dt$ relative to inertial space. However, with such a method of stabilization the platform nevertheless does not possess selectivity ("directional force") with respect to the plane of the horizon, i.e., does not itself end up in horizontal position, if any disturbing torques drive it from the horizontal plane. In order to slave the platform to the horizontal and create in it selectivity relative to this plane, it is necessary to use a system of pendulous correction, analogous to the correction system in a GV (Chapter 6) with a three-degree-of-freedom astatic gyro. For this on platform P there is installed a physical pendulum (accelerometer) Pend, whose plane of oscillation coincides with vertical plane $O\eta\zeta$; the pendulum is connected to torquer TQ of the gyrofloat. Upon deflection of the platform from the horizontal plane from the pendulum a signal proceeds to the torquer, which will start to turn the gyrofloat thanks to which the servo drive is turned on and the stabilizing motor SM will return the platform to horizontal position; then, the control signal from the pendulum will be turned off.

In the considered case the stabilization system also works in a regime of spatial integration, where it is completely self-contained, since it is not necessary to introduce to the torquer a control signal, proportional to angular velocity U cos φ, as this took place without the correcting pendulum.

Let us give as a second example the scheme of a double-axis GS (with an HIG) in which the platform should be stabilized in the horizontal plane relative to two axes. Such a schematic of a double-axis GS or GV of active type is shown in Fig. 9.15. Here, platform P is installed in the outer gimbal ring R and plays the role of inner ring. On the platform are two hermetic integrating gyros HIG_1 and HIG_2.

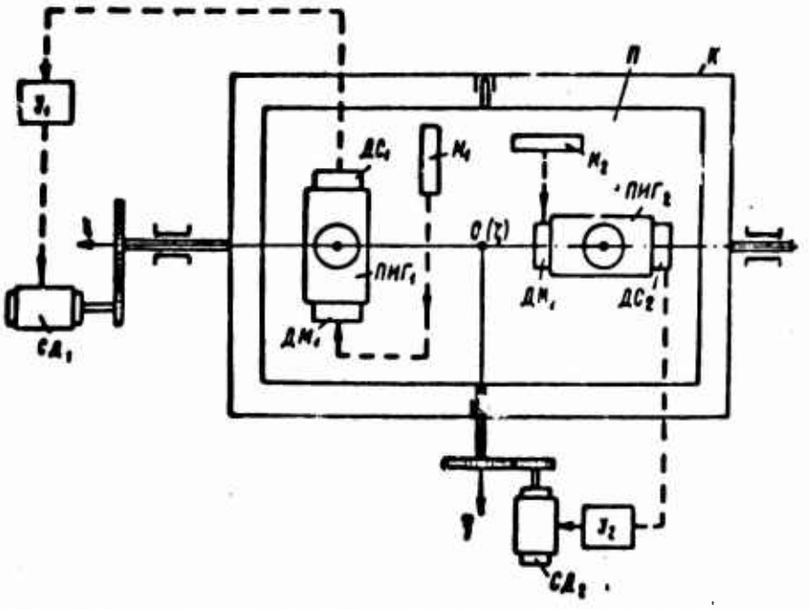

Fig. 9.15. Schematic of a double-axis GS and HIG.

The measuring axis of HIG_1 coincides with the spin axis $O\xi$ of the ring, and the measuring axis of HIG_2 coincides with spin axis $O\eta$ of the platform. Thus, the scheme in Fig. 9.15 is, as it were, a combination of two identical schemes, such as depicted in Fig. 9.14.

The principle of action of a double-axis GS is the same as for a single-axis GS; therefore, we will not stop to describe it. Let us note only that the presence in the system of pendulous correction for two axes ensures automatic work of the system (without external information) in a regime of spatial integration, and this permits holding the platform in a horizontal plane.

On an HIG base it is possible also to stabilize a platform relative to three mutually-perpendicular axes. Such triple-axis active GS's have found wide application in inertial navigational systems (Chapter 11); the schematic of a triple-axis GS with an HIG is given, e.g., in [149].

condition should be satisfied:

$$M_r + M_{a.n} = 0; \qquad (9.3.110)$$

substituting here (9.3.109) and (9.3.2) we obtain

$$H\omega_{ass} = k_1 i,$$

whence

$$i = \frac{H}{k_1}\omega_{ass}. \qquad (9.3.111)$$

Consequently, feeding torquer TQ of the floated IG current i, proportional to angular velocity ω_{ass}, the servo drive will turn platform P with the assigned angular velocity with respect to inertial space. Here, angle α of platform rotation, according to (9.3.111) and inasmuch as $\dot{\alpha} = \omega_{ass}$, will be determined by relationship

$$\alpha = \alpha_0 + \int_0^t \omega_{ass}(\tau)\,d\tau = \alpha_0 + \frac{k_1}{H}\int_0^t i\,d\tau, \qquad (9.3.112)$$

where α_0 – the value of angle α for t = 0.

From expression (9.3.112) it follows that angle α is proportional to the integral with respect to time of the input variable – current i, fed to the torquer by the control signal, which in turn is proportional to the assigned angular velocity $\omega_{ass}(t)$ of rotation of the platform relative to inertial space. In this connection the considered operating conditions of the system are called conditions of integration [149, 180].

Thus, application of an HIG with servo drive with introduction of the corresponding control signals permits stabilization, i.e., constant orientation of the platform in a system of coordinates, accomplishing any assigned motion with respect to inertial space.

Let us consider an example which will explain what has been said. Earlier we already reported that the scheme in Fig. 9.13 is in essence a DG of active type. If it is required that the axis of the gyroscope or platform keep, e.g., constant orientation in azimuth relative to the plane of the meridian for this it is necessary to carry out rotation of the axis of the gyroscope (platform) about the vertical $O\zeta$ with respect to inertial space with angular velocity u_ζ, determined by relationship (1.2.25)

§ 9.4. Gyroscopic Integrators of Linear Accelerations

1. Function and Types of Gyroscopic Integrators of Linear Accelerations

As was shown in § 9.1, one variety of IG is the gyroscopic integrator of linear accelerations of a vehicle. It is usually called [161] a gyroscopic integrator or gyrointegrator; the latter designation we shall use henceforth.

The gyrointegrator is a GD, intended for determination of the component of linear velocity of the c.g., of an object along an assigned direction. In it there is applied an unbalanced gyroscope, sensitive to accelerations of the object. The moment of corresponding forces of inertia of forward motion of the latter causes precession of the gyroscope, the angular velocity of which is proportional to the shown moment, i.e., to the magnitude of acceleration of translational motion of the object. Consequently, the magnitude of the actual angle of precession will be proportional to the corresponding composing of linear velocity of the object, which justifies the accepted designation of the instrument.

One feature of the gyrointegrator is that it measures, in general, the projection of absolute linear velocity of an object on the axis of sensitivity of the instrument. The gyrointegrator, as also the accelerometer, reacts to apparent acceleration of the object, i.e., to the difference between absolute linear acceleration of the object and gravitational acceleration, which leads to corresponding distortions of readings of the instrument.* In connection with what has been said A. Yu. Ishlinskiy [50]

*Accelerations measured by an accelerometer are discussed in detail in § 11.6, Par. 2b.

considers it useful to call such accelerometers newtonmeters.

There exist various types of gyrointegrators, in which there are applied three- and two-degree-of-freedom gyroscopes. As it is known [30, 180, 161], most widely used are the following types of instruments:

1) a gyrointegrator with a three-degree-of-freedom unbalanced gyroscope of gyropendulum type;

2) a gyrointegrator in which there is applied a three-degree-of-freedom gyroscope with additional mass and compensating loads;

3) a gyrointegrator in which there is used an unbalanced two-degree-of-freedom gyroscope with float suspension.

By analogy with gyrotachometers (§ 8.2, Par. 1) it is possible to distinguish: a) gyrointegrators of direct readings, for which the measured linear velocity of the object is fixed in the form of the angle of rotation of the outer gimbal ring relative to its initial position; b) gyrointegrators with feedback or "zero drive," for which measurement of linear velocity of an object is carried out by means of compensation of the moment of forces of inertia (caused by forward motion of the object) by a certain artificially created torque, which ensures retention of the gyroscope axis near the zero position.

We shall give schematics of gyrointegrators, we shall consider principles of their action, and we shall derive their basic relationships.

2. Gyrointegrator of Linear Accelerations of Gyropendulum Type

a) Schematic. Basic Relationships

The schematic of a gyrointegrator of linear accelerations of gyropendulum type [161, 115, 58] is shown in Fig. 9.16. The basic component of the instrument is an unbalanced (heavy) gyroscope. The gyrorotor is fixed in gyrohousing (inner ring) Gh; the rotor and gyrohousing are statically unbalanced with respect to the axis O'x' of oscillation in the outer ring (frame) R. With respect to spin axis $O\eta$ (Oy) of the frame the system is completely balanced. The distance of c.g., O of the gyrohousing from its spin axis O'x' is equal to l. To increase accuracy of work of the gyrointegrator axis Oz of the gyroscope should remain perpendicular to the axis $O\eta$ (y') of frame R. For this we have a correction system, consisting of signal pickoff SP (fixed on the spin axis O'x' of the gyrohousing) and stabilizing motor SM, controlled by it, which is connected by gear transmission with the instrument frame.

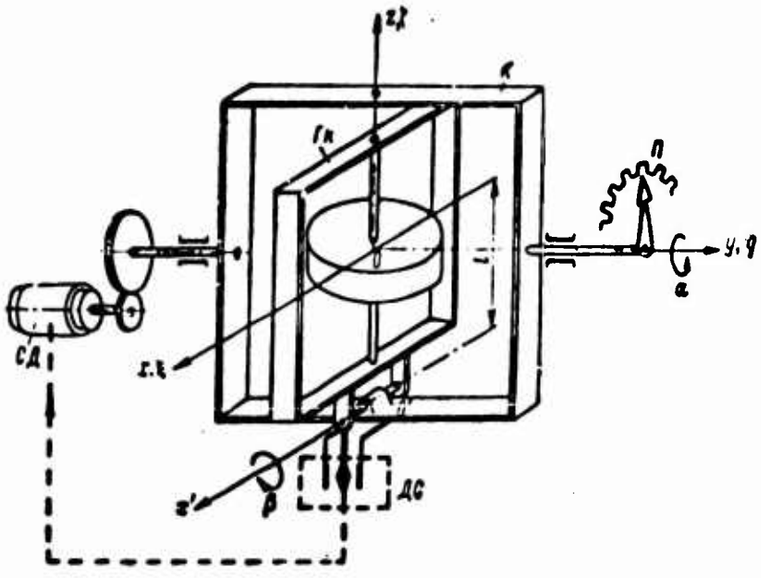

Fig. 9.16. Schematic of a gyrointegrator of linear accelerations of gyropendulum type.

In Fig. 9.16 the signal pickoff is in the form of a contact attachment. In this case the torque, applied by the stabilizing motor to axis Oη (Oy), has a contact (relay) response; this creates a natural oscillating conditions of motion of the gyroscope, in which the center position of axis Oz is perpendicular to axis Oη (Oy), and, consequently, all disturbing torques acting effective on this axis are compensated. Natural oscillations should be with great frequency and small amplitude. The presence of natural oscillations leads also to lowering of the friction torques in axes of the suspension. In principle there is also possible a linear law of control of the stabilizing motor; for this as the signal pickoff it is possible to apply, e.g., an inductive pickoff, which control the motor through an amplifier (not shown in Fig. 9.16).

Let us designate the angle of rotation of gyrohousing about axis O'x' with respect to frame Fr by β, the angle of rotation of frame Fr about axis Oη (Oy) by α. Readings of the gyrointegrator (magnitude of linear velocity of the object), proportional to angle α, are received in the form of voltage from potentiometer Pot.

Axes Oξηζ we consider connected with the vehicle (ship, aircraft); axes Oxyz, combined in initial position (when α = β = 0) with axes Oξηζ, are connected with the gyrohousing; axes Oxyz are in this case Resal's axes (§ 1.1). Euler's angles α and β, determining the position of axis Oz of the gyroscope relative to Oξηζ, can be selected just as for the gyrovertical (Fig. 1.6). Axis Oη is the measuring, input axis, or the axis of sensitivity, since the gyrointegrator reacts to component w linear acceleration $\bar{W}$ of the vehicle on this axis. Axis Oη can also be considered

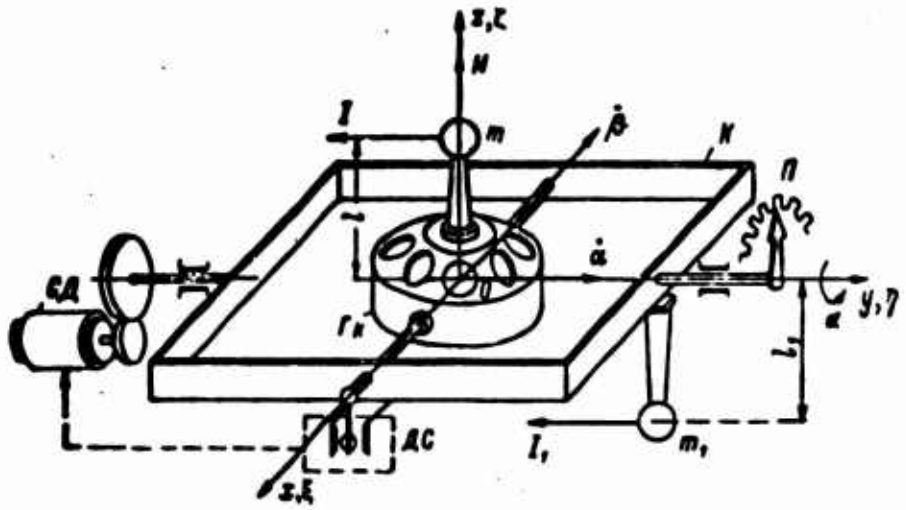

Fig. 9.17. Deriving the basic relationship for a gyrointegrator.

the output axis, since angle α of rotation of frame Fr about it is proportional, as will be shown, to the integral in time of linear acceleration w, i.e., component v of linear velocity of the vehicle along axis Oη.

Let us derive the basic relationship for the considered gyrointegrator, establishing the functional connection between input variable w(t) and output variable α(t). For this we use a modified gyrointegrator schematic, shown in Fig. 9.17. Here it is conditionally shown that gyrohousing Gh has static instability with respect to its spin axis Ox due to the presence of an additional load with mass m,* fixed distance l from the point of suspension O. So that here the system is balanced with respect to axis Oη (Oy) which in reality is carried out in the gyrointegrator (Fig. 9.16), we add to frame Fr another load with mass m_1 at distance l_1 from axis Oη (Oy). The condition of balance of the system with respect to axis Oη (Oy) will be expressed by relationship

$$ml - m_1 l_1 = 0. \tag{9.4.1}$$

Let us assume that axis Oη (Oy) is directed along the longitudinal axis of a vehicle, which moves in a direction, parallel to this axis, with acceleration w. The magnitude of the corresponding force of inertia I will be

$$|I| = mw. \tag{9.4.2}$$

*Considering a real scheme of an instrument (Fig. 9.16), by m one should understand the mass of the rotor and gyrohousing, i.e., the mass of the gyromotor.

The moment of this force with respect to point O is directed along the spin axis Ox of the gyrohousing and is equal to

$$M_x = mlw. \tag{9.4.3}$$

Under the influence of moment M_x frame Fr of the instrument will start to process about axis $O\eta$ (Oy) with angular velocity $\dot{\alpha}$, determined, according to precession theory, by formula (1.6.8); then, considering also (9.4.3), we have

$$\dot{\alpha} = \frac{M_x}{H} = \frac{ml}{H} w. \tag{9.4.4}$$

Since quantities m, l, and H for a given instrument are constant, we introduce designation

$$k = \frac{ml}{H}. \tag{9.4.5}$$

Then we rewrite (9.4.4) in form

$$\dot{\alpha} = kw, \tag{9.4.6}$$

where k — transmission factor of the gyrointegrator.

From (9.4.6) it follows that the angular velocity of rotation of the instrument frame is proportional to the linear acceleration of the vehicle. Integrating (9.4.6) and assuming that in the initial moment, when t = 0, $\alpha(0) = 0$, we have

$$\alpha = k \int_0^t w(\tau)\, d\tau, \tag{9.4.7}$$

i.e., the angle of rotation of the gyrointegrator frame is proportional to the integral in time of linear acceleration of the vehicle. Designating by v the component of linear velocity of the vehicle along axis $O\eta$, we rewrite (9.4.7) for initial conditions t = 0, v(0) = 0 in form

$$\alpha = kv. \tag{9.4.8}$$

Consequently, the angle α of rotation of the instrument frame is proportional to linear velocity v of the vehicle; here the proportionality factor is the transmission factor k of the gyrointegrator, the dimensionality of which, according to (9.4.5), will be [k] = sec/cm.

Let us note that during motion of the vehicle, and consequently also of mass m_1 with acceleration w there appears a force of inertia I_1, equal in magnitude to $m_1 w$. However, it does not affect rotation of the frame about axis $O\eta$ (Oy), but causes

only additional load in the supports of this axis.*

Expressions (9.4.6) and (9.4.8) are approximate, since in them we do not consider inertial terms, and also we do not take into account friction torques in the suspension axes and the torque of the stabilizing motor. Let us first consider the influence of these torques on operation of the gyrointegrator, limiting ourselves to the framework of precession theory. Torque $M_{s.m}$ of the stabilizing motor with respect to axis $O\eta$ (Oy) (if we assume for simplicity a linear characteristic of correction), by analogy with (2.3.2), is determined by formula

$$M_{c.\,m} = -S\beta. \tag{9.4.9}$$

Considering friction in the suspension axes liquid, according to (2.3.72), for friction torques in axes $O\eta$ (Oy) and Ox we have expressions

$$M_{ry} = -n_1\dot{\alpha}; \quad M_{rx} = n_2\dot{\beta}. \tag{9.4.10}$$

Considering (9.4.3), (9.4.9) and (9.4.10), we can express total torques with respect to axes Ox and $O\eta$ (Oy) by relationships

$$M_x = mlw + n_2\dot{\vartheta}; \tag{9.4.11}$$

$$M_y = -S\beta - n_1\dot{\alpha}. \tag{9.4.12}$$

We introduce (9.4.11) and (9.4.12) in equations of precession (1.6.8):

$$\dot{\alpha} = \frac{ml}{H}w + \frac{n_2}{H}\beta; \quad \dot{\beta} = -\frac{S}{H}\beta - \frac{n_1}{H}\dot{\alpha}. \tag{9.4.13}$$

Eliminating from (9.4.13) variable β, we obtain for α the following equation:

$$\ddot{\alpha} + x'\dot{\alpha} = k'\dot{w} + kx'w, \tag{9.4.14}$$

where

$$x' = \frac{x}{1+\frac{n_1n_2}{H^2}}; \quad x = \frac{S}{H}; \quad k' = \frac{k}{1+\frac{n_1n_2}{H^2}}; \tag{9.4.15}$$

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*We correct an inaccuracy in Part I of this work. In (5.5.29) by P one should understand the weight of the outer ring, and by l_{2k} displacement of its c.g., along axis OK (Fig. 3.2). Moment $l_{2k}P$ of unbalance of the outer ring with respect to axis Ox_1 is quenched by reaction of the bearings of this ring. Therefore, during calculation of azimuth drift of a gyroscope due to static unbalance one should not consider component $l_{2k}P$.

$k = \frac{ml}{H}$ [see (9.4.5)]; $\varkappa$ – specific velocity of correction [see (2.3.5)].

We integrate equation (9.4.14) once with respect to time for zero initial conditions

$$\dot{\alpha} + \varkappa'\alpha = k'w + k\varkappa' v. \tag{9.4.16}$$

The integral of this equation for initial conditions t = 0, α(0) = 0 will be

$$\alpha = k' e^{-\varkappa' t} \int_0^t w e^{\varkappa' \tau} d\tau + k\varkappa' e^{-\varkappa' t} \int_0^t v e^{\varkappa' \tau} d\tau. \tag{9.4.17}$$

Performing partial integration and considering initial conditions t = 0, v(0) = = 0, $s(0) = \int_0^t v\,d\tau\big|_{t=0} = 0$, we obtain

$$\alpha = k' \left[v - \varkappa' s + (\varkappa')^2 \int_0^t s e^{-\varkappa'(t-\tau)} d\tau \right] + k\varkappa' s - k(\varkappa')^2 \int_0^t s e^{-\varkappa'(t-\tau)} d\tau, \tag{9.4.18}$$

where $s = \int_0^t v d\tau$ is the path passed by the vehicle.

Analyzing formula (9.4.15), we note that $\varkappa = S/H$ is usually small (see example 6.1); since $\frac{n_1 n_2}{H^2} \ll 1$, then $\varkappa'$ also is small. Therefore, in (9.4.18) it is possible to disregard terms which contain as a factor $(\varkappa')^2$. Then, instead of (9.4.18), we obtain

$$\alpha = k'(v - \varkappa' s) + k\varkappa' s. \tag{9.4.19}$$

With accuracy accepted by us we can replace relationships (9.4.15) by the following ones:

$$\varkappa' \approx \varkappa \left(1 - \frac{n_1 n_2}{H^2}\right); \qquad k' \approx k \left(1 - \frac{n_1 n_2}{H^2}\right). \tag{9.4.20}$$

Considering (9.4.20), we rewrite (9.4.19) in form

$$\alpha = kv \left(1 - \frac{n_1 n_2}{H^2}\right) - k\varkappa \frac{n_1 n_2}{H^2} s. \tag{9.4.21}$$

Comparing (9.4.21) with (9.4.8), we see that friction in the bearings of the gyrointegrator introduces systematic errors in readings of the instrument. Error from component $kv \frac{n_1 n_2}{H^2}$ in a real instrument is usually minute; error caused by $k\varkappa \frac{n_1 n_2}{H^2} s$, will grow with increase of distance s, traveled by the vehicle.

Therefore, in such instruments they seek to substantially decrease friction in the suspension, to decrease the specific velocity of correction $\varkappa$ and to increase angular momentum H of the gyroscope.

Besides the shown ones, there can appear other systematic errors, for instance due to rotation of the vehicle, rotation of the earth, etc.

b) Equations of Motion of a Gyrointegrator Taking into Inertial Terms

Above during composition of equations of motion of a gyrointegrator we started from precession theory, i.e., disregarded inertial terms of the equations. When writing equations of a gyrointegrator taking into account inertial terms one should consider that the position of axis Oz of the gyroscope is determined by the same angles α and β as for a gyrovertical (Fig. 1.6). Therefore, by analogy with equations (6.3.49) of the gyrovertical, considering also liquid friction torques in the suspension axes, we write equations of a gyrointegrator in form

$$\left.\begin{aligned} J_{r.e}\ddot{\beta} - H\dot{\alpha} + n_2\dot{\beta} &= -M_x \\ J_{r\eta}\ddot{\alpha} + H\dot{\beta} + n_1\dot{\alpha} &= M_y \end{aligned}\right\} \qquad (9.4.22)$$

Here: $J_{g.e}$ – total moment of inertia of rotor and inner ring (gyrohousing) with respect to axis O'x' (Fig. 9.16), which instead of (3.2.4) is determined by relationship

$$J_{r.e} = J_e + J_{e.e} + ml^2 \qquad (9.4.23)$$

(m – mass of rotor and inner ring); $J_{g\eta}$ – total moment of inertia of rotor and inner and outer gimbal rings, determined by a relationship of type (3.2.5).

We consider the moment of forces of inertia of translational motion of the vehicle moment M_x [see (9.4.3)]. For moment M_{amp} we take torque $M_{s.m}$, applied by stabilizing motor SM (Fig. 9.16) to axis $O\eta$(Oy). Taking into account what was said we rewrite equations (9.4.22) in form

$$\left.\begin{aligned} J_{r.e}\ddot{\beta} - H\dot{\alpha} + n_2\dot{\beta} &= -mlw \\ J_{r\eta}\ddot{\alpha} + H\dot{\beta} + n_1\dot{\alpha} &= M_{c.z} \end{aligned}\right\} \qquad (9.4.24)$$

If we assume that the stabilizing motor is turned off, instead of (9.4.24) we obtain equations

$$\left.\begin{aligned} J_{r.e}\ddot{\beta} - H\dot{\alpha} + n_2\dot{\beta} &= -mlw \\ J_{r\eta}\ddot{\alpha} + H\dot{\beta} + n_1\dot{\alpha} &= 0 \end{aligned}\right\} \qquad (9.4.25)$$

which differ from the equations of motion of a gyrovertical (6.3.53) only by the presence in the right part of the first equation of the moment mlw of forces of inertia of translational motion of the vehicle.

Equations (9.4.25) characterize the following motions of the gyrointegrator: 1) basic precessional motion of the outer gimbal ring, determined by relationship (9.4.8) or for $w = \text{const}$ by expression $\alpha = kwt$, which can be obtained from the first equation of (9.4.25), if in it we reject inertial term $J_{g.e}\ddot{\beta}$ and do not consider moment of friction $n_2\dot{\beta}$ in the spin axis of the inner gimbal ring; 2) basic precessional motion of the inner ring, determined for $w = \text{const}$ by expression $\beta = -\frac{n_1}{H}kwt$, which is obtained from equations (9.4.25), if in them we reject inertial terms and not to consider small moment $n_2\dot{\beta}$ of forces of friction in the spin axis of the inner gimbal ring; 3) attenuating nutation oscillations of the gyroscope axis, determined by expressions of type (6.3.57). Relationships given in paragraphs 1 and 2 are valid for zero initial conditions.

We shall give certain considerations about analysis of motion of a gyrointegrator taking into account torque $M_{s.m}$, applied by the stabilizing motor. For a relay (contact) characteristic of correction (Fig. 2.18e) torque $M_{s.m}$ is determined by relationship (2.3.17)

$$M_{c.z} = -L_y f_y(\beta), \tag{9.4.26}$$

where $f_{amp}(\beta)$ — a nonlinear function, expressed by a formula of type (2.3.18).

Substituting (9.4.26) in (9.4.24), we have

$$\left.\begin{aligned} J_{r.g}\ddot{\beta} - H\dot{\alpha} + n_2\dot{\beta} &= -mlw \\ J_{r_1}\ddot{\alpha} + H\dot{\beta} + n_1\dot{\alpha} &= -L_y f_y(\beta) \end{aligned}\right\} \tag{9.4.27}$$

Being interested in motion of the gyrointegrator in coordinate β, we obtain when $w = \text{const}$ for β the following equation in operational form ($p = \frac{d}{dt}$):

$$\begin{aligned} \left[J_{r.g}J_{r_1}p^3 + (J_{r.g}n_1 + J_{r_1}n_2)p^2 + (n_1n_2 + H^2)p\right]\beta = \\ = -HL_y f_y(\beta) - n_1 mlw. \end{aligned} \tag{9.4.28}$$

analogous to the second equation of system (6.3.61).

If moment mlw of forces of inertia of translational motion of the vehicle is equal to zero, equation (9.4.28) will take form

$$\begin{aligned} \left[J_{r.g}J_{r_1}p^3 + (J_{r.g}n_1 + J_{r_1}n_2)p^2 + (n_1n_2 + H^2)p\right]\beta = \\ = -HL_y f_y(\beta). \end{aligned} \tag{9.4.29}$$

analogous to the equation (6.3.62) of motion of a gyrovertical with a relay characteristic of correction. Investigation of equation (9.4.29) can be carried out by the same method of harmonic linearization of nonlinearities which we used in § 6.3, Par. 3d during analysis of equation (6.3.62) of a gyrovertical. Solution of an analogous problem in reference to a gyrointegrator is given in the book of Ye. P. Popov and I. P. Pal'tov [115]. On the basis of what was presented in § 6.3, Par. 3d it follows that the gyrointegrator will accomplish natural oscillating motions, the frequency of which, determined by formula (6.3.76), is equal to the frequency of nutation oscillations of a free gyroscope. Amplitude of these natural oscillations is determined by a relationship of type (6.3.78).

From formulas (6.3.76) and (6.3.78) it follows that to obtain natural oscillations with high frequency and small amplitude (which is of essential importance for increasing accuracy of the gyrointegrator) it is useful to [115]: increase angular momentum of the gyroscope; decrease moments of inertia of rings of the Cardan suspension; increase damping (coefficients n_1 and n_2 of viscous friction); as far as possible decrease the torque of the stabilizing motor. In order to satisfy the last requirement it is necessary to decrease moments of forces of friction in the suspension axes, and also instability (unbalance) of the system with respect to axis $O\eta$ (Oy) (Fig. 9.16).

Above we analysed the question of natural oscillations of a gyrointegrator with equality to zero of the moment of forces of inertia of translational motion of the vehicle. This question, taking into account this moment, when the equation of motion of the gyrointegrator is determined by relationship (9.4.28), is considered in the above-indicated book [115]. Referring those interested in a detailed solution of this problem to that book, we indicate the following. In the presence of disturbance m_w the gyrointegrator also accomplishes natural oscillations, whose frequency is determined by the same relationship (6.3.76) as in the absence of an external disturbance. Moment m_w affects the amplitude of natural oscillations, and also causes displacement of the center of oscillations, which leads to additional errors of readings of the gyrointegrator. The limiting value of moment m_w, permissible during use of the instrument, is determined by the possibility of a break in natural oscillations. From what was said above it follows that in examining the theory of the gyrointegrator, and also questions connected with selection of parameters, it is necessary to use complete equations of motion, considering also inertial terms.

c) Application of Gyrointegrator

It is possible to indicate examples of application of the gyrointegrator on vehicles having considerable accelerations with comparatively small time of motion. Such a gyrointegrator was used in the German ballistic rockets of V-2 type. The schematic of it is analogous to the diagram given in Fig. 9.16. The instrument had to determine the component of of linear velocity of the rocket along its longitudinal axis and automatically to send a cut-off signal to the motor upon achievement by the rocket of the assigned design speed. The latter was set before launching on a special mechanism for routing the command signals for turning off the rocket engine.

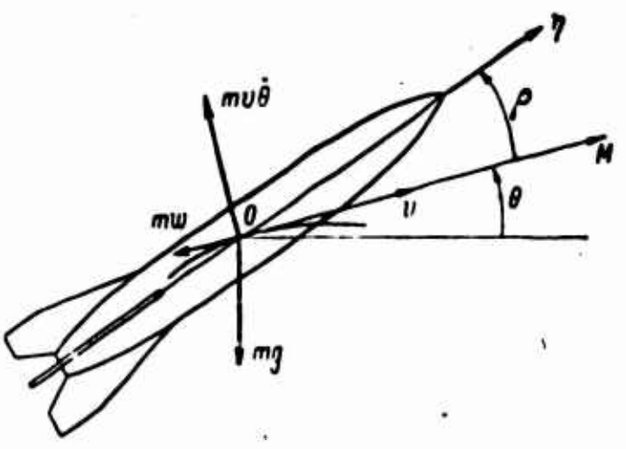

Fig. 9.18. Determining of apparent acceleration.

The principle of action of this integrator does not differ from that considered earlier. Diagrams of the instrument and its description can be found in a number of books and articles (see, e.g., [58, 7, 161, 195, 194, 205]). Referring those who wish to become acquainted with the instrument in detail to this literature, we note only one of its peculiarities, connected with the curvilinear nature of the path of the c.g., of the V-2 on its powered-flight section [161, 7]. For this we first find the expression for the component of the apparent force of gravity, having an effect on the gyrointegrator during curvilinear motion of the rocket.

In Fig. 9.18 there are shown elements of motion of the rocket in the vertical plane. Tangent OM to the path of the rocket c.g. forms with the horizontal plane angle θ; the angle between OM and the longitudinal axis of rocket $O\eta$ is the angle of attack ρ. On the gyrointegrator there act:* force of gravity mg (m – mass of rotor and gyrohousing); force of inertia mw of translational motion of the rocket (w – longitudinal acceleration of the rocket, directed along axis OM); centrifugal force of inertia θ, caused by rolling of the rocket about its transverse axis

*Here for composition of the equation of motion of the gyrointegrator we use D'Alembert's principle (method of kinetostatics [85]), according to which to acting forces there are joined forces of inertia.

cn in the powered-flight trajectory (v – longitudinal velocity of rocket). The resultant force, acting on the gyrointegrator in the direction of longitudinal axis $O\eta$ of the rocket, will be

$$P = m\left[w\cos\rho - v\dot{\theta}\sin\rho + g\sin(\theta+\rho)\right]. \tag{9.4.30}$$

Then the moment M_x of this force with respect to the spin axis Ox (Fig. 9.16) of the instrument frame will be

$$M_x = ml\left[w\cos\rho - v\dot{\theta}\sin\rho + g\sin(\theta+\rho)\right]. \tag{9.4.31}$$

Substituting (9.4.31) in (9.4.4), we obtain

$$\dot{\alpha} = \frac{ml}{H}\left[w\cos\rho - v\dot{\theta}\sin\rho + g\sin(\theta+\rho)\right]$$

or, considering (9.4.5)

$$\dot{\alpha} = k\left[w\cos\rho - v\dot{\theta}\sin\rho + g\sin(\theta+\rho)\right], \tag{9.4.32}$$

from which, for initial conditions t = 0, $\alpha(0) = 0$, we have

$$\alpha = k\int_0^t\left[w\cos\rho - v\dot{\theta}\sin\rho + g\sin(\theta+\rho)\right]d\tau. \tag{9.4.33}$$

Since in the considered case angles of attack ρ are minute, and angular velocity $\dot{\theta}$ is also small, instead of (9.4.33), we obtain approximately,

$$\alpha = kv + k\int_0^t g\sin\theta\,d\tau. \tag{9.4.34}$$

Thus, the gyrointegrator, in general, determines linear velocity v of the rocket, but has methodical error $k\int_0^t g\sin\theta\,d\tau$, i.e., determines the apparent velocity of the rocket.

The considered gyrointegrator possesses deficiencies, which are caused basically by the following factors: 1) low sensitivity of the instrument; 2) influence on accuracy of readings of components of accelerations of the vehicle on axes, not coinciding with the input axis; 3) variations of parameters of the instrument (m, l, H) from their design values; 4) instability, or "drift" of zero; 5) deflections of the real character of the trajectory (for instance, angle θ) from design, etc.

3. Other Types of Gyrointegrators of Linear Accelerations

Let us consider briefly certain other types of gyrointegrators. In Par. 1 of this section we will mention the gyrointegrator in which there is applied a three-degree-of-freedom gyro with additional mass and compensational loads. The schematic of such a gyrointegrator [30, 109] is shown in Fig. 9.19.

In the outer ring R there is fixed gyrohousing Gh. The spin axis $O\eta$ of ring R coincides with the longitudinal axis of the vehicle, which moves with velocity v and acceleration w. To the gyrohousing at distance l from axis $O\eta$ there is fastened a load, whose mass is equal to m. Consequently, the gyrohousing is statically unbalanced with respect to its spin axis. For balancing the system with respect to axis $O\eta$ to ring R there are fastened two compensational loads, whose mass is m'. Springs Sp, connecting the gyrohousing with these loads, serve to keep the axis of the gyroscope in a direction, perpendicular to axis $O\eta$. During movement of the vehicle with acceleration w there appears force of inertia mw, the moment of which with respect to point O will be equal to mlw. This moment causes precession of the gyroscope about axis $O\eta$ with angular velocity $\dot{\alpha}$, determined, by analogy with (9.4.4), by relationship

$$\dot{\alpha} = \frac{ml}{H} w = kw,$$

from which, with zero initial conditions, we obtain

$$\alpha = kv,$$

i.e., the given instrument determines linear velocity v of the vehicle. More detailed investigations of this gyrointegrator are carred out R. Grammel [30].

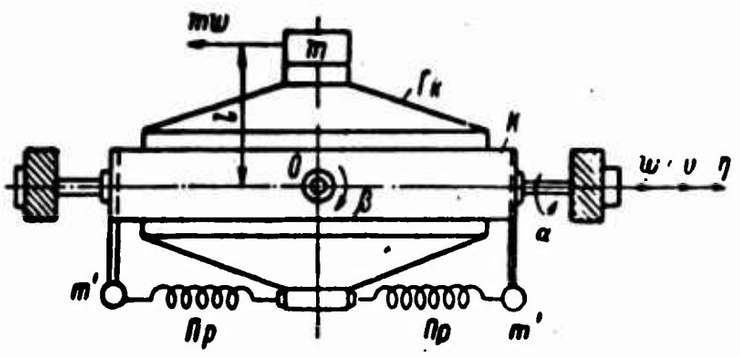

Fig. 9.19. Schematic of a gyrointegrator with an additional mass and compensational loads.

Let us consider a gyrointegrator in which there is used an unbalanced two-degree-of-freedom gyroscope with float suspension. This gyrointegrator is described in a number of articles (for instance, [180, 179, 83]) by the name of floated integrating accelerometer. Let us note that instruments considered by us earlier are gyrointegrators of direct readings, but the floated integrating accelerometer is a gyrointegrator with feedback or with "zero drive" (Par. 1).

The schematic of a floated integrating accelerometer (gyrointegrator of linear accelerations with float suspension) is shown in Fig. 9.20. The basic component of the instrument is the gyrofloat GF, fixed in hydrostatic suspension in the case of the instrument. To the float there is fastened a load, whose mass is m, thanks to which the float is statically unbalanced with respect to its spin axis $O\eta$. On this axis there is fixed signal pickoff SP, controlling amplifier Amp torquer TQ. The latter, through a gear transmission, is mechanically connected with gear Gr, rigidly fastened to the case of the instrument, which is fixed on stabilized platform P. Angle α of rotation of the gyrointegrator about axis $O\zeta$ with respect to platform P is fixed by pickoff PO.

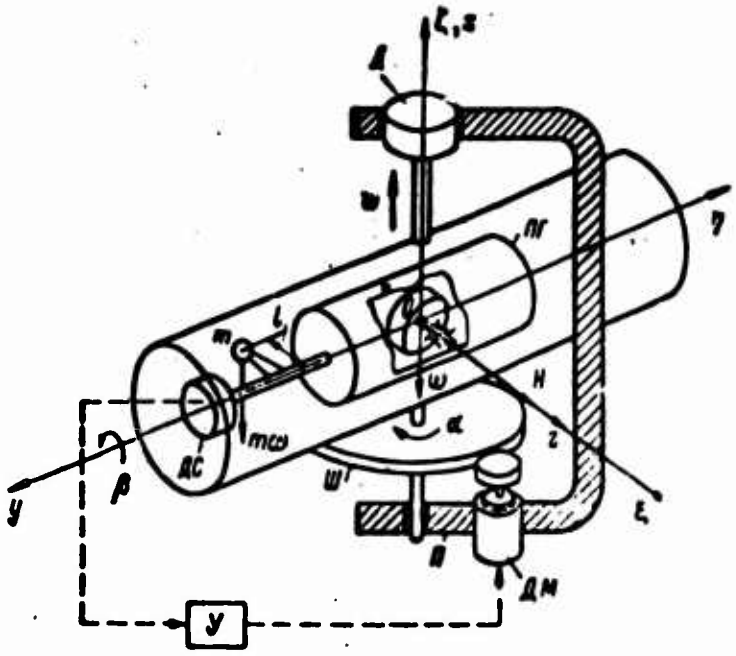

Fig. 9.20. Schematic of a floated integrating accelerometer.

The principle of action of the gyrointegrator is the following. Let us assume that the vehicle on which platform P is placed, together with the gyrointegrator, moves with acceleration w along axis $O\zeta$. Here, there appears force of inertia $-mw$, whose moment of respect to axis $O\eta$ will be $ml w$. This moment causes rotation of the gyrofloat about axis $O\eta$ (Oy) a certain angle β, as a consequence of which signal pickoff SP; through amplifier Amp, turns on torquer TQ. The latter starts to rotate the case of the gyrointegrator about axis $O\zeta$ with a certain angular velocity ω, which causes the appearance of gyroscopic torque $M_g = H\omega$, directed along axis $O\eta$ (Oy); torque $M_g = H\omega$ compensates the moment $ml w$ of forces of inertia of translational motion of the vehicle and thereby returns the gyrofloat to its initial position ($\beta = 0$). Obviously, in the equilibrium position

$$H\omega = ml\dot{w},$$

whence

$$\omega = \frac{ml}{H} w = kw, \qquad (9.4.35)$$

i.e., angular velocity ω of rotation of the gyrointegrator is proportional to the linear acceleration of the vehicle. If by α we denote the angle of rotation of the case of the gyrointegrator about axis $O\zeta$ with angular velocity $\omega = \dot{\alpha}$, then, according to (9.4.35), we have [when $\alpha(0) = 0$]

$$\alpha = k\int_0^t w(\tau)\,d\tau,$$

from which, with zero initial conditions, we have

$$\alpha = kv.$$

Consequently, the angle of rotation of the gyrointegrator case about axis $O\zeta$ is proportional to the linear velocity v of the vehicle.

Merits of the considered gyrointegrator are the following.

1. Since the gyroscope works near zero position ($\beta \sim 0$), there is provided a wide range of measurement by the instrument of accelerations of a vehicle.

2. Application in the instrument of feedback permits us to achieve constancy of zero.

3. Friction in the spin axis of the float is reduced to a minimum, which ensures high sensitivity of the gyrointegrator to linear accelerations of the vehicle. Actually, for any type of gyrointegrator, and in general, any accelerometer, minimum linear acceleration w_{min} is determined by the magnitude of the least moment of forces of inertia $ml\dot{w}_{min}$ of translational motion of the vehicle which overcomes the moment of friction in the spin axis of the gryohousing (float), i.e., w_{min} is found from condition

$$ml\dot{w}_{min} > |M_y|$$

or

$$w_{min} > \frac{K_y}{ml}, \qquad (9.4.36)$$

where K_y — the modulus of dry friction torques in the axis of the gyrohousing (float).

From formula (9.4.36) it is clear that to increase the sensitivity of the gyrointegrator one should decrease dry friction torque in supports of the suspension and increase magnitude ml, i.e., increase the static moment of the pendulum, equal to mgl. Since for a gyrointegrator with float suspension the

residual torque of dry friction is minute, such an instrument possesses greater sensitivity to accelerations than other types of gyrointegrators and accelerometers with supports on ball bearings.

4. Instrument errors due to deviations of parameter from their rated values appear only due to inaccuracy of stabilization of the frequency of the current feeding to the gyr motor, which causes corresponding oscillations of angular momentum H of the gyroscope.

There are indication [180] that a gyrointegrator with float suspension is promising for inertial navigational systems. Certain theoretical research of this gyrointegrator is contained in article [83].

C H A P T E R 1 0

CERTAIN QUESTIONS OF GYROSCOPIC STABILIZATION

Designations Appearing In Cyrillic

в = for = forced

г = g = gyroscope

гр = lo = load

г.э = g.e = gyroequator

к = s = ship

с = s = stabilizing

ст = st = static

т = b = braking

ВГ = AG = Auxiliary Gyroscope

Гр = Lo = Load

ГС = GS = Gyroscopic Stabilizer

ГУ = GD = Gyroscopic Device

ПД = PM = Precessional Motor

Р = F = Frame

С = S = Stand

Т = B = Brake

§ 10.1. Purpose and Types of Gyroscopic Stabilizers

In systems of automatic regulation and control of different installations and systems, in particular for realization of their automatic stabilization, wide application has been found by gyroscopic methods of stabilization. By gyroscopic stabilization we understand stabilization of a system (object) with the help of

gyroscopic devices.

Gyroscopic stabilization is carried out by gyroscopic stabilizers (GS). One of the varieties of such GS by the name of active GS, or gyroframes, was in detail considered in Chapter 7. However, besides activated GS, there exist also other types of gyroscopic stabilizers applied in different areas of technology.

General purpose of gyroscopic stabilization consists of automatic stabilization of separate instruments and devices; conditions of use of them on a rocking and maneuvering object (ship, aircraft) are near to conditions of operation on land. In certain cases there is provided automatic stabilization not only of instruments, devices and installations, but also of the very objects on which they are located. In accordance with this a gyroscopic stabilizer is defined as a gyroscopic device intended for retention of a certain direction or coordinate system connected with the object in constant position with respect to earth or inertial space.

Depending upon its purpose we distinguish: 1) gyroscopic stabilization of separate instruments, devices and installations on oscillating objects (ship, aircraft); 2) gyroscopic stabilization of the objects themselves. Example of the first of these varieties is stabilization of a certain platform on a rocking ship. Examples of the other variety of gyroscopic stabilization are the damper of rolling of a ship and the autopilot, one of the basic tasks of which consists of automatic stabilization of oscillatory motions of aircraft near its center of gravity.

We distinguish full and partial automatic stabilization. The first stipulates stabilization of object relative to all three axes of its rotation and is carried out theoretically exactly, i.e., without systematic errors. Thus, for instance, stabilization on a rocking ship of a certain platform relative to three axes of its rotation (i.e., elimination of the influence of rolling and pitching and yaw of the ship) will be complete if influence on it of oscillatory motions of the ship is compensated without any systematic errors. However, in reality the platform even with full stabilization will not be exactly held in plane of horizon and in the given azimuthal direction due to errors of stabilization appearing under actual conditions.

Partial stabilization stipulates stabilization of the object relative to one or two axes of its rotation; thus stabilization in principle is carried out inaccurately (roughly), first of all due to corresponding systematic errors. Thus, for instance, damping of tossing of a ship usually is produced only relative to its longitudinal axis, but rolling is not completely removed, but only somewhat moderated.

According to the principle of action, gyroscopic stabilizers can be divided into the following varieties:

1) direct GS;

2) active GS, or gyroframes;

3) read-out GS.

Direct gyroscopic stabilizer is defined as a gyroscopic device whose principle of action is based on direct use of stabilizing properties of the astatic gyroscope. Indeed (§ 1.3, Par. 4), if axis of such a gyroscope is forced to change its position in space, then there appears gyroscopic moment which is used for stabilization of the object. For creation of stabilizing effect the gyroscope should have very large angular momentum.

One of the important characteristics of direct GS is the presence of mechanical coupling with the stabilized objects. Therefore, during the analysis of such GS R. Grammel [30] considers " ... systems in which the gyroscope constitutes a considerable part of mass of system and influences, due to the appearance of gyroscopic effects, the position and state of motion of the system directly, without any control elements. Such gyroscopes can be called direct stabilizers."

Let us indicate certain varieties of these GS. Direct stabilization of position or motion of a body is most simply carried out when the body itself represents a gyroscope with sufficiently large angular momentum.* Among such gyroscopes it is possible to include celestial bodies, a gyroscope in Cardan suspension, a rapidly rotating elongated missile and others.

Sometimes for direct gyroscopic stabilization there is used a gyroscope** which constitutes only part of the system (which itself is unstable). An example of such a device is gyroscopic stabilizer of railroad car of monorail railroad. Direct gyroscopic stabilizer can be used also for quenching of oscillations of a stable system. Gyroscopes of such kind are applied in dampers of rolling of a ship.***

Direct GS, along with considerable stabilizing effect and acceptable accuracy, possess a series of important deficiencies connected with large weight and dimensions. Furthermore, for gyroscopes applied in them with large angular momentum, a considerable time of acceleration is characteristic, this considerably increases time of preparation of the entire gyrostabilizer for operation.

Of the enumerated types of direct GS, we will consider subsequently gyroscopes applied in dampers of rolling of a ship. With properties of objects which are

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*Such a gyroscope sometimes is called directional [30].

**Such a gyroscope sometimes is called rectifying [30].

***This gyroscope is frequently called damping [30].

themselves gyroscopes (celestial bodies, fast-rotating elongated missiles and others), it is possible to become acquainted in special literature (see for instance, [30, 65, 20, 190]). Analysis of application of the other type of direct GS, for instance, in gyroscopic stabilizer of railroad car of monorail railroad, is given in a number of books on applied gyroscopy (for instance, [30, 190, 118, 96, 146]*

Another type of gyroscopic stabilizers – active GS, or gyroframe – was considered in Chapter 7. Power GS have certain advantages over direct GS: for instance smaller weight and dimensions, due to application in them of special motors for creation of stabilizing torques which serve for overcoming of influence on object of external perturbing torques. However, active GS (gyroframe) successfully applied for direct stabilization of separate instruments and installations on objects cannot be used as direct stabilizers of the objects themselves, for instance in dampers of rolling of a ship. Therefore, in these cases there are applied only direct GS in which there are used stabilizing properties of astatic gyroscope.

The enumerated two types of GS are stabilizers of active type.

In the third type of GS – read-out gyroscopic stabilizers – there are used gyroscopic devices of indicated type (§ 2.4, Par. 1) which are sensing devices of stabilization system. The latter control through amplifiers the corresponding stabilizing motors, which stabilize the object, which is usually installed in a Cardan suspension.

Thus, read-out gyroscopic stabilizer constitutes a system of automatic control whose sensing or pick-up elements are GD (gyroscope devices) of read-out type, which control servo drives carrying out stabilization of the object.

In read-out stabilizers, GD are connected with the stabilized object not mechanically, but with help of servo systems; due to this these GD do not directly carry a power load with respect to stabilization. In read-out GS there are created certain conveniences for distribution of elements of stabilizer. Application of powerful servo drives makes it possible to carry out statilization of objects possessing considerable inertial masses. Usually read-out GS have considerably smaller dimensions and weight than direct GS or gyroframe. Furthermore, for read-out GS due to application of comparatively small gyroscopes, time of setting the stabilizer in action is small, and power drains are less than for other types of GS. As gyroscopic elements of the indicated GS there are used gyroverticals and directional

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*See Also: Ye. L. Nikolai. Theoretical mechanics. Part III, State United Scientific and Technical Press NKTP USSR, 1939.

gyroscopes of the indicated type, which serve for determination of angular deviations of the object from its assigned position.

With the help of the read-out GS, there may be, for instance, carried out stabilization on a ship of a certain platform relative to three axes of its rotation; thus there is eliminated the influence on it of rolling and pitching and yawing of the ship. In this case, sensing devices of the GS are: 1) gyrovertical, which determines angles of rolling and pitching of the ship; 2) directional gyroscope, which determines angles of yawing and turning of the ship. These angles are transmitted with help of servo systems to servo drives, which introduce pitch and yaw angles of the ship for stabilization of the platform fixed in a Cardan suspension. The given example of read-out stabilization will be more specifically considered in § 10.4.

Active GS (gyroframes) were in detail discussed in Chapter 7; therefore, further there are considered only direct and read-out gyroscopic stablizers – their design, principles of operation and basic relationships.

§ 10.2. Remark on General Principles of Gyroscopic Stabilization

We will make several remarks on certain general principles of gyroscopic stabilization. As was shown, as one of the means of stabilization of objects, there are applied gyroscopic stabilizers, which in a number of cases make it possible to improve stability of system or make an unstable system stable.

If there is selected a system of stabilization and there are assigned parameters of the object and stabilizer, then, by using criteria known from theory of automatic control, for instance of Hurwitz, Mikhaylov, Nyquist and others, it is possible to reveal stability of the "object + stabilizer" system, and also, by using appropriate methods, to find quality of system and its other characteristics. However, during development of GS, before we become occupied with investigations of these questions, it is necessary to distinctly show: 1) what influence is rendered, for instance, by gyroscopic forces, dissipative force or other categories of forces on character of motion of the object; 2) if it is possible to provide or to improve stability of motion of the system by means of addition of some category of forces to forces which act on the system under actual conditions of operations.

The above mentioned and other criteria of stability do not give an answer to such questions. However, in mechanics there are known certain general principles of stabilization of systems which make it possible during designing to correctly approach the discovery of the fundamental possibility of stabilization of the object

and to outline in broad terms the scheme of the stabilizer. These principles were developed in the classical works of Thomson and Tait [203]. Further development of these ideas is contained in a number of works, from which it is necessary first of all to mention works of recent years, in particular those of D. R. Merkin [90] and I. I. Metelitsyna [95].

In them, besides other questions, there is considered the application of methods of Thomson and Tait to nonconservative systems, with which it is necessary to deal in the appendices. Certain questions of gyroscopic stabilization are considered by N. V. Rose [124]. Basic principles of gyroscopic stabilization, along with general theory of stability of motion, are presented in the monograph of N. G. Chetayev [170].

Using these works, we will give certain basic principles of gyroscopic stabilization. To those interested in this question more specifically, including the corresponding mathematical bases, we refer the mentioned literature.

Let us recall certain necessary definitions. In the differential equations of motion of different GD which were considered in the preceding chapters, there were contained gyroscopic torques (gyroscopic reaction torques). Such terms of equations were called gyroscopic. According to the definition of Thomson and Tait [203], force are called gyroscopic, if the sum of their works on any real infinitesimal displacement of the system is equal to zero. Here by gyroscopic force we understand gyroscopic torque $\vec{M}_g$ [see (1.3.33)]. Actually, "force" $\vec{M}_g$ is gyroscopic, since [90] its elementary work on infinitesimal displacement $\vec{\omega}dt$

$$\vec{M}_g \cdot \vec{\omega}\, dt = \left(\vec{G} \times \vec{\omega}\right) \cdot \vec{\omega}\, dt \tag{10.2.1}$$

is identically equal to zero if we consider the known property of the vector-scalar product containing two equal vectors $\vec{\omega}$.

Gyroscopic forces can be [90] real forces applied to the system, as well as simply certain terms of the equations of motion; therefore the term gyroscopic "force" is arbitrary.

In examining of gyroscopic stabilizers we will frequently meet with categories of forces which we call dissipative. Among such forces are included force of fluid drag, which act on points of the system in the direction opposite their velocity, and are proportional to the size of the latter. As it is known (see, for instance, [85]), generalized drag Q_i is called dissipative if it satisfies the relationship

$$Q_i = -\frac{\partial \Phi}{\partial \dot{q}_i}, \tag{10.2.2}$$

where $\dot{q}_1$ — generalized velocity of system;

Φ — a certain positive guadratic form of velocities $\dot{q}_1$.

Function Φ is called dissipative function or Rayleigh dissipative function [155]. If function Φ contains the derivatives of all velocities $\dot{q}_1$, then dissipation is called total; in the opposite case it is called partial [170].

Subsequently it will be necessary for us to meet also with the idea of potential, conservative and nonconservative forces. We assume that force $\vec{F}$ acting on a material particle is such that its projections on coordinate axes can be represented as partial derivatives with respect to the corresponding coordinates of a certain function of the coordinates U (x y z), which is called in this case the force function, i.e.,

$$F_x = \frac{\partial U}{\partial x}; \quad F_y = \frac{\partial U}{\partial y}; \quad F_z = \frac{\partial U}{\partial z}. \tag{10.2.3}$$

Instead of force function U, sometimes we consider potential energy Π, connected with force function by the dependence

$$\Pi = -U. \tag{10.2.4}$$

Then (10.2.3) can be rewritten in the form

$$F_x = -\frac{\partial \Pi}{\partial x}; \quad F_y = -\frac{\partial \Pi}{\partial y}; \quad F_z = -\frac{\partial \Pi}{\partial z}. \tag{10.2.5}$$

Thus, if projections of acting force $\vec{F}$ satisfy relationships (10.2.5), then the field is called a potential field; quantity $\Pi(x, y, z)$ is called potential energy of field; and force $\vec{F}$ is called potential, or conservative. All other forces are called nonconservative.

If on the system there act only potential forces, then the law of conservation of total mechanical energy of system is valid. Among conservative forces it is possible, for instance, to include gravity, force of elasticity, and so forth. Nonconservative forces are, for instance, forces of drag. In accordance with what was said, mechanical systems are subdivided into conservative and nonconservative.

During the analysis of gyroscopic stabilizers there are considered also forces of radial correction. With such "forces," which are corrective torques with proportional characteristic, we met during the study of motion of a gyrovertical, in which there is used a three-degree-of-freedom astatic gyroscope with proportional correction (§ 6.3, Par. 2b). This correction was called radial, since in the presence of it the axis of gyroscope returns to initial position in the shortest direction (trajectories of vertex of gyroscope are a family of straight lines, passing through

origin of coordinates; Fig. 6.3).

In theorems, pertaining to gyroscopic stabilization, sometimes there is used the term degree of stability, where in it there is a different content than in the same concept which we used in § 7.3, Par. 3e. In order to explain the meaning of this term, we will assume that certain conservative system (in the absence of gyroscopic and dissipative forces) is described in normal coordinates q_i by equations of form

$$\ddot{q}_i + c_i q_i = 0 \qquad (i = 1, 2, \ldots, n), \tag{10.2.6}$$

each of which contains only one coordinate.

If any number $c_i > 0$, then motion along the corresponding coordinate will be stable; if $c_i < 0$, then motion along this coordinate is unstable. Therefore, numbers c_i are called (see, for instance, [90]) stability factors of the system, and the number of negative c_i is called degree of instability. Subsequently not the number of unstable coordinates, but its evenness will be significant. Let us designate the determinant consisting of coefficients c_i by $C = [c_{ij}]$; then it is possible to formulate the following rule [90]; if determinant $C > 0$, then number of negative stability factors is even; if $C < 0$, then number of negative stability factors is odd.

Let us give certain theorems pertaining to gyroscopic stabilization.

In the beginning we will consider several theorems of Thomson and Tait [203, 170], which reveal the influence of dissipative forces (formulations of theorems are taken from different works, to which there are given references):

1) dissipative forces do not disturb stability;

2) if equilibrium is stable during the presence of potential forces, then it becomes asymptotically stable with the addition of dissipative forces with total dissipation;

3) equilibrium isolated and unstable in the presence of potential forces cannot be stabilized by dissipative forces.

One of the remarkable theorems of Thomson and Tait pertains to the possibility of stabilization of the position of equilibrium of a system by gyroscopic forces. This theorem states [203, 90]: if instability of isolated position of equilibrium of conservative system has odd degree, then stabilization of equilibrium by gyroscopic forces is impossible.

Sometimes this theorem is formulated in the following way [29]: gyroscopic stabilization is possible only for systems with even number of unstable degrees of

freedom.

This and other theorems were obtained by Thomson and Tait from analysis of first approximation of perturbation equations [170]. The given theorem has important practical applications, which will be discussed further in examining of concrete gyroscopic stabilizers.

Question about stability of system under action of only gyroscopic forces was investigated by D. R. Merkin [90]. He proved the following theorem: in order that position of equilibrium of a system on which there act only gyroscopic forces be stable, it is necessary and sufficient that determinant composed of gyroscopic coefficients not be equal to zero.

In order to explain what gyroscopic coefficients are, we will write expression of gyroscopic torques contained in differential equations of motion of gyroscope in the form [90] $Hg_{ji}q_i$; here H is angular momentum of gyroscope, q_i is generalized (controlling) angular velocity of axis of gyroscope, g_{ji} is gyroscopic coefficient. Thus, for instance, for equations (1.6.1) gyroscopic coefficients will be cos β.

From the above theorem it follows that if on system there act only gyroscopic forces and it has odd number of controlling coordinates, then any position of the system is always unstable.

As an example in which there can take place such motion, it is possible to give the two degree-of-freedom astatic gyroscope which is applied in the gyrotachometer (Chapter 8), if in it there are disconnected spring and damper.

It is necessary to note that the last theorem has important value during resolution of question about the possibility of application, during investigation of motions of a gyroscope, of the so-called simplified (shortened) equations of its motion (§ 1.6, Par. 8).

D. R. Merkin [90] proved also the following theorem of possibility of gyroscopic stabilization of unstable equilibrium of a conservative system: if to an unstable conservative system there are applied gyroscopic forces satisfying the following conditions: 1) the determinant composed of gyroscopic coefficients is not equal to zero; 2) simplified system is stable; 3) of the frequencies of the simplified system there are none which are equal, then for sufficiently large H unstable equilibrium of conservative system will be stabilized by gyroscopic forces.

In real systems, besides gyroscopic forces there will act other dissipative forces. In connection with this we will give corresponding theorems of Thomson and Tait [203, 90, 170] about stability of systems during action on them of gyroscopic and dissipative forces.

1. If motion of conservative system near position of equilibrium is stable, then upon addition of arbitrary gyroscopic and dissipative forces (the latter are not necessarily of total dissipation) stability of equilibrium is conserved.

2. If equilibrium is stable for some potential forces, then it becomes asymptotically stable with addition of arbitrary gyroscopic forces and dissipative forces with total dissipation.

3. Equilibrium which is unstable under action of conservative forces remains unstable after addition of any gyroscopic forces and dissipative forces if the latter possess total dissipation.

From the last theorem it follows [90] that if conservative system is unstable without gyroscopic forces, then in the presence of dissipative forces with total dissipation it is not possible to stabilize it by any gyroscopic forces. If total dissipation does not occur, then in certain cases such stabilization can be carried out.

The mentioned fact is formulated by the following theorem [170].

4. Under certain conditions, equilibrium which is unstable under action only of potential forces can be fixed or stabilized by addition of suitable gyroscopic forces, if degree of instability is not odd and there are not added dissipative forces possessing total dissipation.

The question of the possibility of gyroscopic stabilization in the presence of dissipative forces (damping) is also dealt with by the following theorem [203, 29].

5. If in the system there is no damping, then stabilization of even number of degrees of freedom always can be attained with the help of sufficiently powerful gyroscope; on the contrary, if in the system there are degrees of freedom with damping, then for possibility of gyroscopic stabilization the system should have also artificially disturbed degrees of freedom.

Thus, if of two degrees of freedom one experiences damping, then for possibility of gyroscopic stabilization the second degree of freedom should receive artificial perturbation. This theorem found practical application during realization of gyroscopic stabilization of railroad car of monorail railroad [30].*

As was shown in Theorem of Par. 3, unstable conservative system in the presence of dissipative forces with total dissipation is impossible to stabilize any gyroscopic forces. Under actual conditions there always exist small dissipative

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*See also: Ye. L. Nikolai. Theoretical mechanics, Part III, State United Scientific and Technical Press NKTP USSR, 1939.

forces with total dissipation, as a consequence of which gyroscopic stabilization of unstable equilibrium of conservative system will be disturbed. Therefore, stability existing with only potential forces, Thomson and Tait [203] proposed to call secular, and stability obtained from gyroscopic stabilization – temporary.

To questions of gyrosxopic stabilization of nonconservative systems there are dedicated works of D. R. Merkin [90], I. I. Metelitsyn [95], and others. Not having the possibility to present theorems obtained in them. we will be limited only to certain remarks. In these works, besides conservative, gyroscopic and dissipative forces, there are considered accelerating forces [90], the character of action of which is opposite to dissipative forces, and forces of radial correction. Thus, in them there are considered systems of more complicated structure, and different forces acting under real conditions.

If the system satisfies the earlier enumerated conditions [1) the determinant, composed of gyroscopic coefficients is not equal to zero; 2) simplified system is stable; 3) of the frequencies of the simplified system there are none which are equal], then for sufficiently large H frequencies of the system are divided; half of them become very small (on the order of H^{-1}), and the second half become very large (on the order of H). The first group of frequencies approximately coincides with frequencies of simplified system to which there correspond slow precessional oscillations, but the second group of frequencies to which there correspond fast nutational oscillations, coincides approximately with frequencies of the system, as if it moved under action of only gyroscopic forces (for system with gyroscopes – as if it moved by inertia) [90]. Let us note that each group of frequencies is calculated independently of the other.

Thus if condition of stability is satisfied and gyroscopic forces are dominating, then to oscillations with higher frequencies there corresponds more intense damping than to oscillations with low frequencies.

With such a phenomenon we meet, for instance, in the gyroscopic damper of rolling of a ship (§ 10.3, Par. 2c) and in a gyropendulum (§ 11.4, Par. 3c).

The given theorem makes it possible in applications to be limited by study of slow precessional motions of a gyroscope due to the comparatively fast damping of its nutational oscillations.

Let us give one more theorem important for gyroscopic stabilization [90].

If force function of perturbed motion of conservative system has an isolated minimum (all stability factors are negative), then:

a) for odd number of controlling coordinates and any nonlinear terms, the system

cannot be stabilized by any gyroscopic, radial-corrective, accelerating or dissipative forces;

b) for even number of controlling coordinates and under the condition that on the system there act forces of total dissipation, for stabilization it is necessary to simultaneously add gyroscopic and radial-corrective forces (independently of nonlinear terms).

In the example with the railroad car of the monorail railroad, unstable conservative system having in position of equilibrium minimum of force function is stabilized by addition of gyroscopic, dissipative and accelerating forces without forces of radial correction.

With theorems pertaining to the possibility of obtaining asymptotic stability of nonconservative systems, it is possible to become acquainted in work [90].

Above there were given certain theorems characterizing general principles of gyroscopic stabilization. Proofs of these theorems are contained in the above-indicated literature.

§ 10.3. Direct Gyroscopic Stabilizers

1. Preliminary Remarks

From all types of direct gyroscopic stabilizers we will consider only gyroscopic dampers of rolling of a ship, which can serve as an example of use of gyroscope as a direct stabilizer and which have received the greatest practical application. We will analyze fundamental designs of gyroscopic dampers, principles of their action, basic relationships and their behavior in dynamics.

Gyroscopic dampers, like other types of dampers of rolling of a ship, in principle of operation can be passive or active. As it is known [5],* passive dampers are set in action due to energy created by the rocking of the ship, and do not require additional expenditure of power. Work of active dampers is carried out from a separate source of power in the form of a special mechanism; it is connected with a certain expenditure of power and is usually controlled by an automatic control device. Dampers of rolling of active type, especially if we consider gyroscopic stabilizers, have a series of essential advantages over gyroscopic dampers of passive type. Nevertheless, we in the beginning will consider the devices of passive type in order to reveal their fundamental deficiencies and to compare them with dampers of active type.

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*See Also: A. N. Shmyrev, V. A. Morenshil'dt, S. G. Il'ina. Roll dampers of vessels. Sudpromgiz, 1961.

Creator of gyroscopic roll damper of active type is usually considered to be E. Sperry, who proposed such a gyroscopic stabilizer in 1911. In reality, priority in the matter of development of the design and theory of the gyroscopic roll damper belongs to A. N. Krylov. In connection with this, A. N. Krylov himself wrote [63]: "In 1909 there appeared the question about installation of a gyroscopic damper on the yacht "Strela." Then I developed the detailed theory* of this instrument and designs of it ... If the Naval ministry had appropriated 50,000 rubles for installation and test of the gyroscope-damper on the yacht "Strela," we would be in this matter ahead of Sperry."

2. Gyroscopic Passive-Type Roll Damper of Ship

a) Fundamental Design.

From the preceding paragraph it follows that the gyroscope can serve for energy dissipation of a system which oscillates near position of stable equilibrium, and damping in such a way its oscillations. This property is used in a gyroscopic roll damper of a ship, in which the gyroscope is applied for damping and subsequent dissipation of kinetic energy of oscillatory motion of the ship due to motion of the sea. Actually, gyroscopic torque appearing due to rolling of the ship (jointly with stabilizer) causes oscillations of gyroscope together with its frame with respect to the ship. Consequently, energy of waves rocking the ship will be converted by the damper into energy of oscillations of the frame of the gyroscope. If these oscillations are artificially braked, then there partially is dissipated energy of oscillatory motion of the ship which leads to constant damping of its oscillations, i.e., to calming of rolling.

Gyroscopic roll damper of passive type of a ship was proposed by O. Schlick [197] and subsequently was installed on several ships [63]. The fundamental diagram of the stabilizer is presented in Fig. 10.1. Its basic element is a rate gyroscope whose rotor is fixed in frame F, suspended in supports; the latter are placed in stands S rigidly joined with the ship. Axis of rotation of frame is parallel to transverse axis of ship. In the lower part of the frame there is factened a load Lo, due to which the frame obtains pendulum properties. For damping of oscillations of the frame relative to body of ship there serves a brake B, made in the form of a band brake or hydraulic damper.

For obtaining from the stabilizer of maximum effect, the axis of the gyroscope

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*With theory of gyroscopic roll damper of active type it is possible to become acquainted in the book of A. N. Krylov [65].

in position of equilibrium should be perpendicular to vector of angular velocity of rolling, i.e., to longitudinal axis of the ship. Therefore, axis of gyroscope can be located vertically (Fig. 10.1) or horizontally, parallel to the transverse axis of the ship. O. Schlick took the vertical location of axis of the gyroscope, for which for obtaining of restoring force of the frame there can be used the weight of load Lo. With such location of the axis, the gyroscope is not influenced by yawing of ship, and pitching is influenced only due to work of brake.

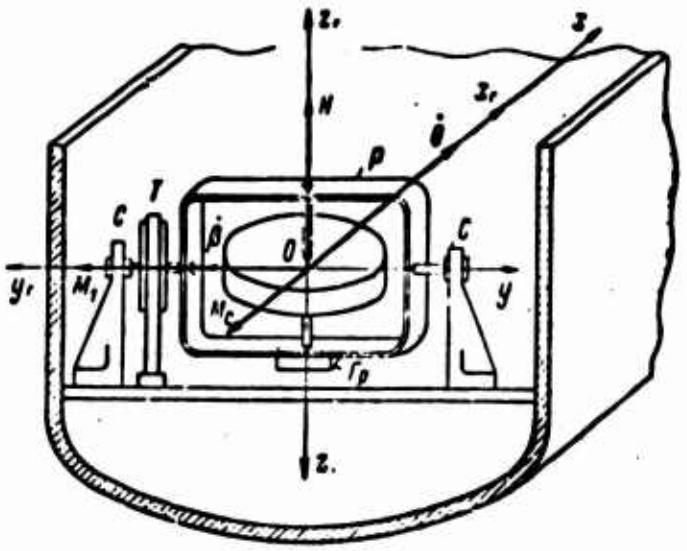

Fig. 10.1. Fundamental diagram of gyroscopic passive-type roll damper.

Brake absorbs that energy which supports the rocking of the ship and appears as a result of action of waves on the ship; furthermore, the brake limits swings of the frame. Indeed [110] in the absence of the brake oscillations of frame and ship occur with phase shift of 90°; between ship and gyroscope there is accomplished continuous exchange of energy without change of total sum of this energy, and damping of oscillations is not observed. With brake turned on, phase shift decreases, and in parallel with the exchange there occurs energy dissipation between gyroscope and ship, for instance in the case of a hydraulic damper — in the form of kinetic and thermal energy of the liquid used in braking.

Principle of action of the considered damper in broad terms consists of the following. During rolling of ship about its longitudinal axis Ox (Fig. 10.1) with angular velocity $\dot{\theta}$, there appears a gyroscopic torque approximately equal to $M_1 = H\dot{\theta}$ and directed along the axis Oy_g of rotation of the frame. Since velocity $\dot{\theta}$ has an oscillatory character, then gyroscopic torque M_1, which causes oscillatory motion of the frame around axis Oy_g, will have the same character. Let us designate angular velocity of rotation of frame around axis Oy_g by $\dot{\beta}$. During this motion of the frame with the gyroscope there appears also a gyroscopic torque which is approximately equal to $M_2 = M_s = H\dot{\beta}$ and directed along the axis Ox; this torque is transmitted through bearings of the body of the ship; we will call it subsequently stabilizing torque. Actually, for the direction of angular velocity $\dot{\theta}$ of rolling shown in Fig. 10.1, the ship rolls to the starboard, i.e., rolling torque caused by action of waves is directed in positive direction of axis Ox. The stabilizing

torque Oy_1 appearing during oscillatory of the frame around M_s is directed in negative direction of axis Ox, i.e., opposite to the rolling torque; this leads to damping of roll of the ship. With more detailed account of principle of action of gyroscopic passive-type roll damper it is possible to become acquainted in special literature (for instance, [110, 26]).

b) Dynamic Characteristics.

Equations of motion. We will form differential equations of motion of considered dynamic "ship + damper" system. Being limited to linearized equations usually applied during investigation of gyroscopic roll dampers, we use for their composition the method Kudrevich (§ 1.6, par. 2). For this we will construct an auxiliary drawing (Fig. 10.2), on which there are drawn the same coordinate axes in Fig. 10.1.

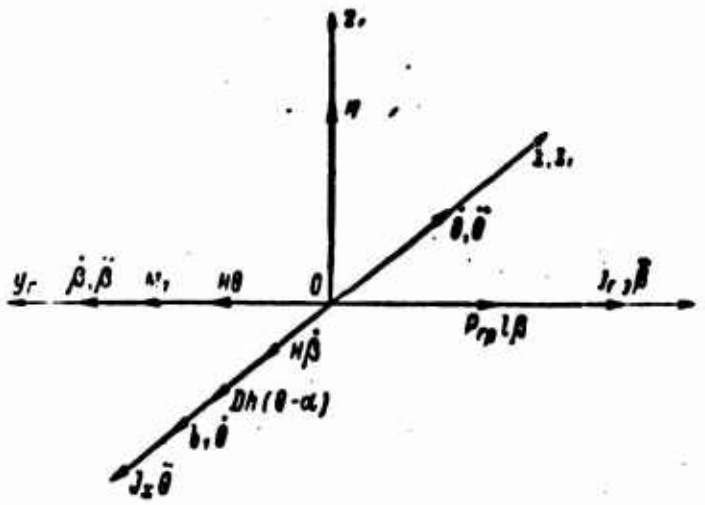

Fig. 10.2. For deviation of equations of motion of ship with roll damper.

Let us introduce the following designations: J_x – moment of inertia of mass of ship relative to its longitudinal axis Ox; θ – angle of roll of ship, taken as positive during roll of the latter to the starboard; α – effective angle of slope of wave; b_1 – drag coefficient of water pertaining to oscillations of the ship; D – weight of ship; h – metacentric height; $J_{g.e}$ – moment of inertia of gyroscope together with frame with relative to axis Oy_g of its rotation; β – angle of rotation of frame around axis Oy_g, taken as positive for slope of axis Oz_g of gyroscope is diametrical plane of ship toward the nose; H – angular momentum of gyroscope; P_{lo} – weight of load fastened to lower part of frame (Fig. 10.1); l – distance from center of gravity load to point O of suspension of gyroscope; M_b – braking torque of frame during its rotation around axis Oy_g caused by action of brake and friction in axis of rotation of frame.

Along longitudinal axis of ship Ox (Fig. 10.2) there are located vectors of torques: $J_x\ddot{\theta}$ – moment of forces of inertia of ship; $b_1\dot{\theta}$ – moment of drag forces of water pertaining to oscillations of the ship; $Dh \sin(\theta - \alpha) \approx (Dh(\theta - \alpha)$ – moment of forces of weight of ship ($Dh\theta$ – restoring torque or moment of stability; $Dh\alpha$ – moment of perturbing action of waves, or rolling moment); $H\dot{\beta}$ – gyroscopic torque. Relative to an axis Oy_g of rotation of frame there act: $J_{g.e}\ddot{\beta}$ – moment of forces of inertia of frame with gyroscope; $P_{lo}l \sin\beta \approx P_{lo}l\beta$ – static moment of load

fastened to frame; M_b – braking torque; $H\dot{\theta}$ – gyroscopic torque.

In accordance with D'Alembert's principle, sums of these torques with respect to axes Ox and Oy_g respectively have to be equal to zero; therefore, using Fig. 10.2, we find differential equation of rolling of ship about axis Ox

$$J_x\ddot{\theta} + b_1\dot{\theta} + Dh(\theta - \alpha) + H\dot{\beta} = 0 \tag{10.3.1}$$

and differential equation of oscillations of frame together with gyroscope about axis Oy_g

$$J_{r.}\ddot{\beta} + P_{rp}\beta - M_r - H\dot{\theta} = 0, \tag{10.3.2}$$

i.e., the system of equations characterizing oscillations of ship (supplied with a gyroscopic damper) and frame of damper,

$$\left.\begin{aligned} J_x\ddot{\theta} + b_1\dot{\theta} + Dh\theta + H\dot{\beta} &= Dh\alpha(t) \\ J_{r.}\ddot{\beta} + P_{rp}\beta - M_r - H\dot{\theta} &= 0 \end{aligned}\right\} \tag{10.3.3}$$

From these equations it follows that action of gyroscope on ship appears in the form of gyroscopic torque $H\dot{\beta}$ directed along longitudinal axis of ship and constituting stabilizing torque M_s.

Braking torque M_b can have different expressions depending upon design of brake. Subsequently we will consider, as this occurs in roll dampers of passive type, that braking forces act according to a definite law independently of the character of waves, and that their torque M_b is determined by the expression

$$M_r = -b_2\dot{\beta}, \tag{10.3.4}$$

where b_2 – braking force coefficient of oscillations of frame around axis Oy_g.

Placing (10.3.4) in (10.3.3), we will obtain

$$\left.\begin{aligned} J_x\ddot{\theta} + b_1\dot{\theta} + Dh\theta + H\dot{\beta} &= Dh\alpha(t) \\ J_{r.}\ddot{\beta} + b_2\dot{\beta} + P_{rp}\beta - H\dot{\theta} &= 0 \end{aligned}\right\} \tag{10.3.5}$$

Transfer Functions. For investigation of behavior of ship (supplied with damper) in transient response, and also for determination of forced oscillations of ship and frame with gyroscope on sea swells, it is necessary to know their transfer functions with respect to disturbance of waves $\alpha(t)$, which subsequently we will designate by $f(t)$. Let us divide first equation of system (10.3.5) by J_x, and the

second equation by $J_{g.e}$. Then, replacing $\alpha(t)$ by $f(t)$, we will rewrite (10.3.5) in the form

$$\left.\begin{aligned}\ddot{\theta}+\frac{b_1}{J_x}\dot{\theta}+\frac{Dh}{J_x}\theta+\frac{H}{J_x}\dot{\beta}&=\frac{Dh}{J_x}f(t)\\ \ddot{\beta}+\frac{b_2}{J_{g.e}}\dot{\beta}+\frac{P_{rp}l}{J_{g.e}}\beta-\frac{H}{J_{g.e}}\dot{\theta}&=0\end{aligned}\right\}\qquad(10.3.6)$$

Let us designate

$$\left.\begin{aligned}n_1^2&=\frac{Dh}{J_x};\quad n_2^2=\frac{P_{rp}l}{J_{g.e}}\\ k_1'&=\frac{H}{J_x};\quad k_2'=\frac{H}{J_{g.e}}\end{aligned}\right\}\qquad(10.3.7)$$

We transform ratio $\frac{b_1}{J_x}$ in the following way:

$$\frac{b_1}{J_x}=\frac{b_1}{J_x}\frac{\sqrt{\frac{Dh}{J_x}}}{\sqrt{\frac{Dh}{J_x}}}=2\zeta_1 n_1,\qquad(10.3.8)$$

$$\zeta_1=\frac{b_1}{2\sqrt{DhJ_x}}.\qquad(10.3.9)$$

Analogously for $\frac{b_2}{J_{g.e}}$ we have

$$\frac{b_2}{J_{g.e}}=\frac{b_2}{J_{g.e}}\frac{\sqrt{\frac{P_{rp}l}{J_{g.e}}}}{\sqrt{\frac{P_{rp}l}{J_{g.e}}}}=2\zeta_2 n_2,\qquad(10.3.10)$$

where

$$\zeta_2=\frac{b_2}{2\sqrt{P_{rp}lJ_{g.e}}}.\qquad(10.3.11)$$

Considering designations of (10.3.7), (10.3.8) and (10.3.10), we will rewrite equation (10.3.6) in the form

$$\left.\begin{aligned}\ddot{\theta}+2\zeta_1 n_1\dot{\theta}+n_1^2\theta+k_1'\dot{\beta}&=n_1^2 f(t)\\ \ddot{\beta}+2\zeta_2 n_2\dot{\beta}+n_2^2\beta-k_2'\dot{\theta}&=0\end{aligned}\right\}\qquad(10.3.12)$$

From equations (10.3.12) it follows that n_1 constitutes frequency of natural sustained oscillations of ship with damper turned off, i.e., with non-rotating rotor

($H = 0$; $k_1' = 0$):

$$n_1 = \sqrt{\frac{Dh}{J_1}}. \tag{10.3.13}$$

Then for period of the indicated oscillations of the ship we have

$$T_1 = \frac{2\pi}{n_1} = 2\pi\sqrt{\frac{J_1}{Dh}}. \tag{10.3.14}$$

Analogousy n_2 is frequency of natural sustained oscillations of frame with gyroscope with non-rotating rotor ($H = 0$; $k_2' = 0$)*

$$n_2 = \sqrt{\frac{P_{rp}l}{J_{r.2}}}. \tag{10.3.15}$$

Then for period of indicated oscillations of frame we have

$$T_2 = \frac{2\pi}{n_2} = 2\pi\sqrt{\frac{J_{r.2}}{P_{rp}l}}. \tag{10.3.16}$$

Quantities ζ_1 and ζ_2 are relative attenuation factors of oscillations of ship and frame of damper (where ζ_2 constitutes braking coefficient).

Let us write equations (10.3.12) in different form; for this we introduce, by analogy with (4.3.23), time constant T_1 of ship and time constant T_2 damper:

$$T_1 = \frac{1}{n_1}; \quad T_2 = \frac{1}{n_2}. \tag{10.3.17}$$

Let us divide first equation of system (10.3.12) by n_1^2, and second equation by n_2^2 and designate

$$k_1 = \frac{k_1'}{n_1^2}; \quad k_2 = \frac{k_2'}{n_2^2}. \tag{10.3.18}$$

Then, considering (10.3.17) and (10.3.18), in place of (10.3.12) we obtain

$$\left.\begin{aligned} T_1^2\ddot{\theta} + 2\zeta_1 T_1\dot{\theta} + \theta + k_1\beta &= f(t) \\ T_2^2\ddot{\beta} + 2\zeta_2 T_2\dot{\beta} + \beta - k_2\theta &= 0 \end{aligned}\right\} \tag{10.3.19}$$

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*Quantity can also be treated as frequency of natural sustained oscillations of frame with rotor rotating ($H \neq 0$), but for $\theta = 0$.

Let us rewrite equations (10.3.19) in operator form ($p = \frac{d}{dt}$):

$$\left.\begin{aligned}\left(T_1^2p^2 + 2\zeta_1T_1p + 1\right)\theta + k_1p\beta = f(t)\\ -k_2p\theta + \left(T_2^2p^2 + 2\zeta_2T_2p + 1\right)\beta = 0\end{aligned}\right\} \qquad (10.3.20)$$

For elimination of variables θ and β we will find principal determinant Δ of system (10.3.20) and determinants Δ_θ and Δ_β, obtained from Δ by means of replacement of corresponding columns by the column of right sides of equation (10.3.20). We have

$$\Delta = \begin{vmatrix} T_1^2p^2 + 2\zeta_1T_1p + 1 & k_1p \\ -k_2p & T_2^2p^2 + 2\zeta_2T_2p + 1 \end{vmatrix} =$$
$$= \left(T_1^2p^2 + 2\zeta_1T_1p + 1\right)\left(T_2^2p^2 + 2\zeta_2T_2p + 1\right) + k_1k_2p^2; \qquad (10.3.21)$$

$$\Delta_\theta = \begin{vmatrix} f(t) & k_1p \\ 0 & T_2^2p^2 + 2\zeta_2T_2p + 1 \end{vmatrix} =$$
$$= \left(T_2^2p^2 + 2\zeta_2T_2p + 1\right)f(t); \qquad (10.3.22)$$

$$\Delta_\beta = \begin{vmatrix} T_1^2p^2 + 2\zeta_1T_1p + 1 & f(t) \\ -k_2p & 0 \end{vmatrix} = k_2pf(t). \qquad (10.3.23)$$

Since $\Delta\cdot\theta = \Delta_\theta$, $\Delta\cdot\beta = \Delta_\beta$, then we obtain following differential equations for θ and β in operator form:

$$\left.\begin{aligned}\left[\left(T_1^2p^2 + 2\zeta_1T_1p + 1\right)\left(T_2^2p^2 + 2\zeta_2T_2p + 1\right) + k_1k_2p^2\right]\theta =\\ = \left(T_2^2p^2 + 2\zeta_2T_2p + 1\right)f(t)\\ \left[\left(T_1^2p^2 + 2\zeta_1T_1p + 1\right)\left(T_2^2p^2 + 2\zeta_2T_2p + 1\right) +\\ + k_1k_2p^2\right]\beta = k_2pf(t)\end{aligned}\right\} \qquad (10.3.24)$$

Applying to (10.3.24) Laplace transform, we find expressions for transfer functions

$$\Phi_{\theta,f}(s) = \Phi_1(s) = \frac{\theta(s)}{f(s)}, \quad \Phi_{\beta,f}(s) = \Phi_2(s) = \frac{\beta(s)}{f(s)} \qquad (10.3.25)$$

in the form

$$\Phi_1(s) = \frac{T_2^2s^2 + 2\zeta_2T_2s + 1}{\left(T_1^2s^2 + 2\zeta_1T_1s + 1\right)\left(T_2^2s^2 + 2\zeta_2T_2s + 1\right) - k_1k_2s^2}; \qquad (10.3.26)$$

$$\Phi_2(s) = \frac{k_2 s}{\left(T_1^2 s^2 + 2\zeta_1 T_1 s + 1\right)\left(T_2^2 s^2 + 2\zeta_2 T_2 s + 1\right) + k_1 k_2 s^2}. \qquad (10.3.27)$$

Above there were obtained differential equations of motion of ship supplied with gyroscopic roll damper of passive type. Let us give differential equation and transfer function of ship without damper. In this case, according to (10.3.19), we have equation of rolling of ship in the form

$$T_1^2 \ddot{\vartheta} + 2\zeta_1 T_1 \dot{\vartheta} + \vartheta = f(t), \qquad (10.3.28)$$

whence for transfer function $\Phi'_{\vartheta \cdot f}(s) = \Phi'_1(s)$ we find expression

$$\Phi'_1(s) = \frac{1}{T_1^2 s^2 + 2\zeta_1 T_1 s + 1}, \qquad (10.3.29)$$

which can be obtained from general formula (10.3.26) if in it we put parameters of damper $T_2 = 0$, $\zeta_2 = 0$, $k_1 = k_2 = 0$.

c) Transient Responses.

Determination of transient responses of "ship + damper" system is a very important problem of the theory of a roll damper and in calculation of its basic parameters. Of the greatest interest is analysis of stability of system and natural oscillations of ship with damper. Let us dwell briefly on these questions.

First of all it is necessary to find characteristic equation of system. It is easy to obtain (§ 7.3, Par. 2), if we know expression of transfer functions $\Phi_1(s)$ or $\Phi_2(s)$; moreover, in both cases this equation is obtained to be the same. If $\Phi_1(s)$ is written, for instance, in the form of (7.3.2), then, by analogy with (7.3.5), characteristic equation of considered system will be

$$N_{11}(\lambda) = 0, \qquad (10.3.30)$$

i.e., for finding of this equation it is necessary to replace in (10.3.26) s by λ and to equate denominator of transfer function $\Phi_1(s)$ to zero. Then we will write characteristic equation in the form

$$\left(T_1^2 \lambda^2 + 2\zeta_1 T_1 \lambda + 1\right)\left(T_2^2 \lambda^2 + 2\zeta_2 T_2 \lambda + 1\right) + k_1 k_2 \lambda^2 = 0. \qquad (10.3.31)$$

After simple transformations we will obtain

$$a_4 \lambda^4 + a_3 \lambda^3 + a_2 \lambda^2 + a_1 \lambda + a_0 = 0, \qquad (10.3.32)$$

where

$$\left.\begin{aligned} a_0 &= 1 \\ a_1 &= 2(\zeta_1 T_1 + \zeta_2 T_2) \\ a_2 &= T_1^2 + T_2^2 + 4\zeta_1\zeta_2 T_1 T_2 + k_1 k_2 \\ a_3 &= 2T_1 T_2(\zeta_1 T_2 + \zeta_2 T_1) \\ a_4 &= T_1^2 T_2^2 \end{aligned}\right\} \qquad (10.3.33)$$

According to the Hurwitz criterion, in case of characteristic equation of fourth degree, necessary and sufficient conditions of stability are the following:

$$a_4 > 0;\quad a_3 > 0;\quad a_2 > 0;\quad a_1 > 0;\quad a_0 > 0;$$
$$\Delta_3 = \begin{vmatrix} a_3 & a_1 & 0 \\ a_4 & a_2 & a_0 \\ 0 & a_3 & a_1 \end{vmatrix} = a_1(a_2 a_3 - a_4 a_1) - a_0 a_3^2 > 0. \qquad (10.3.34)$$

In accordance with (10.3.33), positivity of coefficients a_0 ... a_4 is fulfilled. Let us note only the following. If equation (10.3.32) is divided by a_4, then last term of this equation will be $\frac{a_0}{a_1}$. Taking into account (10.3.33), (10.3.17) and (10.3.7), we have $\frac{a_0}{a_1} = \frac{1}{a_4} = \frac{1}{T_4^2 T_2^2} = n_1^2 n_2^2 = \frac{D}{J_x} \cdot \frac{P_{lo}}{J_{g.e}} hl$. Product hl should be positive, i.e., h and l have to have the same sign. Consequently, load Lo (Fig. 10.1) must be installed lower than point O of gyroscope suspension; this condition is satisfied in design of damper. If frame is made to be balanced ($l = 0$), then $\frac{a_0}{a_4} = 0$ and, according to (10.3.32), one root of characteristic equation will be zero; in other words, position of equilibrium of frame is arbitrary, and such a stabilizer will damp rocking of ship not relative to its straight position ($\theta = 0$), but relative to angle of its slope θ_0 at the moment of turning on the stabilizer.

According to second condition (10.3.34), considering (10.3.33), we have

$$2(\zeta_1 T_1 + \zeta_2 T_2)\left[\left(T_1^2 + T_2^2 + 4\zeta_1\zeta_2 T_1 T_2 + k_1 k_2\right) 2T_1 T_2(\zeta_1 T_2 + \zeta_2 T_1) - \right.$$
$$\left. - 2T_1^2 T_2^2(\zeta_1 T_1 + \zeta_2 T_2)\right] - 4T_1^2 T_2^2(\zeta_1 T_2 + \zeta_2 T_1)^2 > 0. \qquad (10.3.35)$$

Let us note that during development of the damper, it is attempted to make natural frequencies n_1 of oscillations of ship and n_2 of oscillations of frame with gyroscope be equal, since in this case resonance between frequency of waves and frequency of rolling of ship becomes simultaneously resonance between frequency of waves and frequency of oscillations of frame with gyroscope. This promotes enhancement of the effect of damping of rolling [30]. This is all the more expedient

for conditions of irregular roughness of sea, when predominant frequency of rolling oscillations of ship is its natural frequency n_1. Considering (10.3.17), for $n_1 = n_2$, time constants T_1 of ship and T_2 of roll damper are also equal:

$$T_1 = T_2 = T. \tag{10.3.36}$$

Putting (10.3.36) in (10.3.35), we have condition

$$4\zeta_1\zeta_2 T^2 + k_1 k_2 > 0, \tag{10.3.37}$$

which is always satisfied, since ζ_1, ζ_2, k_1 and k_2 are positive values.

It is easy to show that gyroscope would begin to damp natural oscillations of ship even in the case when resistance of water to oscillations of ship is absent, i.e., for $\zeta_1 = 0$. Actually, from (10.3.33) it is clear that for $\zeta_1 = 0$ coefficients of characteristic equation are all positive, and inequality (10.3.37) is satisfied. Consequently, gyroscope will damp not only forced, but also natural oscillations of ship.

Analysis of transient response and basic characteristics of quality of the dynamic "ship + damper" system can be carried out by methods known from theory of automatic control, in particular with help of the frequency method of analysis of quality applied during investigation of transient response of active GS (§ 7.3, par. 3), which is based on use of real frequency-response curve of the system.

Let us note certain general properties of natural oscillations of ship with damper. We use results of analysis of these oscillations obtained in the literature (see, for instance, [30, 20, 146, 110]).

Characteristic equation of system is an equation of fourth degree; using known relationships* between roots and coefficients of characteristic equation, it is possible approximately to determine roots, and consequently to find law of free oscillations of system. Such a problem of analysis of natural oscillations of ship supplied with gyroscopic roll damper of passive type is considered in a concrete

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*If λ_1, λ_2, λ_3, λ_4 are roots of equation of fourth degree

$$x^4 + ax^3 + bx^2 + cx + d = 0,$$

then the mentioned relationships are expressed by equalities [146]

$$\left.\begin{aligned} a &= -(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4) \\ b &= \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_1\lambda_4 + \lambda_2\lambda_3 + \lambda_2\lambda_4 + \lambda_3\lambda_4 \\ c &= -(\lambda_1\lambda_2\lambda_3 + \lambda_1\lambda_2\lambda_4 + \lambda_1\lambda_3\lambda_4 + \lambda_2\lambda_3\lambda_4) \\ d &= \lambda_1\lambda_2\lambda_3\lambda_4 \end{aligned}\right\}$$

numerical example [146], where it is shown that these oscillations have an attenuating character. Presence of gyroscope increases period of natural oscillations of ship and causes more intense decrease of amplitude of oscillations.

This same problem was considered by B. V. Bulgakov [20]. According to (10.3.7), we designate $k' = \sqrt{k_1' k_2'} = \frac{H}{\sqrt{J_x J_{g.e}}}$. If $k' = 0$, which can take place only at $H = 0$ (i.e., when damper does not work), then $k_1' = k_2' = 0$ and characteristic equations corresponding to equations (10.3.12) or (10.3.19) break up into two quadratic equations, which characterize independent oscillations of ship and frame with gyroscope. If $k' = \infty$, which corresponds to a gyroscope with very large angular momentum H, then oscillations of system are separated into slow (precessional) and fast (nutational). On the basis of investigation of this case it is shown [20] that for obtaining of the strongest possible damping of natural oscillations of the ship, it is advantageous to increase braking coefficient ζ_2 of frame of damper, whereas unlimited increase of angular momentum H of gyroscope, even if it were possible, is not at all useful. Let us note that quantity $k' = \frac{H}{\sqrt{J_x J_{g.e}}}$ for sufficiently large H is close to frequency of fast nutational oscillations of gyroscopic system, and quantity $\frac{P_{lo}^l}{H}$ is frequency of its slow precessional oscillations.

Analysis of natural oscillations of ship supplied with gyroscopic roll damper of passive type, under the assumption that it is possible to disregard resistance of water to oscillations of ship, and for equality of frequencies n_1 and n_2, given also in the book of R. Grammel [30]. Under these assumptions there are selected optimum values of angular momentum of gyroscope and braking coefficient of oscillations of frame of damper for the purpose of obtaining of aperiodic damping of natural oscillations of ship.

d) Dynamics of Ship with Roll Damper.

During determination of efficiency of roll damper, most important is calculation of forced oscillations of ship supplied with damper. Such calculations usually were produced for conditions of regular sea swells (see, for instance, [30, 20, 146, 65, 110]), when disturbance $f(t) - \alpha(t)$ due to sea swells is a harmonic function of time. For determination of forced oscillations of ship under these conditions, we will use first equation (10.3.24), which we will write in the form

$$(a_4 p^4 + a_3 p^3 + a_2 p^2 + a_1 p + a_0)\eta = (b_2 p^2 + b_1 p + b_0) f(t). \qquad (10.3.38)$$

where for coefficients $a_0, \ldots a_4, b_0, \ldots b_2$, considering (10.3.38), we have

$$\left.\begin{aligned} a_0 &= 1; & b_0 &= 1 \\ a_1 &= 2(\zeta_1 T_1 + \zeta_2 T_2); & b_1 &= 2\zeta_2 T_2 \\ a_2 &= T_1^2 + T_2^2 + 4\zeta_1\zeta_2 T_1 T_2 + k_1 k_2; & b_2 &= T_2^2 \\ a_3 &= 2T_1 T_2(\zeta_1 T_2 + \zeta_2 T_1); \\ a_4 &= T_1^2 T_2^2; \end{aligned}\right\} \qquad (10.3.39)$$

Disturbance f(t) we will represent in the form

$$f(t) = \alpha_0 \sin \omega t, \qquad (10.3.40)$$

where α_0 – the biggest angle of wave slope;

ω – frequency of wave.

By analogy with (8.2.126), forced oscillations of ship $\theta_{for}(t)$ are determined by relationship

$$\theta_0(t) = \alpha_0 A(\omega) \sin[\omega t + \varphi(\omega)], \qquad (10.3.41)$$

where A(ω) and φ(ω) – amplitude and phase frequency-response curves of "ship + damper" system.

For determination of A(ω) and φ(ω) we will use general method presented in § 3.2, Par. 3d. Comparing (10.3.38) with (3.2.36), for a(ω) ... d(ω), considering (3.2.38), we will obtain

$$\left.\begin{aligned} a(\omega) &= b_0 - b_2\omega^2; & b(\omega) &= b_1\omega \\ c(\omega) &= a_0 - a_2\omega^2 + a_4\omega^4; & d(\omega) &= a_1\omega - a_3\omega^3 \end{aligned}\right\} \qquad (10.3.42)$$

Putting (10.3.42) in (3.2.41), we find

$$A(\omega) = \sqrt{\frac{(b_0 - b_2\omega^2)^2 + b_1^2\omega^2}{(a_0 - a_2\omega^2 + a_4\omega^4)^2 + (a_1\omega - a_3\omega^3)^2}}; \qquad (10.3.43)$$

$$\varphi(\omega) = \operatorname{arctg} \frac{b_1\omega(a_0 - a_2\omega^2 + a_4\omega^4) - (b_0 - b_2\omega^2)(a_1\omega - a_3\omega^3)}{(b_0 - b_2\omega^2)(a_0 - a_2\omega^2 + a_4\omega^4) + b_1\omega(a_1\omega - a_3\omega^3)}. \qquad (10.3.44)$$

Since we will be interested only in amplitudes of forced oscillations of ship, then, according to (10.3.41), we will obtain

$$\theta_0^0 = \alpha_0 A(\omega). \qquad (10.3.45)$$

For determination of final expression for A(ω), we substitute (10.3.39) in (10.3.43); then

$$A(\omega)=\sqrt{\frac{\left(1-T_2^2\omega^2\right)^2+4\zeta_2^2T_2^2\omega^2}{\left[1-\left(T_1^2+T_2^2+4\zeta_1\zeta_2T_1T_2+k_1k_2\right)\omega^2+T_1^2T_2^2\omega^4\right]^2+}\cdots>}$$
$$<\cdots\frac{}{+\left[2\left(\zeta_1T_1+\zeta_2T_2\right)\omega-2T_1T_2\left(\zeta_1T_2+\zeta_2T_1\right)\omega^3\right]^2}. \qquad (10.3.46)$$

Thus, analysis of amplitudes of forced oscillations of ship is reduced to investigation of expression of amplitude-frequency characteristic $A(\omega)$ depending upon parameters of ship and damper. Usually during determination of forced oscillations of ship with damper, we are limited to consideration of a simpler case [30]; namely, we do not consider resistance of water to oscillations of ship ($\zeta_1 = 0$) and consider time constants of ship and damper to be identical ($T_1 = T_2 = T$); then (10.3.46) can be rewritten in the form

$$A(\omega)=\sqrt{\frac{\left(1-T^2\omega^2\right)^2+4\zeta_2^2T^2\omega^2}{\left|1-\left(2T^2+k_1k_2\right)\omega^2+T^4\omega^4\right|^2+\left|2\zeta_2T\omega\left(1-T^2\omega^2\right)\right|^2}}. \qquad (10.3.47)$$

Putting (10.3.47) in (10.3.45), we will obtain

$$\theta_0^0=\sqrt{\frac{\left(1-T^2\omega^2\right)^2+4\zeta_2^2T^2\omega^2}{\left|1-\left(2T^2+k_1k_2\right)\omega^2+T^4\omega^4\right|^2+\left|2\zeta_2T\omega\left(1-T^2\omega^2\right)\right|^2}}\alpha_0. \qquad (10.3.48)$$

With help of formulas to type (10.3.48), in Fig. 10.3 there are constructed [110] curves of amplitudes of forced oscillations of ship supplied with damper.

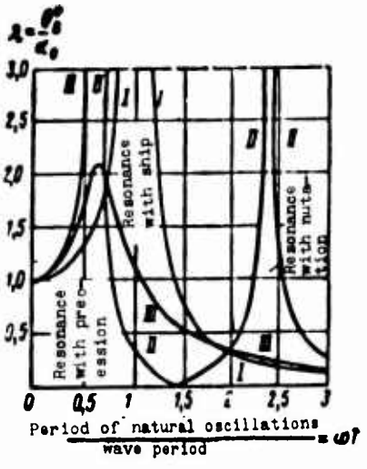

Fig. 10.3. Curves of relative amplitudes of oscillations of ship with damper.

Along the axis of ordinates there is plotted ratio of amplitude θ_{for}^0 of oscillations of ship to its static deviation α_0, i.e., $\frac{\theta_{for}^0}{\alpha_0}$, and along the axis of abscissas – ratio of period of natural oscillations of ship to wave period, or, which is the same, quantity $\frac{\omega}{n}=\omega T$. Let us note that ratio

$$\lambda=\frac{\theta_0^0}{\alpha_0} \qquad (10.3.49)$$

constitutes dynamicity coefficient system (§ 8.2, Par. 6b), which in considered case is equal to $A(\omega)$, i.e., $\lambda(\omega) = A(\omega)$.

In Fig. 10.3 it is shown that for turned off damper ($H = 0$), rolling of ship

occurs according to curve I; however, in reality amplitude of rolling does not attain infinitely large values due to presence of resistance of water to oscillations of ship, which in (10.3.48) was not considered by us. In this case ($k_1 = k_2 = 0$, $\zeta_2 = 0$); according to (10.3.48), dynamicity coefficient will be determined by relationship

$$\lambda = \frac{\theta_0''}{\varepsilon} = \frac{1}{1 - T^2\omega^2}. \qquad (10.3.50)$$

which is represented by curve I in Fig. 10.3.

When damper works without braking ($\zeta_2 = 0$), rolling of ship is determined by curve II. Here there exist two positions of resonance, of which one corresponds to coincidence of wave period with period precession of gyroscope, and the other with period of nutation. It is clear that application of damper without braking is harmful, since there increases probability of appearance of large amplitudes of rolling. In the presence of braking of frame of damper, oscillations of ship follow curve III, which shows that amplitude of rolling under these conditions nowhere attains large magnitude. The most favorable is section between intersection points of curves I and II. Outside this section amplitude of rolling of ship even somewhat is increased. Damper works especially efficiently in the case when wave period is close to period of natural oscillations of ship.

Let us note that graphs similar to the given ones (see, for instance, [30,20]) can be useful for selection of optimum values of angular momentum of gyroscope and braking coefficient of frame. In particular, here increase of angular momentum of gyroscope is expedient. Principal disadvantages of gyroscopic roll damper of passive type, which are revealed already from analysis of curves in Fig. 10.3, will be discussed further on.

Determination of amplitudes of forced oscillations of ship (supplied with damper) on regular sea swells permits us to reveal only certain fundamental peculiarities of the damper. Effectiveness of its action should be calculated for conditions of the irregular roughness of sea, which exists in reality.

Let us give certain general considerations about method of calculations. For carrying out the latter it is necessary first of all to know probability characteristics of disturbance $f(t) = \alpha(t)$ caused by the waves. Methods of their determination for the general case of a three-dimensional irregular agitation are shown by A. A. Sveshnikov [140, 139]. For tentative calculations relative to two-dimensional irregular agitation it is necessary to know spectral density $S_\alpha(\omega)$ of random function

$\alpha(t)$ of angles of wave slope, and thus to take into account effective angle of wave slope with consideration corresponding reduction coefficients [110].

As the dynamic characteristic of a ship with roll damper there serves the amplitude-phase characteristic $\Phi_1(j\omega)$, which is obtained by means of replacement in expression (10.3.26) of s and $j\omega$:

$$\Phi_1(j\omega) = \frac{1 - T_2^2\omega^2 + 2\zeta_2 T_2 j\omega}{\left(1 - T_1^2\omega^2 + 2\zeta_1 T_1 j\omega\right)\left(1 - T_2^2\omega^2 + 2\zeta_2 T_2 j\omega\right) - k_1 k_2 \omega^2}. \qquad (10.3.51)$$

Spectral density $S_\theta(\omega)$ of angles of rolling of ship with damper is determined, by analogy with (4.6.47) by formula

$$S_\theta(\omega) = |\Phi_1(j\omega)|^2 S_\alpha(\omega) \qquad (10.3.52)$$

or, considering (10.3.51),

$$S_\theta(\omega) = \left|\frac{1 - T_2^2\omega^2 + 2\zeta_2 T_2 j\omega}{\left(1 - T_1^2\omega^2 + 2\zeta_1 T_1 j\omega\right)\left(1 - T_2^2\omega^2 + 2\zeta_2 T_2 j\omega\right) - k_1 k_2 \omega^2}\right|^2 S_\alpha(\omega). \qquad (10.3.53)$$

For dispersion D $[\theta]$ and root-mean-square σ_θ value of angles of list of ship we obtain the following expressions:

$$D[\theta] = 2\int_0^\infty S_\theta(\omega)\,d\omega =$$

$$= 2\int_0^\infty \left|\frac{1 - T_2^2\omega^2 + 2\zeta_2 T_2 j\omega}{\left(1 - T_1^2\omega^2 + 2\zeta_1 T_1 j\omega\right)\left(1 - T_2^2\omega^2 + 2\zeta_2 T_2 j\omega\right) - k_1 k_2 \omega^2}\right|^2 S_\alpha(\omega)\,d\omega. \qquad (10.3.54)$$

$$\sigma_\theta = \sqrt{D[\theta]}. \qquad (10.3.55)$$

Thus, knowing parameters of ship T_1, ζ_1, parameters of gyroscope damper T_2, ζ_2, k_1, k_2, and also spectral density $S_\alpha(\omega)$ of random function $\alpha(t)$, we can determine D $[\theta]$ and σ_θ.

Since angles of roll of ship during operation of damper also follow a normal law, then it is very simple to determine mean value E_θ of angles of roll, and also limiting (in the probability sense) value of residual angle of roll of the ship, or to solve the problem of "overshoot" of angles of roll beyond some given value (§ 5.7, ar. 2).

In order to reveal effectiveness of action of roll damper on irregular sea swells, it is also necessary to find probability characteristics of rolling of ship

with turned off damper. For this we will use equation of rolling of ship (10.3.28) and its transfer function (10.3.29). Designating angle of roll of ship without damper by θ', by analogy with (10.3.52) we find expression for spectral density $S_{\theta'}(\omega)$ of angles of roll of ship

$$S_{\theta'}(\omega) = |\Phi_1^*(j\omega)|^2 S_*(\omega). \tag{10.3.56}$$

whence, considering (10.3.29), we have

$$S_{\theta'}(\omega) = \left|\frac{1}{1 - T_1^2\omega^2 + 2\zeta_1 T_1 j\omega}\right|^2 S_*(\omega). \tag{10.3.57}$$

Then for dispersion $D[\theta']$ and root-mean-square value $\sigma_{\theta'}$ of angles of roll of ship, we will obtain following expressions:

$$D[\theta'] = 2\int_0^\infty S_{\theta'}(\omega)\,d\omega = 2\int_0^\infty \left|\frac{1}{1 - T_1^2\omega^2 + 2\zeta_1 T_1 j\omega}\right|^2 S_*(\omega)\,d\omega; \tag{10.3.58}$$

$$\sigma_{\theta'} = \sqrt{D[\theta']}. \tag{10.3.59}$$

Effectiveness of roll damper of ship on irregular sea swells can be characterized by a certain coefficient γ, which is the ratio of root-mean-square value of angle of roll of ship without damper to root-mean-square value of angle of roll of ship with damper:

$$\gamma = \frac{\sigma_{\theta'}}{\sigma_\theta}. \tag{10.3.60}$$

Besides probability characteristics of oscillations of ship, it is of interest to find probability characteristics of oscillations $\beta(t)$ of frame with gyroscope on irregular sea swells. For spectral density $S_\alpha(\omega)$ of these oscillations, considering expression (10.3.27) of the corresponding transfer function $\Phi_2(s)$, we have

$$S_\beta(\omega) = \left|\frac{k_2 j\omega}{(1 - T_1^2\omega^2 + 2\zeta_1 T_1 j\omega)(1 - T_2^2\omega^2 + 2\zeta_2 T_2 j\omega) - k_1 k_2 \omega^2}\right|^2 S_*(\omega), \tag{10.3.61}$$

whence for dispersion $D[\beta]$ and root-mean-square value σ_β of angles β we will obtain following expressions:

$$D[\beta] = 2\int_0^\infty S_\beta(\omega)\,d\omega =$$

$$= 2\int_0^\infty \left|\frac{k_2 j\omega}{(1 - T_1^2\omega^2 + 2\zeta_1 T_1 j\omega)(1 - T_2^2\omega^2 + 2\zeta_2 T_2 j\omega) - k_1 k_2 \omega^2}\right|^2 S_*(\omega)\,d\omega; \tag{10.3.62}$$

$$\sigma_\beta = \sqrt{D[\beta]}.$$

e) Evaluation of Damper.

Passive-type gyroscopic roll damper of a ship was in the beginning of this century installed on several ships [65, 110]; however, subsequently they did not receive wide use in practice due to a series of fundamental deficiencies. Briefly we will indicate dome of them. Usually the damper is calculated for resonance frequency, i.e., for frequency of natural oscillations of the ship, and therefore, for disturbances with different frequency turns out to be little effective, especially during irregular motion of the sea characterized by a certain spectrum of frequencies. Constant displacement of center of gravity of frame relative to axis of its rotation satisfactorily stabilizes rocking of ship only under certain conditions of disturbance. Partially this deficiency is removed by means of braking of the frame; however,action of brake responds to any disturbing influences in an identical manner, while for achievement of best effect it is desirable that braking respond continuously to the changing influence of the waves. The latter is accomplished in roll dampers of active type.

Pitching affects operation of damper insignificantly. More dangerous is yawing of the ship on its course. Investigation of this question (see, for instance, [30, 146]) shows that with deviation of axis of gyroscope from the vertical and during prolonged turns of the ship, the frame with gyroscope under the influence of the thus appearing gyroscopic moment can turn about axis Oy (Fig. 10.1) to the horizontal position, or even be overturned by the additional load upwards. During yawing of the ship on its course, due to the perturbing gyroscopic moment there appear lateral oscillations. Removal of influence of yawing of ship on operation of gyroscopic damper is possible in principle by means of application of two gyroscopes (rotating in opposite directions), axes of housings of which are mechanically connected by toothed sectors (as in a two-gyroscope active GS, § 7.1, Par. 2b). In connection with this one should note that the idea of active gyroscopic frame came into gyroscopic technology from corresponding two-gyroscope devices applied in ship roll dampers and stabilizers of railroad car of monorail railroad [30].

These and other deficiencies of a passive-type gyroscopic roll damper of a ship led to the fact that further development of gyroscopic roll stabilizers went in direction of creation of devices of active type.

3. Gyroscopic Roll Damper of Ship of Active Type

a) Basic Design.

Basic distinction of gyroscopic roll damper of ship of active type from damper

of passive type is that precession of frame and gyroscope is not a consequence of perturbing action of waves, but is created artificially with help of a so-called precessional motor, which is controlled from special auxiliary gyroscope reacting to rolling of the ship. Due to precessional motion of frame with gyroscope there is created gyroscopic (stabilizing) moment, which is directed along longitudinal axis of ship opposite to rolling moment of waves. Due to automatic control of operation of stabilizer, the active damper considerably better reacts to variable conditions of disturbance and does not have the majority of the deficiencies of the passive damper. The biggest effect from damper of active type can be obtained if as the manipulated variable we use not angle or angular velocity of rolling of ship, but angle of wave slope $\alpha(t)$ or its derivative; however, measurement of these parameters directly on ship is connected with large difficulties [30].

Of all active-type gyroscopic roll dampers of a ship the most well known are gyrostabilizers of Sperry [202], which were installed on a number of ships (see, for instance, [110, 30]). Fundamental design of this damper is represented in Fig. 10.4.

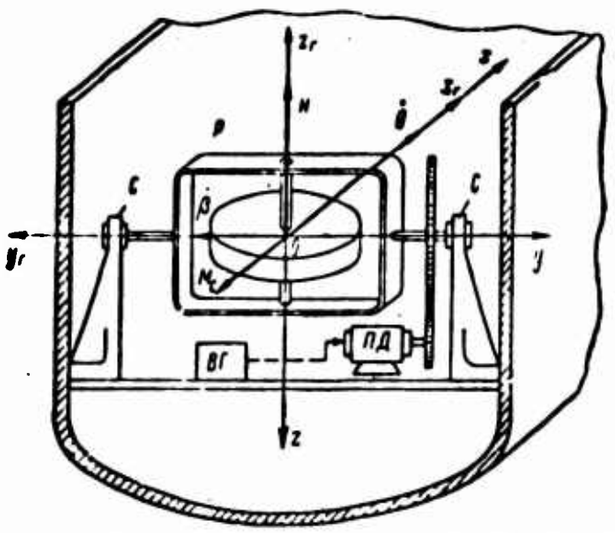

Fig. 10.4. Schematic diagram of active gyroscopic roll damper.

Its basic element is an astatic rate gyroscope, the rotor of which is fixed in frame F, suspended in supports; supports are placed in stands S which are rigidly joined to the ship. Axis of rotation of frame is parallel to transverse axis of ship; axis of gyroscope is located vertically. Frame F is connected through gear transmission with precessional motor PM; which is controlled from auxiliary gyroscope AG. The latter is an astatic gyroscope with two degrees of freedom of the type of a gyrotachometer, therefore, it determines angular velocity $\dot{\theta}(t)$ of rolling of ship, which is the control signal for precessional motor.*

Principle of action of the considered damper in broad terms consists of the following: During rolling of ship about its longitudinal axis Ox (Fig. 10.4) with angular velocity $\dot{\theta}$, auxiliary gyroscope fixes magnitude of $\dot{\theta}$ and transmits it in the

*As will be shown further on, in reality, the torque applied by the precessional motor to the frame is proportional to the sum of $\dot{\theta}$ and $\ddot{\theta}$.

form of a control signal to the precessional motor. The latter through a gear transmission applies to the frame a moment which causes its rotation about axis Oy_g with certain angular velocity $\dot{\beta}$; there appears a gyroscopic (stabilizing) moment $M_s = H\dot{\beta}$ directed along the axis opposite to rolling moment of waves; this compensates roll of ship. Thus, in active damper there always appears stabilizing moment directed opposite to instantaneous rotation of ship during rolling.

Precessional motor rotates frame of gyroscope in the same direction in which it would itself rotate under the influence of disturbance of waves with freedom to turn about axis Oy_g. But with such a possibility precession of frame could occur up to turn of it by an angle of 90° relative to initial position, and stabilizing action of damper would be ceased. It follows from this that in a number of cases precessional motor should render a braking influence, lowering rate of precession of frame to the necessary magnitude. For this, on axis of rotation of motor there is fixed a brake, which regulates its operation. With help of active damper it is possible to cause artificial rocking of ship. This, for instance, protects it from freezing in ice. In more detail the principle of action and design of active roll damper is described in the special literature (see, for instance, [110, 63]).

b) Dynamic Characteristics.

Equations of motion. For obtaining of equations of motion of ship supplied with active gyroscopic roll damper, we will use equations (10.3.3) for the case of a passive damper. In these equations we should set $P_{1o} = 0$, since in active damper frame is balanced; furthermore, braking torque M_B should be replaced by torque M, which precessional motor applies to axis of rotation of frame. Then equations (10.3.3) will be rewritten in the form

$$\left.\begin{aligned} J_x\ddot{\theta} + b_1\dot{\theta} + D\theta + H\dot{\beta} &= Dh_2(t) \\ J_{r.}\ddot{\beta} - H\dot{\theta} &= M \end{aligned}\right\} \qquad (10.3.63)$$

Usually it is considered [30] that moment M is proportional to angular velocity $\dot{\theta}$ of rolling of ship. However, there are grounds to consider [146] that magnitude of moment M should be selected such, that angular velocity $\dot{\beta}$ precession of frame turns out to be proportional to angular velocity $\dot{\theta}$ of rolling of ship, i.e.,

$$\dot{\beta} = m\dot{\theta}, \qquad (10.3.64)$$

where m — constant coefficient.

Then, considering (10.3.63) and (10.3.64), we will obtain following system of

equations of motion of ship with damper:

$$\left.\begin{aligned} J_x\ddot{\theta} + b_1\dot{\theta} + Dh\theta + Hm\dot{\beta} &= Dh\alpha(t) \\ J_{z_1}\ddot{\beta} - H\dot{\theta} &= M \\ \beta &= m\theta \end{aligned}\right\} \qquad (10.3.65)$$

i.e., three equations for determination of variables $\theta(t)$, $\beta(t)$ and $M(t)$.

Transfer functions. According to (10.3.65), for determination of angle θ of roll of ship with damper, by replacing $\alpha(t)$ by $f(t)$ (Par. 2b), we have equation

$$J_x\ddot{\theta} + (b_1 + mH)\dot{\theta} + Dh\theta = Dhf(t). \qquad (10.3.66)$$

from which it follows that action of roll damper (with the selected law control of the damper) reduces to artificial increase of resistance of water to oscillations of ship.

Let us divide (10.3.66) by J_x; we will obtain

$$\ddot{\theta} + \frac{b_1 + mH}{J_x}\dot{\theta} + \frac{Dh}{J_x}\theta = \frac{Dh}{J_x}f(t). \qquad (10.3.67)$$

By analogy with (10.3.7)-(10.3.9), we will designate

$$n_1^2 = \frac{Dh}{J_x};\quad \frac{b_1 + mH}{J_x} = 2\zeta_{1y}n_1;\quad \zeta_{1y} = \frac{b_1 + mH}{2\sqrt{DhJ_x}}, \qquad (10.3.68)$$

where n_1 – frequency of natural sustained oscillations of ship;

ζ_{1_d} – relative attenuation factor of oscillations of ship supplied with damper.

Considering (10.3.68), we will rewrite equation (10.3.67) in the following way:

$$\ddot{\theta} + 2\zeta_{1y}n_1\dot{\theta} + n_1^2\theta = n_1^2 f(t) \qquad (10.3.69)$$

or, by introducing, according to (10.3.17), time constant T_1 of the ship, we have equation

$$T_1^2\ddot{\theta} + 2\zeta_{1y}T_1\dot{\theta} + \theta = f(t). \qquad (10.3.70)$$

which differs from equation (10.3.28) of oscillations of ship without damper only in the magnitude of coefficient ζ_{1_d}. Therefore, transfer function $\Phi_1(s)$ of ship with damper will be determined by formula (10.3.29), if in it we replace ζ_1 by ζ_{1_d}:

$$\Phi_1(s) = \frac{1}{T_1^2 s^2 + 2\zeta_{1y} T_1 s + 1}. \qquad (10.3.71)$$

From (10.3.71) it follows that ship can be considered an oscillating element similar to a physical pendulum (§ 4.3, Par. 2) or gyrotachometer (§ 8.2, Par. 3c). Therefore, frequency-response characteristics $A(\omega)$, $\varphi(\omega)$, $P(\omega)$, $Q(\omega)$ of the ship will be determined by formulas (4.3.34) and (4.3.35), if in them we set $k = 1$, and replace T and ζ by T_1 and ζ_{1d}. Amplitude-phase characteristic $\Phi_1(j\omega)$ of the ship has the same form as for physical pendulum (Fig. 4.14), whence it follows that during action on ship of perturbation $f(t)$ from the sea waves, which changes by a periodic law, steady-state oscillation of ship will be periodic, of the same frequency as $f(t)$. Absolute value of $\Phi_1(j\omega)$ attains maximum for $\omega < n_1$, where n_1 – frequency of natural oscillations of ship.

c) Transient Responses.

Question of transient responses of ship represented as on oscillating element is resolved by the same method as for a gyrotachometer (§ 8.2, Par. 4a). Transfer function $\theta_h(t)$ of ship under disturbance of waves in the form of unit step input [1] (2.3.107) is determined by relationship of type (8.2.97), if in it we replace T and ζ by T_1 and ζ_{1d}, and also to set $k = 1$:

$$\theta_h(t) = \left[1 - e^{-\frac{\zeta_{1y}}{T_1} t}\left(\cos\frac{\sqrt{1-\zeta_{1y}^2}}{T_1} t + \frac{\zeta_{1y}}{\sqrt{1-\zeta_{1y}^2}} \sin\frac{\sqrt{1-\zeta_{1y}^2}}{T_1} t\right)\right]. \qquad (10.3.72)$$

whence it follows that transient function of ship characterized its natural damped oscillations. Damping of these oscillations can be very effective, since, according to (10.3.68), attenuation factor $\zeta_{1d} = \frac{b_1 + mH}{2\sqrt{DhJ_x}}$ by means of selection of corresponding value of angular momentum H of gyroscope can be made quite large.

Indices of quality of the considered system are determined exactly as in the case of a gyrotachometer (§ 8.2, Par. 4b); thus it is of interest to find optimum value of attenuation factor ζ_{1d}. However, for a ship roll damper this value of ζ_{1d} is more expediently found from analysis of its forced oscillations, considering thus desirable degree rolling of ship.

d) Dynamics of Ship with Roll Damper.

For determination of effectiveness of action of a roll damper, most important is calculation of forces oscillations of the ship supplied with the damper. Such

calculations usually were produced for regular sea swells (see, for instance, [30, 146, 63]), when disturbance due to f(t) of the sea waves is a harmonic function of time and is defined by expression (10.3.40). Then forced oscillations of the ship with damper, by analogy with (10.3.41), are determined by relationship

$$\theta_0(t) = \alpha_0 A(\omega) \sin[\omega t + \varphi(\omega)].$$

where α_0 – the biggest angle of wave slope;

$A(\omega)$ and $\varphi(\omega)$ – amplitude and phase frequency-response characteristics of "ship + damper" system.

For $A(\omega)$ and $\varphi(\omega)$ we have expressions which can be obtained from formulas (4.3.34), if in them we set k = 1, and replace T and ζ by T_1 and ζ_{1_d}:

$$A(\omega) = \frac{1}{\sqrt{(1 - T_1^2\omega^2)^2 + 4\zeta_{1y}^2 T_1^2 \omega^2}}; \tag{10.3.73}$$

$$\operatorname{tg} \varphi(\omega) = -\frac{2\zeta_{1y} T_1 \omega}{1 - T_1^2 \omega^2}. \tag{10.3.74}$$

In the future we will be mainly interested in amplitude θ^0_{for} of forced oscillations of ship, for which, according to (10.3.41), we have expression

$$\theta^0_0 = \alpha_0 A(\omega) \tag{10.3.75}$$

or, substituting (10.3.73),

$$\theta^0_0 = \frac{\alpha_0}{\sqrt{(1 - T_1^2\omega^2)^2 + 4\zeta_{1y}^2 T_1^2 \omega^2}}; \tag{10.3.76}$$

from the last formula it follows that amplitude θ^0_{for} of forced oscillations of ship can be made sufficiently small by increasing attenuation factor ζ_{1_d}. The latter can be achieved, according to (10.3.68), by corresponding increase of angular momentum H of gyroscope and coefficient m.

For finding the influence of ζ_{1_d} on amplitude θ^0_{for}, we will introduce dynaminity coefficient λ (§ 8.2, Par. 6b). Since static deviation of ship is $\theta_{st} = \alpha_0$, then by analogy with (8.2.131) and considering (10.3.76), we have

$$\lambda = \frac{\theta^0_0}{\alpha_0} = \frac{1}{\sqrt{(1 - T_1^2\omega^2)^2 + 4\zeta_{1y}^2 T_1^2 \omega^2}}. \tag{10.3.77}$$

If, similarly to (8.2.132), we designate

(10.3.78)

then (10.3.77) can be rewritten in the form

$$\lambda = \frac{1}{\sqrt{(1-\xi_1^2)^2 + \varkappa_{1y}^2 \xi_1^2}}. \tag{10.3.79}$$

Curves $\lambda(\xi_1)$ for different values of attenuation factor ζ_{1_d} are shown in Fig. 8.7. Such curves permit us to substantiate depending upon permissible magnitude of amplitude of forces oscillations of ship with damper, the expedient value of attenuation factor ζ_{1_d}, and consequently also optimum values of angular momentum H of gyroscope and coefficient m [see (10.3.68)].

Having expression (10.3.41) for forced oscillations $\theta_{for}(t)$ of the ship [in which $A(\omega)$ and $\varphi(\omega)$ are defined by formulas (10.3.73) and (10.3.74)], from third equation of system (10.3.65) we can find law $\beta_{for}(t)$ of forced oscillations of frame with gyroscope, and further, according to second equation of this system, law of change of torque M(t) applied by precessional motor to frame with gyroscope.

We will give certain considerations about calculation of effectiveness of an active gyroscopic roll damper of a ship for irregular disturbances. As the dynamic characteristic of ship supplied with roll damper there serves the amplitude-phase characteristic $\Phi_1(j\omega)$, which is obtained by means of replacement in (10.3.71) of s by $j\omega$:

$$\Phi_1(j\omega) = \frac{1}{1 - T_1^2\omega^2 + 2\varkappa_{1y} T_1 j\omega}. \tag{10.3.80}$$

Spectral density $S_\theta(\omega)$ of angles of roll of ship with damper is determined, by analogy with (10.3.52), by the formula

$$S_\theta(\omega) = |\Phi_1(j\omega)|^2 S_\alpha(\omega) \tag{10.3.81}$$

or, considering (10.3.80), by

$$S_\theta(\omega) = \frac{1}{(1 - T_1^2\omega^2)^2 + 4\varkappa_{1y}^2 T_1^2 \omega^2} S_\alpha(\omega), \tag{10.3.82}$$

where $S_\alpha(\omega)$ is spectral density of angle of wave slope.

For dispersion $D[\theta]$ and root-mean-square σ_θ value of angles of roll of ship, we will obtain the following expressions:

$$D[\theta] = 2\int_0^\infty S_\theta(\omega)\, d\omega =$$

$$= 2\int_0^\infty \frac{1}{(1 - T_1^2\omega^2)^2 + 4\varkappa_{1y}^2 T_1^2 \omega^2} S_\alpha(\omega)\, d\omega; \tag{10.3.83}$$

$$\sigma_\theta = \sqrt{D[\theta]}. \tag{10.3.84}$$

Thus, knowing parameters of ship and damper T_1, ζ_{1_d}, and also spectral density $S_\alpha(\omega)$ of random function $\alpha(t)$, we can determine $D[\theta]$ and σ_θ. Since $\theta(t)$ follows a normal law, then further it is possible to calculate mean value E_θ of angles of roll, and also limiting [in the probability sense] value of residual angle of roll of ship, or to solve the problem of "overshoot" of angles of roll beyond a certain assigned value (§ 5.7, Par. 2). Effectiveness of action of roll damper of ship under irregular disturbance from the sea can be characterized by coefficient γ, determined by formula (10.3.60), where σ_θ – root-mean-square value of angle of roll of ship with damper, which is found from relationships (10.3.84) and (10.3.83), and σ_θ – root-mean-square value of angle of roll of ship without damper, determined from formulas (10.3.59) and (10.3.58).

Besides probability characteristics of oscillations of ship, it is also interesting to determine probability characteristics of oscillations $\beta(t)$ of frame of damper under irregular disturbance from the sea. On the basis of third formula of system (10.3.65), we have for correlation function $K_\beta(\tau)$ of random function $\beta(t)$ the expression [for $\beta(0) = 0$]*

$$K_\beta(\tau) = m^2[K_\theta(\tau) + K_\theta(0)]. \tag{10.3.85}$$

Then for dispersion $D[\beta]$ and root-mean-square σ_β of angles β, we will obtain following formulas:

$$D[\beta] = 2m^2 D[\theta]; \tag{10.3.86}$$

$$\sigma_\beta = \sqrt{D[\beta]}. \tag{10.3.87}$$

Let us find probability characteristics of torque $M(t)$, which is applied by precessional motor to axis Oy_g (Fig. 10.4) of rotation of frame with gyroscope. In virtue of second and third equations of system (10.3.65) we have

$$M = mJ_{r.s}\ddot{\theta} - H\dot{\theta}, \tag{10.3.88}$$

whence for spectral density $S_M(\omega)$ of random function $M(t)$ we find

$$S_M(\omega) = H^2 S_\theta(\omega) + m^2 J^2_{r.s} S_{\ddot{\theta}}(\omega) - mHJ_{r.s}\left[S_{\ddot{\theta}\dot{\theta}}(\omega) + S_{\dot{\theta}\ddot{\theta}}(\omega)\right]. \tag{10.3.89}$$

---

*It is assumed that $\theta(t)$ and $\theta(0)$ are uncorrelated (see [120 p. 384]).

where $S_{\theta\dot{\theta}}(\omega)$ and $S_{\dot{\theta}\dot{\theta}}(\omega)$ are mutual spectral densities, whose method of determination was discussed in § 2.1, Par. 3e.

e) Appraisal of Damper.

The considered active roll damper of a ship is installed on a number of ships [65, 30, 110]. In distinction from passive gyroscopic stabilizer, active roll damper retains effectiveness of operation for different frequencies of waves. In active damper it is possible to apply rotors of smaller dimensions than in passive. On irregular sea swells active damper acts more effectively. Yawing leads to different effectiveness of damper during right and left turns of the ship. For elimination of influence of yawing there was developed a stabilizer using two gyroscopes, analogous to the one shown in Par. 2e. Application of gyroscopic roll damper of active type, as experience of operation has shown [110], makes it possible to decrease rolling of ship to several degrees, even under conditions of strong disturbance.

General deficiencies of gyroscopic roll dampers are: complexity of design, large dimensions, difficulty of disposition on ship, high cost, and others.

4. Direct Gyroscopic Stabilization of a Platform on a Ship

In Chapter 7 there was considered application of active gyrostabilizers for stabilization of a certain platform on a ship. Thus, for instance, in Fig. 7.28 and 7.29 are given diagrams of two-gyroscope and four-gyroscope verticals of active type, in which stabilization of platform relative to plane of horizon is carried out with help of stabilizing motors, controlled through amplifiers by signal pickoffs installed on axes of precession of gyroscope housings.

Stabilization of such a platform on a ship relative to plane of horizon can be carried out also with help of direct gyroscopic stabilizers. Different methods of direct gyroscopic stabilization of bodies on ship were shown by A. N. Krylov [65]. He gave the theory of stabilization of a platform fixed in Cardan suspension with the help of gyroscopic stabilizer of Schlick type with an electromagnetic brake. To investigation of passive and active direct gyroscopic stabilizers under conditions of irregular rolling of ship is dedicated article [126].

Above mentioned methods of direct gyroscopic stabilization in practice are quite complicated. Therefore, A. N. Krylov [68] indicated a method of read-out gyroscopic stabilization of a platform. Information about this is given next.

§ 10.4. Read-Out Gyroscopic Stabilizer

1. Fundamental Scheme of Stabilizer

As an example of a read-out gyroscopic stabilization we will consider

stabilization of a platform on a ship. Let us give one of the schemes of such stabilization, developed by A. N. Krylov [68].

During stabilization of some body on a ship it is necessary that it have relative to the ship all three degrees of freedom corresponding to rotation. For this the stabilized body A (Fig. 10.5) is installed in the inner ring of a Cardan suspension, which in turn is suspended in the outer ring C. The latter is installed by its journals in journal bearings D and E, which are rigidly joined with the ship. For retention of the body A in a constant position relative to plane of horizon and at a given azimuthal position, elements of Cardan suspension should turn on angles of yaw φ (Fig. 2.2), pitch ψ and roll θ of the ship. Namely: rotation of body A (Fig. 10.5) relative to inner ring B occurs on angle of yaw φ about axis Oz; rotation of inner ring B relative to outer ring C — on angle of pitch ψ about axis Oy; rotation of outer ring C relative to ship — on angle of roll θ about axis Ox. During continuous introduction of these parameters, body A will under conditions of rolling and yawing of the ship retain constant position with respect to plane of horizon and assigned azimuth. Sensing or pickoff elements of this design of stabilizer are: gyrovertical, which determines angles θ and ψ, and directional gyroscope, which, determines angles φ of yaw and turn of the ship. These angles are transmitted with the help of servo systems to servo drives, which introduce angles φ, ψ and θ into turning of the corresponding elements of the Cardan suspension.

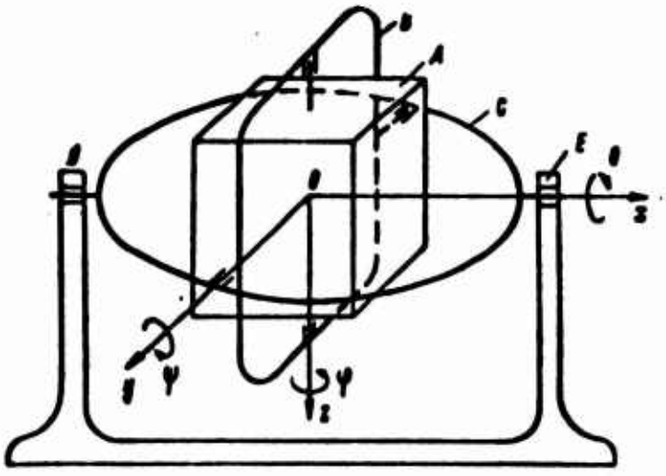

Fig. 10.5. Schematic diagram of read-out gyroscopic stabilizer.

The considered scheme constitutes a read-out gyroscopic stabilizer, since GD applied in it do not carry out actual direct stabilization of body A, but, being devices of read-out type, only control servo drives, which in turn provide stabilization of the body.

2. Dynamic Characteristics of Stabilizer

For determination of dynamic characteristics of system we will consider as an example operation of servo drive turning outer Cardan ring C (Fig. 10.5) by the angle of roll θ. General block diagram of roll stabilization system is shown in

Fig. 10.6; it is a typical scheme of a closed system of automatic control [153].

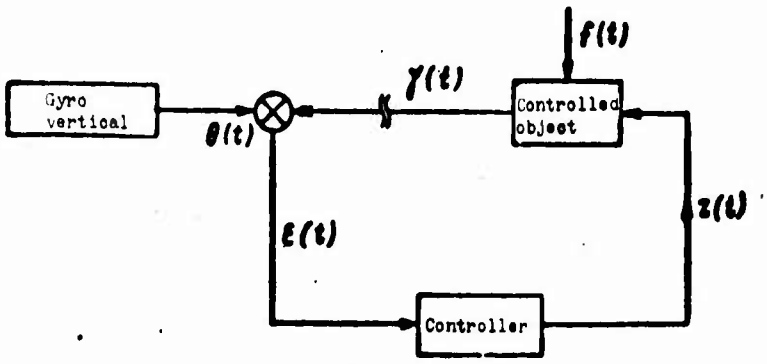

Fig. 10.6. General block diagram of roll stabilization system.

From the pickoff element (gyro - vertical) there proceeds the input variable $\theta(t)$ – angle of roll of the ship. On the adder, from $\theta(t)$ there is subtracted the output variable $\gamma(t)$, i.e., the angle of rotation of the outer Cardan ring C (Fig. 10.5) about axis Ox. Difference $\varepsilon(t) = \theta(t) - \gamma(t)$ is the unbalance, or error of the servo system, i.e., the error in dynamics of stabilization of the body with respect to rolling of the ship.

Unbalance $\varepsilon(t)$ goes to controller (Fig. 10.6), from which the controller action $z(t)$, which is related by a definite functional dependence with $\varepsilon(t)$, goes to the controlled object. Under actual conditions of operation, to the system there will be applied certain disturbances $f(t)$. In the considered case to body A (Fig. 10.5) where will be applied induced inertial torques during rocking of the ship (due to unbalance of body A), frictional torques in axis of rotation of Cardan ring C and others. Furthermore, among disturbances it is possible to include error $\delta\theta(t)$ in determination by gyrovertical (Fig. 10.6) of angle of roll of ship, which goes as interference along with the desired signal $\theta(t)$ to the input of the servo system. Thus, to the input of the servo system of stabilization there proceeds variable $\$(t)$, which is equal to

$$\$(t) = \theta(t) + \delta\theta(t). \tag{10.4.1}$$

For expression of transfer functions of servo drive of statilization we will use general formulas applied in theory of automatic control for investigation of servo systems [153].

Differential equations of stabilization servo drive will be represented in the following form:

1) equation of controlled object

$$D(p)\gamma(t) = M(p)f(t) + K(p)z(t); \tag{10.4.2}$$

2) equation of controller

$$R(p)z(t) = N(p)\varepsilon(t); \tag{10.4.3}$$

3) equation of unbalance or error

$$\varepsilon(t) = \theta(t) - \gamma(t); \tag{10.4.4}$$

here D(p), M(p), K(p) — certain operator polynomials depending on parameters of controlled object;

$\gamma(t)$ — output variable;

$f(t)$ — disturbance;

$z(t)$ — controller action or output variable of controller;

R(p), N(p) — operator polynomials depending on parameters of controller;

$\theta(t)$ — manipulated variable or reference input, determining required law of change of output variable $\gamma(t)$;

$\varepsilon(t)$ — deviation of real change $\gamma(t)$ of output variable from required law of change $\theta(t)$.

To equations (10.4.2)-(10.4.4) we will apply Laplace transform; we will introduce designations

$$\Gamma(s) = L[\gamma(t)] = \int_0^\infty e^{-st}\gamma(t)\,dt \quad (s = \sigma + j\omega;\ \sigma > 0); \tag{10.4.5}$$

$$\Theta(s) = L[\theta(t)] = \int_0^\infty e^{-st}\theta(t)\,dt \tag{10.4.6}$$

etc.

Considering initial conditions to be zero, we find

$$\left.\begin{aligned} D(s)\Gamma(s) &= M(s)F(s) + K(s)Z(s) \\ R(s)Z(s) &= N(s)E(s) \\ E(s) &= \Theta(s) - \Gamma(s) \end{aligned}\right\} \tag{10.4.7}$$

By means of simple transformations, we will obtain

$$\Gamma(s) = \frac{W_0(s)}{1 + W_0(s)}\Theta(s) + \frac{V(s)}{1 + W_0(s)}F(s) \tag{10.4.8}$$

and

$$E(s) = \frac{1}{1 + W_0(s)}\Theta(s) - \frac{V(s)}{1 + W_0(s)}F(s). \tag{10.4.9}$$

where

$$W_0(s) = \frac{N(s)K(s)}{D(s)R(s)}; \quad V(s) = \frac{M(s)}{D(s)}. \tag{10.4.10}$$

Designating

$$\Phi(s) = \frac{W_0(s)}{1 + W_0(s)}, \tag{10.4.11}$$

$$Y(s) = \frac{V(s)}{1 + W_0(s)}, \tag{10.4.12}$$

$$\Phi_0(s) = 1 - \Phi(s) = \frac{1}{1 + W_0(s)}, \tag{10.4.13}$$

we will rewrite (10.4.8) and (10.4.9) in the form

$$\Gamma(s) = \Phi(s)\Theta(s) + Y(s)F(s); \tag{10.4.14}$$

$$E(s) = \Phi_0(s)\Theta(s) - Y(s)F(s). \tag{10.4.15}$$

As it is known [153],

$$\Phi(s) = \frac{\Gamma(s)}{\Theta(s)} \tag{10.4.16}$$

is the transfer function of system of automatic control (SAC) with respect to manipulated variable $\theta(t)$;

$$Y(s) = \frac{\Gamma(s)}{F(s)} \tag{10.4.17}$$

is transfer function of SAC with respect to disturbance f(t);

$$\Phi_0(s) = \frac{E(s)}{\Theta(s)} \tag{10.4.18}$$

is transfer function of error;

$$W_0(s) = \frac{\Gamma_p(s)}{\Theta(s)} \tag{10.4.19}$$

is transfer function open SAC with respect to manipulated variable $\theta(t)$ (element measuring unbalance $\varepsilon(t)$ is disconnected from the output; Fig. 10.6);

$$V(s) = \frac{\Gamma_p(s)}{F(s)} \tag{10.4.20}$$

– transfer function of closed SAC with respect to disturbance f(t)

Thus, for calculation of error $\varepsilon(t)$ of stabilization of the body A it is necessary to have transfer function of the stabilization servo drive, and also characteristics of manipulated variables and disturbances. If we knowing these characteristics of the read-out stabilizer, we can consider transient responses occurring in stabilization servo drive; we use for this general methods applied in theory of automatic control.

3. Method of Determination of Stabilization Errors in Dynamics

We will consider manipulated and variables and disturbances under conditions of

irregular rolling of ship as normal stationary random functions of time. Then, if the considered dynamic system, consisting of servo drive and stabilized object, is linear, then stabilization errors also will be normal stationary random functions of time, for determination of which we should use probability methods.

For finding of probability characteristics of stabilization errors, it is necessary to know probability characteristics of manipulated variables and disturbances, and also dynamic properties of the considered system which are determined by its transfer functions.

As was shown, in the given scheme of stabilization, the manipulated variable is a random function $\theta(t)$ of angles of roll of ship; its probability characteristics were given in § 2.1, Par. 3. Among disturbances one should include random function $\delta\theta(t)$, which characterizes error of gyrovertical in its determination of angle of roll of ship. Probability characteristics of error $\delta\theta(t)$ were given in § 6.5, Par. 2 and 3. Furthermore, among disturbances there are included variables $f(t)$, which are directly applied to the stabilized body A and constitute moments of induced forces of inertia during rolling due to unbalance of body A, frictional torques in axis of rotation of outer Cardan ring, and others. Probability characteristics of random function $f(t)$ have to be determined for concrete conditions.

Error $\varepsilon(t)$ of system in dynamics is expressed by relationship (10.4.4). If considered stabilization system is linear, then general expression for spectral density $S_\varepsilon(\omega)$ or error $\varepsilon(t)$ will be the sum of spectral densities of components of error caused by: 1) reproduction of manipulated variable $\theta(t)$; 2) interference $\delta\theta(t)$; 3) disturbance $f(t)$. Therefore, before we find general expression for $S_\varepsilon(\omega)$, we will consider two particular cases.

1. Interference $\delta\theta(t)$ and disturbance $f(t)$ are absent, $\delta\theta(t) = f(t) = 0$; $\theta(t) \neq 0$.

Spectral density $S_\gamma(\omega)$ of controlled variable $\gamma(t)$ at output of system will be

$$S_\gamma(\omega) = |\Phi(j\omega)|^2 S_\theta(\omega). \qquad (10.4.21)$$

where $\Phi(j\omega)$ – transfer function (amplitude-phase characteristic) of system with respect to manipulated variable;

$S_\theta(\omega)$ – spectral density of manipulated variable $\theta(t)$.

For determination of spectral density $S_\varepsilon(\omega)$ of error of reproduction by servo drive of desired input signal $\theta(t)$, it is sufficient to replace in formula (10.4.21) $\Phi(j\omega)$ by transfer function of error $\Phi_\varepsilon(j\omega)$; then

$$S_\varepsilon(\omega) = |\Phi_\varepsilon(j\omega)|^2 S_\theta(\omega). \qquad (10.4.22)$$

where, according to (10.4.13),

$$\Phi_\varepsilon(j\omega) = 1 - \Phi(j\omega). \tag{10.4.23}$$

2. Disturbance f(t) is absent, $f(t) = 0$; $\delta\theta(t) \neq 0$; $\theta(t) \neq 0$. In this case

$$S_\varepsilon(\omega) = |\Phi_\varepsilon(j\omega)|^2 S_\theta(\omega) + |Y_1(j\omega)|^2 S_{\delta\theta}(\omega) + \\ + \Phi_\varepsilon^*(j\omega) S_{\theta,\delta\theta}(\omega) Y_1(j\omega) + \Phi_\varepsilon(j\omega) S_{\delta\theta,\theta}(\omega) Y_1^*(j\omega). \tag{10.4.24}$$

where $Y_1(j\omega)$ – transfer function of system with respect to interference $\delta\theta(t)$;

$S_{\delta\theta}(\omega)$ – spectral density of interference $\delta\theta(t)$;

$S_{\theta,\delta\theta}(\omega)$, $S_{\delta\theta,\theta}(\omega)$ – mutual spectral densities.

In formula (10.4.24) asterisk designates complex conjugate quantities.

Considering that desired signal $\theta(t)$ and extraneous signal $\delta\theta(t)$ are applied to the same point of the system (to its input) and therefore

$$Y_1(j\omega) = \Phi(j\omega), \tag{10.4.25}$$

we will rewrite (10.4.24) in the form

$$S_\varepsilon(\omega) = |\Phi_\varepsilon(j\omega)|^2 S_\theta(\omega) + |\Phi(j\omega)|^2 S_{\delta\theta}(\omega) + \\ + \Phi_\varepsilon^*(j\omega) S_{\theta,\delta\theta}(\omega) \Phi(j\omega) + \Phi_\varepsilon(j\omega) S_{\delta\theta,\theta}(\omega) \Phi^*(j\omega). \tag{10.4.26}$$

Let us turn to consideration of the general case. On the system there act extraneous signal and external disturbance, $\delta\theta(t) \neq 0$, $f(t) \neq 0$; $\theta(t) \neq 0$.

Random functions $\delta\theta(t)$ and f(t) we will consider for simplicity to be unconnected. Then, by analogy with (10.4.26), we will obtain

$$S_\varepsilon(\omega) = |\Phi_\varepsilon(j\omega)|^2 S_\theta(\omega) + |\Phi(j\omega)|^2 S_{\delta\theta}(\omega) + |Y(j\omega)|^2 S_f(\omega) + \\ + \Phi_\varepsilon^*(j\omega) S_{\theta,\delta\theta}(\omega) \Phi(j\omega) + \Phi_\varepsilon(j\omega) S_{\delta\theta,\theta}(\omega) \Phi^*(j\omega) + \\ + \Phi_\varepsilon^*(j\omega) S_{\theta,f}(\omega) Y(j\omega) + \Phi_\varepsilon(j\omega) S_{f,\theta}(\omega) Y^*(j\omega), \tag{10.4.27}$$

where $Y(j\omega)$ – transfer function of system with respect to disturbance f(t);

$S_f(\omega)$ – spectral density of disturbance f(t);

$S_{\theta,f}(\omega)$ and $S_{f,\theta}(\omega)$ – mutual spectral densities.

Knowing spectral density $S_\varepsilon(\omega)$, it is possible to calculate dispersion D [ε] of error ε(t) by the formula

$$D[\varepsilon] = 2\int_0^\infty S_\varepsilon(\omega)\, d\omega, \tag{10.4.28}$$

where $S_\varepsilon(\omega)$ is defined by relationship (10.4.27).

For mathematical expectation $\bar{\varepsilon}$ of error ε(t), by analogy with (4.6.135) we have

the following general expression:

$$\bar{\varepsilon} = \Phi_0(j\omega)\Big|_{\omega=0}\bar{\theta} + \Phi(j\omega)\Big|_{\omega=0}\overline{\delta\theta} + Y(j\omega)\Big|_{\omega=0}\bar{f}. \qquad (10.4.29)$$

where $\bar{\theta}$, $\overline{\delta\theta}$, $\bar{f}$ are respectively mathematical expectations of manipulated variables interference and disturbance.

Since $\varepsilon(t)$ obeys a normal law, then it is very simple to determine mean value E_ε of error, and also to find its limiting (in the probability sense) value ε_m or to solve the problem of "overshoot" of error ε beyond a certain assigned value ε_0 (§ 5.7, Par. 2).

C H A P T E R 11

GYROSCOPIC NAVIGATIONAL SYSTEMS

Designations Appearing in Cyrillic

Subscripts and Abbreviations

а = a = peak

ак = ac = accelerometer

г = g(gyr) = gyroscope*

г.к = gc = gyrocompass

г.м = g.p = gyropendulum

г.ш = g.l = gyrolatitude

г.э = g.e = gyroequater

д = d = damping

д = dr = drift

и.в = i.v. = inertial vertical

из = sur = surplus

к = cor = corrective

м.м = s.p = simple pendulum

н = nut = nutational

п = red = reduced

п = turn = turn

пр = sp = spring

р = yaw = yaw

*Trans. Ed. Note: Prior to this point the Latin letter "g" has been used in the text in place of the Cyrillic subscript "Г" and has meant "gyroscope." However, on page 311 in this chaper the Russian author has used the Latin letter "g" as a subscript to designate "gravity" or "gravitational"; therefore, from page375 onward "Г" has been replaced by "gyr", and the subscript "g" should be read as "gravitational".

с = v = velocity (superscript)

ст = st = static

т = fr = friction

у = acc = acceleration (superscript)

ф.м = p.p = physical pendulum

ц = circ = circulation

э = e = equatorial

Designations of Materials, Instruments, etc.

Г = G [Gyroscope]

ГВ = GV [Gyrovertical]

ГГК = GHC [Gyrohorizon Compass]

ГК = GC [Gyroscopic Compass]

Гк = Gh [Gyrohousing]

ГМ = GP [Gyropendulum]

ГНС = GNS = [Gyroscopic Navigational System]

Гс = Gs = [Gyrosphere]

ГС = GS [Gyrostabilizer]

ГСП = GSP [Gyrostabilizer platform]

ГУ = GD [Gyro Device]

ГШ = GL [Gyrolatitude]

ГШК = GLC [Gyrolatitude Compass]

Д = MO [Motor]

ДМ = TQ [Torquer]

И = I [Integrator]

ИВ = IV [Inertial Vertical]

ИНС = INS [Inertial Navigation System]

К = H [Housing]

К = C [Cardan Ring (outer)]

П = P [Potentiometer]

П = PL [Platform]

ПВ = VE [Vertical Erector]

Пр = Sp [Spring]

Р = F [Frame]

С = SL [Shaft]

Су = AD [Adder]

§ 11.1. Purpose and Types of Gyroscopic Navigational Systems

For solution of different problems of sea and air navigation it is necessary to

know: 1) position of object (ship, aircraft), characterized by latitude φ and longitude λ; 2) parameters of motion of object: course K, speed v, distance passed over S, altitude h, and so forth; 3) position of vertical of place on moving object.

For determination on ship and on aircraft of these and other parameters, there are applied various methods, the main ones of which are astronomical, radio navigational and self-contained.

Astronomical and radio navigational methods possess high accuracy and are widely used in the navy and in aviation; however, they have an important deficiency – they are not self-contained. Therefore, even greater application is being enjoyed by self-contained methods of navigation using different "mechanical" devices, i.e., devices whose principle of operation is based on laws of mechanics.

Of self-contained navigational systems, a large place is occupied by gyroscopic navigational systems (GNS). Of systems of this type, the most widely known is gyroscopic compass (GC), which determines direction of meridian of the place, and thereby the course of the ship. During solution of individual navigational problems there is used also the gyroscopic pendulum (GP), for instance in sextants, for production in them of artificial horizon. In essence application of GNS was limited up to recent years to these two gyroscopic devices, i.e., they were used basically only for determination of course and position of vertical on the moving object.

However, as will be shown further, on GNS can in principle solve all problems connected with determination of position of an object and parameters of its motion.

Really, it is possible to create a gyroscopic device making it possible to find latitude of a place. Such instruments received the name of gyrolatitudes (GL) or gyrolatitude compasses (GLC); the latter, besides latitude of the place, can also determine course of the object. Combination in one instrument of gyrocompass and gyrovertical (gyrohorizon) gives a system called gyrohorizon compass (GHC). This system makes it possible to determine course and position of vertical on a moving object.

In a number of articles and books (for instance, [10, 48, 50, 117, 121, 162, 142])* there are considered questions of creation of so-called gyroinertial navigation systems. Such systems usually include a gyrostabilized platform, accelerometers, integrators, computer devices, and in principle can determine location and parameters of motion of object, and also position of the vertical. Application of the gyrohorizon compass in combination with accelerometers, integrators and computers also makes it possible to obtain inertial navigational system.

Thus, among GNS there are included: 1) gyropendulums; 2) gyrocompasses; 3) gyrolatitudes and gyrolatitude compasses; 4) gyrohorizon compasses; 5) gyroinertial systems.

*See also: I. A. Gorenshteyn and others. IInertial navigation. "Soviet Radio", 1962; A. A. Yakushenkov. Fundamentals of inertial navigation. "Naval Transport," 1963.

To expound theory of all GNS at present book is impossible – all the more so because to investigations of gyrocompasses and gyropendulums there has been dedicated much literature. Therefore, during account of theory of GC and GP we will give only general information, but will consider in more detail questions of theory of inertial navigational systems, which may find application in sea and air navigation.

Theoretical foundations of the different schemes of other GNS (gyrolatitudes and gyrolatitude compasses) we will not be given, since these systems have still not found practical application.

Before we go on to consideration of GNS, we will dwell on certain ideas of Foucault, who pointed out the possibility of creation of gyrocompasses and gyrolatitudes.

§ 11.2. Foucault Gyroscopes

1. Gyroscope with Three Degrees of Freedom

The first experiment of Foucault was carried out with symmetric astatic gyroscope with three degrees of freedom, fixed in a Cardan suspension in such a manner that on it, insofar as possible, there did not act perturbing torques (friction and others). This experiment, described by Foucault in his reports [187] to the Paris Academy in 1852, had it as its purpose to prove rotation of the Earth. Idea of experiment was based on the fact that for an astatic gyroscope on which do not act moments of external forces (i.e., a free gyroscope), the axis of angular momentum preserves constant direction in motionless (inertial space (§ 3.1, Par. 1). Since in the experiment, axis of rotation of outer Cardan ring was fixed vertically, and axis of gyroscope was fixed horizontally, then visible drift of the latter in azimuth had to occur eastward with the vertical component of angular velocity of terrestrial rotation $U \sin \varphi$ [see (1.2.21)]. In the experiment there really was established visible drift of axis of gyroscope eastward, but is magnitude differed from theoretical. Failure of the experiment of Foucault was basically due to inadequate balance of the gyroscope and the presence of friction in axes of the suspension through which the axis of the gyroscope was carried along by rotation of the Earth. Later, certain investigators successfully used gyroscopes for proof of rotation of Earth [29].

2. Gyroscopes with Two Degrees of Freedom

a) Foucault Gyroscope of the First Kind

Fundamental scheme. Foucault gyroscope of first kind is an astatic gyroscope with two degrees of freedom, axis of which can shift in plane of horizon. Such a

gyroscope can be obtained from a three-degree of freedom astatic gyroscope in the following way.

Let us consider system of coordinates $O\xi\eta\zeta$ (Fig. 11.1); axes $O\xi$ and $O\eta$ are located in plane of horizon and axis $O\zeta$ is directed along the vertical; axis $O\eta$ is oriented to the North. Axes Oxyz are connected with gyroscope. Axis Oy of rotation of outer Cardan ring is located vertically. Inner Cardan ring is rigidly fastened with outer ring in a mutually perpendiuclar position. In initial position axes Oxyz and $O\xi\eta\zeta$ coincide. With turn of outer ring by angle α and axis Oz gyroscope will be displaced in plane of horizon and Resal axes will occupy position Ox_1yz.

We will assume that Foucault gyroscope of first kind is fixed on base motionless with respect to Earth and axis Oz gyroscope is deflected by angle α from direction $O\eta$, i.e., from plane of meridian. Axes $O\xi\eta\zeta$ participate in diurnal rotation of Earth. Projections u_ξ, u_η, u_ζ of induced angular velocity $\vec{u}$ of axes $O\xi\eta\zeta$ determined by relationships (1.2.21)

$$u_\xi = 0; \quad u_\eta = U\cos\varphi; \quad u_\zeta = U\sin\varphi. \tag{11.2.1}$$

In this rotation of axes $O\xi\eta\zeta$ there will participate axis Oz of gyroscope, since it always remains in plane of horizon. Due to this motion of axis Oz, there appears gyroscopic moment $\vec{M}_g$, which is defined by analogy with (1.3.40) by relationship

$$\vec{M}_g = \vec{H} \times \vec{u}, \tag{11.2.2}$$

where $\vec{H}$ is spin angular momentum of gyroscope.

Motion of gyroscope with outer Cardan ring will be influenced only by component $M_{g\zeta}$ of gyroscopic torque along axis $O\zeta$, which, according to (11.2.2), is determined by expression

$$M_\kappa = H_\xi u_\eta - H_\eta u_\xi. \tag{11.2.3}$$

From Fig. 11.1 we have

$$H_\xi = -H\sin\alpha; \quad H_\eta = H\cos\alpha. \tag{11.2.4}$$

Putting (11.2.1) and (11.2.4) in (11.2.3), we will obtain

$$M_\kappa = -HU\cos\varphi\sin\alpha, \tag{11.2.5}$$

whence it follows that gyroscopic torque $M_{g\zeta}$ is directed in negative direction of axis $O\zeta$, and in the absence of perturbing torques along this axis tends to bring axis O_z of the gyroscope into plane of meridian $O\zeta\eta$. Consequently, Foucault gyroscope of first kind possesses properties of a compass. Considering what has been said, torque $M_{g\zeta}$ is called <u>directional torque</u> or <u>directional force</u> of the gyrocompass.

In accordance with this, the component U cos φ sin α [see (11.2.5)] of angular velocity of rotation of Earth is called in the theory of gyrocompasses the useful component of angular velocity of Earth's rotation.

The considered gyroscope can be included in the class of displacement gyroscopes (§ 2.4, Par. 1), since it possesses selectivity with respect to plane of meridian of a place. Appearance in the two-degree-of-freedom astatic gyroscope of properties of a compass is caused by rotation of the Earth, i.e., we have a purely "mechanical" compass. Let us note that "directional force" of gyrocompass [see (11.2.5)]: 1) is increased with increase of angular momentum H of gyroscope; 2) decreases with increase of latitude φ and decrease of angle α of deviation of axis of gyroscope from plane of meridian.

The experiment formulated by Foucault (third experiment of Foucault [187, 29]), with adduction of axis of two-degree-of-freedom gyroscope into plane of meridian turned out to be a failure. Cause of failure was insufficient magnitude of "directional force" [see (11.2.5)] due to smallness of angular momentum H of the applied gyroscope, and the considerable frictional torque in axis of rotation of outer Cardan ring.

Equations of motion and transfer function of GC. For composition of equation of motion of Foucault gyroscope of first kind it is possible to use equations (1.4.2) of motion of gyroscope with three degrees of freedom. Being interested in motion of outer Cardan ring together with rotor about axis Oy (Oζ) (Fig. 11.1), we should turn to the second of these equations

$$J_0\left(\frac{dq}{dt}+pr_1\right)-Hp_1=M_{y_1}=M_y. \tag{11.2.6}$$

For projections p_1, q_1, r_1 of absolute angular velocity of Resal axes on axes Ox_1yz from Fig. 11.1 we find expressions

$$p_1=-u_\eta\sin\alpha;\quad q_1=\dot\alpha+u_\zeta;\quad r_1=u_\eta\cos\alpha \tag{11.2.7}$$

or, substituting (11.2.1)

$$p_1=-U\cos\varphi\sin\alpha;\quad q_1=\dot\alpha+U\sin\varphi;\quad r_1=U\cos\varphi\cos\alpha. \tag{11.2.8}$$

Projections p, q, r of instantaneous angular velocity of gyroscope on axes Ox_1yz, according to (1.2.13), will be

$$p=p_1;\quad q=q_1;\quad r=r_1+\dot\varphi', \tag{11.2.9}$$

where φ' — angular velocity of rotation of rotor relative to inner ring.

Considering (11.2.8), we have

$$p=-U\cos\varphi\sin\alpha;\ q=\dot\alpha+U\sin\varphi;\ r=\dot\varphi'+U\cos\varphi\cos\alpha. \tag{11.2.10}$$

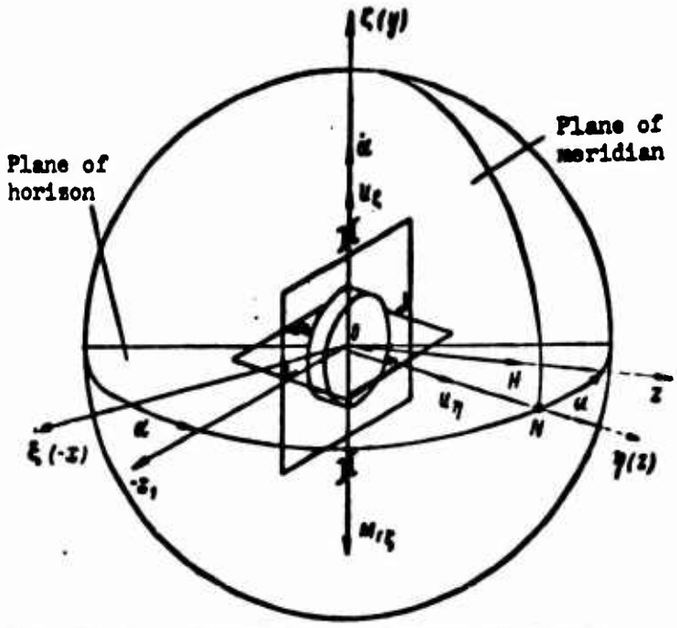

Fig. 11.1. Foucault gyroscope of first kind.

Putting (11.2.8) and (11.2.10) in (11.2.6), we find (for φ = const)

$$J_e\left(\ddot{\alpha} - U^2\cos^2\varphi\sin\alpha\cos\alpha\right) + \\ + HU\cos\varphi\sin\alpha = M_y. \qquad (11.2.11)$$

In the obtained equation it is possible to disregard term $J_e U^2 \cos^2\varphi \sin\alpha\cos\alpha$, which is small as compared to term $HU\cos\varphi\sin\alpha$. Then, replacing J_e by $J_{g\zeta}$,* we will rewrite (11.2.11) in the form

$$J_{r\zeta}\ddot{\alpha} + HU\cos\varphi\sin\alpha = M_y. \qquad (11.2.12)$$

It is necessary to turn attention to the circumstance, that two-degree-of-freedom Foucault gyroscope has one controlling coordinate, and therefore, during its analysis it is impossible to use the corresponding shortened equation (§ 1.6, Par. 8).

Let us assume that relative to an axis $O\zeta(y)$ there acts damping torque, determined by relationship of type (2.3.33)

$$M_\alpha = -b_1\dot{\alpha}. \qquad (11.2.13)$$

Furthermore, let us assume that relative to axis $O\zeta(y)$ there is applied also disturbance f(t). Then torque M_y will be

$$M_y = M_\alpha + f(t) = -b_1\dot{\alpha} + f(t). \qquad (11.2.14)$$

Since in properly designed instrument angles α of deviation of axis of gyroscope from plane of meridian should be small, then in (11.2.12) it is possible to replace $\sin\alpha$ by α. Considering what has been said, and putting in this equation in place of M_y the relationship (11.2.14), we will obtain

$$J_{r\zeta}\ddot{\alpha} + b_1\dot{\alpha} + HU\cos\varphi\cdot\alpha = f(t). \qquad (11.2.15)$$

Thus, we have differential equation of motion of GC with constant coefficients, if latitude φ in a certain interval of time is considered to change little.

We rewrite (11.2.15) in the form

$$\ddot{\alpha} + \frac{b_1}{J_{r\zeta}}\dot{\alpha} + \frac{HU\cos\varphi}{J_{r\zeta}}\alpha = \frac{1}{J_{r\zeta}}f(t); \qquad (11.2.16)$$

*According to (3.2.5), in $J_{g\zeta}$ there are considered equatorial moments of inertia of rotor, of inner ring, and moment of inertia of outer ring relative to axis $O\zeta$.

we designate

$$n_1^2 = \frac{HU\cos\varphi}{J_{rz}}, \tag{11.2.17}$$

whence we will obtain formula for frequency n_1 of natural sustained oscillations of GC relative to plane of meridian:

$$n_1 = \sqrt{\frac{HU\cos\varphi}{J_{rz}}}. \tag{11.2.18}$$

Then for period of these oscillations of GC we have

$$T_{r.z} = \frac{2\pi}{n_1} = 2\pi\sqrt{\frac{J_{rz}}{HU\cos\varphi}}. \tag{11.2.19}$$

Let us transform the ratio $\frac{b_1}{J_{gz}}$:

$$\frac{b_1}{J_{rz}} = \frac{b_1}{J_{rz}} \cdot \frac{\sqrt{\frac{HU\cos\varphi}{J_{rz}}}}{\sqrt{\frac{HU\cos\varphi}{J_{rz}}}} = 2\zeta_1 n_1, \tag{11.2.20}$$

where ζ_1 – relative attenuation factor, determined by relationship

$$\zeta_1 = \frac{b_1}{2\sqrt{J_{rz}HU\cos\varphi}}. \tag{11.2.21}$$

Let us introduce designation

$$\rho_1' = \frac{1}{J_{rz}}. \tag{11.2.22}$$

Considering (11.2.17), (11.2.20) and (11.2.22), we will rewrite equation (11.2.16) in the following way:

$$\ddot{\alpha} + 2\zeta_1 n_1 \dot{\alpha} + n_1^2 \alpha = \rho_1' f(t). \tag{11.2.23}$$

Let us represent (11.2.23) in different form; for this, by analogy with (4.3.23), we will introduce time constant T_1 of gyrocompass

$$T_1 = \frac{1}{n_1} \tag{11.2.24}$$

or, according to (11.2.18),

$$T_1 = \sqrt{\frac{J_{rz}}{HU\cos\varphi}}. \tag{11.2.25}$$

Let us divide (11.2.23) by n_1^2 and designate

$$\rho_1 = \frac{\rho_1'}{n_1^2} = \frac{1}{J_{rz}} \frac{J_{rz}}{HU\cos\varphi} = \frac{1}{HU\cos\varphi}. \tag{11.2.26}$$

Considering (11.2.4) and (11.2.26), we will rewrite (11.2.23) in the form

$$T_1^2\ddot{\alpha} + 2\zeta_1 T_1 \dot{\alpha} + \alpha = \rho_1 f(t). \tag{11.2.27}$$

Let us determine transfer function of GC with respect to disturbance f(t). Let us represent (11.2.23) in operation form ($p = \frac{d}{dt}$):

$$(p^2 + 2\varkappa_1 n_1 p + n_1^2)\alpha = \rho_1' f(t). \qquad (11.2.28)$$

Applying to (11.2.28) Laplace transform, we will obtain following expression for transfer function of GC:

$$W(s) = \frac{\alpha(s)}{f(s)} = \frac{\rho_1'}{s^2 + 2\varkappa_1 n_1 s + n_1^2}. \qquad (11.2.29)$$

This formula can be represented in different form, if its numerator and denominator are divided by n_1^2 and we consider designations of (11.2.24) and (11.2.26):

$$W(s) = \frac{\rho_1}{T_1^2 s^2 + 2\varkappa_1 T_1 s + 1}. \qquad (11.2.30)$$

Comparing (11.2.30) with (4.3.26), we see that transfer function of GC has the same form as transfer function of physical pendulum, i.e., GC with respect to disturbance f(t) can be considered as an oscillating element. Therefore, all other frequency-response characteristics of the GC will be the same as for an oscillating element, for instance a physical pendulum (§ 4.3, Par. 3).

Transient responses of a GC. From (11.2.12) it follows that for $M_y = 0$, position of equilibrium of axis of GC will be $\alpha = 0$ or $\alpha = \pi$. However, it will be stable only at $\alpha = 0$. Consequently, in case of absence of disturbances, axis of Foucault gyroscope of first kind has as its position of equilibrium the plane of the meridian of the place.

We will consider transient response of a GC caused by application to it of disturbance f(t) in the form of a unit step function [1] [see (2.3.107)] for zero initial conditions. Considering in (11.2.27) f(t) = [1], we will rewrite this equation in the form

$$T_1^2\ddot{\alpha} + 2\varkappa_1 T_1\dot{\alpha} + \alpha = \rho_1[1]. \qquad (11.2.31)$$

which is analogous to equation (4.4.1) of oscillations of a physical pendulum.

Particular solution of this equation will be

$$\alpha_* = \rho_1[1]; \qquad (11.2.32)$$

it characterizes error of GC in steady state, i.e., static error $\alpha(\infty)$; consequently [see (11.2.26)],

$$\alpha(\infty) = \rho_1[1] = \frac{1}{HU\cos\varphi}[1]. \qquad (11.2.33)$$

Expression for transfer function $\alpha_h(t)$ of GC for unit step input and zero initial conditions [for t = 0, $\alpha_h(0) = 0$ and $\dot{\alpha}_h(0) = 0$] can be obtained from transfer function

of physical pendulum [see (4.4.13)], if in this formula we replace [cf (4.4.1) and (11.2.31)] k by φ_1, and T and ζ by T_1 and ζ_1:

$$\alpha_k(t) = \rho_1\left[1 - e^{-\frac{\zeta_1}{T_1}t}\left(\cos\frac{\sqrt{1-\zeta_1^2}}{T_1}t + \right.\right.$$

$$\left.\left. + \frac{\zeta_1}{\sqrt{1-\zeta_1^2}}\sin\frac{\sqrt{1-\zeta_1^2}}{T_1}t\right)\right]. \qquad (11.2.34)$$

i.e., axis of gyroscope in the presence of damping ($\zeta_1 \neq 0$) will accomplish damped oscillations. Frequency of these split, according to (4.4.10), is determined by expression

$$n_{1,2} = n_1\sqrt{1-\zeta_1^2}. \qquad (11.2.35)$$

and their period will be

$$T_{r.\,g} = \frac{2\pi}{n_{1,2}} = \frac{2\pi}{n_1\sqrt{1-\zeta_1^2}} = \frac{T_{r.\,g}}{\sqrt{1-\zeta_1^2}}, \qquad (11.2.36)$$

where T_{gc} is period of natural sustained oscillations of GC determined by formula (11.2.19).

In the absence of damping ($\zeta_1 = 0$), according to (11.2.34), we have

$$\alpha_k(t) = \rho_1\left(1 - \cos\frac{t}{T_1}\right) = \rho_1(1 - \cos n_1 t), \qquad (11.2.37)$$

i.e., sustained oscillations with frequency n_1 [see (11.2.18)] and with period $T_{g.c}$ [see (11.2.19)].

<u>The possibility of use of Foucault gyroscope of first kind as a GC</u>. From formula (11.2.19) it follows that period T_{gc} of oscillations of gyroscope is minute; this in practice excludes the possibility of use of Foucault gyroscope of first kind as a GC on a rocking and maneuvering base (ship, aircraft), since, for instance, period of rolling of ship can be of the same order as period T_{gc}. Let us note that increase of period T_{gc} due to increase of moment of inertia $J_{g\zeta}$ of system [see (11.2.19)] is connected with increase of its dimensions and weight; this leads to increase of friction in supports of the suspension. If for increase of T_{gc} we decrease denominator of formula (11.2.19) by means of lowering of angular momentum H of gyroscope, then there decreases the directional torque of the GC [see (11.2.5)].

Another essential deficiency of Foucault gyroscope of first kind during its use as GC is influence of slopes of base on which the instrument is fixed relative to plane of horizon. Slopes of base lead to change of period of oscillations of GC and to appearance of errors in its readings [7].

Oscillations of the base, for instance rolling of a ship, especially greatly affect two-degree-of-freedom GC. Let us assume for simplicity that longitudinal axis of ship coincides with axis $O\eta$ (Fig. 11.1) and that there occurs rolling with angular velocity θ. In this case, instead of (11.2.8) and (11.2.10), we will obtain

$$\left.\begin{aligned} p_1 &= -(U\cos\varphi + \dot\theta)\sin\alpha; \quad q_1 = \dot\alpha + U\sin\gamma; \\ r_1 &= (U\cos\varphi + \dot\theta)\cos\alpha \\ p &= -(U\cos\varphi + \dot\theta)\sin\alpha; \quad q = \dot\alpha + U\sin\gamma; \\ r &= \dot\varphi' + (U\cos\gamma + \dot\theta)\cos\alpha \end{aligned}\right\} \qquad (11.2.38)$$

Putting (11.2.38) in (11.2.6), we find (for φ = const)

$$\begin{aligned} &J_e[\ddot\alpha - (U\cos\varphi + \dot\theta)^2\sin\alpha\cos\alpha] + \\ &+ H(U\cos\varphi + \dot\theta)\sin\alpha = M_\mu \end{aligned} \qquad (11.2.39)$$

or, disregarding terms of higher order of smallness and replacing J_e by $J_{g\zeta}$,

$$J_{g\zeta}\ddot\alpha + H(U\cos\gamma + \dot\theta)\alpha = M_y. \qquad (11.2.40)$$

In torque M_y there is contained frictional torque in axis $Oy(\zeta)$ of rotation of outer Cardan ring during yawing of the ship on its course. For instance, in the presence of liquid friction, by analogy with (2.3.73), for frictional torque $M_{fr\ y}$ for $\alpha \ll \varphi_{yaw}$ we have

$$M_{fry} = -n_1'\dot\varphi_p(t), \qquad (11.2.41)$$

where n_1' – coefficient of fluid friction;

φ_{yaw} – angular velocity of yawing of ship, directed in negative direction of axis $O\zeta(y)$ (Fig. 11.1 and 2.2).

Putting (11.2.41) in (11.2.40), we have ($M_{fr\ y} = M_y$)

$$J_{g\zeta}\ddot\alpha + H[U\cos\gamma + \dot\theta(t)]\alpha = -n_1'\dot\varphi_p(t). \qquad (11.2.42)$$

From equation (11.2.42) it follows that rolling and yawing of ship essentially affect accuracy of readings of GC. It is possible to be convinced of this by means of integration of equation (11.2.42) by one of the methods of approximation; thus, it is necessary to consider that $\theta(t)$ and $\varphi_{yaw}(t)$ are random functions of time.

Thus, Foucault gyroscope of first kind cannot be applied as a navigational GC; in the latter there is used a gyroscope with three degrees of freedom (§ 11.5). However, Foucault gyroscope of first kind has been applied as a GC for orientation of underground mine surveying for its control, for survey of dull holes, and so forth [39]. Such a compass is called a "land" compass. Questions of theory of land GC are presented in a number of articles (for instance, [42, 143, 144]).

B. V. Bulgakov [20] considered the problem of motion of two-degree-of-freedom Foucault gyroscope in the general case, when axis of gyroscope remains in some plane connected with the Earth.* From this general solution there are easily shown particular cases of the gyrocompass and gyrolatitude. To stability of motion of Foucault gyroscopes with two degrees of freedom is dedicated article [156].

b) Foucault Gyroscope of Second Kind

Fundamental scheme. Foucault gyroscope of second kind is an astatic two-degree-of-freedom gyroscope whose axis shifts in plane of meridian. Such a gyroscope can be obtained from three-degree-of-freedom astatic gyroscope in the following way. Let us introduce system of axes $O\xi\eta\zeta$ (Fig. 11.2) oriented just as in Fig. 11.1. Axes Oxyz are connected with the gyroscope. Axis Ox of rotation of outer Cardan ring is located horizontally in plane of prime vertical.** Inner Cardan ring is rigidly fastened with outer ring in mutually perpendicular position. In initial position, axes Oxyz and $O\xi\eta\zeta$ coincide. With turn of outer ring by angle β and axis Oz of gyroscope will be moved in plane of meridian $O\eta\zeta$ and Resal axes will occupy position Oxy_1z.

We will assume that Foucault gyroscope of second kind is fixed on a base motionless with respect to Earth and that its axis Oz is deflected in plane of meridian by angle β from plane of horizon. The object on which gyroscope is fixed is at latitude φ; angular velocity U of diurnal rotation of Earth is directed along the axis of the Earth OP_g, which is inclined to plane of horizon by angle φ. Axes $O\xi\eta\zeta$ participate in diurnal rotation of Earth. Projections of induced angular velocity $\vec{u}$ of axes $O\xi\eta\zeta$ are determined by relationships (11.21.1). In this rotation of the reference system $O\xi\eta\zeta$ there will also participate axis Oz of the gyroscope, since it is always forced to remain in plane of meridian. Due to this there appears gyroscopic torque, which is determined by expression (11.2.2). Motion of gyroscope together with outer Cardan ring will be influenced only by component $M_{g\xi}$ of gyroscopic torque along axis $O\xi$, which, according to (11.2.2), is determined by expression

$$M_{g\xi} = H_\eta u_\zeta - H_\zeta u_\eta. \qquad (11.2.43)$$

From Fig. 11.2 we have

*Analogous problem (polytrope of siff) was considered by Ye. L. Nikolai. Theoretical mechanics, Part III, State United Scientific and Technical Press NKTP USSR, 1939.

**Plane of prime vertical is defined as the vertical plane $O\xi\zeta$, which is perpendicular to plane of meridian $O\zeta\eta$ and plane of horixon $O\xi\eta$.

$$H_\eta = H\cos\beta;\quad H_\zeta = H\sin\beta. \tag{11.2.44}$$

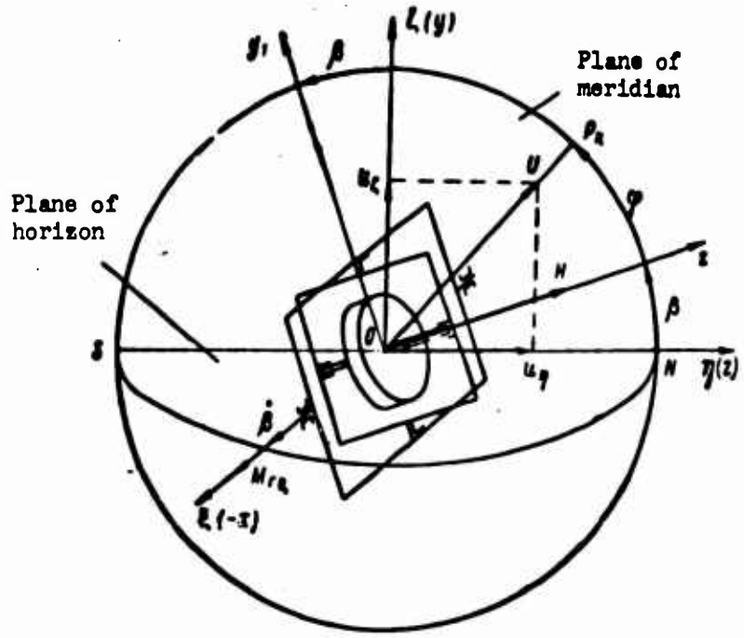

Fig. 11.2. Foucault gyroscope of second kind.

Putting (11.2.1) and (11.2.44) in (11.2.43), we will obtain

$$M_{r\xi} = H\cos\beta\, U\sin\varphi - H\sin\beta\, U\cos\varphi = \\ = HU\sin(\varphi-\beta). \tag{11.2.45}$$

Consequently, gyroscopic torque $M_{g\xi}$ directed along the axis $Ox(\xi)$ of rotation of outer Cardan ring, tends to combine axis of gyroscope Oz with Earth's axis OP_n. Really, according to (11.2.45), gyroscopic torque $M_{g\xi} = 0$, when $\sin(\varphi-\beta) = 0$, i.e., for

$$\beta = \varphi. \tag{11.2.46}$$

Thus, under the influence of torque $M_{g\xi}$, axis of gyroscope Oz will shift in plane of meridian $O\xi\eta$ until it coincides with Earth's axis OP_n, and angle β of turn of axis Oz from plane of horizon is equal to latitude of the place φ. Consequently, Foucault gyroscope of second kind possesses properties of a latitude indicator, i.e., is a gyrolatitude (GL). In this case torque $M_{g\xi}$ is "directional torque" of gyro latitude; due to appearance of this torque, the instrument possesses selectivity with respect to direction of axis of Earth.

The Foucault experiment (second experiment of Foucault [187, 29]), with bringing of axis of two-degree-of-freedom gyroscope into coincidence with Earth's axis, also turned out to be a failure due to impossibility of decreasing friction. Furthermore, even small noncoincidence of center of gravity of gyroscope with point of its suspension involves appearance of gravitational torque, which is comparable with gyroscopic torque which is small in magnitude HU sin $(\varphi-\beta)$ [see (11.2.45)]. This difficulty has not been overcome up to now and is apparently one of the causes of the fact that, in contrast to the gyrocompass, the gyroscopic latitude indicator has still not been realized in practice [20].

Equation of motion of a gyrolatitude. Being interested in motion of outer Cardan ring together with the rotor about axis $Ox(\xi)$ (Fig. 11.2), we should turn to the first of equations of system (1.4.2):

$$J_x\left(\frac{dp}{dt} - qr_1\right) + Hq_1 = M_{x_1} = M_x. \tag{11.2.47}$$

For projections p_1, q_1, r_1 of absolute angular velocity of Resal axes on axis Oxy_1z, from Fig. 11.2 we find

$$p_1 = -\dot{\beta};\ q_1 = u_z\cos\beta - u_y\sin\beta;\ r_1 = u_y\cos\beta + u_z\sin\beta \tag{11.2.48}$$

or, substituting (11.2.1),

$$p_1 = -\dot{\beta};\ q_1 = U\sin(\varphi - \beta);\ r_1 = U\cos(\varphi - \beta). \tag{11.2.49}$$

Projections p, q, r of instantaneous angular velocity of gyroscope on axes Oxy_1z, according to (11.2.9) and (11.2.49), will be

$$p = -\dot{\beta};\ q = U\sin(\varphi - \beta);\ r = \dot{\varphi}' + U\cos(\varphi - \beta). \tag{11.2.50}$$

Putting (11.2.49) and (11.2.50) in (11.4.7), we find

$$J_y\left[-\ddot{\beta} - U^2\sin(\varphi - \beta)\cos(\varphi - \beta)\right] + HU\sin(\varphi - \beta) = M_z. \tag{11.2.51}$$

Let us designate

$$\varphi - \beta = \gamma, \tag{11.2.52}$$

where γ – angular deflection of axis Oz of gyroscope from axis of Earth OP_n.

Then equation (11.2.51) will be rewritten in the form

$$J_y(\ddot{\gamma} - U^2\sin\gamma\cos\gamma) + HU\sin\gamma = M_z. \tag{11.2.53}$$

Discarding small term U^2 sin γ cos γ, replacing J_e by $J_{g\xi}$,* sin γ by γ and considering for simplicity that $M_x = 0$, we obtain equation of small natural oscillations of axis of gyroscope near axis of Earth

$$J_{rt}\ddot{\gamma} + HU\gamma = 0, \tag{11.2.54}$$

which can be rewritten in the form

$$\ddot{\gamma} + n_2^2\gamma = 0, \tag{11.2.55}$$

where

$$n_2^2 = \frac{HU}{J_{rt}}. \tag{11.2.56}$$

Quantity

$$n_2 = \sqrt{\frac{HU}{J_{rt}}} \tag{11.2.57}$$

is frequency of natural sustained oscillation of GL near axis of Earth.

Period of these oscillations of GL will be

$$T_{r.m} = \frac{2\pi}{n_2} = 2\pi\sqrt{\frac{J_{rt}}{HU}}. \tag{11.2.58}$$

*See footnote on page 313.

From equation (11.2.54) it follows that position of equilibrium of instrument occurs at $\gamma = 0$, i.e., when axis of gyroscope coincides with Earth's axis.

Further investigation of motion of Foucault gyroscope of second kind can be carried out by the same method as investigation of motion of Foucault gyroscope of first kind.

Possibility of use of Foucault gyroscope of second kind as a gyrolatitude. Above it was said that Foucault gyroscope of second kind still has not been used as a gyrolatitude in practice due to friction in its suspension and possible instability of the gyroscope. It is also difficult with necessary accuracy to hold axis of gyroscope in plane of meridian. Furthermore, inevitable oscillations of the object can lead to appreciable errors of the instrument. If for removal of influence of oscillations of the object the Foucault gyrolatitude is installed on a platform stabilized relative to planes of horizon and meridian, then errors of it stabilization will nevertheless cause considerable instrument errors [8].

§ 11.3. Generalized Gyropendulum (GP)

1. Fundamental Scheme of Generalized GP

Gyroscopic navigational systems considered further on — GP and GC — are GD of positional type possessing selectivity of "directional force" with respect to directions of vertical and meridian line. Presence in GP and GC of positional properties is explained by the fact that in these instruments there is applied a gyroscope with three degrees of freedom, whose center of gravity is displaced relative to point of suspension. In the GP the axis of gyroscope is located near the vertical, and center of gravity of gyroscope is displaced along its axis below point of suspension. In the GC, axis of gyroscope is located horizontally,* and center of gravity of gyroscope is displaced in its equatorial plane below point of suspension.

However, it is possible to imagine a "generalized" GD in which center of gravity of gyroscope is displaced arbitrarily relative to its point of suspension. Such a GD is called a generalized gyropendulum. This term was introduced by B. I. Kudrevich [71], to whom the first theoretical investigations of such a gyroscope device are due. By generalized GP we will understand a gyroscope with three degrees of freedom, whose center of gravity G (Fig. 11.3) lies in plane of symmetry** of gyroscope at distance l from point of support O; thus axis of gyroscope Oz forms a

*What has been said about direction of axis of gyroscope in GP and GC pertains to idealized conditions, i.e., on the assumption that for these instruments any errors are absent.

**By plane of symmetry of gyroscope we will understand the plane, passing through axis Oz of the gyroscope.

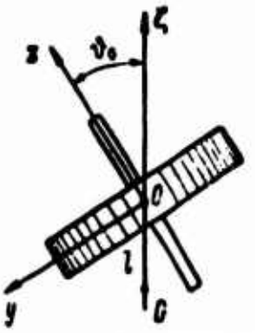

Fig. 11.3. Diagram of generalized GP with displaced center of gravity.

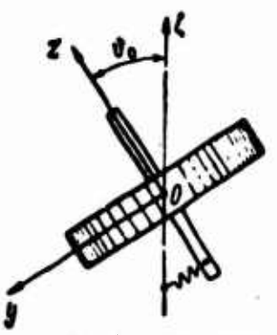

Fig. 11.4. Diagram of generalized GP with spring.

certain constant angle ϑ_0 with line OG, which connects point of suspension with center of gravity and is directed along vertical Oξ. This displacement of center of gravity of gyroscope leads, during deflection of axis of gyroscope by a certain angle relative to initial position ϑ_0, to appearance of corresponding gravitational torque.

Expediency of introduction of concept of generalized GP and its investigation is fully evident. Actually, for $\vartheta_0 = 0$, center of gravity G of gyroscope is located on spin axis Oz of the latter below point of suspension O, and in this case we have a gyropendulum vertical (§ 6.1, Par. 1). At $\vartheta_0 = 90^\circ$, center of gravity G of gyroscope will be in equatorial plane of the latter below point of suspension O, and generalized GP turns out to be transformed into a gyrocompass. For $\vartheta_0 = 90^\circ - \varphi$ (where φ is latitude of the place), as will be shown further, we will have a gyrolatitude. Furthermore, from generalized GP there can be obtained so-called gyrocompass of "East-West" type. B. I. Kudrevich composed a table [8] of different types of gyroscope devices obtained from generalized GP and their basic characteristics. To the development of theory of generalized GP are dedicated individual works (see, for instance, [123]).

Generalized GP may also be created [71] not by means of displacement of center of gravity of gyroscope, but on the basis of a balanced gyroscope whose turn relative to its initial position ϑ_0 is limited by a spring (Fig. 11.4), which introduces a torque analogous to gravitational torque in generalized GP with displaced center of gravity (Fig. 11.3).

§ 2. Equations of Motion of Generalized GP

We will compose differential equations of motion of a generalized GP. As system of reference we choose axes Oξηζ (Fig. 11.5), which are oriented just as in Fig. 11.2: axis Oη is directed in plane of horizon to the north, and axis Oζ is along the true vertical. Axes Oxyz are connected with the gyroscope. Upon azimuthal turn by angle α in plane of horizon, axes Oxyz will occupy position $Ox_1y_0z_0$, for which axis Oz_0 forms with the vertical Oζ a constant angle ϑ_0. Then center of gravity of the gyroscope is at point G_0 on axis Oζ; $OG_0 = l$. Let us impart to the gyroscope rotation about axis Ox_1 by angle δ. Then Resal axes connected with gyrohousing will occupy position Ox_1yz, which relative to reference system Oξηζ is characterized by angles α and $(\vartheta_0 + \delta)$ or α and β, where

$$\beta = 90^\circ - (\vartheta_0 + \delta). \tag{11.3.1}$$

During turn of gyroscope by angle δ, its center of gravity will shift to point G, i.e., line OG will turn relative to vertical $O\zeta$ by angle δ.

During account of theory of generalized GP we will consider only the particular case of installation of it on a base which is motionless with respect to Earth. Axes $O\xi\eta\zeta$ participate in diurnal rotation of Earth; projections u_ξ, u_η, u_ζ of angular velocity $\vec{u}$ axes of $O\xi\eta\zeta$ are determined by relationships (11.2.1). For composition of equations of motion of generalized GP we will use equations (1.4.2) of motion of gyroscope with three degrees of freedom; mass of Cardan rings for simplicity will not be considered. Let us use first two equations of system (1.4.2):

$$\left.\begin{aligned} J_0\left(\frac{dp}{dt} - qr_1\right) + Hq_1 &= M_{x_1} \\ J_0\left(\frac{dq}{dt} + pr_1\right) - Hp_1 &= M_{y_1} \end{aligned}\right\} \qquad (11.3.2)$$

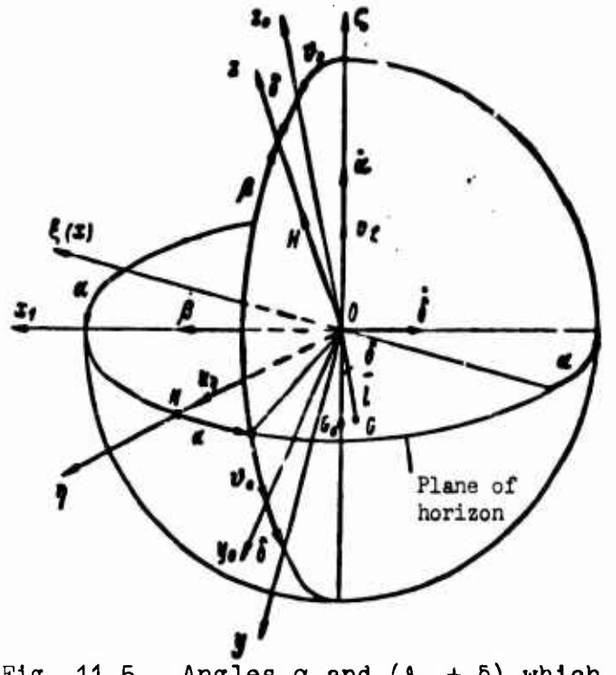

Fig. 11.5. Angles α and $(\vartheta_0 + \delta)$ which characterize position of generalized GP.

For projections p_1, q_1, r_1 of Resal axes on axes Ox_1yz of absolute angular velocity, from Fig. 11.5 we find expressions

$$\left.\begin{aligned} p_1 &= -\dot{\delta} + u_\xi \sin\alpha \\ q_1 &= -\left(\dot{\alpha} + u_\zeta\right)\sin\left(\vartheta_0 + \delta\right) + u_\xi \cos\alpha\cos\left(\vartheta_0 + \delta\right) \\ r_1 &= u_\xi \cos\alpha\sin\left(\vartheta_0 + \delta\right) + \left(\dot{\alpha} + u_\zeta\right)\cos\left(\vartheta_0 + \delta\right) \end{aligned}\right\} \qquad (11.3.3)$$

Projections p, q, r of instantaneous angular velocity of gyroscope onto axes Ox_1yz, according to (1.2.13) and (11.3.3), will be

$$\left.\begin{aligned} p &= -\dot{\delta} + u_\xi \sin\alpha \\ q &= -\left(\dot{\alpha} + u_\zeta\right)\sin\left(\vartheta_0 + \delta\right) + u_\xi \cos\alpha\cos\left(\vartheta_0 + \delta\right) \\ r &= \dot{\varphi}' + u_\xi \cos\alpha\sin\left(\vartheta_0 + \delta\right) + \left(\dot{\alpha} + u_\zeta\right)\cos\left(\vartheta_0 + \delta\right) \end{aligned}\right\} \qquad (11.3.4)$$

where φ' – angular velocity of rotation of rotor relative to the inner ring.

As torques M_{x_1} and M_{y_1}, contained in equations (11.3.2), we will consider only the components of gravitational torque $\vec{M}_g$ caused by displacement of center of gravity G of gyroscope relative to point O of its suspension (Fig. 11.5). For torque $\vec{M}_g$, by analogy with (2.3.25), we have expression

$$\vec{M}_g = \vec{l} \times \vec{P}. \qquad (11.3.5)$$

where $\vec{P} = -P\zeta^0$ – weight of gyroscope.

By projecting (11.3.5) onto axes $O\xi\eta\zeta$ we find

$$\left.\begin{aligned} M_{g\xi} &= l_\eta P_\zeta - l_\zeta P_\eta \\ M_{g\eta} &= l_\zeta P_\xi - l_\xi P_\zeta \\ M_{g\zeta} &= l_\xi P_\eta - l_\eta P_\xi \end{aligned}\right\} \qquad (11.3.6)$$

For projections of vectors $\vec{P}$ and $\vec{l}$ on the same axis we have

$$P_\xi = 0; \quad P_\eta = 0; \quad P_\zeta = -P; \qquad (11.3.7)$$

$$l_\xi = l\sin\delta\sin\alpha; \quad l_\eta = -l\sin\delta\cos\alpha; \quad l_\zeta = -l\cos\delta. \qquad (11.3.8)$$

Putting (11.3.7) and (11.3.8) in (11.3.6), we will obtain

$$M_{g\xi} = lP\sin\delta\cos\alpha; \quad M_{g\eta} = lP\sin\delta\sin\alpha; \quad M_{g\zeta} = 0. \qquad (11.3.9)$$

For determination of projections M_{gx_1} and M_{gy} interesting us, we will use transformation of axes $0\xi\eta\zeta$ to $0x_1yz$, which we will write in matrix form

$$[x_1, y, z] = C[\xi, \eta, \zeta], \qquad (11.3.10)$$

where transformation matrix C, Fig. 11.5, will be

$$C = \begin{Vmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha\cos(\vartheta_0+\delta) & \cos\alpha\cos(\vartheta_0+\delta) & -\sin(\vartheta_0+\delta) \\ -\sin\alpha\sin(\vartheta_0+\delta) & \cos\alpha\sin(\vartheta_0+\delta) & \cos(\vartheta_0+\delta) \end{Vmatrix} \qquad (11.3.11)$$

For determination of M_{gx_1} and M_{gy} we will use expressions (11.3.9) and elements of matrix C; then we will obtain

$$M_{gx_1} = lP\sin\delta; \quad M_{gy} = 0. \qquad (11.3.12)$$

Putting (11.3.3), (11.3.4) and (11.3.12) in (11.3.2), we will obtain differential equations of motion of generalized gyropendulum in the form (for φ = const)

$$\left.\begin{aligned} &J_0\Big\{-\ddot{\delta} + u_\eta\dot{\alpha}\cos\alpha - \Big[-\big(\dot{\alpha}+u_\zeta\big)\sin(\vartheta_0+\delta) + \\ &+ u_\eta\cos\alpha\cos(\vartheta_0+\delta)\Big]\Big[u_\eta\cos\alpha\sin(\vartheta_0+\delta) + \\ &+\big(\dot{\alpha}+u_\zeta\big)\cos(\vartheta_0+\delta)\Big]\Big\} + H\Big[-\big(\dot{\alpha}+u_\zeta\big)\sin(\vartheta_0+\delta) + \\ &+ u_\eta\cos\alpha\cos(\vartheta_0+\delta)\Big] = lP\sin\delta \\ &J_0\Big\{-\ddot{\alpha}\sin(\vartheta_0+\delta) - \big(\dot{\alpha}+u_\zeta\big)\dot{\delta}\cos(\vartheta_0+\delta) - \\ &- u_\eta\Big[\dot{\alpha}\sin\alpha\cos(\vartheta_0+\delta) + \dot{\delta}\cos\alpha\sin(\vartheta_0+\delta)\Big] + \\ &+\big(-\dot{\delta}+u_\eta\sin\alpha\big)\Big[u_\eta\cos\alpha\sin(\vartheta_0+\delta) + \\ &+\big(\dot{\alpha}+u_\zeta\big)\cos(\vartheta_0+\delta)\Big]\Big\} - H\big(-\dot{\delta}+u_\eta\sin\alpha\big) = 0 \end{aligned}\right\} \qquad (11.3.13)$$

Approximately with accuracy up to terms of second order of smallness we have*

*It is necessary to consider that angle ϑ_0 is not small. We will consider generalized coordinates α, δ, angular velocities $\dot{\alpha}$, $\dot{\delta}$, and also components u_ξ, u_η, u_ζ of induced angular velocity of reference system $0\xi\eta\zeta$ to be small.

$$\left.\begin{aligned}
&J_0\left[-\ddot{\alpha}\sin(\theta_0+\delta)-2\left(\dot{\alpha}+u_\zeta\right)\dot{\delta}\cos(\theta_0+\delta)-\right.\\
&\left.-2u_\eta\dot{\delta}\cos\alpha\sin(\theta_0+\delta)\right]-H\left(-\dot{\delta}+u_\eta\sin\alpha\right)=0\\
&J_0\left[-\ddot{\delta}+u_\eta\dot{\alpha}\cos\alpha+\left(\dot{\alpha}+u_\zeta\right)u_\eta\cos\alpha\sin^2(\theta_0+\delta)+\right.\\
&+\left(\dot{\alpha}+u_\zeta\right)^2\sin(\theta_0+\delta)\cos(\theta_0+\delta)-\\
&-u_\eta^2\cos^2\alpha\sin(\theta_0+\delta)\cos(\theta_0+\delta)-\\
&\left.-u_\eta\left(\dot{\alpha}+u_\zeta\right)\cos\alpha\cos^2(\theta_0+\delta)\right]+\\
&+H\left[-\left(\dot{\alpha}+u_\zeta\right)\sin(\theta_0+\delta)+u_\eta\cos\alpha\cos(\theta_0+\delta)\right]=\\
&=lP\sin\delta.
\end{aligned}\right\} \qquad (11.3.14)$$

Being limited subsequently to consideration only of basic precessional motion of generalized GP, we will discard in (11.3.14) inertial terms; then we have

$$\left.\begin{aligned}
&H\left(\dot{\delta}-u_\eta\sin\alpha\right)=0\\
&H\left[u_\eta\cos\alpha\cos(\theta_0+\delta)-\left(\dot{\alpha}+u_\zeta\right)\sin(\theta_0+\delta)\right]=lP\sin\delta
\end{aligned}\right\} \qquad (11.3.15)$$

Designating by analogy with (2.3.30)

$$k=\frac{lP}{H},$$

we will rewrite (11.3.15) in the form

$$\left.\begin{aligned}
&\dot{\delta}-u_\eta\sin\alpha=0\\
&u_\eta\cos\alpha\cos(\theta_0+\delta)-\left(\dot{\alpha}+u_\zeta\right)\sin(\theta_0+\delta)-k\sin\delta=0
\end{aligned}\right\} \qquad (11.3.16)$$

We have obtained equations which with accuracy up to designations coincide with equations of motion of generalized GP, which were found by B. I. Kudrevich [71] by another method.

Equations (11.3.16) by means of simple trigonometric transformations with accuracy up to small terms of second order can be represented in the form

$$\left.\begin{aligned}
&\dot{\delta}-u_\eta\alpha=0\\
&u_\eta\left(\cos\theta_0-\sin\theta_0\cdot\delta\right)-\left(\dot{\alpha}+u_\zeta\right)\left(\sin\theta_0+\cos\theta_0\cdot\delta\right)-k\delta=0
\end{aligned}\right\} \qquad (11.3.17)$$

3. Analysis of Certain Particular Cases

For investigation of motion of generalized GP it is necessary to integrate equation (11.3.17). Without giving the solution of this general problem, we will consider only certain particular cases. First of all we will determine position of static equilibrium of axis of gyroscope, which is characterized by angles α_r, δ_r (or β_r). For this in equations (11.3.17) we will set $\dot{\alpha}=\dot{\delta}=0$; first equation gives

$$\alpha_r=0, \qquad (11.3.18)$$

i.e., axis of gyroscope of generalized GP in position of static equilibrium is located in plane of meridian, and the considered GD is in general a gyrocompass.

Let us turn to analysis of certain particular cases.

Case of gyrocompass. In this $\$_0 = 90^\circ$. Considering that in (11.3.17) $\$ = 90^\circ$ and taking into account (11.2.1), we will obtain

$$\left.\begin{aligned} \dot{\beta} - U\cos\varphi\cdot\alpha &= 0 \\ \dot{\alpha} + U\sin\varphi + (U\cos\varphi + k)\beta &= 0 \end{aligned}\right\} \tag{11.3.19}$$

or, considering (11.3.1), for $\$_0 = 90^\circ$

$$\left.\begin{aligned} \dot{\beta} + U\cos\varphi\cdot\alpha &= 0 \\ \dot{\alpha} + U\sin\varphi - (U\cos\varphi + k)\beta &= 0 \end{aligned}\right\} \tag{11.3.20}$$

i.e., we have equations of motion of single-rotor gyrocompass [20] located on a base motionless with respect to Earth [see (11.5.43)].

In position of equilibrium we have

$$\left.\begin{aligned} \alpha_1 &= 0 \\ \beta_1 &= \frac{U\sin\varphi}{U\cos\varphi + k} \end{aligned}\right\} \tag{11.3.21}$$

i.e., in this position axis of gyroscope is located in plane of meridian, but raised above plane of horizon in northern hemisphere ($\varphi > 0$) by angle β_r.

For the usually taken structural parameters of the GC $k = \frac{lP}{H} \gg U\cos\varphi$. Therefore for β_r, according to (11.3.21), we approximately have [see (11.5.46)]

$$\beta_1 = \frac{HU}{lP}\sin\varphi, \tag{11.3.22}$$

whence it follows that to angle β_r there corresponds gravitational torque $lP\beta_r = HU\sin\varphi$ and angular velocity of precession $\frac{lP\beta_r}{H}$, which is equal to $U\sin\varphi$. This rate of precession of GC is necessary for compensation of vertical component $U\sin\varphi$ of Earth's rotation and for holding thereby of axis of gyroscope in plane of meridian.

Using equations (11.3.20), we can reveal character of sustained oscillations of GC, their period, trajectory of vertex of gyroscope, etc. (see § 11.5).

Case of gyropendulum. In this case $\$_0 = 0^\circ$. Considering that in (11.3.17) $\$_0 = 0^\circ$ and taking into account (11.2.1), we will obtain

$$\left.\begin{aligned} \dot{\beta} - U\cos\varphi\cdot\alpha &= 0 \\ -(\dot{\alpha} + U\sin\varphi + k)\beta + U\cos\varphi &= 0 \end{aligned}\right\} \tag{11.3.23}$$

Let us show that in the considered case instrument possesses properties of the gyrovertical, i.e., axis of gyroscope will follow the local vertical, which rotates around meridian line in outer space eastward with angular velocity $U\cos\varphi$ with the horizontal component of the Earth's rotation (Fig. 1.8). Actually, from the second

equation of system (11.3.23) we find coordinate δ_r of position of equilibrium of axis of gyroscope:

$$\delta_1 = \frac{U\cos\varphi}{U\sin\varphi + k}. \qquad (11.3.24)$$

Since for gyropendulum usually $k = \frac{lP}{H} \gg U \sin\varphi$, then for δ_r we have approximately

$$\delta_1 = \frac{H}{lP} U\cos\varphi. \qquad (11.3.25)$$

From (11.3.25) it follows that in position of equilibrium axis of gyroscope will be deflected in plane of meridian* toward the north by angle δ_r. For this slope δ_r there appears gravitational torque $lP\delta_1 = HU\cos\varphi$ and angular velocity of precession $\frac{lP\delta_r}{H}$, which is equal to $U\cos\varphi$. This angular velocity of precession is necessary for compensation of horizontal component of angular velocity of Earth's rotation, i.e., for automatic following of axis of gyroscope after vertical of the place, which travels eastward with angular velocity $U\cos\varphi$. Thus, for $\$_0 = 0$, generalized GP becomes a gyropendulum vertical.

Case of gyrolatitude. In this case $\$_0 = 90^\circ - \varphi$ (φ is latitude of place), i.e., axis of gyropendulum is directed along axis of Earth. Considering in (11.3.17) that $\$_0 = 90^\circ - \varphi$ and taking into account (11.2.1), we will obtain

$$\left.\begin{aligned} &\dot\delta - U\cos\varphi\cdot\alpha = 0\\ &U\cos\varphi(\sin\varphi - \cos\varphi\cdot\delta) - (\dot\alpha + U\sin\varphi)(\cos\varphi + \\ &\quad + \sin\varphi\cdot\delta) - k\dot\delta = 0 \end{aligned}\right\}$$

whence

$$\left.\begin{aligned} &\dot\delta - U\cos\varphi\cdot\alpha = 0\\ &\dot\alpha\cos\varphi + (\dot\alpha\sin\varphi + U + k)\delta = 0 \end{aligned}\right\} \qquad (11.3.26)$$

In position of equilibrium ($\dot\alpha = \dot\delta = 0$)

$$\alpha_1 = 0, \quad \delta_1 = 0, \qquad (11.3.27)$$

i.e., axis of gyroscope is established in direction of axis of Earth, and such a GP can be an indicator of geographic latitude of place of observation.

Other cases of application of generalized GP can be met in the corresponding literature (for instance, [71, 123]).

After clarifying certain general properties of GD with center of gravity of the gyroscope displaced relative to point of its suspension, we will consider in more detail the most important varieties of generalized GP — gyropendulum and gyrocompass.

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*Indeed, from first equation of system (11.3.23) it follows that in position of equilibrium coordinate $\alpha_r = 0$, i.e., axis of gyroscope is located in place of meridian.

Equations (11.3.23) differ in form from equations of GP obtained by B. V. Bulgakov [20, p. 34, formula 33]. However, from the latter it is easy to go over to equations (11.3.23), if for considered case of a base motionless with respect to Earth we set $W_1 = W_\xi = W_2 = W_\eta = 0$, $u_1 = u_\xi = 0$, $u_2 = u_\eta = U\cos\varphi$, $u_3 = u_\xi = U\sin\varphi$, and taking into account different selection of Euler angles (cf. Figures 11.5 and 2.19) we replace β by δ, $-u_\zeta\alpha$ by $+u_\eta\alpha$, $\dot\alpha$ by $-\dot\alpha\sin\delta \approx -\dot\alpha\delta$, and also take the restoring torque $lP\alpha = 0$.

§ 11.4. Gyropendulums (GP)

1. Purpose of GP. Fundamental Scheme of the Instrument

a) Types of GP

As was shown earlier (§ 6.1, Par. 1), gyropendulum (GP) is defined as a gyroscope with three degrees of freedom whose center of gravity lies on figure axis (axis of rotor) at a certain distance from point of support. Gyropendulum can be considered as a particular case of generalized GP, (§ 11.3, Par. 3). Gyroverticals in which there are used GP, are called gyropendulum verticals. Such gyroverticals in principle possess essential advantages over GV based on application of three-degree-of-freedom astatic pendulum gyroscope (Chapter 6). However, realization in practice of gyropendulum verticals having acceptable accuracy of readings during maneuvering of object (ship, aircraft) is very complicated technically, since it is difficult to realize M. Schuler's condition of undisturbability (§ 4.5, Par. 5) of the gyropendulum by accelerations of the object. In practice this condition is satisfied in so-called inertial verticals (the basic element of inertial navigational systems, see § 11.6) by means of artificial simulation of Schuler's pendulum with help of a system containing a gyrostabilized platform, accelerometers, integrators, computer and other elements.

In gyropendulum verticals there are applied spherical, or two-component, GP, possessing two controlling coordinates. In different GD there are used also plane, or single-component GP, having one controlling coordinate. Depending upon number of gyroscopes, we distinguish single-gyroscope, two-gyroscope and four-gyroscope GP.

During account of theory of GP we will consider usual scheme of gyropendulum vertical with one gyroscope. Regarding plane, two-gyroscope and four-gyroscope GP, we will indicate only their basic peculiarities.

b) Fundamental Scheme of GP. Simple Relationships

Schematic diagram of GP is shown in Fig. 11.6. Gyrorotor is installed in gyrohousing (in inner Cardan ring) Gh, which is suspended in outer Cardan ring C. Center of gravity G of gyrohousing with rotor is displaced along axis Oz of gyroscope by the amount l relative to point of suspension O.

Let us consider principle of action of GP and derive simple relationships. In Fig. 11.7 axis Oz of gyropendulum is deflected from vertical $O\zeta$ by angle α about axis $O\eta$ of rotation of outer Cardan ring. Torque M_g due to weight P of rotor with

gyrohousing is directed along axis Oη and in magnitude is equal to

$$M_g = lP \sin\alpha \approx lP\alpha. \qquad (11.4.1)$$

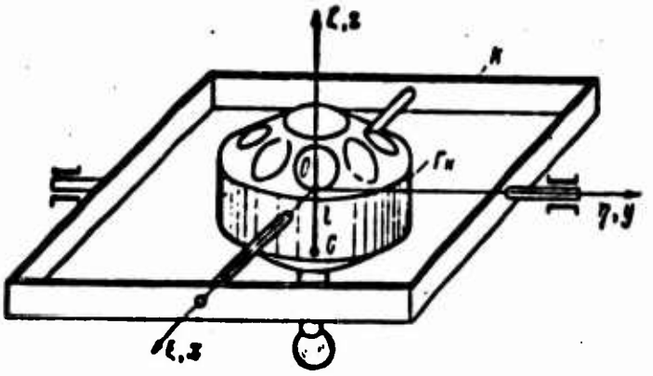

Fig. 11.6. Schematic diagram of GP.

For determination of basic precessional motion of axis Oz of gyroscope under action of torque M_g we will use Resal's theorem (§ 1.3, Par. 2). According to (1.3.23) we have:

$$u = M_g, \qquad (11.4.2)$$

where u is linear velocity of end of vector $\vec{H}$ of angular momentum of gyroscope; it is determined by relationship

$$u = kH \sin\alpha \approx kH\alpha, \qquad (11.4.3)$$

k is angular velocity of precession of GP.

Substituting (11.4.1) and (11.4.3) in (11.4.2), we have formula

$$k = \frac{lP}{H}, \qquad (11.4.4)$$

which coincides with (2.3.30).

With deflection of axis Oz of gyroscope from vertical Oζ, the GP starts under action of gravitational torque M_g to precess about Oζ; then axis Oz will describe a cone with vertex at point O. Angular velocity of precession k is determined by formula (11.4.4), and its period $T_{g.p}$ will be

$$T_{g.p} = \frac{2\pi}{k} = 2\pi\frac{H}{lP}. \qquad (11.4.5)$$

In the absence either of friction in the suspension or a special damping attachment, precessional oscillations of axis of gyroscope will be undamped. For transformation of GP into a GV, it is necessary to quench these oscillations with help of a special damper. Use of frictional force for quenching of natural oscillations of axis of gyroscope is impossible, since friction generates additional errors in the GP; therefore, in every possible way it is attempted to decrease frictional torque in the suspension.

Comparing (11.4.5) with formula (4.1.2) for period of oscillations of simple pendulum, we will find length l_{red} of simple pendulum, which is isochronous with the considered gyropendulum, i.e., "reduced" length of the GP:

$$l_{red} = \frac{g}{k^2} = g\left(\frac{H}{lP}\right)^2. \qquad (11.4.6)$$

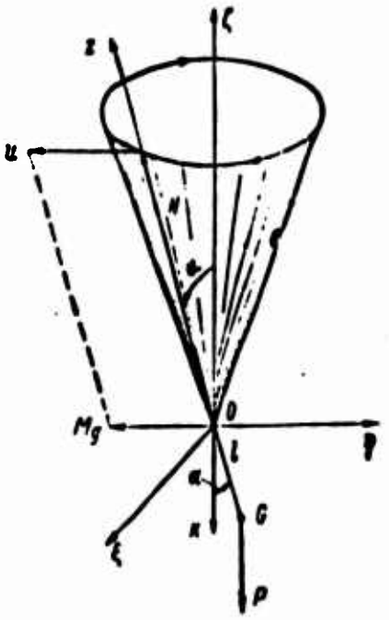

Fig. 11.7. Precessional motion GM.

Thus, if we take a gyroscope of comparatively small weight P, but possessing large angular momentum H and small static torque lP, then we can obtain a GP with large period, i.e., equivalent to a simple pendulum of very great length.

A GP, if in it the gyroscope does not rotate (H = 0), becomes a physical pendulum whose period is determined by formula (4.1.4)

$$T_{\phi.u} = 2\pi\sqrt{\frac{J_e}{lP}}, \qquad (11.4.7)$$

where J_e is equatorial moment of inertia of gyrorotor.

Then length l_e of the equivalent simple pendulum which has the same period as the considered physical pendulum is expressed by relationship (4.1.5):

$$l_e = \frac{J_e}{lP}g. \qquad (11.4.8)$$

whence it follows that physical pendulum is equivalent to a simple pendulum of considerably smaller length than the GP.

Verticals of gyropendulum type constructed according to the considered scheme, for several reasons, which will be shown further, have found comparatively little application in practice. The first more or less useful artificial horizon of gyropendulum type was built by Flerie [29, 20] for stabilization relative to plane of horizon of the optical system of a sextant. There are also known the naval gyrohorizon and aviation gyrohorizon of Anschutz [29, 20].

Let us give an example for determination of certain parameters characterizing motion of the gyropendulum aviation gyrohorizon of Anschutz.

Example 11.1: To calculate parameters characterizing motion of gyropendulum aviation horizon of Anchutz, for the following initial data [29, 20]: rpm rate of rotor N = 20,000 rpm; angular velocity of spin of rotor Ω = 2094 1/sec; angular momentum of gyroscope H = $1.708\cdot10^5$ g·cm·sec; weight of gyrohousing with rotor P = 5 kg; distance of center of gravity of gyroscope from point of suspension l = 0.25 cm; equatorial moment of inertia of gyrorotor J_e = 48.94 g·cm·sec^2.

Solution: 1. By formula (11.4.4) we find angular velocity k of precession of the GP:

$$k = \frac{lP}{H} = \frac{0{,}25\cdot5000}{1{,}708\cdot10^5} = 7{,}319\cdot10^{-3}\ \text{1/sec}.$$

2. According to (11.4.5) we determine period $T_{g.p}$ of natural sustained oscillations of GP:

$$T_{r.u} = \frac{2\pi}{k} = \frac{6,28}{7,319 \cdot 10^{-3}} = 858,5 \text{ sec} = 4 \text{ min } 19 \text{ sec.}$$

3. Using (11.4.6), we calculate "reduced" length of GP l_{red}:

$$l_{red} = \frac{g}{k^2} = \frac{9,81}{(7,319 \cdot 10^{-3})^2} = 183 \cdot 100 \text{ m} = 183,1 \text{ km.}$$

4. By formula (11.4.7) we determine period $T_{p.p}$ of oscillations of physical pendulum obtained from GM with non-rotating rotor (H = 0):

$$T_{ph.u} = 2\pi \sqrt{\frac{J_0}{lP}} = 6,28 \sqrt{\frac{48,94}{0,25 \cdot 5000}} = 1,243 \text{ sec.}$$

5. According to (11.4.8) length l_e of the equivalent simple pendulum

$$l_e = \frac{J_0}{lP} g = \frac{48,94}{0,25 \cdot 5000} 981 = 27,84 \text{ cm} = 0,2784 \text{ m.}$$

From example 11.1 it follows that the GP has considerably larger period of oscillations than the corresponding physical pendulum. Therefore, accuracy of readings of GP on a rocking ship or aircraft will be considerably higher than accuracy of physical pendulum.

2. Dynamic Characteristics of GP

a) Geometric and Kinematic Parameters

Investigation of GP starts with composition of differential equations of motion and corresponding transfer functions. Initial position of axes $Oxyz$, which are connected with gyroscope, and reference system $O\xi\eta\zeta$, where axis $O\zeta$ is directed along geocentric vertical, is shown in Fig. 11.6. Position of axis Oz of gyroscope relative to its assigned direction $O\zeta$ (Fig. 2.19), or, which is the same, position of Resal axes Ox_1y_1z relative to $O\xi\eta\zeta$, is determined by angles α and β, which are considered subsequently as errors of the gyropendulum.

Transformation of axes $O\xi\eta\zeta$ to Ox_1y_1z is written in the form of matrix equality (6.2.1), in which matrix of transformation $A_1 = \|a'_{\mu,\nu}\|$ ($\mu,\nu = 1, 2, 3$) is defined by Table (6.2.2), or for small angles α and β – by (6.2.3).

For projections p_1, q_1, r_1 of angular velocity $\vec{\omega}_1$ of Resal axes Ox_1y_1z onto these same axes we have formulas (1.2.15). With consideration of induced motion of axes $O\xi\eta\zeta$ caused by rotation of Earth and vehicular motion for projections of absolute angular velocity of axes Ox_1y_1z onto the same axes there were found formulas (1.2.34) or (1.2.35) (for case of small α and β).

b) Differential Equations of Motion and Transfer Functions of the GP

For composition of differential equations of motion of the GP we will use generalized Euler equations (without considering, for simplicity, the mass of the Cardan rings); i.e., we will use first two equations of system (1.4.2)

$$\left.\begin{aligned} J_0\left(\frac{dp}{dt} - qr_1\right) + Hq_1 &= M_{x_1} \\ J_0\left(\frac{dq}{dt} + pr_1\right) - Hp_1 &= M_{y_1} \end{aligned}\right\} \qquad (11.4.9)$$

For small angles α and β quantities p_1, q_1, r_1 are determined by relationships (1.2.35)

$$\left.\begin{aligned} p_1 &= -\dot{\beta} + u_\xi - u_\zeta\alpha \\ q_1 &= \dot{\alpha} + u_\eta - u_\zeta\beta \\ r_1 &= u_\xi\alpha + u_\eta\beta + u_\zeta \end{aligned}\right\} \qquad (11.4.10)$$

For p, q, r, according to (1.2.13), we have

$$p = p_1; \quad q = q_1; \quad r = \dot{\gamma} + r_1. \qquad (11.4.11)$$

Putting (11.4.10) and (11.4.11) in (11.4.9) and considering components u_ξ, u_η, u_ζ of induced angular velocity $\vec{u}$ of axes $O\xi\eta\zeta$ to be constant (for comparatively small time intervals of operation of the GP), with accuracy up to terms of the second order of smallness we will obtain

$$\left.\begin{aligned} J_0\ddot{\beta} - H\left(\dot{\alpha} + u_\eta - u_\zeta\beta\right) &= -M_{x_1} \\ J_0\ddot{\alpha} - H\left(-\dot{\beta} + u_\xi - u_\zeta\alpha\right) &= M_{y_1} \end{aligned}\right\} \qquad (11.4.12)$$

Moments M_{x_1} and M_{y_1} will be represented in the form

$$\left.\begin{aligned} M_{x_1} &= M_{(g.w)_x} + M_{dx} + f_2(t) \\ M_{y_1} &= M_{(g.w)_y} + M_{dy} + f_1(t) \end{aligned}\right\} \qquad (11.4.13)$$

where $M_{(g.w)_x}$ and $M_{(g.w)_y}$ are components of torque $\vec{M}_{(g.w)}$, which is due to action of gravity and forces of inertia due to accelerations of the object;

M_{dx} and M_{dy} are damping torques;

$f_1(t)$ and $f_2(t)$ are other perturbing torques relative to axes of suspension of the GP.

Components $M_{(g.w)_x}$ and $M_{(g.w)_y}$ are determined by formulas (2.3.56); we will at first consider moments of forces of damping (damping torques) to be equal to zero; then we will rewrite (11.4.13) in the form

$$\left.\begin{aligned} M_{x_1} &= -ml\left[W_\eta - (g_0 + W_\zeta)\beta\right] + f_2(t) \\ M_{y_1} &= ml\left[W_\xi - (g_0 + W_\zeta)\alpha\right] + f_1(t) \end{aligned}\right\} \qquad (11.4.14)$$

Substituting (11.4.14) into (11.4.12), we have

$$\left.\begin{aligned} J_0\ddot{\beta} - H\left(\dot{\alpha} + u_\eta - u_\zeta\beta\right) &= ml\left[W_\eta - \left(g_0 + W_\zeta\right)\beta\right] - f_2(t) \\ J_0\ddot{\alpha} - H\left(-\dot{\beta} + u_\xi - u_\zeta\alpha\right) &= ml\left[W_\xi - \right. \\ &\left. - \left(g_0 + W_\zeta\right)\alpha\right] + f_1(t) \end{aligned}\right\} \qquad (11.4.15)$$

For the case of a fixed base, if we do not consider also rotation of Earth, $u_\xi = u_\eta = u_\zeta = 0$, $W_\xi = W_\eta = W_\zeta = 0$, and in place of (11.4.15) we will obtain

$$\left.\begin{aligned} J_0\ddot{\beta} - H\dot{\alpha} + mg_0l\beta &= -f_2(t) \\ J_0\ddot{\alpha} + H\dot{\beta} + mg_0l\alpha &= f_1(t) \end{aligned}\right\} \qquad (11.4.16)$$

Considering [see (2.2.3)]

$$mg_0 \approx mg = P, \qquad (11.4.17)$$

we will rewrite (11.4.16) in the form

$$\left.\begin{aligned} J_0\ddot{\beta} + lP\beta - H\dot{\alpha} &= -f_2(t) \\ J_0\ddot{\alpha} + lP\alpha + H\dot{\beta} &= f_1(t) \end{aligned}\right\} \qquad (11.4.18)$$

Let us divide left and right sides of these equations by H:

$$\left.\begin{aligned} \frac{J_0}{H}\ddot{\beta} + \frac{lP}{H}\beta - \dot{\alpha} &= -\frac{1}{H}f_2(t) \\ \frac{J_0}{H}\ddot{\alpha} + \frac{lP}{H}\alpha + \dot{\beta} &= \frac{1}{H}f_1(t) \end{aligned}\right\} \qquad (11.4.19)$$

Let us designate

$$T_0 = \frac{J_0}{H}; \quad k = \frac{lP}{H}; \quad h = \frac{1}{H}. \qquad (11.4.20)$$

Then we will write equations (11.4.19) in the form

$$\left.\begin{aligned} T_0\ddot{\beta} + k\beta - \dot{\alpha} &= -hf_2(t) \\ T_0\ddot{\alpha} + k\alpha + \dot{\beta} &= hf_1(t) \end{aligned}\right\} \qquad (11.4.21)$$

or in operator form ($p = \frac{d}{dt}$)

$$\left.\begin{aligned} (T_0p^2 + k)\beta - p\alpha &= -hf_2(t) \\ p\beta + (T_0p^2 + k)\alpha &= hf_1(t) \end{aligned}\right\} \qquad (11.4.22)$$

Eliminating variables β and α, we will obtain

$$\left.\begin{aligned} [(T_0p^2 + k)^2 + p^2]\alpha &= (T_0p^2 + k)hf_1(t) + hpf_2(t) \\ [(T_0p^2 + k)^2 + p^2]\beta &= -(T_0p^2 + k)hf_2(t) + hpf_1(t) \end{aligned}\right\} \qquad (11.4.23)$$

Applying to (11.4.23) Laplace transform, we find following expressions for transfer functions of the gyropendulum:

$$\left.\begin{aligned} W_{11}(s) &= \frac{\alpha(s)}{f_1(s)} = \frac{(T_0 s^2 + k)h}{(T_0 s^2 + k)^2 + s^2} \\ W_{12}(s) &= \frac{\alpha(s)}{f_2(s)} = \frac{hs}{(T_0 s^2 + k)^2 + s^2} \\ W_{21}(s) &= \frac{\beta(s)}{f_1(s)} = \frac{hs}{(T_0 s^2 + k)^2 + s^2} \\ W_{22}(s) &= \frac{-\beta(s)}{f_2(s)} = \frac{(T_0 s^2 + k)h}{(T_0 s^2 + k)^2 + s^2} \end{aligned}\right\} \qquad (11.4.24)$$

As will be shown further, on during pulse action, in accordance with (11.4.24), the GP, in distinction from a three-degree-of-freedom astatic gyroscope (§ 3.2, Par. 3), will accomplish two forms of undamped harmonic oscillations: 1) nutational oscillations with frequency $\frac{1}{T_0}$, where T_0 is time constant of GP (for its nutational oscillations); 2) precessional oscillations with frequency k.

In the case of a balanced gyroscope $l = 0$, and consequently, $k = 0$. Then transfer functions (11.4.24) of the GP will be the same as transfer functions (3.2.31) of a three-degree-of-freedom astatic gyroscope, if in the last expressions we take $k = h$, $k_1 = k_2 = hT_0$, $T = T_0$.

Differential equations and transfer functions of the GP, taking into account damping and other factors, will be composed further on.

3. Transient Responses of the GP on a Fixed Base

a) Transient Response of Nutational and Precessional Oscillations of Axis of Gyroscope

Knowing expressions of transfer functions of GP, we can find transient response of gyropendulum on fixed base with help of the same methods of the theory of automatic control which we used in § 3.2, Par. 3, during the analysis of motion of three-degree-of-freedom astatic gyroscope, when disturbances $f_1(t)$ and $f_2(t)$ were considered in the form of moments of instantaneous forces (impacts) or of torques constant in magnitude. Without producing detailed analysis of transient responses of the GP, we will be limited to their general consideration, proceeding from the form of the transfer functions. Thus we will use one important circumstance which introduces an essential simplification.

According to one of the theorems (§ 10.2), if condition of stability is observed and gyroscopic forces dominate over others (for large H), then frequencies of oscillations of gyroscope are separated and can be calculated independently of each other.

Actually, for analysis of basic precessional motion of the GP, in differential

equations (11.4.21) it is possible to discard inertial terms; this is equivalent to disregarding time constant T_0 of the gyroscope in expressions (11.4.24) for transfer functions. Then instead of (11.4.24) we will obtain

$$\left.\begin{aligned} W_{11}(s) &= \frac{\alpha(s)}{f_1(s)} = -\frac{hk}{s^2+k^2} \\ W_{12}(s) &= \frac{\alpha(s)}{f_2(s)} = \frac{hs}{s^2+k^2} \\ W_{21}(s) &= \frac{\beta(s)}{f_1(s)} = \frac{hs}{s^2+k^2} \\ W_{22}(s) &= \frac{-\beta(s)}{f_2(s)} = \frac{hk}{s^2+k^2} \end{aligned}\right\} \qquad (11.4.25)$$

As an example we will give the characteristic of motion of gyroscope relative to coordinate α. From the expression for transfer function $W_{11}(s)$, it follows that the GP with respect to disturbance $f_1(t)$ relative to analogous axis (§ 3.2, Par. 3) can be considered as a conservative network, and therefore motion $\alpha(t)$ during disturbance $f_1(t)$ in the form of instantaneous torque (impact) will represent undamped harmonic precessional oscillations with frequency k. In this case trajectory of vertex of gyroscope on the representative plane is a circle.

On the other hand, the form of transfer function $W_{12}(s)$ shows that the GP with respect to disturbance $f_2(t)$ relative to the cross axis can be considered as a conservative network with introduction of derivative [114], i.e., motion of gyroscope $\alpha(t)$ during disturbance $f_2(t)$ will constitute undamped harmonic precessional oscillations with frequency k. Frequency k of precessional oscillations of GP is determined by relationship (11.4.4), and their period $T_{g.p}$ — by formula (11.4.5).

For analysis of nutational oscillations of GP in accordance with the above-indicated theorem, in differential equations (11.4.21) it is possible to discard terms containing as a factor the quentity k, or to do the same in expressions (11.4.24) for transfer functions of the GP; then the latter will take the form

$$\left.\begin{aligned} W_{11}(s) &= \frac{\alpha(s)}{f_1(s)} = \frac{T_0 h}{T_0^2 s^2+1} \\ W_{12}(s) &= \frac{\alpha(s)}{f_2(s)} = \frac{h}{(T_0^2 s^2+1)s} \\ W_{21}(s) &= \frac{\beta(s)}{f_1(s)} = \frac{h}{(T_0^2 s^2+1)s} \\ W_{22}(s) &= \frac{-\beta(s)}{f_2(s)} = \frac{T_0 h}{T_0^2 s^2+1} \end{aligned}\right\} \qquad (11.4.26)$$

Consequently, transfer functions of GP will be the same as transfer functions (3.2.31) of astatic gyroscope with three degrees of freedom, if in (3.2.31) we take $k = h$, and $k_1 = k_2 = hT_0$, $T = T_0$. Therefore, transient responses of GP here are the same as for the indicated gyroscope (§ 3.2, par. 3). In general, for instance, motion of axis of gyroscope $\alpha(t)$ constitutes undamped harmonic oscillations with frequency $1/T_0$ and shift of axis of gyroscope superimposed on them, which is proportional to the integral of input disturbance $f_2(t)$. These sustained oscillations of axis of gyroscope are nutational oscillations; their frequency, by analogy with (3.2.62), will be

$$n = \frac{1}{T_0} \tag{11.4.27}$$

or, considering (11.4.20),

$$n = \frac{H}{J_0}.$$

Period T_{nut} of nutational oscillations of gyropendulum is determined by expression

$$T_0 = \frac{2\pi}{n} = 2\pi \frac{J_0}{H}. \tag{11.4.28}$$

Example 11.2: To calculate time constant T_0, frequency n and period T_{nut} of nutational oscillations of a GP whose parameters are given in example 11.1.

Solution: 1. By formula (11.4.27) we find frequency n of nutational oscillations:

$$n = \frac{H}{J_0} = \frac{1,708 \cdot 10^5}{48,94} = 3,49 \cdot 10^3 \text{ 1/sec.}$$

2. Period T_{nut} of nutational oscillations is determined by formula (11.4.28):

$$T_0 = \frac{2\pi}{n} = \frac{6,28}{3,49 \cdot 10^3} = 1,8 \cdot 10^{-3} \text{ sec.}$$

3. According to (11.4.27), we find time constant T_0 of the GP by considering its nutational oscillations:

$$T_0 = \frac{1}{n} = \frac{1}{3,49 \cdot 10^3} = 0,2865 \cdot 10^{-3} \text{ sec.}$$

Comparing period of nutation ($T_{nut} = 1.8 \cdot 10^{-3}$ sec.) of the GP with period of its basic precessional motion ($T_{g.p} = 858.5$ sec, Example 11.1), we see that the latter is many times larger than period of nutational oscillations. Amplitudes of nutational oscillations of axis of GP, just as for the astatic gyroscope with three degrees of freedom (§ 3.2, Par. 3), are obtained to be negligible and many times less than amplitudes of precessional oscillations of the GP.

Comparison [76, 20, 135] of amplitudes of precessional oscillations of GP with amplitudes of oscillations of physical pendulum (for the same initial conditions) shows that the first is considerably less than the second. This also indicates

advantages of gyroscopic pendulum over the usual physical pendulum.

Using the earlier indicated theorem, we found frequencies k and n of precessional and nutational oscillations of GP separately. Exact value of these frequencies can be determined according to (11.4.23) by means of solution of following characteristic equation:

$$(T_0\lambda^2 + k)^2 + \lambda^2 = 0. \qquad (11.4.29)$$

As calculations show [90], relative error of separate calculation of frequencies k and n as compared to their exact determination proceeding from (11.4.29) is a fraction of a percent; this is fully acceptable for practice.

In connection with above considered question, it is important to present the following theorem [90]: if frequencies of system with sufficiently large H are separated, then solution of simplified equations is acceptable. Therefore, during analysis of GP there are usually used simplified (shortened) equations, and basic precessional motion of gyroscope is investigated.

b) Stability of GP

Above considered oscillations of GP are stable for $l > 0$, i.e., when center of gravity of gyropendulum is located lower than point of its suspension (Fig. 11.7). However, it is possible to show that for sufficiently large value of angular momentum H, GP will be stable and also for $l < 0$, when its center of gravity is located higher than point of suspension, i.e., in the case of a statically unstable system ("inverted" GP). Actually, characteristic equation (11.4.29) corresponding to homogeneous differential equation of natural oscillations of GP can be written in the form

$$T_0^2\lambda^4 + (1 + 2T_0k)\lambda^2 + k^2 = 0. \qquad (11.4.30)$$

Since $k = \frac{lP}{H}$ [see (11.4.20)], then for $l < 0$ we have $k < 0$. Analysis of characteristic equation (11.4.30) shows that its roots are real, and motion of GP is stable if the following inequality [20] is satisfied:

$$H > 2\sqrt{J_0|l|P}. \qquad (11.4.31)$$

It is easy to show that this condition is sufficient. Stability of "inverted" GP if condition (11.4.31) is satisfied gives an example of possibility of realization of gyroscopic stabilization of a statically unstable mechanical system (§ 10.2).

c) Influence on GP of Frictional Torques in the Suspension

Influence of fluid frictional torques in axes of suspension of GP on character

of its natural oscillations is considered in a number of works (for instance, [20, 71]). It is shown that in this case nutational and precessional oscillations of axis of gyroscope under the influence of friction attenuate; nutation attenuates faster than precession [20]. In connection with this it is interesting to present the statement of B. V. Bulgakov [18]: "this selective action of resisting forces, which rapidly quenches high-frequency oscillations, and is the true cause of the fact that usually in all technical applications of the gyroscope it is necessary for us to deal only with precession, while nutation appears only during starting or stopping of the gyroscope, during shocks and vibrations of supports, etc."

The question of the influence on GP of dry frictional torques in axes of suspension was considered by certain authors [18, 57]. Here also there occurs damping of nutational and precessional oscillations of axis of gyroscope; thus, as in the case of fluid friction, nutation attenuates faster than precession. Trajectory of vertex of gyroscope on representative plane constitutes a converging spiral with a dead zone [45]. A characteristic peculiarity of the case of dry friction, as it was shown in [18], is full quenching of precessional and nutational motions in a finite time.

d) Quenching of Natural Oscillations of GP

As was shown in Par. 1b, for transformation of gyropendulum into a gyrovertical it is necessary to quench its natural precessional oscillations. Use for this of forces of friction in suspension leads to additional errors of the GP, especially in case of vehicle maneuvers and oscillations. Therefore, it is necessary to decrease frictional force in suspension by all possible means, and to quench natural oscillation of the GP with help of special devices.

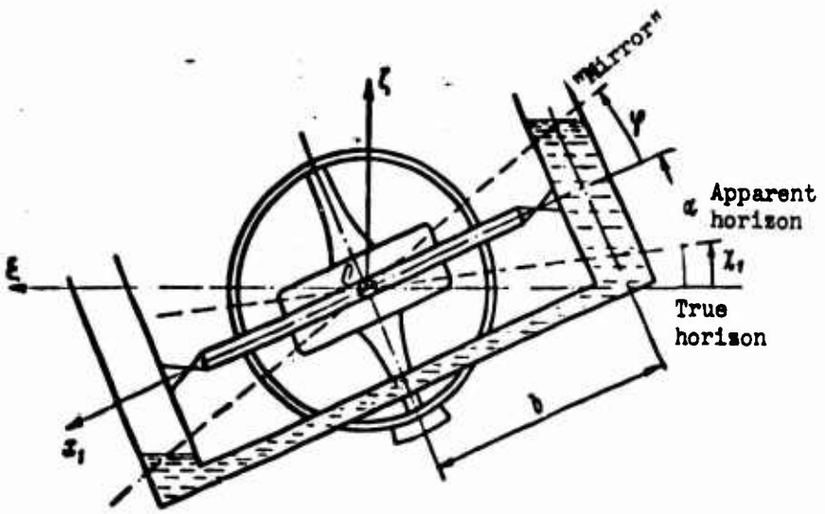

Fig. 11.8. Hydraulic damper of GP.

In certain GD, for instance in GC and GP, for quenching of natural oscillations of axis of gyroscope we apply a hydraulic damper, which is made according to the model of Frahm cisterns intended for damping of rolling of a ship. The damper consists of two vessels containing liquid joined with each other by a thin tube and connected with the corresponding ring of the Cardan suspension of the GP. Action of such a damper is based on the fact that the liquid during shift in the vessels lags in phase relative to oscillations of the gyroscope and by its weight counteracts oscillatory motion of the gyroscope.

Let us consider the behavior of a GP supplied with hydraulic damper; for this we will use the account of this question which is due to B. V. Bulgakov [20]. Schematic diagram of the damper connected with the external Cardan ring is shown in Fig. 11.8; on it there is designated: α – angle of inclination of axis of gyroscope together with outer Cardan ring; φ – angle of inclination of "mirror," i.e., of the straight line connecting centers of gravity of free surfaces of the liquid in both vessels with plane of the outer Cardan ring; χ_1 – angle of inclination of apparent horizon to true, horizon which by analogy with relationship (4.5.22) is determined by formula

$$\chi_1 = \frac{w_1}{g}. \tag{11.4.32}$$

Magnitude of torque introduced by surplus of liquid in vessel,

$$M_{sur} = c\varphi, \tag{11.4.33}$$

where $c = \frac{1}{2}b^2 S_\gamma$;

b – distance from center of gravity of vessel to axis of rotation;

S – area of cross section of each vessel;

γ – specific gravity of liquid.

Rate $\dot{\varphi}$ of flow of liquid in vessels on the basis of Poiseuille's law is expressed by relationship

$$\dot{\varphi} = -F(\alpha + \varphi), \tag{11.4.34}$$

where F is so-called flow factor [71], which is defined in terms of structural constants of the system.

Let us compose differential equations of motion of a GP supplied with a hydraulic damper. Being interested in analysis only of basic precessional motion of the gyroscope, we will use equations (11.4.21), discarding in them inertial terms $T_0\ddot{\alpha}$ and $T_0\ddot{\beta}$; torques $f_1(t)$ and $f_2(t)$ of disturbances will be taken equal to zero; we will consider torque M_{sur} [see (11.4.33)] applied relative to axis Oy_1 (Fig. 2.19). Then we will obtain equations

$$\left.\begin{aligned}&\dot{\alpha}-k_1^2\beta=0\\&\dot{\beta}+k\alpha+(1-\rho)k_2\tau=0\end{aligned}\right\}\qquad(11.4.35)$$

where ρ, if we consider (11.4.20), is determined by relationship

$$\rho=1-\frac{c}{lP}.\qquad(11.4.36)$$

Equations (11.4.35) should be supplemented with equation (11.4.34) of oscillations of liquid in the damper; then we will obtain system of equations [20]

$$\left.\begin{aligned}&\dot{\alpha}-k_1^2\beta=0\\&\dot{\beta}+k\alpha+(1-\rho)k_2\tau=0\\&\dot{\tau}=-F(\alpha+\tau)\end{aligned}\right\}\qquad(11.4.37)$$

characterizing motion of GP supplied with hydraulic damper.

Characteristic equation of system (11.4.37) we will write in the form

$$a_3\lambda^3+a_2\lambda^2+a_1\lambda+a_0=0,\qquad(11.4.38)$$

where

$$a_0=1;\quad a_1=\frac{1}{F\rho};\quad a_2=\frac{1}{k^2\rho};\quad a_3=\frac{1}{Fk^2\rho}.\qquad(11.4.39)$$

For equation the cubic, according to the Hurwitz criterion, conditions of stability have the form

$$\left.\begin{aligned}&a_3>0;\quad a_2>0;\quad a_1>0;\quad a_0>0\\&\Delta_2=\begin{vmatrix}a_2 & a_0\\ a_3 & a_1\end{vmatrix}=a_2a_1-a_3a_0>0\end{aligned}\right\}\qquad(11.4.40)$$

whence, considering (11.4.39) and (11.4.36), we will obtain the following condition of stability [20]:

$$0<\rho<1\qquad(11.4.41)$$

or

$$0<c<lP,\qquad(11.4.42)$$

if this condition is satisfied natural oscillations of the GP will be attenuated.

The solution of more general problem about hydraulic quenching of oscillations of GP in two planes [20] is known.

Let us note that during investigation of question about damping of natural oscillations of GP in the presence of accelerations of the vehicle, we should originate from equations (11.4.15), supplementing them with expression (11.4.33) of torque M_{sur}, and should write in place of (11.4.34) the equation of oscillations of fluid in the damper in the form

$$\dot{\tau}=-F(\alpha+\tau+\chi_1),\qquad(11.4.43)$$

where χ_1 (Fig. 11.8) is determined by relationship (11.4.32), in which W_ξ is component

of acceleration along axis $O\xi$.

Besides the hydraulic damper, there exist other means for quenching of natural oscillations of GP. One of the wide-spread methods is based on principle of radial correction, which also is applied in GV based on the astatic pendulum gyroscope (§ 6.2, Par. 2). During analysis of these GV there was considered case of radial correction and taken into account possible unbalance of the gyroscope; this is equivalent to imposition on the GP of radial correction torques.

If we consider what was said, differential equations of precessional oscillations of GP with radial correction in the absence of disturbances can be obtained from equations (6.2.21), if we set in them $f_1(t) = f_2(t) = \chi_1(t) = \chi_2(t) = 0$:

$$\left.\begin{aligned} T_1\dot{\alpha} + \alpha - kT_1\beta = 0 \\ T_1\dot{\beta} + \beta + kT_1\alpha = 0 \end{aligned}\right\} \qquad (11.4.44)$$

where T_1 is time constant of correction, and k is defined by relationship (11.4.20).

Characteristic equation of system (11.4.44) has the form

$$T_1^2\lambda^2 + 2T_1\lambda + 1 + k^2T_1^2 = 0 \qquad (11.4.45)$$

or

$$T_д^2\lambda^2 + 2\zeta T_д\lambda + 1 = 0, \qquad (11.4.46)$$

where

$$T_д = \frac{T_1}{\sqrt{1 + k^2T_1^2}}; \quad \zeta = \frac{1}{\sqrt{1 + k^2T_1^2}}. \qquad (11.4.47)$$

According to (11.4.46), the GP with radial correction can be considered as an oscillating circuit. Consequently, natural oscillations of the GP will be attenuated. Quantity T_d is time constant of this circuit, and ζ is relative attenuation factor. Frequency of natural sustained oscillations of a GP with radial correction is determined by relationship*

$$k_д = \frac{1}{T_д} = \frac{\sqrt{1 + k^2T_1^2}}{T_1} = \sqrt{k^2 + \frac{1}{T_1^2}}. \qquad (11.4.48)$$

In the absence of radial correction $T_1 = \infty$, and then $k_d = k$. Thus, use in GP of radial correction ensures quenching of natural precessional oscillations of axis of gyroscope exactly as in the case of application of a hydraulic damper. Theory of GP with radial correction was developed in detail by B. V. Bulgatrov [20].

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*Let us note that in part I of the book on pp. 283 and 425, quantity $\frac{1}{T_d}$ should be called not frequency of damped oscillations of the gyroscope, but frequency of undamped oscillations of the gyroscope with damping.

4. Static Errors of GP Due to Rotation of the Earth and Vehicular Motion (Rate Errors of GP). Ballistic Errors of GP and Their Compensation

a) Preliminary Remark

Most important is analysis of motion of GP with consideration of rotation of Earth and vehicular motion; thus there are determined:

1) errors due to rotation of Earth and rectilinear and uniform motion of the object (vehicle), or rate errors of the GP;

2) errors due to longitudinal accelerations of the object and its circulation (turn), called ballistic errors of the GP.

The question about calculations and methods of compensation of rate errors of GP is comparatively simple. The most complicated turns out to be determination of ballistic errors of GP and establishment of main possibilities of their compensation. Solution of this problem has occupied many investigators, among which it is necessary to mention M. Schuler [199], A. N. Krylov [65], B. V. Bulgakov [20], B. I. Kudrevich [71], A. Yu. Ishlinskiy [45, 53], Ya. N. Roytenberg [127], R. I. Chertkov [168, 135], E. I. Sliv [135], L. I. Tkachev [159].

Below there is given short information on the theory of rate and ballistic errors of GP, and also there are considered general principles of their compensation.

b) Velocity Errors of GP

During determination of velocity and ballistic errors of GP it is necessary to agree on orientation of the axes $O\xi\eta\zeta$ (Fig. 2.19) taken as the reference system. Usually during investigation of GP axes $O\xi$ and $O\eta$ are disposed geographically [20, 71]. However, another approach is possible (see, for instance [127]), when during the analysis of this instrument axes $O\xi$ and $O\eta$ are connected with the trajectory of the object (vehicle). In this case angles α and β (Fig. 2.19) characterize instrument errors relative to axes connected with the trajectory of the object; this is of the greatest interest.* Such an orientation of axes $O\xi\eta\zeta$ we used earlier during the analysis of the gyrovertical (Chapter 6); we will also apply it in examining the theory of GP.

For determination of rate errors of GP we will use equations (11.4.15): we will discard in them inertial terms and will take disturbances $f_1(t)$ and $f_2(t)$ equal to zero; then these equations will have the form

*In the absence of rolling and yawing of the ship, angle α characterizes error of the GP in plane of a rib of the ship, and angle β characterizes error of the GP in diametrical plane of ship.

$$\left.\begin{aligned}\dot{\alpha}-\left[\frac{ml}{H}(g_0+W_\zeta)+u_\zeta\right]\beta&=-u_\eta-\frac{ml}{H}W_\eta\\ \dot{\beta}+\left[\frac{ml}{H}(g_0+W_\zeta)+u_\zeta\right]\alpha&=u_\xi+\frac{ml}{H}W_\xi\end{aligned}\right\}\qquad(11.4.49)$$

Velocity errors of GP are determined during rectilinear and uniform motion of the object. In this case center of gravity of the object will move on the terrestrial sphere along the arc of a great circle, i.e., by orthodromy, and for the above adopted orientation of axes $O\xi\eta\zeta$, projections u_ξ, u_η, u_ζ of induced angular velocity $\vec{u}$ of these axes are determined by relationships (1.2.32):*

$$\left.\begin{aligned}u_\xi&=-U\cos\varphi\sin K-\frac{v}{R}\\ u_\eta&=U\cos\varphi\cos K\\ u_\zeta&=U\sin\varphi\end{aligned}\right\}\qquad(11.4.50)$$

For the considered case of orientation of axes $O\xi\eta\zeta$, projections W_ξ, W_η, W_ζ are determined by relationships (2.2.42), in which we will take $W_s = W_\eta$, $W_n = W_\xi$; considering in them $v_s = v$ and $h \ll R$, we will obtain

$$\left.\begin{aligned}W_\xi&=-U^2R\cos\varphi\sin\varphi\sin K-2Uv\sin\varphi+2Uv_\zeta\cos\varphi\cos K\\ W_\eta&=\dot{v}+\frac{vv_\zeta}{R}+U^2R\sin\varphi\cos\varphi\cos K+2Uv_\xi\cos\varphi\sin K\\ W_\zeta&=\dot{v}_\zeta-\frac{v^2}{R}-U^2R\cos^2\varphi-2Uv\cos\varphi\sin K\end{aligned}\right\}\qquad(11.4.51)$$

For a ship with rectilinear and uniform motion $\dot{v} = 0$, $v_\zeta = 0$, $\dot{v}_\zeta = 0$; then, disregarding in (11.4.51) the minute remaining terms, we will approximately consider that $W_\xi = W_\eta = W_\zeta = 0$. Considering what has been said, we will rewrite (11.4.49) in the form

$$\left.\begin{aligned}\dot{\alpha}-\left(\frac{ml}{H}g_0+u_\zeta\right)\beta&=-u_\eta\\ \dot{\beta}+\left(\frac{ml}{H}g_0+u_\zeta\right)\alpha&=u_\xi\end{aligned}\right\}\qquad(11.4.52)$$

or, taking into account (11.4.17) and (11.4.20),

$$\left.\begin{aligned}\dot{\alpha}-(k+u_\zeta)\beta&=-u_\eta\\ \dot{\beta}+(k+u_\zeta)\alpha&=u_\xi\end{aligned}\right\}\qquad(11.4.53)$$

Considering for a comparatively small time of use of the instrument that latitude φ is constant, according to (11.4.50) we will designate

$$k'=k+u_\zeta=k+U\sin\varphi.\qquad(11.4.54)$$

*For the considered case of motion of the object there were also determined rate errors of a GV using three-degree-of-freedom astatic pendulum gyroscope (§ 6.4, par. 2a). Motion of object is assumed to be along arc of great circle (by orthodromy); thus the course of object is variable. In general rate errors of GP, just as of GC (§ 11.5, Par. 4b), during change of speed and course of the object will be variable.

Then in place of (11.4.53) we will obtain

$$\left.\begin{aligned}\dot{\alpha}-k'\beta&=-u_1\\ \dot{\beta}+k'\alpha&=u_2\end{aligned}\right\} \qquad (11.4.55)$$

whence we find static errors of the GP due to rotation of Earth and vehicular motion, i.e., rate errors of the GP

$$\alpha^{c}_{ст}=\frac{1}{k'}u_2;\quad \beta^{c}_{ст}=\frac{1}{k'}u_1 \qquad (11.4.56)$$

or, considering (11.4.50),

$$\alpha^{c}_{ст}=-\frac{1}{k'}\left(U\cos\varphi\sin K+\frac{v}{R}\right);\quad \beta^{c}_{ст}=\frac{1}{k'}U\cos\varphi\cos K. \qquad (11.4.57)$$

Formulas (11.4.57) can be somewhat simplified. Actually, for the usually taken parameters of the GP $k \gg U\sin\varphi$. Thus, for instance, for conditions of example 11.1 there was obtained $k = 7.319\cdot10^{-3}$ 1/sec, while $U = 7.29\cdot10^{-5}$ 1/sec, i.e., the magnitude of k was approximately 100 times greater than that of U. Therefore, considering, according to (11.4.54), $k' \approx k$, in place of (11.4.57) we have*

$$\alpha^{c}_{ст}=-\frac{1}{k}\left(U\cos\varphi\sin K+\frac{v}{R}\right);\quad \beta^{c}_{ст}=\frac{1}{k}U\cos\varphi\cos K. \qquad (11.4.58)$$

Example 11.3: To calculate rate errors of the GP, whose parameters are given in example 11.1 for installation of it on an aircraft flying with speed v = 200 m/sec with course $K = 45^\circ$ at latitude $\varphi = 0^\circ$.

Solution: By the formulas (11.4.58) we find

$$\alpha^{c}_{ст}=-\frac{1}{k}\left(U\cos\varphi\sin K+\frac{v}{R}\right)=-\frac{1}{7{,}319\cdot10^{-3}}\left(7{,}29\cdot10^{-5}\cdot0{,}7071+\frac{200}{6371\cdot10^{3}}\right)=-0{,}01133\text{ rad}=-39';$$

$$\beta^{c}_{ст}=\frac{1}{k}U\cos\varphi\cos K=\frac{1}{7{,}319\cdot10^{-3}}7{,}29\cdot10^{-5}\cdot0{,}7071=0{,}007043\text{ rad}=24'.$$

From Example 11.3 it follows that velocity errors of GP can be considerable. In this case they were obtained to be larger than for the gyrovertical in which there is used an astatic gyroscope with radial correction (Example 6.4).

In the presence of rate errors rate α^{v}_{st} and β^{v}_{st} there appear corresponding gravitational torques, which cause precessional motion of GP with velocity equal to

*Formulas (11.4.57) for rate errors of GP are of the same type as the analogous expressions of B. V. Bulgakov [20], and contain as a factor the quantity $\frac{1}{k'} = \frac{1}{k+U\sin\varphi}$ [see (11.4.54)]; thus in them there is contained geographic latitude of the place. Since usually $k \gg U\sin\varphi$, then from (11.4.57) we went over to simplified formulas (11.4.58), which contain quantity $\frac{1}{k}$. These relationships for rate errors will be used in further presentation of elementary information on the theory of ballistic errors of GP. Developing refined theory of GP, R. I. Chertkov [135] obtained expressions for rate errors of GP somewhat differing from those mentioned above, in them as a factor there is contained quantity $\frac{1}{k}$, but there is considered not geographic, but geocentric latitude of the place.

velocity of rotation in space of the vertical of the place (§ 6.4, Par. 2).

Let us note that in principle there is possible creation of a GP not having rate errors. Thus, for instance, B. V. Bulgakov [20] considered a GP consisting of two gyroscopes rotating in different directions. For them there is satisfied the condition of undisturbability by accelerations of the object. Rate errors of such gyroscopes are equal in magnitude, but are opposite in sign and can be in principle mutually compensated. If we realize materially the bisector of axes of these two gyroscopes, then it during any motion of the object can accomplish only oscillation near the true vertical, without experiencing any perturbations from velocity and accelerations; thus the instrument can also indicate in a certain scale absolute velocity of object relative to inertial space [20]. Detailed theoretical investigation of two-gyroscope vertical was carried out by A. Yu. Ishlinskiy [53] and R. I. Chertkov [135].

Ya. N. Roytenberg [127] showed that for the so-called four-gyroscope vertical (Par. 6) during realization of the condition of undisturbability, there also are absent rate and ballistic errors;* thus principal vector of spin angular momenta of gyroscopes is proportional to absolute velocity of the ship in space.

c) Ballistic Errors of GP

General remark. In the presence of accelerations of the object, due to noncoincidence of center of gravity of GP with point of its suspension, on the gyroscope there will act moments of forces of inertia, which cause forced precessional motion of axis of gyroscope, which due to this deviates from position of equilibrium determined by rate errors (Par. b). Thus appearing additional errors of gyroscope caused by accelerations of object are called ballistic. These errors can considerably exceed rate errors. When action of accelerations on GP is ceased, axis of gyroscope accomplishes natural oscillations near position of equilibrium determined by velocity and course of the object.

During finding of ballistic errors of GP we will consider two particular, but most characteristic cases of maneuver of the object: rectilinear uniformly accelerated motion and circulation (turn).

Rectilinear uniformly accelerated motion. Let us use equations (11.4.49) of precessional motion of GP. As the object we will consider a ship accomplishing rectilinear uniformly accelerated motion by orthodromy. In this case projections u_ξ, u_η, u_ζ of induced angular velocity $\vec{u}$ of axes $O\xi\eta\zeta$ are determined by formulas

*This circumstance for the first time was indicated by M. Schuler [199].

(11.4.50), and projections W_ξ, W_η, W_ζ of linear acceleration $\overline{W}$ of the ship – by relationships (11.4.51). It is easy to show that in the considered case for the ship $W_\zeta \ll g_0$. Actually, according to (11.4.51), for $\dot{v}_\zeta = 0$

$$W_\zeta = -\frac{v^2}{R} - U^2 R\cos^2\varphi - 2Uv\cos\varphi\sin K. \qquad (11.4.59)$$

Let us estimate the biggest values of terms contained in (11.4.59) (example 11.4).

Example 11.4: To calculate values of quantities contained in formula (11.4.59), $\frac{v^2}{R}$, U^2R and $2Uv$, during motion of ship with velocity $v = 10$ m/sec; $R = 6371$ km, $U = 7.29\cdot10^{-5}$ 1/sec.

Solution: 1. Let us determine magnitude of $\frac{v^2}{R}$:

$$\frac{v^2}{R} = \frac{10^2}{6371\cdot10^3} = 0{,}157\cdot10^{-4}\ \text{m/sec}^2.$$

2. We find value of U^2R:

$$U^2R = 7{,}29^2\cdot10^{-10}\cdot6371\cdot10^3 = 3{,}386\cdot10^{-2}\ \text{m/sec}^2.$$

3. We calculate the value of $2Uv$:

$$2Uv = 2\cdot7{,}29\cdot10^{-5}\cdot10 = 0{,}1458\cdot10^{-2}\ \text{m/sec}^2.$$

4. By formula (11.4.59) we find the biggest value of vertical acceleration W_ζ:

$$|W_\zeta| < \left|\frac{v^2}{R} + U^2R + 2Uv\right| = 0{,}157\cdot10^{-4} + 3{,}386\cdot10^{-2} + 0{,}1458\cdot10^{-2} =$$

$$= 3{,}533\cdot10^{-2}\ \text{m/sec}^2.$$

From Example 11.4 it follows that in this case $W_\zeta \ll g_0$, and therefore in formulas (11.4.49), by analogy with (11.4.17) and (11.4.20), it is possible to take

$$\frac{ml}{H}(g_0 + W_\zeta) \approx \frac{ml}{H}g_0 \approx \frac{ml}{H}g = \frac{lP}{H} = k. \qquad (11.4.60)$$

Further, according to (11.4.54), $k + u_\zeta = k + U\sin\varphi = k'$, but since usually $k \gg U$, then we will take approximately $k' = k$. Then equations (11.4.49), if we consider (11.4.60) and (11.4.50), can be rewritten in the form

$$\left.\begin{aligned}\dot{\alpha} - k\beta &= -U\cos\varphi\cos K - \frac{k}{g}W_\eta\\ \dot{\beta} + k\alpha &= -U\cos\varphi\sin K - \frac{v}{R} + \frac{k}{g}W_\xi\end{aligned}\right\} \qquad (11.4.61)$$

Let us eliminate from these equations in the beginning β, and then α; thus we should consider that for orthodromy $K = \frac{v}{R}\sin K\tan\varphi$ [see (2.2.34)]. Taking latitude φ for a comparatively small time of maneuver of the ship to be constant, considering $k \gg U$ and being limited to calculation of terms of first order of smallness, we will obtain*

*In formulas (11.4.62) we disregarded terms containing factor $U\frac{v}{R}$, which for conditions of Example 11.4 is $U\frac{v}{R} = 7.29\cdot10^{-5}\frac{10}{6371\cdot10^3} = 1.14\cdot10^{-10}$ m/sec^2, which is negligible.

$$\left.\begin{aligned}\ddot{\alpha}+k^2\alpha&=-k\left(U\cos\varphi\sin K+\frac{v}{R}\right)+\frac{k^2}{g}W_\xi-\frac{k}{g}\dot{W}_\eta\\ \ddot{\beta}+k^2\beta&=kU\cos\varphi\cos K+\frac{k^2}{g}W_\eta-\frac{\dot{v}}{R}+\frac{k}{g}\dot{W}_\xi\end{aligned}\right\}\qquad(11.4.62)$$

Let us find for the considered case expressions of accelerations W_ξ, W_η and their derivatives contained in (11.4.62). Since for the ship $v_\zeta = 0$, then, according to (11.4.51), we have

$$\left.\begin{aligned}W_\xi&=-U^2R\cos\varphi\sin\varphi\sin K-2Uv\sin\varphi\\ W_\eta&=\dot{v}+U^2R\sin\varphi\cos\varphi\cos K\end{aligned}\right\}\qquad(11.4.63)$$

whence, considering (2.2.34), we will obtain

$$\left.\begin{aligned}\dot{W}_\xi&=-U^2v\sin^2\varphi\sin K\cos K-2U\dot{v}\sin\varphi\\ \dot{W}_\eta&=\ddot{v}-U^2v\sin^2\varphi\sin^2K\end{aligned}\right\}\qquad(11.4.64)$$

From Example 11.4 it follows that quantities U^2R and $2Uv$ contained in (11.4.63) as compared to v are negligible ($U^2R = 3.386\cdot10^{-2}$ m/sec^2; $2Uv = 0.1458\cdot10^{-2}$ m/sec^2); therefore, we should approximately take $W_\xi = 0$, $W_\eta = \dot{v}$. Considering formula (11.4.64), we notice that during uniformly accelerated motion of the ship $\dot{v} = 0$; quantities U^2v and $2U\dot{v}$ are also minute. Actually, for the conditions of Example 11.4 and $\dot{v} = 0.05$ m/sec^2 we have $U^2v = 53.14\cdot10^{-9}$ m/sec. $2U\dot{v} = 0.729\cdot10^{-5}$ m/sec^3 which is negligible. Consequently, with fully sufficient accuracy $\dot{W}_\xi = \dot{W}_\eta = 0$. Thus, in the considered case instead of (11.4.63) and (11.4.64) it is possible to take

$$W_\xi=0;\quad W_\eta=\dot{v};\quad \dot{W}_\xi=\dot{W}_\eta=0.\qquad(11.4.65)$$

Then we will rewrite equation (11.4.62) in the form

$$\left.\begin{aligned}\ddot{\alpha}+k^2\alpha&=-k\left(U\cos\varphi\sin K+\frac{v}{R}\right)\\ \ddot{\beta}+k^2\beta&=kU\cos\varphi\cos K+\left(\frac{k^2}{g}-\frac{1}{R}\right)\dot{v}\end{aligned}\right\}\qquad(11.4.66)$$

Quantity $\frac{k^2}{g} - \frac{1}{R}$, which is constant for a given GP, will be designated b_0:

$$b_0=\frac{k^2}{g}-\frac{1}{R}.\qquad(11.4.67)$$

Formula (11.4.67) can be represented in the form

$$b_0=\frac{k^2}{g}\left(1-\frac{v^2}{k^2}\right),\qquad(11.4.68)$$

where

$$v^2=\frac{g}{R}.\qquad(11.4.69)$$

Considering (11.4.67), we will rewrite equations (11.4.66) in the following way:

$$\left.\begin{aligned}\ddot{\alpha}+k^2\alpha&=-k\left(U\cos\varphi\sin K+\frac{v}{R}\right)\\ \ddot{\beta}+k^2\beta&=kU\cos\varphi\cos K+b_0\dot{v}\end{aligned}\right\}\qquad(11.4.70)$$

whence it follows that coefficient b_0 characterizes sensitivity of GP to acceleration $\dot{v}$ of motion of ship.

Solution of corresponding homogeneous equations

$$\ddot{\alpha} + k^2\alpha = 0; \quad \ddot{\beta} + k^2\beta = 0 \tag{11.4.71}$$

shows that on a non-rotating and fixed base (U = 0, v = 0, $\dot{v}$ = 0), the axis of gyroscope accomplishes conical precessional motion with angular velocity k; trajectory of vertex of gyroscope on representative plane is a circle. With introduction of damping, precessional motion of axis of gyroscope acquires an attenuating character, i.e., trajectory of its vertex is a spiral.

For determination of ballistic errors of GP we must integrate equations (11.4.70) for any law of change of velocity v(t) and acceleration $\dot{v}(t)$ of motion of the ship.

Taking into account formulas (11.4.58) for rate errors $\alpha^v_{st} = \alpha^v$, $\beta^v_{st} = \beta^v$ of the GP, we will rewrite equations (11.4.70) in the form*

$$\left.\begin{aligned} \ddot{\alpha} + k^2\alpha &= k^2\alpha^c \\ \ddot{\beta} + k^2\beta &= k^2\beta^c + b_0\dot{v} \end{aligned}\right\} \tag{11.4.72}$$

or

$$\left.\begin{aligned} \ddot{\alpha} + k^2(\alpha - \alpha^c) &= 0 \\ \ddot{\beta} + k^2(\beta - \beta^c) &= b_0\dot{v} \end{aligned}\right\} \tag{11.4.73}$$

Let us designate

$$\alpha^y = \alpha - \alpha^c; \quad \beta^y = \beta - \beta^c. \tag{11.4.74}$$

It is easy to see that α^{acc} and β^{acc} are ballistic errors of the GP caused by accelerations of ship. Then for determination of α^{acc} and β^{acc} we obtain the following system of differential equations:

$$\left.\begin{aligned} \ddot{\alpha}^y + k^2\alpha^y &= 0 \\ \ddot{\beta}^y + k^2\beta^y &= b_0\dot{v} \end{aligned}\right\} \tag{11.4.75}$$

Integrating these equations, we find ballistic errors of GP for any law of change of velocity of movement v(t) of ship. If equations (11.4.75) are integrated for certain given initial conditions, then we will obtain ballistic errors of the GP caused by its initial deviations and change of velocity of motion of the ship. For determination of α^{acc} and β^{acc}, during integration of equations (11.4.75) we will consider initial conditions to be zero [69]; i.e., for t = 0 $\alpha^{acc}(0) = \beta^{acc}(0) = 0$. This permits us to determine ballistic errors of GP caused by change of velocity of ship, and not by initial deflections of the gyropendulum.

Then from first equation of system (11.4.75) we obtain

*Here velocity error α^v_{st} is variable, since v = v(t); subsequently rate errors will be designated α^v and β^v.

$$\alpha^y = 0, \tag{11.4.76}$$

i.e., uniformly accelerated motion of ship does not cause error in coordinates α.*

Using the method of variation of arbitrary constants, we find solution of second equation of system (11.4.75) in the form

$$\beta^y = \frac{k_0}{k}\int_0^t \dot{v}(\tau)\sin k(t-\tau)\,d\tau. \tag{11.4.77}$$

As we see, acceleration $\dot{v}(t)$ of ship causes ballistic error β^{acc} of the GP. Formula (11.4.77) can be transformed in the following way:

Let us designate

$$b = 1 - \frac{v^2}{k^2}; \tag{11.4.78}$$

then, considering (11.4.68), in place of (11.4.77) we will obtain

$$\beta^y = k\frac{b}{g}\int_0^t \dot{v}(\tau)\sin k(t-\tau)\,d\tau. \tag{11.4.79}$$

Coefficient b can be called <u>coefficient of ballistic error</u> of the GP, since it characterizes ballistic instrument error during acceleration $\dot{v}(t)$ of the ship.

Let us consider the most simple case, when ship moves with constant acceleration $\dot{v}$ = const. Then from (11.4.79) we have

$$\beta^y = b\frac{\dot{v}}{g}(1-\cos kt). \tag{11.4.80}$$

<u>Example 11.5</u>: To calculate ballistic error β^{acc} of GP during rectilinear uniformly accelerated motion of ship; acceleration $\dot{v}$ = 0.05 m/sec^2; parameters of instrument are the same as in Example 11.1. To calculate error β^{acc} by the moment of termination of a maneuver continuing for 3 minutes.

<u>Solution</u>: 1. By formula (11.4.78), considering (11.4.69), we find value of coefficient b of ballistic error of GP:

$$b = 1 - \frac{v^2}{k^2} = 1 - \frac{g}{Rk^2} = 1 - \frac{9{,}81}{6371\cdot 10^3 (7{,}319\cdot 10^{-3})^2} = 0{,}9713.$$

2. According to (11.4.80) we calculate ballistic error β^{acc} of the GP:

$$\beta^y = b\frac{\dot{v}}{g}(1-\cos kt) = 0{,}9713\,\frac{0{,}05}{9{,}81}\left[1-\cos(7{,}319\cdot 10^{-3}\cdot 180)\right] = 0{,}00371 \text{ rad} = 0°13'.$$

We will compare ballistic error β^{acc} of GP with static deviation χ_{st}^{acc} of short-period physical pendulum. According to (4.5.22), this deviation will be

$$\chi_{cr}^{y} = \frac{\overline{w}_y}{g}$$

or, considering (11.4.65)

$$\chi_{cr}^{y} = \frac{\dot{v}}{g}. \tag{11.4.81}$$

In this case coefficient of ballistic error of physical pendulum is equal to

*What has been said is true only for zero initial conditions. For non-zero initial conditions, the vertex of the gyroscope describes on the representative plane a circle; i.e., there will occur ballistic errors also with respect to coordinate α.

unity, i.e., short-period physical pendulum is established in the direction of the apparent vertical, whereas coefficient of ballistic error of GP $b = 1 - \frac{\nu^2}{k^2}$ [see (11.4.78)] is less than unity. Consequently, a GP, for which $k \gg \nu$ and therefore $b \approx 1$, during rectilinear uniformly accelerated motion of ship has ballistic error β^{acc}, constant component of which is equal to deviation χ_{st}^{acc} of physical pendulum, i.e., axis of GP will accomplish oscillation near the apparent vertical. However, if for decrease of deviation of short-period pendulum it is difficult in practice to undertake anything, ballistic error of GP can be considerably decreased by means of selection of its parameters, and in principle can even be reduced to zero, since for $k = \nu$ coefficient $b = 0$ and error $\beta^{acc} = 0$.

The above mentioned formula can be written in another form. Let us divide equation (11.4.66) by k^2 and introduce time constant T of GP, which characterizes its basic precessional motion,

$$T = \frac{1}{k}. \tag{11.4.82}$$

Equations (11.4.66) will take the form

$$\left.\begin{aligned} T^2\ddot{\alpha} + \alpha &= -T\left(U\cos\varphi\sin K + \frac{v}{R}\right) \\ T^2\ddot{\beta} + \beta &= TU\cos\varphi\cos K + \left(\frac{1}{g} - \frac{T^2}{R}\right)\dot{v} \end{aligned}\right\} \tag{11.4.83}$$

By analogy with (11.4.67), considering (11.4.68) and (11.4.78), we will designate

$$b_0' = \frac{1}{g} - \frac{T^2}{R} = \frac{1}{g}\left(1 - \frac{T^2 g}{R}\right) = \frac{1}{g}(1 - \nu^2 T^2) = \frac{b}{g}. \tag{11.4.84}$$

Then we will write (11.4.83) in the form

$$\left.\begin{aligned} T^2\ddot{\alpha} + \alpha &= -T\left(U\cos\varphi\sin K + \frac{v}{R}\right) \\ T^2\ddot{\beta} + \beta &= TU\cos\varphi\cos K + b_0'\ddot{v} \end{aligned}\right\} \tag{11.4.85}$$

whence it follows that GP without damping can be considered as a harmonically oscillating (conservative) circuit. Time constant T of GP, which characterizes its harmonic oscillations, i.e., its basic precessional motion, is very large. Thus, for conditions of Example 11.1 we have $T = \frac{1}{k} = \frac{1}{7.319\cdot10^{-3}} = 137$ sec, which by many times exceeds time constant $T_0 = 0.2865\cdot10^{-3}$ sec. (Example 11.2), which characterizes nutational oscillations of the GP.

Considering (11.4.82), we can represent formulas (11.4.77) and (11.4.80) in the form

$$\beta^v = Tb_0\int_0^t \dot{v}(\tau)\sin\frac{1}{T}(t-\tau)\,d\tau, \tag{11.4.86}$$

$$\beta^v = T^2 b_0\dot{v}\left(1 - \cos\frac{t}{T}\right). \tag{11.4.87}$$

if we consider that, according to (11.4.68) and (11.4.78), $b_0 = \frac{k^2}{g}b = \frac{1}{T^2 g}b$.

Coefficients obtained above, b_0 [see (11.4.68)] and b_0 [see (11.4.78)], are expressed in terms of time constant T of gyropendulum in the following way:

$$b_0 = \frac{1}{T^2 g}(1 - v^2 T^2); \tag{11.4.88}$$

$$b = 1 - v^2 T^2. \tag{11.4.89}$$

Circulation (turn). Let us consider case of circulation of a ship with constant angular velocity ω_{circ} during constant speed of motion v. As in the preceding case, we will start with formulas (11.4.49). During circulation, the current value of heading of ship is determined by formula

$$K = K_0 - \omega_{circ} t, \tag{11.4.90}$$

where K_0 – initial heading;

$\omega_{circ} = -\dot{K}$ – angular velocity of circulation; during left circulation $\omega_{circ} > 0$.

In the considered case, projections u_ξ and u_η of induced angular velocity $\vec{u}$ of axes $O\xi\eta\zeta$ are determined by relationships (11.4.50); thus, in them heading K should be considered to be variable [see (11.4.90)]. Regarding projection u_ζ, in general it is expressed by formula (1.2.33):

$$u_\zeta = U \sin\varphi + \frac{v_E}{R} \operatorname{tg}\varphi - \dot{K}.$$

During circulation of ship it is permissible to take

$$u_\zeta \approx -\dot{K} = \omega_{circ}, \tag{11.4.91}$$

since usually for middle latitudes $|\omega_{circ}| >> \left|U \sin\varphi + \frac{v_E}{R} \tan\varphi\right|$.

For accelerations W_ξ and W_η during circulation of ship with constant velocity, disregarding, as before, components of centripetal acceleration from rotation of Earth and Coriolis acceleration, by analogy with (2.2.43), for $\dot{v} = 0$ we have

$$W_\xi = -v\omega_{circ}; \quad W_\eta = 0. \tag{11.4.92}$$

Since rate errors of GP were considered earlier in detail, subsequently we will not consider gyroscopic torques due to rotation of Earth. Then, putting relationships (11.4.60), (11.4.50), (11.4.91), (11.4.92) in (11.4.49) and considering U = 0, we will obtain

$$\left.\begin{aligned} \dot{\alpha} - (k + \omega_{circ})\beta &= 0 \\ \dot{\beta} + (k + \omega_{circ})\alpha &= -\frac{v}{R} - \frac{k}{g} v\omega_{circ} \end{aligned}\right\} \tag{11.4.93}$$

Eliminating at first β, and then α, we have

$$\left.\begin{aligned} \ddot{\alpha} + (k + \omega_{circ})^2 \alpha &= -(k + \omega_{circ})\frac{v}{R} - \frac{k + \omega_{circ}}{g} kv\omega_{circ} \\ \ddot{\beta} + (k + \omega_{circ})^2 \beta &= 0 \end{aligned}\right\} \tag{11.4.94}$$

whence it follows that axis of gyroscope accomplishes conical precessional motion with frequency $(k + \omega_{circ})$ and period

$$T'_{r.u} = \frac{2\pi}{k + \omega_u}, \tag{11.4.95}$$

which differs from period $T_{g.p}$ [see (11.4.5)] of precessional oscillations of axis of gyroscope in the absence of circulation of ship.

Ballistic errors α^{acc} and β^{acc} of GP during circulation of ship will be determined by relationships of type (11.4.74). For rate errors α^v, β^v of the GP, according to (11.4.58), for U = 0 we will obtain

$$\alpha^c = -\frac{v}{kR}; \quad \beta^c = 0. \tag{11.4.96}$$

Then for ballistic errors of GP in the considered case of left circulation of ship, considering (11.4.94), (11.4.74), (11.4.96) and (11.4.78), we have

$$\left.\begin{aligned} \ddot{\alpha}^v + (k + \omega_u)^2 \alpha^v &= -k(k + \omega_u) b \frac{v\omega_u}{g} \\ \ddot{\beta}^v + (k + \omega_u)^2 \beta^v &= 0 \end{aligned}\right\} \tag{11.4.97}$$

For determination of α^{acc} and β^{acc}, during integration of equations (11.4.97) we will consider initial conditions to be zero; i.e., for t = 0 $\alpha^{acc}(0) = \beta^{acc}(0) = 0$, $\dot{\alpha}^{acc}(0) = \dot{\beta}^{acc}(0) = 0$. Then from second equation of system (11.4.97) we obtain

$$\beta^v = 0, \tag{11.4.98}$$

i.e., circulation of ship does not cause error in coordinate β.*

Using method of variation of arbitrary constants, we find particular solution of first equation of system (11.4.97) in the form

$$\alpha^v = -\frac{kb}{g} \int_0^t v\omega_u \sin[(k + \omega_u)(t - \tau)]\, d\tau.$$

Since we assumed that ω_{circ} = const and v = const, then

$$\alpha^v = -\frac{k}{k + \omega_u} b \frac{v\omega_u}{g} [1 - \cos(k + \omega_u) t]. \tag{11.4.99}$$

It is possible in principle to make the same conclusions regarding expression (11.4.99) as regarding formula (11.4.80) for ballistic error β^{acc}. It is possible to decrease error α^{acc} by means of proper selection of parameters of the GP for which coefficient b will be sufficiently small.

Above there was considered case of left circulation of ship, when directions of vectors $\vec{H}$ and $\vec{\omega}_{circ}$ coincide. During right circulation, when $\omega_{circ} < 0$, vectors $\vec{H}$ and $\vec{\omega}_{circ}$ are directed in opposite directions. For this case in place of (11.4.99) we will obtain

$$\alpha^v = \frac{k}{k - \omega_u} b \frac{v\omega_u}{g} [1 - \cos(k - \omega_u) t]; \tag{11.4.100}$$

then period of precessional oscillations of gyroscope will be

*See footnote on p. 348.

$$T'_{r.u} = \frac{2\pi}{k - \omega_u}. \qquad (11.4.101)$$

Comparing (11.4.99) with (11.4.100), we see that for right circulation error α^{acc} is obtained to be larger than for left circulation. Especially large can be error of GP for right circulation in the case when ω_{circ} is close to quantity k, i.e., when period of circulation $T_{circ} = \frac{2\pi}{\omega_{circ}}$ is close to period $T_{g.p} = \frac{2\pi}{k}$ of undamped precessional oscillations of axis of gyroscope. Therefore, period $T_{g.p}$ should be much larger than possible value of period T_{circ} of circulation of ship. Let us note that in a real instrument having damping, error α^{acc} of GP will have definite value even for $k = \omega_{circ}$.

Example 11.6: To calculate ballistic error α^{acc} of GP during left circulation of a ship proceeding with velocity v = 10 m/sec; period of circulation T_{circ} = 5 minutes; parameters of instrument are the same as in Example 11.1; coefficient of ballistic error b = 0.9713 (Example 11.5). To calculate error α^{acc} by the moment of termination of circulation continuing for 50 sec.

Solution: 1. We determine angular velocity of circulation ω_{circ}:

$$\omega_u = \frac{2\pi}{T_u} = \frac{6,28}{300} = 0,02094 \text{ 1/sec}.$$

2. By formula (11.4.99) we determine ballistic error α^{acc}:

$$\alpha^y = -\frac{k}{k+\omega_u} b \frac{\omega_u}{g} [1 - \cos(k + \omega_u) t] = -\frac{7,319 \cdot 10^{-3}}{7,319 \cdot 10^{-3} + 20,94 \cdot 10^{-3}} \times$$

$$\times 0,9713 \frac{10 \cdot 20,94 \cdot 10^{-3}}{9,81} [1 - \cos(7,319 \cdot 10^{-3} + 20,94 \cdot 10^{-3}) 50] = -0,00453 \text{ rad} = -16'.$$

From Example 11.6 it follows that magnitude of error α^{acc} can be very significant.

Above there were briefly considered problems of determination of ballistic errors of GP under simple conditions of motion: during rectilinear uniformly accelerated motion of ship and for circulation. The reader can become acquainted with more detailed investigations of ballistic errors in a number of works (for instance, [45, 30, 20, 71, 127, 168, 135, 159]). Let us note that during the analysis of ballistic errors of GP it is necessary to consider [20, 41] damping existing in the instrument, realized, for instance, with the help of a hydraulic damper or by means of imposition on the gyroscope of radial corrective torques.

Ballistic errors of GP appearing during separate maneuvers of ship can be accumulated. B. V. Bulgakov [20] investigated these errors during prolonged maneuvering of ship carried out according to any law, where the only general condition was that absolute value of velocity of ship was bounded by a known upper limit. In this same work there is resolved question about determination of law of change of

magnitude and direction of velocity of ship at which deviation of GP attains maximum (accumulated) value as compared to values of deviations of GP for any other law, and there are calculated magnitudes of accumulated deviations; this same question is investigated [127] in reference to a four-gyroscope vertical.

d) Compensation of Ballistic Errors

The question of compensation of ballistic errors of GP has very important significance, since on its successful solution depends possibility of creation of such instruments working with acceptable accuracy under conditions of maneuvering of the ship.

Compensation of ballistic errors of GP is based on the theorem of M. Schuler. In § 4.5, Par. 5 it was shown that, according to principle of M. Schuler, a simple pendulum of length l equal to radius R of Earth, during any motion of the object (vehicle) over surface of terrestrial sphere, in its position of equilibrium is disposed along radius of Earth at the given point. This direction coincides with vertical of the site (with direction of resultant force of terrestrial gravitation and centrifugal force due to diurnal rotation of the Earth) at the poles and on the equator. Period of natural oscillations of such a simple pendulum is equal to

$$T_{n.n} = 2\pi\sqrt{\frac{l}{g}} = 2\pi\sqrt{\frac{R}{g}} = 5064 \text{ sec.} = 84{,}4 \text{ min.} \qquad (11.4.102)$$

In the same place it was shown that for a physical pendulum with period $T_{p.p}$ = = 84.4 minutes, not for all motions of the object over surface of terrestrial sphere is there ensured undisturbability of the pendulum by accelerations of the object. However, realization of the condition of M. Schuler for a physical pendulum is impossible in practice, since due to large value of R the necessary value of length of arm of pendulum is obtained to be insignificantly small.*

M. Schuler extended his theorem to the gyrocompass and gyropendulum, while indicating necessity of realization in them of a period of natural sustained

*It is easy to show that large value of period $T_{p.p}$ of oscillations of physical pendulum can be obtained also for a very large arm of the pendulum. Actually, for $T_{p.p}$ we have expression (4.1.4): $T_{p.p} = 2\pi\sqrt{\frac{J}{mgl}}$; replacing in it $J = J_0 + ml^2$ [see (4.6.19)], where J_0 is moment of inertia of pendulum relative to its center of gravity, we find

$$T_{n.n} = 2\pi\sqrt{\frac{J_0 + ml^2}{mgl}} = 2\pi\sqrt{\frac{J_0}{mgl} + \frac{l}{g}},$$

from which it is clear that for production of large value of period $T_{p.p}$ it is necessary to make quantity l very small or very great.

oscillations equal to 84.4 minutes. In 1923 M. Schuler patented his "gyroscopic device for increase of period of natural oscillations of a solid body, i.e., a gyrovertical or gyrohorizon" [172] in two variants: four-gyroscope and three-gyroscope pendulum systems, which had as their purpose the compensation of rate errors [127].

According to the theorem of M. Schuler, for ensuring of undisturbability of the GP by accelerations of the object, period $T_{g.p}$ [see (11.4.5)] of its natural precessional sustained oscillations should be

$$T^{\circ}_{r.u} = \frac{2\pi}{k} = 2\pi\sqrt{\frac{R}{g}} = 5064 \text{ sec.} = 84 \text{ min. } 24 \text{ sec.} = 84{,}4 \text{ min.}, \qquad (11.4.103)$$

whence frequency k of precessional oscillations of GP should be equal to frequency ν of oscillations of simple pendulum with length equal to radius of Earth, i.e.,

$$k = \nu = \sqrt{\frac{g}{R}} = 1{,}24065 \cdot 10^{-3} \text{ 1/sec.} \qquad (11.4.104)$$

Let us note that the value of $\sqrt{gR}$ = 7.90446 km/sec is that limiting initial velocity at which a body cast in a horizontal direction in a vacuum does not drop to Earth, but becomes its satellite, moving around it in a circular orbit.

Practical realization of condition k = ν is extraordinarily complicated, since displacement l of center of gravity of the GP from point of suspension which is necessary for this is obtained to be insignificantly small. Actually, for parameters of GP given in Example 11.1, according to (11.4.4) we obtain

$$l = \frac{kH}{P} = \frac{\nu H}{P} = \frac{1{,}24065 \cdot 10^{-3} \cdot 1{,}708 \cdot 10^{8}}{5 \cdot 10^{9}} = 0{,}0425 \text{ cm.}$$

Ensuring constancy of so small a value of l under real conditions of operation of the instrument is extraordinarily difficult. In contrast to a GC, condition (11.4.103) for a GP up to now has been impossible to realize. Period of precession $T_{g.p}$ for known GP has been increased to 15-20 minutes [30]. Practical realization of "Schuler pendulum" with period of 84.4 minutes is carried out in inertial navigational systems (§ 11.6) by means of application of integrated correction of an astatic gyroscope from a short-period physical pendulum (accelerometer).

As will be shown further on, for a GP with period of 84.4 minutes possessing accuracy fully acceptable for the majority of practically important cases, there are absent ballistic errors caused by horizontal accelerations of the object during its motion over the terrestrial sphere. The only systematic errors of such a GP are rate errors, i.e., axis of gyroscope makes with the vertical of the site an angle which is proportional to absolute velocity of the object. Since rate errors of GP have a systematic character, they can be eliminated from readings of the instrument. In the presence of accelerations of the object and changes of velocity, and

heading connected with them, axis of gyroscope aperiodically, i.e., without excitation of natural precessional oscillations, will pass to a new position of equilibrium, which is determined by current values of velocity and heading of the object. In connection with this, the condition of undisturbability of the GP is sometimes called the condition of "aperiodic transitions."

We will show that when condition (11.4.103) or (11.4.104) is satisfied, the GP in the absence of damping does not have ballistic errors, and its only systematic errors are rate errors. For this we will use equations (11.4.61) and will consider the general case of circulation of a ship with angular velocity ω_{circ} and constant linear acceleration $\dot{v}$. Replacing in these equations during left circulation quantity k by $(k + \omega_{circ})$ and taking $W_r = \dot{v}$ and $W_\xi = - v\omega_{circ}$, we will obtain

$$\left.\begin{aligned} \dot{\alpha} - (k + \omega_u)\beta &= -U\cos\varphi\cos K - \frac{k}{g}\dot{v} \\ \dot{\beta} + (k + \omega_u)\alpha &= -U\cos\varphi\sin K - \frac{v}{R} - k\frac{v\omega_u}{g} \end{aligned}\right\} \qquad (11.4.105)$$

By analogy with (6.3.4) we will introduce quantity z, which is complex deviation of axis of gyroscope

$$z = \alpha + j\beta. \qquad (11.4.106)$$

Multiplying the second equation (11.4.105) by $j = \sqrt{-1}$ and adding it with the first, according to (11.4.106) we will obtain

$$\dot{z} + j(k + \omega_u)z = -U\cos\varphi e^{jK} - j\frac{v}{R} - k\frac{\dot{v} + jv\omega_u}{g}. \qquad (11.4.107)$$

During rectilinear motion of ship ($\omega_{circ} = 0$) with constant velocity v = const, $\dot{v} = 0$, particular solution of equation (11.4.107) will be written in the form

$$\begin{aligned} z_1 = \alpha_1 + j\beta_1 &= -\frac{1}{jk}U\cos\varphi e^{jK} - \frac{1}{k}\cdot\frac{v}{R} = \\ &= -\frac{1}{jk}U\cos\varphi(\cos K + j\sin K) - \frac{1}{k}\cdot\frac{v}{R}. \end{aligned} \qquad (11.4.108)$$

Separating imaginary and real parts, we will obtain expressions

$$\left.\begin{aligned} \alpha_1 &= -\frac{1}{k}\left(U\cos\varphi\sin K + \frac{v}{R}\right) \\ \beta_1 &= \frac{1}{k}U\cos\varphi\cos K \end{aligned}\right\}$$

which coincide with relationships (11.4.58) for rate errors of the GP.

Let us assume that v(t) changes by any law. Equation (11.4.107), if we consider (11.4.69), can be reduced to the form

$$\dot{z} + j(k + \omega_u)z = -U\cos\varphi e^{jK} - \frac{k}{g}\left[\dot{v} + j\left(\omega_u + \frac{v^2}{k}\right)v\right]. \qquad (11.4.109)$$

If in the instrument there is satisfied condition (11.4.104) of M. Schuler, then in place of (11.4.109) we have

$$\dot{z} + j(k+\omega_u)z = -U\cos\varphi e^{jK} - \frac{k}{g}[\dot{v} + j(\omega_u + k)v], \qquad (11.4.110)$$

whence it follows that for any form of function v(t) this equation has as its particular solution the quantity z, which is determined by the same formula (11.4.108) as in the case of rectilinear and uniform motion of the ship, but now z_r will be variable, since velocity v is changed. Thus, during observance of condition (11.4.104) for GP, no errors besides rate errors appear.

This circumstance can be clarified more graphically by the method which was used by B. V. Bulgakov and Ya. N. Roytenberg [22] during investigation of an analogous question in reference to an active gyrohorizon. Since ballistic error z^{acc} may be, by analogy with (11.4.74), represented in the form

$$z^v = z - z^c,$$

where z^v is rate error determined by relationship (11.4.108), then equation (11.4.107) can be transformed in the following way ($K = -\omega_{circ}$):

$$\dot{z}^v + j(k+\omega_u)z^v = -\frac{k}{g}\left(1 - \frac{\nu^2}{k^2}\right)(\dot{v} + j\omega_u v). \qquad (11.4.111)$$

Introducing coefficient of ballistic error $b = 1 - \frac{\nu^2}{k^2}$ [see (11.4.78)], we will rewrite (11.4.111) in the form

$$\dot{z}^v + j(k+\omega_u)z^v = -\frac{k}{g}b(\dot{v} + j\omega_u v), \qquad (11.4.112)$$

whence it follows that when condition of M. Schuler is satisfied ($k = \nu$, $b = 0$) particular solution of equation (11.4.112), i.e., value of ballistic error of GP, $z^{acc} = 0$; consequently, GP does not have ballistic errors for any form of function v(t).

For the earlier considered particular case of rectilinear uniformly accelerated motion of a ship, when equations of GP are determined by relationships (11.4.70) and condition (11.4.104) is satisfied, coefficient b_0 [see (11.4.68)] becomes zero and acceleration $\dot{v}$ of motion of ship will not affect gyropendulum. Thus, according to (11.4.78), coefficient of ballistic error $b = 0$ and ballistic error of GP β^{acc} [see (11.4.77)] also is equal to zero. The same occurs and in other earlier considered particular case, i.e., during circulation of ship. Actually, from expression (11.4.99) it follows that when the condition of M. Schuler is satisfied $k = \nu$, or which is the same, $b = 0$, ballistic error of GP $\alpha^{acc} = 0$.

For GP for which condition of aperiodic transitions, is satisfied time constant T [see (11.4.82)] is

$$T = \frac{1}{k} = \frac{1}{\nu} = \frac{1}{1{,}24065 \cdot 10^{-3}} = 806 \text{ sec}. \qquad (11.4.113)$$

M. Schuler's condition of undisturbability considered above, applied to GP is approximate. Existence of this condition is connected with the following assumptions [159, 134]; 1) condition of M. Schuler is valid within the framework of linear theory, since it is possible to substantiate it proceeding from linearized differential equations of motion of the GP; 2) it is assumed that Earth is a sphere; 3) it is considered that vertical accelerations are absent [$W_\zeta = 0$; see (11.4.59)]; 4) it is assumed that induced and Coriolis accelerations due to rotation of Earth and motion of the object (vehicle) over surface of Earth [see (11.4.63)] are small as compared to accelerations due to force of terrestrial gravitation and accelerations of object in its relative motion; 5) there is not considered vertical component u_ζ of induced rotation of system of reference relative to inertial space, which is small as compared to angular velocity k of precession of the GP [see (11.4.53)], and others.

In connection with this, from theoretical point of view in the general case of motion of the object, condition of M. Schuler in application to GP in principle is not valid, and even when it is realized this instrument nevertheless will have ballistic errors. In reference to GP, B. V. Bulgakov [20] and R. I. Chertkov [168, 135] obtained conditions of undisturbability of more general form;* however, in practice their realization is very difficult.

From what was said above it follows that GP, even when M. Schuler's condition of undisturbability is satisfied, will in the presence of accelerations have ballistic errors. Thus investigation of errors of GP shows [168, 159] that: 1) for the comparatively small angles α and β of deviation of axis of gyroscope from the vertical which occur in practice, ballistic errors of GP connected with linearization of equations of motion are obtained to be minute; 2) for comparatively small vertical accelerations of object when vertical velocity is of the order of 20-30 m/sec, errors of GP do not exceed several minutes of arc; 3) inexactness of satisfaction of condition of aperiodicity, for instance, by 2-3%, leads to errors not exceeding 1-2'.

Thus, realization in gyropendulum verticals of condition of aperiodic transitions ensures compensation of ballistic errors of gyrovertical with accuracy fully acceptable for majority of practical problems.

As A. Yu. Ishlinskiy [52] indicated, A. Schuler's condition plays in applied gyroscopy a role absolutely analogous to that of the condition of invariance in theory of automatic control. Soviet scientists have introduced an essential contribution into investigation of conditions of undisturdability of GD. In reference to gyropendulums,

*In [135] there are obtained conditions of undisturbability of GP in which there are not made the assumptions enumerated in the above mentioned Par. 4 and 5.

here a considerable role has been played by work of A. Yu. Ishlinskiy [53], R. I. Chertkov [168, 135], L. I. Tkachev [159], S. S. Tikhmenev [157], and others.

Let us indicate one more circumstance hampering the obtaining in GP of a period of 84.4 minutes. In the gyropendulum, just as in any gyroscopic device, it is necessary to introduce damping with help of appropriate attachments, as which, as it was earlier said, there is applied a hydraulic damper or devices imposing on the GP radial corrective torques. Thus, period of damped oscillations of GP will differ from period of its natural sustained oscillations by 84.4 minutes when condition of aperiodic transitions is satisfied. Due to this, GP, besides rate errors, will also have ballistic errors, which would be equal to zero in the absence of damping. Consequently, in the presence in the instrument of damping, it is not possible to select parameters of GP in such a way that period of its natural damped oscillations is equal to 84.4 minutes. However, as B. V. Bulgakov indicated [20], the general assertion of the impossibility of obtaining a gyroscopic pendulum or gyrocompass accomplishing damped oscillations and not having any errors besides rate errors has not been proven by anyone up to the present time.

We will show that for GP with damping realized by means of imposition of pendulum radial corrective torques, when condition of aperiodic transitions is satisfied, there are besides rate errors also ballistic errors. For this we will introduce in equations (11.4.105) radial corrective torques $M_{cor\ x}$ and $M_{cor\ y}$ referred to angular momentum H of the gyroscope. Then, taking into account relationships (2.3.66) and (2.3.5), we will rewrite equations (11.4.105) in the form

$$\left.\begin{aligned} \dot{\alpha} + \varkappa\alpha - (k+\omega_u)\beta &= -U\cos\varphi\cos K - \frac{k}{g}\dot{v} - \varkappa\frac{v\omega_u}{g} \\ \dot{\beta} + \varkappa\beta + (k+\omega_u)\alpha &= -U\cos\varphi\sin K - \frac{v}{R} - \\ &\quad - k\frac{v\omega_u}{g} + \varkappa\frac{\dot{v}}{g} \end{aligned}\right\} \qquad (11.4.114)$$

where $\varkappa$ is specific rate of correction.

Let us write these equations, by analogy with (11.4.107), in the following way:

$$\dot{z} + [\varkappa + j(k+\omega_u)]z = -U\cos\varphi e^{jK} - j\frac{v}{R} - (k - j\varkappa)\frac{\dot{v} + jv\omega_u}{g}. \qquad (11.4.115)$$

For v = const and K = const (ω_{circ} = 0), particular solution of this equation has the form

$$z_1 = \alpha_1 + j\beta_1 = -\frac{U\cos\varphi}{\varkappa + jk}(\cos K + j\sin K) - \frac{j\frac{v}{R}}{\varkappa + jk}, \qquad (11.4.116)$$

whence we obtain expression for rate errors of GP with damping

$$\left.\begin{aligned} \alpha_r = \alpha^c = -\frac{1}{\varkappa^2+k^2}\left[U\cos\varphi(\varkappa\cos K + k\sin K) + k\frac{v}{R}\right] \\ \beta_r = \beta^c = -\frac{1}{\varkappa^2+k^2}\left[U\cos\varphi(\varkappa\sin K - k\cos K) + \varkappa\frac{v}{R}\right] \end{aligned}\right\} \qquad (11.4.117)$$

which for $\varkappa = 0$ become the rate errors of a GP without damping which are determined by relationships (11.4.58).

Equation (11.4.115), by analogy with (11.4.109), can be written in the form

$$\dot{z} + [\varkappa + j(k+\omega_u)]z = -U\cos\varphi\cdot e^{jK} - \frac{k}{g}\left[\dot{v} + j\left(\omega_u + \frac{v^2}{k}\right)v\right] + \frac{\varkappa}{g}(\dot{v} + jv\omega_u). \qquad (11.4.118)$$

Let us assume now that v(t) is changed by any law and that in the instrument there is satisfied the condition of M. Schuler (11.4.116). Then instead of (11.4.118) we will obtain equation

$$\dot{z} + [\varkappa + j(k+\omega_u)]z = -U\cos\varphi\cdot e^{jK} - \frac{k}{g}[\dot{v} + j(k+\omega_u)v] + \frac{\varkappa}{g}(\dot{v} + jv\omega_u), \qquad (11.4.119)$$

particular solution of which does not coincide with (11.4.116). Therefore, a GP with damping and with condition of aperiodic transitions satisfied in it will have, besides rate errors, also ballistic errors. If damping is ceased ($\varkappa = 0$), then equation (11.4.119) becomes (11.4.110); then, as was shown earlier, the GP will not have ballistic errors.

Ballistic errors of GP with radial correction were investigated in detail by B. V. Bulgakov. He showed that during weak and average damping, a GP with period of 84.4 minutes should be in principle more favorable than an aperiodic vertical not having a pendulum effect at all; with strong radial correction both instruments have practically identical ballistic errors. On the basis of corresponding calculations there has been established the most expedient relationship between parameters $\varkappa$ and k for GP.

5. Dynamic Errors of GP

a) General Characteristics of Dynamic Errors

Among dynamic errors of GP there are included errors appearing under real conditions of use of instrument during variable external disturbances. Such errors appear during oscillations of the object, during vibrations of the place of installation of GP, and so forth. For concreteness we will consider dynamic errors of the GP under conditions of rocking of the ship. Oscillatory motions of ship on sea swells affect GP due to unbalance of the gyroscope, and also through friction in axes of its suspension. Since influence on gyroscope of dry and fluid frictional torques

was considered in § § 5.7 and 6.5, then during calculation of dynamic errors of GP we will consider influence of rocking of ship due to unbalance of the gyroscope.

b) Differential Equations of a GP in a Rocking Ship

Let us compose differential equations of motion of GP during rocking of a ship; we will not consider gyroscopic torques due to induced rotation of reference system $O\xi\eta\zeta$ relative to inertial space caused by rotation of Earth and motion of the ship. Let us assume that for realization of damping there is applied system of pendulum correction analogous to the system in the gyrovertical, in which there is used an astatic gyroscope (Chapter 6). Let us use equations (11.4.15), discarding from them inertial terms $J_e\alpha$, $J_e\beta$ and above-indicated gyroscopic torques. Quantity u_ζ we will replace by angular velocity $\dot{\varphi}(t)$ of yawing of ship, considering that $u_\xi = -\dot{\varphi}(t)$ (cf. Figures 2.2 and 2.19: external disturbing torques $f_1(t)$ and $f_2(t)$ we will take equal to zero. To the remaining equations we will add pendulum corrective torques, which under conditions of rocking of ship are determined by relationships (2.3.70). Then instead of (11.4.15) we will obtain following differential equations of GP during rocking of ship:

$$\left.\begin{aligned}\dot{\alpha}+\frac{S}{H}\alpha+\dot{\varphi}(t)\beta-\frac{ml}{H}(g_0+W_\eta)\beta&=\\ =-\frac{ml}{H}W_\eta+\frac{S}{H}\chi_1(t)&\\ \dot{\beta}+\frac{S}{H}\beta-\dot{\varphi}(t)\alpha+\frac{ml}{H}(g_0+W_\eta)\alpha&=\\ =\frac{ml}{H}W_\xi+\frac{S}{H}\chi_2(t)&\end{aligned}\right\}\qquad(11.4.120)$$

or, considering (11.4.17), (11.4.20) and (2.3.5),

$$\left.\begin{aligned}\dot{\alpha}+\varkappa\alpha-\left[k\left(1+\frac{W_\zeta}{g}\right)-\dot{\varphi}(t)\right]\beta&=-\frac{k}{g}W_\eta+\varkappa\chi_1(t)\\ \dot{\beta}+\varkappa\beta+\left[k\left(1+\frac{W_\zeta}{g}\right)-\dot{\varphi}(t)\right]\alpha&=\frac{k}{g}W_\xi+\varkappa\chi_2(t)\end{aligned}\right\}\qquad(11.4.121)$$

During rocking of ship W_ξ, W_η, W_ζ are random functions characterizing place of installation of instrument, and $\chi_1(t)$ and $\chi_2(t)$ are random functions characterizing oscillation of corrective pendulums utilized in the GP for creation of radial corrective damping torques.

For accelerations W_ξ, W_η, W_ζ, considering only terms of first order of smallness in expressions (2.2.48) and different orientation of axes $O\xi\eta\zeta$ in Figures 2.19 and 2.2, with y = 0 (GP is located in diametrical plane of ship), we will obtain following relationships:

$$W_\xi = \ddot{\eta}_0 + x\ddot{\varphi} - z\ddot{\theta}; \quad W_\eta = \ddot{\xi}_0 + z\ddot{\psi}; \quad W_\zeta = -\ddot{\zeta}_0 + x\ddot{\psi}. \tag{11.4.122}$$

Putting (11.4.122) in (11.4.121), we have

$$\left.\begin{aligned} \dot{\alpha} + x\alpha - \left[k\left(1 - \frac{\ddot{\zeta}_0 - x\ddot{\psi}}{g}\right) - \dot{\varphi}(t)\right]\beta = \\ = -\frac{k}{g}(\ddot{\xi}_0 + z\ddot{\psi}) + x\chi_1(t) \\ \dot{\beta} + x_1\beta + \left[k\left(1 - \frac{\ddot{\zeta}_0 - x\ddot{\psi}}{g}\right) - \dot{\varphi}(t)\right]\alpha = \\ = \frac{k}{g}(\ddot{\zeta}_0 + x\ddot{\varphi} - z\ddot{\theta}) + x\chi_2(t) \end{aligned}\right\} \tag{11.4.123}$$

In equations (11.4.123), for a certain simplification of solution of the problem we were limited to consideration of only terms of first order of smallness. Let us note that if for W_ξ, W_η, W_ζ we use expressions (2.2.49), then we will obtain equations of GP on a rocking ship taking into account terms the second order of smallness. These equations are analogous to equation (4.6.98) of oscillations of a physical pendulum.

For analysis of dynamic errors of GP on rocking ship we will take into account only transverse motion of ship, i.e., transverse horizontal shifts $\eta_0(t)$ of its center of gravity and rolling $\theta(t)$. Consequently, dynamic errors of GP on a rocking ship are found for the same conditions for which dynamic errors of pendulum bank indicator were determined.

In accordance with what has been said, in equations (11.4.123) we will set $\ddot{\psi} = 0$, $\dot{\varphi} = 0$, $\ddot{\varphi} = 0$, $\ddot{\xi}_0 = 0$, $\ddot{\zeta}_0 = 0$; deviation χ_2 of corrective pendulum in diametrical plane of ship we will take equal to zero, and deviation $\chi_1 = \chi$ of corrective pendulum in plane of rib of the ship we will find from equation (4.6.49) of oscillations in this plane of the physical pendulum, in which coefficient k is determined by relationship (4.6.48), and function f(t) – by formula (4.6.41). For usually applied short-period corrective pendulum, for the instance pendulum flap of an aircraft GV (Example 4.2), time constant T = 0.045 sec., i.e., is minute. Therefore, according to (4.6.49), (4.6.48) and (4.6.41), we have approximately $\chi_1 = \chi \approx \frac{1}{g}(\ddot{\eta}_0 - z\ddot{\theta})$. Taking into account what has been said, we will reqrite equations (11.4.123) in the form

$$\left.\begin{aligned} \dot{\alpha} + x\alpha - k\beta = \frac{x}{g}(\ddot{\zeta}_0 - z\ddot{\theta}) \\ \dot{\beta} + x_1\beta + k\alpha = \frac{k}{g}(\ddot{\zeta}_0 - z\ddot{\theta}) \end{aligned}\right\} \tag{11.4.124}$$

whence, eliminating at first β, and then α, we find

$$\left.\begin{aligned}\ddot{\alpha}+2\varkappa\dot{\alpha}+(\varkappa^2+k^2)\alpha&=\frac{\varkappa^2+k^2}{g}\left(\ddot{\eta}_0-\ddot{z}^{(l)}\right)+\frac{\varkappa}{g}\left(\dddot{\eta}_0-\dddot{z}^{(l)}\right)\\ \ddot{\beta}+2\varkappa\dot{\beta}+(\varkappa^2+k^2)\beta&=\frac{k}{g}\left(\dddot{\eta}_0-\dddot{z}^{(l)}\right)\end{aligned}\right\}\qquad(11.4.125)$$

Since disturbances $\eta(t)$ and $\theta(t)$ are normal stationary random functions of time (§ 2.1, Par. 3), then, as follows from (11.4.125), $\alpha(t)$ and $\beta(t)$, which characterize angles of displacement of axis of GP from vertical, also will be normal stationary random functions. Due to damping, natural oscillations of GP in coordinates α and β attenuate and a GP on a rocking ship will accomplish only forced oscillations. From equations (11.4.125) it follows that damping, besides quenching of natural oscillations, generates forced oscillations in coordinate α. This leads to errors analogous in nature to errors of damping-acceleration of a GC (§ 11.5).

In order to compare errors of GP during transverse rocking of ship with errors under the same conditions of a pendulum bank indicator, we will calculate errors $\alpha(t)$ of GP in the plane of a rib. For this we will use first equation of system (11.4.125):

$$\ddot{\alpha}+2\varkappa\dot{\alpha}+(\varkappa^2+k^2)\alpha=\frac{\varkappa^2+k^2}{g}\left(\ddot{\eta}_0-\ddot{z}^{(l)}\right)+\frac{\varkappa}{g}\left(\dddot{\eta}_0-\dddot{z}^{(l)}\right).\qquad(11.4.126)$$

Let us determine mathematical expectation $\bar{\alpha}$ of random function $\alpha(t)$. For this we will apply to (11.4.126) the operation of mathematical expectation. Considering that $\overline{\ddot{\theta}(t)}=0$, $\overline{\dddot{\theta}(t)}=0$, $\overline{\ddot{\eta}_0(t)}=0$, $\overline{\dddot{\eta}_0(t)}=0$, we obtain,

$$\ddot{\bar{\alpha}}+2\varkappa\dot{\bar{\alpha}}+k_d^2\bar{\alpha}=0,\qquad(11.4.127)$$

whence;

$$\bar{\alpha}=e^{-\varkappa t}(C_1\cos k_d t+C_2\sin k_d t);\qquad(11.4.128)$$

here C_1 and C_2 — constants of integration, determined by initial conditions;

k_d — frequency of undamped precessional oscillations of a GP with radial correction

$$k_d=\sqrt{\varkappa^2+k^2}=k\sqrt{1+\left(\frac{\varkappa}{k}\right)^2}.\qquad(11.4.129)$$

Expression (11.4.129) coincides with relationship (11.4.48), if in the latter time constant of correction T_1 is replaced by $\frac{1}{\varkappa}$ [see (2.3.7)].

According to (11.4.128), due to presence of damping, upon completion of transient response mathematical expectation of random process $\alpha(t)$ will be equal to zero

$$\bar{\alpha}=0,\qquad(11.4.130)$$

i.e., on the average axis GP on a rocking ship will retain its vertical position. This conclusion is valid only in reference to the considered case, when in

differential equation (11.4.126) of GP on rocking ship there were not taken into account terms of the second order of smallness, including vertical accelerations of rocking. In general, if we consider these circumstances, $\bar{\alpha} \neq 0$. Quantity $\bar{\alpha}$ can be found by the same method which we used in determining mathematical expectation $\bar{\chi}$ of random motions of physical pendulum for rocking with consideration of terms the second order of smallness (§ 4.6, Par. 12).

Let us turn to determination of dispersion $D[\alpha]$ of random motions $\alpha(t)$ of axis of GP during irregular rocking of ship. For this at first we will find spectral density $S_\alpha(\omega)$ of random function $\alpha(t)$. Let us designate in formula (11.4.126) by analogy with (4.6.41)

$$f(t) = \ddot{\tau}_{\omega}(t) - z\ddot{\vartheta}(t). \tag{11.4.131}$$

Then equation (11.4.126) we will write in the form

$$\ddot{\alpha} + 2\varkappa\dot{\alpha} + (\varkappa^2 + k^2)\alpha = \frac{\varkappa^2 + k^2}{g} f(t) + \frac{\varkappa}{g}\dot{f}(t). \tag{11.4.132}$$

Let us divide left and right sides of this equation by $(x^2 + k^2)$ and introduce designations

$$\left.\begin{aligned} T_\Delta^2 = \frac{1}{\varkappa^2 + k^2};\quad T_\Delta\zeta = \frac{\varkappa}{\varkappa^2 + k^2};\quad \zeta = \frac{\varkappa}{\sqrt{\varkappa^2 + k^2}} \\ q = \frac{1}{g};\quad q_1 = \frac{1}{g} T_\Delta\zeta \end{aligned}\right\} \tag{11.4.133}$$

Considering these designations, we will rewrite (11.4.132) in the following way:

$$T_\Delta^2\ddot{\alpha} + 2T_\Delta\zeta\dot{\alpha} + \alpha = qf(t) + q_1\dot{f}(t). \tag{11.4.134}$$

In this equation T_d is time constant of GP with radial correction, and ζ is relative damping factor.

Let us define transfer function of GP with respect to disturbance f(t):

$$W(s) = \frac{q_1 s + q}{T_\Delta^2 s^2 + 2\zeta T_\Delta s + 1}, \tag{11.4.135}$$

whence for amplitude-phase characteristic of the GP we will obtain

$$W(j\omega) = \frac{q_1 j\omega + q}{-T_\Delta^2\omega^2 + 2\zeta T_\Delta j\omega + 1}. \tag{11.4.136}$$

By analogy with (4.6.47), for spectral density $S_\alpha(\omega)$ of random function $\alpha(t)$ we have

$$S_\alpha(\omega) = |W(j\omega)|^2 S_f(\omega), \tag{11.4.137}$$

where $S_f(\omega)$ is spectral density of random function f(t).

For the considered case there was obtained expression (4.6.64) of spectral density $S_f(\omega)$

$$S_f(\omega)=\frac{2A_0\mu_0}{\pi}\gamma_2^2\frac{b_0^6}{\omega^4+2a_0\omega^2+b_0^4}+\frac{2A_2\mu_2}{\pi}z^2\frac{b_2^2\omega^4}{\omega^4+2a_2\omega^2+b_2^4}, \qquad (11.4.138)$$

where, by analogy with (2.1.56),

$$a_0=\mu_0^2-\lambda_0^2;\quad b_0^2=\mu_0^2+\lambda_0^2;\quad a_2=\mu_2^2-\lambda_2^2;\quad b_2^2=\mu_2^2+\lambda_2^2. \qquad (11.4.139)$$

Putting (11.4.136) and (11.4.138) in (11.4.137), we will obtain formula

$$S_\alpha(\omega)=\frac{q_1\omega^2+q^2}{\left(1-T_A^2\omega^2\right)^2+4\zeta^2T_A^2\omega^2}\left[\frac{2A_0\mu_0}{\pi}\gamma_2^2\frac{b_0^6}{\omega^4+2a_0\omega^2+b_0^4}+\frac{2A_2\mu_2}{\pi}z^2\frac{b_2^2\omega^4}{\omega^4+2a_2\omega^2+b_2^4}\right], \qquad (11.4.140)$$

which becomes expression (4.6.65) for spectral density $S_\alpha(\omega)$ of random oscillations of pendulum bank indicator, if in (11.4.140) we set $T_d = T$, $q = k$, $q_1 = 0$.

For dispersion D[α] we have expression

$$D[\alpha]=2\int_0^\infty S_\alpha(\omega)\,d\omega, \qquad (11.4.141)$$

where $S_\alpha(\omega)$ is determined by formula (11.4.140).

During calculation of D[α] we will use the same approach which we applied during calculation of dispersion of errors of a pendulum bank indicator for rocking (§ 4.6, Par. 9). In accordance with equation (11.4.132), term $\frac{\alpha^2+k^2}{g}f(t)$ in its right side characterizes the disturbing effect of rocking on GP due to unbalance of the gyroscope and second term $\frac{x}{g}\dot{f}(t)$ characterizes disturbing effect of rocking on GP due to damping caused by application to the gyroscope of a radial pendulum correction torque.

If now we turn to equation (4.6.39) of oscillations of a pendulum bank indicator, then analogously the first term $k_1(\ddot{\eta}_0-z\ddot{\theta}) = k_1f(t)$ of its right side characterizes disturbing effect of rocking on bank indicator itself, and second term $2\zeta n\dot{\theta}$ characterize disturbing effect of rocking on bank indicator due to damping. During calculation of dispersion of random motions of bank indicator for rocking, term $2\zeta n\dot{\theta}$ was discarded; therefore for comparative calculation of dispersion of random motions of GP for rocking we should also not take into account the disturbing effect of rocking due to damping of the GP. For this, in equation (11.4.132) it is necessary to disregard term $\frac{x}{g}\dot{f}(t)$ or, considering (11.4.133), to set in equation (11.4.134) $q_1 = 0$. Then for spectral density $S_\alpha(\omega)$, considering in (11.4.140) that $q_1 = 0$, we will obtain formula

$$S_\alpha(\omega)=\frac{q^2}{\left(1-T_A^2\omega^2\right)^2+4\zeta^2T_A^2\omega^2}\left[\frac{2A_0\mu_0}{\pi}\gamma_2^2\frac{b_0^6}{\omega^4+2a_0\omega^2+b_0^4}+\frac{2A_2\mu_2}{\pi}z^2\frac{b_2^2\omega^4}{\omega^4+2a_2\omega^2+b_2^4}\right], \qquad (11.4.142)$$

which coincides with accuracy up to the designations with expression (4.6.65) for spectral density $S_\eta(\omega)$ of random motions of pendulum bank indicator for rocking.

Therefore, for obtaining of dispersion D[α] it is possible to directly use

formula (4.6.80)* for dispersion D[χ], if in the latter we replace T by T_d:

$$D[\alpha] = \frac{A_0 b_0^2 \cdot T_\lambda^2}{g^2} \cdot \frac{1 + 4T_\lambda\mu_0(\zeta + T_\lambda\mu_0) + T_\lambda^2 \frac{\mu_0}{\zeta} b_0^2}{(1 - T_\lambda^2 b_0^2)^2 + 4\left[T_\lambda^2 \cdot^2 b_0^2 + T_\lambda\mu_0(\zeta + T_\lambda\mu_0) + T_\lambda^2 \cdot^2 b_0^2\right]} +$$

$$+ \frac{A_2 b_2^2 \cdot z^2}{g^2} \cdot \frac{b_2^2 + \frac{\mu_0}{\zeta T_\lambda}}{(1 - T_\lambda^2 b_2^2)^2 + 4\left[T_\lambda^2 \cdot^2 b_2^2 + T_\lambda\mu_2(\zeta + T_\lambda\mu_2) + T_\lambda^2 \cdot^2 b_2^2\right]}, \qquad (11.4.143)$$

where $b_{2'} = b_2$; $b_{0'} = b_0$ [see (11.4.139)].

For root mean square σ_α and mean E_α of values of error of the GP we have

$$\sigma_\alpha = \sqrt{D[\alpha]}; \qquad (11.4.144)$$

$$E_\alpha = \rho\sqrt{2D[\alpha]} \quad (\rho = 0{,}4769\ldots). \qquad (11.4.145)$$

Example 11.7: To calculate probability characteristics D[α], σ_α, E_α and limiting value α_m of error α(t) of GP for irregular rocking of ship for the conditions of Example 4.6 and 4.7 taking parameters of GP from Example 11.1; specific rate of radial pendulum correction = 0.01 1/sec [127].

Solution: 1. By formulas (11.4.133) we find time constant T_d of GP with radial correction and also relative damping factor ζ:

$$T_\lambda = \frac{1}{\sqrt{n^2 + k^2}} = \frac{1}{\sqrt{0{,}01^2 + (7{,}319 \cdot 10^{-3})^2}} = 1{,}366 \cdot 10^2 \text{ sec.},$$

$$\zeta = \frac{n}{\sqrt{n^2 + k^2}} = \frac{0{,}01}{\sqrt{0{,}01^2 + (7{,}319 \cdot 10^{-3})^2}} = 1{,}366.$$

2. Using expression (11.4.143), we determine dispersion:

$$D[\alpha] = 0{,}0281 \cdot 10^{-4}.$$

3. By formulas (11.4.144) and (11.4.145) we calculate root mean square σ_α and mean E_α values of error of GP:

$$\sigma_\alpha = 0{,}001676 \text{ rad.} = 5'{,}8, \quad E_\alpha = 3'{,}9.$$

4. For conditions of Example 4.7 we find limiting value of α_m of error of GP:

$$\alpha_m = 3{,}3\sigma_\alpha = 3{,}3 \cdot 5{,}8 = 19'{,}1.$$

Comparing results of Examples 11.7 and 4.7, we see that error of GP, on rocking ship is considerably less than error of pendulum bank indicator. This indicates essential advantages of GP over physical pendulum under the conditions of a moving and rocking vehicle.

Investigation of random oscillations of GP on a rocking ship, taking into account vertical accelerations, is given in the article of A. A. Sveshnikov [137]. Besides considered errors of the GP on a rocking ship, there can also occur errors from other

*Let us note that in formula (4.6.80) 1/g = k = q [see (4.6.48) and (11.4.133)].

disturbances, for instance from vibrations of place of installation of instrument. Methods of calculation of these errors, due to their random character, are kept the same as methods of calculation of errors due to rocking.

6. Certain Remarks

In different GD there are applied so-called plane GP. Schematic diagram of such a GP is shown in Fig. 11.9.

Gyrorotor is fixed in housing Gh, which is suspended on axis 1 in Cardan ring C. The ring constitutes a physical pendulum, which can be rocked about axis of suspension 2, thereby turning by a certain angle α. Turn of gyrohousing about axis 1 by angle β is limited by spring Sp.

Equations of motion of plane GP in the absence of external influences can be written in the form [20]

$$\left.\begin{aligned} J_g\ddot{\alpha} + lP\alpha + H\dot{\beta} = 0 \\ J_{g.e}\ddot{\beta} + c\beta - H\dot{\alpha} = 0 \end{aligned}\right\} \qquad (11.4.146)$$

where J_g – moment of inertia of entire system with respect to axis of suspension;

$J_{g.e}$ – moment of inertia of rotor and gyrohousing with respect to axis 1;

l – distance of center of gravity of system from axis of suspension of GP;

P – weight of system;

H – angular momentum of gyroscope;

c – stiffness coefficient of spring.

B. V. Bulgakov [20] showed that the considered system is oscillatory, where nutational oscillations, which are small as compared to the basic slow precessional oscillations, can be disregarded. The latter can be found from analysis of shortened equations obtained from (11.4.146) by means of discarding inertial terms:

$$\left.\begin{aligned} H\dot{\beta} + lP\alpha = 0 \\ H\dot{\alpha} - c\beta = 0 \end{aligned}\right\} \qquad (11.4.147)$$

or

$$\left.\begin{aligned} \dot{\beta} + k\alpha = 0 \\ \dot{\alpha} - \rho\beta = 0 \end{aligned}\right\} \qquad (11.4.148)$$

where

$$k = \frac{lP}{H}; \quad \rho = \frac{c}{H}. \qquad (11.4.149)$$

Equations (11.4.148) we will write in the following way:

$$\left.\begin{aligned} \ddot{\alpha} + n^2\alpha = 0 \\ \ddot{\beta} + n^2\beta = 0 \end{aligned}\right\} \qquad (11.4.150)$$

where

$$n^2 = kp. \tag{11.4.151}$$

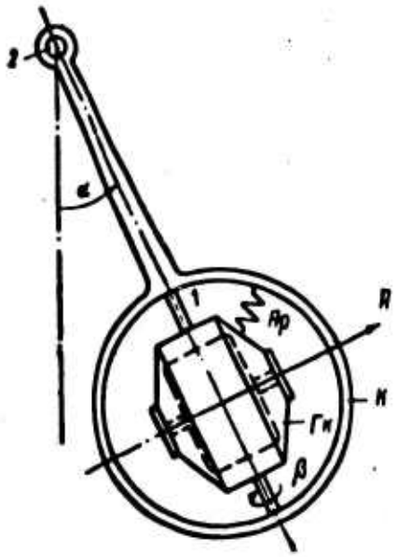

Fig. 11.9. Plane GP.

Integrating the first of these equations, we have

$$\alpha = C_1 \cos nt + C_2 \sin nt.$$

Taking initial conditions t = 0, $\alpha(0) = \alpha_0$, $\dot{\alpha}(0) = \dot{\alpha}_0$, we will obtain

$$\alpha = d \sin (nt + \varepsilon), \tag{11.4.152}$$

where

$$a = \sqrt{\alpha_0^2 + \frac{\dot{\alpha}_0^2}{n^2}}; \quad \operatorname{tg} \varepsilon = \frac{n\alpha_0}{\dot{\alpha}_0}. \tag{11.4.153}$$

Differentiating (11.4.152) and substituting the obtained expression in second equation of system (11.4.148), we have

$$\beta = \frac{an}{p} \cos (nt + \varepsilon). \tag{11.4.154}$$

Thus, solution of equations (11.4.150) for the given initial conditions has the form

$$\left.\begin{aligned} \alpha &= a \sin (nt + \varepsilon) \\ \beta &= \frac{an}{p} \cos (nt + \varepsilon) \end{aligned}\right\} \tag{11.4.155}$$

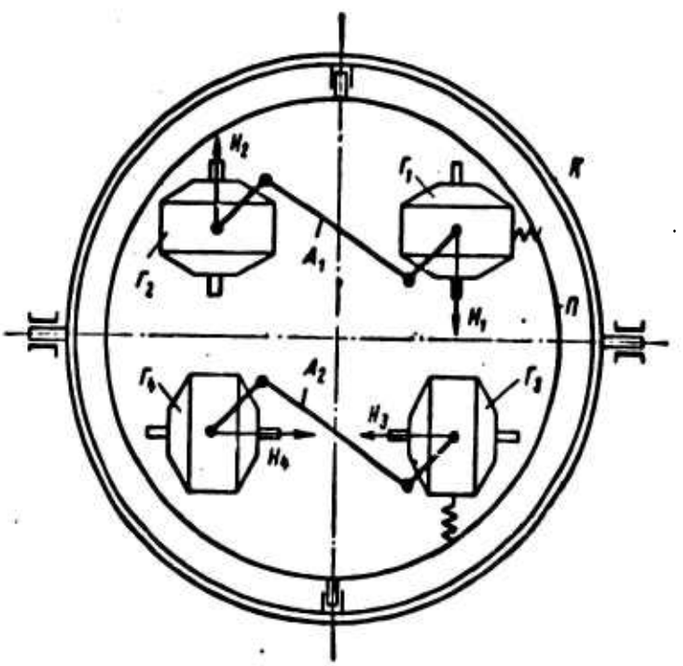

Fig. 11.10. Schematic diagram of four-gyroscope vertical.

It characterizes coupled oscillations of pendulum and gyroscope, which are shifted in phase by 90°; thus, when pendulum passes through position of equilibrium, then axis of gyroscope obtains maximum deflection, and conversely. When the condition of M. Schuler is satisfied, the plane GP also will be undisturbed by accelerations of the object. Behavior of plane GP during irregular rocking of the ship is considered in article [130].

We will give briefly the characteristics of one more type of GP – the so-called four-gyroscope vertical [199, 127]. Schematic diagram of this vertical is shown in Fig. 11.10. It resembles the diagram (Fig. 7.29) of the four-gyroscope

active gyrovertical (§ 7.6, Par. 2a). On platform Pl there are located two two-gyroscope frames, each of which contains two gyroscopes G_1, G_2 and G_3, G_4; axes of precession of every pair of gyroscopes are connected by anti-parallelograms (shafts) A_1 and A_2. Platform Pl is suspended in outer Cardan ring C, axis of rotation of which is fixed parallel to longitudinal axis of ship. Gyroscopes G_1 and G_3 are connected with platform Pl with the help of springs. Center of gravity of entire system is located below point of its suspension; consequently, it constitutes a spherical pendulum.

In the considered GP there is also a device for imposition of radial corrective torques; in Fig. 11.10 it is not shown. Essence of its operation it that during turn of gyroscopes G_1 and G_3 about their axes of precession, correcting devices apply relative to axes of rotation of Cardan ring C and platform Pl torques proportional to angles of turns of these gyroscopes.

Theoretical investigation of four-gyroscope vertical was conducted by Ya. N. Roytenberg [127]. We will indicate certain peculiarities of this system. Its equations of motion are composed by the same method which was used during composition of equations of motion of the four-gyroscope actice GV (§7.6, Par. 2a).

If the base is motionless, then vertices of gyroscopes describe converging spirals in corresponding vertical planes. With consideration of rotation of Earth and rectilinear and uniform motion of the ship itself, for the GV there are revealed velocity errors analogous to errors of single-gyroscope GP. During rocking the amplitude of oscillations of four-gyroscope GP is many times less than amplitude of oscillations of single-gyroscope GP. This is one of the essential advantages of multigyroscope systems as compared to single-gyroscope systems.

There have been investigated ballistic errors of a four-gyroscope GV, and its behavior under the condition that period of its natural oscillations is equal to 84.4 minutes has been considered in special detail. It has been shown that in this case spherical pendulum oscillates around true vertical* for any law of motion of the ship; i.e., the instrument has neither rate nor ballistic errors. Deviation of gyroscopes from their position of equilibrium are proportional to components of absolute velocity of ship along its longitudinal and transverse axes. Investigation of question about accumulation of ballistic errors of GV during prolonged maneuvering of ship showed that maximum ballistic errors of multigyroscope GV are twice as small as corresponding errors of single-gyroscope GP. Those interested in the theory of multigyroscope

*More exact investigation of the question [135] shows that these oscillations occur about the geocentric vertical.

GV we refer to the mentioned article [127]. Refined precessional theory of such GV, taking into account small terms, is considered in [135].

A. Yu. Ishlinskiy constructed within the framework of precessional theory the strict theory of a number of gyroscopic devices, in particular of the GP [45, 53]. Let us note a series of important statements given in his articles. First of all in them it is shown that known works on the theory of GP, including the thorough investigation of B. V. Bulgakov [20], do not resolve the question with proper degree of rigor. Cause of this is that, motion of point of suspension was considered relative to Earth, the rotation of which considerably complicated the investigation. There were considered not always exactly forces of inertia of induced motion. Furthermore, the question about forces acting in the Cardan suspension was considered without sufficient attention.

In his account of theory of GP, A. Yu. Ishlinskiy at first in detail investigates forces acting on the system "rotor + inner ring + outer ring." The essence of this question was considered by us in § 1.6, Par. 7. Analysis of GP is much simplified by introduction of a so-called non-rotating sphere S surrounding the Earth and preserving constant orientation relative to inertial space. The basic problem of theory of GP is investigation of behavior of an instrument whose point of suspension arbitrarily shifts over the surface of the Earth, and consequently, also over the surface of non-rotating sphere S. Thus, motion of axis of GP is studied with respect to translationally moving system $O\xi^*\eta^*\zeta^*$ which preserves its orientation relative to inertial space. As external forces acting on the GP there are considered gravity force F, directed to center of Earth, and force of inertia Q of translational induced motion of axes $O\xi^*\eta^*\zeta^*$; coriolis forces of inertia are absent, since axes $O\xi^*\eta^*\zeta^*$ move translationally.

Choosing axes Oxyz connected with internal Cardan ring, we write equations of GP in the following general form [45]:

$$\omega_x H = l(Q_x + F_x); \quad \omega_y H = l(Q_y + F_y). \tag{11.4.156}$$

where ω_x, ω_y – projections of angular velocity of axes Oxyz relative to system $O\omega^*\eta^*\zeta^*$, i.e., relative to motionless sphere S, on axes Ox and Oy;

F_x, Q_x and F_y, Q_y – projections of forces F and Q on the same axes;

H and l – earlier adopted designations.

Further there is introduced into consideration the Darboux trihedron $Ox^0y^0z^0$ (with vertex at center of Cardan suspension), for which axis Oz^0 is normal to sphere S, and axis Ox^0 is directed along velocity vector v of point of suspension of GP.

Position of axes Oxyz relative to $Ox^0y^0z^0$ is determined by angles ψ and θ. As a result there are obtained the following differential equations of motion of the GP [45]:

$$\left.\begin{aligned}\left(\frac{v}{R}\sin\theta\sin\psi-\omega\cos\theta\sin\psi+\frac{d\theta}{dt}\cos\psi\right)H=\\ =l\left[\left(F-\frac{mv^2}{R}\right)\cos\theta\sin\psi-m\left(\frac{dv}{dt}\cos\psi+mv\sin\theta\sin\psi\right)\right]\\ \left(\frac{v}{R}\cos\theta+\omega\sin\theta+\frac{d\psi}{dt}\right)H=\\ =l\left[-\left(F-\frac{mv^2}{R}\right)\sin\theta-m\omega v\cos\theta\right]\end{aligned}\right\}\qquad(11.4.157)$$

where ω – projection of angular velocity of Darboux trihedron relative to axes $O\xi^*\eta^*\zeta^*$ (relative non-rotating sphere S) onto axis Oz^0;

v – velocity of object relative to sphere S;

R – radius of sphere S.

Thus, if motion of point of suspension of GP, and consequently also of Darboux trihedron relative to sphere S is given, then functions v(t) and ω(t) are known.

For small angles ψ and θ, equations (11.4.157) have the form

$$\left.\begin{aligned}\frac{d\theta}{dt}-\left[\omega+\frac{l}{H}\left(F-\frac{mv^2}{R}\right)\right]\psi=-\frac{ml}{H}\cdot\frac{dv}{dt}\\ \frac{d\psi}{dt}+\left[\omega+\frac{l}{H}\left(F-\frac{mv^2}{R}\right)\right]\theta=-\frac{v}{R}-\frac{ml}{H}\omega v\end{aligned}\right\}\qquad(11.4.158)$$

which allows integration in quadratures.

A. Yu. Ishlinskiy for solution of equations (11.4.158) uses the substitution $\theta=-\frac{ml}{H}v+\$$ and reduces the indicated equations to the following system:

$$\left.\begin{aligned}\frac{d\theta}{dt}-p(t)\psi=0\\ \frac{d\psi}{dt}+p(t)\theta=q(t)\end{aligned}\right\}\qquad(11.4.159)$$

where

$$\begin{aligned}p(t)&=\omega+\frac{l}{H}\left(F-\frac{mv^2}{R}\right);\\ q(t)&=\left[\frac{ml^2}{H^2}\left(F-\frac{mv^2}{R}\right)-\frac{1}{R}\right]v.\end{aligned}\qquad(11.4.160)$$

Of the greatest interest is the case when q(t) ≡ 0. For small velocity of point of suspension relative to Earth, it is possible to take

$$F-\frac{mv^2}{R}=mg,\qquad(11.4.161)$$

where g – acceleration due to gravity.

Then, according to (11.4.160), condition q(t) = 0 exists for

$$\frac{mgl}{H}=\sqrt{\frac{g}{R}}.$$

i.e., when the condition of M. Shuler is satisfied.

It is easy to see that for zero initial conditions [$t = 0$, $\vartheta(0) = 0$, $\psi(0) = 0$], axis of GP will make with the goecentric vertical angle θ, which is determined by relationship

$$\theta = -\frac{ml}{H}v, \qquad (11.4.162)$$

which characterizes rate error of the GP.

In this case the GP does not have ballistic errors. Expression (11.4.162) for known velocity and heading of the object can be used for determination of direction of the geocentric vertical.

Those interested more deeply in the theory of GP we refer to the mentioned articles [45, 53]. Their method of investigation of GP on non-rotating sphere S was earlier used by A. Yu. Ishlinskiy in the investigation of the theory of other gyroscopic devices [44, 49].

§ 11.5. Gyrocompasses (GC)

1. Purpose of the GC. Principles of Construction Designs of the Instrument

a) Types of GC

A gyrocompass determines on a moving object the direction of the local meridian, i.e., the heading of the object (vehicle). GC have found wide application on ships, basically for purposes of navigation. It is included in the class of positional GD, i.e., it possesses a "directional force" (directional torque). The GC can be considered as a variety of the generalized GP (§ 11.3, Par. 3).

In the simplest scheme of GC there is used a three-degree-of-freedom gyroscope, the center of gravity of which is displaced in its equatorial plane below the point of suspension. Due to this the GC is subject to disturbance due to accelerations of the object, which cause ballistic errors of compass which have the same nature as for the GP. These errors of the GC are partially compensated by means of realization in the instrument of M. Schuler's condition of undisturbability, which was by him validated in reference to GC in 1910 [200]. Realization of this condition made it possible to create models of GC which are useful also under conditions of maneuvering of the ship.

Up to the present of time gyroscopic technology has had at its disposal tens of different models of first-class GC, which representing very complicated and highly developed types of gyroscopic devices. Theory of gyrocompass is the most highly

developed area of contemporary theory of GD. In its creation an outstanding role was played by the works of A. N. Krylov, B. V. Bulgakov, B. I. Kudrevich, M. Shuler, J. Geckeler, O. Martinssen, A. Rawlings, Ye. L. Nikolai, A. Yu. Ishlinskiy, Ya. N. Roytenberg, D. R. Merkin, V. N. Koshlyakov, and others. To questions of theory and practice of GC there have been dedicated a huge amount of monographic and educational literature. Therefore, in the present book there is given only the briefest information from the theory of GC.

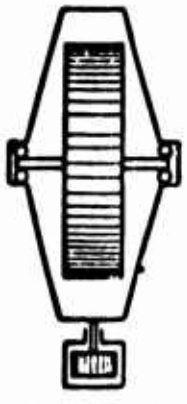

Fig. 11.11. Diagram of sensing device of GC with pendulum.

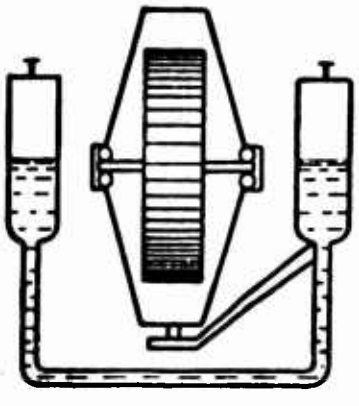

Fig. 11.12. Diagram of sensing device of GC with mercury vessels.

There are many types and varieties of GC. Without attempting to give their classification, we will indicate only three basic types. Depending upon method of creation of directional force, we distinguish: 1) GC with pendulum; 2) GC with mercury vessels; 3) GC with electromagnetic control.

In GC with pendulum, rotor is contained in gyrohousing, to the lower part of which there is fastened a pendulum (Fig. 11.11). Due to this, center of gravity of rotor with gyrohousing turns out to be displaced in the equatorial plane below point of suspension.

In GC with mercury vessels, rotor and gyrohousing are balanced in such a way that their common center of gravity coincides with point of suspension. To the gyrochamber there is fastened a mercury ballistic system consisting of two vessels (Fig. 11.12) which are connected by means of a connecting tube and partially filled with mercury. To the right vessel there is fastened a so-called arm, which connects vessels with the gyrochamber. When axis of gyroscope is deflected from plane of horizon, surplus of mercury in one of vessels causes application to the gyroscope of a gravitational analogous to the corresponding torque in a GC with a pendulum.

In recent years GC with electromagnetic control have come into use (see, for instance, [59]). In GC of this type the gyroscope is brought into the meridian not due to action of gravitational torque created by pendulum, mercury ballistic vessels or analogous devices, but by means of application to it with the help of a special torque pickoff (TP) of a correcting torque; thus the TP is controlled from a pendulum (accelerometer) fixed on gyrohousing of GC. Rate errors due to motion of ship are compensated in the instrument by means of application of corrective torques calculated

by a computer. In GC with electromagnetic control, it is sufficiently simple to carry out by a purely electrical method damping of natural oscillations of the gyroscope [59, 99]. Let us note that such GC can also work as a directional gyroscope [59]; this is important for decrease of ballistic instrument errors during maneuvers of the ship.

Depending upon number of applied gyroscopes, we distinguish single-rotor, two-rotor and three-rotor GC. The widest use has been obtained by two-rotor GC — Anschütz gyrocompasses [75] and the Soviet gyrocompass "Kurs" [6].

There are known instruments in which there are combined the gyrovertical (gyrohorizon) and gyrocompass; these are called gyrohorizon compasses [75]; they make it possible in principle to determine the heading of the ship, and also angles of pitching and rolling. Such a system makes it possible to obtain a so-called space gyrocompass [20, 184], i.e., an instrument whose three principal axes are gyroscopically stabilized in the east - north - zenith directions.

It is also possible to classify GC according to other criteria; for instance: by method of damping of its natural oscillations; in dependence upon methods of obtaining an aperiodic compass (undisturbed by accelerations of the ship) at any latitude; according to method of suspension of sensing device; according to different design features, and so forth.

Briefly expounding the theory of GC, in the beginning we will reveal certain general properties of it in the example of a single-rotor gyrocompass with pendulum, and then will consider mainly the scheme of two-rotor GC.

b) Essence of Single-Rotor GC

Principle of action of gyrocompass is assumed to be known. Let us remember that in § 11.2, Par. 2a, there was considered a Foucault gyroscope of the first kind, for which axis of rotation is located vertically (Fig. 11.1), and inner ring is rigidly fastened to outer ring, and therefore axis of gyroscope is in plane of horizon. Thus, due to rotation of Earth there appears gyroscopic torque $M_{gyr\zeta}$, determined by relationship (11.2.5). This torque, called directional torque, tends to bring axis of gyroscope into plane of meridian. Consequently, Foucault gyroscope of first kind possesses properties of a compass, which appear in it due to rotation of Earth.

In the considered GC with the pendulum there is a certain analogy. Actually, here there is applied a gyroscope with three degrees of freedom; axis of rotation of outer Cardan ring is fixed vertically. With the inner ring (gyrochamber) there

is connected a pendulum, which during deflection of axis of gyroscope from plane of horizon creates a restoring torque, which tends to hold axis of gyroscope in horizontal position. Such a device A. N. Krylov [65] called a gyroscope "with incomplete coupling;" due to its application, it was possible to realize the gyrocompass. Thus, the appearance in the considered gyrocompass of a directional torque tending to bring axis of gyroscope into plane of meridian (just as in a Foucault gyroscope of first kind, i.e., in two-degree-of-freedom GC), is caused by rotation of Earth and presence of coupling of the gyroscope with the Earth, which is realized through the pendulum.

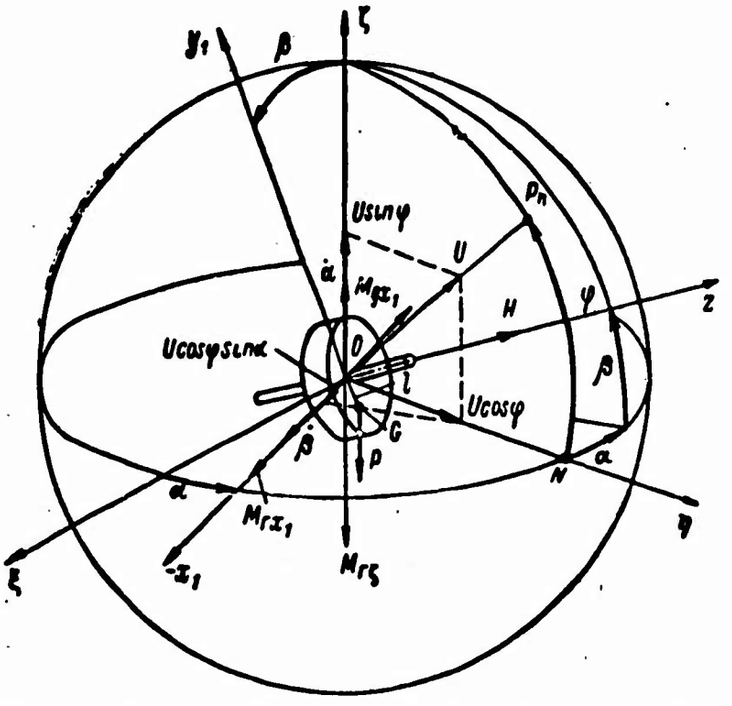

Fig. 11.13. Angles α and β of deflection of axis of GC.

Let us select axes Oξηζ (Fig. 11.13) oriented geographically, i.e., in such a manner that axis Oη is directed in plane of horizon to north, axis Oξ – to east and axis Oζ – along the vertical. During turn of outer Cardan ring by angle α, and inner ring by angle β, Resal axes will occupy position Ox_1y_1z. By analogy with Fig. 3.12 we will decompose angular velocity U of diurnal rotation of Earth into vertical U sin φ and horizontal U cos φ components and will show the desired (§ 11.2, Par. 2a) U cos φ sin α component of the latter. Vertical component U sin φ creates gyroscopic torque M_{gyrx_1},* which is directed along the axis Ox_1 of rotation of inner Cardan ring and equal to

$$M_{rx_1} = -HU \sin\varphi \cos\beta \approx -HU \sin\varphi. \qquad (11.5.1)$$

This moment causes rise of axis of gyroscope by angle β above plane of horizon; thus there appears gravitational torque M_{gx_1}, which is also directed along the axis Ox_1:

$$M_{gx_1} = lP \sin\beta \approx lP\beta. \qquad (11.5.2)$$

Restoring torque M_{gx_1} balances gyroscopic torque M_{gyrx_1}. Angle β_r of rise of axis of gyroscope above plane of horizon in position of equilibrium is determined from relationship

$$M_{rx_1} + M_{gx_1} = 0 \quad \text{or} \quad -HU \sin\varphi + lP\beta_r = 0,$$

*See note on p.307 – [Trans. Ed. Note].

whence we have expression

$$\beta_r = \frac{HU}{lP} \sin \varphi. \qquad (11.5.3)$$

which coincides with (11.3.22).

As was shown during the analysis of generalized GP (§ 11.3, Par. 3; case of the gyrocompass), to angle β_r there correspond gravitational torque $lP\beta_r = HU \sin \varphi$ and rate of precession $\frac{lP\beta_r}{H}$, which is equal to U sin φ. This rate of precession of the GC is necessary for compensation of vertical component U sin φ of terrestrial rotation and for retention thereby of axis of gyroscope in plane of meridian $O\zeta P_n N$.

The desired component U cos φ sin α gives gyroscopic torque $M_{gyr\zeta}$, directed along the axis Oζ and determined by relationship (11.2.5):

$$M_{r\zeta} = -HU \cos \varphi \sin \alpha \approx -HU \cos \varphi \cdot \alpha. \qquad (11.5.4)$$

In the theory of GC this torque is called <u>directional torque of the gyrocompass</u>. It is equal to zero at α = 0. Therefore, considering also (11.5.3), position of equilibrium of axis of GC on a base which is motionless with respect to Earth is determined by coordinates

$$\alpha_r = 0; \quad \beta_r = \frac{HU}{lP} \sin \varphi. \qquad (11.5.5)$$

In order to determine character of motion of axis of gyroscope near the indicated position of equilibrium, we will assume that axis of gyroscope, having slope β_r to plane of horizon, is deflected eastward by angle α. Since due to diurnal rotation of Earth the eastern edge of the plane of the horizon is lowered in space, then northern end of axis of gyroscope is visibly raised above plane of horizon. Thus angle of inclination β will be increased. Now restoring torque $M_{gx_1} = lP\beta$ will be greater than gyroscopic torque $|M_{gyrx_1}| = HU \sin \varphi = lP\beta_r$; due to the difference in these torques $M_{gx_1} - M_{gyrx_1} = lP\beta - HU \sin \varphi = lP(\beta - \beta_r)$, which is directed toward the west, there will appear precessional motion of axis of gyroscope toward plane of meridian. Thus, axis of gyroscope rises above plane of horizon and shifts in azimuth. When axis of gyroscope crosses plane of meridian $O\zeta P_n N$ to the west, it will start to be lowered toward plane of horizon, and under the action of the difference in torques $M_{gx_1} - M_{gyrx_1} = lP\beta - lP\beta_r = -lP(\beta_r - \beta)$, which is directed toward the east (for $\beta < \beta_r$), will further precess eastward toward plane of meridian. Consequently, axis of gyroscope accomplishes sustained oscillations near position of equilibrium, which is determined by coordinates $\alpha_r = 0$ and β_r [see (11.5.5)]; thus, vertex of gyroscope will describe on representative plane (tangent to point N of intersection of axis

$O\eta$ with unit sphere) an ellipse which is strongly elongated in the horizontal direction. For quenching of these oscillations, in the GC there are applied special damping devices.

Axis of gyroscope is brought into plane of meridian in an analogous way in the GC with mercury vessels (see, for instance, [71, 7, 6]).

c) Essence of Two-Rotor GC

Two-rotor GC is one of the most highly developed of gyroscopic devices in design and accuracy.

The sensing device of this GC constitutes a hollow sphere Gs (Fig. 11.14a), called a gyrosphere or float. In the gyrosphere there are placed gyroscopes G_1 and G_2, a hydraulic damper and other elements of the instrument. Spin axes of gyroscopes G_1 and G_2 are located horizontally; axes of precession of these gyroscopes are located vertically and are connected by a hinged mechanism — a shaft C, similarly to the way this is done in a two-gyroscope active gyroframe (Fig. 7.2). Shaft Sh with the help of springs Sp is connected with body of gyrosphere. In the initial position (with non-rotating rotors), springs hold the gyroscopes in a position in which their axes form with the N-S* direction of the gyrosphere equal angles $\varepsilon = 45^\circ$. Due to the shaft gyroscopes G_1 and G_2 can turn about axes of procession in various directions by identical angles δ (Fig. 11.14b). Inside the gyrosphere is filled with hydrogen. Center of gravity of gyrosphere is on its vertical axis below its geometric center; this ensures, as and in the single-rotor GC, the necessary pendulum torque. Gyrosphere is placed in a fluid, and therefore in the suspension there exists only viscous friction.

The gyrosphere is surrounded by the so-called slave sphere. In order to center the gyrosphere relative to the slave sphere, in the lower part of gyrosphere there is placed an "electromagnetic blowout" coil, through which there is passed an alternating current. The latter induces in the slave sphere eddy currents, which "repel" the coil and center the gyrosphere in vertical and horizontal directions, so that its only motionless point is its geometric center. Parameters of the system are selected in such a way that period of precessional oscillations of gyrosphere in the absence of damping is equal to 84.4 minutes. This practically removes ballistic errors of GC. Constancy of the period of 84.4 minutes independently of local latitude is ensured in a two-rotor GC of Anschütz [75] by corresponding change of angle of

*As the N-S direction there is taken direction coinciding with bisector Oz of the angle formed by spin axes of gyroscopes G_1 and G_2.

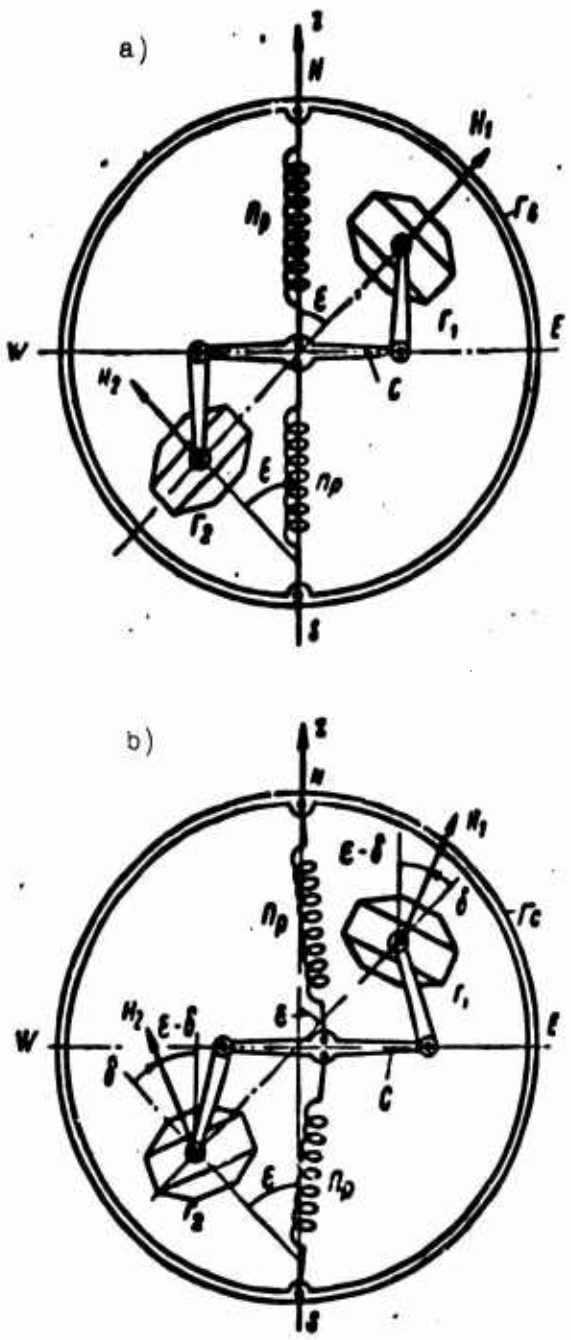

Fig. 11.14. Diagram of sensing device of two-rotor GC.

separation of the gyroscopes.* Presence in the compass of two gyroscopes makes it possible to increase period of oscillations of gyrosphere around N-S axis to 10-15 minutes. This considerably lowers instrument error during rocking of ship. Natural oscillations of gyrosphere are damped by a hydraulic damper installed in its upper part.

Total angular momentum of gyroscopes G_1 and G_2 is equal to geometric sum of their spin angular momenta H_1 and H_2. If we designate by 2ε (Fig. 11.14a) the angle between axes of these gyroscopes, then total angular momentum H of the system will be

$$H = 2H' \cos \varepsilon \quad (H' = H_1 = H_2). \qquad (11.5.6)$$

Angular-momentum vector $\vec{H}$ is directed along the bisector of angle 2ε, i.e., coincides with direction of N-S axis of gyrosphere, which during operation of the compass is established in plane of meridian. With turn of gyroscopes G_1 and G_2 around their axes of precession by angle δ, direction of vector $\vec{H}$ as this is easy to establish from Fig. 11.14b, will not be changed, but its magnitude will be changed, and instead of by (11.5.6) will be determined by relationship

$$H = 2H' \cos(\varepsilon - \delta). \qquad (11.5.7)$$

Principle of operation of two-rotor GC in broad terms is the same as that of the single-rotor GC. However, here there are certain peculiarities. Besides the pendulum torque, which causes precessional motion of gyrosphere in azimuth, on the gyroscopes through the shaft there act the torques of the springs. This causes precessional motion of gyrosphere about the N-S axis, due to which the axis perpendicular to it (E-W diameter of the gyrosphere) in its position of equilibrium will be horizontal.

*Other methods of regulation of period in two-rotor GC can be realized by change of angular velocity of spin of rotors or magnitude of maximum pendulum torque due to change of metacentric height. This problem in the GC with mercury vessels can be solved by changing angular velocity of spin of rotor or magnitude of maximum pendulum torque due to change of area of cross section of vessels or their moment arm.

If relative to the N-S axis there acts external disturbing torque, then there appears precessional motion of gyroscopes G_1 and G_2 inside the gyrosphere; due to this there appears a gyroscopic torque compensating the external disturbing torque.

Description of device and principle of action of the two-rotor GC can be found in extensive literature on gyroscopic compasses (see, for instance, [75, 6, 30]).

2. Dynamic Characteristics of GC

a) Geometric and Kinematic Parameters

Investigation of GC is started with composition of differential equations of motion and corresponding transfer functions. As the sensing device of the GC we will consider the gyrosphere (Fig. 11.14). Let us select system of coordinates $O\xi\eta\zeta$ (Fig. 11.15), oriented geographically along the east - north - zenith directions. Axes of coordinate system Oxyz are principal axes of inertia of the gyrosphere; thus axis Oz is parallel to bisector of angle formed by axes of gyroscopes G_1 and G_2 (Fig. 11.14); center of gravity G of gyrosphere (Fig. 11.15) is located below its geometric center O on axis Oy. Turn of the gyrosphere, i.e., of axes Oxyz relative to reference system $O\xi\eta\zeta$ can be represented as consecutive rotation of it by angles α, β and γ where α is angle of azimuthal turn of gyrosphere, β is angle of use of end of N-S axis above plane of horizon, and γ is angle of rotation of gyrosphere about N-S axis. With turn by angles α and β, the moving axis will occupy position Ox_1y_1z (Resal axes).

Transformation of axes $O\xi\eta\zeta$ to Ox_1y_1z we will write in the form of the following matrix equality:

$$[x_1,\ y_1,\ z] = A_1[\xi,\ \eta,\ \zeta]. \tag{11.5.8}$$

where matrix $A_1 = \|a'_{\mu\nu}\|$ ($\mu,\ \nu = 1,\ 2,\ 3$) is determined by Table (5.2.2) (cf. Figs. 11.15 and 3.2). For small angles α and β for matrix A_1 we have Table (5.2.3)

$$A_1 = \begin{Vmatrix} -1 & -\alpha & 0 \\ 0 & -\beta & 1 \\ -\alpha & 1 & \beta \end{Vmatrix} \tag{11.5.9}$$

Transformation of axes $O\xi\eta\zeta$ to Oxyz can be written in matrix form

$$[x,\ y,\ z] = A[\xi,\ \eta,\ \zeta], \tag{11.5.10}$$

where matrix $A = \|a_{\mu\nu}\|$ ($\mu,\ \nu = 1,\ 2,\ 3$) can be represented in the form of product of three corresponding to turns of axes Oxyz by angles α, β and γ; finally we will obtain

$$A = \begin{Vmatrix} -\cos\alpha\cos\gamma + \sin\alpha\sin\beta\sin\gamma & -\sin\alpha\cos\gamma - \cos\alpha\sin\beta\sin\gamma & \cos\beta\sin\gamma \\ \cos\alpha\sin\gamma + \sin\alpha\sin\beta\cos\gamma & \sin\alpha\sin\gamma - \cos\alpha\sin\beta\cos\gamma & \cos\beta\cos\gamma \\ -\sin\alpha\cos\beta & \cos\alpha\cos\beta & \sin\beta \end{Vmatrix} \tag{11.5.11}$$

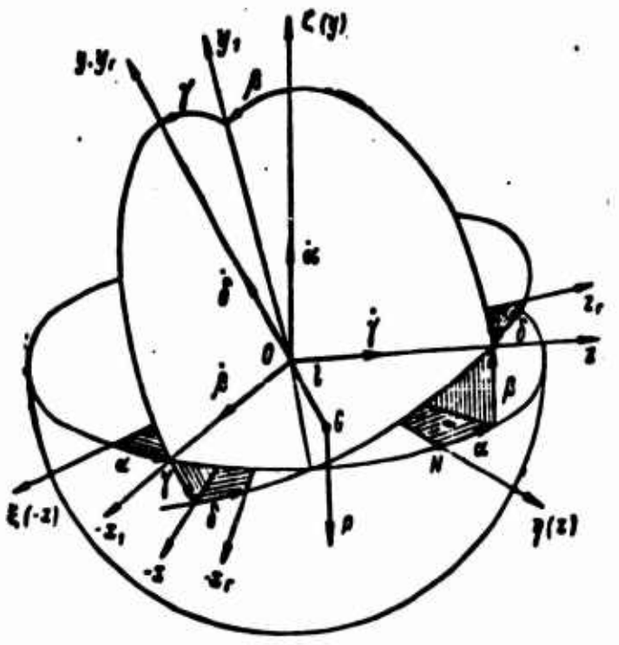

Fig. 11.15. Angles α, β, γ, δ, determining position of sensing device of GC.

With accuracy up to terms of first order of smallness we have

$$A = \begin{Vmatrix} -1 & -\alpha & \gamma \\ \gamma & -\beta & 1 \\ -\alpha & 1 & \beta \end{Vmatrix} \qquad (11.5.12)$$

Thus, position of gyrosphere is determined by three generalized coordinates: α, β, γ. Gyroscopes G_1 and G_2 (Fig. 11.14b) can turn about axes of precession by identical angles δ in different directions. Let us connect with gyrohousing of gyroscope G_1 axes $Ox_1y_1z_{gyr}$; turn of these axes by angle δ is shown in Fig. 11.15.

Transformation of axes Oxyz to $Ox_{gyr}y_{gyr}z_{gyr}$ will be represented in matrix form

$$[x_r,\ y_r,\ z_r] = B[x,\ y,\ z], \qquad (11.5.13)$$

where matrix $B = \|b_{\mu\nu}\|$ (μ, ν = 1, 2, 3) will be

$$B = \begin{Vmatrix} \cos\delta & 0 & -\sin\delta \\ 0 & 1 & 0 \\ \sin\delta & 0 & \cos\delta \end{Vmatrix} \qquad (11.5.14)$$

or approximately

$$B = \begin{Vmatrix} 1 & 0 & -\delta \\ 0 & 1 & 0 \\ \delta & 0 & 1 \end{Vmatrix} \qquad (11.5.15)$$

Let us note that angle δ is small (usually does not exceed 7° [6]) as compared to angle ε (Fig. 11.14), which is equal to 45°.

Projections p_1, q_1, r of angular velocity of gyrosphere onto Resal axes Ox_1y_1z, taking into account induced motion of axes Oξηζ caused by rotation of Earth and motion of the object (vehicle), are determined for small α and β by relationships (5.2.6), if in the last of them we consider angular velocity γ of rotation of the float:

$$\left.\begin{aligned} p_1 &= -\dot\beta - u_\zeta - u_\eta\alpha \\ q_1 &= \dot\alpha - u_\eta\beta + u_\xi \\ r &= \dot\gamma - u_\zeta\alpha + u_\eta + u_\xi\beta \end{aligned}\right\} \qquad (11.5.16)$$

b) Differential Equations of Motion of GC

During composition of differential equations of motion of GC we will use precessional theory, disregarding small nutational oscillations of system. For reduction of subsequent computations we will compose in the beginning the equation of motion of gyrosphere relative to axes Ox_1 and $O\zeta$ (Oy) or Oy_1 (for small β). For this we will use shortened Euler equations (1.6.9), replacing in them, according to (11.5.7), H by $2H' \cos(\varepsilon-\delta)$ and considering formulas (11.5.16) for p_1, q_1:

$$\left.\begin{aligned} 2H'\cos(\varepsilon-\delta)\left(\dot{\alpha}-u_{y_1}\beta+u_{z_1}\right)&=M_{x_1}\\ 2H'\cos(\varepsilon-\delta)\left(\dot{\beta}+u_{x_1}+u_{z_1}\alpha\right)&=M_{y_1}\end{aligned}\right\} \qquad (11.5.17)$$

where M_{x_1} and M_{y_1} are external torques applied to gyrosphere.

Let us turn to composition of equation of motion of gyrosphere about axis Oz, proceeding from condition of equilibrium of gyroscopic torque and external torque relative to this axis. Since gyroscopes G_1 and G_2 (Fig. 11.14) are connected with each other by a shaft, then for the eastern gyroscope G_1 angular velocity $\dot{\delta}$ is directed in positive direction of axis Oy (Fig. 11.15), and for the western gyroscope G_2 — in negative direction of axis Oy. Projections of angular momenta of gyroscopes G_1 and G_2 on axis Ox will be: $-H_1 \sin(\varepsilon-\delta) = -H' \sin(\varepsilon-\delta)$ and $H_2 \sin(\varepsilon-\delta) = H' \sin(\varepsilon-\delta)$. As a result along the axis Oz there will act gyroscopic torque

$$M_{z_1} = -2H' \sin(\varepsilon-\delta)\cdot\dot{\delta}. \qquad (11.5.18)$$

This torque is balanced by external torque M_z along the same axis, if we disregard in the equation the corresponding inertial term. Consequently, equation of motion of gyrosphere about axis Oz will be

$$2H' \sin(\varepsilon-\delta)\cdot\dot{\delta} = M_z. \qquad (11.5.19)$$

Let us note that component q_1 of angular velocity of Resal axes along the axis Oy_1 (Oy) does not create gyroscopic torques relative to axis Oz, since rotors of gyroscopes G_1 and G_2 rotate in different directions; i.e., as just in the two-gyroscope active frame, rotation of system of reference around axes of precession of gyroscopes does not render an influence on motion of the frame (§ 7.1, Par. 2b).

Further it is necessary to compose equation of motion of gyroscopes G_1 and G_2 around their axes of precession Oy (Oy_{gyr}). For this we will use the same method which we applied during composition of equations of motion of gyroscopes of the two-gyroscope active frame around their axes of precession. During finding of equation of motion of eastern gyroscope G_1 relative to axis Oy, it is necessary to consider gyroscopic torque

$$M_{ry}^{(1)} = H_1 \cos(\varepsilon - \delta)\cdot p_1 + H_1 \sin(\varepsilon - \delta)\cdot r, \qquad (11.5.20)$$

where angular velocities p_1 and r are determined by relationships (11.5.16).

Designating by $M_y^{(1)}$ the external torque applied to gyroscope G_1, we will obtain equation of motion of this gyroscope in the form

$$H_1 \cos(\varepsilon - \delta)\cdot p_1 + H_1 \sin(\varepsilon - \delta)\cdot r = -M_y^{(1)}. \qquad (11.5.21)$$

During composition of equation of motion of western gyroscope G_1 around axis Oy, it is necessary to consider gyroscopic torque

$$M_{ry}^{(2)} = H_2 \cos(\varepsilon - \delta)\cdot p_1 - H_2 \sin(\varepsilon - \delta)\cdot r. \qquad (11.5.22)$$

Designating by $M_y^{(2)}$ the external torque applied to gyroscope G_2, we will obtain equation of motion of this gyroscope in the form

$$H_2 \cos(\varepsilon - \delta)\cdot p_1 - H_2 \sin(\varepsilon - \delta)\cdot r = -M_y^{(2)}. \qquad (11.5.23)$$

For composition of equations of motion of system of gyroscopes G_1 and G_2 around axis Oy, we should consider that gyrohousings of gyroscopes are connected by a kinematic transmission with transmission ratio -1. Therefore, subtracting from (11.5.21) relationship (11.5.23) and considering that $H_1 = H_2 = H'$, $M_y^{(1)} - M_y^{(2)} = M_y$, we will obtain

$$2H' \sin(\varepsilon - \delta)\cdot r = -M_y \qquad (11.5.24)$$

or, considering (11.5.16),

$$2H' \sin(\varepsilon - \delta)\left(\dot{\gamma} - u_\xi \alpha + u_\eta + u_\zeta \beta\right) = -M_y. \qquad (11.5.25)$$

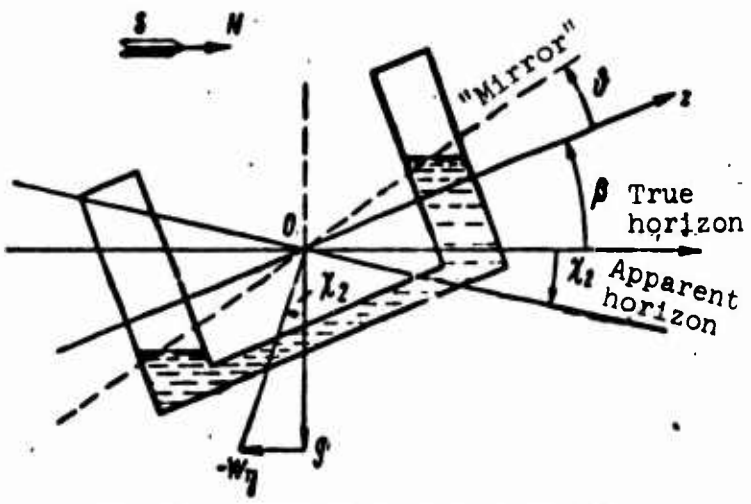

Fig. 11.16. Hydraulic damper of GC.

Taking together (11.5.17), (11.5.19) and (11.5.25), we will obtain system of equation characterizing motion of gyrosphere relative to reference system $O\xi\eta\zeta$ and rotation of gyrochambers of gyroscopes G_1 and G_2 inside the gyrosphere:

$$\left.\begin{aligned} 2H' \cos(\varepsilon - \delta)\cdot\left(\dot{\alpha} - u_\xi \beta + u_\zeta\right) &= M_{x_1} \\ 2H' \cos(\varepsilon - \delta)\cdot\left(\dot{\beta} + u_\xi + u_\eta \alpha\right) &= M_{y_1} \\ 2H' \sin(\varepsilon - \delta)\cdot\dot{\xi} &= M_z \\ 2H' \sin(\varepsilon - \delta)\cdot\left(\dot{\gamma} - u_\eta \alpha + u_\eta + u_\zeta \beta\right) &= -M_y \end{aligned}\right\} \qquad (11.5.26)$$

It is necessary to supplement these equations with equation of flow of liquid in the hydraulic damper. The scheme of the damper (Fig. 11.16) is the same as that of hydraulic damper of GP (Fig. 11.8), the principle of action of which is known (§ 11.4, Par. 3e). Damper is fixed in the N-S direction of the gyrosphere. In

Fig. 11.16 there are designated: β – angle of inclination of gyrosphere to plane of horizon; $\$$ – angle of inclination of "mirror;"* χ_2 – angle of inclination of apparent horizon to true horizen, which, by analogy with relationship (4.5.22), is determined by formula

$$\chi_2 = \frac{W_\eta}{g}, \tag{11.5.27}$$

where W_η – component of total acceleration of object along axis $O\eta$.

Magnitude of torque, introduced by surplus of liquid in right vessel, similarly to (11.4.33), is determined by expression

$$M_{ж} = c\vartheta. \tag{11.5.28}$$

Equation of flow of liquid in the damper, by analogy with (11.4.43), will be written in the form

$$\dot{\vartheta} = -F(\beta + \vartheta + \chi_2), \tag{11.5.29}$$

where F – flow factor [71].

Joining (11.5.29) to (11.5.26), we will obtain:

$$\left.\begin{aligned} 2H'\cos(\varepsilon-\delta)\cdot\left(\dot{\alpha}-u_2\beta+u_3\right) &= M_{x_1}\\ 2H'\cos(\varepsilon-\delta)\cdot\left(\dot{\beta}+u_1+u_2\alpha\right) &= M_{y_1}\\ \dot{\vartheta} &= -F(\beta+\vartheta+\chi_2)\\ 2H'\sin(\varepsilon-\delta)\cdot\dot{\delta} &= M_z\\ 2H'\sin(\varepsilon-\delta)\cdot\left(\dot{\gamma}-u_1\alpha+u_2+u_3\beta\right) &= -M_y \end{aligned}\right\} \tag{11.5.30}$$

Single angle δ is small, it is possible to replace $\cos(\varepsilon-\delta) \approx \cos\varepsilon + \delta\sin\varepsilon$, $\sin(\varepsilon-\delta) \approx \sin\varepsilon - \delta\cos\varepsilon$ and, considering (11.5.6) for $\varepsilon = 45^\circ$ and omitting in (11.5.30) small products of δ with $\dot{\alpha}$, $\dot{\beta}$, and so forth, to rewrite these equations in the form

$$\left.\begin{aligned} H\left(\dot{\alpha}-u_3\beta+u_1+u_1\delta\right) &= M_{x_1}\\ H\left(\dot{\beta}+u_1+u_3\alpha+u_1\delta\right) &= M_{y_1}\\ \dot{\vartheta} &= -F(\beta+\vartheta+\chi_2)\\ H\dot{\delta} &= M_z\\ H\left(\dot{\gamma}-u_1\alpha+u_2+u_3\beta-u_1\delta\right) &= -M_y \end{aligned}\right\} \tag{11.5.31}$$

Let us find expression of torques applied to gyrosphere and to gyroscopes. Torques M_{x_1}, M_{y_1}, M_z and M_y we will represent in the form

*By this angle we understand the angle formed by the line connecting centers of surfaces of liquid in vessels of the damper and axis Oz of the sensing device.

$$\left.\begin{aligned} M_{x_1} &= M_{(g.w)_x} + M_{dx} + f_1(t) \\ M_{y_1} &= M_{(g.w)_y} + f_2(t) \\ M_z &= M_{(g.w)_z} + f_3(t) \\ M_y &= M_{sp} + M_{fr} + f_4(t) \end{aligned}\right\} \qquad (11.5.32)$$

where $M_{(g,\ w)_x}$ – components of torque $\vec{M}_{g,\ w}$, due to action of gravity and forces of inertia as a result of accelerations of the object;

M_{dx} – damping torque, introduced by hydraulic damper;

M_{sp} – torque due to tension of springs;

M_{fr} – fluid frictional torque in axes of precession of gyroscopes;

$f_1(t),\ldots f_4(t)$ – other disturbing torques.

Torque $\vec{M}_{g,\ w}$, by analogy with (2.3.50), can be written in the form

$$\vec{M}_{g.w} = \vec{l} \times \left(m\vec{g}_0 - m\vec{W}\right). \qquad (11.5.33)$$

where m – mass of gyrosphere;

$\vec{l}$ – displacement of center of gravity of gyrosphere from its geometric center along axis Oy;

$\vec{g}_0$ – acceleration due to force of terrestrial gravitation;

$\vec{W}$ – total acceleration of object.

Projecting (11.5.33) onto axes Ox_1y_1z, we have:

$$\left.\begin{aligned} M_{(g.w)_x} &= ml_y(g_{0z} - W_z) - ml_z(g_{0y} - W_y) \\ M_{(g.w)_y} &= ml_z(g_{0x} - W_x) - ml_x(g_{0z} - W_z) \\ M_{(g.w)_z} &= ml_x(g_{0y} - W_y) - ml_y(g_{0x} - W_x) \end{aligned}\right\} \qquad (11.5.34)$$

For projections of vector $\vec{l}$ onto axes Ox_1y_1z (Fig. 11.15), we find

$$l_x = l\gamma; \quad l_y = -l; \quad l_z = 0. \qquad (11.5.35)$$

Projections g_{0x}, g_{0y}, g_{0z}, according to Table (11.5.9), will be

$$g_{0x} = -g_{0\xi} - g_{0\eta}\alpha; \quad g_{0y} = -g_{0\eta}\beta + g_{0\zeta};$$
$$g_{0z} = -g_{0\xi}\alpha + g_{0\eta} + g_{0\zeta}\beta.$$

Since $g_{0\xi} = g_{0\eta} = 0$, $g_{0\zeta} = -g_0$, then

$$g_{0x} = 0; \quad g_{0y} = -g_0; \quad g_{0z} = -g_0\beta. \qquad (11.5.36)$$

For W_x, W_y, W_z, according to Table (11.5.9), we have

$$W_x = -W_\xi - W_\eta\alpha; \quad W_y = -W_\eta\beta + W_\zeta; \qquad (11.5.37)$$
$$W_z = -W_\xi\alpha + W_\eta + W_\zeta\beta.$$

Putting (11.5.35), (11.5.36), (11.5.37) in (11.5.34) and considering that approximately $mg_0 \approx mg = P$ [see (11.4.17)], we will obtain:

$$\left.\begin{aligned} M_{(g.w)_x} &= lP\left[\left(1+\frac{W_\zeta}{g}\right)\beta-\frac{W_\xi}{g}\alpha+\frac{W_\eta}{g}\right] \\ M_{(g.w)_y} &= lP\left[\left(1+\frac{W_\zeta}{g}\right)\beta-\frac{W_\xi}{g}\alpha+\frac{W_\eta}{g}\right]\gamma \\ M_{(g.w)_z} &= -lP\left[\left(1+\frac{W_\zeta}{g}\right)\gamma-\frac{W_\eta}{g}\beta\gamma-\frac{W_\xi}{g}-\frac{W_\eta}{g}\alpha\right] \end{aligned}\right\} \qquad (11.5.38)$$

Damping torque M_{dx} created by surplus of liquid in one of the vessels of hydraulic damper, if we take into account (11.5.28), will be

$$M_{xx} = M_{xy} = c\vartheta.$$

Torque due to tension of springs Sp (Fig. 11.14) connecting shaft with gyrosphere, for small δ is determined by relationship

$$M_{xp} = -c_1\delta, \qquad (11.5.39)$$

where coefficient c_1 characterizes stiffness of springs.

Fluid frictional torque in axes of precession of gyroscopes will be*

$$M_\tau = -n_1\dot{\delta}, \qquad (11.5.40)$$

where n_1 – coefficient of fluid friction.

Putting (11.5.28) and (11.5.38) - (11.5.40) in (11.5.32), we will obtain

$$\left.\begin{aligned} M_{x_1} &= lP\left[\left(1+\frac{W_\zeta}{g}\right)\beta-\frac{W_\xi}{g}\alpha+\frac{W_\eta}{g}\right]+c\vartheta+f_1(t) \\ M_{y_1} &= lP\left[\left(1+\frac{W_\zeta}{g}\right)\beta-\frac{W_\xi}{g}\alpha+\frac{W_\eta}{g}\right]\gamma+f_2(t) \\ M_z &= -lP\left[\left(1+\frac{W_\zeta}{g}\right)\gamma-\frac{W_\eta}{g}\beta\gamma-\frac{W_\xi}{g}-\frac{W_\eta}{g}\alpha\right]+f_3(t) \\ M_\delta &= -c_1\delta-n_1\dot{\delta}+f_4(t) \end{aligned}\right\} \qquad (11.5.41)$$

Let us introduce (11.5.41) into equations (11.5.31); considering (11.5.27), we have

$$\left.\begin{aligned} H\left(\dot{\alpha}-u_\zeta\beta+u_\zeta+u_\eta\delta\right)-lP\left[\left(1+\frac{W_\zeta}{g}\right)\beta-\frac{W_\xi}{g}\alpha+\frac{W_\eta}{g}\right]-c\vartheta &= f_1(t) \\ H\left(\dot{\beta}+u_\xi+u_\eta\alpha+u_\zeta\delta\right)-lP\left[\left(1+\frac{W_\zeta}{g}\right)\beta-\right. & \\ \left.-\frac{W_\xi}{g}\alpha+\frac{W_\eta}{g}\right]\gamma &= f_2(t) \\ \delta = -F\left(\beta+\vartheta+\frac{W_\eta}{g}\right) & \\ H\dot{\delta}+lP\left[\left(1+\frac{W_\zeta}{g}\right)\gamma-\frac{W_\zeta}{g}\beta\gamma-\frac{W_\xi}{g}-\frac{W_\eta}{g}\alpha\right] &= f_3(t) \\ H\left(\dot{\gamma}-u_\xi\alpha+u_\eta+u_\zeta\beta-u_\eta\delta\right)-n_1\dot{\delta}-c_1\delta &= -f_4(t) \end{aligned}\right\} \qquad (11.5.42)$$

*Here we do not consider dry frictional torque in bearings of vertical axis of precession.

We have obtained differential equations of precessional motion of two-rotor GC in general form. Certain possible simplifications of them will be made in examining of the different cases of operation of GC.

3. Transient Responses of GC on Fixed Base

a) Sustained Oscillations of GC

During the analysis of sustained oscillations of GC we consider the case of installation of it on a base which is motionless with respect to Earth. Subsequently, during our short account of theory of GC we will be interested in azimuthal deflections α of gyrosphere, and also in deflections β of N-S direction of gyrosphere from plane of horizon. Let us note that angle γ of transverse slope of gyrosphere is minute; angle δ of turn of gyrohousings around their axes of precession also is small, but may be approximately 7-10 times greater than angle γ.

In the considered case, according to (1.2.21), $u_\xi = 0$, $u_\eta = U \cos \varphi$, $u_\zeta = U \sin \varphi$; components of total acceleration $\vec{W}$ of ship, if we disregard induced and Coriolis accelerations, will be $W_\xi = W_\eta = W_\zeta = 0$. Then, using for determination of sustained oscillations of gyrosphere with respect to coordinates α and β the first two equations of system (11.5.42) and considering in them also that $\gamma = \delta = 0$, $c = 0$ (hydraulic damper is turned off), in the absence of disturbances [$f_1(t) = f_2(t) = 0$)] we will obtain

$$\left.\begin{aligned} H(\dot{\beta} + U\cos\varphi \cdot \alpha) &= 0 \\ H(\ddot{\alpha} - U\cos\varphi \cdot \beta + U\sin\varphi) - lP\beta &= 0 \end{aligned}\right\} \qquad (11.5.43)$$

Let us determine the position of static equilibrium of N-S axis of gyrosphere; for this we will set in (11.5.43) $\dot{\alpha} = \dot{\beta} = 0$ and will introduce designation [20]

$$k = \sqrt{\frac{U\cos\varphi \cdot lP}{H}}. \qquad (11.5.44)$$

Then for coordinates α_r, β_r of position of equilibrium we have

$$\left.\begin{aligned} \alpha_r &= 0 \\ \beta_r &= \frac{\frac{HU}{lP}\sin\varphi}{1 + \frac{HU}{lP}\cos\varphi} = \frac{U^2}{k^2} \cdot \frac{\sin\varphi\cos\varphi}{1 + \frac{HU}{lP}\cos\varphi} \end{aligned}\right\} \qquad (11.5.45)$$

whence it follows that in position of equilibrium N-S axis of gyrosphere is in plane of meridian ($\alpha_r = 0$) and is raised above plane of horizon by angle β_r (in northern hemisphere).

Since usually for GC $\frac{HU}{lP}\cos\varphi \ll 1$, then for β_r we have approximately expression

$$\beta_r = \frac{HU}{lP}\sin\varphi = \frac{U^2}{k^2}\sin\varphi\cos\varphi, \qquad (11.5.46)$$

which coincides with the earlier found relationship (11.5.3). As was shown (par. 1b), rise β_r of N-S axis of gyrosphere is necessary for compensation of vertical component of terrestrial rotation and retention thereby of the N-S axis in plane of meridian.

For finding of law of motion of N-S axis of gyrosphere on a base motionless relative to Earth in the absence of damping, we will use equations (11.5.43), discarding in them the term HU cos $\varphi\cdot\beta$, which is small as compared to $lP\beta$; then we have

$$\left.\begin{aligned} \beta + U\cos\varphi\cdot\alpha &= 0 \\ \dot{\alpha} + U\sin\varphi - \frac{lP}{H}\beta &= 0 \end{aligned}\right\} \qquad (11.5.47)$$

or, considering (11.5.44) and (11.5.46),

$$\left.\begin{aligned} \beta + U\cos\varphi\cdot\alpha &= 0 \\ \dot{\alpha} - \frac{k^2}{U\cos\varphi}(\beta - \beta_r) &= 0 \end{aligned}\right\} \qquad (11.5.48)$$

whence

$$\left.\begin{aligned} \ddot{\alpha} + k^2\alpha &= 0 \\ \ddot{\beta} + k^2\beta &= k^2\beta_r \end{aligned}\right\} \qquad (11.5.49)$$

General solution of first equation will be

$$\alpha = C_1\cos kt + C_2\sin kt,$$

where constants of integration C_1 and C_2 are determined by initial conditions; as initial conditions we will take: at t = 0, $\alpha(0) = \alpha_0$, $\dot{\alpha}(0) = 0$. Then we will obtain

$$\alpha = \alpha_0\cos kt. \qquad (11.5.50)$$

Finding $\dot{\alpha}$ and putting it in second equation of system (11.5.48), we will obtain

$$\beta - \beta_r = -\alpha_0\frac{U\cos\varphi}{k}\sin kt. \qquad (11.5.51)$$

Thus, oscillations of N-S axis of gyrosphere are determined by expressions

$$\alpha = \alpha_0\cos kt, \quad \beta - \beta_r = -\alpha_0\frac{U\cos\varphi}{k}\sin kt, \qquad (11.5.52)$$

which characterize its undamped harmonic oscillations with frequency k, which are shifted from each other in phase by angle $\frac{\pi}{2}$. Period $T_{g.c}$ of sustained oscillations of GC, if we consider (11.5.44), is determined by formula

$$T_{g.c} = \frac{2\pi}{k} = 2\pi\sqrt{\frac{H}{lPU\cos\varphi}}. \qquad (11.5.53)$$

Let us find trajectory of "vertex" of gyrosphere on representative plane, considering α and β as rectangular coordinates of vertex on this plane. Eliminating for this from (11.5.52) time t, we will obtain

$$\frac{\alpha^2}{\alpha_0^2} + \frac{(\beta - \beta_r)^2}{\left(\alpha_0 \frac{U \cos \varphi}{k}\right)^2} = 1, \qquad (11.5.54)$$

whence it follows that trajectory of vertex of gyrosphere is an ellipse with semiaxes α_0 and $\alpha_0 \frac{U \cos \varphi}{k}$ and with center at point $(0, \beta_r)$. Ellipse will be strongly elongated in horizontal direction. Actually, ratio of its semiaxes $\frac{\alpha_0}{\alpha_0 \frac{U \cos \varphi}{k}} = \frac{k}{U \cos \varphi}$, if we consider (11.5.44), is a large quantity $\sqrt{\frac{lP}{HU \cos \varphi}}$, since for the gyrocompass $lP \gg HU \cos \varphi$.

Example 11.8: To calculate parameters of sustained oscillations of gyrocompass [75] for latitude $\varphi = 60^\circ$, for which $H = 1.55 \cdot 10^8$ g·cm·sec, weight of gyrosphere $P = 8500$ g, magnitude $l = 0.66$ cm, pendulum torque $lP = 5610$ g·cm, $U = 7.29 \cdot 10^{-5}$ 1/sec.

Solution: 1. By formula (11.5.4) we find directional torque $M_{gyr\zeta}$ for deflection of axis of gyrosphere in azimuth by angle $\alpha = 1^\circ$:

$$|M_{gr}| = HU \cos \varphi \cdot \alpha = 1{,}55 \cdot 10^8 \cdot 7{,}29 \cdot 10^{-5} \cdot 0{,}5 \cdot 0{,}01745 = 0{,}1 \text{ g. cm.}$$

2. According to (11.5.46) we calculate angle β_r of rise of N-S axis of gyrosphere above plane of horizon:

$$\beta_r = \frac{HU}{lP} \sin \varphi = \frac{1{,}55 \cdot 10^8 \cdot 7{,}29 \cdot 10^{-5}}{5610} 0{,}866 = 0{,}001744 \text{ rad.} = 6'.$$

3. Using expression (11.5.44), we determine frequency of natural sustained oscillations of gyrosphere:

$$k = \sqrt{\frac{U \cos \varphi \cdot lP}{H}} = \sqrt{\frac{7{,}29 \cdot 10^{-5} \cdot 0{,}5 \cdot 5610}{1{,}55 \cdot 10^8}} = 1{,}148 \cdot 10^{-3} \text{ 1/sec.}$$

4. By formula (11.5.53) we calculate period $T_{g.c}$ of oscillations of gyrosphere:

$$T_{r.g} = \frac{2\pi}{k} = \frac{6{,}28}{1{,}148 \cdot 10^{-3}} = 5473{,}17 \text{ sec.} = 91{,}2 \text{ min.}$$

5. We determine relation $\frac{k}{U \cos \varphi}$ of semiaxes of ellipse of sustained oscillations of vertex of gyrosphere:

$$\frac{k}{U \cos \varphi} = \frac{1{,}148 \cdot 10^{-3}}{7{,}29 \cdot 10^{-5} \cdot 0{,}5} = 31{,}5.$$

For bringing N-S axis of gyrosphere into plane of meridian, it is necessary to damp the above-indicated elliptic oscillations; this is done by application of different damping devices, in particular the hydraulic damper.

b) Damped Oscillations of GC

For revealing the general character of motion of a GC supplied with a hydraulic

damper, we will use equations (11.5.42); thus we will be interested in changes of coordinates α and β. Just as during the analysis of sustained oscillations of GC (par. a), we will consider the case of installation of it on a base which is motionless with respect to Earth. Considering what has been said, we will use the first three equations of system (11.5.42), considering in them $W_\xi = W_\eta = W_\zeta = 0$, $u_\xi = 0$, $u_\eta = U \cos \varphi$, $u_\zeta = U \sin \varphi$ [see (1.2.21)], $\gamma = \delta = 0$ and $f_1(t) = f_2(t) = 0$; then we will obtain equations

$$\left.\begin{aligned} H(\dot\beta + U\cos\varphi\cdot\alpha) &= 0\\ H(\dot\alpha - U\cos\varphi\cdot\beta + U\sin\varphi) - lP\beta - c\vartheta &= 0\\ \dot\vartheta &= -F(\beta+\vartheta)\end{aligned}\right\} \qquad (11.5.55)$$

which for $c = 0$ and $\$ = 0$, $\dot\$ = 0$ become equations (11.5.43) of sustained oscillations of the GC.

Disregarding in (11.5.55) the term $HU \cos \varphi\cdot\beta$, which is small as compared to $lP\beta$, we will obtain

$$\left.\begin{aligned} H(\dot\beta + U\cos\varphi\cdot\alpha) &= 0\\ H(\dot\alpha + U\sin\varphi) - lP\beta - c\vartheta &= 0\\ \dot\vartheta &= -F(\beta+\vartheta)\end{aligned}\right\} \qquad (11.5.56)$$

Let us determine the position of static equilibrium of N-S axis of gyrosphere and levels of liquid in vessels of damper relative to center line. For this in (11.5.56) we will set $\dot\alpha = \dot\beta = \dot\$ = 0$. Then for coordinates α_r, β_r, $\$_r$ of position of equilibrium, considering designations (11.5.44) and (11.4.36), we will obtain [20]:

$$\left.\begin{aligned} \alpha_r &= 0\\ \beta_r &= \frac{HU\sin\varphi}{lP-c} = \frac{1}{\rho}\cdot\frac{U^2}{h^2}\sin\varphi\cos\varphi\\ \vartheta_r &= -\beta_r\end{aligned}\right\} \qquad (11.5.57)$$

Thus, in the position of equilibrium in the presence of damping, N-S axis of gyrosphere is in plane of meridian; angle of use of N-S axis above plane of horizon in the presence of damping will be larger than in the absence of it [see (11.5.46) and (11.4.36)]. Since in position of equilibrium angle $\$_r = -\beta_r$ [see (11.5.57)], then in the considered case of absence of acceleration of the object, level of liquid in vessels of damper is parallel to plane of horizon.

Taking into account (11.5.57) and (11.5.44), we rewrite equations (11.5.56) in the form

$$\left.\begin{aligned} \dot\beta + U\cos\varphi\cdot\alpha &= 0\\ \dot\alpha - \frac{h^2}{U\cos\varphi}(\beta-\beta_r) - \frac{c}{H}(\vartheta-\vartheta_r) &= 0\\ \dot\vartheta + F(\vartheta+\beta) &= 0\end{aligned}\right\} \qquad (11.5.58)$$

or, designating deflections from position of equilibrium

$$\beta_1 = \beta - \beta_n, \quad \vartheta_1 = \vartheta - \vartheta_n, \tag{11.5.59}$$

we will obtain

$$\left.\begin{aligned} \dot{\beta}_1 + U\cos\varphi \cdot \alpha &= 0 \\ \dot{\alpha} - \frac{k^2}{U\cos\varphi}\beta_1 - \frac{\epsilon}{H}\vartheta_1 &= 0 \\ \dot{\vartheta}_1 + F(\vartheta_1 + \beta_1) &= 0 \end{aligned}\right\} \tag{11.5.60}$$

Considering (11.4.36) and (11.5.44), we have

$$\left.\begin{aligned} \dot{\beta}_1 + U\cos\varphi \cdot \alpha &= 0 \\ \dot{\alpha} - \frac{k^2}{U\cos\varphi}\beta_1 - \frac{k^2}{U\cos\varphi}(1-\rho)\vartheta_1 &= 0 \\ \dot{\vartheta}_1 + F(\vartheta_1 + \beta_1) &= 0 \end{aligned}\right\} \tag{11.5.61}$$

For analysis of natural oscillations of GC near position of equilibrium, it is necessary to find characteristic equation of system (11.5.61), which is obtained if determinant Δ of this system is equated to zero. For finding Δ we will write (11.5.61) in operator form ($p = \frac{d}{dt}$):

$$\left.\begin{aligned} p\alpha - \frac{k^2}{U\cos\varphi}\beta_1 - \frac{k^2}{U\cos\varphi}(1-\rho)\vartheta_1 &= 0 \\ U\cos\varphi \cdot \alpha + p\beta_1 &= 0 \\ F\beta_1 + (p+F)\vartheta_1 &= 0 \end{aligned}\right\} \tag{11.5.62}$$

Then determinant Δ of this system of equations will be

$$\Delta = \begin{vmatrix} p & -\frac{k^2}{U\cos\varphi} & -\frac{k^2}{U\cos\varphi}(1-\rho) \\ U\cos\varphi & p & 0 \\ 0 & F & p+F \end{vmatrix} = \tag{11.5.63}$$

$$= p^3 + Fp^2 + k^2 p + Fk^2\rho.$$

Consequently, from condition $\Delta = 0$, replacing p by λ, we will obtain characteristic equation of system (11.5.61) in the form

$$a_3\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 = 0, \tag{11.5.64}$$

where

$$a_0 = 1; \quad a_1 = \frac{1}{F\rho}; \quad a_2 = \frac{1}{k^2\rho}; \quad a_3 = \frac{1}{Fk^2\rho}. \tag{11.5.65}$$

Comparing (11.5.64) and (11.5.65) with (11.4.38) and (11.4.39), we see that characteristic equations of oscillations of GC and GP were obtained to be identical. Therefore, when the earlier found condition (11.4.42) is satisiied:

$$0 < \epsilon < 1P$$

natural oscillations of GC with respect to coordinates α, β, ϑ will attenuate. Knowing characteristic equation (11.5.64) of the system, we can simply find laws of

oscillations $\alpha(t)$, $\beta_1(t)$ and $\$_1(t)$ of gyrosphere and liquid in vessels of the damper. Since for parameters of the GC one root of characteristic equation (11.5.64) is a negative real root, and the two others are complex conjugate with the negative real part, then oscillations $\alpha(t)$, $\beta_1(t)$ and $\$_1(t)$ have the character of attenuating harmonic oscillations on which, there is superimposed attenuating aperiodic motion, i.e., in general they have the same character as natural damped oscillations of active GS (§ 7.3, Par. 2b). Detailed investigations and calculations of damped oscillations of GC can be met in the literature (see, for instance, [20, 6, 71, 64, 69, 75]).

4. Static Errors of GC Due to Vehicular Motion. Ballistic Errors of GC and Their Compensation

a) Preliminary Remark

During the analysis of accuracy of GC, of the greatest interest is calculation of its errors due to vehicular motion; thus we distinguish:

1) errors due to motion of vehicle (object) with constant velocity and heading, or rate errors of the GC;

2) errors due to longitudinal accelerations of object and its circulation (turn), or ballistic errors of the GC.

The most complicated is investigation of ballistic errors of GC and establishment of fundamental possibilities of their compensation. Solution of this problem has occupied many researchers, among which we should mention M. Schuler [200], I. Geckeler [183], A. Rawlings [196], A. N, Krylov [64, 69], Ye. L. Nikolai [105], B. V. Bulgakov [20], B. I. Kudrevich [71], A. Yu. Ishlinskiy [44], Ya. N. Roytenberg [128], D. R. Merkin [91], V. N. Koshlyakov [60, 61], E. I. Sliv [135], and others.

Further there is given short information on the theory of rate and ballistic errors of GC, and there are also considered general principles of their compensation.

b) Errors of GC Due to Motion of Vehicle with Constant Velocity and Heading

For determination of errors of GC during motion of the object with constant velocity and heading we will use first two equations of system (11.5.42); we will not consider damping. By analogy with par. 3b, disregarding components of induced and Coriolis acceleration, we will take $W_\xi = W_\eta = W_\zeta = 0$, $\gamma = \delta = 0$ and $c = 0$, $f_1(t) = f_2(t) = 0$. Then the first two equations of system (11.5.42) will be written in the form

$$\left.\begin{aligned} H\left(\dot{\beta} + u_1 + u_2\alpha\right) &= 0 \\ H\left(\ddot{\alpha} - u_2\beta + u_3\right) - lP\beta &= 0 \end{aligned}\right\} \qquad (11.5.66)$$

where components u_ξ, u_η, u_ζ of induced angular velocity of axes $O\xi\eta\zeta$ (Fig. 11.15) for the case of motion of object with constant velocity and heading and with geographic orientation of these axes are determined by relationships (1.2.25):

$$u_\xi = -\frac{v_N}{R}; \quad u_\eta = U\cos\varphi + \frac{v_E}{R}; \quad u_\zeta = U\sin\varphi + \frac{v_E}{R}\operatorname{tg}\varphi. \tag{11.5.67}$$

Expression for velocity error α_{st}^v of GC, which determines position of static equilibrium of N-S axis of gyrosphere relative to plane of meridian during motion of a ship with constant velocity and heading, can be found from first equation of system (11.5.66), if in it we set $\dot{\beta} = 0$; then

$$\alpha_{cr}^c = -\frac{u_\xi}{u_\eta} \tag{11.5.68}$$

or considering (11.5.67),

$$\alpha_{cr}^c = -\frac{-\frac{v_N}{R}}{U\cos\varphi + \frac{v_E}{R}} = \frac{v_N}{RU\cos\varphi + v_E}. \tag{11.5.69}$$

Since $v_n = v\cos K$ [see (1.2.22)], then for northern headings error α_{st}^v is western, and for southern courses — it is eastern. As B. V. Bulgakov showed [20], this error "... does not at all depend on construction of gyrocompass, but occurs due to the fact that for a moving support the figure axis tends to be established along the horizontal component of unknown angular velocity of the Darboux trihedron, which is formed in turn from eastern component u_ξ and northern component u_η." It was already said that velocity error α_{st}^v determines position of static equilibrium of N-S axis of gyrosphere during motion of ship with constant velocity and heading. During motion of ship with variable velocity and heading, we also will have velocity error α^v, determined by formula (11.5.69), but it will be variable and at every given moment will correspond to current values of velocity and course (for greater detail see Par. 4b).

Let us note that expression (11.5.69) for velocity error α_{st}^v of GC is approximate, since it was obtained from linearized equations of motion of GC. It is easy to obtain exact formula for α_{st}^v from geometric construction, considering what was said above about physical essence of error α_{st}^v. Actually, N-S axis (Oz) (Fig. 11.17) of gyrosphere tends to be established along horizontal component $(u_\xi\vec{\xi}^0 + u_\eta\vec{\eta}^0)$ of angular velocity u of axes $O\xi\eta\zeta$, i.e., it will be deflected in azimuth from direction $O\eta$ to the north by angle α_{st}^v, which determined from Fig. 11.17 by relationship

$$\operatorname{tg}(-\alpha_{cr}^c) = \frac{u_\xi}{u_\eta}$$

or

$$\operatorname{tg}\alpha_{ст}^{c}=-\frac{u_{\xi}}{u_{\eta}}. \tag{11.5.70}$$

which becomes for small α_{st}^{V} formula (11.5.68).

Substituting in (11.5.70) expressions (11.5.67), we will obtain)

$$\operatorname{tg}\alpha_{ст}^{c}=\frac{v_{N}}{RU\cos\varphi+v_{E}}. \tag{11.5.71}$$

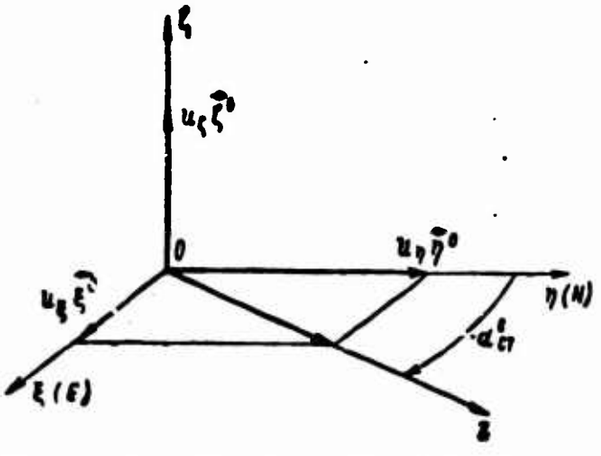

Fig. 11.17. For determination of velocity error α_{st}^{V} of GC.

Error α_{st}^{V} can attain large magnitudes, especially at high latitude.

Example 11.9: To calculate velocity error α_{st}^{V} of a GC fixed on a ship proceeding with speed v = 10 m/sec with constant heading K = 45° (the ship moves along a loxodrome) at latitude φ = 75°; angular velocity of rotation of Earth U = $7.29\cdot10^{-5}$ 1/sec; radius of Earth R = 6371 km.

Solution: 1. By formulas (1.2.22) we find v_E and v_N:

$$v_E=v\sin K=10\cdot0{,}707=7{,}07\ \text{m/sec};\quad v_N=v\cos K=10\cdot0{,}707=7{,}07\ \text{m/sec}.$$

2. According to (11.5.71) we calculate velocity error α_{st}^{V}:

$$\operatorname{tg}\alpha_{ст}^{c}=\frac{v_N}{RU\cos\varphi+v_E}=\frac{7{,}07}{6371\cdot10^{3}\cdot7{,}29\cdot10^{-5}\cdot0{,}2588+7{,}07}=0{,}0556;$$

$$\alpha_{ст}^{c}=3^{\circ}11'.$$

From Example 11.9 it is clear that velocity error α_{st}^{V} can attain large magnitudes. Therefore, in GC there are applied corresponding corrective devices, which consider magnitude of α_{st}^{V} in readings of heading of ship determined by this instrument. In the absence of such devices for calculation of velocity error, there are used special tables or graphs [71, 6].

During motion of object there is also changed magnitude of β_r of rise above plane of horizon of N-S axis of gyrosphere; this quantity for v = 0 is determined by formulas (11.5.45) or (11.5.46). Expression for quantity β_r, which for $v \neq 0$ we will designate β_{st}^{V}, can be obtained from second equation of system (11.5.66), if in it we set $\dot{\alpha} = 0$:

$$\beta_{ст}^{c}=\frac{Hu_{\zeta}}{Hu_{\eta}+lP} \tag{11.5.72}$$

or, taking into account (11.5.67),

$$\beta_{cr}^{c} = -\frac{H\left(U\sin\varphi + \frac{v_E}{R}\operatorname{tg}\varphi\right)}{H\left(U\cos\varphi + \frac{v_E}{R}\right) + lP}. \qquad (11.5.73)$$

Since usually $\frac{H\left(U\cos\varphi + \frac{v_E}{R}\right)}{lP} \ll 1$, then instead of (11.5.73) we approximately have formula

$$\beta_{cr}^{c} = -\frac{H}{lP}\left(U\sin\varphi + \frac{v_E}{R}\operatorname{tg}\varphi\right), \qquad (11.5.74)$$

which for $v_E = 0$ becomes (11.5.46). Necessity of rise of N-S axis of gyrosphere above plane of horizon by angle β_{st}^{v}, which is determined in the presence of motion of ship by formula (11.5.74), is fully evident.

It is necessary to note that during motion of ship, period of natural sustained oscillations of GC is changed as compared to its value in the case of a base which is motionless with respect to Earth. This value was determined by formula (11.5.53). Thus, for instance, if we consider that the object moves with constant speed and heading and do not consider change of latitude, then from (11.5.66) we can find expression for period $T_{g.c}$ of natural sustained oscillations of gyrosphere with respect to coordinates α and β in the form

$$T_{g.c} = 2\pi\sqrt{\frac{H}{Hu_\eta^2 + u_\eta lP}} \qquad (11.5.75)$$

or, considering as before that $\frac{Hu_\eta}{lP} \ll 1$, approximately

$$T_{g.c} = 2\pi\sqrt{\frac{H}{lPu_\eta}}; \qquad (11.5.76)$$

then, considering (11.5.67), we have formula

$$T_{g.c} = 2\pi\sqrt{\frac{H}{lP\left(U\cos\varphi + \frac{v_E}{R}\right)}}, \qquad (11.5.77)$$

which for $v_E = 0$ becomes (11.5.53).

In the more general case, period $T_{g.c}$ depends on latitude φ and eastern component v_E of velocity of ship,* which in formula (11.5.77) are variable. This circumstance should be considered during solution of problem of compensation of ballistic errors of GC. Let us note that if we take into account eastern component v_E of velocity

*In the most general case period $T_{g.c}$ depends also on northern component v_N of velocity of ship [cf. further (11.5.109) and (11.5.110)].

of ship, there is changed magnitude of directional torque of GC; thus, for v_E = const the expression for this torque instead of (11.5.4) will take the form

$$M_{r.}=-H\left(U\cos\gamma+\frac{v_E}{R}\right)\sin\alpha\approx-H\left(U\cos\gamma+\frac{v_E}{R}\right)\alpha. \qquad (11.5.78)$$

c) Ballistic Errors of GC and Their Compensation

General remark. During maneuvering of the object (which we will subsequently consider to be a ship), due to changes of its velocity and heading, in GC there appear ballistic errors, the nature of which in principle is the same as those in GP (§ 11.4, Par. 4c).

Analysis of ballistic errors of GC during change of speed of movement and heading of ship is the most important and complicated question of the theory of this instrument. During investigation of ballistic errors of GC, to which there are dedicated numerous works (see Par. 4a), usually: 1) these errors are determined for different forms of maneuvers of ship; 2) possibilities of their compensation are revealed, i.e., there are established conditions of undisturbability of GC by accelerations of the ship; 3) there are errors found appearing due to inexact satisfaction in GC of the condition of undisturbability, presence of damping, and so forth. Analysis of all these questions goes beyond the scope of the present book. Therefore, considering also earlier presented material on ballistic errors of GP (§ 11.4, Par. 4c), we will clarify only the essence of influence of accelerations of ship on errors of GC and fundamental possibilities of compensation of the latter by means of realization in the GC of corresponding condition of undisturbability.

Forms of ballistic errors of GC. During motion of ship with constant speed and heading the GC has velocity error in azimuth, which is determined by relationship (11.5.69) or (11.5.71). Instantaneous value of velocity error during change of speed and course is continuously changed, and at every given moment of time corresponds to their current values.

Furthermore, during maneuvering under the action of forces of inertia, in the GC there appears ballistic precession (this term was introduced by B. I. Kudrevich [71]), which tends to bring the axis of the gyroscope to a new equilibrium position corresponding to new values of velocity and heading of the ship. As B. I. Kudrevich indicated [71],"ballistic precessional motion is that most useful phenomenon, which, if one basic requirement, is observed ... can make it possibile to use readings of the gyroscopic compasses during variable conditions of motion of the ship, i.e., during change of motion and heading ... Ballistic precessional motion caused by

change of meridional component of velocity is always directed toward the new position of equilibrium which corresponds to the new values of velocity and heading."

In the presence of ballistic precession, axis of gyroscope in a finite interval of time will obtain a corresponding ballistic shift [71]. It is easy to see that if ballistic shift is equal to difference in velocity errors of GC for final and initial values of velocity and heading of ship, then due to ballistic precession the axis of the gyroscope will pass to new position of equilibrium a periodically, i.e., without excitation of natural oscillations. This circumstance is fully analogous to that which was earlier noted with respect to GP (§ 11.4, Par. 4). As will be shown further on, for observance of the condition of aperiodic transitions, the period of natural sustained oscillations of GC should be in first approximation equal to 84.4 minutes. Then the GC will not have any errors other than velocity errors, i.e., in it during maneuvering of the ship ballistic errors will be absent.

If in the GC the condition of aperiodic transitions is not observed, then when the ship attains new values of velocity and heading, axis of gyroscope will not be in position of equilibrium; i.e., in the GC there will appear ballistic error. From the preceding it follows that ballistic error can be defined [71] as the difference between magnitude of ballistic shift and difference in velocity errors of axis of gyroscope for final and initial values of velocity and heading of ship.

Ballistic error appearing due to nonsatisfaction of condition of aperiodic transitions in azimuth is called ballistic error of the first kind [71]. A characteristic peculiarity of this error is the fact that it attains the biggest magnitude at the time of termination of the maneuver, and subsequently axis of gyroscope (N-S axis of gyrosphere) accomplishes oscillation with amplitude corresponding to this value of error.

However, even when in the GC the indicated condition of aperiodic transitions is satisfied, nevertheless there appear ballistic errors caused by the device for damping of natural oscillations of the gyroscope. These errors are called errors of damping of acceleration or errors of the second kind [71].

Further we will give certain general relationships for ballistic errors of the first kind appearing during uniformly accelerated motion of ship with constants heading and during circulation; thus we will mainly be interested in deviations of axis of gyroscope in azimuth and in height.

Rectilinear uniformly accelerated motion of ship. Being interested in ballistic errors in coordinates α and β, which characterize position of N-S axis of gyrosphere,

we will use first two equations of system (11.5.42); we will consider that damping is ceased. For simplification we will disregard eastern component v_E of velocity of ship as compared to component of velocity RU cos φ due to rotation of Earth, which on equator (φ = 0) is equal to 900 knots; then instead of (11.5.67) we will obtain

$$u_\xi = -\frac{v_N}{R}; \quad u_\eta \approx U\cos\gamma; \quad u_\zeta \approx U\sin\gamma. \tag{11.5.79}$$

Under these conditions, disregarding components of induced and Coriolis acceleration, we will take

$$W_\xi = 0; \quad W_\eta = \dot{v}_N; \quad W_\zeta = 0. \tag{11.5.80}$$

Then, considering in first two equations of system (11.5.42) that $\gamma = \delta = 0$, $c = 0$, $f_1(t) = f_2(t) = 0$ and considering (11.5.79) and (11.5.80), we will rewrite them in the form ($U\cos\varphi \ll \frac{lP}{H}$)

$$\left.\begin{aligned} \dot{\beta} + U\cos\gamma\cdot\alpha - \frac{v_N}{R} &= 0 \\ \dot{\alpha} + U\sin\gamma - \frac{lP}{H}\beta - \frac{lP}{H}\cdot\frac{\dot{v}_N}{g} &= 0 \end{aligned}\right\} \tag{11.5.81}$$

or, considering (11.5.44), (11.5.69) and (11.5.74) (considering in them by condition that $v_E = 0$ and replacing α_{st}^V, β_{st}^V by the designations α^V, β^V)

$$\left.\begin{aligned} \dot{\beta} + U\cos\gamma\cdot(\alpha - \alpha^c) &= 0 \\ \dot{\alpha} = \frac{k^2}{U\cos\gamma}\left(\beta - \beta^c + \frac{\dot{v}_N}{g}\right) \end{aligned}\right\} \tag{11.5.82}$$

Let us designate

$$\alpha^V = \alpha - \alpha^c; \quad \beta^V = \beta - \beta^c. \tag{11.5.83}$$

In accordance with the above introduced designations, α^{acc} and β^{acc} are ballistic errors of the GC. Then for determination of these errors we obtain the following system of differential equations:

$$\left.\begin{aligned} \dot{\beta}^V + U\cos\gamma\cdot\alpha^V &= 0 \\ \dot{\alpha}^V - \frac{k^2}{U\cos\gamma}\beta^V &= \left(\frac{k^2}{\nu^2} - 1\right)\frac{\dot{v}_N}{RU\cos\gamma} \end{aligned}\right\} \tag{11.5.84}$$

where ν^2 is defined by relationship (11.4.69).

Eliminating from (11.5.84) at first β^{acc}, and then α^{acc}, we will obtain

$$\left.\begin{aligned} \ddot{\alpha}^V + k^2\alpha^V &= \frac{1}{RU\cos\gamma}\left(\frac{k^2}{\nu^2} - 1\right)\ddot{v}_N \\ \beta^V + k^2\beta^V &= \left(1 - \frac{k^2}{\nu^2}\right)\frac{\dot{v}_N}{R} \end{aligned}\right\} \tag{11.5.85}$$

Integrating these equations, we find ballistic errors α^{acc}, β^{acc} of the GC for an arbitrary law of change of meridional component $v_N(t)$ of velocity of movement of ship. For determination of α^{acc} and β^{acc}, during integration of equations (11.5.85) we will assume zero initial conditions [69], i.e., at $t = 0$, $\alpha^{acc}(0) = \beta^{acc}(0) = 0$, $\dot{\alpha}^{acc}(0) = \dot{\beta}^{acc}(0) = 0$; this will allow us to find ballistic errors α^{acc}, β^{acc} caused by change of velocity of movement of ship, and not by initial deflections of the GC.

Solution of second equation of system (11.5.85) has the form

$$\beta^{y} = \frac{1}{Rk}\left(1 - \frac{k^2}{\nu^2}\right)\int_0^t \dot{v}_N(\tau)\sin k(t-\tau)\,d\tau; \qquad (11.5.86)$$

here t is time of action of acceleration $\dot{v}_N(t)$, i.e., formula (11.5.86) is valid in the interval $0 < t < \xi$, where ξ is limiting value of time t.

For determination of α^{acc} it is possible to differentiate (11.5.86) and to place the obtained result in the first equation of system (11.5.84), whence there is directly determined α^{acc}. We have

$$\beta^{y} = \frac{1}{R}\left(1 - \frac{k^2}{\nu^2}\right)\int_0^t \dot{v}_N(\tau)\cos k(t-\tau)\,d\tau; \qquad (11.5.87)$$

putting (11.5.87) in the first equation of system (11.5.84), we will obtain

$$\alpha^{y} = \frac{1}{RU\cos\varphi}\left(\frac{k^2}{\nu^2} - 1\right)\int_0^t \dot{v}_N(\tau)\cos k(t-\tau)\,d\tau. \qquad (11.5.88)$$

Thus, for α^{acc} and β^{acc} we have the following expressions:

$$\left.\begin{aligned} \alpha^{y} &= \frac{1}{RU\cos\varphi}\left(\frac{k^2}{\nu^2} - 1\right)\int_0^t \dot{v}_N(\tau)\cos k(t-\tau)\,d\tau \\ \beta^{y} &= \frac{1}{Rk}\left(1 - \frac{k^2}{\nu^2}\right)\int_0^t \dot{v}_N(\tau)\sin k(t-\tau)\,d\tau \end{aligned}\right| \qquad (11.5.89)$$

When M. Schuler's condition of aperiodic transitions of GC is satisfied, $k = \nu$, which is analogous to the corresponding condition (11.4.104) for GP, ballistic errors of GC $\alpha^{acc} = \beta^{acc} = 0$. As will be shown further on, this condition is valid in reference to any character of maneuvering of the ship only in the plane of the meridian.

Formulas (11.5.89) can be rewritten in somewhat different form, if in them we introduce coefficient b ballistic error, which is determined by relationship (11.4.78); then we will obtain

$$\left.\begin{aligned} \alpha^{y} &= \frac{1}{RU\cos\varphi}\cdot\frac{k^2}{\nu^2}b\int_0^t \dot{v}_N(\tau)\cos k(t-\tau)\,d\tau \\ \beta^{y} &= -\frac{1}{Rk}\cdot\frac{k^2}{\nu^2}b\int_0^t \dot{v}_N(\tau)\sin k(t-\tau)\,d\tau \end{aligned}\right| \qquad (11.5.90)$$

Into equations (11.5.85) and their solutions (11.5.89) or (11.5.90), in place of quantity k it is possible to introduce time constant T of the GC, which is determined by analogy with (11.4.82) by expression [see (11.5.44)]

$$T = \frac{1}{k} = \sqrt{\frac{H}{U \cos \varphi \cdot IP}}. \tag{11.5.91}$$

Of the greatest interest is determination of ballistic error α^{acc} of the GC in azimuth. Let us consider the most simple case, when ship moves with constant heading and with constant acceleration $\dot{v}_N$ = const. Then from (11.5.88) we have

$$\alpha^y = \frac{v_N}{RU \cos \varphi \cdot k} \left(\frac{k^2}{\nu^2} - 1 \right) \sin kt, \tag{11.5.92}$$

where as before t is in the interval $0 < t < \xi$, and ξ is limiting time of action of acceleration $\dot{v}_N$.

If acceleration $\dot{v}_N$ acts during a short interval of time, which is small as compared to period of natural oscillations of GC $T_{g.c} = \frac{2\pi}{k}$ [see (11.5.53)], then it is possible to set sin kt $\approx$ kt and to rewrite (11.5.92) in the form*

$$\alpha^y = \frac{\dot{v}_N t}{RU \cos \varphi} \left(\frac{k^2}{\nu^2} - 1 \right). \tag{11.5.93}$$

Since $\dot{v}_N t$ constitutes the difference between current velocity $v_N(t)$ of the ship and its value $v_N(0)$ at the time of beginning of the action of acceleration, i.e.,

$$\dot{v}_N t = v_N(t) - v_N(0), \tag{11.5.94}$$

that instead of (11.5.93) we will obtain

$$\alpha^y = \left(\frac{k^2}{\nu^2} - 1 \right) \frac{v_N(t) - v_N(0)}{RU \cos \varphi} \tag{11.5.95}$$

or, considering (11.5.69) (for v_E = 0),

$$\alpha^y = \left(\frac{k^2}{\nu^2} - 1 \right) [\alpha^c(t) - \alpha^c(0)], \tag{11.5.96}$$

i.e., ballistic error α^{acc} is proportional to difference in velocity errors of GC by the end and by the beginning of maneuver of the ship.

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*Formula (11.5.93) can be used for determination of error α^{acc} during the action of acceleration. For finding of α^{acc} upon completion of the action of acceleration, i.e., at $t > \xi$, it is possible to apply formula

$$\alpha^y = \frac{\dot{v}_N \xi}{RU \cos \varphi} \left(\frac{k^2}{\nu^2} - 1 \right) \cos k \left(t - \frac{\xi}{2} \right),$$

where t is current time, and ξ is the small time of action of acceleration. This formula can be obtained from the corresponding relationship (see [135, p. 211]).

Example 11.10: To determine ballistic error α^{acc} of GC with increase of velocity of motion of the ship in plane of meridian (K = 0) from $v_N(0)$ to $v_N(t)$ = 10 m/sec in 10 min. Parameters of GC are taken from Example 11.8; $k = 0.89 \cdot 10^{-3}$ 1/sec, $\nu = 1.24 \cdot 10^{-3}$ 1/sec, $R = 6371 \cdot 10^3$ m, $U = 7.29 \cdot 10^{-5}$ 1/sec, $\varphi = 75°$. Error α^{acc} by the moment of termination of maneuver is calculated.

Solution: 1. We determine acceleration $\dot{v}_N$ in plane of meridian:

$$\dot{v}_N = \frac{v_N(t) - v_N(0)}{t} = \frac{10}{600} = \frac{1}{60} \text{ m/sec}^2.$$

2. By formula (11.5.93) we calculate ballistic error α^{acc} of the GC:

$$\alpha^y = \frac{\dot{v}_N t}{RU\cos\varphi}\left(\frac{k^2}{\nu^2} - 1\right) = \frac{\frac{1}{60} \cdot 600}{6371 \cdot 10^3 \cdot 7{,}29 \cdot 10^{-5} \cdot 0{,}2588}\left(\frac{0{,}893^2 \cdot 10^{-6}}{1{,}24^2 \cdot 10^{-6}} - 1\right) =$$
$$= -0{,}04006 \text{ rad.} = -2°18'.$$

Circulation. Let us assume that at the time of beginning of circulation, the ship was moving with heading K_0. During circulation of ship with constant angular velocity ω_{circ} and constant velocity v, the current value of course K during left (counter-clockwise) circulation is determined by relationship (11.4.90): $K = K_0 - \omega_{circ} t$. Then for northern component v_N of velocity of ship and for acceleration $\dot{v}_N$ we have

$$v_N = v\cos(K_0 - \omega_u t); \quad \dot{v}_N = v\omega_u \sin(K_0 - \omega_u t). \tag{11.5.97}$$

Let us be limited to consideration of the particular case when in the beginning of circulation $K_0 = 0$; then

$$v_N = v\cos\omega_u t; \quad \dot{v}_N = -v\omega_u \sin\omega_u t. \tag{11.5.98}$$

For determination in this case of ballistic error α^{acc} we will use expression (11.5.88), putting in it formula (11.5.98); then we will obtain expression

$$\alpha^y = -\frac{1}{RU\cos\varphi}\left(\frac{k^2}{\nu^2} - 1\right)\int_0^t v\omega_u \sin\omega_u \tau \cos k(t-\tau)\,d\tau, \tag{11.5.99}$$

integration of which by parts gives

$$\alpha^y = \frac{\frac{k^2}{\nu^2} - 1}{RU\cos\varphi} \cdot \frac{v\omega_u^2}{\omega_u^2 - k^2}(\cos\omega_u t - \cos kt). \tag{11.5.100}$$

From (11.5.100) it follows that ballistic error of GC when the condition of aperiodic transitions is satisfied ($k = \nu$) is equal to zero. If this condition is not satisfied, then error α^{acc} constitutes a superposition of two periodic oscillations

with frequency ω_{circ} equal to angular velocity of circulation of the ship, and with frequency k of natural sustained oscillations of the GC. The first motion A. N. Krylov [69] called forced motion, and the second — accompanying motion.

When frequency $\omega_{circ} = k$ or period T_{circ} of circulation is equal to period $T_{g.c}$ of natural sustained oscillations of the GC, then there occurs resonance; however, the probability of this equality of periods is extraordinarily small. Furthermore, natural oscillation of the GC attenuate. Usually $\frac{k^2}{\omega_u^2} = \frac{T_u^2}{T_{r.u}^2} \ll 1$; therefore, instead of (11.5.100) we approximately have

$$\alpha^y = v\frac{\frac{k^2}{v^2} - 1}{RU\cos\varphi}(\cos\omega_u t - \cos kt). \tag{11.5.101}$$

If time of circulation, as usually occurs, is comparatively small, then it is possible to take cos kt ≈ 1; then instead of (11.5.101) will obtain

$$\alpha^y = \frac{\frac{k^2}{v^2} - 1}{RU\cos\varphi}(v\cos\omega_u t - v). \tag{11.5.102}$$

Since at the time of beginning of circulation the ship moved with heading to the north, then according to (11.5.98) $v = v_N(0)$ and $v\cos\omega_{circ}t = v_N(t)$; then (11.5.102) will be written in the form (11.5.95):

$$\alpha^y = \left(\frac{k^2}{v^2} - 1\right)\frac{v_N(t) - v_N(0)}{RU\cos\varphi}. \tag{11.5.103}$$

Consequently, for ballistic error α^{acc} of the GC during circulation of the ship, in the first approximation we have the same relationship (11.5.96) as for error of GC during change of velocity of movement of the ship:

$$\alpha^y = \left(\frac{k^2}{v^2} - 1\right)[\alpha^c(t) - \alpha^c(0)]. \tag{11.5.104}$$

Example 11.11: To determine ballistic error α^{acc} of GC by the end of circulation of a ship accomplishing turn for 3 minutes from the heading 0° to the heading 180°; velocity of movement of ship v = 10 m/sec; other initial data are the same as in Example 11.10.

Solution: 1. We determine period T_{circ} and angular velocity ω_{circ} of circulation of the ship:

$$T_u = 3 \cdot 2 = 6 \text{ min};$$

$$\omega_u = \frac{2\pi}{T_u} = \frac{6{,}28}{6} = 1{,}047 \text{ 1/min} = 0{,}01745 \text{ 1/sec}.$$

2. By formula (11.5.101) we find ballistic error α^{acc} of the GC:

$$e^{\gamma} = \varepsilon \frac{\frac{M}{\nu^2} - 1}{RU\cos\varphi} (\cos\omega_{\mu} t - \cos kt) =$$

$$= 10 \frac{\frac{0{,}693^2 \cdot 10^{-6}}{1{,}94^2 \cdot 10^{-6}} - 1}{6371 \cdot 10^3 \cdot 7{,}29 \cdot 10^{-5} \cdot 0{,}5} (\cos 0{,}01745 \cdot 180 - \cos 0{,}000693 \cdot 180) = 0{,}07858 \text{ rad.} = 4°33'.$$

Certain remarks. Earlier there were briefly considered simple cases of maneuvering of ship and presented ~~ ural relationships characterizing the appearing ballistic errors of the GC. More detailed information about such errors of GC for different character of maneuvering of ship can be found in the earlier indicated literature on the theory of gyrocompasses. Here we will be limited to only certain general remarks.

Above there were considered ballistic errors of GC on the assumption that the device for damping of natural oscillations is turned off. With damping present and during maneuvering of the ship there appear, as was shown, ballistic errors of the second kind, or damping errors. For determination of these errors it is necessary to start with equations of GC on a moving base, taking into account operation of the damper, i.e., with equations of type (11.5.56), if in them we further introduce terms due to motion of the ship; such equations can be obtained directly from system (11.5.42). Integration of equations found in such a way permits us to determine ballistic errors of the second kind [64, 71, 20, 135, 6].

Presence in the GC of a device for damping of its natural oscillations is connected with one more peculiarity. Let us assume that in the GC there is satisfied the condition of aperiodic transitions, and that the period of its natural sustained oscillations is 84.4 minutes. With use of the damper, period of damped oscillations of GC will be larger than indicated; therefore, in GC with damper it is not possible to exactly satisfy conditions of aperiodic transitions. In connection with this, during maneuvering of the ship, besides ballistic errors of second kind there further appear errors of the first kind due to lack of satisfaction in the GC of the above-indicated condition. Therefore, during maneuvers of the ship sometimes there is applied "turning off of damping" [71]. However, this means does not make it possible to fully remove ballistic errors of GC during maneuvering of the ship [71, 135, 7, 6].

Above we discussed ballistic errors of GC during isolated maneuvers of the ship. Under real conditions of prolonged maneuvering, consiting of a series of consecutive maneuvers with change of heading and speed of movement of the ship, there occurs accumulation of ballistic errors of the GC. This is, as A. N. Krylov [64] indicated, the phenomenon of "heredity of oscillations." To this question there have been

dedicated a number of investigations [64, 20, 135].

d) Compensation of Ballistic Errors

Compensation of ballistic errors of GC is based mainly on the use of the theorem of M. Schuler. From formulas (11.5.89) and (11.5.100) it follows that when the condition $k = \nu$ is satisfied, the GC with damping turned off does not have ballistic errors; for realization of condition $k = \nu$, the period of natural precessional sustained oscillations of GC, if we consider (11.4.104), should be

$$T_{r.k} = \frac{2\pi}{k} = \frac{2\pi}{\nu} = 2\pi\sqrt{\frac{R}{g}} = 5064 \text{ sec.} = 84.4 \text{ min.} \qquad (11.5.105)$$

Thus, the GC with period of 84.4 minutes in the absence of damping does not have ballistic errors, and the effect of action on it of acceleration of the ship will appear in transition of axis of the gyroscope (N-S axis of gyrosphere) to a new position of equilibrium (corresponding to new values of velocity and heading of the ship) aperiodically, without excitation of natural oscillations. Relationship $k = \nu$, if we consider (11.5.44) and (11.4.104), can be represented in the form

$$H = \frac{lP}{g} RU\cos\varphi \qquad (11.5.106)$$

or

$$\frac{lP}{H} = \frac{g}{RU}\sec\varphi, \qquad (11.5.107)$$

i.e., for realization of condition of aperiodic transitions the ratio $\frac{lP}{H}$ should be changed in the instrument depending upon latitude of the place.

In the majority of GC there is absent a device for control of latitude, and therefore in them the condition of aperiodic transitions is satisfied for one so-called calculating latitude. In certain GC the period is regulated according to latitude by means of change of rpm rate of the gyroscopes [75].*

For GC with period of 84.4 minutes, angle β_r of deflection of N-S axis of gyrosphere from plane of horizon will be according to (11.5.46),

$$\beta_r = \frac{U^2}{k^2}\sin\varphi\cos\varphi = \frac{U^2}{\nu^2}\sin\varphi\cos\varphi = \frac{1}{2}\cdot\frac{RU^2}{g}\sin 2\varphi. \qquad (11.5.108)$$

Comparing (11.5.108) with (2.2.5), we see that β_r is angle of deflection of the force of gravity from the direction of the usual force of terrestrial gravitation due to the influence of centrifugal force arising from rotation of the Earth (Fig. 2.11). Consequently, in aperiodic GC ($k = \nu$) the axis of the gyroscope (N-S axis of gyrosphere) is located in the plane of the geocentric horizon.

*For possible methods of control of period of GC see p. 378.

M. Schuler's condition of aperiodic transitions, which was mentioned above, is approximate. It is valid only during motion of the ship according to any law in the plane of the meridian, i.e., when eastern component v_E of velocity movement is equal to zero. With consideration of v_E, which is especially important for high latitudes, the condition of aperiodic transitions of GC was obtained by Ya. N. Roytenberg (see [44]) and E. I. Sliv [135] in the form

$$H = \frac{IP}{g}(RU\cos\varphi + v_E). \tag{11.5.109}$$

Calculations show [135] that this condition in practice ensures compensation of ballistic errors of the GC.

A. Yu. Ishlinskiy [44] obtained instead of (11.5.109) the exact condition of compensation of ballistic errors of GC in the following form:*

$$H = \frac{IP}{g}\sqrt{(RU\cos\varphi + v_E)^2 + v_N^2}. \tag{11.5.110}$$

This condition is absolutely exact, and valid not only for small, but also for all finite angles of deflection of axis of gyroscope from plane of meridian. In reference to two-rotor GC, condition (11.5.110) can be written in the following way:

$$H = 2H'\cos\varepsilon = \frac{IP}{g}\sqrt{(RU\cos\varphi + v_E)^2 + v_N^2}. \tag{11.5.111}$$

For the case of a two-rotor GC, this condition will be automatically satisfied during observance of relationship [44]

$$M_{sp} = -\frac{(2H')^2\sin\varepsilon\cos\varepsilon}{IPR}g, \tag{11.5.112}$$

which determines magnitude of torque which should be applied relative to the axis of precession of one of the gyroscopes. In connection with this, let us note that in existing two-rotor GC, to axes of precession of gyroscopes there are applied torques developed by springs and determined by relationship (11.5.39).

When conditions (11.5.111) and (11.5.112) are satisfied, deflection of N-S axis of gyrosphere from plane of meridian is determined only by velocity error, independently of character of maneuvering of the ship; thus angle γ (Fig. 11.15) of turn of gyrosphere around the N-S axis is equal to zero; i.e., $\gamma = 0$. This question will be considered in more detail in Par. 6. Let us note that conditions (11.5.106) of M. Schuler and (11.5.109) of Ya. N. Roytenberg and E. I. Sliv can be obtained as

*As A. Yu. Ishlinskiy [44] indicated, condition (11.5.110) in the approximate theory of GC was formulated by V. G. Zheleznov.

particular cases from the general condition (11.5.110) of aperiodic transitions due to A. Yu. Ishlinskiy.

5. Dynamic Errors of GC

a) General Characteristics of Dynamic Errors

Among dynamic errors of GC there are included errors appearing under conditions of operation of the instrument during variable external disturbances. Such errors take place in cases of oscillations of the object, vibration of place of installation of the GC, and so forth.

For concreteness we will consider dynamic errors of GC under conditions of rocking of a ship. Oscillatory motion of a ship on sea swells shows up basically due to unbalance of the gyroscope and friction in axes of the suspension. Since frictional torques in axes of suspension of contemporary GC are reduced to a minimum, we will consider only dynamic errors of GC on a rocking ship due to unbalance of the sensing device. In the beginning we will consider these errors for single-rotor GC, and then for two-rotor GC.

b) Dynamic Errors of Single-Rotor GC on a Rocking Ship

Single-rotor GC, which began to be used on ships in the beginning of the present century (for instance, GC of Anschutz type and Sperry type [71, 20]), had appreciable errors due to rocking. Appearance of these errors is connected with the fact that sensing device of such GC, constituting a rotor with displaced c.g. (center of gravity), does not have stabilization relative to axis of rotor, i.e., in the E-W plane (with location of axis of rotor in plane of meridian). Therefore, oscillations of sensing device in this plane have the same character as oscillations of the usual physical pendulum with a small period of several tenths of a second.

Analysis of errors of single-rotor GC on a rocking ship usually [20] is produced for the case of pure rolling of the ship, which most essentially affects these errors. It is easy to see that during motion of ship in plane of meridian ($K = 0^\circ, 180^\circ$), rolling does not cause errors of the GC; the same is true during motion of ship along a parallel ($K = 90^\circ, 270^\circ$). Errors of GC due to rolling appear during motion of the ship with intermediate headings. They attain the biggest values on quarter (intercardinal) headings ($K = 45^\circ, 135^\circ, 225^\circ, 315^\circ$), and therefore errors of the GC on a rocking ship are called quarter, or intercardinal errors [20].*

*It is necessary to note that in the general case of rocking, for instance during simultaneous rolling and pitching oscillations of the ship, magnitude of error of GC due to rocking in principle is determined by the same method as error of GC due to rolling, but then in formulas (11.5.125) below, for accelerations w_x and w_y it is necessary to consider components not only due to rolling, but also due to pitching of the ship.

We will consider intercardinal error of GC due to rolling of ship which have random character.* For this we will use equations of motion of axis of gyroscope, supplementing them with equation of oscillatory motion of sensing device (around axis of rotor), the character of oscillations of which is analogous to oscillations of physical pendulum.

We will take equations (11.5.48) of motion of GC on a base which is motionless with respect to Earth and into the first of them will introduce torque $lP\frac{W_\eta}{g}\gamma$ [see second equation of system (11.5.42)], which is caused by oscillations of sensing device about axis of gyroscope; then instead of (11.5.48) we will obtain

$$\left.\begin{aligned}\dot\beta + U\cos\varphi\cdot\alpha - \frac{lP}{H}\cdot\frac{W_\eta}{g}\gamma = 0\\ \dot\alpha - \frac{H^0}{U\cos\varphi}(\beta-\beta_r) = 0\end{aligned}\right\}$$

or, considering (11.5.44) (where H is angular momentum of gyroscope),

$$\left.\begin{aligned}\dot\beta + U\cos\varphi\cdot\alpha - \frac{H^0}{U\cos\varphi}\cdot\frac{W_\eta}{g}\gamma = 0\\ \dot\alpha - \frac{H^0}{U\cos\varphi}(\beta-\beta_r) = 0\end{aligned}\right\} \qquad (11.5.113)$$

Let us introduce damping of natural oscillations of the gyrocompass. B. V. Bulgakov [20] distinguishes two methods of damping of oscillations of GC: 1) with the help of a couple with a vertical axis; 2) with the help of a couple with a norizontal axis. In the first of them there are included the method of air streams, which is applied in the single-rotor GC of Anschütz type [20, 71], and the method of eccentric suspending of the pendulum, which is utilized in single-rotor GC of Sperry type [20, 71]. In the second method there is applied a hydraulic damper, just as in two-rotor GC.

Let us assume that in single-rotor GC there is applied method of damping with the help of a couple with vertical axis, the moment of which with respect to axis Oy_1 (Fig. 11.15) is determined by expression $M_{y_1} = -S(\beta + \frac{W_\eta}{g})$, which is analogous to relationship (2.3.65). Introducing M_{y_1} in the first equation of system (11.5.113) and designating

$$\varkappa = \frac{S}{H}, \qquad (11.5.114)$$

we will obtain

$$\left.\begin{aligned}\dot\beta + U\cos\varphi\cdot\alpha + \varkappa\left(\beta + \frac{W_\eta}{g}\right) - \frac{H^0}{U\cos\varphi}\cdot\frac{W_\eta}{g}\gamma = 0\\ \dot\alpha - \frac{H^0}{U\cos\varphi}(\beta-\beta_r) = 0\end{aligned}\right\} \qquad (11.5.115)$$

whence for the coordinate interesting us α we have equation

*Analogous problem in reference to the single-rotor GC with mercury vessels is considered in [129].

$$\ddot{\alpha} + \varkappa\dot{\alpha} + k^2\alpha = -\varkappa\frac{k^2}{U\cos\varphi}\left(\beta_r + \frac{\dot{W}_\eta}{g}\right) + \frac{k^2}{U^2\cos^2\varphi}\cdot\frac{W_\xi}{g}\gamma. \qquad (11.5.116)$$

The first term $-\varkappa\frac{k^2}{U\cos\varphi}(\beta_r + \frac{W_\eta}{g})$ of the right side of this equation is caused by damping-acceleration error, and therefore, we will not consider it;* then equation (11.5.116) will be written in the form

$$\ddot{\alpha} + \varkappa\dot{\alpha} + k^2\alpha = \frac{k^2}{U^2\cos^2\varphi}\cdot\frac{W_\xi}{g}\gamma, \qquad (11.5.117)$$

where variable α denotes error of GC due to rocking of the ship.

Let us designate

$$\varkappa = 2\zeta k, \qquad (11.5.118)$$

where ζ is relative attenuation factor.

Then equation (11.5.117) will take the form

$$\ddot{\alpha} + 2\zeta k\dot{\alpha} + k^2\alpha = \frac{k^2}{U^2\cos^2\varphi}\cdot\frac{W_\xi}{g}\gamma. \qquad (11.5.119)$$

Besides this, it is necessary to compose equation for angle γ, which characterizes oscillations of sensing device about axis Oz (Fig. 11.15) of the rotor due to perturbing acceleration W_ξ acting along axis Oξ.

Since the sensing device of single-rotor GC oscillates around axis Oz as an ordinary physical pendulum, then the equation of its oscillations with respect to angle γ, by analogy with (4.3.12), will be written in the following way:

$$J_z\ddot{\gamma} + b\dot{\gamma} + lP\gamma = lP\frac{W_\xi}{g}, \qquad (11.5.120)$$

where J_z is moment of inertia of sensing device of GC relative to axis Oz:

b is damping factor of oscillations of pendulum around this axis.

Introducing by analogy with (4.3.14), (4.3.16) and (4.3.17) the designations

$$n^2 = \frac{lP}{J_z}, \quad \frac{b}{J_z} = 2\zeta_1 n, \quad \zeta_1 = \frac{b}{2\sqrt{J_z lP}}, \qquad (11.5.121)$$

we will rewrite (11.5.120) in the form

$$\ddot{\gamma} + 2\zeta_1 n\dot{\gamma} + n^2\gamma = n^2\frac{W_\xi}{g}, \qquad (11.5.122)$$

where n is frequency of natural sustained oscillations of sensing device of GC around axis Oz;

---

*Let us note that quantity $-\varkappa\frac{k^2}{U\cos\varphi}\beta_r$ is not connected with influence of rocking of ship; term $-\varkappa\frac{k^2}{U\cos\varphi}\cdot\frac{W_\eta}{g}$ due to acceleration W_η of rocking has mathematical expectation equal to zero (as it is shown further on, $\overline{W}_\eta = 0$); therefore, it does not affect the mathematical expectation $\overline{\alpha}$ determined subsequently.

ζ_1 is relative attenuation factor.

Thus, for determination of error α of GC during rocking of the ship, we have the following system of equations:

$$\left.\begin{aligned} \ddot{\alpha} + 2\varkappa_1 k\dot{\alpha} + k^2\alpha &= \frac{k^4}{U^2\cos^2\varphi}\cdot\frac{W_\eta}{g}\gamma \\ \ddot{\gamma} + 2\varkappa_2 n\dot{\gamma} + n^2\gamma &= n^2\frac{W_\xi}{g} \end{aligned}\right\} \qquad (11.5.123)$$

From these equations it is easy to perceive the cause of appearance of azimuthal error of single-rotor GC during rocking of the ship. Actually, due to eastern component W_ξ of acceleration of rocking there appears torque $lP\frac{W_\xi}{g}$, which is directed along the axis Oz of the gyroscope and cuases inclination γ of sensing device of GC. Inclination γ in the presence of northern component W_η of acceleration of rocking causes torque $lP\frac{W_\eta}{g}\gamma$, which is directed along the vertical axis. This torque causes precessional motion with respect to height and thereupon azimuthal motion $\alpha(t)$ of axis of gyroscope. If we consider only rolling, then $W_\eta(t)$ and $\gamma(t)$ are coupled random functions of time. Mathematical expectation $lP\frac{W_\eta}{g}\gamma$ of the above-indicated torque causes systematic deviation $\bar{\alpha}$ of GC on a rocking ship.

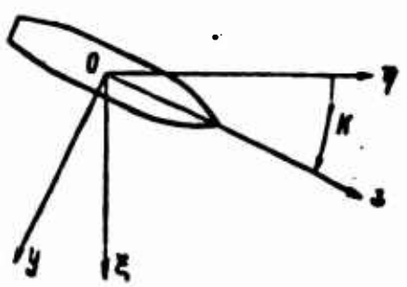

Fig. 11.18. For determination of components of accelerations of rocking W_ξ, W_η.

We will determine accelerations W_ξ and W_η due to rolling of ship decoding Fig. 11.18 we have

$$\left.\begin{aligned} W_\xi &= w_x\sin K + w_y\cos K \\ W_\eta &= w_x\cos K - w_y\sin K \end{aligned}\right\} \qquad (11.5.124)$$

For w_x and w_y, if we consider (2.2.45), during pure rolling of the ship we obtain

$$w_x = 0;\quad w_y = -z\ddot{\theta}. \qquad (11.5.125)$$

Substituting (11.5.125) in (11.5.124), we find

$$W_\xi = -z\ddot{\theta}\cos K;\quad W_\eta = z\ddot{\theta}\sin K. \qquad (11.5.126)$$

Let us introduce (11.5.126) into (11.5.123); then

$$\left.\begin{aligned} \ddot{\alpha} + 2\varkappa_1 k\dot{\alpha} + k^2\alpha &= \frac{k^4}{U^2\cos^2\varphi}\cdot\frac{z\sin K}{g}\ddot{\theta}\gamma \\ \ddot{\gamma} + 2\varkappa_2 n\dot{\gamma} + n^2\gamma &= -n^2\frac{z\cos K}{g}\ddot{\theta} \end{aligned}\right\} \qquad (11.5.127)$$

From (11.5.127) it follows that the disturbance is random function $\ddot{\theta}(t)$; therefore, motion $\alpha(t)$ of axis of gyroscope in azimuth also is a random function of time, probability characteristics of which can be determined from system of equations (11.5.127). Of the greatest interest is finding of mathematical expectation $\bar{\alpha} = M[\alpha(t)]$ of azimuthal drift of GC on a rocking ship.

First we will determine $\bar{\alpha}$ approximately, and then we will indicate the exact

solution. In the considered case, oscillations of sensing device of the GC around axis Oz (Fig. 11.15) are similar to oscillations of short-period physical pendulum. Therefore, by analogy with (4.6.137), from second equation of system (11.5.127), disregarding in the first approximation the dynamic correction, we can assume that

$$\gamma(t) \approx -\frac{z\cos K}{g}\ddot{\theta}. \tag{11.5.128}$$

Putting (11.5.128) into first equation of system (11.5.127), we will obtain

$$\ddot{\alpha} + 2k\dot{\alpha} + k^2\alpha = -\frac{k^4}{U^2\cos^2\varphi}\cdot\frac{z^2}{2g^2}\dot{\theta}^2\sin 2K, \tag{11.5.129}$$

whence it follows that $\alpha(t)$ is a steady-state random function.

For finding of $\bar{\alpha}$ we will apply to (11.5.129) the operation of mathematical expectation:

$$\ddot{\bar{\alpha}} + 2k\dot{\bar{\alpha}} + k^2\bar{\alpha} = -\frac{k^4}{U^2\cos^2\varphi}\cdot\frac{z^2}{2g^2}D[\ddot{\theta}]\sin 2K, \tag{11.5.130}$$

whence after termination of transient response we have

$$\bar{\alpha} = -\frac{k^4}{U^2\cos^2\varphi}\cdot\frac{z^2}{2g^2}D[\ddot{\theta}]\sin 2K. \tag{11.5.131}$$

We have obtained an approximate expression for intercardinal error of a single-rotor GC during irregular rocking of the ship. From (11.5.131) it follows that error $\bar{\alpha}$ is proportional to dispersion $D[\ddot{\theta}]$ of angular acceleration of rocking. The biggest value of $\bar{\alpha}$ is attained on quarter headings.

Replacing in (11.5.131) $D[\ddot{\theta}]$ by an expression of type (2.1.77), we will obtain

$$\bar{\alpha} = -\frac{k^4}{U^2\cos^2\varphi}\cdot\frac{z^2}{2g^2}\left(\mu_2^2+\lambda_2^2\right)^2 D[\theta]\sin 2K. \tag{11.5.132}$$

If in the GC there is realized the period of M. Schuler, then, according to (11.5.105), $k^2 = \nu^2 = \frac{g}{R}$, and instead of (11.5.132) we have

$$\bar{\alpha} = -\frac{1}{RU^2\cos^2\varphi}\cdot\frac{z^2}{2g}\left(\mu_2^2+\lambda_2^2\right)^2 D[\theta]\sin 2K. \tag{11.5.133}$$

Example 11.12: To determine intercardinal error $\bar{\alpha}$ of single-rotor GC; z = -3 m; latitude $\varphi = 60^\circ$; parameters of rocking of ship: $D[\theta] = 3.79\cdot10^{-2}$, $\lambda_2 = 0.42$ 1/sec, $\mu_2 = 0.04$ 1/sec (Example 4.6); heading of ship $K = 45^\circ$.

Solution: By formula (11.5.133) we determine intercardinal error $\bar{\alpha}$ of the GC:

$$\bar{\alpha} = -\frac{1}{RU^2\cos^2\varphi}\frac{z^2}{2g}\left(\mu_2^2+\lambda_2^2\right)^2 D[\theta]\sin 2K =$$

$$= -\frac{1}{6371\cdot10^3\,(7{,}29)^2\cdot10^{-10}\cdot0{,}25}\cdot\frac{9}{2\cdot9{,}81}\,(0{,}04^2+0{,}42^2)^2\cdot3{,}79\cdot10^{-2}\cdot1 =$$

$$= -0{,}06506 \text{ rad.} = -3'44''.$$

From Example 11.12 it follows that error $\bar{\alpha}$ of single-rotor GC can attain very

large values.

Near position of equilibrium, which is determined by mathematical expectation $\bar{\alpha}$, axis Oz of gyroscope accomplishes random motions, which are characterized by dispersion D[α] of random function α(t). For finding D[α], in the first approximation it is possible to use equation (11.5.129). Without giving the solution of this problem, we will note that the value of D[α] is obtained to be small, since in the right side of equation (11.5.129) the disturbance is $[\ddot{\theta}(t)]^2$ – a small quantity of second order. Therefore, error of GC on a rocking ship is characterized basically by mathematical expectation $\bar{\alpha}$ determined by formulas (11.5.131) or (11.5.132). These formulas are approximate.

Let us show how to find exact formula for $\bar{\alpha}$. Let us use for this equations (11.5.127). Dividing the first of them by k^2, and the second by n^2 and introducing designations

$$T = \frac{1}{k}, \tag{11.5.134}$$

$$T_1 = \frac{1}{n}, \tag{11.5.135}$$

we will obtain

$$T^2\ddot{\alpha} + 2\xi T\dot{\alpha} + \alpha = \frac{k^2}{U^2\cos^2\varphi}\cdot\frac{z\sin K}{g}\ddot{\theta}\gamma; \tag{11.5.136}$$

$$T_1^2\ddot{\gamma} + 2\xi_1 T_1\dot{\gamma} + \gamma = -\frac{z\cos K}{g}\ddot{\theta}, \tag{11.5.137}$$

where T – time constant of GC;

T_1 – time constant of oscillations of GC with respect to angle γ.

Let us represent disturbance $\ddot{\theta}(t)$, by analogy with (4.6.149), in the form of spectral expansion

$$\ddot{\theta}(t) = \int_{-\infty}^{\infty} e^{j\omega t}\, d\Phi(\omega), \tag{11.5.138}$$

where dΦ(ω) possesses properties of (4.6.150).

Putting (11.5.138) in (11.5.137), we will obtain upon completion of transient response, by analogy with (4.6.153), the steady-state solution of this equation in the form

$$\gamma(t) = -\int_{-\infty}^{\infty} W_1(j\omega)\, e^{j\omega t}\, d\Phi(\omega), \tag{11.5.139}$$

where amplitude-phase characteristic of oscillation of gyrosphere with respect to coordinate γ, according to equation (11.5.137), will be

$$W_\gamma(j\omega) = \frac{\frac{s\cos K}{g}}{1 - T_1^2\omega^2 + 2\varkappa_1 T_1 j\omega}. \qquad (11.5.140)$$

Then

$$\gamma(t) = -\int_{-\infty}^{\infty} \frac{\frac{s\cos K}{g}}{1 - T_1^2\omega^2 + 2\varkappa_1 T_1 j\omega} e^{j\omega t}\, d\Phi(\omega). \qquad (11.5.141)$$

In virtue of equation (11.5.136), considering (4.6.135), for mathematical expectation $\bar{\alpha}$ of random function $\alpha(t)$ we have

$$\bar{\alpha} = W_\alpha(j\omega)\Big|_{\omega=0} \bar{\theta}_\gamma, \qquad (11.5.142)$$

where $W_\alpha(jw)$ — amplitude-phase characteristic of oscillations of GC with respect to coordinate α — is determined by relationship

$$W_\alpha(j\omega) = \frac{\frac{k^2}{U^2\cos^2\varphi} \cdot \frac{s\sin K}{g}}{1 - T^2\omega^2 + 2\varkappa T j\omega}, \qquad (11.5.143)$$

whence

$$W_\alpha(j\omega)\Big|_{\omega=0} = \frac{k^2}{U^2\cos^2\varphi} \cdot \frac{s\sin K}{g}. \qquad (11.5.144)$$

Putting (11.5.144) in (11.5.142), we will obtain

$$\bar{\alpha} = \frac{k^2}{U^2\cos^2\varphi} \cdot \frac{s\sin K}{g} \bar{\theta}_\gamma \qquad (11.5.145)$$

or, considering (11.5.138) and (11.5.141),

$$\bar{\alpha} = -\frac{k^2}{U^2\cos^2\varphi} \cdot \frac{s^2}{2g^2} \sin 2K \int_{-\infty}^{\infty} \frac{1}{1 - T_1^2\omega^2 + 2\varkappa_1 T_1 j\omega} S_{\ddot{\theta}}(\omega)\, d\omega. \qquad (11.5.146)$$

Let us designate

$$I = \int_{-\infty}^{\infty} \frac{1}{1 - T_1^2\omega^2 + 2\varkappa_1 T_1 j\omega} S_{\ddot{\theta}}(\omega)\, d\omega, \qquad (11.5.147)$$

consequently,

$$\bar{\alpha} = -\frac{k^2}{U^2\cos^2\varphi} \cdot \frac{s^2}{2g^2} \sin 2K \cdot I. \qquad (11.5.148)$$

For small time constant T_1, which exists for a single-rotor GC, instead of (11.5.147) we will obtain

$$I \approx \int_{-\infty}^{\infty} S_{\ddot{\theta}}(\omega)\, d\omega = D[\ddot{\theta}].$$

Substituting $I \approx D[\ddot{\theta}]$ in (11.5.148), we find for $\bar{\alpha}$ the approximate formula (11.5.131). In order to obtain exact formula for $\bar{\alpha}$, it is necessary to calculate

integral (11.5.147).

In the beginning we will find $S_{\ddot{\theta}}(\omega)$. According to (2.1.84) and (2.1.58) we have*

$$S_{\ddot{\theta}}(\omega) = \frac{2A_2\mu_2}{\pi} \cdot \frac{b_2^2\omega^4}{\omega^4 + 2a_2\omega^2 + b_2^4}, \qquad (11.5.149)$$

where $A_2 = D[\ddot{\theta}]$; $a_2 = \mu_2^2 - \lambda_2^2$; $b_2^2 = \mu_2^2 + \lambda_2^2$ [see (2.1.59)].

Putting (11.5.149) in (11.5.147), we will obtain

$$I = \int_{-\infty}^{\infty} \frac{2\mu_2 b_2^2}{\pi} D[\theta] \frac{\omega^4}{\left(1 - T_1^2\omega^2 + 2\zeta_1 T_1 j\omega\right)\left(\omega^4 + 2a_2\omega^2 + b_2^4\right)} d\omega. \qquad (11.5.150)$$

Reducing (11.5.140) to an integral of the type (4.6.71) and using for its calculation tables [153], by the same method which we repeatedly applied earlier we will obtain

$$I = b_2^2 \frac{b_2^2\left(1 - 4\zeta_1 T_1\mu_2 - T_1^2 b_2^2\right) - 4\mu_2^2}{\left(1 - T_1^2 b_2^2\right)^2 + 4\left[T_1^2 b_2^2\zeta_1^2 + T_1\mu_2\left(\zeta_1 + T_1\mu_2\right) + T_1^3\zeta_1\mu_2 b_2^2\right]} D[\theta]. \qquad (11.5.151)$$

Let us introduce (11.5.151) into (11.5.148); then we will obtain exact formula for intercardinal error $\bar{\alpha}$ of single-rotor GC during irregular rocking of ship:

$$\bar{\alpha} = -\frac{k^2}{U^2\cos^2\varphi} \cdot \frac{g^2}{2g^2} b_2^2 \times \\ \times \frac{b_2^2\left(1 - 4\zeta_1 T_1\mu_2 - T_1^2 b_2^2\right) - 4\mu_2^2}{\left(1 - T_1^2 b_2^2\right)^2 + 4\left[T_1^2 b_2^2\zeta_1^2 + T_1\mu_2\left(\zeta_1 + T_1\mu_2\right) + T_1^3\zeta_1\mu_2 b_2^2\right]} D[\theta] \sin 2K. \qquad (11.5.152)$$

Example 11.13: To calculate intercardinal error $\bar{\alpha}$ of single-rotor GC for the conditions of Example 11.12; period of natural oscillations of sensing device around axis Oz $T_\gamma = 1$ sec, frequency n and time constant T_1 of these oscillations $n = \frac{2\pi}{T_\gamma} = \frac{6{,}28}{1} = 6{,}28$ 1/sec. $T_1 = \frac{1}{n} = \frac{1}{6{,}28} = 0{,}16$ sec. $k = \nu = 1{,}24 \cdot 10^{-3}$ 1/sec, $\zeta_1 = 0{,}5$.

Solution: By formula (11.5.152) we find

$$\bar{\alpha} = -0{,}0610 \text{ rad.} = -3°30'.$$

Comparing results of calculations of Examples 11.12 and 11.13, we see that values of $\bar{\alpha}$ calculated by approximate and exact formulas little differ. Therefore, during practical calculations of intercardinal error $\bar{\alpha}$ of single-rotor GC for conditions of irregular rocking of ship, it is possible, in general, to use the approximate formula.

The considerable magnitudes of intercardinal errors of single-rotor GC on a rocking ship lead to the necessity to apply in these types of GC appropriate measures for decrease of these errors. Thus, for instance, in the Sperry single-rotor GC [71],

*For determination of spectral density $S_{\ddot{\theta}}(\omega)$ see [120, p. 80].

there is realized gyroscopic stabilization of the place of connection of the pendulum with the gyrohousing relative to the vertical. This considerably decreases angles γ of turn of sensing device around axis Oz, and therefore error $\bar{\alpha}$ is lowered.

Another method of decrease of error of GC on a rocking ship consists of application of multigyroscope systems, among which the two-rotor GC is included. Here, due to gyroscopic stabilization of gyrosphere around N-S axis, angles γ of its inclination will be very small, and therefore magnitude $\bar{\alpha}$ of intercardinal error of two-rotor GC, as will be shown below, is minute. There is known, furthermore, the following method of decrease of influence of rocking of ship on GC: there is realized phase shift of torque, applied to GC on the rocking ship relative to oscillations of sensing device around axis Oz. This method is applied in GC having mercury vessels, where by selection of parameters of system (period of natural oscillations of mercury in ballistic vessels) we make phase shift of oscillations of liquid in vessels relative to oscillations of sensing device of GC equal to 90°.

c) Dynamic Errors of Two-Rotor GC on a Rocking Ship

One of the most effective methods of removal of influence of rocking of ship on GC is the creation of multigyroscope, in particular two-gyroscope, compasses. As was shown during determination of intercardinal error of single-rotor GC (Par. b), appearance of considerable error on a rocking ship is due to the fact that because of smallness of period of oscillations of sensing device around axis Oz, angles γ of its slope relative to plane of horizon are obtained on the rocking ship to be large. This circumstance, as can be seen from first equation of system (11.5.123), leads to considerable errors α of the GC.

In two-rotor GC, due to application in it of two gyroscopes, the sensing device — the gyrosphere — has gyroscopic stabilization with respect to the N-S axis. Period of oscillations of gyrosphere around this axis is 12-15 minutes [71], i.e., by many times exceeds periods of rocking of the ship. Therefore, angles γ of turn of gyrosphere around N-S axis, i.e., angles of inclination of its E-W diameter with respect to plane of horizon, are negligible, and consequently, the error of a two-rotor GC during rocking of a ship is minute. Gyroscopic stabilization of gyrosphere around N-S axis, as has been shown, is caused by presence in it of two gyroscopes connected by a shaft. Actually, if disturbing torque due to rocking is applied relative to N-S axis (Fig. 11.14b), then this causes precessional motion of gyroscopes around their axes of precession by angle δ; the thus appearing component of gyroscopic torque along N-S axis will compensate the external disturbing torque.

Let us determine the random error during rocking of a ship of a two-rotor GC.* For this we will first compose differential equations of motion of two-rotor GC during rocking of the ship. Let us use equations (11.5.42); disregarding in them insignificant terms and considering disturbances $f_1(t),\ldots, f_4(t)$ to be absent, we will obtain

$$\left.\begin{aligned} &H\left(\dot{\alpha}+u_\zeta\right)-lP\beta-lP\frac{\nabla_\eta}{g}-c\vartheta=0\\ &H\left(\dot{\beta}+u_\eta+u_\zeta\alpha\right)-lP\frac{\nabla_\eta}{g}\gamma=0\\ &\dot{\vartheta}=-F\left(\beta+\vartheta+\frac{\nabla_\eta}{g}\right)\\ &H\dot{\delta}+lP\gamma-lP\frac{\nabla_\xi}{g}=0\\ &H\left(\dot{\gamma}+u_\eta\right)-n_1\delta-c_1\delta=0 \end{aligned}\right\} \qquad (11.5.153)$$

For u_ξ, u_η, u_ζ in the case of a motionless ship, according to (11.5.67), we have $u_\xi = 0$, $u_\eta = U\cos\varphi$; $u_\zeta = U\sin\varphi$; then we will rewrite equations (11.5.153) in the form

$$\left.\begin{aligned} &\dot{\beta}+U\cos\varphi\cdot\alpha=\frac{lP}{H}\cdot\frac{\nabla_\eta}{g}\gamma\\ &\dot{\alpha}+U\sin\varphi-\frac{lP}{H}\beta-\frac{c}{H}\vartheta=\frac{lP}{H}\cdot\frac{\nabla_\eta}{g}\\ &\dot{\vartheta}+F\vartheta+F\beta=-F\frac{\nabla_\eta}{g}\\ &\dot{\delta}+\frac{lP}{H}\gamma=\frac{lP}{H}\cdot\frac{\nabla_\xi}{g}\\ &\dot{\gamma}-\frac{n_1}{H}\delta-\frac{c_1}{H}\delta=-HU\cos\varphi \end{aligned}\right\} \qquad (11.5.154)$$

Being interested only in errors of GC due to rocking of ship, we will omit in second and in last equations terms U sin φ and − HU cos φ, which affect position of equilibrium of sensing device of GC due to rotation of Earth; considering designations of (11.5.44) and (11.4.36), instead of (11.5.154) we will obtain

$$\left.\begin{aligned} &\dot{\beta}+U\cos\varphi\cdot\alpha=\frac{k^2}{U\cos\varphi}\cdot\frac{\nabla_\eta}{g}\gamma\\ &\dot{\alpha}-\frac{k^2}{U\cos\varphi}\beta-\frac{k}{U\cos\varphi}(1-\rho)\vartheta=\frac{k^2}{U\cos\varphi}\cdot\frac{\nabla_\eta}{g}\\ &\dot{\vartheta}+F\vartheta+F\beta=-F\frac{\nabla_\eta}{g}\\ &\ddot{\gamma}+2\zeta_1 n\dot{\gamma}+n^2\gamma=2\zeta_1 n\frac{\dot{\nabla}_\xi}{g}+n^2\frac{\nabla_\xi}{g} \end{aligned}\right\} \qquad (11.5.155)$$

*An analogous problem is considered in [130].

where

$$n^2 = \frac{c_1 lP}{H^2}; \quad 2\zeta_1 n = \frac{n_1 lP}{H^2}; \quad \zeta_1 = \frac{n_1}{2H}\sqrt{\frac{lP}{c_1}}. \tag{11.5.156}$$

Discarding in (11.5.155) term $2\zeta_1 n\frac{\dot{W}_\xi}{g}$, which is caused by presence of damping and characterizes damping-acceleration error of the GC, and replacing W_ξ, W_η by relationships (11.5.126), we will rewrite (11.5.155) in the form

$$\left.\begin{aligned} \dot{\beta} + U\cos\varphi\cdot\alpha &= \frac{k^2}{U\cos\varphi}\cdot\frac{z\sin K}{g}\ddot{\theta}\gamma \\ \dot{\alpha} - \frac{k^2}{U\cos\varphi}\beta - \frac{k^2}{U\cos\varphi}(1-\rho)\theta &= \frac{k^2}{U\cos\varphi}\cdot\frac{z\sin K}{g}\ddot{\theta} \\ \dot{\theta} + F\theta + F\beta &= -F\frac{z\sin K}{g}\ddot{\theta} \\ \ddot{\gamma} + 2\zeta_1 n\dot{\gamma} + n^2\gamma &= -n^2\frac{z\cos K}{g}\theta \end{aligned}\right\} \tag{11.5.157}$$

Since disturbance $\ddot{\theta}(t)$ is a random function of time, then error $\alpha(t)$ of GC, and also angles $\beta(t)$, $\gamma(t)$ and angle $\psi(t)$ of slope of the "mirror" are also random functions.

Of the greatest interest is determination of mathematical expectation $\bar{\alpha} = M[\alpha(t)]$ of azimuthal drift of the GC on a rocking ship. For this we will apply to equations of system (11.5.157) the operation of mathematical expectation; considering that $\ddot{\theta} = 0$, we will obtain

$$\left.\begin{aligned} \dot{\bar{\beta}} + U\cos\varphi\cdot\bar{\alpha} &= \frac{k^2}{U\cos\varphi}\cdot\frac{z\sin K}{g}\overline{\ddot{\theta}\gamma} \\ \dot{\bar{\alpha}} - \frac{k^2}{U\cos\varphi}\bar{\beta} - \frac{k^2}{U\cos\varphi}(1-\rho)\bar{\theta} &= 0 \\ \dot{\bar{\theta}} + F\bar{\theta} + F\bar{\beta} &= 0 \\ \ddot{\bar{\gamma}} + 2\zeta_1 n\dot{\bar{\gamma}} + n^2\bar{\gamma} &= 0 \end{aligned}\right\} \tag{11.5.158}$$

From the last equation it follows that upon completion of transient response

$$\bar{\gamma} = 0. \tag{11.5.159}$$

For determination of $\bar{\alpha}$ from first equation it is necessary at first to find $\overline{\ddot{\theta}\gamma}$. Analogous problem was solved by us during the analysis of intercardinal error of single-rotor GC [see (11.5.145)]. In order to use the earlier obtained result, we will rewrite last equation of system (11.5.157) in the form

$$T_1^2\ddot{\gamma} + 2\zeta_1 T_1\dot{\gamma} + \gamma = -\frac{z\cos K}{g}\ddot{\theta}, \tag{11.5.160}$$

where

$$T_1 = \frac{1}{n} \tag{11.5.161}$$

constitutes time constant of oscillations of gyrosphere around N-S axis.

Equation (11.5.160) is analogous to equation (11.5.137). Therefore, for quantity $\overline{\ddot{\theta}\gamma}$, considering (11.5.145) and (11.5.146), we have

$$\overline{\ddot{\theta}_\gamma} = -\frac{z\cos K}{g}\int_{-\infty}^{\infty}\frac{1}{1-T_1^2\omega^2+2\zeta_1 T_1\omega}S_{\ddot{\gamma}}(\omega)\,d\omega \qquad (11.5.162)$$

or

$$\overline{\ddot{\theta}_\gamma} = -\frac{z\cos K}{g}I, \qquad (11.5.163)$$

where I is determined by formula (11.5.147) or (11.5.151).

System of homogeneous equations (for $\overline{\ddot{\theta}\gamma} = 0$) of the first three equations of (11.5.158) is analogous to system (11.5.61), the characteristic equation (11.5.64) of which determines natural damped oscillations of the GC. Therefore, in accordance with (11.5.158), upon completion of transient responses we have

$$\left.\begin{aligned}\bar{\alpha} &= \frac{k^2}{U^2\cos^2\varphi}\cdot\frac{z\sin K}{g}\overline{\ddot{\theta}_\gamma}\\ \bar{\beta} &= 0;\quad \bar{\delta} = 0\end{aligned}\right\} \qquad (11.5.164)$$

Quantity $\bar{\alpha}$ constitutes mathematical expectation of error of GC during rocking of ship. Putting in $\bar{\alpha}$ the value of $\overline{\ddot{\theta}\gamma}$ from (11.5.163), we will obtain the expression

$$\bar{\alpha} = -\frac{k^2}{U^2\cos^2\varphi}\cdot\frac{z^2}{2g^2}\sin 2K\cdot I, \qquad (11.5.165)$$

which coincides with formula (11.5.148) of intercardinal error of single-rotor GC. Introducing (11.5.151) into (11.6.165), we will obtain exact formula for intercardinal error $\bar{\alpha}$ of two-rotor GC:

$$\bar{\alpha} = -\frac{k^2}{U^2\cos^2\varphi}\cdot\frac{z^2}{2g^2}b_2^2\times$$
$$\times\frac{b_2^2\left(1-4\zeta_1T_1\mu_2-T_1^2b_2^2\right)-4\mu_2^2}{\left(1-T_1^2b_2^2\right)^2+4\left[T_1^2b_2^2\mu_1^2+T_1\mu_2\left(\zeta_1+T_1\mu_2\right)+T_1^2\mu_2^2b_2^2\right]}D(\theta)\sin 2K. \qquad (11.5.166)$$

Example 11.14: To calculate intercardinal error of two-rotor GC. Initial data: $k = \nu = 1.24\cdot10^{-3}$ 1/sec, $U = 7.29\cdot10^{-5}$ 1/sec, $\varphi = 60^\circ$, period of oscillations of gyrosphere around N-S axis $T_\gamma = 15$ minutes, frequency n and time constant T_1 of these oscillations $n = \frac{2\pi}{T_\gamma} = \frac{6.28}{900} = 0.698\cdot10^{-2}$ 1/sec $T_1 = \frac{1}{n} = \frac{900}{6.28} = 143.3$ sec, $\zeta_1 = 0.05$; data on rolling of ship are taken from Example 11.12.

Solution: By formula (11.5.166) we find intercardinal error $\bar{\alpha}$:

$$\bar{\alpha} = 0{,}1718\cdot10^{-4}\ \text{rad.} = 4''.$$

Comparing values of $\bar{\alpha}$ obtained in Examples 11.13 and 11.14, we see that

intercardinal error of two-rotor GC is considerably less than error of single-rotor GC. This result, as it was shown earlier, is explained by the large magnitude of period of oscillations of gyrosphere around N-S axis. Actually, magnitude T_γ of this period, if we consider (11.5.156), will be

$$T_1 = \frac{2\pi}{n} = 2\pi \frac{H}{\sqrt{c_1 l P}}. \qquad (11.5.167)$$

i.e., is proportional to angular momentum H. Period T_γ in the known models of two-rotor GC is 12-15 minutes [71].

This analysis of influence of rocking of a ship on the two-rotor GC is valid if hydraulic damper is made in the form of two interconnected vessels with round cross section (Fig. 11.16).

Above there were considered errors of GC during pure rolling of the ship. Problem of determination of error of GC in the general case of rocking of the ship, taking into account vertical accelerations and terms of the second order of smallness, requires special consideration.

6. Gyrohorizon Compass

Briefly we will consider one of the varieties of GC, which has acquired the name of space gyrocompass, or gyrohorizon compass (GHC). As was shown in Par. 1a this device represents a combination of the gyrohorizon and gyrocompass, i.e., it permits us to determine angles of rocking of ship (gyrohorizon) and its heading (gyrocompass); three principla axes of space GC are gyroscopically stabilized in the east-north-zenith directions.

Such a system was realized in the space gyrocompass of Anschütz [30, 184, 75]. Its main peculiarity was that the period of natural sustained oscillations of the gyrosphere about the three axes of its rotation was 84.4 minutes. Therefore, here it was possible to completely remove intercardinal error of the GC. Besides heading of the ship, the instrument determined angles of rolling and pitching.

Sensing device (sensitive element) of space GC also is the gyrosphere; the location in it of the two gyroscopes is shown in Fig. 11.19. Axes of gyroscopes G_1 and G_2 in initial position (with non-rotating rotors) are directed along E-W diameter of gyrosphere; gyrochambers are connected by shaft Sh.

Approximate theory of space GC was initially developed by I. Geckeler [184] (see also [30]). He showed that period of oscillations of gyrosphere around the three axes is equal to 84.4 minutes. This is attained by application to axes of

precession of gyroscopes G_1 and G_2 of an appropriate torque. Here, in distinction from single-rotor and two-rotor GC, period of oscillations [see (11.5.53)] of sensing device does not depend on latitude of the place. Space GC has the same velocity error as the usual GC [see (11.5.71)]. Ballistic errors of space GC are small [30], and considerably less than for two-rotor GC.

Designs and parameters of space GC of Anschütz type are given in a number of works (see, for instance, [30]).

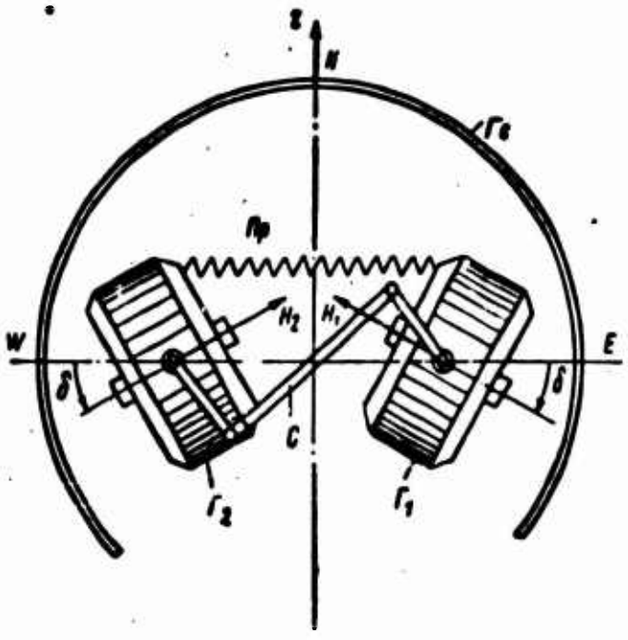

Fig. 11.19. Diagram of sensing device (sensitive element) of space GC.

Strict theory of gyrohorizon compass is given by A. Yu. Ishlinskiy [44]. Due to the importance of the results obtained by him, we will consider briefly peculiarities of the theory of GHC, using this work, in which as the sensing device of the GHC there is used a gyrosphere analogous to the gyrosphere of the two-gyroscope GC (Fig. 11.14).

During investigation [44] of GHC there is used the method of the non-rotating sphere S (see § 11.4, Par. 6) surrounding Earth and preserving constant orientation relative to inertial space. Motion of the instrument is considered during arbitrary shift of its point of suspension relative to sphere S. At the basis of the investigation there is assumed precessional theory. Sensing device (Fig. 11.20) is considered as a set of gyroscopes G_1 and G_2 whose axes of precession are connected by frame F; coupling of the gyrochambers is arbitrarily shown in the form of a gear transmission, which in its action is identical to shaft Sh (Fig. 11.14). This system will form a gyroscopic frame. Total angular momentum H of gyroframe, by analogy with (11.5.6), is determined by expression $H = 2H' \cos \varepsilon$ (where $H' = H_1 = H_2$) and is directed along the bisector of angle 2ε. With frame F there is connected coordinate system Oxyz, axis Oy of which is parallel to vector $\vec{H}$, and axis Oz — to axes of precession of gyroscopes. Besides this system of coordinates, there is introduced also a translationally moving system $O\xi^*\eta^*\zeta^*$, the origin of which is located at point of suspension of frame, and axes ξ^*, η^*, ζ^* are constantly oriented relative to inertial space, i.e., relative to non-rotating sphere S.

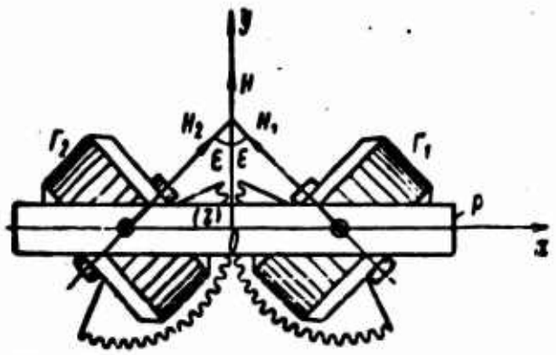

Fig. 11.20. Diagram of frame of sensing device (sensitive element).

In the considered case, the gyroframe constitutes a mechanical system with four degrees of freedom; equations of its motion relative to axes $O\xi^*\eta^*\zeta^*$ are obtained on the basis of precessional theory in the form [44].

$$\left.\begin{aligned} -\omega_x 2H'\cos\varepsilon = M_x;\quad \omega_z 2H'\cos\varepsilon = M_z \\ \frac{d}{dt}(2H'\cos\varepsilon) = M_y;\quad -\omega_y 2H'\sin\varepsilon = N \end{aligned}\right\} \quad (11.5.168)$$

where M_x, M_y, M_z — external torquer acting on the frame;

$N = M_{z'} - M_{z''}$; $M_{z'}$, and $M_{z''}$ are torques acting relative to axes of precession of gyroscopes; torque N is created by spring device Sp (Fig. 11.14) and is a function of angle ε;

ω_x, ω_y, ω_z — projections of angular velocity $\vec{\omega}$ of axes Oxyz relative to non-rotating sphere S on to corresponding axes.

As external forces acting on the frame, there are considered gravity force F which is directed toward center of Earth, reactive force of suspension and inertia force Q of translational induced motion of axes $O\xi^*\eta^*\zeta^*$. Since gravity force F, reactive force of suspension and component Q of inertia force of induced motion are directed along the axis Oz, then their moment relative to point of suspension of frame is equal to zero, and for finding M_x, M_y, M_z [see (11.5.168)] there are considered only components Q_x and Q_y. As a result, instead of (11.5.168) there is obtained

$$\left.\begin{aligned} -\omega_x 2H'\cos\varepsilon = mlR\left(\frac{d\omega_x}{dt} - \omega_y\omega_z\right);\quad \omega_z 2H'\cos\varepsilon = 0 \\ \frac{d}{dt}(2H'\cos\varepsilon) = mlR\left(\frac{d\omega_y}{dt} + \omega_x\omega_z\right);\quad -\omega_y 2H'\sin\varepsilon = N \end{aligned}\right\} \quad (11.5.169)$$

where m — mass of frame together with gyroscopes;

l — distance of center of gravity of frame from point of its suspension;

R — radius of Earth taken as a sphere.

Thus linear velocities v_x, v_y, v_z of point of suspension of frame relative to sphere S will be

$$v_x = \omega_y R;\quad v_y = -\omega_x R;\quad v_z = 0. \quad (11.5.170)$$

From third equation of system (11.5.169) we have

$$\omega_z = 0. \quad (11.5.171)$$

From first two equations there can be obtained

$$2H'\cos\varepsilon = H = mlR\omega_y, \quad (11.5.172)$$

whence, taking into account the fourth equation, we find condition

$$N = -\frac{(2H')^2}{mlR}\cos\varepsilon\sin\varepsilon, \tag{11.5.173}$$

which is analogous to (11.5.112).

Resulting expressions permit us to obtain the following relationships:

$$v_y = 0, \tag{11.5.174}$$

$$2H'\cos\varepsilon = mlv, \tag{11.5.175}$$

where v is velocity of point of suspension of frame relative to sphere S.

From (11.5.170) and (11.5.174) it follows that at the initial moment axis Ox should be directed along tangent to trajectory of point of suspension of frame during its motion over sphere S. According to (11.5.175), for initial velocity v_0 initial angle ε_0 is determined by expression

$$\cos\varepsilon_0 = \frac{ml}{2H'}v_0. \tag{11.5.176}$$

Investigation [44] of motion of frame relative to Earth shows that axis Oy of frame (Fig. 11.20) in position of equilibrium forms with the N-S direction angle Φ, which is velocity error and is determined by the same formula (11.5.71) as that for the two-rotor compass.

Since absolute velicity v of point of suspension of frame (i.e., its velocity relative to non-rotating sphere S) is determined by the evident relationship

$$v = \sqrt{(RU\cos\varphi + v_E)^2 + v_N^2}, \tag{11.5.177}$$

where v_E and v_N are eastern and northern components of velocity relative to Earth, then, putting (11.5.177) in (11.5.175), we will obtain condition

$$H = 2H'\cos\varepsilon = ml\sqrt{(RU\cos\varphi + v_E)^2 + v_N^2}. \tag{11.5.178}$$

which is analogous to (11.5.111).

If there are satisfied condition (11.5.173) [for which (11.5.178) is valid] and the earlier indicated initial conditions, then the GHC does not have any errors besides the velocity errors determined by formula (11.5.71). Consequently, for GHC during maneuvering of the ship there will be absent ballistic errors. Thus axis Oz of frame is directed along geocentric vertical, i.e., along radius of Earth.

Further there is introduced [44] the Darboux trihedron $Ox^oy^oz^o$ with vertex at point of suspension of frame. Its axis Oz^o is normal to sphere S, and axis Ox^o is directed along vector $\vec{v}$ of point of suspension. Position of axes Oxyz of frame relative to axes $Ox^oy^oz^o$ is defined by angles α, β, γ, where α is angle of azimuthal turn of frame, β is angle of its rise relative to plane of horizon, and γ is angle of rotation of frame around axis Oy, i.e., around N-S axis. As a result, there were

obtained the following differential equations of motion of frame relative to Darboux trihedron:*

$$\left.\begin{aligned}
&-\left[\frac{v}{R}(\sin\alpha\sin\gamma-\cos\alpha\sin\beta\cos\gamma)+\right.\\
&\left.+\left(\omega+\frac{d\alpha}{dt}\right)\cos\beta\cos\gamma+\frac{d\beta}{dt}\sin\gamma\right]2H'\cos\varepsilon=\\
&=ml\frac{dv}{dt}\sin\alpha\cos\beta-ml\omega v\cos\alpha\cos\beta+\\
&+\left(ml\frac{v^2}{R}-lF\right)\sin\beta\\
&\frac{d}{dt}2H'\cos\varepsilon=ml\frac{dv}{dt}(\cos\alpha\cos\gamma-\sin\alpha\sin\beta\sin\gamma)+\\
&+ml\omega(\sin\alpha\cos\gamma+\cos\alpha\sin\beta\sin\gamma)+\\
&+\left(ml\frac{v^2}{R}-lF\right)\cos\beta\sin\gamma
\end{aligned}\right\}\qquad(11.5.179)$$

$$\left.\begin{aligned}
&\left[\frac{v}{R}(\sin\alpha\cos\gamma+\cos\alpha\sin\beta\sin\gamma)-\left(\omega+\frac{d\alpha}{dt}\right)\cos\beta\sin\gamma+\right.\\
&\left.+\frac{d\beta}{dt}\cos\gamma\right]2H'\cos\varepsilon=0\\
&-\left[\frac{v}{R}\cos\alpha\cos\beta+\left(\omega+\frac{d\alpha}{dt}\right)\sin\beta+\frac{d\gamma}{dt}\right]2H'\sin\varepsilon=N(\varepsilon)
\end{aligned}\right\}\qquad(11.5.179)$$

These equations are identically satisfied if in them: we set $\alpha = \beta = \gamma = 0$, replace ε by relationship (11.5.175), and $N(\varepsilon)$ by formula (11.5.173). Thus independently of the form of functions $v(t)$ and $\omega(t)$, which characterize motion of point of suspension of frame over sphere S, vector $\vec{H}$ which is directed along the axis Oy, will be perpendicular to vector $\vec{v}$ of velocity of this point relative to sphere S, and axis Oz is directed along radius of the Earth.

Thus, if there are satisfied conditions (11.5.173) and (11.5.175), then $\alpha = \beta = \gamma = 0$, and consequently equatorial plane Oxy of frame will be always horizontal (gyrohorizon), and axis Oy with accuracy up to the velocity error will indicate north (gyrocompass), i.e., the considered instrument indeed is a gyrohorizon compass. If the above-indicated conditions are satisfied and at the initial moment of time $\alpha = \beta = \gamma = 0$, then the instrument will always indicate the vertical and the direction to the north.

For small angles α, β, γ and angle δ of turn of gyroscopes from their undisturbed position ε_0 (current angle $\varepsilon = \varepsilon_0 + \delta$), considering (11.5.173) and (11.5.175), we will obtain perturbation equation of frame in the form

*In these equations we considered only gravitational torques, inertial torques and torque $N(\varepsilon)$, created by the spring device.

$$\left.\begin{aligned} &-mlv\frac{d\alpha}{dt}-ml\frac{dv}{dt}\alpha+lF\beta=-\omega 2H'\sin\varepsilon_0\cdot\delta\\ &\frac{d\beta}{dt}+\frac{v}{R}\alpha=\omega\gamma;\quad \frac{d\gamma}{dt}+\frac{2H'\sin\varepsilon_0}{mlR}\delta=-\omega\beta\\ &-\frac{d}{dt}(2H'\sin\varepsilon_0\cdot\delta)+l\left(F-\frac{mv^2}{R}\right)\gamma=\omega mlv\alpha \end{aligned}\right\} \qquad (11.5.180)$$

If v and ω are constant, then expressions (11.5.180) become differential equations with constant coefficients. If we approximately set $F - \frac{mv^2}{R} \approx F = mg$, then roots of characteristic equation of system (11.5.180) will be $\pm j(\nu + \omega)$, $\pm j(\nu - \omega)$, where $\nu = \sqrt{\frac{g}{R}}$ is frequency corresponding to period $T_{g.c} = 2\pi\sqrt{\frac{R}{g}}$ of M. Schuler.

From this short account of the content of the article of A. Yu. Ishlinskiy [44], it can be seen that it has important value for explanation of the properties of a GHC and more precise solution of a number of problems of the theory of GC.

Some of these properties of GHC were later explained by D. R. Merkin [91] by another method. The question of the reducibility of equations of disturbed motion of GHC to a system with constant coefficients was considered by V. N. Koshlyakov [62].

§ 11.6. Gyro Inertial Navigational Systems

1. General Information about Inertial Navigation Systems (INS).* Principles of Construction of INS

a) Purpose of INS

Among the different gyroscopic navigational systems, in the last decades there have been developed so-called gyro inertial navigational systems — one of the types of INS. The appearance of INS first of all is explained by the desire to carry out with high accuracy self-contained determination of position of an object and parameters of its motion. Practical realization of INS on ships and aircraft turned out to be possible after certain successes achieved in production of different elements of inertial systems: gyroscopes, accelerometers, servo systems, computers, etc., to the accuracy of which there are presented very high requirements.

An INS is designed for determination of position of an object, parameters of its motion, and, if necessary, for automatic control of its motion. In connection with this, INS can solve the following problems:

1) determine position of object, which is characterized, for instance, by latitude φ and longitude λ;

2) develop certain parameters for stabilization of the object about its center of gravity and retention of center of gravity of object on its assigned trajectory;

3) carry out automatic control of motion of object, and others.

For solution of these problems, in the INS there have to be determined the following data: geographic coordinates of position of object, components of its linear velocity, heading, distance passed over, lateral displacement of center of gravity from assigned trajectory, altitude, and others. From what has been said it follows that an INS can operate in the readout mode, when it produces data necessary for navigation of the object, and in the control mode, when data determined by it are used for automatic control of motion of the object. In accordance with this, we distinguish inertial navigational systems and inertial control systems. In the present book we will consider only inertial navigational systems.

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*Such an abbreviated designation is used also in the book: V. G. Denisov. Navigational equipment of aircraft. Oborongiz, 1963.

b) Essence and Types of INS

Principle of inertial navigation. Inertial navigation is a method of determination of position of an object and control of its motion based on measurement of accelerations of the object, their subsequent integration, and calculation of coordinates of the object and control parameters.

Accelerations of object and all other self-contained measurements are carried out in INS relative to an inertial system of coordinates, which preserves constant orientation with respect to fixed stars. This is the reason for introduction of the term "inertial navigation."*

The essence of INS consists of the following: Input variables of the system are accelerations of the object, which are measured by sensitive elements — accelerometers — which are usually installed on a gyro-stabilized platform (GSP), which holds them in a definite position relative to the inertial system of coordinates. Measured accelerations enter integrators, at the output of which there are obtained components of linear velocity of object relative to the inertial system. With the help of computers these components are converted into components of velocity of the object relative to the Earth. Subsequent integration of components of velocity of object, taking into account a series of corrections introduced by the computer for the action of gravity force,** for non-sphericity of the Earth and for the influence of other factors, makes it possible to determine coordinates of the object. From what has been said, it follows that an inertial system in principle makes it possible to solve the problem of an "absolute log," since in it there is determined the actual motion of the object relative to the Earth.

The main elements of INS are accelerometers, gyroscopic devices, integrators, computers. Taking into account the fact that earlier there were given dynamic

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*Translation Ed. Note: This footnote in the Russian text pertains to the relative merit of using either of two Russian words for "inertial."

Certain authors [117] consider the term "inertsionnaya [inertial] navigation," to be more correct [than "inertsial'naya navigation"], since determination by accelerometers of accelerations of the object is carried out by means of measurement of corresponding inertial forces. However, it has not obtained wide usage.

**As will be shown further on, accelerometers measure only acceleration caused by non-gravitational forces (engine thrust, lift, drag of the atmosphere), and do not measure acceleration due to forces of gravitation.

characteristics of different types of GD, we will consider below also certain designs and dynamic characteristics of accelerometers.

Certain relationships. Essence of the above-stated method of determination of coordinates and parameters of motion of an object can be clarified by certain simple relationships. Let us assume that the gyrostabilized platform applied in INS is held in plane of horizon, and also has azimuthal stabilization in plane of meridian; we will not yet consider stabilization errors. On the platform there are located two accelerometers, axes of sensitivity of which are directed towards north and on east; these accelerometers will arbitrarily be called "northern" and "eastern." Disregarding components of induced and Coriolis accelerations, we will consider that the northern accelerometer measures component $\dot{v}_N$, and the eastern – component $\dot{v}_E$ of acceleration of object relative to Earth along north – south and east – west axes. Variables $\dot{v}_N$ and $\dot{v}_E$ go to integrators, at the output of which for zero initial conditions we will obtain

$$v_N = \int_0^t \dot{v}_N d\tau; \quad v_E = \int_0^t \dot{v}_E d\tau. \tag{11.6.1}$$

After second integration we find displacements of object S_N and S_E in northern and eastern directions; for zero initial conditions we have

$$S_N = \int_0^t v_N d\tau = \int_0^t \int_0^t \dot{v}_N d\tau^2; \quad S_E = \int_0^t v_E d\tau = \int_0^t \int_0^t \dot{v}_E d\tau^2. \tag{11.6.2}$$

Consequently, displacements of object are found by means of double integration of its accelerations, which are measured by accelerometers.

According to relationships (1.2.22), heading K of the object can be calculated in a computer by the formula

$$\operatorname{tg} K = \frac{v_E}{v_N}. \tag{11.6.3}$$

Since by virtue of (1.2.26), during motion over sphere of radius R, rate of change of latitude

$$\dot{\varphi} = \frac{v_N}{R}, \tag{11.6.4}$$

then latitude φ is determined in computer by the formula

$$\varphi = \varphi_0 + \frac{1}{R}\int_0^t v_N d\tau = \varphi_0 + \frac{1}{R}\int_0^t \int_0^t \dot{v}_N d\tau^2. \tag{11.6.5}$$

where φ_0 is initial value of latitude, which should be introduced into the system.

For rate of change of longitude, according to (1.2.27) we have

$$\dot{\lambda} = \frac{v_E}{R\cos\varphi}. \tag{11.6.6}$$

Then for λ the computer gives

$$\lambda = \lambda_0 + \int_0^t \frac{v_E}{R\cos\varphi}\,d\tau = \lambda_0 + \int_0^t \frac{1}{R\cos\varphi}\int_0^t \dot{v}_E\,d\tau^2, \tag{11.6.7}$$

where λ_0 is initial value of longitude which should be introduced into the system.

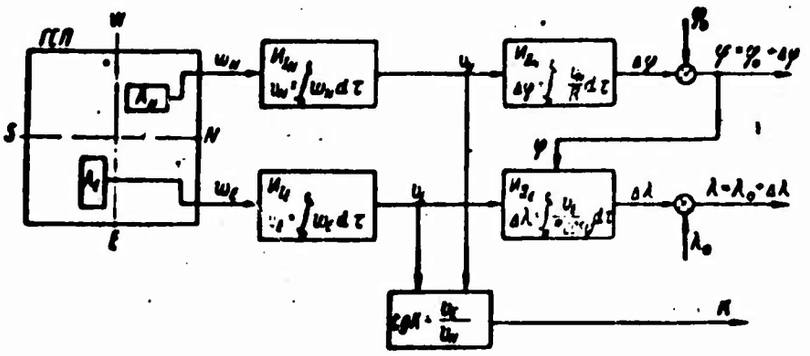

Fig. 11.21. Diagram of a simple INS.

The presence in the inertial navigational system of a GSP permits us to determine also angles of rotation of object relative to its center of gravity. If on the GSP we install a third accelerometer and direct its axis of sensitivity along the vertical, then we can determine vertical acceleration of the object, and then by means of integration we can determine vertical velocity and altitude of the object. Thus, application in the system of these elements permits us to find necessary coordinates and parameters of motion of the object.

Inertial system of open type. A simple inertial system of open type which realizes the above mentioned analytic dependences for coordinates of position and parameters of motion of the object is shown in Fig. 11.21. On the GSP, which preserves horizontal position and orientation which is constant in azimuth relative to the N-S direction, there are installed northern A_N and eastern A_E accelerometers.

The latter measure northern w_N and eastern w_E components of acceleration of the object, which are intorduced into the first integrators I_{I_N} and I_{I_E}. At the output of these integrators there are obtained northern v_N and eastern v_E components of velocity of the object, and by them there is calculated its heading K. Components

v_N and v_E are introduced into the second integrators I_{II_N} and I_{II_E}, at the output of which there are obtained increments $\Delta\varphi$ and $\Delta\lambda$ of latitude and longitude. After summation of them with initial values φ_0 and λ_0, there are found the current values φ and λ of latitude and longitude of the object.

It is easy to see that with ideally exact stabilization of the platform, the given system in principle makes possible self-contained determination of geographic coordinates and parameters of motion of the object. However, in reality the GSP, due to different disturbances, will deviate from the plane of the horizon. Due to this, readings of accelerometers will be distorted by magnitudes equal to components of acceleration due to gravity force. This leads to the appearance of errors of geographic coordinates φ, λ and value of heading K which are accumulated in time. Furthermore, in the scheme there are not considered corrections for rotation of the Earth and other factors. The accuracy of such a scheme is very low, and therefore, such simplified open systems of inertial navigation have not found practical application. Let us note that such systems are called open because in them there is absent a connection between accelerations measured by the accelerometers and the position of the GSP.

Principle of construction of closed inertial system. Simulation of the pendulum of M. Schuler. For solution of problem of inertial navigation, it is necessary first of all to obtain the exact vertical, i.e., the exact horizontal platform on which the accelerometers are installed. Creation of an exact gyro-vertical (gyro horizon) is extraordinarily complicated. The basic problem of obtaining a precision gyro-vertical consists of realization of the condition of its undisturbability by accelerations of the object. For this, as it is known, the period of natural sustained oscillations of the system should be 84.4 minutes. In § 4.5, Par. 5 it was shown that it is impossible in practice to create a physical pendulum with such a period. Construction of a gyroscopic pendulum undisturbed by accelerations is a quite complicated problem (§ 11.4, Par. 5).

At present, practical solution of this problem has proceeded by the way shown in 1932 by Soviet engineer Ye. B. Levental'. He proposed a scheme of artificial simulation of an undisturbable physical pendulum with period of natural oscillations of 84.4 minutes by means of use of gyroscopes, accelerometers and integrators.

This model of M. Schuler's pendulum is a closed dynamic system. Continuous erection in it of the vertical is based on turning of the GSP by an angle equal to

angle of rotation of the local vertical during motion of the object along surface of the Earth. For this on the GSP there are installed accelerometers, which measure horizontal accelerations of the object. The latter are integrated, and corresponding signals are introduced to the torquers of the gyroscopic system; this causes precessional motion of platform following the local vertical. Such a method of correction of GSP has the name of integral correction. Thus, by means of selection of transmission coefficients of the system, it is possible in practice to realize the condition of its undisturbability by horizontal accelerations of the object.

The artificial vertical obtained by such a method is called an inertial vertical (IV). It is the basis of the inertial navigational system, since simultaneously with erection of the vertical it determines components of linear velocity of the object, by which the computer calculates coordinates of position of the object and parameters of its motion.

Let us note that the IV, in distinction from the physical pendulum, indicates direction of vertical not due to action on it of gravity, but as a result of double integration of horizontal acceleration of the object measured by an accelerometer. Actually, acceleration goes to the first integrator. The signal from the output of the integrator, through the torquer, causes precessional motion of the platform. The angle of rotation of the platform is proportional to the integral of readings of the first integrator, i.e., to the double integral of horizontal acceleration of the object. Consequently, angle of rotation of platform is equal to angle of rotation of vertical during motion of object over surface of Earth, which is proportional to the path passed over. Therefore, IV will without oscillations follow the local vertical. If, however, the IV did not occupy in the beginning a vertical position, or has drifts in virtue of different circumstances, then in it there appear oscillations with period of 84.4 minutes; thus the average position of it coincides with the local vertical. Oscillations of the IV are not affected by horizontal accelerations of the object.

Subsequently, we will consider mainly questions connected with methods of construction and analysis of errors of IV, and will only briefly touch upon the operation of other elements of the INS.

Requirements made of inertial systems. Certain of their peculiarities. An inertial system should have: 1) very high accuracy; 2) high sensitivity; 3) possibility to work for a prolonged period of time, especially in the case of its use

in navigation; 4) wide range of change of input variables; 5) sufficiently accurate initial setting of GSP relative to vertical and in azimuth; 6) high noise immunity and independence of the presence of external information; 7) high shock and vibration resistance, and also small weight and dimensions, especially for use in air navigation.

An essential advantage of INS as compared to other navigational systems is that it is completely self contained: INS determines coordinates of object and parameters of its motion in the absence of any connection with other objects or reference points. Inertial system makes it possible to completely automate processes of navigation and can in principle be used under different conditions of motion of the object. System does not require ground installations; in it all radio emission is absent.

Advantages of INS are evident, but the question about their creation and practical application still has by far not been solved. This is explained by the exceptionally high requirements for accuracy of separate elements and subassemblies, especially in the GD, and in the entire system as a whole. Inertial systems in a certain sense are analogous to navigational systems in which there is realized the method of dead reckoning,* and their errors are accumulated in time. Therefore, the question of creation of inertial systems operating for a long time, for instance for navigation, is exceptionally complicated in practical solution. In connection with this, along with further development and improvement of INS, there has appeared a tendency to create combined systems, which combine INS with other navigational systems. One of such systems consists of INS and a Doppler navigational system [142, 149]. Here the INS is supplemented with radar using the Doppler effect. The latter makes it possible continuously or discretely to correct velocity of object determined by the inertial system; this considerably decreases accumulated errors of inertial system in determination of coordinates and parameters of motion of the object. Let us note that Doppler system makes it possibile to measure velocity of object with a high accuracy, reaching tenths of a percent. Correction of velocity of object with help of Doppler system, i.e., introduction into the INS of external information, has a series of peculiarities: it permits change of frequency of natural

*Method of dead reckoning is based on obtaining of necessary data about motion of object by means of continuous or discrete time integration of acceleration or velocity of the object [142].

sustained oscillations of the system with retention of property of its undisturbability by accelerations of the object; it ensures damping of oscillations, etc.

In another type of combined system there is used INS and astronavigational system [142, 149]. Such systems have obtained the name of <u>astroinertial navigational</u> systems. The main element of such a system is the optical telescope, which is usually installed on a GSP and carries out tracking of celestial bodies (stars). Due to this there is, for instance, the possibility to correct drift of the GSP, and consequently, to eliminate accumulation of errors of the inertial system. Requirements for accuracy of gyroscopic devices of astroinertial navigational systems can be considerably lowered. If there is temporary loss of the celestial body, the presence of gyroscopic device provides "memory" of the navigational system.

There are also possible other types of combined navigational systems. All of them are deprived of the basic advantage of inertial systems – their self-contained character. However, in a number of cases, application of combined systems permits solution of problem of navigation of objects with smaller requirements for accuracy of separate elements and subassemblies of the system; this leads to decrease of weights, dimensions, and also cost of the system as a whole. Subsequently, we will consider only gyro inertial navigational systems. Those wishing to become acquainted with combined navigational systems we refer to the corresponding literature (see, for instance, [142, 149, 117]).

<u>Types of inertial systems</u>. The classification given below of INS is to a certain extent arbitrary and is based on propositions accepted in a number of works on inertial navigation (see, for instance, [149, 121, 142, 162, 117]). As the basis of classification of INS it is possible to use:

1) purpose of system;
2) applied type of GD;
3) location of GSP;
4) method of erection of vertical;
5) orientation of axes of sensitivity of accelerometers in azimuth;
6) magnitude of period of natural oscillations of the vertical;
7) method of damping of oscillations of vertical.

Depending upon purpose, we distinguish the following INS:

a) working in readout mode;
b) working in mode of control of object.

Depending upon type of GD applied, to provide for reproduction of terrestrial or inertial systems of coordinates,* we distinguish:

a) INS using astatic gyroscopes with two or three degrees of freedom, operating in readout mode or in mode of active stabilization;

b) INS using positional gyroscopic devices, for instance the gyro horizon compass.

In inertial systems of the first type, the condition of undisturbability of system by accelerations of object is ensured by means of artificial simulation of physical pendulum with period of 84.4 minutes. In inertial systems of the second type, the condition of undisturbability is realized by means of realization in the gyro horizon compass of the condition of aperiodic transitions by means of selection of appropriate parameters of the instrument.

In inertial systems one of basic elements is gyro-stabilized platform. Depending upon location of GSP, we distinguish:

a) INS with gyro-stabilized platform oriented along axes of terrestrial system of coordinates; here GSP coincides with plane of horizon and retains its assigned azimuthal direction;

b) INS with GSP oriented along axes of inertial system of coordinates; here GSP retains constant position in inertial space, which is connected with the motionless stars.

Basic characteristic of INS is method of construction of vertical; depending upon this, we distinguish the following INS:**

a) with gyroscopic vertical;

b) with mechanical (geometric) vertical erector;

c) with analytic vertical erector.

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*By terrestrial system of coordinates we will understand the coordinate system, two axes of which are located in plane of geocentric horizon, and whose third axis is directed along radius of Earth, i.e., along geocentric vertical. Orientation of axes of inertial system of coordinates will be stipulated further on.

**The division of inertial systems into the types shown in Par. a and b is to a certain extent arbitrary, since the gyroscope also is a mechanical device. However, here there is stressed the circumstance that the vertical in one case (Par. a) is reproduced with the help of gyroscopes, and in the other (Par. b) with the help of an erector without the use of gyroscopes.

In INS with gyroscopic vertical, accelerometers (usually two) are installed on GSP and are connected with it by an integral correction circuit, thanks to which the GSP is oriented in terrestrial system of coordinates, i.e., preserves its horizontal position and also its assigned orientation in azimuth. Parameters of motion of the object and its coordinates are determined analytically with help of a computer. Considering what has been said, such an inertial system sometimes is called semianalytic [121].

In INS with mechanical (geometric) vertical erector, accelerometers are installed on the platform, which composes together with them a vertical erector in which the vertical is reproduced on the basis of readings of the accelerometers without the application of gyroscopes. The GSP in the system is oriented along axes of inertial system of coordinates; GSP is used for "memorization" of position of vertical at the initial moment.* Coordinates of object are determined by means of measurement of angles of turn of erector of vertical relative to GSP. There to this, such an inertial system is sometimes called geometric [121].

In INS with analytic erector of vertical, accelerometers (usually three) are installed on the GSP, which is oriented along axes of inertial system of coordinates. Corrections in readings of accelerometers for accelerations due to forces of gravitation are taken into account when necessary with help of external correction or by means of autocompensation. On the basis of accelerations measured by accelerometers and corrected for accelerations due to forces of gravitation, there is produced determination of vertical by the analytic method, which is used also for finding of coordinates of the object. Therefore, the considered inertial system is sometimes called analytic [121, 142].

In an INS with gyroscopic vertical, axes of sensitivity of accelerometers can have different orientation in plane of horizon, i.e., different azimuthal orientation. Depending upon it, we distinguish the following INS:

a) with geographic orientation;

b) with orthodromic orientation;

c) with free orientation in azimuth;

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*GSP are not always used in INS with mechanical vertical erector for "memorization" of plane of horizon at the initial moment. If the GSP has equatorial orientation, then it indicates plane of the equator.

d) with compass and arbitrary orientations of axes.

In INS with geographic orientation axes of sensitivity of accelerometers are directed along north-south and west-east axes, and determine components of acceleration of the object along these axes. By means of subsequent integration of these accelerations and further mathematical transformations with help of a computer, we obtain latitude and longitude of position of object.

In INS with orthodiomic orientation, axes of sensitivity of accelerometers are directed along axes of assigned orthodrome of motion of the object and perpendicular to it and determine components of acceleration of the object along these axes. Subsequent integration of these accelerations and mathematical transformations with the help of a computer make it possible to determine path passed over by the object along the assigned orthodrome, and also shift of object in direction perpendicular to the orthodrome.

In INS with free orientation in azimuth, axes of sensitivity of accelerometers have in plane of horizon constant orientation relative to inertial space. Thus they will revolve relative to plane of meridian around the local vertical with angular velocity $\dot{\rho}$ expressed by relationship (1.2.28)

$$\dot{\rho} = -u_z = -\left(U \sin\varphi + \frac{v_E}{R} \operatorname{tg}\varphi\right). \tag{11.6.8}$$

Accelerometers determine components of acceleration of the object along axes of inertial system of coordinates. According to readings of accelerometers the computer produces shifts of the object along the given inertial axes.

For transition from a system of coordinates which is free in azimuth to a geographic system of coordinates, we must take into account angle ρ, which is determined, according to (11.6.8), by expression

$$\rho = -\int_0^t \left(U \sin\varphi + \frac{v_E}{R} \operatorname{tg}\varphi\right) dt. \tag{11.6.9}$$

In INS with compass orientation, axes of sensitivity of accelerometers are directed along the axis which forms the angle of velocity error (§ 11.5, Par. 4b) with the plane of the true meridian, and along the axis perpendicular to it.

In INS with arbitrary orientation, axes of sensitivity of accelerometers can rotate in plane of horizon according to an arbitrarily assigned law.

Depending upon period of oscillations of inertial vertical, we distinguish the following INS:

a) with period of natural oscillations of vertical of 84.4 minutes;

b) with period of natural oscillations of vertical differing from 84.4 min.

By means of us of external information, in principle it is possible to create an inertial system with period of natural oscillation of vertical differing from 84.4 min. which retains the property of undisturbability by accelerations of the object.

Finally, in dependence upon the method of damping of natural oscillations of the vertical, we can distinguish the following INS:

a) with damping of oscillations of vertical with the help of internal forces;

b) with damping of oscillations of the vertical by means of use of external information.

The given division of INS has basically a methodological purpose and is given to make the subsequent account more systematic.

c) Methods of Construction of Vertical in INS

As was shown, a basic characteristic of INS is method of construction in it of the local vertical. Depending upon this, we distinguish inertial systems with gyroscopic vertical, with mechanical (geometric) vertical erector, with analytic vertical erector.

Let us turn to characteristics of corresponding schemes of INS and to a short mathematical explanation of them.

Inertial system with gyroscopic vertical. This type of INS is characterized by the fact that the vertical in it is constructed with the help of gyroscopes by means of artificial simulation of an undisturbed physical pendulum with period of natural oscillations of 84.4 minutes according to the scheme of Ye. B. Levental', which consists of gyroscopes, accelerometers and integrators and constitutes a closed dynamic system. On the GSP there are installed accelerometers (usually two) which measure horizontal accelerations of the object; the latter are integrated, and corresponding signals are introduced to torquers of the GSP; due to this it precesses following after the local vertical; i.e., it retains its horizontal position. As was indicated, such a method of correction of the GSP has the name of integral correction. By means of selection of corresponding transmission coefficients of

the system, it is possible in practice to realize the condition of undisturbability of the vertical by horizontal accelerations of the object. The obtained gyroscopic vertical by horizontal accelerations of the object. The obtained gyroscopic vertical, which is frequently called the inertial vertical, is the basis of the considered type of INS.

For explanation of principle of construction of gyroscopic vertical with use of integral correction, we will use a simple scheme of gyrovertical (Fig. 2.16), which is based on three-degree-of-freedom astatic pendulum gyroscope, often modifying it and somewhat supplementing it with new elements.

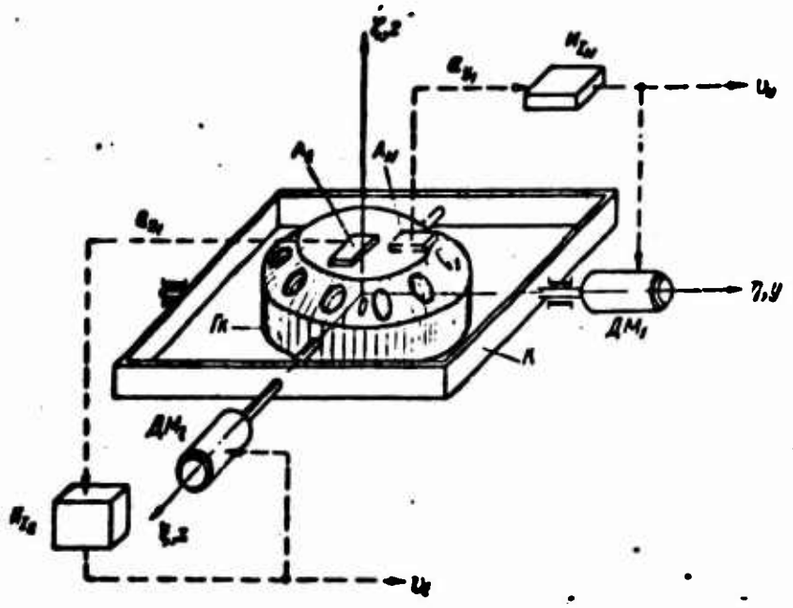

Fig. 11.22. Fundamental scheme of undisturbable GV with integral correction.

Fundamental scheme of undisturbable GV with integral correction, which simulates physical pendulum with period of 84.4 minutes, is shown in Fig. 11.22. Let us assume that as the gyroscopic device there is used a sufficiently "powerful" astatic gyroscope with three degrees of freedom. On its gyrohousing there are located accelerometers A_N and A_E. Since axes $O\xi\eta\zeta$ will be considered to be criented geographically (axis $O\eta$ — to the north, axis $O\xi$ — to the east, axis $O\zeta$ — along the geocentric vertical, i.e., along radius of the Earth), then axis of sensitivity of accelerometer A_N will be directed towards north, and that of accelerometer A_E — to the east. Initial position of axes Oxyz, which are connected with gyrorotor, and of axes $O\xi\eta\zeta$ is shown in Fig. 11.22. Position of axis Oz of gyroscope relative to vertical direction $O\zeta$ assigned to it, or, which is the same, position of Resal axes Ox_1y_1z (Fig. 1.6) relative to $O\xi\eta\zeta$ is, determined by angles α and β, which are errors

of the GV.

For composition of equations of motion of the GV we will use shortened Euler equations (1.6.9):

$$Hq_1 = M_{x_1}; \quad -Hp_1 = M_{y_1}. \tag{11.6.10}$$

where for p_1 and q_1, taking into account induced motion of axes $O\xi\eta\zeta$, and considering α and β to be small, we have

$$\left.\begin{aligned} p_1 &= -\dot{\beta} + u_\xi - u_\zeta\alpha \\ q_1 &= \dot{\alpha} + u_\eta - u_\zeta\beta \end{aligned}\right\} \tag{11.6.11}$$

Projections u_ξ, u_η, u_ζ of induced angular velocity $\vec{u}$ of axes $O\xi\eta\zeta$ for geographic orientation of axes $O\xi$ and $O\eta$ are determined by relationships (1.2.25).

For clarification of principle of integral correction of a GV we will consider the Earth to be non-rotating; then, according to (1.2.25), we will obtain*

$$u_\xi = -\frac{v_N}{R}; \quad u_\eta = \frac{v_E}{R}; \quad u_\zeta = \frac{v_E}{R}\operatorname{tg}\varphi. \tag{11.6.12}$$

Putting (11.6.12) in (11.6.11) and disregarding for not too large latitudes the small terms $u_\zeta\alpha$ and $u_\zeta\beta$, we have approximately

$$p_1 = -\dot{\beta} - \frac{v_N}{R}; \quad q_1 = \dot{\alpha} + \frac{v_E}{R}. \tag{11.6.13}$$

As torques M_{x_1} and M_{y_1} we will consider only torques $M_{cor\ x}$ and $M_{cor\ y}$, which are applied to the GV by the system of integral correction, i.e.,

$$M_{x_1} = M_{kx}; \quad M_{y_1} = M_{ky}. \tag{11.6.14}$$

Let us find expressions for torques $M_{cor\ x}$ and $M_{cor\ y}$. For this, in accordance with the diagram in Fig. 11.22, it is necessary to know accelerations determined by accelerometers A_N and A_E; then correcting torques $M_{cor\ x}$ and $M_{cor\ y}$ will be proportional to integrals of readings of the accelerometers. Accelerations a_{x_1} and a_{y_1}, which are measured by accelerometers along Resal axes Ox_1 and Oy_1 (Fig. 1.6), if we consider matrix (6.2.3), will be determined by relationships

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* For illustration of the basic properties of an inertial vertical, the Earth is taken as a sphere with radius R; consideration of the non-spherical nature of the Earth is discussed further on.

$$a_{x_1} = W_\xi - (W_\zeta + g_0)\alpha; \quad a_{y_1} = W_\eta - (W_\zeta + g_0)\beta. \tag{11.6.15}$$

where W_ξ, W_η, W_ζ – components of total linear acceleration of point O;

g_0 – acceleration due to force of terrestrial gravitation.

Since during horizontal motion of the object $W_\zeta \ll g_0$, then instead of (11.6.15) we will obtain

$$a_{x_1} = W_\xi - g_0\alpha; \quad a_{y_1} = W_\eta - g_0\beta. \tag{11.6.16}$$

Disregarding components of induced and Coriolis accelerations, according to (2.2.28), for W_ξ and W_η we have approximately

$$W_\xi \approx \dot{v}_E, \quad W_\eta \approx \dot{v}_N, \tag{11.6.17}$$

where $\dot{v}_E$ and $\dot{v}_N$ are components of acceleration of object relative to Earth.

Considering (11.6.17), we will rewrite (11.6.16) in the form

$$a_{x_1} = \dot{v}_E - g_0\alpha \quad a_{y_1} = \dot{v}_N - g_0\beta. \tag{11.6.18}$$

Accelerations a_{x_1} and a_{y_1} proceed to integrators I_{I_E} and I_{I_N} (Fig. 11.22); output signals from integrators are introduced into torquers TQ_2 and TQ_1. The latter apply to the gyroscope corrective torques $M_{cor\ x}$ and $M_{cor\ y}$, which, if we consider what has been said and relationship (11.6.18), will be determined by the following expressions:

$$\left.\begin{aligned} M_{cx} &= \mu \int_0^t a_{x_1} d\tau = \mu \int_0^t (\dot{v}_E - g_0\alpha)\, d\tau \\ M_{cy} &= \mu \int_0^t a_{y_1} d\tau = \mu \int_0^t (\dot{v}_N - g_0\beta)\, d\tau \end{aligned}\right\} \tag{11.6.19}$$

where μ is some proportionality factor.

Taking into account (11.6.13), (11.6.14) and (11.6.19), we will rewrite equations (11.6.10) in the form

$$\left.\begin{aligned} \dot{\alpha} + \frac{v_E}{R} &= \frac{\mu}{H} \int_0^t (\dot{v}_E - g_0\alpha)\, d\tau \\ \dot{\beta} + \frac{v_N}{R} &= \frac{\mu}{H} \int_0^t (\dot{v}_N - g_0\beta)\, d\tau \end{aligned}\right\} \tag{11.6.20}$$

whence after differentiation we will obtain

$$\left.\begin{aligned}\ddot{\alpha}+\frac{\mu}{H}g_0\alpha&=\dot{v}_E\left(\frac{\mu}{H}-\frac{1}{R}\right)\\ \ddot{\beta}+\frac{\mu}{H}g_0\beta&=\dot{v}_N\left(\frac{\mu}{H}-\frac{1}{R}\right)\end{aligned}\right\} \qquad (11.6.21)$$

If the following condition is satisfied:

$$\frac{\mu}{H}=\frac{1}{R} \qquad (11.6.22)$$

then instead of (11.6.21) we have

$$\left.\begin{aligned}\ddot{\alpha}+\frac{g_0}{R}\alpha&=0\\ \ddot{\beta}+\frac{g_0}{R}\beta&=0\end{aligned}\right\} \qquad (11.6.23)$$

It follows from this that when condition (11.6.22) is satisfied, the GV is not disturbed by horizontal accelerations of the object. Frequency of natural sustained oscillations of GV

$$\nu=\sqrt{\frac{g_0}{R}}; \qquad (11.6.24)$$

period $T_{i.v.}$ of oscillations of GV, i.e., of the inertial vertical, will be

$$T_{i.v.}=\frac{2\pi}{\nu}=2\pi\sqrt{\frac{R}{g_0}}=5064 \text{ sec.}=84{,}4 \text{ min.} \qquad (11.6.25)$$

Consequently, integral correction, when there is satisfied the condition of undisturbability (11.6.22), provides simulation of the pendulum of M. Schuler. If at the initial moment axis Oz of the gyroscope is established exactly along the geocentric vertical $O\zeta$, i.e., at t = 0 $\alpha(0) = \beta(0) = 0$ and $\dot{\alpha}(0) = \dot{\beta}(0) = 0$ then in the future the axis of the gyroscope will follow the indicated vertical independently of the accelerations of the object. If initial conditions differ from zero, then axis of gyroscope accomplishes sustained oscillations with the period of M. Schuler near the vertical.

Besides erection of the vertical, in the considered scheme after the first integrators we obtain northern v_N and eastern v_E components of velocity of the object. This permits us (by means of application in the inertial system of second integrators and computer), by the same method as in the scheme in Fig. 11.21, to determine latitude φ and longitude λ of position of the object and its heading.

In the above considered case, we for simplicity did not consider diurnal rotation of the Earth. If we take it into account, then the inertial vertical will have certain velocity (systematic) errors, which in systems of inertial navigation usually are compensated. Analysis of these errors is given further on.

In order to realize undisturbability of the inertial vertical by horizontal accelerations of the object, it is necessary to satisfy condition (11.6.22), i.e., to select corresponding value of transmission coefficient μ; this is fully possible in practice. In distinction from this, the condition of undisturbability of GV of the gyropendulum type is difficult to satisfy (§ 11.4, Par. 4). This fact constitutes one of the essential advantages of inertial verticals over verticals of the gyropendulum type. We will note that equations (11.6.23) of motion of inertial vertical (Fig. 11.22) coincide with equations (11.4.71) of natural oscillations of the gyropendulum for which velocity and ballistic errors are compensated.

More detailed analysis of inertial system with GV will be given further on.

Inertial system with mechanical (geometric) vertical erector. This type of inertial system is characterized by the fact that the local vertical in it is obtained not with the help of gyroscopes, but with help of a geometric erector, on the platform of which there are located accelerometers. The gyroscopic device used in the system, in the form of a GSP, is oriented along the axes of the inertial system of coordinates. Thus, the GSP is used for "memorization" of the position of the vertical at the initial moment.

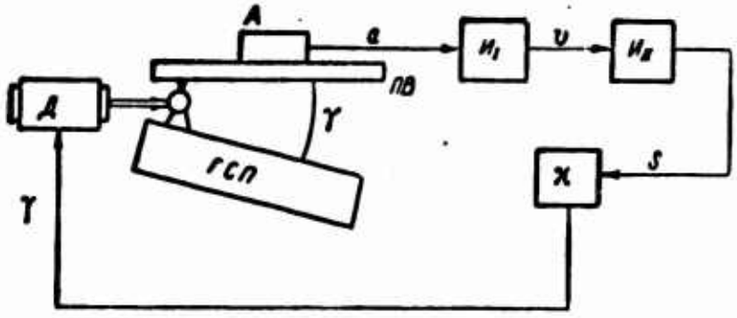

Fig. 11.23. Fundamental diagram of mechanical VE.

Let us consider the principle of operation of the mechanical (geometric) erector of the vertical (VE) and clarify the condition of undisturbability of the system by accelerations of the object. For simplicity we will consider erection of the vertical only in one plane. Let us assume that the object moves along the

arc of a great circle, i.e., along an orthodrome; rotation of the Earth we will not consider in the beginning.

Simplified fundamental scheme of mechanical erector of the vertical is shown in Fig. 11.23. On the platform of the VE there is installed an accelerometer A, axis of sensitivity of which is directed along trajectory of motion of the object. The signal taken from the accelerometer proceeds to the first and second integrators I_I and I_{II} and further, through the computer, to motor Mo; the latter rotates the platform of the VE relative to the GSP by angle γ, which is proportional to the second integral of the horizontal accelerations of the object (i.e., to path passed over by the object S), if we assume that at the initial moment both platforms are horizontal.

Actually, if at the initial moment platform of VE coincides with the GSP and occupies horizontal position, then acceleration measured by the accelerometer a is equal to horizontal acceleration $\dot{v}$ of the object:

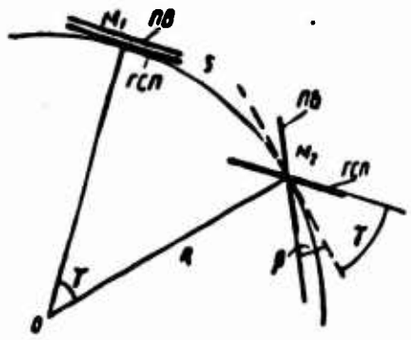

Fig. 11.24. Position of VE and GSP.

$$a = \dot{v}. \tag{11.6.26}$$

At the output of first and second integrators respectively we will obtain

$$v = \int_0^t \dot{v}\,d\tau, \quad S = \int_0^t\int_0^t \dot{v}\,d\tau^2, \tag{11.6.27}$$

where S is path passed over by the object.

Angle γ of turn of platform of the VE relative to the GSP, according to the diagram in Fig. 11.23, will be

$$\gamma = x\int_0^t\int_0^t \dot{v}\,d\tau^2, \tag{11.6.28}$$

where x is proportionality factor.

From Fig. 11.24 it follows that in order to make platform of VE horizontal during motion of the object, angle γ should be determined by relationship

$$\gamma = \frac{S}{R}. \tag{11.6.29}$$

Then, according to (11.6.28) and (11.6.29), we have

$$x\int_0^t\int_0^t \dot{v}\,d\tau^2 = \frac{S}{R}$$

or, considering (11.6.27),

$$\varkappa = \frac{1}{R}. \tag{11.6.30}$$

Consequently, when condition (11.6.30) is satisfied, platform of VE will remain horizontal, independently of accelerations of the object; i.e., equality (11.6.30) is the condition of undisturbability of the VE by horizontal accelerations of the object.

It is easy to show that the VE constitutes an oscillatory system with period of 84.4 minutes; i.e., it simulates the pendulum of M. Schuler. Let us assume that at initial point M_1 (Fig. 11.24) platform of the VE and the GSP coincide with plane of horizon. At a certain current point M_2 the GSP retains constant its initial position in inertial space, but platform of VE deviates from plane of horizon (depicted at point M_2 of the dotted line) by angle β. Then the reading of the accelerometer, by analogy with (11.6.16), will be

$$a = \dot{v} - g_0\beta.$$

At the output of the second integrator we will obtain the quantity

$$\int_0^t\int_0^t (\dot{v} - g_0\beta)\, d\tau^2,$$

which should be proportional to the angle between platform of VE and GSP; the latter is equal to $(\gamma + \beta)$. Then, by analogy with (11.6.28), we have

$$\gamma + \beta = \varkappa \int_0^t\int_0^t (\dot{v} - g_0\beta)\, d\tau^2 \tag{11.6.31}$$

or, considering (11.6.29),

$$\beta + \frac{S}{R} = \varkappa \int_0^t\int_0^t (\dot{v} - g_0\beta)\, d\tau^2. \tag{11.6.32}$$

After differentiating twice we obtain

$$\ddot{\beta} + \frac{\ddot{S}}{R} = \varkappa(\dot{v} - g_0\beta)$$

or, since $\ddot{S} = \dot{v}$,

$$\ddot{\beta} + \varkappa g_0\beta = \dot{v}\left(\varkappa - \frac{1}{R}\right). \tag{11.6.33}$$

If condition (11.6.30) is satisfied, then instead of (11.6.33) we will obtain equation of sustained oscillations of platform of VE with the period of M. Schuler

[see (11.6.25)]

$$\beta + \frac{g_0}{R}\beta = 0. \qquad (11.6.34)$$

Consequently, when condition (11.6.30) is satisfied, the VE is undisturbable by accelerations of the object, i.e., in inertial systems with mechanical (geometric) VE, there is artificially simulated the pendulum of M. Schuler, just as in inertial systems with GV. If at initial moment the platform of the VE is horizontal and in the system there is satisfied condition (11.6.30), then subsequently the platform; independently of accelerations of the object, will retain its horizontal position, i.e., we obtain a mechanical, or geometric, erector of the local vertical.

In an actual system on the platform of the VE there have to be installed two accelerometers, axes of sensitivity of which are mutually perpendicular. Furthermore, platform of VE should ensure the given orientation of axes of sensitivity of accelerometers in plane of horizon. Consequently, inertial system with mechanical (geometric) VE should have two Cardan suspensions: one for installation of platform of the VE, and the other for suspension of the GSP. Let us note that in an inertial system with gyroscopic vertical there is applied only one Cardan suspension.

In the considered inertial system, there are made very high requirements of the accuracy of the GSP. Here the gyroscopic system operates under more complicated conditions, which are caused by the variable action of gravitational field of the Earth, since the GSP should retain constant orientation in inertial space. A considerable increase of accuracy of GSP can be obtained in the given system with the help of so-called astro-correction. For this, on the GSP we install two telescopes with servo systems, which correct position of the GSP by means of sighting of celestial bodies. In this case, requirements of accuracy of the GD can be lower than in a gyroinertial navigational system.

In an inertial system with mechanical (geometric) VE coordinates of the object are determined by means of measurement of angles of turn of platform of VE relative to the GSP. In reference to the scheme represented in Fig. 11.23, this can be explained in the following way: Let us assume that the object moves in plane of meridian; then the path S passed over by it is proportional to change $\Delta\varphi$ of local latitude:

$$S = R\Delta\varphi, \tag{11.6.35}$$

or, if we consider (11.6.29) (see Fig. 11.24),

$$\Delta\varphi = \gamma, \tag{11.6.36}$$

i.e., angle γ of turn of platform of VE relative to GSP (Fig. 11.23) is equal to change of latitude $\Delta\varphi$ of position of the object. Therefore, quantity $\Delta\varphi$ is directly taken from the VE. In connection with this, the considered inertial system is sometimes called geometric [121]. Application in it of GSP provides a reference for measurement of angle γ of turn of platform of the VE, which is proportional to path passed over by the object. Question about determination of coordinates of an object in the general case of its motion with the help of inertial system of the geometric type will be considered further on.

Inertial system with analytic vertical erector. In this system with analytic vertical erector. In this system there is applied a GSP oriented along axes of inertial system of coordinates; on the GSP there are installed three accelerometers. Readings of accelerometers are corrected for accelerations due to forces of gravitation with the help of external correction of auto compensation. Erection of the vertical is carried out by the analytic method. Coordinates of object are determined on the basis of readings of accelerometers corrected for accelerations due to forces of gravitation.

Let us consider the principle of action of the analytic VE. For simplicity we will consider that the object moves along the arc of a great circle; rotation of the Earth will not be considered. In Fig. 11.25 there are shown axes $Ox_0y_0z_0$ of the inertial system of coordinates, relative to which the GSP maintains constant orientation. In the considered case we assume that on the GSP there are installed two accelerometers, axes of sensitivity of which coincide respectively with axes Oy_0 and Oz_0. On the same figure there are shown axes $O\xi\eta\zeta$ of the terrestrial system of coordinates (axis $O\zeta$ — the geocentric vertical — is directed along the radius of the Earth); angle of rotation of axes $O\xi\eta\zeta$ relative to $Ox_0y_0z_0$ we will designate by γ_0. Let us introduce also axes Oxyz, which are turned relative to terrestrial axes $O\xi\eta\zeta$ by angle β; and relative to inertial axes $Ox_0y_0z_0$ by angle γ; axis Oz is the instrument vertical, which is reproduced by the analytic VE; i.e., Oz is the "analytic" vertical. Consequently, angle β is error of analytic VE; thus,

$$\gamma = \gamma_0 + \beta. \tag{11.6.37}$$

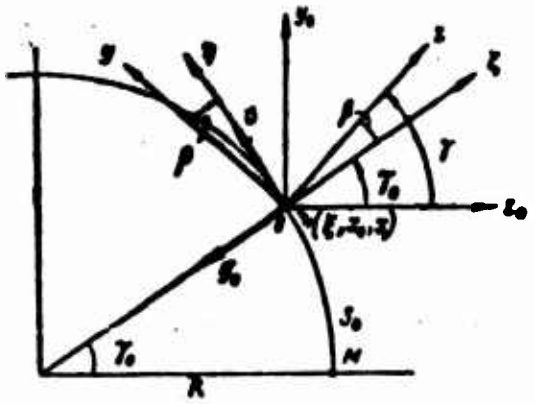

Fig. 11.25. Erection of analytic vertical.

In the considered case the object moves with acceleration $\dot{v}$ along axis $O\eta$; acceleration due to force of terrestrial gravitation directed along axis $O\zeta$ will be designated by g_0.

We will compose equation of motion of inertial mass m of accelerometer relative to the motion system of coordinates:

$$m\dot{\vec{v}} = m\vec{g}_0 + \vec{f},$$

where $m\vec{g}_0$ — force of terrestrial gravitation;

$\vec{f}$ — force of elastic deformation of spring connecting mass m with housing of accelerometer.

We will designate acceleration measured by the accelerometer by $\vec{a}$; it is proportional to force $\vec{f}$, i.e., $\vec{a} = \frac{1}{m}\vec{f}$. Then the above equation can be written in the form

$$\vec{a} = \dot{\vec{v}} - \vec{g}_0.$$

By projecting the obtained vector equality onto axes Oy_0 and Oz_0, we find expressions for accelerations a_{y_0} and a_{z_0}, measured by the accelerometers along inertial axes Oy_0 and Oz_0, in the form

$$a_{y_0} = \dot{v}\cos\gamma_0 + g_0\sin\gamma_0; \quad a_{z_0} = -\dot{v}\sin\gamma_0 + g_0\cos\gamma_0. \tag{11.6.38}$$

Accelerations a_{y_0} and a_{z_0} are projected onto axes Oy and Oz of the analytic VE, and with the help of the computer is there determined angle γ of turn of "analytic" vertical relative to inertial system of coordinates. Really, by projecting a_{y_0} and a_{z_0} into axis Oy, we will obtain acceleration a_y in the form

$$a_y = a_{y_0}\cos\gamma - a_{z_0}\sin\gamma. \tag{11.6.39}$$

By means of double integration of acceleration a_y we find angle γ of turn of analytic vertical relative to its initial position, when the object is at point M, i.e., by analogy with (11.6.28) we have

$$\gamma = x\int_0^t\int_0^t a_y d\tau^2. \tag{11.6.40}$$

where x — proportionality factor.

Angle γ obtained in such a way is used in relationship (11.6.39) for determination of acceleration a_y. Thus, analytic construction in computer of angle γ, which characterizes position of vertical, is carried out on the basis of equations (11.6.39) and (11.6.40) with the bulk of the well known method of a "reduction of balance;" this is directly seen from the block diagram in Fig. 11.26.

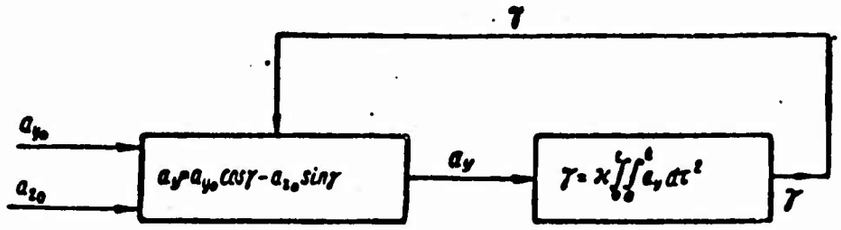

Fig. 11.26. Block diagram of analytic VE.

It is easy to show that analytic VE is an oscillatory system with period of 84.4 minutes, i.e., it simulates the pendulum of M. Schuler. For this we will substitute in (11.6.40) in place of a_y its expression (11.6.39); considering (11.6.38) and (11.6.37), we will obtain for small β the relationship

$$\gamma_0 + \beta = \chi \int_0^t \int_0^t (\dot{v} - g_0 \beta)\, d\tau^2, \qquad (11.6.41)$$

which is analogous to (11.6.31).

Formula (11.6.41), if we take into account the fact that (Fig. 11.25)

$$\gamma_0 = \frac{S_0}{R},$$

can by the same method as (11.6.32) easily be transformed to the form (11.6.33):

$$\ddot{\beta} + \chi g_0 \beta = \dot{v}\left(\chi - \frac{1}{R}\right).$$

If there is satisfied the condition of undisturbability (11.6.30)

$$\chi = \frac{1}{R},$$

then we obtain

$$\ddot{\beta} + \frac{g_0}{R}\beta = 0,$$

i.e., analytic vertical accomplishes sustained oscillations with period of M. Schuler, and consequently, is undisturbed by horizontal accelerations of the object.

Besides erection of the vertical, in the system there is determined path passed over by the object S. Indeed, the value of S worked out by the system is expressed by relationship

$$S = R\gamma, \tag{11.6.42}$$

or, if we consider (11.6.37) and the above formula $\gamma_0 = \frac{S_0}{R}$,

$$S = R(\gamma_0 + \beta) = R\gamma_0 + R\beta = S_0 + R\beta, \tag{11.6.43}$$

where S_0 — true value of path;

$R\beta$ — error in determination of path caused by error β in erection of vertical.

As was shown, in inertial systems with analytic VE, there appears the problem of correction of readings of accelerometers for accelerations due to forces of gravitation. More detail about methods of solution of this problem in the general case of motion of the object will be given further on.

Above there were given characteristics of different methods of construction of the local vertical in inertial systems. In examining of certain questions of theory of inertial verticals and their errors, we will mainly consider inertial systems with GV.

2. Dynamic Characteristics of Inertial Verticals (IV)

a) Preliminary Remark

Dynamic characteristics of IV determined by its differential equations and transfer functions, will be found in reference to an inertial system with gyroscopic vertical, which will subsequently be called an inertial vertical. Orientation of axes will be taken as geographic, in which axes of sensitivity of accelerometers are directed along the north-south and east-west lines.

Geometric and kinematic parameters of IV, if we consider selection of Euler angles according to Fig. 1.6, will be the same as for GV using astatic gyroscope with ordinary (not integral) pendulum correction (§ 6.2, Par. 1). Only expressions for projections u_ξ, u_η, u_ζ of angular velocity $\vec{u}$ of axes $O\xi\eta\zeta$, which for IV are assumed to be oriented geographically, will differ, since for the GV considered in § 6.2 GV they were connected with the trajectory of the object.

In the beginning we will give dynamic characteristics of the accelerometer which is the sensing device of the inertial vertical.

b) Dynamic Characteristics of Accelerometer

Accelerations measured by accelerometer. Accelerometer measures accelerations

caused by non-gravitational forces, among which, for instance in reference to aircraft, there are included: thrust of motors, drag, lift, and others. Accelerations due to forces of gravitation are not measured by the accelerometer, since they act equally on its housing and on its sensitive element. As the sensitive element there is applied a certain mass connected with housing of the instrument with the help of a spring. Therefore, the accelerometer does not register, for example, free fall acceleration of the body. Due to this peculiarity of the accelerometer, inertial systems determine distance passed over by the object (vehicle) under the action of nongravitational forces, but do not take into account distance passed over under the action of forces of gravitation. This leads to the necessity to compensate in inertial systems accelerations due to forces of gravitation.

We will clarify what has been said with several simple formulas; thus, we consider that the object moves near surface of Earth. Let us assume that sensing device of accelerometer is an inertial mass connected with housing of the instrument with the help of a spring. Readings of accelerometer are proportional to displacement x of mass m relative to the housing of the instrument: $x = ka_x$, where a_x is acceleration measured by accelerometer in the direction of input axis Ox, i.e., acceleration of its mass relative to the housing in the direction of this axis. By analogy with the preceding, the equation of motion of mass m of accelerometer relative to motionless system of coordinates in projections onto axis Ox will be written in the form

$$(m\vec{w})_x = (m\vec{g}_0 + \vec{f})_x,$$

or

$$(\vec{f})_x = m(\vec{v} - \vec{g}_0)_x. \qquad (11.6.44)$$

Since $(\vec{f})_x = cx$ (c is stiffness coefficient of spring, and x is displacement of inertial mass), then, designating acceleration measured by accelerometer along axis Ox by $a_x = \frac{c}{m}x = \frac{1}{k}x$, we have

$$a_x = W_x - g_{0x}, \qquad (11.6.45)$$

where $W_x = (\dot{\vec{v}})_x$ — projection of absolute acceleration of object (vehicle) onto axis Ox;

$g_{0x} = (\vec{g}_0)_x$ — projection of acceleration due to force of terrestrial gravitation (gravitational acceleration) onto the same axis.

From formula (11.6.45) it is clear that accelerometer measures so-called

apparent acceleration, i.e., difference between absolute linear acceleration of object and gravitational acceleration. If axis Ox is directed vertically upwards, then $g_{0_x} = -g_0$. During motion of object vertically with acceleration $W_x > 0$ the accelerometer measures acceleration $a_x = W_x + g_0$. During motion of object downwards with acceleration $W_x < 0$, if we designate in this case the projection of acceleration of object onto axis Ox by -W, we will write the expression for acceleration measured by the accelerometer in the form $a_x = -W + g_0$. During free fall of body $|W| = g_0$, and consequently $a_x = 0$, i.e., the accelerometer, as was shown earlier, does not register free fall acceleration of the body.

Let us add to (11.6.44) the equation of motion of the object relative to inertial system of coordinates; this equation can be written in the form

$$M\vec{v} = M\vec{g}_0 + \vec{F},$$

where M – mass of object (vehicle);

$\vec{v}$ – absolute acceleration of center of gravity of object;

$\vec{F}$ – resultant of all non-gravitational forces acting on the object.

From this equation we find

$$\vec{v} - \vec{g}_0 = \frac{\vec{F}}{M}.$$

By substituting the last equality into (11.6.44) and considering (11.6.45), we obtain

$$a_x = W_x - g_{0x} = \left(\frac{\vec{F}}{M}\right)_x.$$

As was shown, the accelerometer measures difference between absolute acceleration of object and gravitational acceleration. This difference $\left(\frac{\vec{F}}{M}\right)_x$ is sometimes called "thrust acceleration," since the greatest of the forces acting, for instance, on an aircraft, is the thrust of the engine.

If, in accordance with what has been said, the axes of sensitivity of the three accelerometers is directed along axes Oξ, Oη, Oζ of coordinate system Oξηζ, then accelerations measured by these accelerometers a_ξ, a_η, a_ζ will be

$$a_\xi = W_\xi - g_{0\xi}; \quad a_\eta = W_\eta - g_{0\eta}; \quad a_\zeta = W_\zeta - g_{0\zeta}. \tag{11.6.46}$$

where W_ξ, W_η, W_ζ – components of absolute acceleration of object along axes Oξηζ;

$g_{0\xi}$, $g_{0\eta}$, $g_{0\zeta}$ – components of gravitational acceleration along the same axes.

From formulas (11.6.46) it directly follows that for determination of components W_ξ, W_η, W_ζ of absolute acceleration of the object along axes $0\xi\eta\zeta$ it is necessary to compensate components $g_{0\xi}$, $g_{0\eta}$, $g_{0\zeta}$ of gravitational acceleration. In INS operating in a terrestrial system of coordinates, this compensation is carried out by means of installation of the accelerometers on a platform gyro-stabilized in the plane of the horizon. In INS operating in inertial system of coordinates, accelerations due to forces of gravity are taken into account with the help of external correction or auto compensation.

In connection with the above noted peculiarity of accelerometers, sometimes we say that they determine not real accelerations of the object, but only apparent accelerations; therefore, after integration of signals from accelerometers we obtain not real velocities and displacements of the object, but apparent. For finding of real values of velocities and displacements of the object, it would be necessary for instance, to introduce into the input of the first integrators correcting signals proportional to corresponding components of accelerations due to forces of gravitation.

Principle of operation of accelerometer and its dynamic characteristics. As was said, sensitive element of accelerometer is the inertial mass, which usually with the help of a spring is connected with housing of the instrument, and consequently also with the body of the object (vehicle). Accelerometers are divided into linear, or axial, and pendulum. In the linear accelerometer, acceleration of the object is determined by deflection of the inertial mass along axis of sensitivity of the accelerometer; in the pendulum accelerometer, acceleration is found by angle of displacement of pendulum. Subsequently we will consider the linear accelerometer, since accelerometers of pendulum type were considered in Chapter 4.

Fundamental scheme of linear accelerometer is shown in Fig. 11.27. Housing of the instrument H is fixed on gyro-stabilized platform. Sensitive element of the accelerometer – mass M – is connected with housing H by spring Sp. Axis 0η, along which mass M can shift is axis of sensitivity, or input axis. Displacement of mass M along axis 0η, which is designated further by x, is taken in the form of a voltage from potentiometer P; consequently, x is the output variable. Acceleration of object along axis 0η will be designated $a_\eta = \ddot{\eta}$; it is for the accelerometer the input variable. For damping of natural oscillations of mass M there serves damper D.

Differential equation of motion of inertial mass M can be written in a form analogous to the equation of motion (4.3.12) of a physical pendulum* (pendulum accelerometer),

$$m\ddot{x} + b\dot{x} + cx = ma_\eta. \tag{11.6.47}$$

where m – magnitude of inertial mass;

b – damping factor;

c – stiffness coefficient of spring.

In this equation there is not contained acceleration due to forces of gravitation, since it is assumed that axis Oη is located strictly horizontally. In (11.6.47) there is also not considered friction or influence of certain other factors, for instance accelerations due to rotation of the object around its center of gravity, the difference between accelerations due to forces of gravitation at the center of gravity of the object and at the place of installation of the accelerometer, and others [142]. Let us represent equation (11.6.47) in the form

$$\ddot{x} + \frac{b}{m}\dot{x} + \frac{c}{m}x = a_\eta. \tag{11.6.48}$$

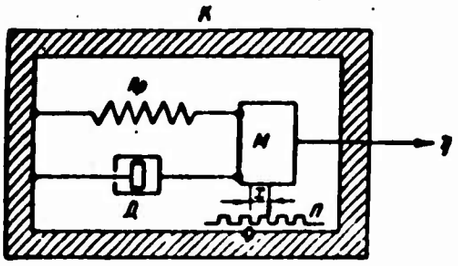

Fig. 11.27. Schematic diagram of linear accelerometer.

Let us designate

$$n^2 = \frac{c}{m}, \tag{11.6.49}$$

where n is frequency of natural sustained oscillations of the accelerometer

$$n = \sqrt{\frac{c}{m}}. \tag{11.6.50}$$

Period of natural oscillations of the accelerometer will be

$$T_{ск} = \frac{2\pi}{n} = 2\pi\sqrt{\frac{m}{c}}. \tag{11.6.51}$$

Ratio b/m will be transformed in the following way:

---

*The positive direction of displacement x of mass M in the presence of acceleration a_η of the object is chosen in such a manner that for $a_\eta > 0$ we have displacement $x > 0$.

$$\frac{b}{m}=\frac{b}{m}\frac{\sqrt{\frac{c}{m}}}{\sqrt{\frac{c}{m}}}=2\zeta n, \tag{11.6.52}$$

where

$$\zeta=\frac{b}{2\sqrt{cm}}; \tag{11.6.53}$$

quantity ζ constitutes relative damping factor.

Considering (11.6.49) and (11.6.52), we rewrite equation (11.6.48) in the following way:

$$\ddot{x}+2\zeta n\dot{x}+n^2x=a_\eta. \tag{11.6.54}$$

Let us represent (11.6.54) in another form; for this, by analogy with (4.3.23), we will intorduce time constant T of the accelerometer

$$T=\frac{1}{n}. \tag{11.6.55}$$

Dividing equation (11.6.54) by n^2 and considering (11.6.55), we have

$$T^2\ddot{x}+2\zeta T\dot{x}+x=T^2a_\eta. \tag{11.6.56}$$

Applying to (11.5.56) the Laplace transform, we will obtain the expression for transfer function of the accelerometer

$$W(s)=\frac{x(s)}{a_\eta(s)}=\frac{T^2}{T^2s^2+2\zeta Ts+1}, \tag{11.6.57}$$

whence it follows that the accelerometer, similarly to the physical pendulum (Chapter 4) and the gyrotachometer (Chapter 8), is an oscillatory network.

In the position of static equilibrium (for α_η = const)

$$x_{ст}=T^2a_\eta, \tag{11.6.58}$$

i.e., displacement of inertial mass of accelerometer relative to its housing is proportional to acceleration a_η of the object.

Considering what has been said, for frequency-response characteristics of the accelerometer we can easily obtain formulas of the same form as for frequency-response characteristics of physical pendulum and gyrotachometer (see § 4.3, Par. 3 and § 8.2, Par. 3).

There exist various types and designs of accelerometers. In recent years

accelerometers of componsational type have come into use, analogous in concept to gyrotachometers with zero drive (§ 8.2, Par. 7). Creation of accelerometers intended for INS is a very complicated problem if we consider that of such accelerometers there are made requirements of high sensitivity, large range of measured accelerations, strict linearity of readings, insensitivity to vibration and shock, and so forth. Different schemes of accelerometers can be met in the literature (see, for instance, [149, 117, 142, 108]).

c) Block Diagram of an IV

As was shown (Par. 1, c), the position of the IV relative to the geocentric vertical is determined by angles α and β. In connection with this, block diagram of IV consists of two identical channels, each of which simulates a plane physical pendulum with period of 84.4 minutes.

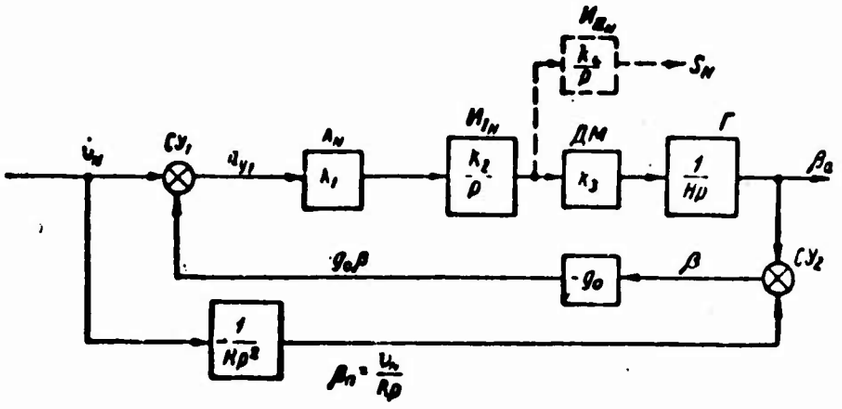

Fig. 11.28. Block diagram of "northern" channel of IV.

Let us consider the block diagram of the "northern" channel, the input variable of which is the northern component of acceleration of the object,* and the output variable of which is angle β of turn of the gyroscope. Block diagram of this channel [142, p. 458] is shown in Fig. 11.28. At the input of the system there enters the northern component of acceleration $\dot{v}_N = pv_N$ ($p = \frac{d}{dt}$ — differential operator), which is measured by the northern accelerometer A_N with transfer coefficient $k_1 = T^2$ [T is time constant of accelerometer, see (11.6.58)]. Readings of the accelerometer enter the first integrator I_{I_N}, the transfer characteristic of which is

*Here there is considered a simplified scheme of IV, in which there are considered only relative accelerations of the object.

$\frac{k_2}{p}$ (k_2 — transfer coefficient of integrator). Output signal from integrator is transmitted to torquer TQ, which has torque transfer coefficient k_3. The TQ applies a corrective torque to gyroscope G, the transfer function of which is $\frac{1}{Hp}$ (H is angular momentum of the gyroscope), i.e., the gyroscope, in accordance with precessional theory, is considered as an integrating network. Under the action of the applied torque the gyroscope precesses, turning relative to inertial space by angle β_a, which enters adder AD_2.

On the diagram the feedback line with transfer function $-\frac{1}{Rp^3}$ characterizes turn of terrestrial system of coordinates relative to the inertial system. At the input of this line there enters acceleration $\dot{v}_N = pv_N$; at the output there is obtained angle $\beta_{turn} = \frac{v_N}{Rp}$, which determines angle of rotation of terrestrial system of coordinates relative to the inertial system. Angle β_{turn} enters adder AD_2, from which there is taken the relative angle $\beta = \beta_a - \beta_{turn}$, i.e., the angle of inclination of the gyroscope together with the accelerometer A_N relative to plane of the horizon. Due to this, into the adder AD, there proceeds component $g_0\beta$ of acceleration due to gravity. As a result, accelerometer A_N registers acceleration $a_{y_1} = \dot{v}_N - g_0\beta$ [see (11.6.18)].

Using the block diagram shown in Fig. 11.28, we can compose the equation of the IV in the form

$$(pv_N - g_0\beta)\frac{k_1k_2k_3}{Hp^2} = \beta_a. \tag{11.6.59}$$

Let us replace β_a by $(\beta_{turn} + \beta)$; considering that $\beta_{turn} = \frac{v_N}{Rp}$, and introducing the designation

$$\mu = k_1k_2k_3, \tag{11.6.60}$$

instead of (11.6.59) we will obtain

$$\left(p^2 + \frac{\mu}{H}g_0\right)\beta = pv_N\left(\frac{\mu}{H} - \frac{1}{R}\right) \tag{11.6.61}$$

or equation

$$\ddot{\beta} + \frac{\mu}{H}g_0\beta = \dot{v}_N\left(\frac{\mu}{H} - \frac{1}{R}\right). \tag{11.6.62}$$

which coincides with the second equation (11.6.21).

If condition (11.6.22) is satisfied, we have

$$\ddot{\beta} + \frac{g_0}{R}\beta = 0, \qquad (11.6.63)$$

i.e., the IV is not disturbed by horizontal accelerations of the object (vehicle).

d) Equations of motion of IV

Composition of differential equations of motion of an IV, which consists in the considered case of a gyro-stabilized platform, accelerometers and integrators, is a comparatively complicated problem [142, 162, 165]. However, in the first approximation, for demonstration of the basic properties of an IV it is possible to be limited to differential equations of motion of simple IV, the fundamental scheme of which is shown in Fig. 11.22. Description of this scheme and composition of equations on the assumption that the Earth does not rotate was given in Par. 1 of the present paragraph.

For composition of equations of motion of IV taking into account rotation of the Earth and all accelerations acting on the IV we will use the shortened equations of Euler (11.6.10), in which p_1 and q_1 are expressed by formulas (11.6.11). In the latter, projections u_ξ, u_η, u_ζ of induced angular velocity $\vec{u}$ of axes $O\xi\eta\zeta$ for the geographic orientation of axes $O\xi$ and $O\eta$ assumed by us are determined by relationship (1.2.25). As torques M_{x_1} and M_{y_1} we will consider only torques $M_{cor\ x}$ and $M_{cor\ y}$, which are applied to the IV by the system of integral correction; i.e., $M_{x_1} = M_{cor\ x}$, $M_{y_1} = M_{cor\ y}$ [see (11.6.14)]; thus $M_{cor\ x}$ and $M_{cor\ y}$ will be considered to be proportional to the integrals of readings of the accelerometers a_{x_1} and a_{y_1}. The latter determined for $W_\zeta \ll g_0$ by relationships (11.6.16). By substituting in them in place of W_ξ and W_η components of total accelerations of the object along axes $O\xi$ and $O\eta$, which are oriented geographically, from formulas (2.2.28) we will obtain for readings a_{x_1} and a_{y_1} of the accelerometers the following expressions (for $R \gg h$):

$$\left.\begin{aligned} a_{x_1} &= \dot{v}_E - \frac{v_E v_N}{R}\operatorname{tg}\varphi - 2Uv_N\sin\varphi - g_0\alpha \\ a_{y_1} &= \dot{v}_N + \frac{v_E^2}{R}\operatorname{tg}\varphi + U^2R\cos\varphi\sin\varphi + \\ &\quad + 2Uv_E\sin\varphi - g_0\beta \end{aligned}\right\} \qquad (11.6.64)$$

Then for torques $M_{cor\ x}$ and $M_{cor\ y}$ of integral correction of the IV, by analogy with formulas (11.6.19), we will obtain

$$\left.\begin{aligned} M_{xz} &= \mu \int_0^t a_{x_1} d\tau = \mu \int_0^t \left(\dot{v}_E - \frac{v_E v_N}{R} \operatorname{tg} \varphi - \right. \\ &\left. - 2U v_N \sin \varphi - g_0 \alpha \right) d\tau \\ M_{zy} &= \mu \int_0^t a_{y_1} d\tau = \mu \int_0^t \left(\dot{v}_N + \frac{v_E^2}{R} \operatorname{tg} \varphi + \right. \\ &\left. + U^2 R \cos \varphi \sin \varphi + 2U v_E \sin \varphi - g_0 \beta \right) d\tau \end{aligned}\right\} \qquad (11.6.65)$$

where μ is a certain proportionality factor, taken to be indentical for both correction circuits of the IV.

Let us introduce (1.2.25) into (11.6.11):

$$\left.\begin{aligned} p_1 &= -\beta - \frac{v_N}{R} - \left(U \sin \varphi + \frac{v_E}{R} \operatorname{tg} \varphi \right) \alpha \\ q_1 &= \dot{\alpha} + U \cos \varphi + \frac{v_E}{R} - \left(U \sin \varphi + \frac{v_E}{R} \operatorname{tg} \varphi \right) \beta \end{aligned}\right\} \qquad (11.6.66)$$

Disregarding in (11.6.66), for not too large latitudes φ, the last two small terms, we will obtain

$$\left.\begin{aligned} p_1 &= -\beta - \frac{v_N}{R} \\ q_1 &= \dot{\alpha} + U \cos \varphi + \frac{v_E}{R} \end{aligned}\right\} \qquad (11.6.67)$$

Let us place (11.6.67) and (11.6.65) in (11.6.10):

$$\left.\begin{aligned} \dot{\alpha} + U \cos \varphi + \frac{v_E}{R} &= \frac{\mu}{H} \int_0^t \left(\dot{v}_E - \frac{v_E v_N}{R} \operatorname{tg} \varphi - \right. \\ &\left. - 2U v_N \sin \varphi - g_0 \alpha \right) d\tau \\ \beta + \frac{v_N}{R} &= \frac{\mu}{H} \int_0^t \left(\dot{v}_N + \frac{v_E^2}{R} \operatorname{tg} \varphi + U^2 R \cos \varphi \sin \varphi + \right. \\ &\left. + 2U v_E \sin \varphi - g_0 \beta \right) d\tau \end{aligned}\right\} \qquad (11.6.68)$$

whence after differentiation, considering (1.2.26), we find

$$\left.\begin{aligned}
\ddot{\alpha}+\frac{\mu}{H}g_0\alpha &= \left(\dot{v}_E - Uv_N\sin\varphi\right)\left(\frac{\mu}{H}-\frac{1}{R}\right)- \\
&-\frac{\mu}{H}\left(\frac{v_E v_N}{R}\operatorname{tg}\varphi + Uv_N\sin\varphi\right) \\
\ddot{\beta}+\frac{\mu}{H}g_0\beta &= \dot{v}_N\left(\frac{\mu}{H}-\frac{1}{R}\right)+ \\
&+\frac{\mu}{H}\left(\frac{v_E^2}{R}\operatorname{tg}\varphi + U^2R\cos\varphi\sin\varphi + 2Uv_E\sin\varphi\right)
\end{aligned}\right\} \qquad (11.6.69)$$

When the condition of undisturbability (11.6.22) is satisfied, we will obtain equations of the IV in the form

$$\left.\begin{aligned}
\ddot{\alpha}+\frac{g_0}{R}\alpha &= -\frac{1}{R}\left(\frac{v_E v_N}{R}\operatorname{tg}\varphi + Uv_N\sin\varphi\right) \\
\ddot{\beta}+\frac{g_0}{R}\beta &= \frac{1}{R}\left(\frac{v_E^2}{R}\operatorname{tg}\varphi + U^2R\cos\varphi\sin\varphi + 2Uv_E\sin\varphi\right)
\end{aligned}\right\} \qquad (11.6.70)$$

from which it is clear that equations of motion of the IV even when condition (11.6.22) is satisfied are nonhomogeneous, and consequently the IV, for geographic orientation of axes of sensitivity of the accelerometers, will not be directed along the geocentric vertical. Thus, the IV has a deviation from the geocentric vertical which is caused by rotation of Earth and vehicular motion (i.e., induced and coriolis accelerations), and remains undisturbed only with respect to accelerations $\dot{v}_N$ and $\dot{v}_E$ of the object. These deviations of the IV are usually called its systematic errors. Methods of compensation of these errors in INS are expounded in Par. 8 of the present section.

From what has been said, it follows that in the scheme of IV based on the gyroscopic method of artificial simulation of a physical pendulum with period of 84.4 minutes, in the case of geographic orientation of axes of sensitivity of the accelerometers, there is not ensured complete undisturbability of the IV by accelerations due to rotation of the Earth and motion of the vehicle (object). This circumstance is due to the fact that, as was shown earlier (§ 4.5, Par. 5), the physical pendulum in the general case of motion of the object over the spherical surface of the Earth, even when the condition of M. Schuler is satisfied, will be disturbed by accelerations of the point of its suspension.

Equations (11.6.70) can be rewritten in different form. Let us replace $\frac{g_0}{R}$ by ν^2 [see (11.6.24)]; we will introduce, by analogy with (11.4.113) time constant $T = \frac{1}{\nu}$; the quantity $\frac{1}{g_0} \approx \frac{1}{g}$ will be designated by transfer coefficient k [see

(4.3.24)]; then

$$\left.\begin{aligned} T^2\ddot{\alpha}+\alpha &= -k\left(\frac{v_E v_N}{R}\operatorname{tg}\varphi + Uv_N\sin\varphi\right)\\ T^2\ddot{\beta}+\beta &= k\left(-\frac{v_E}{R}\operatorname{tg}\varphi + U^2R\cos\varphi\sin\varphi + 2Uv_E\sin\varphi\right)\end{aligned}\right\} \qquad (11.6.71)$$

In the general case, equations of the IV, if we consider also deflection of the GSP in azimuth, are obtained to be more complicated [162, 165, 142]. However, if we disregard certain small terms, then they become equation (11.6.70).

3. Transient Responses of the IV

a) Transient Responses in the Absence of Damping

In the absence of damping, if we assume that systematic errors of the IV are compensated, instead of (11.6.70) we will obtain differential equations (11.6.23) of oscillations of the IV. Since motions of the vertical along coordinates α and β are of the same kind, then we will consider only transient response with respect to coordinate β, which characterizes deviation of the IV from the geocentric vertical in plane of meridian. We will rewrite the second equation (11.6.23), considering (11.6.24), in the form

$$\ddot{\beta} + \nu^2\beta = 0, \qquad (11.6.72)$$

where ν is the frequency corresponding to the period of M. Schuler.

General solution of equation (11.6.72) has the form

$$\beta = C_1\cos\nu t + C_2\sin\nu t, \qquad (11.6.73)$$

where C_1 and C_2 are constants of integration determined by initial conditions.

If we assume that at $t = 0$, $\beta(0) = \beta_0$, $\dot{\beta}(0) = \dot{\beta}_0$, then in place of (11.6.73) we will obtain

$$\beta = \beta_0\cos\nu t + \frac{\dot{\beta}_0}{\nu}\sin\nu t. \qquad (11.6.74)$$

Expression (11.6.74) can also be represented in the form

$$\beta = B\cos(\nu t + \varepsilon), \qquad (11.6.75)$$

where amplitude of oscillations B and initial phase ε are determined by relationships

$$B = \sqrt{\beta_0^2 + \frac{\dot{\beta}_0^2}{\nu^2}}; \quad \operatorname{tg}\varepsilon = -\frac{\dot{\beta}_0}{\nu\beta_0}. \qquad (11.6.76)$$

Thus, in the absence of damping, the IV accomplishes harmonic oscillations with frequency ν, whose amplitude and initial phase depend on errors β_0, $\dot{\beta}_0$ is the initial orientation of the IV. These oscillations do not appear only in the case when $\beta_0 = 0$, $\dot{\beta}_0 = 0$, i.e., when at the initial moment the IV is directed along the geocentric vertical; thus, during the entire time of motion $\beta(t) \equiv 0$. As will be shown further on, oscillations $\beta(t)$ of the IV considerably affect the accuracy of determination of the inertial system of coordinates of the object and parameters of its motion. Therefore, in INS there usually are made very high requirements for accuracy of initial orientation of the IV.

Let us note that β_0 and $\dot{\beta}_0$ up to the moment of start of operation of the system are random variables. If we consider that they are independent and have zero mathematical expactations, then, in accordance with (11.6.74), we obtain the following expression for mathematical expectation $\overline{\beta}$ and root mean square value σ_β of deviation of the IV from the geocentric vertical:

$$\overline{\beta} = 0; \tag{11.6.77}$$

$$\sigma_\beta = \sqrt{\sigma_{\beta_0}^2 \cos^2 \nu t + \frac{\sigma_{\dot{\beta}_0}^2}{\nu^2} \sin^2 \nu t}\,, \tag{11.6.78}$$

where σ_{β_0} and $\sigma_{\dot{\beta}_0}$ are root mean square values of quantities β_0 and $\dot{\beta}_0$.

b) Transient Responses in the Presence of Damping

As was shown above, for non-zero initial conditions the IV accomplishes sustained oscillations (system is on the boundary of stability). Instrument errors of its separate elements also lead to such oscillations of the IV. These oscillations cause corresponding errors in determination by the INS of position of the object and parameters of its motion. In connection with this, during erection of the IV there appears the problem of damping of its natural oscillations.

Methods of damping of IV are varied. Schemes of damping of IV are divided basically into two types: the scheme using internal damping and the scheme using external information.

Let us consider several schemes of internal damping. The simplest scheme of this type was proposed by Ye. B. Levental'. The essence of this method of damping is that

besides the corrective torque which is proportional to the integral of the acceleration of the object measured by the acceleromter, there is applied a torque on the gyroscope which is proportional to the measured acceleration itself.

Let us explain how there is carried out damping, for instance, of oscillations of IV with respect to coordinate β. For this we will use the second equation of system (11.6.10)

$$-Hp_1 = M_{y_1}, \tag{11.6.79}$$

where p_1 is defined by relationship (11.6.13).

We will represent torque M_{y_1} in the form of the sum of corrective torque $M_{cor\ y}$ and damping torque M_d. Corrective torque $M_{cor\ y}$ is expressed by formula (11.6.19); damping torque M_d, in accordance with the applied method of damping, will be [see (11.6.18)]

$$M_д = ba_{y_1} = b\left(\dot{v}_N - g_0\beta\right), \tag{11.6.80}$$

where b is damping factor.

Then for moment M_{y_1}, considering (11.6.19) and (11.6.80), we have

$$M_{y_1} = \mu\int_0^t\left(\dot{v}_N - g_0\beta\right)dt + b\left(\dot{v}_N - g_0\beta\right). \tag{11.6.81}$$

Putting (11.6.81) and (11.6.13) in (11.6.79), we will obtain

$$\beta + \frac{v_N}{R} = \frac{\mu}{H}\int_0^t\left(\dot{v}_N - g_0\beta\right)dt + \frac{b}{H}\left(\dot{v}_N - g_0\beta\right), \tag{11.6.82}$$

whence after differentiation we find

$$\ddot{\beta} + \frac{b}{H}g_0\dot{\beta} + \frac{\mu}{H}g_0\beta = \dot{v}_N\left(\frac{\mu}{H} - \frac{1}{R}\right) + \frac{b}{H}\ddot{v}_N. \tag{11.6.83}$$

When the condition of undisturbability (11.6.22) is satisfied, instead of (11.6.83) we will obtain

$$\ddot{\beta} + \frac{b}{H}g_0\dot{\beta} + \frac{g_0}{R}\beta = \frac{b}{H}\ddot{v}_N. \tag{11.6.84}$$

Let us designate

$$c' = \frac{b}{H}. \tag{11.6.85}$$

Let us transform $\frac{b}{H} g_0$ in the following way:

$$\frac{b}{H} g_0 = \frac{b}{H} g_0 \frac{\sqrt{\frac{g_0}{R}}}{\sqrt{\frac{g_0}{R}}} = 2\zeta\nu, \tag{11.6.86}$$

where $\nu = \sqrt{\frac{g_0}{R}}$ [see (11.6.24)];

ζ is relative attenuation factor

$$\zeta = \frac{b\sqrt{Rg_0}}{2H}. \tag{11.6.87}$$

Considering designations (11.6.24), (11.6.85) and (11.6.86), we will rewrite equation (11.6.84) in the form

$$\ddot{\beta} + 2\zeta\nu\dot{\beta} + \nu^2\beta = c'\ddot{v}_N \tag{11.6.88}$$

or, introducing by analogy with (11.4.113) the time constant $T = \frac{1}{\nu}$, we will obtain

$$T^2\ddot{\beta} + 2\zeta T\dot{\beta} + \beta = c\ddot{v}_N, \tag{11.6.89}$$

where

$$c = \frac{c'}{\nu^2} = \frac{2\zeta}{g_0\nu}. \tag{11.6.90}$$

Equation (11.6.89) characterizes damped oscillations of the IV, which are analogous to oscillations of physical pendulum with damping [see (4.3.25)]. Therefore, transfer function of the IV with damping and its transfer characteristics will be the same as for the physical pendulum (§§ 4.3 and 4.4).

From equation (11.6.89) it follows that application of the method of internal damping of the IV of leads to loss by the vertical of the property of undisturbability, since it is influenced by $\ddot{v}_N$, and this causes corresponding dynamic errors of the IV which are analogous in nature to damping-acceleration errors of the GC (§ 11.5, Par. 4, c). Thus, there appear corresponding errors in determination by the inertial system of position of object and parameters of its motion. For decrease of errors of the IV it is possible to decrease coefficient c, i.e., relative attenuation factor ζ [see (11.6.90)], but this leads to weakening of damping of the IV. Another possibility of decrease of such errors is application of so-called limited damping of the IV [163]. It consists of the fact that damping torque M_d

[see (11.6.80)] is given in the form of a mixed characteristic instead of a proportional characteristic (Fig. 2.18b). This form of damping is especially useful under conditions of prolonged change of velocity of the object.

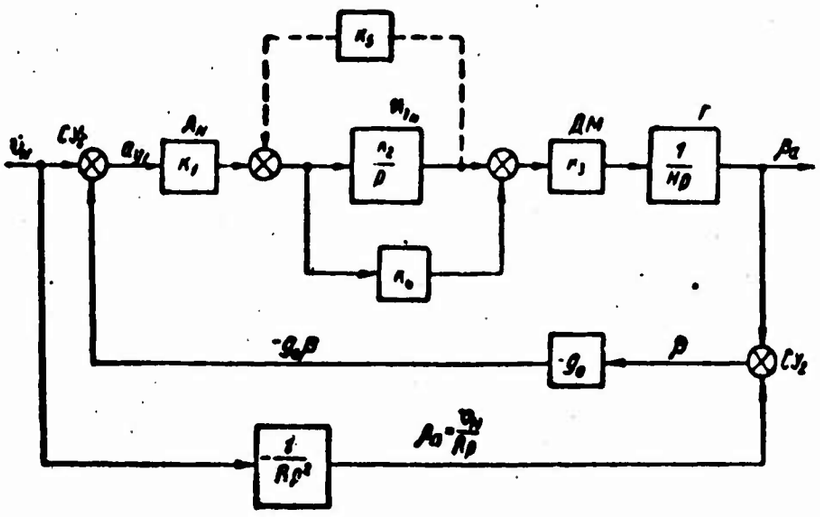

Fig. 11.29. Block diagram of "northern" channel of the IV with damping with the help of a direct connection including the first integrator.

The above considered method of damping usually is called [142] damping with the help of direct connection including the first integrator. Thus, the block diagram of the "northern" channel of the IV, which is shown in Fig. 11.28, if we take into account damping, will take the form shown in Fig. 11.29.* According to this diagram, equation of IV with damping will be [142] the following equation instead of (11.6.61):

$$\left(p^2+\frac{k_1k_2k_4}{H}g_0p+\frac{k_1k_2k_3}{H}g_0\right)\beta=pv_N\left(\frac{k_1k_2k_3}{H}-\frac{1}{R}\right)+\frac{k_1k_2k_4}{H}p^2v_N. \qquad (11.6.91)$$

If we designate

$$\mu=k_1k_2k_3,\quad b=k_1k_2k_4. \qquad (11.6.92)$$

then equation (11.6.91) will take the form

$$\ddot{\beta}+\frac{b}{H}g_0\dot{\beta}+\frac{\mu}{H}g_0\beta=\dot{v}_N\left(\frac{\mu}{H}-\frac{1}{R}\right)+\frac{b}{H}\ddot{v}_N.$$

*In this case, in the diagram in Fig. 11.29, feedback with transfer coefficient k_5 should be disconnected.

which coincides with (11.6.83).

There are also possible other methods of internal damping of IV. One of them is damping with the help of feedback including the first integrator. An idea of the scheme of IV with this method can be given by Fig. 11.29, if in it we eliminate the direct coupling with transfer coefficient k_4 and connect feedback (designated by dotted line) with transfer coefficient k_5.

With this scheme of damping, equation of IV will have the form [142]

$$\left(p^2 + k_2k_3p + \frac{k_1k_2k_3}{H}g_0\right)\beta = pv_N\left(\frac{k_1k_2k_3}{H} - \frac{1}{R}\right) - k_2k_3\frac{v_N}{R}. \qquad (11.6.93)$$

Designating, by analogy with (11.6.92),

$$\mu = k_1k_2k_3, \quad \frac{b}{H}g_0 = k_2k_3, \qquad (11.6.94)$$

we will rewrite (11.6.93) in the form

$$\ddot{\beta} + \frac{b}{H}g_0\dot{\beta} + \frac{\mu}{H}g_0\beta = \dot{v}_N\left(\frac{\mu}{H} - \frac{1}{R}\right) - \frac{b}{H}g_0\frac{v_N}{R}. \qquad (11.6.95)$$

With observance of condition of undisturbability (11.6.22), we have

$$\ddot{\beta} + \frac{b}{H}g_0\dot{\beta} + \nu^2\beta = -\frac{b}{H}g_0\frac{v_N}{R}; \qquad (11.6.96)$$

considering designation (11.6.86), we will obtain equation of damped oscillations of IV

$$\ddot{\beta} + 2\zeta\nu\dot{\beta} + \nu^2\beta = -2\zeta\nu\frac{v_N}{R}, \qquad (11.6.97)$$

which is analogous to (11.6.88). Introducing time constant $T = \frac{1}{\nu}$ [see (11.4.113)], instead of (11.6.97) we have

$$T^2\ddot{\beta} + 2\zeta T\dot{\beta} + \beta = -2\zeta T\frac{v_N}{R}, \qquad (11.6.98)$$

whence it follows that even with the given method of damping with the help of internal couplings, the IV loses the property of undisturbability.

What has been considered above permits us to formulate the following general statement [16]: damping of an inertial system by means of any internal couplings leads to disturbance of the condition of undisturbability.

Let us note one more peculiarity of damping of IV with the help of internal couplings. If the inertial system is used for control of motion of the object

(i.e., operates in the control mode), then damping introduced into the system with the help of internal couplings disappears, and the system becomes unstable [16, 10]. Thus [16], damping introduced into the inertial system by means of internal couplings in the readout mode leads to disturbance of the condition of undisturbability, and in the control mode completely disappears.

Due to this, we resort to other methods of damping of IV. One of the effective methods is based on use of external information. This includes data concerning velocity and coordinates of the object, which are obtained on the latter with the help of devices not contained in the INS. Furthermore, with help of external information obtained from these systems it is possible to carry out damping of the IV, and also to change period of natural oscillations of the latter while preserving properties of undisturbability by accelerations of the object.

For explanation of this circumstance, we will consider oscillations of the IV with respect to coordinate β during use of external information [162]. Let us use equation (11.6.79) of motion of the IV in plane of the meridian.

Let us assume that as information, into the inertial system there enters the component of displacement of the object S_N^* in plane of meridian and component of velocity v_N^* in this same plane. In the inertial system there are produced instrument values of these quantities S_N and v_N; with help of a computer there are found differences $\frac{n_1}{R}(S_N^* - S_N)$ and $\frac{b_1}{R}(v_N^* - v_N)$ (where n_1 and b_1 are certain coefficients, and R is radius of Earth), which together with the reading a_{y_1} of the "northern" accelerometer A_N (Fig. 11.28) are introduced into the first integrator I_{I_N}. For simplification of further computations we will assume that external information does not contain errors. Then, by analogy with (11.6.43), error of IV $\beta = \frac{S_N^* - S_N}{R}$, and $\dot{\beta} = \frac{v_N^* - v_N}{R}$. Consequently, the signal proceeding from the external information into the first integrator will be $(n_1\beta + b_1\dot{\beta})$. As a result, for torque $M_{cor\ y}$ of integral correction, instead of (11.6.19) we will obtain

$$M_{cor\ y} = \mu \int_0^t \left(\dot{v}_N - g_0\beta - n_1\beta - b_1\dot{\beta}\right) d\tau. \qquad (11.6.99)$$

Putting (11.6.99) and (11.6.13) in (11.6.79), we have

$$\beta + \frac{v_N}{R} = \frac{\mu}{H} \int_0^t \left(\dot{v}_N - g_0\beta - n_1\beta - b_1\dot{\beta}\right) d\tau. \qquad (11.6.100)$$

whence after differentiation, considering the condition of undisturbability (11.6.22), we will obtain

$$\ddot{\beta} + \frac{h_1}{R}\dot{\beta} + \frac{g_0 + n_1}{R}\beta = 0, \qquad (11.6.101)$$

i.e., the external information provides damping of the IV. Furthermore, the period of natural oscillations of the IV, which is equal to

$$T'_{n.o} = 2\pi\sqrt{\frac{R}{g_0 + n_1}}, \qquad (11.6.102)$$

differs from the period of M. Schuler [see (11.6.25)]. Thus the IV preserves its property of undisturbability by accelerations of the object. This makes it possible to considerably decrease, for instance by 10-15 times [16], the period of natural oscillations of the IV; this in a number of cases has great value.

It is possible to note [142] that introduction of external information is equivalent to "change" of the radius of the earth while preserving the intensity of its gravitational field. Really, the same period $T'_{i.v.}$ of oscillations of the IV could have been obtained in the absence of external information, if the radius of the Earth were $r < R$, i.e.,

$$T'_{n.o} = 2\pi\sqrt{\frac{r}{g_0}}, \qquad (11.6.103)$$

whence, according to (11.6.102), we have

$$\frac{r}{R} = \frac{g_0}{g_0 + n_1}. \qquad (11.6.104)$$

The possibility of decrease of period of undisturbed oscillations of IV means of introduction of external information is a property of not only inertial systems, but in general, of gyroscopic systems possessing positional properties (gyropendulum, gyrocompass). The basis of this property in reference to the indicated gyroscopic systems was for the first time given by K. Glitscher [185].

Investigation of the INS using external information permits us to formulate the following important statement [16]: "for construction of a damped inertial system with arbitrary period which is invariant with respect to maneuvering of the object, it is necessary to introduce into it external information on velocity, and coordinates of the object, and on the basis of comparison of this information with corresponding information given by the inertial system, to determine angles and

angular velocities of the vertical."

External information can be introduced into the inertial system in discrete steps. Interval of time between introductions of information depends on period of oscillations of the IV. Another peculiarity of damping of IV with the help of external information is that error in determination of the informational data (path, velocity) leads to corresponding errors in the IV (Par. 7). One of the deficiencies of the given method of damping is that with its use the inertial system becomes non-self-contained. Detailed investigations of the method of damping of IV with the help of external information can be met in special works (for instance, [162, 165, 16, 10]).

4. General Characteristics of Errors of the IV

Elements of INS are presented with exceptionally high requirements for accuracy of their readings. Due to this during theoretical investigation of INS, of especially important value is the analysis of accuracy. However, the question about estimation of error of INS has up til now been insufficiently well developed; thus, is explained first of all by the complexity of the actual system, and also by the insufficiency of experimental data.

Let us give brief information an errors of IV and about the influence of them on the accuracy of the inertial system. Errors of IV, just as of other GD, will be subdivided into instrument and systematic errors, and errors in statics and in dynamics will be distinguished.

5. Instrument Errors of IV

a) General Characteristics of Instrument Errors

Instrument errors of an IV are caused by errors of its separate elements and subassemblies: gyroscopes, accelerometers, integrators, computers, and others, and are due to imperfections of these devices: their structural deficiencies, deviations of characteristics of their parameters from rated values, etc.

Among instrument errors of IV it is possible to include the following:

1) errors caused by inaccurate initial setting of the GSP;

2) errors due to disturbances applied to gyroscopes: unbalance torques; frictional torques in axes of suspension; inertial torques due to accelerations during oscillations and vibrations of the place of installation of the instrument, which are connected with elastic deformations of the gyroscopes, etc.:

3) errors caused by variations of parameters of gyroscopes and their correction circuits;

4) errors caused by errors in accelerometers and first integrators;

5) errors caused by inaccuracy of azimuthal stabilization of the GSP;

6) errors caused by damping of oscillations of the IV, etc.

Instrument errors of IV are different depending upon character of external influences, i.e., under static and dynamic conditions. Here there are considered instrument errors in statics – in the absence of external disturbances. Certain errors of the IV in dynamics will be determined further on (Par. 7). Thus there is considered the IV applied in inertial system with gyroscopic vertical. We will assume that systematic errors of IV are compensated, and will consider only their random components, which under static conditions are treated as random variables.

b) Errors Caused by Inaccurate Initial Setting of the GSP

One of the essential factors determining error of IV is inaccurate initial installation of the GSP relative to the reference system $O\xi\eta\zeta$, which in the case considered further on has geographic orientation. Errors of initial installation of GSP relative to plane of horizon are characterized by angular deflections $\alpha(0) = \alpha_0$, $\beta(0) = \beta_0$ (Fig. 1.6) and angular velocities $\dot{\alpha}(0) = \dot{\alpha}_0$, $\dot{\beta}(0) = \dot{\beta}_0$, and also by a certain angle $\delta(0) = \delta_0$ and angular velocity $\dot{\delta}(0) = \dot{\delta}_0$ of deflection of the platform in azimuth. If in the first approximation we disregard azimuthal deflection of the platform, then error β of the IV, for instance in plane of the meridian, is characterized by expression (11.6.74)

$$\beta = \beta_0 \cos \nu t + \frac{\dot{\beta}_0}{\nu} \sin \nu t.$$

Thus, errors β_0 and $\dot{\beta}_0$ of initial orientation of the IV cause oscillations of it with period of 84.4 minutes. Root mean square value σ_β of angle of deviation of the IV from geocentric vertical is determined by relationship (11.6.78)

$$\sigma_\beta = \sqrt{\sigma_{\beta_0}^2 \cos^2 \nu t + \frac{\sigma_{\dot{\beta}_0}^2}{\nu^2} \sin^2 \nu t}.$$

Limiting value β_m of error of IV, by analogy with (4.6.87), will be (for $\overline{\beta} = 0$)

$$\beta_m = z\sigma_\beta, \qquad (11.6.105)$$

where z is determined from Table 1 (§ 4.6, Par. 10) by assigned probability Q_z of the current angle β exceeding limiting value β_m.

Example 11.15: To calculate root mean square σ_β and limiting β_m values of error of IV due to initial deviation of it from vertical, 32 minutes after inclusion

of the system, for the biggest values of initial deflection and angular velocity $\beta_{0m} = 3'$, $\beta_{0m} = 0.6$ angular minutes/minute. During determination of limiting value β_m of error of IV, we should assume probability Q_z that current angle β will exceed limit β_m by not more than 0.1%.

Solution: 1. We find root mean square values σ_{β_0} and $\sigma_{\dot{\beta}_0}$, considering them to be equal to 1/3 of the biggest deviations:

$$\sigma_{\beta_0} = \frac{\beta_{0m}}{3} = \frac{3}{3} = 1' = 0{,}0002909 \text{ rad.};$$

$$\sigma_{\dot{\beta}_0} = \frac{\dot{\beta}_{0m}}{3} = \frac{0{,}6}{3} = 0{,}2 \text{ ang. min/min} = 0{,}9696 \cdot 10^{-6} \text{ 1/sec.}$$

2. By formula (11.6.24) we calculate frequency ν of oscillations of the IV:

$$\nu = \sqrt{\frac{g_0}{R}} = \sqrt{\frac{9{,}81}{6371 \cdot 10^3}} = 1{,}24 \cdot 10^{-3} \text{ 1/sec.}$$

3. According to (11.6.78) we find root mean square value σ_β of error of IV:

$$\sigma_\beta = \sqrt{\sigma_{\beta_0}^2 \cos^2 \nu t + \frac{\sigma_{\dot{\beta}_0}^2}{\nu^2} \sin^2 \nu t} =$$

$$= \sqrt{(2{,}909)^2 \cdot 10^{-8} \cos^2 1{,}24 \cdot 10^{-3} \cdot 1920 + \frac{(9{,}696)^2 \cdot 10^{-14}}{(1{,}24)^2 \cdot 10^{-6}} \sin^2 1{,}24 \cdot 10^{-3} \cdot 1920} =$$

$$= 0{,}0005787 \text{ rad.} = 2'.$$

4. For $Q_z = 0.1\%$, by Table 1 (§ 4.6, Par. 10) we determine $z = 3.3$; then by (11.6.105) we find limiting value β_m of error of the IV:

$$\beta_m = z\sigma_\beta = 3{,}3 \cdot 2 = 6{,}6.$$

From Example 11.15 it follows that error of IV caused by inaccurate initial installation of GSP may be considerable.

b) Errors Due to Disturbances Applied to GSP

For determination of errors of IV due to disturbances applied to GSP it is necessary to consider a concrete design of GSP. We will show certain peculiarities of the influence of disturbances, considering the design of IV shown in Fig. 11.22. Let us use shortened equations (11.6.10), where p_1 and q_1, if we do not consider rotation of Earth, are determined by formulas (11.6.13). As torques M_{x_1} and M_{y_1} we will consider torques $M_{cor\ x}$ and $M_{cor\ y}$, which are applied to the gyroscope by the system of integral correction, and perturbing torques $f_2(t)$ and $f_1(t)$, applied to axes of Cardan suspension of the gyroscope:

$$M_{x_1} = M_{cor\ x} + f_2(t); \quad M_{y_1} = M_{cor\ y} + f_1(t). \tag{11.6.106}$$

Substituting here relationships (11.6.19), we will obtain

$$\left.\begin{aligned} M_{x_1} &= \mu \int_0^t \left(\dot{v}_E - g_0\alpha\right) d\tau + f_2(t) \\ M_{y_1} &= \mu \int_0^t \left(\dot{v}_N - g_0\beta\right) d\tau + f_1(t) \end{aligned}\right\} \qquad (11.6.107)$$

We will introduce (11.6.107) and (11.6.13) into equations (11.6.10):

$$\left.\begin{aligned} \dot{\alpha} + \frac{v_E}{R} &= \frac{\mu}{H} \int_0^t \left(\dot{v}_E - g_0\alpha\right) d\tau + \frac{f_2(t)}{H} \\ \dot{\beta} + \frac{v_N}{R} &= \frac{\mu}{H} \int_0^t \left(\dot{v}_N - g_0\beta\right) d\tau + \frac{f_1(t)}{H} \end{aligned}\right\} \qquad (11.6.108)$$

whence after differentiation we will obtain

$$\left.\begin{aligned} \ddot{\alpha} + \frac{\mu}{H} g_0\alpha &= \dot{v}_E\left(\frac{\mu}{H} - \frac{1}{R}\right) + \frac{\dot{f}_2(t)}{H} \\ \ddot{\beta} + \frac{\mu}{H} g_0\beta &= \dot{v}_N\left(\frac{\mu}{H} - \frac{1}{R}\right) + \frac{\dot{f}_1(t)}{H} \end{aligned}\right\} \qquad (11.6.109)$$

When condition of undisturbability (11.6.22) is satisfied, we find

$$\left.\begin{aligned} \ddot{\alpha} + \frac{g_0}{R}\alpha &= \frac{1}{H} \dot{f}_2(t) \\ \ddot{\beta} + \frac{g_0}{R}\beta &= \frac{1}{H} \dot{f}_1(t) \end{aligned}\right\} \qquad (11.6.110)$$

whence it follows that perturbing torques which are constant in magnitude $f_2(t)$ and $f_1(t)$ do not change the position of equilibrium of the IV, but cause, as will be shown further on, oscillations of it near the position of equilibrium.

As perturbing torques $f_2(t)$ and $f_1(t)$ we will consider at first static unbalance of the gyroscope, when its center of gravity lies on axis Oz (Fig. 11.22) at a certain distance l from point of support O. In this case weight of gyroscope $P = mg_0$ introduces a torque whose components relative to axes Ox_1 and Oy_1 are determined by relationships (2.3.56)

$$\left.\begin{aligned} f_2(t) &= -\frac{P}{g_0} l \left[W_y - (W_z + g_0)\beta\right] \\ f_1(t) &= \frac{P}{g_0} l \left[W_x - (W_z + g_0)\alpha\right] \end{aligned}\right\} \qquad (11.6.111)$$

Considering that $W_\zeta \ll g_0$ and taking into account (11.6.17), we will rewrite (11.6.111) in the form

$$\left.\begin{aligned} f_2(t) &= -lP\left(\frac{\dot{v}_N}{g_0} - \beta\right) \\ f_1(t) &= lP\left(\frac{\dot{v}_E}{g_0} - \alpha\right) \end{aligned}\right\} \qquad (11.6.112)$$

Substituting (11.6.112) into (11.6.107), we will obtain

$$\left.\begin{aligned} M_{y_1} &= \mu\int_0^t\left(\dot v_E - g_0\alpha\right)dt - lP\left(\frac{\dot v_N}{g_0} - \beta\right)\\ M_{y_2} &= \mu\int_0^t\left(\dot v_N - g_0\beta\right)dt + lP\left(\frac{\dot v_E}{g_0} - \alpha\right)\end{aligned}\right| \qquad (11.6.113)$$

Taking into account (11.6.13) and (11.6.113), we will rewrite equations (11.6.10) in the form

$$\left.\begin{aligned} \dot\alpha + \frac{v_E}{R} &= \frac{\mu}{H}\int_0^t\left(\dot v_E - g_0\alpha\right)dt - \frac{lP}{H}\left(\frac{\dot v_N}{g_0} - \beta\right)\\ \dot\beta + \frac{v_N}{R} &= \frac{\mu}{H}\int_0^t\left(\dot v_N - g_0\beta\right)dt + \frac{lP}{H}\left(\frac{\dot v_E}{g_0} - \alpha\right)\end{aligned}\right| \qquad (11.6.114)$$

whence, differentiating and considering designation (2.3.20)

$$k = \frac{lP}{H}, \qquad (11.6.115)$$

we will obtain equations

$$\left.\begin{aligned} \ddot\alpha + \frac{\mu}{H}g_0\alpha - k\dot\beta &= \dot v_E\left(\frac{\mu}{H} - \frac{1}{R}\right) - k\frac{\ddot v_N}{g_0}\\ \ddot\beta + \frac{\mu}{H}g_0\beta + k\dot\alpha &= \dot v_N\left(\frac{\mu}{H} - \frac{1}{R}\right) + k\frac{\ddot v_E}{g_0}\end{aligned}\right| \qquad (11.6.116)$$

which for k = 0 (i.e., for l = 0) became equations (11.6.21).

When condition (11.6.22) is satisfied, we have

$$\left.\begin{aligned} \ddot\alpha + \frac{g_0}{R}\alpha - k\dot\beta &= -k\frac{\ddot v_N}{g_0}\\ \ddot\beta + \frac{g_0}{R}\beta + k\dot\alpha &= k\frac{\ddot v_E}{g_0}\end{aligned}\right| \qquad (11.6.117)$$

whence it follows that in the presence of static unbalance of the gyroscope, even when condition (11.6.22) is satisfied, the IV is disturbed by maneuvering of the object (vehicle).

For determination of errors α and β of the IV which are caused by static unbalance of the gyroscope, in virtue of (11.6.117) we obtain the following differential equations in operator form ($p = \frac{d}{dt}$):

$$\left[\left(p^2 + \frac{g_0}{R}\right)^2 + k^2p^2\right]\alpha = -k\left(p^2 + \frac{g_0}{R}\right)p^2\frac{v_N}{g_0} + \frac{k^2p^2v_E}{g_0}; \qquad (11.6.118)$$

$$\left[\left(p^2 + \frac{g_0}{R}\right)^2 + k^2p^2\right]\beta = k\left(p^2 + \frac{g_0}{R}\right)p^2\frac{v_E}{g_0} + \frac{k^2p^2v_N}{g_0}. \qquad (11.6.119)$$

From these formulas it follows that for constant v_N and v_E, the IV will be undisturbed by accelerations of the object. However, in the process of establishment of these values of v_N and v_E, the IV will be disturbed by maneuvering of the

object. Furthermore, according to the left sides of equations (11.6.118) and (11.6.119), during static unbalance of the gyroscope there will be changed transient responses of the IV.

Therefore, gyroscopes applied in INS have to possess high accuracy of stabilization of center of gravity. These questions are considered in detail by G. O. Fridlender [162].

As another form of disturbances we will consider frictional torques in axes of suspension of the gyroscope. Then, considering that $f_2(t) = M_{fr\ x}$ and $f_1(t) = M_{fr\ y}$, we will rewrite (11.6.107) in the form

$$\left.\begin{aligned} M_{x_1} &= H\int_0^t \left(\dot{v}_E - g_0\alpha\right)d\tau + M_{\tau x} \\ M_{y_1} &= H\int_0^t \left(\dot{v}_N - g_0\beta\right)d\tau + M_{\tau y} \end{aligned}\right\} \qquad (11.6.120)$$

Substituting here in place of $M_{fr\ x}$ and $M_{fr\ y}$ for the case of dry (Coulomb) friction the expressions (2.3.84), we will obtain

$$\left.\begin{aligned} M_{x_1} &= H\int_0^t \left(\dot{v}_E - g_0\alpha\right)d\tau + M^0_{\tau x} + K_x \operatorname{sign}\dot{\beta} \\ M_{y_1} &= H\int_0^t \left(\dot{v}_N - g_0\beta\right)d\tau + M^0_{\tau y} - K_y \operatorname{sign}\dot{\alpha} \end{aligned}\right\} \qquad (11.6.121)$$

Let us introduce (11.6.121) and (11.6.13) into equations (11.6.10):

$$\left.\begin{aligned} \dot{\alpha} + \frac{v_E}{R} &= \frac{1}{H}\int_0^t \left(\dot{v}_E - g_0\alpha\right)d\tau + \frac{M^0_{\tau x}}{H} + \frac{K_x}{H}\operatorname{sign}\beta \\ \dot{\beta} + \frac{v_N}{R} &= \frac{1}{H}\int_0^t \left(\dot{v}_N - g\beta\right)d\tau + \frac{M^0_{\tau y}}{H} - \frac{K_y}{H}\operatorname{sign}\dot{\alpha} \end{aligned}\right\} \qquad (11.6.122)$$

whence, if we disregard sign-alternating components of frictional torques ($K_x = K_y = 0$) and consider (11.6.22), we will obtain

$$\ddot{\alpha} + \frac{g_0}{R}\alpha = 0; \quad \ddot{\beta} + \frac{g_0}{R}\beta = 0. \qquad (11.6.123)$$

i.e., components of frictional torques with fixed sign in axes of the suspension of the gyroscope do not change the position of equilibrium of the IV,* but cause, in general, oscillations of it near the vertical.

Let us explain this in an analysis of motion of the IV in plane of meridian, i.e., with respect to coordinate β. Let us assume that at $t = 0$ $\beta(0) = \beta_0 = 0$; in accordance with the second equation of system (11.6.122), initial angular velocity $\dot{\beta}(0) = \dot{\beta}_0$ is determined by the fixed-sign component $M^0_{fr\ y}$ of frictional torque

*What has been said is valid if we assume that quantities $M^0_{fr\ x}$, $M^0_{fr\ y}$ of fixed-sign components of frictional torques in axes of suspension of the gyroscope are constants.

[see (2.3.90)]:

$$\dot{\beta}_0 = \frac{M^0_{fr\,y}}{H} = \eta^0_1. \tag{11.6.124}$$

Then oscillations of the IV, according to (11.6.74), for $\beta_0 = 0$ and $\dot{\beta}_0 = \eta^0_1$ are determined by the relationship

$$\beta = \frac{\eta^0_1}{\nu} \sin \nu t. \tag{11.6.125}$$

If we consider a group of IV made according to the same model, then fixed-sign component $M^0_{fr\,y}$ of frictional torque is a random variable. We will consider $M^0_{fr\,y}$ to be a normal random variable with root mean square value $\sigma_{M^0_{fr\,y}}$; then, in virtue of (11.6.124), root mean square value $\sigma_{\eta^0_1}$ of random rate of precession η^0_1 of gyroscope under the influence of friction will be

$$\sigma_{\eta^0_1} = \frac{1}{H} \sigma_{M^0_{fr\,y}}. \tag{11.6.126}$$

For root mean square value σ_β of error of the IV, if we consider (11.6.125), we will obtain

$$\sigma_\beta = \frac{\sigma_{\eta^0_1}}{\nu} \sin \nu t, \tag{11.6.127}$$

where $\sigma_{\eta^0_1}$ is determined by formula (11.6.126).

Example 11.16: To calculate root mean square σ_β and limiting β_m values of error of the IV due to the fixed-sign component of frictional torque in axis of suspension 21.1 minutes after turning on the system, for the following initial data: root mean square value of fixed-sign component of frictional torque $\sigma_{M^0_{fr\,y}} = 0.01$ g·cm, angular momentum of gyroscope [4, p. 147] $H = 6.45 \cdot 10^3$ g·cm sec, frequency of oscillations of the IV $\nu = 1.24 \cdot 10^{-3}$ 1/sec (Example 11.15). Limiting value β_m of error of the IV is determined for the conditions of Example 11.15.

Solution: 1. By formula (11.6.126) we find root mean square value $\sigma_{\eta^0_1}$ of random rate of precession under the influence of friction:

$$\sigma_{\eta^0_1} = \frac{1}{H} \sigma_{M^0_{fr\,y}} = \frac{1}{6{,}45 \cdot 10^3} 0{,}01 = 0{,}155 \cdot 10^{-5} \text{ 1/sec}.$$

2. According to (11.6.127) we calculate root mean square value σ_β of error of the IV:

$$\sigma_\beta = \frac{\sigma_{\eta^0_1}}{\nu} \sin \nu t = \frac{0{,}155 \cdot 10^{-5}}{1{,}24 \cdot 10^{-3}} 1 = 0{,}00125 \text{ rad} = 4',3.$$

3. By the same method as in Example 11.15, we find limiting value β_m of error of the IV (for $\overline{\beta} = 0$):

$$\beta_m = z\sigma_\beta = 3{,}3 \cdot 4{,}3 = 14',2.$$

From Example 11.16 it follows that even comparatively small frictional torque causes considerable deviations of the IV.

d) Errors Caused by Variations of Parameters of the Gyroscope and its Correction Circuit

Variations of parameters of the gyroscope and its correction circuit also cause errors of the IV. For explanation of what has been said, we will consider corresponding errors of the in plane of meridian, i.e., with respect to coordinates β. Let us use second equation of system (11.6.21)

$$\ddot{\beta}+\frac{\mu}{H}g_0\beta=\dot{v}_N\left(\frac{\mu}{H}-\frac{1}{R}\right).$$

We will consider that in the considered system there can be changed angular momentum of gyroscope H, and also coefficient μ.

Since according to (11.6.60) $\mu = k_1k_2k_3$, then it is clear that μ is changed due to deviations from rated values of transfer coefficients of the accelerometer k_1, of the integrator k_2 and of the torquer k_3 (Fig. 11.28). In the considered case we consider change of μ due to deviation of transfer coefficient k_3 of the torquer with respect to the corrective torque applied to the gyroscope.

From the above equation it follows that for constant acceleration $\dot{v}_N$ = const in case of zero initial conditions

$$\beta=\frac{H}{\mu g_0}\dot{v}_N\left(\frac{\mu}{H}-\frac{1}{R}\right)\left(1-\cos\sqrt{\frac{\mu}{Hg_0}}t\right). \qquad (11.6.128)$$

Let us designate peak value of error of the IV

$$\beta_a=\frac{H}{\mu g_0}\dot{v}_N\left(\frac{\mu}{H}-\frac{1}{R}\right). \qquad (11.6.129)$$

Let us find change of quantity β_{peak} due to change of H and μ. Let us designate true (rated) values of H and μ by H_0 and μ_0. According to (11.6.129), to them there will correspond peak value $\beta_{peak(0)}$ of error

$$\beta_{a(0)}=\frac{H_0}{\mu_0 g_0}\dot{v}_N\left(\frac{\mu_0}{H_0}-\frac{1}{R}\right).$$

Let us assume that in the IV, by analogy with (11.6.22), there is realized the condition

$$\frac{\mu_0}{H_0}=\frac{1}{R}, \qquad (11.6.130)$$

when it is satisfied we have

$$\beta_{a(0)}=0. \qquad (11.6.131)$$

We will assume that in the actual instrument these parameters differ from rated values by δH and $\delta\mu$ and are respectively

$$H = H_0 + \delta H; \quad \mu = \mu_0 + \delta\mu;$$

thus β_{peak} will be determined by formula (11.6.129), i.e., will differ from value $\beta_{peak(0)}$ by

$$\delta\beta_0 = \beta_0 - \beta_{0(0)}, \tag{11.6.132}$$

which represents the change of amplitude of oscillations of the IV due to changes of H and μ, or the amplitude of oscillations (error) of the IV, since $\beta_{peak(0)} = 0$ [see (11.6.131)].

Since δH and $\delta\mu$ are small quantities as compared to their rated values, then for determination of $\delta\beta_{peak}$ we will use method of solution of the analogous problem given in § 5.5, Par. 2c. Then, in virtue of (11.6.129), we will obtain

$$\delta\beta_0 = \frac{\partial\beta_{0(0)}}{\partial H_0}\delta H + \frac{\partial\beta_{0(0)}}{\partial\mu_0}\delta\mu. \tag{11.6.133}$$

For partial derivatives $\frac{\partial\beta_{peak(0)}}{\partial H_0}$ and $\frac{\partial\beta_{peak(0)}}{\partial\mu_0}$ we have

$$\frac{\partial\beta_{0(0)}}{\partial H_0} = -\frac{\dot{\theta}_N}{\varepsilon_0}\cdot\frac{1}{\mu_0 R}; \quad \frac{\partial\beta_{0(0)}}{\partial\mu_0} = \frac{\dot{\theta}_N}{\varepsilon_0}\cdot\frac{H_0}{\mu_0^2 R}.$$

Putting them in (11.6.133), we will obtain

$$\delta\beta_0 = \frac{\dot{\theta}_N}{\varepsilon_0}\cdot\frac{H_0}{\mu_0 R}\left(\frac{\delta\mu}{\mu_0} - \frac{\delta H}{H_0}\right). \tag{11.6.134}$$

When the condition of undisturbability (11.6.130) is satisfied we have $\frac{H_0}{\mu_0 R} = 1$, and in place of (11.6.134) we find

$$\delta\beta_0 = \frac{\dot{\theta}_N}{\varepsilon_0}\left(\frac{\delta\mu}{\mu_0} - \frac{\delta H}{H_0}\right). \tag{11.6.135}$$

If we consider that $\delta\mu$ and δH are independent random variables with zero mathematical expectations, then according to (11.6.135) we will obtain the following expressions for mathematical expectation $\delta\overline{\beta}_{peak}$ and root mean square value $\sigma_{\delta\beta_{peak}}$ of amplitude of oscillations of the IV:

$$\delta\overline{\beta}_0 = 0; \tag{11.6.136}$$

$$\sigma_{\delta\beta_0} = \frac{|\dot{\theta}_N|}{\varepsilon_0}\sqrt{\frac{1}{\mu_0^2}\sigma_{\delta\mu}^2 + \frac{1}{H_0^2}\sigma_{\delta H}^2}. \tag{11.6.137}$$

where $\sigma_{\delta\mu}$ and $\sigma_{\delta H}$ are root mean square values of errors $\delta\mu$ and δH.

Formula (11.6.137) can also be represented in the form

$$\sigma_{\delta\beta_0} = \frac{|\dot{\theta}_N|}{\varepsilon_0}\sqrt{\frac{\sigma_{\delta\mu}^2}{\mu_0} + \frac{\sigma_{\delta H}^2}{H_0}}, \tag{11.6.138}$$

where $\sigma_{\frac{\delta\mu}{\mu_0}}$ and $\sigma_{\frac{\delta H}{H_0}}$ are root mean square values of relative errors $\frac{\delta\mu}{\mu_0}$ and $\frac{\delta H}{H_0}$.

Peak value $\delta\beta_{peak}$ of error of the IV due to variations of parameters δH and $\delta\mu$ can be very high. However, such a value of error of the IV is scarcely attained under real conditions of use of the IV. Thus, for instance, in the considered case error of IV will obtain value $\delta\beta_{peak}$ under the condition that the object moves with constant acceleration $\dot{v}_N$ for a quarter of the period of M. Schuler, i.e., approximately 21.1 minutes. Such duration of the action of acceleration which is constant in magnitude is hardly possible. In practice, time of action of acceleration will be considerably smaller, and therefore error of IV will be much less than its peak value $\delta\beta_{peak}$. Due to this, calculation of error of IV due to variation of parameters δH, $\delta\mu$ should be conducted not for peak value $\delta\beta_{peak}$, but for the current value $\delta\beta$, which exists at a given time of action of the acceleration $\dot{v}_N$. For error $\delta\beta$, if we proceed from formula (11.6.128), by the same method used during determination of $\delta\beta_{peak}$ we obtain the following expression:

$$\delta\beta = \delta\beta_{max}(1-\cos\nu t), \tag{11.6.139}$$

where peak value $\delta\beta_{peak}$ of error of the IV is determined by relationship (11.6.135).

Then for root mean square value $\sigma_{\delta\beta}$ of error of the IV we have expression

$$\sigma_{\delta\beta} = \sigma_{\delta\beta_{max}}(1-\cos\nu t), \tag{11.6.140}$$

where root mean square value $\sigma_{\delta\beta_{peak}}$ of amplitude of error of the IV is determined by formula (11.6.137) or (11.6.138).

Errors $\delta\mu$ and δH also change the frequency of natural oscillations of the IV, and consequently disturb the condition of undisturbability (11.6.130).

Analogously, deviation $\delta\nu$ of frequency of oscillations of the IV will be obtained in the form

$$\delta\nu = \frac{1}{2}\nu\left(\frac{\delta\mu}{\mu_0} - \frac{\delta H}{H_0}\right) \tag{11.6.141}$$

or the root mean square value will be $\sigma_{\delta\nu}$

$$\sigma_{\delta\nu} = \frac{1}{2}\nu\sqrt{\frac{1}{\mu_0^2}\sigma^2_{\delta\mu} + \frac{1}{H_0^2}\sigma^2_{\delta H}}. \tag{11.6.142}$$

Formula (11.6.142), by analogy with (11.6.138), can also be represented in the form

$$\sigma_{\delta\nu} = \frac{1}{2}\nu\sqrt{\sigma^2_{\frac{\delta\mu}{\mu_0}} + \sigma^2_{\frac{\delta H}{H_0}}}. \tag{11.6.143}$$

Using (11.6.125), we will obtain expression for root mean square value $\sigma_{\delta T}$ of error in a period of oscillations of the IV

$$\sigma_{\delta T} = T_{n.s}\frac{\sigma_{\delta\nu}}{\nu}. \tag{11.6.144}$$

Example 11.17: To calculate root mean square $\sigma_{\delta\beta_{peak}}$, $\sigma_{\delta\beta}$, $\sigma_{\delta\nu}$, $\sigma_{\delta T}$ and limiting $\delta\beta_{peak\ m}$, $\delta\beta_m$, $\delta\nu_m$, δT_m values of errors of the IV, and also of errors in frequency and period of its oscillations for the following initial data: $T_{i.v.}$ = 5064 sec; $\nu = 1.24 \cdot 10^{-3}$ 1/sec; the constant value of acceleration of the object $\dot{v}_N$ = 2 m/sec^2 acts for t = 10 minutes; the biggest values of relative errors of parameters of the IV $\left(\frac{\delta\mu}{\mu_0}\right)_m = \left(\frac{\delta H}{H_0}\right)_m = 1\%$. Limiting values $\delta\beta_{peak\ m}$, $\delta\beta_m$, $\delta\nu_m$, δT_m are determined for the conditions of Example 11.15.

Solution: 1. We find root mean square values $\sigma_{\frac{\delta\mu}{\mu_0}}$, $\sigma_{\frac{\delta H}{H_0}}$ of relative errors $\frac{\delta\mu}{\mu_0}$ and $\frac{\delta H}{H_0}$:

$$\sigma_{\frac{\delta\mu}{\mu_0}} = \frac{\left(\frac{\delta\mu}{\mu_0}\right)_m}{3} = \frac{0{,}01}{3} = 0{,}0033;$$

$$\sigma_{\frac{\delta H}{H_0}} = \frac{\left(\frac{\delta H}{H_0}\right)_m}{3} = \frac{0{,}01}{3} = 0{,}0033.$$

2. By formula (11.6.138) we calculate $\sigma_{\delta\beta_{peak}}$:

$$\sigma_{\delta\beta_0} = \frac{|\dot{v}_N|}{g_0} \sqrt{\sigma^2_{\frac{\delta\mu}{\mu_0}} + \sigma^2_{\frac{\delta H}{H_0}}} = \frac{2}{9{,}81} \sqrt{(0{,}0033)^2 + (0{,}0033)^2} = 0{,}0009516 \text{ rad} = 3',27.$$

3. Using (11.6.140), we find $\sigma_{\delta\beta}$:

$$\sigma_{\delta\beta} = \sigma_{\delta\beta_0}(1 - \cos \nu t) = 3{,}27\,(1 - \cos 1{,}24 \cdot 10^{-3} \cdot 600) = 0',83.$$

4. According to (11.6.143) and (11.6.144), we determine $\sigma_{\delta\nu}$ and $\sigma_{\delta T}$:

$$\sigma_{\delta\nu} = \frac{1}{2} \nu \sqrt{\sigma^2_{\frac{\delta\mu}{\mu_0}} + \sigma^2_{\frac{\delta H}{H_0}}} =$$

$$= \frac{1}{2} \cdot 1{,}24 \cdot 10^{-3} \sqrt{(0{,}0033)^2 + (0{,}0033)^2} = 0{,}2894 \cdot 10^{-5} \text{ 1/sec};$$

$$\sigma_{\delta T} = T_{i.v.} \frac{\sigma_{\delta\nu}}{\nu} = 5064 \frac{0{,}2894 \cdot 10^{-5}}{1{,}24 \cdot 10^{-3}} = 11{,}82 \text{ sec}.$$

5. By the same method as in Example 11.15, we find limiting values of errors (for $\overline{\delta\beta}_{peak} = \overline{\delta\beta} = \overline{\delta\nu} = \overline{\delta T} = 0$):

$$\delta\beta_{0m} = 3\sigma_{\delta\beta_0} = 3{,}3 \cdot 3{,}27 = 10',79;$$

$$\delta\beta_m = 3\sigma_{\delta\beta} = 3{,}3 \cdot 0{,}83 = 2',74;$$

$$\delta\nu_m = 3\sigma_{\delta\nu} = 3{,}3 \cdot 0{,}2894 \cdot 10^{-5} = 0{,}955 \cdot 10^{-5} \text{ 1/sec}$$

$$\delta T_m = 3\sigma_{\delta T} = 3{,}3 \cdot 11{,}82 = 39{,}01 \text{ sec}.$$

From Example 11.17 it follows that deviaticns of parameters of the IV can have a considerable influence on its accuracy. Therefore, it is necessary to make high

requirements of the stability of H and μ.

e) Errors Caused by Errors of Accelerometers and First Integrators

First we will consider influence of errors of the accelerometer on IV. To determine errors of IV with respect to coordinate β, we will use the second equation of system (11.6.20), which, taking into account (11.6.18), we will rewrite in the form

$$\beta + \frac{v_N}{R} = \frac{\mu}{H}\int_0^t a_{y_1} d\tau. \qquad (11.6.145)$$

Accelerometers have two forms of errors: 1) error of transfer coefficient k_1 (Fig. 11.28), which causes deviation in magnitude of coefficient μ [see (11.6.60)]; 2) error δa_{y_1} in measurement of acceleration a_{y_1} by the accelerometer.

Change of transfer coefficient k_1 of the accelerometer, which leads to change $\delta\mu$, has the name of scale error; the latter linearly depends on measured acceleration. Influence of $\delta\mu$ on error of IV was considered earlier (Par. e); thus, for example, change of k_1 causes through $\delta\mu$ error of the IV, which is determined by relationship (11.6.140).

Therefore, subsequently we will be limited to estimate of influence of error δa_{y_1} of accelerometer, which is caused by displacement of its zero. Taking into account δa_{y_1}, we will rewrite equation (11.6.145) in the form

$$\beta + \frac{v_N}{R} = \frac{\mu}{H}\int_0^t (a_{y_1} + \delta a_{y_1})\, d\tau, \qquad (11.6.146)$$

or, taking into account (11.6.18),

$$\beta + \frac{v_N}{R} = \frac{\mu}{H}\int_0^t (\dot{v}_N - g_0\beta + \delta a_{y_1})\, d\tau, \qquad (11.6.147)$$

whence, after differentiation, if (11.6.22) is satisfied, we have

$$\ddot{\beta} + \frac{g_0}{R}\beta = \frac{1}{R}\delta a_{y_1}. \qquad (11.6.148)$$

Integrating (11.6.148), we obtain ($\frac{g_0}{R} = \nu^2$)

$$\beta = C_1 \cos \nu t + C_2 \sin \nu t + \frac{1}{g_0}\delta a_{y_1}. \qquad (11.6.149)$$

For initial conditions t = 0, $\beta(0) = \beta_0 = 0$, $\dot{\beta}(0) = \dot{\beta}_0 = 0$, we have

$$\beta = \frac{1}{g_0}\delta a_{y_1}(1 - \cos \nu t), \qquad (11.6.150)$$

whence it follows that the IV accomplishes oscillation near the displaced position determined by quantity $\frac{1}{g_0}\delta a_{y_1}$.

Root mean square value σ_β of deviation of IV will be

$$\sigma_\beta = \frac{1}{g_0}\sigma_{\delta a_{y_1}}(1 - \cos \nu t), \qquad (11.6.151)$$

where $\sigma_{\delta a_{y_1}}$ is root mean square value of displacement of the zero of the accelerometer.

Example 11.18: To calculate root mean square σ_β and limiting β_m values of error of the IV for the following initial data: root mean square value of displacement of zero of accelerometer $\sigma_{\delta a_{y_1}} = 0.5$ cm/sec^2; $\nu = 1.24 \cdot 10^{-3}$ 1/sec; value of moment of time $t = \frac{\pi}{2\nu} = 21.1$ minutes. Limiting value β_m is determined for the conditions of Example 11.15.

Solution: 1. By formula (11.6.151) we determine σ_β:

$$\sigma_\beta = \frac{1}{g_0}\sigma_{\delta a_{y_1}}(1-\cos \nu t) = \frac{1}{981}0{,}5\left(1-\cos\frac{\pi}{2}\right) = 0{,}0005097 \text{ rad} = 1',75.$$

2. By the same method as in Example 11.15, we find limiting value β_m of error:

$$\beta_m = 3\sigma_\beta = 3{,}3 \cdot 1{,}75 = 5',78.$$

From Example 11.18 it follows that accelerometers applied in INS have to possess extremely small displacement of the zero.

Let us turn to analysis of influence of errors of the first integrator on accuracy of IV; these appear due to: 1) error in transfer coefficient k_2 of the integrator (Fig. 11.28), which causes deviation in the magnitude of coefficient μ [see (11.6.60)]; 2) "drift" εt of integrator* (ε is deviation of integrator in 1 sec).

Influence of the first of these factors, which manifests itself in change of coefficient μ, is the same as that due to variation of parameters of correction of gyroscope and accelerometer. Therefore, we will demonstrate the influence of drift of the zero of the first integrator on error of the IV. For this we will use the second equation of system (11.6.20), which, if we take into account drift of the integrator, will be written in the form

$$\beta + \frac{v_N}{R} = \frac{\mu}{H}\left[\int_0^t (\dot{v}_N - g_0\beta)\,d\tau + \varepsilon t\right], \qquad (11.6.152)$$

whence after differentiation, if the condition of undisturbability (11.6.22) is satisfied, we will obtain equation

$$\ddot{\beta} + \frac{g_0}{R}\beta = \frac{1}{R}\varepsilon, \qquad (11.6.153)$$

which is analogous to equation (11.6.148) of the IV in the presence of displacement of zero of the accelerometer. Therefore, by analogy with (11.6.151), for root mean

*Drift of the integrator is defined [112] as the appearance of an output signal in the absence of an input signal from the accelerometer.

square value σ_β of deviation of the IV we have

$$\sigma_\beta = \frac{1}{g_0}\sigma_\varepsilon(1 - \cos \nu t), \qquad (11.6.154)$$

where σ_ε is root mean square value of drift of the integrator in 1 sec.

Thus, drift of first integrator, just as displacement of zero of the accelerometer, causes displacement of IV and oscillations of it near this deflected position.

f) Errors Caused by Inaccuracy of Azimuthal Stabilization of the GSP

In the given (Par. a) list of instrument errors of IV it was shown that one of the causes of these errors is inaccuracy of azimuthal stabilization of the GSP. For investigation of this question it is necessary to compose equations of the triaxial GSP applied in the considered type of INS (see, for instance, [50, 142, 162, 175]).

Errors of the GSP are caused by: 1) errors of initial azimuthal orientation of the GSP; 2) drift of the platform in azimuth; 3) change of azimuthal orientation of axes of sensitivity of accelerometers. Analysis of influence of the first two factors on the accuracy of IV shows [175],* that errors of initial azimuthal orientation of platform and drift of azimuthal gyroscope lead to errors of the IV. Change of azimuthal orientation of axes of sensitivity of accelerometers also causes error of the IV.

Due to what has been said, there are made very high requirements of the accuracy of initial azimuthal orientation of platform and drift of the azimuthal gyroscope.

g) Errors Caused by Damping of Oscillations of the IV**

Let us consider possible errors in IV during damping of it with the help of internal couplings; errors during damping with help of external information in dynamics will be discussed in Par. 7. Let us assume that damping of the IV is carried out by means of feedback, including the first integrator (Fig. 11.29). For this case, motion of the IV with respect to coordinate β is described by equation (11.6.97):

$$\ddot{\beta} + 2\varkappa_\nu \dot{\beta} + \nu^2 \beta = -2\varkappa_\nu \frac{v_N}{R}. \qquad (11.6.155)$$

For v_N = const, error of IV upon completion of transient response is determined by relationship

$$\beta_{st} = -\frac{\varkappa_\nu}{\nu} \cdot \frac{v_N}{R}. \qquad (11.6.156)$$

Example 11.19: To determine error β_{st} of IV due to damping by means of

*See also: A. A. Yakushenkov. Fundamentals of inertial navigation "Naval Transport," 1963.

**These errors may also be included among systematic errors.

inclusion of first integrator by feedback for the following initial data: relative damping factor $\zeta = 0.5$; Schuler frequency $\nu = 1.24 \cdot 10^{-3}$ 1/sec; constant value of velocity of ship in plane of meridian $v_N = 10$ m/sec; R = 6371 km.

Solution: By formula (11.6.156) we find magnitude of error β_{st}:

$$|\beta_{cr}| = \frac{\zeta}{\nu} \cdot \frac{v_N}{R} = \frac{2 \cdot 0{,}5}{1{,}24 \cdot 10^{-3}} \cdot \frac{10}{6371 \cdot 10^3} = 0{,}001266 \text{ rad} = 4',35.$$

From Example 11.19 it follows that error β_{st} is considerable. It will be still larger if IV is installed, for instance, on an aircraft. Thus, for speed of aircraft $v_N = 200$ m/sec for the same initial data we will obtain error $|\beta_{st}| = 87'$, which is absolutely impermissible. Therefore, the most expedient is damping of IV with the help of external information.

h) Influence of Errors of IV on Error of Determination of Coordinates of the Object (Vehicle) by the INS

Considering the operation of INS with geographic orientation, we will estimate influence of error of IV with respect to coordinate β in the plane of the meridian on error δS in determination by the inertial system of path S, passed over by the object in this plane.* Thus we will not consider the influence of other errors on δS, for instance due to error of initial establishment of local latitude φ, errors of the second integrator, and others. Of all possible errors of the IV, we will take into account errors caused by inaccurate initial setting of the IV, by disturbing torques applied to the IV, by variation of parameters of accelerometers, gyroscopes, first integrators and by inaccuracies of accelerometers and first integrators. For this we will use the second equation of system (11.6.20), in which for simplification we will consider that $v_N = 0$, $\dot{v}_N = 0$; then we will obtain

$$\beta = \frac{\mu}{H} \int_0^t (-g_0 \beta) \, d\tau. \qquad (11.6.157)$$

Taking into account the above mentioned errors of the IV, according to (11.6.108), (11.6.147), (11.6.152), and replacing H by $H_0 + \delta H$, μ by $\mu_0 + \delta\mu$, in place of (11.6.157) we will obtain

$$\beta = \frac{\mu_0 + \delta\mu}{H_0 + \delta H} \left[\int_0^t (-g_0 \beta + \delta a_{y_1}) \, d\tau + \varepsilon t \right] + \frac{f_1(t)}{H_0 + \delta H}, \qquad (11.6.158)$$

whence with accuracy up to terms of the second order of smallness we have

$$\beta = \left[\frac{\mu_0}{H_0} \left(1 - \frac{\delta H}{H_0} \right) + \frac{\delta\mu}{H_0} \right] \left[\int_0^t (-g_0 \beta + \delta a_{y_1}) \, d\tau + \varepsilon t \right] + \frac{f_1(t)}{H_0} \left(1 - \frac{\delta H}{H_0} \right). \qquad (11.6.159)$$

*Such questions are considered in a number of the above cited works on inertial navigation (see also: N. I. Yefremov. On the theory of invariance of inertial systems based on the method of dead-reckoning. "News of Higher Educ. Institution. Instrument Manufacture," Vol. V, Issue 6, 1962).

Let us designate

$$k_H = 1 - \frac{\delta H}{H_0}. \qquad (11.6.160)$$

Then we will rewrite (11.6.159) in the form

$$\beta = \left(\frac{\mu_0}{H_0} k_H + \frac{\delta\mu}{H_0}\right)\left[\int_0^t (-g_0\beta + 2a_{y_1})\,d\tau + \varepsilon t\right] + k_H \frac{f_1(t)}{H_0}. \qquad (11.6.161)$$

Since in contemporary precision gyroscopic devices applied in inertial systems, oscillations δH of angular momentum are minute, then in accordance with (11.6.160) we will take $k_H \approx 1$ and instead of (11.6.161) we will obtain

$$\beta = \left(\frac{\mu_0}{H_0} + \frac{\delta\mu}{H_0}\right)\left[\int_0^t (-g_0\beta + \delta a_{y_1})\,d\tau + \varepsilon t\right] + \frac{f_1(t)}{H_0} \qquad (11.6.162)$$

or

$$\dot\beta + \left(\frac{\mu_0}{H_0} + \frac{\delta\mu}{H_0}\right) g_0\beta = \left(\frac{\mu_0}{H_0} + \frac{\delta\mu}{H_0}\right)(2a_{y_1} + \varepsilon) + \frac{\dot f_1(t)}{H_0}. \qquad (11.6.163)$$

We will transform coefficient $(\frac{\mu_0}{H_0} + \frac{\delta\mu}{H_0})$ in the following way:

$$\frac{\mu_0}{H_0} + \frac{\delta\mu}{H_0} = \frac{\mu_0}{H_0}\left(1 + \frac{\delta\mu}{\mu_0}\right). \qquad (11.6.164)$$

According to condition (11.6.130) of undisturbability of the IV

$$\frac{\mu_0}{H_0} = \frac{1}{R};$$

we will designate

$$k_\mu = 1 + \frac{\delta\mu}{\mu_0}. \qquad (11.6.165)$$

Then instead of (11.6.164) we will obtain

$$\frac{\mu_0}{H_0} + \frac{\delta\mu}{H_0} = \frac{1}{R} k_\mu.$$

Considering this expression, we will rewrite (11.6.163) in the form

$$\ddot\beta + k_\mu \frac{g_0}{R}\beta = \frac{k_\mu}{R}(2a_{y_1} + \varepsilon) + \frac{\dot f_1(t)}{H_0}. \qquad (11.6.166)$$

In the first approximation for $\frac{\delta\mu}{\mu_0} \ll 1$, according to (11.6.165), we will take $k_\mu \approx 1$; then, considering (11.6.24), we will obtain

$$\ddot\beta + \nu^2\beta = \frac{1}{R}(2a_{y_1} + \varepsilon) + \frac{\dot f_1(t)}{H_0}. \qquad (11.6.167)$$

Designating angular velocity of precession (drift) of the gyroscope

$$\omega_\Delta = \frac{f_1(t)}{H_0}, \qquad (11.6.168)$$

we will rewrite (11.6.167) in the form

$$\ddot\beta + \nu^2\beta = \frac{1}{R}(2a_{y_1} + \varepsilon) + \dot\omega_\Delta. \qquad (11.6.169)$$

Further it is necessary to establish the dependence between error β of the IV and error caused by it δS in determination by the inertial system of path passed over by the object. Using the block diagram of the IV shown in Fig. 11.28, and

supplementing it with the second integrator I_{II_N} for obtaining of path passed over by the object $S_N \equiv S$ in the plane of the meridian, we will compose equation of the channel for determination of path in the form

$$\left(pv_N - g_0\beta + \delta a_{y_1} + \varepsilon\right)\frac{k_1k_2k_4}{p^2} = S. \qquad (11.6.170)$$

Considering that $k_1k_2k_4 = 1$ and not considering changes of these coefficients relative to their computed values, for error of path δS we have

$$\delta S = S - \frac{v_N}{p} = \frac{1}{p^2}\left(-g_0\beta + \delta a_{y_1} + \varepsilon\right)$$

or

$$\delta\ddot{S} = -g_0\beta + \delta a_{y_1} + \varepsilon, \qquad (11.6.171)$$

where β is determined by equation (11.6.169).

During integration of equations (11.6.169) and (11.6.171) for determination of error δS, we will use the operational method.* For this we will find the transform of equation (11.6.169) for initial conditions** $t = 0$, $\beta(0) = \beta_0$, $\dot{\beta}(0) = \dot{\beta}_0$:

$$(s^2 + \nu^2)\beta(s) = \frac{1}{R}\left(\delta a_{y_1} + \varepsilon\right) + s\omega_n + s\beta_0 + \dot{\beta}_0 \qquad (11.6.172)$$

and will consider subsequently that δa_{y_1}, ε and ω_{dr} are random variables which are not time-dependent.

Then we will find the transform of equation (11.6.171) for initial conditions $t = 0$, $\delta S(0) = \dot{\delta S}(0) = 0$:

$$s^2\delta S(s) = -g_0\beta(s) + \delta a_{y_1} + \varepsilon. \qquad (11.6.173)$$

Substituting here $\beta(s)$ from (11.6.172), we will obtain

$$\delta S(s) = -\frac{g_0\beta_0}{s^2 + \nu^2} + \frac{\delta a_{y_1} + \varepsilon}{s^2 + \nu^2} - \frac{g_0\left(\beta_0 + \omega_n\right)}{(s^2 + \nu^2)s}. \qquad (11.6.174)$$

Passing over from transforms to inverse transforms, we have

$$\delta S = \beta_0 R(\cos\nu t - 1) + \frac{1}{g_0}\delta a_{y_1} R(1 - \cos\nu t) +$$

$$+ \frac{1}{g_0}\varepsilon R(1 - \cos\nu t) - R\left(\beta_0 + \omega_n\right)t\left(1 - \frac{\sin\nu t}{\nu t}\right). \qquad (11.6.175)$$

We will designate components error δS:

$$\delta S_1 = \beta_0 R(\cos\nu t - 1); \qquad (11.6.176)$$

$$\delta S_2 = \frac{1}{g_0}\delta a_{y_1} R(1 - \cos\nu t); \qquad (11.6.177)$$

$$\delta S_3 = \frac{1}{g_0}\varepsilon R(1 - \cos\nu t); \qquad (11.6.178)$$

*See [88], and also: V. A. Ditkin and P. I. Kuznetsov. Reference book on operational calculus. State Press for Technical and Theoretical Literature, 1951.

**It is assumed that initial angular velocity $\dot{\beta}_0$ is caused by sampling action on the gyroscope.

$$\delta S_4 = -R\dot{\beta}_0 t\left(1-\frac{\sin \nu t}{\nu t}\right); \tag{11.6.179}$$

$$\delta S_5 = -R\omega_{dr} t\left(1-\frac{\sin \nu t}{\nu t}\right). \tag{11.6.180}$$

Then

$$\delta S = \delta S_1 + \delta S_2 + \delta S_3 + \delta S_4 + \delta S_5. \tag{11.6.181}$$

From formula (11.6.176) it follows that component δS_1 is caused by inaccuracy of initial setting of the GSP. Considering β_0 to be a random variable, for root mean square value $\sigma_{\delta S_1}$ of error δS_1 we will obtain

$$\sigma_{\delta S_1} = R\sigma_{\beta_0}(\cos \nu t - 1). \tag{11.6.182}$$

Component δS_2 [see (11.6.177)] is caused by error δa_{y_1} of the accelerometer. For small time t formula (11.6.177) can approximately be represented in the form

$$\delta S_2 = \frac{1}{g_0}\delta a_{y_1} R(1-\cos \nu t) \approx \frac{\delta a_{y_1} t^2}{2}, \tag{11.6.183}$$

i.e., at first error δS_2 is increased proportionally to the square of time, since in the IV there is carried out double integration of acceleration measured by the accelerometer. However, subsequently (for large t), increase δS_2 is limited due to the presence in the circuit of feedback, and the error acquires an oscillatory character.

For root mean square value $\sigma_{\delta S_2}$ of error δS_2 we will obtain

$$\sigma_{\delta S_2} = \frac{R}{g_0}\sigma_{\delta a_{y_1}}(1-\cos \nu t). \tag{11.6.184}$$

What has been said above completely pertains also to error δS_3 [see (11.6.178)] due to drift of the first integrator; therefore, for root mean square value $\sigma_{\delta S_3}$ of error δS_3 we have

$$\sigma_{\delta S_3} = \frac{R}{g_0}\sigma_{\varepsilon_0}(1-\cos \nu t). \tag{11.6.185}$$

Error δS_4 [see (11.6.179)] is caused by initial angular velocity $\dot{\beta}_0$ of the GSP. This error, besides an oscillatory component, contains a component which increases proportionally to time.

For root mean square value $\sigma_{\delta S_4}$ of error δS_4 we have

$$\sigma_{\delta S_4} = R\sigma_{\dot{\beta}_0} t\left(1-\frac{\sin \nu t}{\nu t}\right). \tag{11.6.186}$$

From formula (11.6.180) it follows that error δS_5 is caused by drift of the gyroscope with angular velocity ω_{dr}. Thus, gyroscope drift, besides oscillatory change of error δS_5, causes continuous build-up of it in proportion to time. For

small interval of time t, formula (11.6.180) can be approximately represented in the form

$$\delta S_5 = -R\omega_{\Delta}\left(t - \frac{\sin \nu t}{\nu}\right) \approx -\frac{g_0 \omega_{\Delta} t^3}{6}. \qquad (11.6.187)$$

From what has been said above, it follows that there have to be made very high requirements of the accuracy of gyroscope applied in INS.

For root mean square value $\sigma_{\delta S_5}$ of error δS_5, according to (11.6.180), we have

$$\sigma_{\delta S_5} = R\sigma_{\omega_{dr}}\left(t - \frac{\sin \nu t}{\nu}\right), \qquad (11.6.188)$$

where $\sigma_{\omega_{dr}}$ is root mean square value of gyroscope drift velocity.

Considering in (11.6.181) all components of error δS to be independent, and taking into account (11.6.182), (11.6.184), (11.6.185), (11.6.186) and (11.6.188), for root mean square value $\sigma_{\delta S}$ of error δS caused by errors of the IV in determination by the INS of path of the object, we have

$$\sigma_{\delta S}^2 = R^2\sigma_{\Delta_x}^2(\cos \nu t - 1)^2 + \frac{R^2}{g_0^2}\left(\sigma_{\omega_{dr_y}}^2 + \sigma_{\Delta_z}^2\right)(1 - \cos \nu t)^2 + \\ + R^2\left(\sigma_{\Delta_x}^2 + \sigma_{\omega_{dr}}^2\right)t^2\left(1 - \frac{\sin \nu t}{\nu t}\right)^2. \qquad (11.6.189)$$

Example 11.20: To calculate root mean square $\sigma_{\delta S}$ and limiting δS_m values of error in determination by the inertial system of path of the object, which is due to errors of the IV, for the conditions of Examples 11.15 and 11.18; root mean square value of angular velocity of gyroscope drift $\sigma_{\omega_{dr}} = 0.2$ deg/hr; period of operation of system $t = \frac{2\pi}{\nu} \cdot \frac{1}{8} = 633$ sec; $\nu = 1.24 \cdot 10^{-3}$ 1/sec; R = 6371 km; $\sigma_{\Delta} = 0.5$ cm/sec^2.

Solution: 1. By formula (11.6.189) we determine $\sigma_{\delta S}$:

$$\sigma_{\delta S} = 1{,}55 \text{ km}.$$

2. By the same method as in Example 11.15, we find limiting value of error δS_m (for $\overline{\delta s} = 0$):

$$\delta S_m = 3\sigma_{\delta S} = 3.3 \cdot 1{,}55 = 5{,}12 \text{ km}.$$

From Example 11.20 it follows that for the accepted values of errors of elements of the IV and for t ≈ 10 minutes, error in determination by the inertial system of path passed over by the object is very great.

Error δS causes corresponding error $\delta\varphi$ in determination by the inertial system of latitude of the position of the object. According to (11.6.35)

$$\delta\varphi = \frac{1}{R}\delta S, \qquad (11.6.190)$$

whence for root mean square value $\sigma_{\delta\varphi}$ of error $\delta\varphi$, we will obtain

$$\sigma_{\delta\varphi} = \frac{1}{R}\sigma_{\delta S}. \qquad (11.6.191)$$

Thus, for the conditions of Example 11.20, we have $\sigma_{\delta\varphi} = 0'.83$, and limiting value of error $\delta\varphi_m = 2'.74$.

During finding of error δS we did not take into account instrument error due to azimuthal deflection of the GSP. This question is considered in separate works (for instance, [162, 175]*). Let us note only that inaccuracy of initial azimuthal orientation of GSP and drift of azimuthal gyroscope cause error in the IV, and also error in determination by the inertial system of velocity and path passed over by the object. Therefore, very high requirements are made of the accuracy of initial azimuthal orientation of the GSP and azimuthal gyroscope. Besides what has been indicated, during motion, for instance, of an aircraft along an orthodrome, azimuthal drift of the platform leads to lateral displacement of center of gravity of the aircraft from its assigned orthodrome.

Above there were considered only certain basic instrument errors of IV and their influence on accuracy of determination by the inertial system of coordinates of the object. It is possible to become acquainted with analysis of other instrument errors of IV in the literature (see, for instance, [142, 162, 117] and the books indicated in the note on p. 494).

6. Systematic Errors of IV

a) General Characteristics of Systematic Errors

Among systematic errors of IV there are included:*

1) errors due to rotation of Earth and vehicular motion with respect to the Earth caused by induced and Coriolis accelerations;

2) errors due to non-sphericity of the Earth and non-centrality of its gravitational field;

3) errors caused by vertical component of velocity of the object, and others.

b) Errors Caused by Induced and Coriolis Accelerations

During the analysis of IV in the case of geographic orientation of axes of sensitivity of accelerometers, and with consideration of induced and Coriolis

*See also: A. A. Yakushenkov. Fundamentals of inertial navigation, "Naval Transport," 1963; J. Carpentier, J. Radix, J. Bouvet, G. Bonnevalle. Navigation par inertie. Dunod, Paris, 1962.

**Errors considered in Par. 5g, which are connected with damping of oscillations of IV, may also be counted as systematic errors.

accelerations, there were obtained equations (11.6.70) of motion of the considered inertial vertical

$$\left.\begin{aligned}\ddot{\alpha}+\frac{g_0}{R}\alpha&=-\frac{1}{R}\left(\frac{v_E v_N}{R}\operatorname{tg}\varphi+Uv_N\sin\varphi\right)\\ \ddot{\beta}+\frac{g_0}{R}\beta&=\frac{1}{R}\left(\frac{v_E^2}{R}\operatorname{tg}\varphi+U^2R\cos\varphi\sin\varphi+2Uv_E\sin\varphi\right)\end{aligned}\right\}$$

Since these equations are nonhomogeneous, then consequently the IV during geographic orientation of axes of sensitivity of accelerometers will not be directed along the geocentric vertical, i.e., it has a deviation from it caused by induced and Coriolis accelerations. These deviations, which are usually called <u>systematic errors</u>, are determined, according to (11.6.70), for zero initial conditions by the following relationships:

$$\left.\begin{aligned}\alpha^{y}&=-\frac{1}{Rv}\int_0^t\left(\frac{v_E v_N}{R}\operatorname{tg}\varphi+Uv_N\sin\varphi\right)\sin v(t-\tau)\,d\tau\\ \beta^{y}&=\frac{1}{Rv}\int_0^t\left(\frac{v_E^2}{R}\operatorname{tg}\varphi+U^2R\cos\varphi\sin\varphi+\right.\\ &\qquad\left.+2Uv_E\sin\varphi\right)\sin v(t-\tau)\,d\tau\end{aligned}\right\}\qquad(11.6.192)$$

Errors α^{acc} and β^{acc} are analogous to errors of a physical pendulum in a Cardan suspension [see (4.5.23)*]; thus the fundamental circumstance is that the IV is not influenced by relative accelerations $\dot{v}_N$, $\dot{v}_E$ of the object. Errors α^{acc} and β^{acc} are determined to be quantities of the same order as in Examples 4.3 and 4.4 (§ 4.5, Par. 6). Since such errors are for inertial systems impermissibly large, they are usually compensated.**

*Formulas (4.5.23) characterize static values of errors of the short-period physical pendulum.

**Above it was shown that induced and Coriolis accelerations cause error in the IV. Let us note that errors in determination of coordinates of the object depend not on these accelerations themselves, but on their derivatives. Actually, latitude φ is determined as the double integral of readings α_{y_1} of the northern accelerometer (Fig. 11.21). Considering (11.6.64), we have

$$\varphi=\varphi_0+\frac{1}{R}\int_0^t\int_0^t\left(\dot{v}_N+\frac{v_E^2}{R}\operatorname{tg}\varphi+U^2R\sin\varphi\cos\varphi+2Uv_E\sin\varphi-g_0\beta\right)d\tau^2.$$

Constant component β^0 of error of the IV, according to (11.6.70), will be $\beta^0=\frac{1}{g_0}\left(\frac{v_E^2}{R}\operatorname{tg}\varphi+U^2R\cos\varphi\sin\varphi+2Uv_E\sin\varphi\right)$. Substituting $\beta=\beta^0$ in the expression for φ, we have $\varphi=\varphi_0+\frac{1}{R}\int_0^t\int_0^t\dot{v}_N d\tau^2$; this coincides with (11.6.5), i.e., induced and Coriolis accelerations do not cause error in latitude φ (with accuracy up to the oscillatory component of error). If, however, now we consider the alternating character of accelerations, then it is possible to show that error in φ will depend on the derivative of these accelerations.

For compensation of systematic errors of IV due to induced and Coriolis accelerations there are directly used values of components v_N and v_E of velocity of the object and latitude φ of the position of the object which are produced by the inertial system itself. One of the methods of compensation in IV of these systematic errors is that, corrections produced by the computer $2Uv_N \sin \varphi$, $2Uv_E \sin \varphi$,...[see (11.6.68)] are introduced together with readings of accelerometers into the input of the first integrators; due to this, from readings of the accelerometers there are eliminated components of induced and Coriolis accelerations. Let us note that certain terms of (11.6.68), for instance $U^2 R \cos \varphi \sin \varphi$, can turn out to be small as compared to other terms, and therefore in a number of cases it is not necessary to compensate them.

From the first equation of system (11.6.68) it follows that it is also necessary to compensate the horizontal component $U \cos \varphi$ of angular velocity of diurnal rotation of the Earth. For this it is necessary to apply to the eastern gyroscope of the GSP corrective torque $HU \cos \varphi$, where H is angular momentum of the gyroscope. The scheme of compensation in IV of the considered systematic errors will be given in Par. 8.

Besides INS with geographic orientation of axes of sensitivity of the accelerometers, there are also applied inertial systems with orthodromic orientation of these axes (Par. 1b), when the latter are directed along the assigned orthodrome of motion of the object and perpendicular to it. With such orientation of axes of sensitivity of accelerometers, the IV also has systematic errors due to induced and Coriolis accelerations which are analogous to the corresponding errors of a physical pendulum in a Cardan suspension during motion of the object along an orthodrome [see (4.5.24)]. These errors of the IV must also be compensated. Detailed analysis of systematic errors of the IV for orthodromic orientation of axes of sensitivity of accelerometers is given in a number of works (for instance, [142, 162]).

It is of interest to find if there exists an orientation of axes of sensitivity of the accelerometers for which the IV would not have systematic errors due to induced and Coriolis accelerations. One of such possibilities is indicated by the analysis of errors of the physical pendulum with period of 84.4 minutes, which was given in § 4.5, Par. 5. There it was shown that if such a pendulum is installed on a platform which is stabilized in azimuth from a free gyroscope, then it is undisturbed by relative, induced and Coriolis accelerations, i.e., it does not have

systematic errors due to rotation of the Earth and motion of the vehicle, and for zero initial conditions conserves the direction of the geocentric vertical given to it at the initial moment. Such a method of installation of the physical pendulum corresponds to an inertial system with free orientation in azimuth (Par. 1b), when axes of sensitivity of accelerometers have in plane of horizon constant azimuthal orientation relative to inertial space.

Let us show that in this case the IV does not have systematic errors due to induced and Coriolis accelerations. For this we will compose equation of motion of the IV, using shortened Euler equations (11.6.10), in which p_1 and q_1 are determined by formulas (11.6.11). If in the latter we disregard small terms $u_\zeta\alpha$ and $u_\zeta\beta$ for not too large latitudes φ, then we have approximately

$$p_1 = -\dot{\beta} + u_\xi; \quad q_1 = \dot{\alpha} + u_\eta. \tag{11.6.193}$$

Since in the considered case the GSP preserves horizontal position, then components u_ξ and u_η of induced angular velocity of axes $O\xi\eta\zeta$ are connected with linear velocities v_ξ and v_η by relationships (4.5.10)

$$u_\xi = -\frac{v_\eta}{R}; \quad u_\eta = \frac{v_\xi}{R}. \tag{11.6.194}$$

Then instead of (11.6.193) we will obtain

$$p_1 = -\beta - \frac{v_\eta}{R}; \quad q_1 = \dot{\alpha} + \frac{v_\xi}{R}. \tag{11.6.195}$$

As torques M_{x_1} and M_{y_1} we will consider only torques $M_{cor\ x}$ and $M_{cor\ y}$, applied to the IV by the system of integral correction, i.e., we will assume that $M_{x_1} = M_{cor\ x}$, $M_{y_1} = M_{cor\ y}$ [see (11.6.14)]; thus, we will consider $M_{cor\ x}$ and $M_{cor\ y}$ to be proportional to integrals of readings of the accelerometers α_{x_1} and α_{y_1}. The latter are determined for $W_\zeta \ll g_0$ by relationships (11.6.16). Substituting in them in place W_ξ and W_η expressions (4.5.9), we will obtain

$$a_{x_1} = \dot{v}_\xi - v_\eta u_\zeta - g_0\alpha; \quad a_{y_1} = \dot{v}_\eta + v_\xi u_\zeta - g_0\beta. \tag{11.6.196}$$

Then for torques $M_{cor\ x}$ and $M_{cor\ y}$ of integral correction of the IV, by analogy with (11.6.19), we will obtain

$$M_{cor\ x} = \mu\int a_{x_1}d\tau = \mu\int\left(\dot{v}_\xi - v_\eta u_\zeta - g_0\alpha\right)d\tau; \quad M_{cor\ y} = \mu\int a_{y_1}d\tau = \mu\int\left(\dot{v}_\eta + v_\xi u_\zeta - g_0\beta\right)d\tau. \tag{11.6.197}$$

Putting (11.6.195) and (11.6.197) in (11.6.10), we have

$$\left.\begin{aligned}\dot{\alpha} + \frac{v_\xi}{R} &= \frac{\mu}{H}\int\left(\dot{v}_\xi - v_\eta u_\zeta - g_0\alpha\right)d\tau\\ \dot{\beta} + \frac{v_\eta}{R} &= \frac{\mu}{H}\int\left(\dot{v}_\eta + v_\xi u_\zeta - g_0\beta\right)d\tau\end{aligned}\right\} \tag{11.6.198}$$

whence after differentiation we find

$$\left.\begin{aligned}\ddot{\alpha}+\frac{\mu}{H}g_0\alpha&=\dot{v}_\eta\left(\frac{\mu}{H}-\frac{1}{R}\right)-\frac{\mu}{H}v_\eta u_\zeta\\ \ddot{\beta}+\frac{\mu}{H}g_0\beta&=\dot{v}_\eta\left(\frac{\mu}{H}-\frac{1}{R}\right)+\frac{\mu}{H}v_\xi u_\zeta\end{aligned}\right\}\qquad(11.6.199)$$

When condition of undisturbability (11.6.22) is satisfied, we will obtain equations of the IV in the form

$$\left.\begin{aligned}\ddot{\alpha}+\frac{g_0}{R}\alpha&=-\frac{1}{R}v_\eta u_\zeta\\ \ddot{\beta}+\frac{g_0}{R}\beta&=\frac{1}{R}v_\xi u_\zeta\end{aligned}\right\}\qquad(11.6.200)$$

For the considered inertial system with free orientation in azimuth, component u_ζ of angular velocity of trihedron $O\xi\eta\zeta$ relative to inertial space along the axis $O\zeta$ is equal to zero ($u_\zeta = 0$). Then, instead of (11.6.200) we will obtain

$$\left.\begin{aligned}\ddot{\alpha}+\frac{g_0}{R}\alpha&=0\\ \ddot{\beta}+\frac{g_0}{R}\beta&=0\end{aligned}\right\}$$

i.e., homogeneous equations, and, consequently, the IV does not have systematic errors due to induced and Coriolis accelerations.

Thus, for inertial system with free orientation in azimuth, when axes of sensitivity of accelerometers have in plane of horizon constant azimuthal orientation relative to inertial space, there are absent systematic errors caused by rotation of Earth and vehicular motion. Application of such a system ensures automatic compensation of the indicated errors.

However, we should consider that an inertial system with free orientation in azimuth has an essential deficiency connected with the absence in the system of a directional torque; this can lead in a real system to drift of the GSP in azimuth.

c) The Influence of Non-Sphericity of the Earth

One of the systematic errors of IV is error due to non-sphericity of the Earth and non-centrality of its gravitational field. Detailed investigation of this error and the bases of methods of its compensation goes beyond the scope of the present account of the theory of IV. Therefore, we will be limited only to certain short explanations.

Earlier during investigation of different GU we assumed the Earth to be a sphere of average radius R = 6371.228 km. In reality, the Earth has the form of an ellipsoid of rotation, which is called the terrestrial spheriod [160]. On the terrestrial spheroid (Fig. 11.30), the equator and the parallel drawn through the

place of observation A constitute circles. The meridian passing through axis $P_N P_S$ (which coincides with axis of rotation of Earth) and place of observation A is an ellipse. Semimajor axis of the ellipsoid will be designated a; the semiminor axis will be b; radius of parallel on the sphere will be designated ρ. At point A on the terrestrial spheroid we will assign a system of coordinates $O\xi\eta\zeta$ having geographic orientation: axis $O\xi$ is directed along the tangent to the parallel to the east; axis $O\eta$ — along the tangent to the meridian to the north; axis $O\zeta$ — along the line AB at the angle of geographic latitude φ to the plane of the equator, i.e., along the normal to the ellipsoidal surface of the Earth (along the line of action of gravity). Geocentric latitude φ' is determined by the angle between line AC, which connects place of observation A with center C of the ellipsoid, and the plane of the equator. The cross section of the terrestrial spheroid along the plane of the meridian is represented in Fig. 11.31.

In the Soviet Union, as the terrestrail spheroid there is accepted the spheroid of Krasovskiy, for which the semimajor axes a = 6378.245 km and the semiminor axis b = 6356.863 km. The shape of the terrestrial spheroid is characterized by its flattening α, which is determined by relationship [160]

$$\alpha = \frac{a-b}{a}. \qquad (11.6.201)$$

For the spheroid of Krasovskiy $\alpha = \frac{1}{298.3} \approx \frac{1}{300}$. Frequently instead of flattening α there is used the magnitude of eccentricity of the terrestrial spheroid

$$e = \frac{\sqrt{a^2-b^2}}{a}. \qquad (11.6.202)$$

Between α and e there exists the relationship

$$e^2 = 2\alpha - \alpha^2 \approx 2\alpha.$$

Difference γ between geographic φ and geocentric φ' latitudes is called the <u>reduction</u> [160]

$$\gamma = \varphi - \varphi'; \qquad (11.6.203)$$

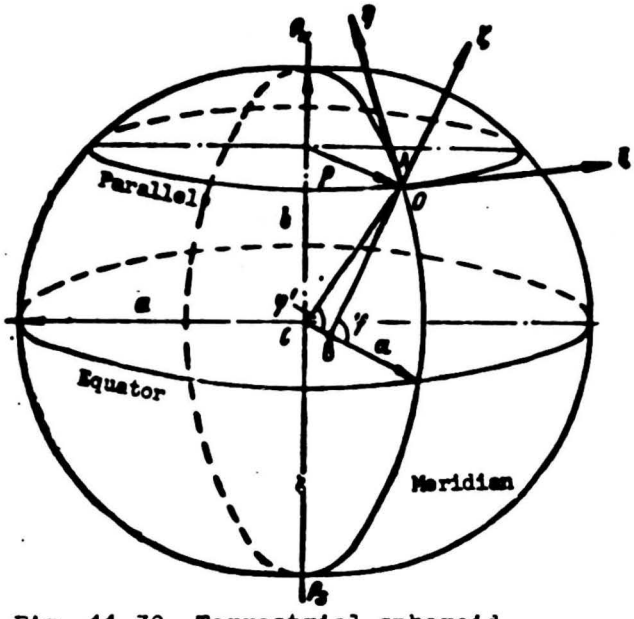

Fig. 11.30. Terrestrial spheroid.

it is determined by relationship [66]

$$\operatorname{tg}\gamma \approx \alpha \sin 2\varphi \qquad (11.6.204)$$

or, considering that $\alpha = \frac{1}{300}$,

$$\gamma \approx 11'27'' \sin 2\varphi. \qquad (11.6.205)$$

The largest value, 11'27" is attained by reduction γ at latitude $\varphi = 45^\circ$.

Radius ρ of parallel on the spheroid (Fig. 11.30 and 11.31) is determined by relationship [160]

$$\rho = \frac{a\cos\varphi}{\sqrt{1-e^2\sin^2\varphi}}. \qquad (11.6.206)$$

Instead of ρ, B. V. Bulgakov [20] uses the quantity $R_1 = AC'$ (Fig. 11.31), which constitutes the main principal radius of curvature of the surface of the spheroid in the plane which is perpendicular to plane of meridian. Since

$$R_1 = \frac{\rho}{\cos\varphi}, \qquad (11.6.207)$$

then in virtue of (11.6.206)

$$R_1 = \frac{a}{\sqrt{1-e^2\sin^2\varphi}}. \qquad (11.6.208)$$

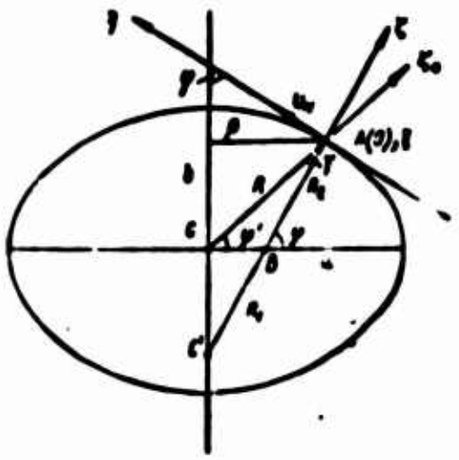

Fig. 11.31. Cross section of terrestrial spheroid along plane of meridian.

We will introduce principal radius of curvature $R_2 = AB$ of meridianal cross section of the ellipsoid. For it we have the following expression [20, 160]:

$$R_2 = \frac{a(1-e^2)}{(1-e^2\sin^2\varphi)^{\frac{3}{2}}}. \qquad (11.6.209)$$

Latitudes φ and φ' are connected through eccentricity e by relationship [20]

$$\operatorname{tg}\varphi' = (1-e^2)\operatorname{tg}\varphi. \qquad (11.6.210)$$

During motion of the object over the terrestrial spheroid, there are changed latitude φ and longitude λ of its position. By analogy with (1.2.26) and (1.2.27), we have

$$\dot{\varphi} = \frac{v_N}{R_2}, \quad \dot{\lambda} = \frac{v_E}{\rho} = \frac{v_E}{R_1\cos\varphi}, \qquad (11.6.211)$$

where v_N and v_E are components of relative velocity of the object along the meridian and the parallel.

For corresponding components of absolute velocity of the object, by analogy with (1.2.29) we have

$$V_E = R_1 U\cos\varphi + v_E; \quad V_N = v_N. \qquad (11.6.212)$$

We will give formulas for components u_ξ, u_η, u_ζ of angular velocity $\vec{u}$ of axes $O\xi\eta\zeta$ which is caused by rotation of the Earth and motion of the vehicle, taking into account oblateness of the Earth. For u_ξ, u_η, u_ζ, with geographic orientation of axes $O\xi$ and $O\eta$, instead of formulas (1.2.28) we easily obtain for the terrestrial spheroid the relationships

$$\left.\begin{aligned} u_\xi &= -\dot{\varphi} = -\frac{v_N}{R_1} = -\frac{V_N}{R_1} \\ u_\eta &= (U+\lambda)\cos\varphi = U\cos\varphi + \frac{v_E}{R_1} = \frac{V_E}{R_1} \\ u_\zeta &= (U+\lambda)\sin\varphi = U\sin\varphi + \frac{v_E}{R_1}\operatorname{tg}\varphi \end{aligned}\right| \qquad (11.6.213)$$

which for $R_1 = R_2 = R$ became (1.2.28). Let us note that in (11.6.213) there is contained geographic latitude φ, while in formulas (1.2.28) by φ there is denoted, in general, geocentric latitude φ'. This distinction is due to the fact that earlier (Fig. 1.9) axis $O\zeta$ was considered by us to be directed along radius of Earth, which was assumed to be a sphere. In the case of the terrestrial spheroid (Fig. 11.31) axis $O\zeta$ is directed along the normal to the ellipsoidal surface of the Earth, i.e., along the line of action of gravity. Formulas (11.6.213) can be used during the analysis of precision GD, in particular IV, if it is necessary to consider non-sphericity of the Earth.

For this purpose it is necessary to obtain expressions for components of total acceleration $\overline{W}$ of the object, while taking into account non-sphericity of the Earth. Let us use general formulas (2.2.37) for components W_ξ, W_η, W_ζ of acceleration $\overline{W}$ along axes $O\xi\eta\zeta$ (Fig. 11.30)

$$\left.\begin{aligned} W_\xi &= \dot{v}_\xi + v_\eta u_\zeta - v_\zeta u_\eta \\ W_\eta &= \dot{v}_\eta + v_\zeta u_\xi - v_\xi u_\zeta \\ W_\zeta &= \dot{v}_\zeta + v_\xi u_\eta - v_\eta u_\xi \end{aligned}\right| \qquad (11.6.214)$$

where u_ξ, u_η, u_ζ – projections of angular velocity $\overline{u}$ of axes $O\xi\eta\zeta$ which are determined by formulas (11.6.213);

v_ξ, v_η, v_ζ – projections of absolute velocity of the object.

Projections $v_\xi = V_E$ and $v_\eta = V_N$ are determined by formulas (11.6.212). If the object, for instance a ship, moves over the terrestrial spheroid, i.e., its height above sea level $\zeta = h = 0$, then its "vertical" velocity $v_\zeta = 0$.*

*We will show that indeed $v_\zeta = 0$. Let us introduce in Fig. 11.31 the geocentric vertical $O\zeta_0$, directed along radius of Earth $R = CA$. Quantity R is determined with accuracy up to e^2 (e is eccentricity) by relationship (160) $R = a\left(1 - \frac{e^2}{3}\sin^2\varphi\right)$, whence $\dot{R} = \frac{dR}{dt} = -a\frac{e^2}{2}\sin 2\varphi\cdot\dot{\varphi}$. If we consider (11.6.211), (11.6.209) and the formula for reduction (160) $\sin\gamma = \gamma = \frac{e^2}{2}\sin 2\varphi$, then for $\dot{R}$ with accuracy up to e^2 we have $\dot{R} = -\frac{e^2}{2}v_N\sin 2\varphi = -v_N\sin\gamma$. Let us find absolute velocity of object v_{ζ_0} along axis $O\zeta_0$; from Fig. 11.31 we obtain $v_{\zeta_0} = \dot{R} = -v_N\sin\gamma + v_\zeta\cos\gamma$. Since

Putting (11.6.213), (11.6.212) and $v_\zeta = 0$, $\dot{v}_\zeta = 0$ in (11.6.214), we will obtain

$$\left.\begin{aligned} W_\xi &= \dot{v}_\xi - \frac{v_\xi v_N}{R_1} \operatorname{tg}\varphi - Uv_N \sin\varphi\left(1+\frac{R_1}{R_2}\right) + U\dot{R}_1\cos\varphi \\ W_\eta &= \dot{v}_N + \frac{v_\xi}{R_1}\operatorname{tg}\varphi + U^2 R_1 \cos\varphi\sin\varphi + 2Uv_\xi\sin\varphi \\ W_\zeta &= -\frac{v_\xi}{R_1} - \frac{v_N}{R_2} - U^2 R_1\cos^2\varphi - 2Uv_\xi\cos\varphi \end{aligned}\right| \qquad (11.6.215)$$

where R_1 can be represented in the form

$$\dot{R}_1 = \frac{dR_1}{dt} = \frac{dR_1}{d\varphi}\cdot\frac{d\varphi}{dt} = \frac{v_N}{R_2}\cdot\frac{dR_1}{d\varphi}. \qquad (11.6.216)$$

Derivative $\frac{dR_1}{dt}$ can be determined from (11.6.208), (11.6.209) and (11.6.211); we have $\dot{R}_1 = \frac{v_N e^2 \sin\varphi\cos\varphi}{1-e^2}$ and $U\dot{R}_1\cos\varphi = \frac{Uv_N e^2\sin\varphi\cos^2\varphi}{1-e^2}$; then instead of (11.6.215) we will obtain

$$\left.\begin{aligned} W_\xi &= \dot{v}_\xi - \frac{v_\xi v_N}{R_1}\operatorname{tg}\varphi - 2Uv_N\sin\varphi \\ W_\eta &= \dot{v}_N + \frac{v_\xi}{R_1}\operatorname{tg}\varphi + U^2 R_1\cos\varphi\sin\varphi + 2Uv_\xi\sin\varphi \\ W_\zeta &= -\frac{v_\xi}{R_1} - \frac{v_N}{R_2} - U^2 R_1\cos^2\varphi - 2Uv_\xi\cos\varphi \end{aligned}\right| \qquad (11.6.217)$$

Comparing (11.6.217) with analogous formulas (2.2.28) for $h = 0$, which were obtained when non-sphericity of the Earth was disregarded, we see that in (11.6.217) there are contained radii of curvature R_1 and R_2 of the spheroidal surface of the Earth, which differ from radius of Earth R in (2.2.28). Furthermore, in formulas (2.2.28) by φ there is denoted, in general, geocentric latitude φ, while in (11.6.217) there is directly contained quantity φ'. In other respects, formulas (11.6.217) do not differ from (2.2.28). For $R_1 = R_2 = R$ and $\varphi = \varphi'$ formulas (11.6.217) become (2.2.28).

Using during investigation of IV relationships (11.6.217), we must take into account the following circumstance. For the terrestrial spheroid, axis $O\zeta$ is directed along the normal to the ellipsoidal surface of the Earth, i.e., along the line of action of gravity, if by the latter we understand resultant of the force of terrestrial gravitation and the centrifugal force due to rotation of the Earth. Therefore, during investigation of IV, thus centrifugal force can be included in the force of gravity, and then in expressions (11.6.217) for W_η, W_ζ it is not necessary to take into account projections of centripetal acceleration from rotation

[FOOTNOTE CONT'D FROM PRECEDING PAGE].

$-v_N\sin\gamma = -v_N\frac{\varphi^2}{2}\sin 2\varphi = \dot{R}$, then, consequently, $v_\zeta = 0$, which it was required to prove.

of the Earth (terms containing e^2). In this case instead of (11.6.217) we will obtain

$$\left.\begin{aligned} W_\xi &= \dot v_\xi - \frac{v_\xi v_N}{R_0}\operatorname{tg}\varphi - 2\Omega v_N \sin\varphi \\ W_\eta &= \dot v_N + \frac{v_\xi^2}{R_0}\operatorname{tg}\varphi + 2\Omega v_\xi \sin\varphi \\ W_\zeta &= -\frac{v_\xi^2}{R_0} - \frac{v_N^2}{R_0} - 2\Omega v_\xi \cos\varphi \end{aligned}\right| \qquad (11.6.218)$$

Let us note one more important circumstance. During investigation of IV on the assumption that Earth is a sphere, it was assumed that gravitational field of the Earth is central, and therefore we directed acceleration $\vec g_0$ due to force of terrestrial gravitation along the geocentric vertical (Fig. 2.11). However, in reality, due to oblateness of the Earth, direction of gravitational field somewhat deviates from the geocentric vertical, mainly in the plane of the geographic meridian, by angle $\Delta\gamma$, which is determined by relationship [142]

$$\Delta\gamma = \delta' \beta \sin 2\varphi', \qquad (11.6.219)$$

where φ' is geocentric latitude.

Therefore, for projections $g_{0\xi}$, $g_{0\eta}$, $g_{0\zeta}$ of acceleration $\vec g_0$ onto axes $O\xi\eta\zeta$ (Fig. 2.11), if we take into account angle $\Delta\gamma$ of deviation of vector $\vec g_0$ from the geocentric vertical $O\zeta$, instead of (2.3.53) we should use formulas

$$\left.\begin{aligned} g_{0\xi} &= 0;\ g_{0\eta} = -g_0 \sin\Delta\gamma \approx -g_0\Delta\gamma; \\ g_{0\zeta} &= -g_0\cos\Delta\gamma \approx -g_0. \end{aligned}\right| \qquad (11.6.220)$$

These relationships are sometimes taken [142] into account during the analysis of IV on the assumption that Earth has form of sphere.

In conclusion let us note that disregarding of non-sphericity of Earth causes for the IV a series of undesirable phenomena: change of period of its natural oscillations, deviation of position of equilibrium from the vertical, oscillations near the equilibrium position. This leads to corresponding errors in the INS during its determination of coordinates of the object and parameters of the object's motion. The above relationships permit us to show the influence of non-sphericity of the Earth on accuracy of IV and of the inertial system as a whole, and also to substantiate possible methods of taking into account oblateness of the Earth in readings of the INS. These questions are considered in a number of works (for instance, [142, 162, 121, 117]).

d) The Influence of the Vertical Component of Velocity of the Object

During analysis of operation of the IV we obtained, for instance for readings of accelerometers in the case of geographic orientation of their axes of sensitivity, relationships (11.6.64), in which there were not considered vertical components of velocities and accelerations of the object. With consideration of the latter, in accordance with (2.2.27) for h = 0, readings of accelerometers a_{x_1} and a_{y_1} will, if we take into account (11.6.15), be determined by the following relationships instead of by (11.6.64):

$$\left.\begin{aligned} a_{x_1} &= \dot{v}_E - \frac{v_E v_N}{R}\operatorname{tg}\varphi - 2Uv_N\sin\varphi + v_\zeta\left(\frac{v_E}{R} + 2U\cos\varphi\right) - \\ &- \left(g_0 + \dot{v}_\zeta - \frac{v_E^2 + v_N^2}{R} - U^2R\cos^2\varphi - 2Uv_E\cos\varphi\right)\alpha, \\ a_{y_1} &= \dot{v}_N + \frac{v_E^2}{R}\operatorname{tg}\varphi + U^2R\cos\varphi\sin\varphi + \\ &+ 2Uv_E\sin\varphi + \frac{v_N v_\zeta}{R} - \\ &- \left(g_0 + \dot{v}_\zeta - \frac{v_E^2 + v_N^2}{R} - U^2R\cos^2\varphi - 2Uv_E\cos\varphi\right)\beta. \end{aligned}\right| \qquad (11.6.221)$$

From these formulas it follows that terms proportional vertical component v_ζ of velocity of the object will cause additional dynamic errors of the IV; i.e., there will be changed its position of equilibrium, and also there will be observed deviation from the equilibrium position. Furthermore, vertical accelerations of the object and also vertical components of induced and Coriolis accelerations disturb the condition of undisturbability of the IV.

Analysis of influence of vertical component of velocity of the object on systematic errors of the IV is given by G. O. Fridlender [162], who also proposed one of the methods of their compensation by means of satisfaction by the condition of undisturbability (11.6.22), but with consideration of change of magnitude of R in accordance with actual change of distance of object from center of Earth. Influence of vertical accelerations of object, and also of vertical components of induced and Coriolis accelerations is less important. What has been said is valid for velocities of the object less than the tripled value of the speed of sound [16]. For higher velocities of the object, if we consider (11.6.15), period of natural oscillations of the IV will be determined by the following relationship (for W_ζ = = const) instead of by (11.6.25):

$$T_{в.з} = 2\pi \sqrt{\frac{R}{g_0 + W_\zeta}}, \qquad (11.6.222)$$

where vertical acceleration W_ζ of the object is expressed by formula (2.2.27), which for $h \ll R$ has the form

$$W_\zeta = \dot{v}_\zeta - \frac{v_E^2 + v_N^2}{R} - U^2 R \cos^2 \varphi - 2U v_E \cos \varphi. \qquad (11.6.223)$$

For very high horizontal velocity of the object, for instance for aircraft at velocity $v > UR \approx 450$ m/sec, in formula (11.6.223) the second term has predominant value. Since $v_E^2 + v_N^2 = v^2$, where v is relative horizontal velocity of the object, then in the indicated case $W_\zeta \approx -\frac{v^2}{R} = \text{const}$ and instead of (11.6.222) we will obtain

$$T_{в.з} = 2\pi \sqrt{\frac{R}{g_0 - \frac{v^2}{R}}} = 2\pi \frac{R}{\sqrt{Rg_0 - v^2}}. \qquad (11.6.224)$$

For the first cosmic velocity $v = \sqrt{Rg_0}$, period of oscillations of the IV tends to infinity. Consequently, it is obtained that the inertial system cannot function on an object which moves with first cosmic velocity. However, if initial deviations of the system are eliminated with the help of external information, then the presence of a large period after cessation of correction is not an important deficiency of the system [16]. Here application of an inertial system with analytic vertical is possible.

Above there were indicated certain systematic errors of the IV. They cause corresponding systematic errors of the inertial system in its determination of position of the object and parameters of motion of the object. Analysis of these errors is given in works of G. O. Fridlender [162], V. P. Seleznev [142] and others.

7. Dynamic Errors of the IV

a) General Characteristics of Dynamic Errors

Errors of IV are most fully characterized by dynamic errors appearing under actual conditions of operation of the system during continuous change of input parameters (accelerations) and during variable external disturbances. Among dynamic errors of the IV we include:

1) errors due to oscillations of the GSP and deflections of axes of sensing elements of the system of integral correction – accelerometers – caused by them;

2) errors due to disturbances applied to the gyroscopes: unbalance torques; frictional torques in axes of the suspension; inertial torques due to accelerations during oscillations and vibrations of the place of installation of the instrument which are connected with elastic deformations of the gyroscopes, etc.;

3) errors caused by damping of oscillations of the IV, for instance those appearing due to errors in the external information which is used for damping;

4) instrument errors in dynamics which we caused by deviations of parameters of the IV from their rated values, etc.

Under actual conditions of operation of the IV, the majority of the above-mentioned errors have a random character, since they are caused by disturbances which are random functions of time.

Before we go on to analysis of certain of the above-indicated dynamic errors, we will consider dynamics of the IV, considering thus time constants of their elements. The latter is very important for calculation of errors of IV and for establishment of requirements for dynamic characteristics of its elements.

b) Dynamics of IV Taking into Account Time Constants of its Elements

Earlier during composition of differential equations of motion of IV we considered all its elements to be inertialess, i.e., we did not consider their time constants. Now we will compose differential equation of motion of IV, for instance in the plane of the meridian, i.e., with respect to coordinate β, taking into account time constants of its elements: gyroscopes, accelerometers, integrators, actuating motors. The basic component part of the IV is the gyro-stabilized platform. It is considered [180, 6] that the most promising for inertial navigational systems are GSP in which there are used floating integrating gyroscopes with servo drives, which make it possible to ensure high accuracy of gyroscopic stabilization.

In Fig. 11.3 there is given a schematic diagram of the "northern" channel of the IV with a uniaxial GSP which is based on use of the floating integrating

gyroscope (FIG) working jointly with a servo drive. An analogous scheme of a uniaxial platform with FIG and pendulum correction was described in § 9.3, par. 6b. In the diagram in Fig. 9.14 the correction signal from pendulum P proceeds directly to the torquer TQ of the floating gyro. In the diagram in Fig. 11.32, the signal from the accelerometer A_N is introduced at first into the integrator I_{I_N}, and from the output of the integrator, in accordance with the principle of integral correction applied in the IV, enters the torquer TQ of the floating gyro.

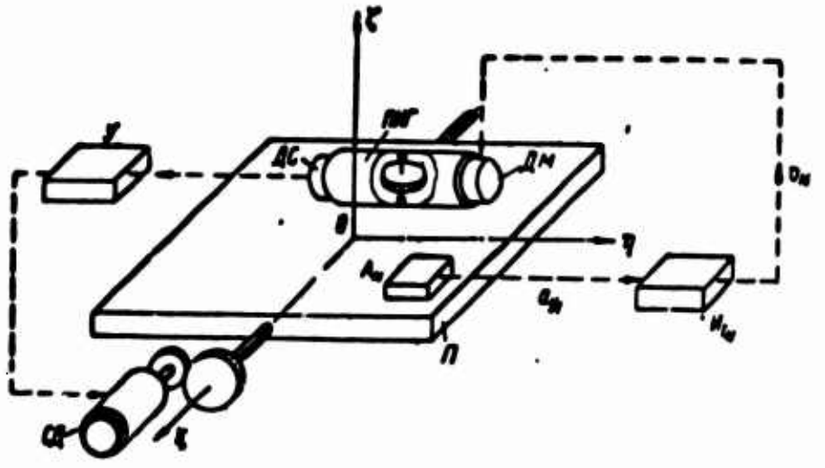

Fig. 11.32. Schematic diagram of "northern" channel of IV with GSP based on the use of FIG.

We will briefly explain the operation of the design shown in Fig. 11.32. The FIG is located on the platform in such a way that the axis of the gyroscope in its initial position coincides with the vertical $O\zeta$. Input axis $O\xi$, which is directed from west to east, is horizontal and coincides with the axis of rotation of platform P; output axis $O\eta$ is also horizontal and directed towards the north. During rotation of platform P around axis $O\xi$, the FIG through the signal pickoff SP and amplifier AM switches on stabilizing motor SM, which turns the platform in the opposite direction; as a result, the platform retains its constant position relative to inertial space. In this case, the system operates in the mode of geometric stabilization (§ 9.3, par. 6b). For retention of the side in the horizontal position there is applied an integral correction system, elements of which are accelerometer A_N, integrator I_{I_N} and torquer TQ. During rotation of platform P around axis $O\xi$ by angle β relative to the plane of the horizon, the accelerometer will register component $g_0\beta$ [see (11.6.16)] of the force of gravity and transmit the corresponding signal to the integrator I_{I_N}. Output signal from the integrator, which is proportional to the integral of readings of the accelerometer, enters the torquer TQ. The latter turns the floating gyro assembly; due to this, through the signal pickoff SP and amplifier

AM there is switched on the stabilizing motor SM, which returns the platform to horizontal position; thus the controlling signal from the accelerometer will be turned off. In the considered case, the floating gyro with servo drive operates in the mode of spatial integration (§ 9.3, Par. 6b).

If the object moves with acceleration $\dot{v}_N$, then readings $a_{y_1} = \dot{v}_N - g_0\beta$ [see (11.6.18)] of accelerometer A_N enter integrator I_{I_N}, and then enter the torquer TQ. This provides integral correction of the platform, which acquires the properties of an oscillatory system and if a certain condition is satisfied becomes undisturbable by accelerations of the object.

Let us compose the equation of the IV with respect to angle β, taking into account time constants of its separate elements.

1. Floating integrating gyroscope.

By analogy with (9.3.14), we will write the equation of the FIG in the form

$$T_{f.g}\ddot{\psi} + \dot{\psi} = k\left\{\omega_i + \frac{\rho}{k}\left[M_{t.q} + f_1(t)\right]\right\}. \qquad (11.6.225)$$

where $T_{f.g}$ – time constant of FIG [see (9.3.12)];

ψ – angle of rotation of floating gyro assembly;

k – transfer coefficient of FIG [see (9.3.13)];

ω_ξ – absolute angular velocity of platform around axis $O\xi$;

ρ – transfer coefficient of FIG with respect to controlling and disturbing torques applied relative to the axis of rotation of the float [see (9.3.13)];

$M_{t.q}$ – torque applied by the torquer;

$f_1(t)$ – disturbing torque relative to axis of rotation of the float.

Angular velocity ω_ξ can be written in the form

$$\omega_i = \omega_{ind} + \omega_{rel}, \qquad (11.6.226)$$

where ω_{ind} – induced angular velocity of the platform, which is equal to absolute angular velocity of the object (vehicle) relative to inertial space;

ω_{rel} – relative angular velocity of the platform (with relative to the object).

In the considered case, when axis $O\eta$ (Fig. 11.32) is directed towards north, for ω_{ind} we have

$$\omega_{ind} = \frac{v_N}{R} = \int_0^t \frac{\dot{v}_N}{R}\,d\tau. \qquad (11.6.227)$$

Designating angle of rotation of platform P around axis $O\xi$ relative to the object by β, we will obtain

$$\omega_{rel} = \dot{\beta}. \qquad (11.6.228)$$

Putting (11.6.227) and (11.6.228) in (11.6.226), we have

$$\alpha_1 = \int_0^t \frac{\dot{v}_N}{R} d\tau + \beta. \qquad (11.6.229)$$

2. Signal pickoff.

Voltage U taken from the signal pickoff is related with angle ψ of turn of the floating gyro assembly by relationship

$$U = -\varkappa_1 \psi, \qquad (11.6.230)$$

where $\varkappa_1$ – transfer coefficient.

3. Amplifier.

Voltage U_{am} taken from the amplifier is determined by equation

$$T_y \dot{U}_y + U_y = \varkappa_2 U, \qquad (11.6.231)$$

where T_{am} – time constant of amplifier;

U – voltage from signal pickoff, which goes to the input of the amplifier,

$\varkappa_2$ – amplifier gain.

4. Stabilizing motor and platform.

Equation of motion of platform around axis $O\xi$ is the following:

$$I_\xi \dot{\omega}_{\text{отн}} = M_c + f_2(t), \qquad (11.6.232)$$

where I_ξ – moment of inertia of platform relative to axis $O\xi$, with consideration of the given moment of inertia of the armature of the motor;

M_s – stabilizing torque on axis $O\xi$;

$f_2(t)$ – disturbing torques acting relative to axis $O\xi$, including inertial torques of induced motion.

For moment of inertia I_ξ we have relationship

$$I_\xi = J_{pl} + \gamma^2 J_{ar}. \qquad (11.6.233)$$

where J_{pl} – moment of inertia of platform relative to axis $O\xi$;

J_{ar} – moment of inertia of armature;

γ – gear ratio of gear transmission from axis of stabilizing motor to axis of rotation of platform.

Stabilizing torque M_s, by analogy with (7.2.23), will be

$$M_c = \gamma \eta M_{c.m}, \qquad (11.6.234)$$

where $M_{s.m}$ – torque of motor;

η – efficiency of transmission.

Considering for simplicity $\eta \approx 1$, we will obtain

$$M_c = \gamma M_{c.m}. \qquad (11.6.235)$$

For direct current motor with independent excitation, by analogy with (7.2.44)

we have*

$$M_{e.\,a} = \Phi i, \tag{11.6.236}$$

where Φ – magnetic flux of excitation windings;

i – current intensity in armature of motor.

Current intensity i is contained in the equation of balance of voltages in the armature circuit of the motor in the following way [see (7.2.46)]:

$$l\frac{di}{dt} + ri + c\gamma\omega_{отн} = U_y, \tag{11.6.237}$$

where l – self-inductance coefficient of armature circuit (including self-inductance coefficient of amplifier);

r – resistance of armature circuit (including internal resistance of amplifier);

s – coefficient characterizing dependence of counterelectromotive force induced in armature of motor on relative angular velocity $\gamma\omega_{rel}$ of its rotation;

U_{am} – voltage brought from the amplifier to the armature of the stabilizing motor.

Taking into account relationships (11.6.232), (11.6.235), (11.6.236), (11.6.237), we will obtain

$$I'_\xi l\ddot{\omega}_{отн} + I'_\xi r\dot{\omega}_{отн} + c\Phi\omega_{отн} = \frac{\Phi}{\gamma}U_y + \frac{1}{\gamma^2}\left[lf'_2(t) + rf_2(t)\right]. \tag{11.6.238}$$

where I'_ξ – total moment of inertia of armature of motor and platform, reduced to axis of armature of the motor [see (11.6.233)]

$$I'_\xi = \frac{I_\xi}{\gamma^2} = J_a + \frac{J_{пл}}{\gamma^2}. \tag{11.6.239}$$

We will divide (11.6.238) by s^Φ and introduce designations

$$T_м = \frac{I'_\xi r}{c\Phi}, \quad T_э = \frac{l}{r}, \quad x_3 = \frac{1}{c\gamma}, \quad x_4 = \frac{r}{c\Phi\gamma^2}, \tag{11.6.240}$$

where T_m – electromechanical time constant of motor;

T_e – electromagnetic time constant of armature circuit of motor;

x_3 – transfer coefficient of motor;

x_4 – transfer coefficient with respect to disturbance $f_2(t)$.

Then instead of (11.6.238) we will obtain

$$T_э T_м\ddot{\omega}_{отн} + T_м\dot{\omega}_{отн} + \omega_{отн} = x_3U_y + x_4\left[T_э f'_2(t) + f_2(t)\right] \tag{11.6.241}$$

or, considering (11.6.228),

$$T_э T_м\dddot{\beta} + T_м\ddot{\beta} + \dot{\beta} = x_3U_y + x_4\left[T_э\dot{f}_2(t) + f_2(t)\right]. \tag{11.6.242}$$

5. Accelerometer.

Equation of accelerometer, by analogy with (11.6.56), will be written in the form

$$T_a^2\ddot{x} + 2\zeta T_a\dot{x} + x = k_1a_{y_1} + k_1f(t), \tag{11.6.243}$$

*See footnote on p. 20 .

where T_a – time constant of accelerometer [see (11.6.55)];

ζ – relative damping factor [see (11.6.53)];

x – linear displacement of inertial mass of accelerometer;

$k_1 = T_a^2$ – transfer coefficient;

a_{y_1} – acceleration measured by accelerometer;

$f(t)$ – disturbing acceleration, for instance due to oscillatory motion of the object.

Taking a_{y_1} the expression (11.6.18), we will obtain

$$T_a^2\ddot{x} + 2\zeta T_a\dot{x} + x = k_1\left(\dot{v}_N - g_0\beta\right) + k_1 f(t) \tag{11.6.244}$$

Voltage U_a at the output of the accelerometer

$$U_a = k_2 x, \tag{11.6.245}$$

where k_2 is transfer coefficient.

6. Integrator.

Equation of integrator will be presented in the form

$$T_1\dot{U}_1 + \dot{U}_1 = k_3 U_a. \tag{11.6.246}$$

where T_1 – time constant of integrator;

U_1 – voltage at the output of the integrator;

k_3 – transfer coefficient.

7. Torquer.

Current intensity $i_{t.q.}$ in the torquer is proportional to voltage taken from the integrator

$$i_{t.q} = k_4 U_1, \tag{11.6.247}$$

where k_4 – transfer coefficient.

The torque of the torquer, according to (9.3.2), can be written in the form

$$M_{t.q} = -k_5 i_{t.q}, \tag{11.6.248}$$

where k_5 is transfer coefficient.

Applying to the above obtained equation the Laplace transform, we can easily find transfer functions of elements of the IV and construct its block diagram (Fig. 11.33).

Thus, considering (11.6.225), (11.6.229), (11.6.230), (11.6.231), (11.6.242), (11.6.244)-(11.6.248), we have the following system of equations of the considered inertial vertical:

$$\left.\begin{aligned}
&T_{\text{ф.у}}\ddot{\psi}+\dot{\psi}=k\left\{\int_0^t\frac{\dot{\vartheta}_N}{R}\,d\tau+\beta+\frac{\rho}{k}\left[-k_1k_2U_{\text{ф}}+f_1(t)\right]\right\}\\
&\text{(floating gyroassembly)}\\
&T_y\dot{U}_y+U_y=-\varkappa_1\varkappa_2\psi\\
&\text{(amplifier)}\\
&T_0T_{\text{ст}}\dddot{\beta}+T_{\text{ст}}\ddot{\beta}+\dot{\beta}=\varkappa_0U_y+\varkappa_0\left[T_0\dot{f}_0(t)+f_0(t)\right]\\
&\text{(stabilizing motor and platform)}\\
&T_{\text{а}}^2\ddot{x}+2\zeta T_{\text{а}}\dot{x}+x=k_1\left(\dot{v}_N-g_0\beta\right)+k_1f(t).\\
&\text{(accelerometer)}\\
&T_{\text{и}}\dot{U}_{\text{и}}+U_{\text{и}}=k_2k_3x\\
&\text{(integrator)}
\end{aligned}\right\}\qquad(11.6.\ldots)$$

Let us compose at first the equation of the IV with respect to coordinate β, disregarding time constants of its elements and not considering disturbances. Then system (11.6.249) in operational form ($p=\frac{d}{dt}$) can be written in the following way:

$$\left.\begin{aligned}
p\psi&=k\left(\frac{\vartheta_N}{R}+p\beta-\frac{\rho}{k}k_1k_2U_{\text{ф}}\right)\\
U_y&=-\varkappa_1\varkappa_2\psi;\quad p\beta=\varkappa_0U_y\\
x&=k_1\left(pv_N-g_0\beta\right);\quad pU_{\text{и}}=k_2k_3x
\end{aligned}\right\}\qquad(11.6.250)$$

whence by means of successive elimination of variables it is easy to obtain equation of IV with respect to coordinate β in the form

$$\left(p^3+\varkappa_0kp^2+\varkappa_0k_0g_0\right)\beta=pv_N\left(\varkappa_0k_0-\frac{\varkappa_0k}{R}\right).\qquad(11.6.251)$$

where

$$\varkappa_0=\varkappa_1\varkappa_2\varkappa_3.\qquad(11.6.252)$$

$$k_0=\rho k_1k_2k_3k_4k_5.\qquad(11.6.253)$$

From equation (11.6.251) it follows that when the following condition is satisfied:

$$\varkappa_0k_0-\frac{\varkappa_0k}{R}=0\qquad(11.6.254)$$

or

$$\frac{k_0}{k}=\frac{1}{R}\qquad(11.6.255)$$

the IV is undisturbed by accelerations $\dot{v}_N$ of the object.

When condition (11.6.255) is satisfied, equation (11.6.251) takes the form

$$\left(\frac{1}{\varkappa_0k}p^2+p^2+\frac{g_0}{R}\right)\beta=0\qquad(11.6.256)$$

or

$$\frac{1}{\varkappa_0k}\dddot{\beta}+\ddot{\beta}+\frac{g_0}{R}\beta=0.\qquad(11.6.257)$$

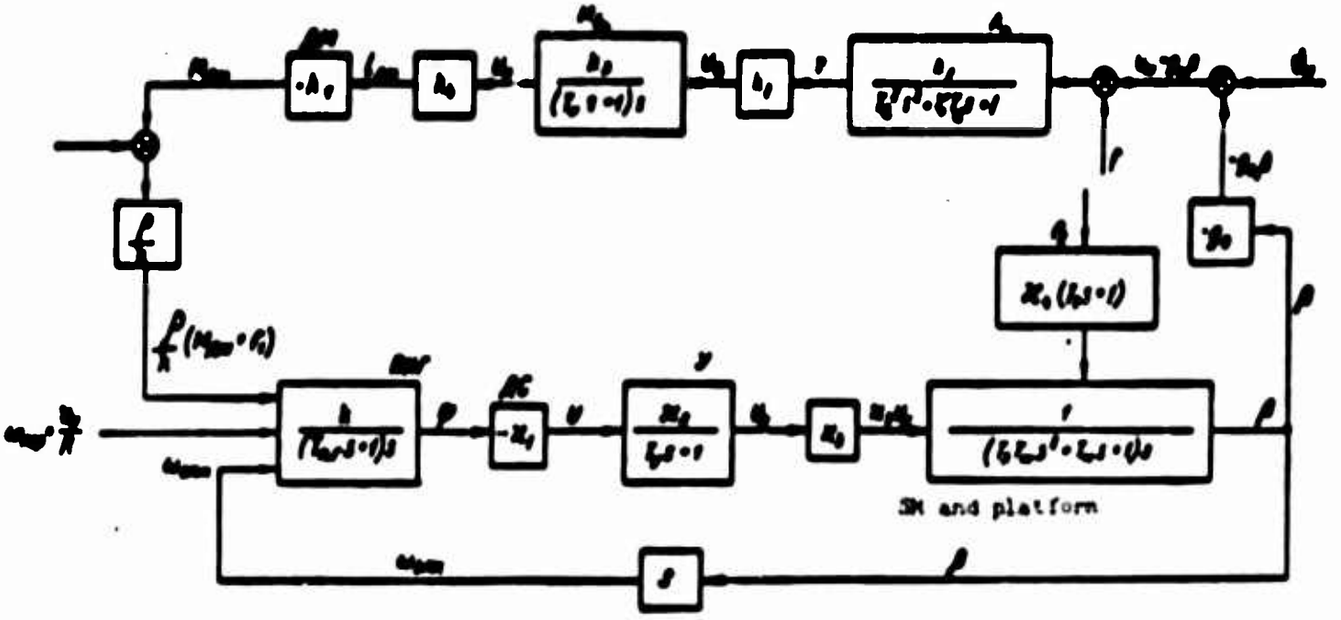

Fig. 11.35. Block diagram of "northern" channel of IV with GSP based on use of PIG.

If the total amplification factor $k_0 k$ of the IV is made sufficiently large, then the fundamental oscillation of the IV, according to (11.6.257), will be determined by the equation already known to us

$$\beta + \frac{g_0}{R}\beta = 0, \tag{11.6.258}$$

which characterized undamped harmonic oscillations of the IV with Schuler frequency $\nu = \sqrt{\frac{g_0}{R}}$ and period $T_{sh} = 2\pi\sqrt{\frac{R}{g_0}}$, equal to 84.4 min.

We will give equation of the IV, taking into account time constants of its elements. Using system (11.6.249), we can simply obtain the following equation for angle of displacement β of the platform from the plane of the geocentric horizon:

$$\begin{aligned}
&\left[p^3(T_{s.s}p+1)(T_0T_{ma}p^2+T_{ma}p+1)(T_yp+1)+k_0kp^2+\right.\\
&\left.+\frac{k_0k_{ma}k_0}{(T_{ma}p+1)(T_{ma}^2p^2+2\xi T_{ma}p+1)}\right]\beta=\\
&=pg_N\left[\frac{k_0k_0}{(T_{ma}p+1)(T_{ma}^2p^2+2\xi T_{ma}p+1)}-\frac{k_0k}{R}\right]+\\
&+\frac{k_0k_0}{(T_{ma}p+1)(T_{ma}^2p^2+2\xi T_{ma}p+1)}f(t)-\\
&-\frac{k_0}{b}pf_1(t)+k_0(T_{s.s}p+1)(T_yp+1)p^2f_2(t),
\end{aligned} \tag{11.6.259}$$

where [see (9.3.13)]

$$b=\frac{1}{g}. \tag{11.6.260}$$

If in (11.6.259) we set time constants of elements of the IV $T_{f.g} = T_m = T_e = T_{am} = T_a = T_i = 0$ and disturbances $f(t) = f_1(t) = f_2(t) = 0$, then this equation becomes (11.6.261). From (11.6.259) it follows that for guarantee of undisturbability of the IV by acceleration $pv_N = \dot{v}_N$ of the object, it is necessary, in general, that expression $\left[\frac{k_0 k_m}{(T_g p+1)(T_g^2 p^2+2\xi T_g p+1)} - \frac{k_0 k_a}{R}\right]$ be equal to zero; the latter in principle cannot be realized due to the influence of time constants of accelerometer and integrator. Therefore, for guarantee of undisturbability of the IV by accelerations of the object it is necessary that accelerometers, integrators, and also torquers possess minimum time constants. Furthermore, it is desirable to have small time constants of the floating gyro, amplifier and stabilizing motor, in order to bring frequency of natural oscillations of the IV closer to Schuler frequency.

Let us set time constants of elements of the IV equal to zero; then instead of (11.6.259) we will obtain equation

$$\left(p^3 + k_m k_a p^2 + k_m k_a k_0\right)\beta - p v_N\left(k_m k_a - \frac{k_m k_a}{R}\right) + k_m k_a f(t) - \frac{k_m}{k_0} p f_1(t) + k_m p^2 f_2(t). \tag{11.6.261}$$

If there is satisfied the condition of undisturbability (11.6.255), then

$$\left(\frac{1}{k_0} p^3 + p^2 + \frac{k_a}{R}\right)\beta = \frac{1}{R} f(t) - \frac{1}{k_0} p f_1(t) + \frac{k_a}{k_0} p^2 f_2(t) \tag{11.6.262}$$

or for sufficiently large value of coefficient $k_0 k$ we will obtain

$$\beta + \frac{k_a}{R}\beta = \frac{1}{R} f(t) - \frac{1}{k_0} f_1(t) + \frac{k_a}{k_0} f_2(t). \tag{11.6.263}$$

Since, according to (11.6.255), $k = Kk_0$, then, designating

$$\frac{1}{k_0} = k_I, \quad \frac{k_a}{k_0} = k_{II}, \tag{11.6.264}$$

we will rewrite equation (11.6.263) in the form

$$\beta + \frac{k_a}{R}\beta = \frac{1}{R} f(t) - \frac{k_I}{R} f_1(t) + \frac{k_{II}}{R} f_2(t). \tag{11.6.265}$$

Disturbances $f(t)$, $f_1(t)$, $f_2(t)$ are random functions of time. Knowing their probability characteristics, we can determine probability characteristics of error $\beta(t)$ of the IV in dynamics. Individual problems of dynamics of GSP in which there are used FIG, are discussed in a number of works (for instance, [165, 28, 55, 103, 92]).

c) Certain Dynamic Errors in IV

Let us consider errors of the IV in dynamics. If we do not take into account

time constants of elements of the IV, then its error β can be determined from equation (11.6.265), in the right side of which there are contained disturbances: 1) f(t), applied to the accelerometer; 2) $f_1(t)$, applied to the PIG; 3) $f_2(t)$, applied to the platform. Usually f(t), $f_1(t)$, $f_2(t)$ are random functions of time. If there are known their probability characteristics, then, by using (11.6.265), we can determine probability characteristics of random function β(t), which is the error of the IV in dynamics.

As an example we will consider error β(t) due to disturbance f(t) on the accelerometer. For this in (11.6.265) we set $f_1(t) = f_2(t) = 0$; then we will obtain

$$\ddot{\beta} + \frac{k_2}{k}\beta = \frac{1}{k} f(t). \tag{11.6.266}$$

Error β(t) of the IV causes corresponding error δS(t) in determination by the inertial system of the path passed over by the object. Error δS(t), which is a random function of time, by analogy with (11.6.171) is related to β(t) by the relationship

$$\delta S(t) = -g\beta(t) + f(t) \tag{11.6.267}$$

Then, considering (11.6.24), we will obtain the equation for determination of error δS(t):

$$\delta\ddot{S} + \nu^2 \delta S = f(t). \tag{11.6.268}$$

Let us assume that f(t) is normal stationary random function with mathematical expectation $\overline{f(t)} = 0$. Then mathematical expectation of error δS(t) will also be equal to zero:

$$\overline{\delta S(t)} = 0. \tag{11.6.269}$$

Therefore, random function δS(t) is characterized by its dispersion D[δS] or root mean square value $\sigma_{\delta S}$. During determination of D[δS] we should consider that random function δS(t) is not stationary, since in (11.6.268) there is absent the term characterizing damping of the system. Consequently, for finding D[δS] from (11.6.268), it is impossible to use basic relationships of the theory of stationary random functions. Solution of the analogous problem by determination of probability characteristics of the random function at the output of a linear dynamic system without damping, which is given in several works [193, 138, 167], shows that dispersion of solution of equation of type (11.6.268) contains a term proportional to time, constant terms, and terms which are harmonic functions of time with frequency 2ν [138]. During prolonged operation of the system, the term proportional to time has predominant value for large t, if we disregard other terms, we can determine dispersion D[δS] by formula [138]

$$D[\delta S] = \frac{\pi}{\nu^2} S_f(\nu)\, t. \tag{11.6.270}$$

where $S_f(\omega)|_{\omega=\nu}$ is spectral density of disturbance f(t).

For determination of $S_f(\nu)$ we will assume that a ship moves westward, and accelerometer A_N (Fig. 11.32) measures component of acceleration in the plane of a rib of the ship. Then disturbance f(t) will be caused by orbital motion of center of gravity of the ship and its rolling. In this case f(t) is determined by relationship (4.6.41).

If we consider only rolling, then

$$f(t) = -\ddot{\beta}. \tag{11.6.271}$$

Then spectral density $S_f(\omega)$ will be

$$S_f(\omega) = \omega^4 S_\beta(\omega) \tag{11.6.272}$$

or, considering (2.1.58) and (2.1.84),

$$S_f(\omega) = \frac{2A_\beta \sigma_\beta^2}{\pi} \cdot \frac{b_2^2 \omega^4}{\omega^4 + 2a_2\omega^2 + b_2^2}, \tag{11.6.273}$$

where a_2 and b_2^2 are determined by formulas (2.1.59).

Consequently, for $S_f(\nu)$ we have (for $\nu \ll b_2$)

$$S_f(\nu) = \frac{2A_\beta \sigma_\beta^2}{\pi} \cdot \frac{b_2^2 \nu^4}{\nu^4 + 2a_2\nu^2 + b_2^2} \approx \frac{2A_\beta \sigma_\beta^2}{\pi} \cdot \frac{\nu^4}{\mu_2^2 + \lambda_2^2} D[\theta]. \tag{11.6.274}$$

Putting (11.6.274) in (11.6.270), we will obtain

$$D[\delta S] = 2A_\beta \sigma_\beta^2 \frac{\nu^4}{\mu_2^2 + \lambda_2^2} D[\theta] t. \tag{11.6.275}$$

For root mean square value $\sigma_{\delta S}$ of error $\delta S(t)$ we have

$$\sigma_{\delta S} = \nu^2 \sigma_\beta \sqrt{\frac{2A_\beta}{\mu_2^2 + \lambda_2^2} t}. \tag{11.6.276}$$

i.e., $\sigma_{\delta S}$ is proportional to $\sqrt{t}$.

The fact of increase of dispersion of errors of the inertial system due to random errors of accelerometer (the same thing occurs for random errors of the gyroscope, integrator, etc.) is important in principle and indicates the fact that it is necessary to considerably lower random errors of elements of the IV, and in individual cases — to introduce damping. Actually, if into the left side of equation (11.6.268) we introduce a term proportional to $\dot{\delta S}$ (system becomes damped), then dispersion $D[\delta S]$ no longer will increase in time, as this was shown during calculation of dispersion $D[\chi]$ of random motions of the physical pendulum with damping [see (4.6.80)].

What has been said about influence of errors of the accelerometer on accuracy of the inertial system also completely pertains to influence of errors of gyroscopes

[193, 167], integrators and other elements of the IV. Certain probability characteristics of random drifts of gyroscopes are given (on the basis of experimental data) in individual articles (for instance, [193, 122]).

As was shown (par. a), dynamic errors of the IV are caused also by damping of its oscillations. One of the effective methods of damping is based on use of external information about coordinates of object and its speed. In this case errors of the IV in dynamics are caused by corresponding errors in determination of the data of the information, i.e., of path and velocity of the object. In order to demonstrate the method of finding these dynamic errors, we will compose the equation of the IV with respect to coordinate β, taking into account errors of external information δS in determination of path and δv of velocity of the object. Then the equation of the IV, which in the absence of errors of external information was expressed by relationship (11.6.100), will have the form

$$\beta + \frac{\dot{\vartheta}_N}{R} = \frac{1}{H}\int_0^t \left(\dot{v}_N - g_0\beta - n_1\beta - n_1\frac{\delta S}{R} - b_1\beta - \frac{b_1}{R}\delta v\right)dt, \qquad (11.6.277)$$

whence after differentiation, if we consider the condition of undisturbability (11.6.22), we will obtain

$$\ddot{\beta} + \frac{b_1}{R}\dot{\beta} + \frac{g_0 + n_1}{R}\beta = -\frac{n_1}{R^2}\delta S(t) - \frac{b_1}{R^2}\delta v(t). \qquad (11.6.278)$$

Usually errors $\delta S(t)$ and $\delta v(t)$ are random functions of time. Then error of the IV $\beta(t)$ is also a random function, for determination of probability characteristics of which it is necessary to know probability characteristics of random functions $\delta S(t)$ and $\delta v(t)$. If mathematical expectations $\overline{\delta S(t)} = 0$, $\overline{\delta v(t)} = 0$, then in accordance with (11.6.278), after termination of transient response, mathematical expectation $\bar{\beta}$ of error $\beta(t)$ also is equal to zero:

$$\bar{\beta} = 0. \qquad (11.6.279)$$

If we consider that $\delta S(t)$ and $\delta v(t)$ are stationary random functions of time, then it is easy to find spectral density $S_\beta(\omega)$ of error $\beta(t)$. According to (11.6.278) we have*

$$S_\beta(\omega) = |W_{\beta,\delta S}(j\omega)|^2 S_{\delta S}(\omega) + |W_{\beta,\delta v}(j\omega)|^2 S_{\delta v}(\omega), \qquad (11.6.280)$$

where $S_{\delta S}(\omega)$ and $S_{\delta v}(\omega)$ — spectral densities of random functions $\delta S(t)$ and $\delta v(t)$;

$W_{\beta,\delta S}(j\omega)$, $W_{\beta,\delta v}(j\omega)$ — amplitude-phase responses, determined from (11.6.278) by the following relationships:

*It is assumed that random functions $\delta S(t)$ and $\delta v(t)$ are not related.

$$\left.\begin{aligned} W_{\beta,\delta S}(j\omega) &= \frac{\frac{a_1}{R^2}}{-\omega^2 + \frac{b_1}{R} j\omega + \frac{g_0 + a_1}{R}} \\ W_{\beta,\delta v}(j\omega) &= \frac{\frac{b_1}{R^2}}{-\omega^2 + \frac{b_1}{R} j\omega + \frac{g_0 + a_1}{R}} \end{aligned}\right\} \qquad (11.6.281)$$

Then for dispersion of error $\beta(t)$ we have

$$D[\beta(t)] = \int_{-\infty}^{\infty} |W_{\beta,\delta S}(j\omega)|^2 S_{\delta S}(\omega)\, d\omega + \int_{-\infty}^{\infty} |W_{\beta,\delta v}(j\omega)|^2 S_{\delta v}(\omega)\, d\omega. \qquad (11.6.282)$$

Thus, knowing dynamic characteristics of the IV and spectral densities $S_{\delta S}(\omega)$ and $S_{\delta V}(\omega)$ of errors of the external information, we can determine dispersion $D[\beta]$ of corresponding error $\beta(t)$ of the IV. Detailed investigations and calculations of errors of the inertial systems in the presence of damping with use of external information are given in a number of works (see, for instance, [142, 16, 10]).

If it is necessary to determine dynamic errors of the IV taking into account time constants of its elements, then there should be used equations of the IV of type (11.6.259).

During investigation of an inertial system intended for use in the control mode, there appears the problem of analysis of it taking into account dynamic properties of the controlled object, for instance an aircraft. Solution of this problem is given in a number of works (see, for instance, [162, 9, 10, 163]).

8. Application of IV in Inertial Systems

a) Preliminary Remark

Let us consider several fundamental schemes of INS which explain peculiarities of use of inertial verticals in these systems. In accordance with the earlier given classification, we will consider inertial systems: 1) with gyroscopic vertical; 2) with mechanical (geometric) erection of the vertical; 3) with analytic erection of the vertical.

b) Fundamental Scheme of an INS with Gyroscopic Vertical

It is known (Par. 1c) that in INS with GV, erection of the vertical is carried out with help of gyroscopes by means of artificial simulation of the undisturbed physical pendulum with period of 84.4 min by the scheme of Ye. B. Levental', which

consists of gyroscopes, accelerometers, integrators and is a closed dynamic system. Schematic diagram of an INS with gyroscopic vertical and geographic orientation of axes of sensitivity of the accelerometers [121, 175, 142] is shown in Fig. 11.34.

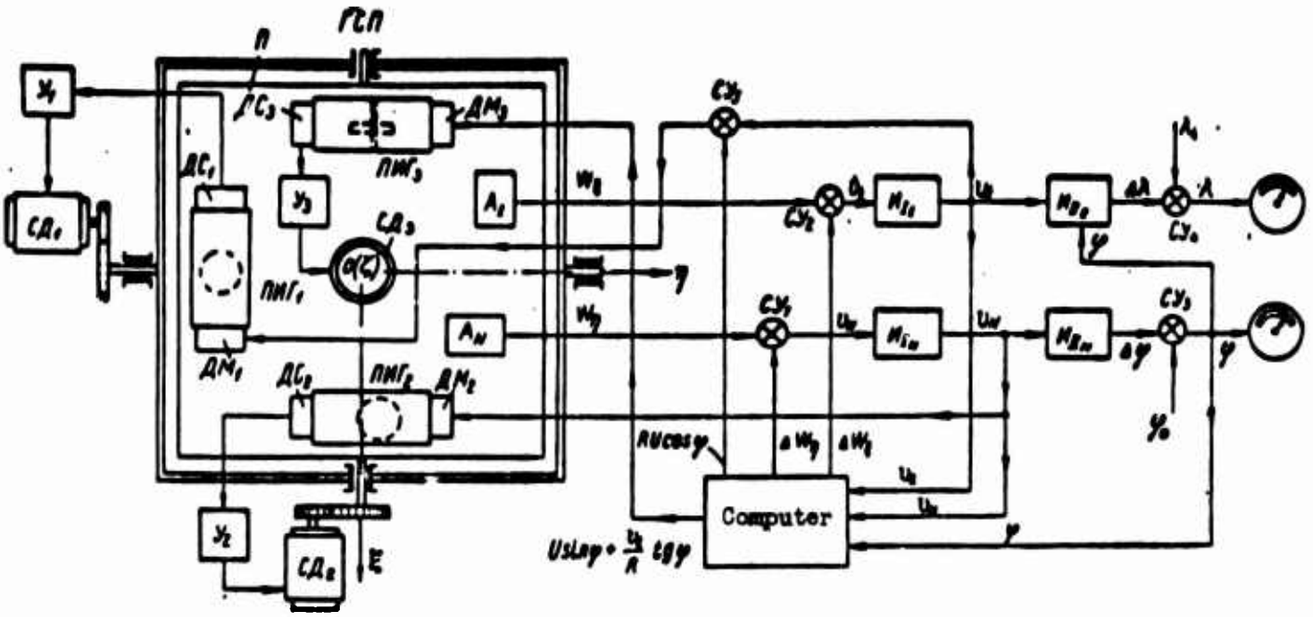

Fig. 11.34. Schematic diagram of INS with GV.

Basic component part of the system is the inertial vertical. The IV consists of a gyro-stabilized platform GSP with accelerometers A_N and A_E, integrators I_{I_N}, I_{I_E} and computer located on it. On platform P, which is installed in a complete Cardan suspension having three degrees of freedom (third ring of the suspension is not shown in Fig. 11.34), there are located three floating integrating gyroscopes FIG_1, FIG_2, FIG_3. The latter through signal pickoffs SP_1, SP_2, SP_3 and amplifiers AM_1, AM_2, AM_3 are connected with stabilizing motors SM_1, SM_2, SM_3, which carry out stabilization of the platform relative to plane of horizon and azimuthal direction to the north. Due to this, axes of sensitivity of accelerometers A_N and A_E have geographic orientation. Accelerometers measure components W_η and W_ξ of total acceleration of the object; from these components, on adders AD_1 and AD_2 there are eliminated corrections ΔW_η and ΔW_ξ for induced and Coriolis accelerations produced by the computer. Relative accelerations $w_N = \dot{v}_N$ and $w_E = \dot{v}_E$ obtained from adding devices AD_1 and AD_2 enter first integrators I_{I_N} and I_{I_E}. From the output of the integrators components v_N and v_E of relative velocity of the object enter torquers TQ_2 and TQ_1 of the floating gyros FIG_2 and FIG_1; this proves for integral correction of the IV. Thus, in the eastern channel, into adder AD_5, there is introduced the correction for horizontal component U cos φ of rotation of the Earth which is obtained from the computer. The latter also produces angular velocity U sin φ + $\frac{v_E}{R}$ tan φ of

induced rotation of the plane of the meridian; this velocity is introduced into the torquer TQ_3 of the floating gyro FIG_3; due to this there is carried out azimuthal orientation of platform P in the plane of the meridian.

Components v_N and v_E of relative velocity of the object are introduced into the second integrators I_{II_N} and I_{II_E}, from which there are taken changes of latitude $\Delta\varphi$ and longitude $\Delta\lambda$ of the position of the object. The latter are added in the adders AD_3 and AD_4 with initial values φ_0 and λ_0. As a result there are obtained current latitude φ and longitude λ. Values of φ, v_N and v_E are introduced also into the computer for determination in it of the earlier mentioned corrections. More detailed information about this INS and its investigation are given in a number of works (for instance, [121, 175, 142]).

c) Fundamental Scheme of INS with Mechanical (Geometric) Vertical Erector (VE)

As it is known (Par. 1c), in INS with mechanical VE obtaining of the local vertical is carried out without use of gyroscopes, with the help of geometric vertical erector, on the platform of which there are located accelerometers. Gyroscopic device of the system in the form of a GSP is oriented along axes of inertial system of coordinates. Thus the GSP is used for "memorization" of position of vertical at the initial moment.* Schematic diagram of mechanical (geometric) VE for one channel is shown in Fig. 11.23. Schematic diagram of an INS with mechanical VE [142, 149] is given in Fig. 1.35.

Basic component parts of the system are the GSP and VE. Erection of the GSP here in principle is the same as in the scheme in Fig. 11.34; i.e., it contains three floating integrating gyroscopes utilized by the unloading servo drives. The difference is only the fact that in this system the GSP does not have correction, and therefore the platform preserves constant position in inertial space, for instance perpendicular to axis of the Earth $P_N P_S$.

The platform of the VE is fixed so that it can turn relative to frame F around axis $O\xi$, which is directed towards the east; axis $O\eta$ is oriented to the north, and axis $O\zeta$ is along the vertical. Due to this, accelerometers A_N and A_E fixed on the platform of the VE have geographic orientation. From the accelerometers there are components $\dot{v}_N$ and $\dot{v}_E$ of relative acceleration** of the object, which are introduced into the first integrators I_{I_N} and I_{I_E}. Components v_N and v_E of relative velocity of

*See footnote 2 on p. 432.

**Induced and Coriolis components of acceleration of the object can be compensated just as in the scheme shown in Fig. 11.34.

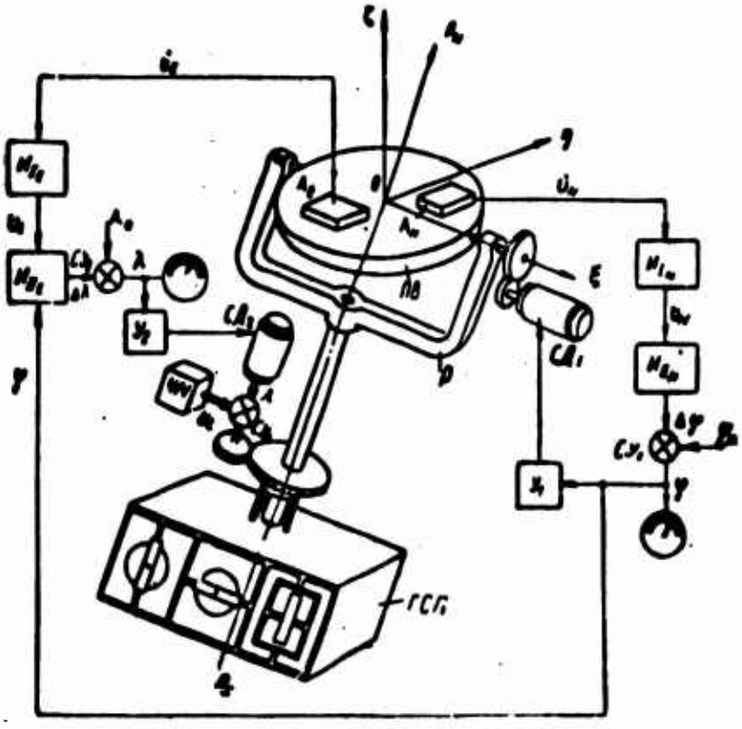

Fig. 11.35. Schematic diagram of INS with mechanical (geometric) VE.

the object obtained at the output of these integrators enter the second integrators I_{II_N} and I_{II_E}, from which there are taken changes of latitude $\Delta\varphi$ and longitude $\Delta\lambda$ of position of the object. The latter are added in adders AD_1 and AD_2 with initial values φ_0 and λ_0. Latitude φ through amplifier AM_1 enters stabilizing motor SM_1,* which turns the platform of the VE by angle φ around axis $O\xi$. Since the GSP and frame F are perpendicular to axis of Earth P_NP_S, then turn of the VE by angle φ ensures retention of platform of VE in horizontal position. Longitude λ through amplifier AM_2 enters stabilizing motor SM_2, which turns frame F together with VE around axis P_NP_S. In order to take into account rotation of Earth with velocity U around axis P_NP_S, there is used a clockwork mechanism CM, which introduces the quantity Ut to adder AD_3, where there is obtained the quantity $\lambda + Ut$; this quantity in turn proceeds to rotate frame F around axis P_NP_S. The latter provides for azimuthal orientation of platform of VE in geographic system of coordinates. From what has been said, it follows that geographic coordinates φ and λ of the object are characterized in the considered inertial system by angle of turn φ of the platform of the VE around axis $O\xi$ relative to frame F and by angle of rotation λ of frame F together with platform of VE around axis P_NP_S relative to the GSP; due to this, the system is sometimes called a geometric system [121]. More detailed information about this inertial system and its investigation can be found in a number of works (for instance, [142, 149]).

d) Fundamental Scheme of INS with Analytic VE

Earlier it was shown (Par. 1c) that in INS with analytic VE there is applied GSP oriented along axes of inertial system of coordinates; on the GSP there are fixed three accelerometers. Readings of these accelerometers are corrected for

*For motor SM_1 to turn platform by angle φ, it must operate jointly with a servo system, which in Fig. 11.35 is not shown. In the simple case represented in Fig. 11.35, as motor SM_1 it is possible to consider, for instance, a selsyn.

accelerations due to forces of gravitation with the help of external correction or autocompensation. Erection of the vertical is carried out by the analytic method. Coordinates of the object are determined on the basis of readings of the accelerometers, corrected for accelerations due to forces of gravitation.

In Fig. 11.36 there is shown the installation of the three accelerometers A_{x_0}, A_{y_0}, A_{z_0} on the GSP, which is stabilized relative to axes $0x_0y_0z_0$, which preserve constant position in inertial space. Axis of sensitivity of the accelerometers are directed along axes of coordinate system $0x_0y_0z_0$. Readings of the accelerometers a_x, a_y, a_z are integrated twice due to which there are determined displacements (coordinate) of the object S_x, S_y, S_z in inertial system of coordinates. Readings of accelerometers a_x, a_y, a_z, by analogy with (11.6.46), are expressed by relationships

$$a_x = \ddot{x} - g_x, \quad a_y = \ddot{y} - g_y, \quad a_z = \ddot{z} - g_z, \tag{11.6.283}$$

where $\ddot{x}$, $\ddot{y}$, $\ddot{z}$ – components of absolute acceleration of the object along axes $0x_0y_0z_0$;

g_x, g_y, g_z – components of gravitational acceleration along the same axes.

If we designate initial coordinates of the object by S_{x_0}, S_{y_0}, S_{z_0}, and initial components of its velocity by v_{x_0}, v_{y_0}, v_{z_0}, then for current coordinates S_x, S_y, S_z we have the evident relationships

$$\left.\begin{aligned} S_x &= S_{x_0} + v_{x_0}t + \iint_0^t (a_x + g_x)\,dt^2 \\ S_y &= S_{y_0} + v_{y_0}t + \iint_0^t (a_y + g_y)\,dt^2 \\ S_z &= S_{z_0} + v_{z_0}t + \iint_0^t (a_z + g_z)\,dt^2 \end{aligned}\right\} \tag{11.6.284}$$

From the given formulas it follows that readings of accelerometers a_x, a_y, a_z must be corrected for components g_x, g_y, g_z of gravitational acceleration. For this purpose, as it was said, there is applied external correction or autocompensation. Let us consider a scheme of INS with autocompensation of gravitational acceleration [142, 13]. This scheme usually consists of three identical channels for determination of coordinates of object S_x, S_y, S_z; therefore, we will give fundamental scheme of only one channel – for determination of S_x (Fig. 11.37).

Reading a_x of accelerometer A_{x_0} is introduced into adder AD_1, where it is added with quantity g_x, which is determined by the autocompensation system, i.e., on the basis of the data themselves which are produced by the inertial system. The obtained

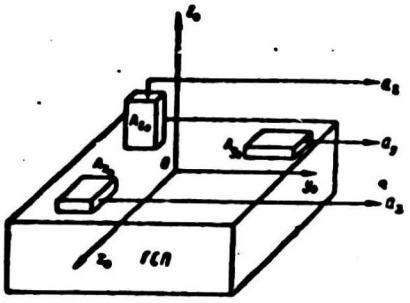

Fig. 11.36. Location of accelerometers on a GSP in an INS with analytic VE.

sum $\ddot{x} = a_x + g_x$ [see (11.6.283)] is introduced into the first integrator I_{I_x}. Readings taken from it are added in adder AD_2 with the component v_{x_0} of initial velocity of the object, and the result is introduced into second integrator I_{II_x}. Increment obtained at the output ΔS_x is added in adder AD_3 with initial coordinate S_{x_0}. As a result there is obtained current value of coordinate $S_x = S_{x_0} + \Delta S_x$. Analogously there are found current values of other coordinates S_y and S_z. Coordinates S_x, S_y, S_z are introduced into the computer, which produces autocompensation signals g_x, g_y, g_z. For determination of g_x, g_y, g_z there exist various approximate analytic dependences [142, 13, 117]. For instance, for g_x the simplest formula has the form

$$g_x = \frac{g_0 R^2 S_x}{(R+h)^3} \approx \frac{g_0}{R} S_x \text{ (for } h \ll R\text{).} \qquad (11.6.285)$$

where g_0 — acceleration due to force of terrestrial gravitation;

R — radius of Earth;

h — altitude of flight of the object, for instance an aircraft, above surface of Earth.

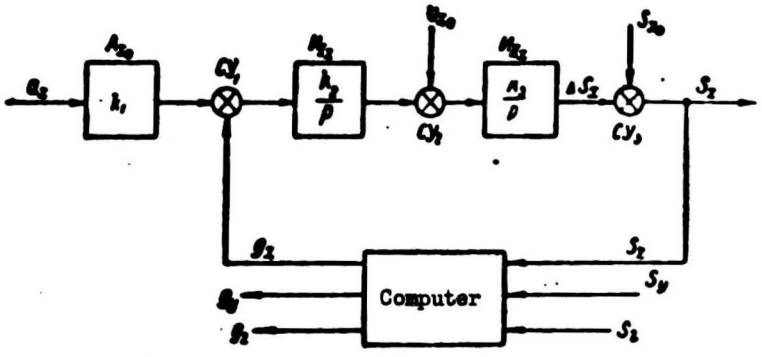

Fig. 11.37. Fundamental scheme of determination of S_x in an INS with analytic VE.

If it is necessary to go from coordinates S_x, S_y, S_z to the coordinates of the object in the terrestrial system, then erection of the vertical necessary for this transition is carried out analytically by the same method which is realized in the scheme in Fig. 11.26. Detailed theoretical investigation of such an inertial system is contained in a number of works [142, 13, 14, 12, 40].

Above there were considered INS containing a gyro-stabilized platform; however, in principle there is possible creation of INS without gyroscopic devices. One of such systems has the name of inertial-pendulum system of navigation [142]. In it

there is used an inertial body as stabilizer, with which there are connected accelerometers. Signal from accelerometer is given through the amplifier to the motor, which acts on the inertial body; due to this the accelerometer retains its horizontal position. Accelerations measured by accelerometers enter the integrators for determination of coordinates of the object.

There are possible inertial systems without stabilizers [142, 12]. In this case the coordinate system which fixes directions of the axes of sensitivity of accelerometers is reproduced analytically. For this purpose it was proposed [12] to use six accelerometers rigidly installed on the object at a certain distance from each other. They are designed for determination of linear accelerations of center of gravity of the object, and also angles of its rotation relative to the inertial system of coordinates. For practical realization of systems without stabilizers there are necessary especially accurate accelerometers and computers. It is possible to become acquainted with design of the inertial system based on this principle and with theoretical investigations of it in works [142, 12].

9. Certain Remarks About Other Types of Gyroscopic Navigational Systems

Among gyroscopic navigational systems, as was shown in § 11.1, there are included also gyrohorizon compasses, gyrolatitude compasses and gyrolatitudes.

The theory of the gyrohorizon compass (GHC) was considered in § 11.5, Par. 6. In reference to the use of this instrument as a navigational system, it is possible to say the following. The GHC gives plane of horizon and indicates direction to the north; thus, thanks to satisfaction of the condition of aperiodic transitions, the system is undisturbed by accelerations of the object. Consequently, the sensing element of the GHC is identical in its properties to the inertial vertical obtained in the INS by means of artificial simulation of the pendulum of M. Schuler with help of gyroscopes, accelerometers and integrators. Let us place on the sensing element of the GHC two accelerometers with axes sensitivity directed toward north and east.* Accelerometers will measure accelerations of the object along these directions. By means of double integration of readings of the accelerometers, one can determine latitude φ and longitude λ of position of the object. Consequently, by supplementing the GHC with two accelerometers, integrators and computer according to a scheme analogous to that shown in Fig. 11.34, we will obtain an inertial navigational system of the gyrohorizon compass type.

*In reality axes of sensitivity of accelerometers deviate from the indicated axes by the angle of velocity error of the GC (§ 11.5, Par. 4b).

Such an inertial system with two accelerometers is investigated in work [166]. One of the peculiarities of this system is that undisturbability of azimuthal motion of the sensing element of the GHC is attained by appropriate control of velocity of rotation of the rotors. As initial data for operation of the control system there serve current values of readings of accelerometers, which are used for subsequent determination of latitude and longitude of position of the object.

Application in GHC of two accelerometers makes it possible instead of a sensing element with displaced center of gravity to use a balanced sensing element, which for retention of it in the plane of the horizon and in direction of the local meridian is supplied with a system of correction (from the same accelerometers) which is analogous to the one used in the GC with electromagnetic control [59] (see § 11.5, Par. 1a). In this case for obtaining in the instrument of the vertical (plane of horizon) there is applied essentially the principle of integral correction from the accelerometers. One of such INS of the gyrohorixon compass type [2] is based on the property of the GHC to measure vector of absolute velocity of the object [44] and on the property of the GC to indicate direction of vector of its absolute horizontal velocity.

The system consists [2] of two astatic gyroscopes located mutually perpendicularly to each other on a platform installed in a Cardan suspension. On the platform there are placed two accelerometers. The system of correction using readings of its sensing elements — accelerometers — applies correcting torques relative to the horizontal and vertical axes of the corresponding gyroscopes; these torques are proportional to the integrals of readings of the accelerometers. By selection of appropriate coefficients there is provided undisturbability of the system by accelerations of the object. Thus, one of the gyroscopes acquires properties of the gyrocompass, and the other — properties of the gyrohorizon. If these gyroscopes are connected to the control of actuating motors fixed on the axes of the Cardan suspension of the platform, then the latter will be stabilized in plane of horizon and will preserve azimuthal direction to the north. Consequently, readings of one of the accelerometers will make it possible to determine horizontal component of absolute velocity of the object, and readings of the other — vertical component of this velocity. Presence of the indicated data, for known initial coordinates and with consideration of diurnal rotation of the Earth, permits us to determine current coordinates of the object in terrestrial system and parameters of its motion. Merit of this inertial system is that it has a directional torque in azimuth; this decreases influence of disturbances.

System does not have systematic errors during measurement by it of absolute horizontal velocity of the object. A detailed theoretical investigation of this system is contained in article [2].

There is possible another variant of INS of the gyrohorizon compass type: on sensing element of the GHC there is installed only one accelerometer, which is oriented in the north direction; due to this, after integration of its readings there will be obtained the northern component v_N of velocity of the object. Eastern component v_E of velocity is determined in the following way: As B. V. Bulgakov [20] showed, angle δ of rotation of gyroscopes inside the gyrosphere (Fig. 11.19) with accuracy up to harmonic oscillations determines in a certain scale the absolute eastern velocity of the ship $V_E = RU \cos \varphi + v_E$, whence it is easy to obtain quantity v_E. Let us note that practical realization of this scheme of INS of the gyrohorizon compass type is very complicated.

It is of interest to indicate the scheme of the gyroscopic navigational system proposed by A. Yu. Ishlinskiy [48] for determination of latitude φ and longitude λ of position of the object using a gyrohorizon compass, directional gyroscope and a computer. The latter determines φ and λ on the bases of integration of the following nonlinear equations [48]:

$$\dot{\varphi} = \omega_y \sin \vartheta; \quad \dot{\vartheta} = \omega_z - \omega_y \cos \vartheta \operatorname{tg} \varphi; \quad \dot{\lambda} = \omega_y \frac{\cos \vartheta}{\cos \varphi} - U, \tag{11.6.286}$$

where ϑ — angle between the tangent to the trajectory of object and the parallel; it is equal to velocity error of the GC [see (11.5.71)];

ω_y, ω_z — components of angular velocity of the trihedron Oxyz, axes Ox and Oy of which are tangent to the earth's surface, and axis Oz of which is directed along radius of Earth.

Quantity ω_y is determined by the GHC, since, according to (11.5.172), angular velocity ω_y is connected with angle ε of separation of the gyroscopes by the relationship

$$\omega_y = \frac{2H' \cos \varepsilon}{mlR}. \tag{11.6.287}$$

Angular velocity ω_z is determined by the directional gyroscope, the sensitive axis of which is connected with the compass trihedron Oxyz. If into the computer there are introduced ω_y, ω_z and initial values of coordinates φ_0, λ_0, ϑ_0, then in accordance with formulas (11.6.286) there will be obtained current coordinates $\varphi(t)$, $\lambda(t)$, $\vartheta(t)$. Certain investigations of this system are contained also in [38].

Some of the types of gyroscopic navigational systems are gyrolatitude compasses and gyrolatitudes. In a number of works there are given fundamental schemes of these systems and their theoretical foundations. We refer those interested in such systems to the appropriate literature (which will be given below), and will ourselves be limited to the following remarks.

The most well known scheme of gyrolatitude compass is the scheme with the sliding pendulum [7, 145], which was for the first time considered by A. P. Korzhov. The basic element of the instrument is the astatic gyroscope with three degrees of freedom, fixed in Cardan suspension on a stabilized platform. With the platform there is connected an annular vessel, the plane of which passes through axis of gyroscope. In the vessel there is liquid; the period of its flow can be regulated with help of a diaphragm with a hole fixed in the vessel. The system containing a vessel with liquid is called a sliding pendulum. By means of investigation of differential equations of motion of the system it is possible to show that in position of equilibrium the axis of the gyroscope is established in plane of meridian at the angle of latitude φ to plane of horizon. Consequently, this gyrolatitude compass determines heading of the object K and latitude φ of its position. It has velocity and ballistic errors. For compensation of ballistic errors, period of natural sustained oscillations of the gyrolatitude compass should be equal to 84.4 min, since the instrument is a variety of the gyropendulum. Let us note that period of natural oscillations of the gyrolatitude compass does not depend on local latitude. The question of practical creation of such an instrument is very complicated; in particular, there is necessary stabilization of its sensing element, and this will require not one, but two gyroscopes. The scheme and analysis of the considered gyrolatitude compass are given in a number of works (for instance, [7, 92, 145]).

Of interest is the scheme of gyrolatitude compass with indirect correction [98, 7]. Sensing element of the instrument is an astatic gyroscope with three stabilized relative to the plane of the horizon. For transformation of the gyroscope into a gyrolatitude compass there serves a torquer located on the horizontal axis of the suspension; it applies to the gyroscope a corrective torque proportional to angular velocity of deflection of axis of gyroscope from plane of horizon. Angular velocity is determined by differentiation of the indicated angle of displacement, which is taken from the signal pickoff which is fixed on the horizontal axis of the suspension of the gyroscope. Consequently, we have here a system of automatic derivative control; due to this there is not required introduction of special damping. Analysis shows [98, 7], that in the position of equilibrium the axis of the gyroscope is established in plane of meridian at the angle of latitude φ to plane of horizon. It is possible to become acquainted with the scheme and investigation of the gyrolatitude compass with indirect correction in the above-indicated works.

There are known a number of schemes of gyrolatitudes which make it possible to determine the local latitude. The gyrolatitude using the Foucault gyroscope of second kind was considered by us in § 11.2; application of such an instrument on a rocking base is quite difficult. One of the modifications of this scheme was investigated by D. R. Merkin [92]. There have been proposed schemes of gyrolatitudes using gyroscopes with three degrees of freedom [7]. One of such schemes consists of a gyroscope in a Cardan suspension; axis of rotation of outer ring is horizontal; to the latter there is fastened a load. The platform on which the gyroscope is fixed has geographic orientation by means of connection with a GC. Investigation shows [7] that in the position of equilibrium the axis of the gyroscope is established at the angle of latitude φ to plane of horizon. Accurac of readings of this instrument is greatly influenced by change of velocity of the object and its circulations. These and other circumstances hamper the practical application of such systems. Other schemes of gyrolatitudes are given in the article [101].

For determination of local latitude there may also be applied the method of C. Fox [188], which is based on measurement of gyroscopic torques. The device consists of two two-degree-of-freedom gyroscopes located on a platform which is stabilized relative to plane of horizon and azimuthal direction in plane of meridian. On the axes of precession of the gyroscopes there are fixed torquers which serve for compensation of the corresponding gyroscopic torques appearing about these axes. Since the platform is stabilized relative to plane of horizon and azimuthal direction in plane of meridian, then rotation of it relative to inertial space causes appearance of gyroscopic torques. The latter tend to deflect gyroscopes from their initial position in plane of the platform, but are balanced by corresponding torques applied by the torquers. This makes it possible to measure these gyroscopic torques, the ratio of which determines [188] the local latitude φ. With the help of a computer one can determine longitude λ, by using relationship $\dot{\lambda} = \frac{v \sin K}{R \cos \varphi}$, where v and K are assumed to be known.

It is necessary to note that the scheme of C. Fox can hardly be useful in practice, since here, like the Foucault gyroscope of second kind (§ 11.2, Par. 2b), errors of stabilization of the platform — more exactly, angular velocities of its oscillation relative to the given position — will cause appreciable error of two-degree-of-freedom gyroscopes.

In conclusion let us note that for self-contained determination of local latitude and longitude it is possible also to use the astatic gyroscope fixed on a horizontal

base [135]. In [135] there is investigated a three-degree-of-freedom astatic gyroscope, the axis of the outer Cardan ring of which is fixed horizontally. There are composed equations of motion analogous to (3.3.51). The obtained system of nonlinear differential equations is integrated within finite angles α and β of rotation of outer and inner Cardan rings.* As a result there are found expressions whose realization in the computer makes it possible to calculate latitude and longitude of position of the object. For certain particular cases, the corresponding formulas are obtained to be quite simple. Thus, for instance, if with the earlier shown installation of the astatic gyroscope, its axis at the initial moment is located exactly along the vertical ($\alpha_0 = \beta_0 = 0$), then for latitude φ and longitude λ we have the relations [135]

$$\left.\begin{aligned} \lambda &= \lambda_0 - \arcsin\frac{\sin\beta}{\cos\varphi_0} - Ut \\ \varphi &= \alpha + \arcsin\frac{\cos\varphi_0}{\cos\beta} \end{aligned}\right\} \qquad (11.6.288)$$

Consequently, by taking from the gyroscope angles α and β, by formulas (11.6.288) it is possible to calculate current values of φ and λ.

If axis of gyroscope is parallel to plane of equator, i.e., $\alpha_0 = \varphi_0$, $\beta_0 = 0$, then

$$\left.\begin{aligned} \lambda &= \lambda_0 - \beta - Ut \\ \varphi &= \alpha \end{aligned}\right\} \qquad (11.6.289)$$

i.e., values of φ and λ can directly be taken from axes of Cardan suspension of gyroscope. If in the considered case the object is motionless, then $\lambda = \lambda_0$ and $\beta = -Ut$; i. e., the free gyroscope can serve as a chronometer. It is necessary to note that practical realization of this method of determination of coordinates φ and λ requires creation of an "ideal" free gyroscope and accurate computers.

*Integration of analogous equations was for the first time performed by A. M. Letov [81], D. R. Merkin [90], R. I. Chertkov [169, 135], E. I. Sliv [135].

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