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INTERACTION ENERGY OF A DIELECTRIC IN AN ELECTROSTATIC FIELD

Victor Gilinsky and Dennis Holliday

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PREFACE

This report clarifies a paradox in the calculation of the energy of a dielectric in an electrostatic field. The connection with a recent paper by M. S. Plesset and G. Venezian is discussed. This work should be of interest to persons teaching electromagnetic theory.

A paper based on this report is being submitted to the American Journal of Physics.

SUMMARY

A straightforward computation of the interaction field energy of a dielectric in a uniform external electrostatic field appears to yield a result different from that given by a general theorem. It is shown here that the discrepancy can be resolved by considering the uniform field of infinite extent to be the limit of a field of finite extent. A recent paper by M. S. Plesset and G. Venezian is discussed.

CONTENTS

I. INTRODUCTION

We are going to use the familiar problem of a dielectric sphere in a uniform electric field to illustrate some ideas about the interaction energy of a dielectric in a uniform field of infinite extent. Our conclusions also apply to magnetostatic problems.

The field energy in a linear electrostatic system is given by

$$U_{field} = \tfrac{1}{2} \int d^3x \; E \cdot D. \tag{1.1}$$

When a dielectric object is placed in an external field whose sources are fixed,[1] the interaction energy of the dielectric and field is the resulting change in field energy ΔU:

$$\Delta U_{field} = \tfrac{1}{2} \int d^3x \; (E \cdot D - E_o \cdot D_o) \; , \tag{1.2}$$

where E_o and D_o are the initial fields and E and D are the fields with the dielectric in place. The interaction energy calculated from (1.2) can be used in the principle of virtual work to find the forces or torques on the dielectric object.

Many books[2] show that the change in field energy can be expressed as an integral over the volume of the dielectric

[1] The sources of the field can, in general, be taken as charges on conductors. "Fixed sources" means that these conductors are fixed in position with a fixed amount of charge.

[2] For example, L. D. Landau and E. M. Lifshitz, Electrodynamics of Continuous Media, Addison-Wesley, Reading, Mass., 1960, p. 54.

$$\Delta U_{field} = -\tfrac{1}{2}\int d^3x\, P\cdot E_o \qquad \text{(fixed sources)} \qquad (1.3)$$

where P is the polarization of the dielectric. In a uniform field (1.3) becomes

$$\Delta U_{field} = -\tfrac{1}{2}\, p\cdot E_o \qquad \text{(fixed sources)} \qquad (1.4)$$

where p is the total induced dipole moment of the body. However, a straightforward calculation of ΔU_{field} can yield a result different from (1.4).

For example, consider a dielectric sphere (radius a, dielectric constant k) which is brought into a region of uniform field E_o of very large extent (see Fig. 1). After the dielectric is in the field, the potentials in spherical coordinates are given by

$$\varphi(r,\theta) = \begin{cases} -\frac{3}{k+2} E_o\, r\cos\theta & r < a \\ -E_o\, r\cos\theta + \frac{k-1}{k+2} E_o \frac{a^3}{r^2}\cos\theta & r > a \end{cases} \qquad (1.5)$$

and the electric fields are

$$E_r(r,\theta) = \begin{cases} \frac{3}{k+2} E_o \cos\theta & r < a \\ E_o\cos\theta + 2\frac{k-1}{k+2} E_o \frac{a^3}{r^3}\cos\theta & r > a \end{cases} \qquad (1.6)$$

$$E_\theta(r,\theta) = \begin{cases} -\frac{3}{k+2} E_o \sin\theta & r < a \\ -E_o\sin\theta + \frac{k-1}{k+2} E_o \frac{a^3}{r^3}\sin\theta & r > a\,. \end{cases} \qquad (1.7)$$

A direct integration using the fields (1.6) and (1.7) in (1.2) suggests that the difference in field energy before and after the dielectric sphere is in the field is

$$\Delta U = \frac{1}{6} 4\pi \epsilon_o \frac{k-1}{k+2} a^3 E_o^2 = \frac{1}{6} p E_o, \qquad (1.8)$$

where p is the dipole moment of the sphere.

In order to reconcile (1.8) with (1.4) we are going to solve the problem of a dielectric sphere in a finite capacitor composed of two hemispheres. Then we will let the radius of the capacitor become infinitely large so that the electric field becomes constant, and we will see where the field energy goes.

However, before we go on to the calculation let us make a few general remarks about the interaction energy and its use in the principle of virtual work. If we imagine that the constant electric field is produced by a very large capacitor, then the condition that the sources stay fixed while the dielectric is placed in the field corresponds to the constraint of constant charge Q on the plates. That is, (1.4) can be written as

$$\Delta U_{field} = - \tfrac{1}{2} p \cdot E_o \qquad \text{(constant Q)}. \qquad (1.9)$$

Another possible constraint is that the voltage V between the plates is held constant, for example, by batteries. In this case it can be shown generally that

$$\Delta U_{field} = \tfrac{1}{2} p \cdot E_o \qquad \text{(constant V)}. \qquad (1.10)$$

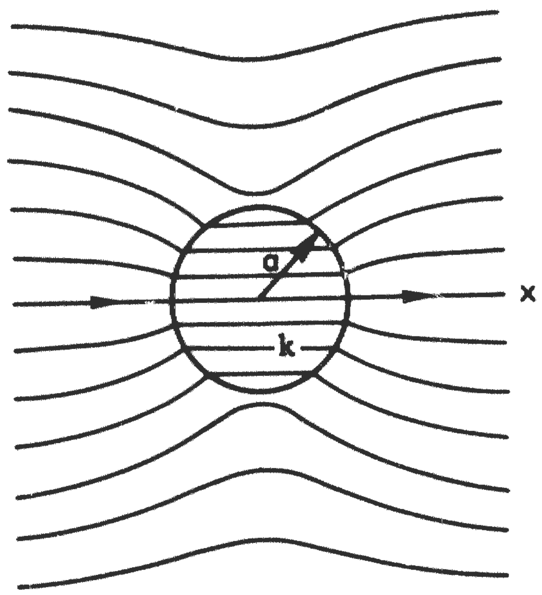

Fig. 1—Dielectric sphere in a uniform field

The difference between (1.9) and (1.10) occurs because in the "constant V" case there is also a change in battery energy when the dielectric is placed in the field. In general, as long as the dielectric is linear, one can show that

$$\Delta U_{batt} = - 2\ \Delta U_{field} \qquad \text{(constant V)} \qquad (1.11)$$

and so the <u>total</u> energy change

$$\Delta U_{total} = \Delta U_{field} + \Delta U_{batt} \qquad (1.12)$$

is the same in both cases, as it must be.

Note that if we now apply the principle of virtual work,

$$F\ \delta x = - \delta(\Delta U_{tot}), \qquad (1.13)$$

we obtain the force on a dielectric body in an inhomogeneous field

$$F = p \cdot \frac{\partial E_o}{\partial x}. \qquad (1.14)$$

The result does not depend on the constraints imposed during the virtual displacement. An alternate rule, which is often stated in textbooks, is that the force can be obtained from the change in field energy alone (if one remembers to change the sign when keeping the potential V constant)

$$F = - \frac{\partial \Delta U_{field}}{\partial x} \qquad \text{(constant Q)} \qquad (1.15)$$

$$F = + \frac{\partial \Delta U_{field}}{\partial x} \qquad \text{(constant V).} \qquad (1.16)$$

Equation (1.16) holds because the energy lost by the batteries is always twice the energy gained by the field.

It is clear that the change in field energy given in (1.8) for the case of an infinite uniform external field is not by itself suitable for use with the principle of virtual work. The reason for this is not that the principle of virtual work in the form of (1.15) or (1.16) fails in fields of infinite extent. Instead, as we shall see in the next two sections, the change in field energy calculated in (1.8) is only a part of the total change in field energy ΔU_{field}.

II. SPHERICAL CAPACITOR

In Fig. 2 we show the arrangement of the spherical capacitor and dielectric sphere. The potentials can be written in the form

$$\varphi_1 = \sum_{n\ \mathrm{odd}} A_n r^n P_n(\cos\theta) \qquad r < a \qquad (2.1)$$

$$\varphi_2 = \sum_{n\ \mathrm{odd}} \left(B_n r^n + \frac{C_n}{r^{n+1}}\right) P_n(\cos\theta) \qquad a < r < R \qquad (2.2)$$

$$\varphi_3 = \sum_{n\ \mathrm{odd}} \frac{D_n}{r^{n+1}} P_n(\cos\theta) \qquad r > R \qquad (2.3)$$

with

$$A_n = \frac{V}{R^n} \lambda_n \frac{(2n+1)^2}{2(nk+n+1)} \left[1 - \left(\frac{a}{R}\right)^{2n+1} \frac{n(k-1)}{nk+n+1}\right]^{-1} \qquad (2.4)$$

$$B_n = \frac{V}{R^n} \lambda_n \frac{2n+1}{2} \left[1 - \left(\frac{a}{R}\right)^{2n+1} \frac{n(k-1)}{nk+n+1}\right]^{-1} \qquad (2.5)$$

$$C_n = -\frac{V}{R^n} \lambda_n \frac{2n+1}{2} \frac{n(k-1)}{nk+n+1} \left[1 - \left(\frac{a}{R}\right)^{2n+1} \frac{n(k-1)}{nk+n+1}\right]^{-1} a^{2n+1} \qquad (2.6)$$

and

$$D_n = V R^{n+1} \lambda_n \frac{2n+1}{2}. \qquad (2.7)$$

The coefficient λ_n is given by

$$\lambda_n = \int_0^1 dx\, P_n(x) \qquad n\ \mathrm{odd}. \qquad (2.8)$$

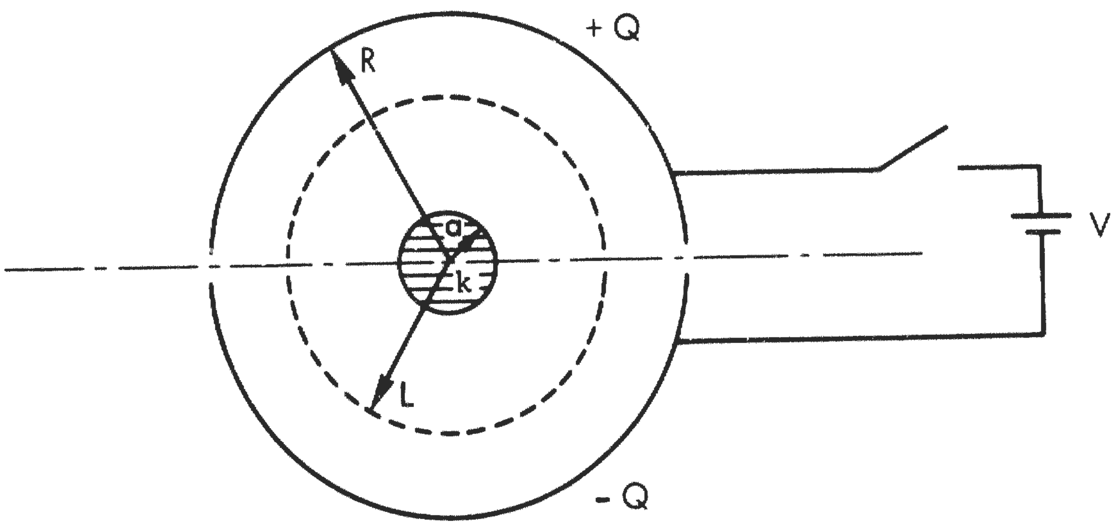

Fig. 2—Spherical capacitor

The electric fields are:

within dielectric sphere $(r < a)$

$$E_r = -\sum_{n \text{ odd}} n A_n r^{n-1} P_n(\cos\theta)$$

$$E_\theta = -\sum_{n \text{ odd}} A_n r^{n-1} P_n^1(\cos\theta) \tag{2.9}$$

inside capacitor $(a < r < R)$

$$E_r = -\sum_{n \text{ odd}} \left(n B_n r^{n-1} - \frac{(n+1)C_n}{r^{n+2}}\right) P_n(\cos\theta)$$

$$E_\theta = -\sum_{n \text{ odd}} \left(B_n r^{n-1} + \frac{C_n}{r^{n+2}}\right) P_n^1(\cos\theta) \tag{2.10}$$

outside capacitor $(r > R)$

$$E_r = \sum_{n \text{ odd}} (n+1) \frac{D_n}{r^{n+2}} P_n(\cos\theta)$$

$$E_\theta = -\sum_{n \text{ odd}} \frac{D_n}{r^{n+2}} P_n^1(\cos\theta). \tag{2.11}$$

We can now use (1.2) to compute the field energy change brought about by bringing the dielectric sphere in from an infinite distance. The result will depend on the conditions under which this is done, but all processes can be reduced to linear combinations of two cases: constant voltage between the plates or constant charge on the plate.

Let us first consider the constant voltage case. To simplify the calculations of the field energy change we will imagine that a is a small quantity compared with R and we will keep only terms independent of a and terms which do not vanish as $R \to \infty$. With the dielectric sphere in place the field energy is

$$U_{field} = U_1 + U_2 + U_3, \tag{2.12}$$

where U_1, U_2, and U_3 are the field energies in the regions $r < a$, $a < r < R$, and $r > R$, respectively:

$$U_1 = K \epsilon_o \pi \sum_{n\ odd} A_n^2 \frac{2n}{2n+1} a^{2n+1} = \epsilon_o \pi \left(\frac{3V}{4R}\right)^2 \frac{6k}{(k+2)^2} a^3 + \left(\frac{V}{R}\right)^2 0 \left(\frac{a^3}{R^3}\right)$$

$$U_2 = \epsilon_o \pi \sum_{n\ odd} \frac{2}{2n+1} \left(1 - \left(\frac{a}{R}\right)^{2n+1}\right) \left[nB_n^2 R^{2n+1} + (n+1) \frac{C_n^2}{a^{2n+1}}\right] \tag{2.13}$$

$$\approx \frac{1}{2} \epsilon_o \pi \left(\frac{V}{R}\right)^2 R^3 \sum_{n\ odd} \lambda_n^2 n(2n+1) + \epsilon_o \pi \left(\frac{3V}{4R}\right)^2 \frac{2k^2-4k-4}{(k+2)^2} a^3$$

$$+ \left(\frac{V}{R}\right)^2 0 \left(\frac{a^3}{R^3}\right) \tag{2.14}$$

and

$$U_3 = \frac{1}{2} \epsilon_o \pi \left(\frac{V^2}{R^2}\right) R^3 \sum_{n\ odd} \lambda_n^2 (n+1)(2n+1). \tag{2.15}$$

Therefore, at constant plate voltage, the change in field energy is

$$\Delta U_{field} = U_1+U_2+U_3 - \left[\frac{1}{2} \epsilon_o \pi \left(\frac{V}{R}\right)^2 R^3 \sum_{n\ odd} \lambda_n^2 n(2n+1) + U_3 \right]$$

$$\approx 2\pi\, \epsilon_o \left(\frac{3V}{4R}\right)^2 \frac{k-1}{k+2}\, a^3 + \left(\frac{V}{R}\right)^2 0 \left(\frac{a^3}{R^3}\right) \quad \text{(V constant).} \qquad (2.16)$$

But, $3V/4R$ is E_o, the field at the origin before the dielectric is put in, so

$$\Delta U_{field} = \frac{1}{2}\, p \cdot E_o \left[1 + 0 \left(\frac{a^3}{R^3}\right)\right], \qquad \text{(V constant)} \qquad (2.17)$$

where p, the dipole moment of the dielectric sphere in a uniform field, is

$$p = 4\pi\, \epsilon_c \frac{k-1}{k+2}\, a^3\, E_o. \qquad (2.18)$$

This result is the true change in field energy and agrees exactly with (1.10) when $R \rightarrow \infty$.

The energy gained or lost by the batteries in keeping the voltage constant ΔU_{batt} is found from the equation

$$\Delta U_{batt} = -\, V\, \Delta\, Q. \qquad (2.19)$$

We can compute the change in charge ΔQ by integrating over the surface of the capacitor the difference in radial components of the electric fields before and after the dielectric sphere is in the field (Gauss' Law):

$$\Delta Q = V \left(\frac{3}{4R}\right)^2 4\pi \epsilon_o \frac{k-1}{k+2} a^3 + \frac{V}{R^2} 0 \left(\frac{a^3}{R^3}\right). \tag{2.20}$$

From (2.20) and (2.19) we see that the batteries lose energy:

$$\Delta U_{batt} = - \left(\frac{3V}{4R}\right)^2 4\pi \epsilon_o \frac{k-1}{k+2} a^3 + \left(\frac{V}{R}\right)^2 0 \left(\frac{a^3}{R^3}\right) = - p \cdot E_o \left[1+0 \left(\frac{a^3}{R^3}\right)\right]. \tag{2.21}$$

The total energy change or interaction energy is then

$$\Delta U_{tot} = \Delta U_{field} \text{ (constant V)} + \Delta U_{batt} = - \frac{1}{2} p \cdot E_o \left[1+0 \left(\frac{a^3}{R^3}\right)\right]. \tag{2.22}$$

We can also compute ΔU_{field} (constant Q). It is, of course, equal to ΔU_{tot} because there is no change in battery energy in this case:

$$\Delta U_{field} = - \frac{1}{2} p \cdot E_o \left[1+0 \left(\frac{a^3}{R^3}\right)\right] \quad \text{(constant Q)}. \tag{2.23}$$

III. INFINITE CAPACITOR

So far we have shown that in the limit of an infinitely large capacitor the results of (1.9) and (1.10) remain true. We will now show the significance of the energy change given by (1.8).

Let us consider the constant voltage case. We saw that the change in field energy occurs in the field within the spherical capacitor. Imagine now that we calculate the change in the field energy in the two regions $0 < r < L$ and $L < r < R$:

$$\Delta U_{field} = \Delta U_{field}\ (0 \to L) + \Delta U_{field}\ (L \to R). \tag{3.1}$$

If we now let $R \to \infty$ in both terms while keeping $3V/4R = E_o$, we get

$$\begin{aligned} \Delta U_{field}\ (0 \to L) = k\ \epsilon_o \pi \frac{2}{3} a^3 A_1^2 \Big|_{R \to \infty} \\ - \ \epsilon_o \pi \frac{2}{3} a^3 B_1^2 \Big|_{R \to \infty} \\ + \ \epsilon_o \pi \frac{4}{3} \frac{1}{a^3} C_1^2 \Big|_{R \to \infty} + 0\left(\frac{a^3}{L^3}\right) \end{aligned} \tag{3.2}$$

$$\Delta U_{field}(L \to \infty) = \epsilon_o \pi \frac{2}{3} (B_1^2 - E_o^2) R^3 \Big|_{R \to \infty} + 0\left(\frac{a^3}{L^3}\right).$$

Using (2.4) - (2.6) and taking $L \to \infty$ we find (with obvious notation) that

$$\left.\begin{aligned} \Delta U_{field}\ (\text{local}) &= \frac{1}{6}\, p \cdot E_o \\ \Delta U_{field}(\text{distant}) &= \frac{1}{3}\, p \cdot E_o . \end{aligned}\right\} \text{(constant V)} \tag{3.3}$$

ΔU_{field} (local), the change in field energy in the region $0 < r < L$ as $L \to \infty$, is what was calculated in (1.8) using (1.2). This is not the total change in field energy because there is also a change ΔU_{field} (distant) in the region $L < r < \infty$ as $L \to \infty$. ΔU_{field} (local) + ΔU_{field} (distant) add up, of course, to the total change in field energy $\frac{1}{2}$ $p \cdot E_o$ (constant V) as calculated in the previous section.

We see, therefore, that a direct calculation of the field energy from the fields given in (1.6) and (1.7) will net only the quantity ΔU_{field} (local). In order to get the <u>entire</u> field energy change it is necessary to calculate a finite case and take the limit correctly. This resolves the discrepancy between (1.8) and (1.9) and (1.10).

IV. COMMENT

We have found that so long as the field of infinite extent is considered as the limit of a field of finite extent there is no discrepancy between the general results for the interaction energy (1.9) and (1.10) and results directly calculated from (1.2). This means that the principle of virtual work in the form (1.15) or (1.16) can be used to calculate the forces on a dielectric in an infinite field if the total field energy change is used.

In a recent paper[3] M. S. Plesset and G. Venezian discussed the problem of calculating the force on a dielectric in a field of infinite extent using the principle of virtual work. They obtained the correct total energy change during an infinitesimal displacement of the dielectric by subtracting the directly calculated "field energy change" $\frac{1}{6}\, p \cdot E_o$ of (1.8) (which we now recognized as ΔU_{field} (local)) from the energy crossing a closed surface at infinity. This energy which amounted to $\frac{2}{3}\, p \cdot E_o$ was described as the energy change in the sources at infinity. Our previous discussion shows that this description is not correct. At constant potential (constant V) the energy lost by the sources at infinity was given by (2.21) as $-\Delta U_{batt} = p \cdot E_o$. According to (3.3) a third of this energy $\frac{1}{3}\, p \cdot E_o$ goes into the distant field outside the surface at infinity leaving $\frac{2}{3}\, p \cdot E_o$ to pass through the surface. At constant charge (constant Q) the $\frac{2}{3}\, p \cdot E_o$ cannot come from the sources at infinity because they are held constant. Instead this energy comes from the change in field

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[3] M. S. Plesset and G. Venezian, Am. J. Phys., Vol. 32, 1964, p. 860.

energy in the region outside the surface at infinity. The total energy change ΔU_{total} is, of course, $-\frac{1}{2}\, p \cdot E_o$ in both cases.

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A discussion and clarification of a paradox in the calculation of the energy of a dielectric in an electrostatic field. A straightforward computation of the interaction field energy of a dielectric in a uniform external electrostatic field appears to yield a result different from that given by a general theorem. It is shown, however, that the discrepancy can be resolved by considering the uniform field of infinite extent to be the limit of a field of finite extent.