ESD-TDR-66-41

Technical Note 1966-9

L. A. Gardner, Jr.

A Method for Estimating the Parameters of a Stable Difference Equation Driven by Sinusoids and Observed in the Presence of Nonwhite Noise

11 February 1966

Prepared under Electronic Systems Division Contract AF 19(628)-5167 by

Lincoln Laboratory
MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Lexington, Massachusetts

The work reported in this document was performed at Lincoln Laboratory, a center for research operated by Massachusetts Institute of Technology, with the support of the U.S. Air Force under Contract AF 19(628)-5167.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

LINCOLN LABORATORY

A METHOD FOR ESTIMATING THE PARAMETERS OF A STABLE DIFFERENCE EQUATION DRIVEN BY SINUSOIDS AND OBSERVED IN THE PRESENCE OF NONWHITE NOISE

L. A. GARDNER, JR.

Group 22

TECHNICAL NOTE 1966-9

11 FEBRUARY 1966

LEXINGTON MASSACHUSETTS

ABSTRACT

A computationally feasible and strongly consistent method is considered for estimating the coefficients of

$$\gamma_0 x_t + \gamma_1 x_{t-1} + \cdots + \gamma_p x_{t-p} = u_t .$$

The steady-state solution is observed in the presence of nonwhite noise, and the system is driven by a certain selected superposition of sinusoids, u_t. When the noise is white, the estimates are shown to possess an asymptotic normal distribution with covariance matrix which depends on the coefficients only via the values of

$$\left| \sum_{k=0}^{p} \gamma_k e^{ik\omega} \right|^2$$

evaluated at the input frequencies.

Accepted for the Air Force
Franklin C. Hudson
Chief, Lincoln Laboratory Office

1. INTRODUCTION

We observe the steady state solution of a p^{th} order difference equation in the presence of additive noise. The deterministic input sequence is at our disposal, up to an unknown gain. Corresponding to a superposition of sinusoidal inputs, we exhibit an estimate of the $p+1$ parameters having the following properties.

(i) On-line computation of the estimates can be done recursively in the incoming data by operations which entail only first-order storage requirements. (ii) As the observation time increases to infinity the estimate is strongly consistent under very weak conditions on the noise process, which include stationarity but not the existence of a power spectrum. (iii) The standardized estimates tend to joint normality with a covariance matrix which is relatively easy to calculate when the noise is white. If, in addition, the noise is Gaussian, the estimate has minimal (generalized) variance.

The initial step is "naive" least squares, i.e. solving normal equations (and hence (i)). Simple instantaneous nonlinear operations are then performed on the least squares solution. The result is mapped by a fixed matrix into the estimate of the ratio of coefficients to gain.

2. THE MODEL AND ESTIMATE

Corresponding to a given deterministic sequence $\{u_t\}$, let $\{x_t\}$ satisfy

$$x_t = \theta_1 x_{t-1} + \cdots + \theta_p x_{t-p} + K u_t \qquad (t = \ldots, -1, 0, 1, \ldots)$$

for some input gain $K \neq 0$. We will assume stability, i.e. that the p roots of the characteristic equation

$$\lambda^p - \theta_1 \lambda^{p-1} - \cdots - \theta_p = 0$$

are each in modulus less than unity. We will also assume, but only for the sake of simplicity, that the roots are distinct.

We observe

$$y_t = x_t + \xi_t \qquad (t = 1, 2, \ldots, n, \ldots),$$

where $\{\xi_t\}$ is some zero mean stochastic process, and suppose that the regression sequence was arbitrarily initialized in the remote past. In statistical terminology we are faced with a particular nonlinear regression problem:

$$x_t = F_t(K, \theta_1, \ldots, \theta_p)$$

is known up to the values of $p+1$ parameters to be estimated.

The input. To estimate K and $\theta_1, \ldots, \theta_p$ we distinguish between odd and even values of $p+1$, and accordingly take for the input

$$u_t = \begin{cases} \dfrac{1}{\sqrt{2}} + \displaystyle\sum_{j=1}^{q} \cos \omega_j t & \text{if } p = 2q \\ \\ \dfrac{1}{\sqrt{2}} + \displaystyle\sum_{j=1}^{q} \cos \omega_j t + \dfrac{(-1)^t}{\sqrt{2}} & \text{if } p = 2q+1. \end{cases}$$

The angular frequencies

$$0 < \omega_1 < \cdots < \omega_q < \pi$$

are to be chosen. Thus, the input is $1/\sqrt{2}$ when $p=0$, giving the constant regression $K/\sqrt{2}$ for the trivial case of no θ's. The input is $1/\sqrt{2} + (-1)^t/\sqrt{2}$ when $p=1$, $1/\sqrt{2} + \cos \omega_1 t$ when $p=2$, and so on.

We present the estimation procedure for an even number of parameters. After doing so, we indicate the modifications to be made for an odd number, as well as for the case when the gain is known. Thus, in what follows,

$$p+1 = 2q + 2$$

for some integer $q \geq 0$.

Step 1. Choose q distinct frequencies interior to $(0, \pi)$, and define

$$h'_t = \left[\frac{1}{\sqrt{2}}, \cos\omega_1 t, \sin\omega_1 t, \ldots, \cos\omega_q t, \sin\omega_q t, \frac{(-1)^t}{\sqrt{2}} \right]$$

for $t = 1, 2, \ldots, n$ with $n > p$. Define matrices

$$H_n = [h_1, \ldots, h_n] \qquad B_n = (H_n H'_n)^{-1}$$

where prime denotes transposition. Let $\vec{y}_n$ be the column vector of the first n observations $y_1, \ldots, y_n$, and let

$$\varphi_n = B_n H_n \vec{y}_n .$$

It is known that successive vectors so defined (the solutions of normal equations for increasing sample size) satisfy the difference equation

$$\varphi_n = \varphi_{n-1} + k_n(y_n - h'_n \varphi_{n-1}) \qquad (n = 1, 2, \ldots ; \varphi_0 = 0)$$

where the gain vectors can be computed recursively without matrix inversions. We are not interested in the details of such computations - only that they can be done.

Step 2. At each n, then, we have a $(p+1)$ - vector whose components (for reasons which will become clear) we label as

$$\varphi'_n = \left[a_0^{(n)}, a_1^{(n)}, b_1^{(n)}, \ldots, a_q^{(n)}, b_q^{(n)}, a_{q+1}^{(n)} \right].$$

From it we compute a new vector

$$\psi'_n = \left[c_0^{(n)}, c_1^{(n)}, d_1^{(n)}, \ldots, c_q^{(n)}, d_q^{(n)}, c_{q+1}^{(n)} \right].$$

by means of the relations

$$c_j^{(n)} = \frac{a_j^{(n)}}{\left(a_j^{(n)}\right)^2 + \left(b_j^{(n)}\right)^2}$$

$$d_j^{(n)} = - \frac{b_j^{(n)}}{\left(a_j^{(n)}\right)^2 + \left(b_j^{(n)}\right)^2} .$$

These hold for $j = 0, 1, \ldots, q, q+1$ with $b_0^{(n)} = b_{q+1}^{(n)} \equiv 0$.

<u>Step 3.</u> Next form the $p+1$ by $p+1$ $(p = 1, 3, \ldots)$ matrix

$$M = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \cos \omega_1 & \cos 2\omega_1 & & \cos p\omega_1 \\ 0 & \sin \omega_1 & \sin 2\omega_1 & & \sin p\omega_1 \\ & \vdots & & & \vdots \\ 1 & \cos \omega_q & \cos 2\omega_q & & \cos p\omega_q \\ 0 & \sin \omega_q & \sin 2\omega_q & & \sin p\omega_q \\ 1 & -1 & +1 & \cdots & +1 \end{bmatrix}$$

This matrix is nonsingular, and we can compute

$$\gamma_n = \begin{bmatrix} \gamma_0^{(n)} \\ \gamma_1^{(n)} \\ \cdot \\ \cdot \\ \cdot \\ \gamma_p^{(n)} \end{bmatrix} = M^{-1}\psi_n$$

for $n > p$.

Step 4. The estimate of K based on the first n observations is $1/\gamma_0^{(n)}$, and that of θ_k is $-\gamma_k^{(n)}/\gamma_0^{(n)}$ $(k = 1, 2, \ldots, p)$. If the original system is rewritten

$$\gamma_0 x_t + \gamma_1 x_{t-1} + \cdots + \gamma_p x_{t-p} = u_t ,$$

and it is the γ's which are to be estimated, then the procedure terminates with Step 3.

Odd number of parameters. When

$$p + 1 = 2q + 1 \quad (q \geq 0)$$

we modify the preceding by

(i) deleting the last entry of h_t in Step 1,

(ii) deleting the last entries of φ_n and ψ_n in Step 2,

(iii) deleting the last row of M in Step 3,

and leave everything else unchanged.

Choice of input frequencies. The inversion of M is obviated when we use the particular angular frequencies

$$\omega_j = \frac{2\pi j}{p+1} \qquad (j = 1, 2, \ldots, q).$$

For $p+1 = 2q+2$, put

$$S = \begin{bmatrix} \frac{1}{p+1} & & & & & 0 \\ & \frac{2}{p+1} & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \frac{2}{p+1} & \\ 0 & & & & & \frac{1}{p+1} \end{bmatrix}$$

The matrix $S^{\frac{1}{2}}$ M is orthogonal, so it is any easy matter to solve the equations in Step 3. We get for $k=0, 1, \ldots, p$

$$\gamma_k^{(n)} = \frac{c_0^{(n)}}{p+1} + \frac{2}{p+1} \sum_{j=1}^{q} \left(c_j^{(n)} \cos k\omega_j + d_j^{(n)} \sin k\omega_j \right) + \frac{(-1)^k}{p+1} c_{q+1}^{(n)} .$$

For $p+1 = 2q+1$ the last term is dropped.

Known input gain. When K is known, we replace the input u_t by u_t/K and everywhere read p for $p+1$. In Step 3 the first column of M is deleted, and the estimate of θ_k is the k^{th} component of $-M^{-1} \psi_n$ $(k = 1, 2, \ldots, p)$, i.e. $\gamma_0^{(n)} \equiv 1$.

3. THE ASYMPTOTIC BEHAVIOR OF THE ESTIMATES

The following two lemmas preceding the theorems of interest contain known results, and are included only to make the report self-contained. Proofs of the lemmas and theorems are given in Sec. 4 and Sec. 5 respectively.

Lemma 1. Let $\theta_1, \ldots, \theta_p$ be any real numbers such that the roots

$$\lambda_1, \ldots, \lambda_p$$

of

$$g(\lambda) = \lambda^p - \theta_1 \lambda^{p-1} - \cdots - \theta_p = 0$$

are in modulus less than unity and distinct. Let $\{v_t\}$ be any bounded number sequence. Then the steady state solution of

$$x_t = \theta_1 x_{t-1} + \cdots + \theta_p x_{t-p} + v_t$$

is

$$x_t = \sum_{n=0}^{\infty} \beta_n v_{t-n}$$

where

$$\beta_n = \sum_{k=1}^{p} \alpha_k \lambda_k^n \qquad \alpha_k = \frac{\lambda_k^{p-1}}{\prod_{j \neq k} (\lambda_k - \lambda_j)} \quad .$$

Lemma 2. For the particular input

$$v_t = K \cos \omega t \qquad (K \neq 0,\ \omega \text{ arbitrary})$$

the conclusion of Lemma 1 reads

$$x_t = a \cos \omega t + b \sin \omega t$$

where

$$\frac{a}{a^2 + b^2} = \sum_{k=0}^{p} \gamma_k \cos k\omega$$

$$\frac{b}{a^2 + b^2} = -\sum_{k=0}^{p} \gamma_k \sin k\omega$$

and

$$\gamma_0 = \frac{1}{K} \qquad \gamma_k = -\frac{\theta_k}{K} \quad (k = 1, 2, \ldots, p).$$

Theorem 1. In the notation of Sec. 2 let

$$\theta' = [K, \theta_1, \ldots, \theta_p]$$

$$\theta'_n = \left[\frac{1}{\gamma_0^{(n)}}, -\frac{\gamma_1^{(n)}}{\gamma_0^{(n)}}, \ldots, -\frac{\gamma_p^{(n)}}{\gamma_0^{(n)}} \right]$$

with $p+1$ even or odd as the case may be. Then

$$P\{\lim_n \theta_n = \theta\} = 1$$

for any distinct interior points $\omega_1, \ldots, \omega_q$ of $(0, \pi)$ and any 0 mean stochastic process $\{\xi_t\}$ for which

$$\mathcal{E} \xi_t \xi_{t+|h|} = \sigma_h = O(1/h^{\epsilon})$$

as $h \to \infty$ for some $\epsilon > 0$.

Theorem 2. Let $\{\xi_t\}$ be a sequence of independent zero mean random variables with common variance σ^2, and suppose $\sup_t \mathcal{E}|\xi_t|^{2+\delta} < \infty$ for some $\delta > 0$. Choose

$$0 = \omega_0 < \omega_1 < \cdots < \omega_q < \omega_{q+1} = \pi,$$

which are otherwise unrestricted.

(a) The $(p+1)$-vector $\sqrt{n}(\gamma_n - \gamma)$ has a large sample normal distribution about the origin with covariance matrix, for $p+1 = 2q+2$ even,

$$\Sigma = 2\sigma^2 M^{-1} P^2 M^{-1\prime}$$

$$P = \mathrm{diag}[\rho_0, \rho_1, \rho_1, \ldots, \rho_q, \rho_q, \rho_{q+1}].$$

M is given in Step 3, and the entries of P are

$$\rho_j = \left| \sum_{k=0}^{p} \gamma_k e^{ik\omega_j} \right|^2 \qquad (j = 0, 1, \ldots, q, q+1)$$

(b) The covariance matrix of the asymptotic normal distribution of $\sqrt{n}(\theta_n - \theta)$ is

$$\Delta' \Sigma \Delta$$

where

$$\Delta = -K \left[\begin{array}{c|cccc} K & \theta_1 & \theta_2 & \cdots & \theta_p \\ \hline 0 & & & I & \end{array} \right]$$

and I is the $p \times p$ identity. The entries of P are also expressible as

$$\rho_j = \frac{1}{K^2} \left| 1 - \sum_{k=1}^{p} \theta_k e^{ik\omega_j} \right|^2 = \frac{1}{K^2} \left| g(e^{i\omega_j}) \right|^2 \qquad (g = g_p),$$

i.e. in terms of the coefficients $\theta_1, \ldots, \theta_p$ or the characteristic roots $\lambda_1, \ldots, \lambda_p$.

(c) For $p + 1 = 2q + 1$ odd, we delete the last row of M and the last entry of P under (a).

Corollary. For

$$\omega_j = \frac{2\pi j}{p+1} \qquad (j = 1, 2, \ldots, q)$$

the covariance matrix Σ in Theorem 2(a) becomes a Toeplitz matrix with entries

$$\sigma_{ab} = \frac{2\sigma^2}{(p+1)^2} \left[\rho_0^2 + 4 \sum_{j=1}^{q} \rho_j^2 \cos(b-a)\omega_j + \rho_{q+1}^2 (-1)^{b-a} \right] \qquad (a, b = 0, 1, \ldots, p).$$

The generalized variance of the asymptotic distribution of $\sqrt{n}(\gamma_n - \gamma)$ is

$$\det \Sigma = \tfrac{1}{4}\left(\frac{4\sigma^2}{p+1}\right)^{p+1} \rho_0^2 \rho_1^4 \cdots \rho_q^4 \rho_{q+1}^2 ,$$

and that of $\sqrt{n}(\theta_n - \theta)$ is

$$\det(\Delta' \Sigma \Delta) = \tfrac{1}{4}\left(\frac{4\sigma^2}{p+1}\right)^{p+1} \frac{1}{K^{2p}} (K^2\rho_0)^2 (K^2\rho_1)^4 \cdots (K^2\rho_q)^4 (K^2\rho_{q+1})^2 .$$

Each of the factors $K^2\rho_j$ depends only on the θ's, and is no larger than 4^p.

Confidence intervals free of unknowns. We can consistently estimate ρ_j by

$$\rho_j^{(n)} = \frac{1}{\left(a_j^{(n)}\right)^2 + \left(b_j^{(n)}\right)^2} \qquad (j=0, 1, \ldots, q, q+1;\ b_0^{(n)} = b_{q+1}^{(n)} \equiv 0)$$

resulting from Step 1. Letting P_n be the diagonal matrix $[\rho_0^{(n)}, \rho_1^{(n)}, \rho_1^{(n)}, \ldots, \rho_q^{(n)}, \rho_q^{(n)}, \rho_{q+1}^{(n)}]$, the vector of $p+1 = 2q+2$ random variables

$$\frac{P_n^{-1} M \sqrt{n}(\gamma_n - \gamma)}{\sqrt{\dfrac{2}{n-1} \sum_{t=1}^{n} (y_t - h_t' \varphi_n)^2}}$$

has a standardized multivariate normal distribution in large samples.

Again, it is only necessary to delete all terms involving ρ_{q+1} when $p+1 = 2q+1$, as well as the last row of M.

K known. The matrix Σ is now $p \times p$ with $2q+2$ is redefined as p when it is even. Under Theorem 2(a) the first column of M is (has been) deleted, and in the definition of ρ_j we set $\gamma_0 = 1$ and $\gamma_k = -\theta_k$ $(k = 1, 2, \ldots, p)$. Under Theorem 2(b), we set $-\Delta$ equal to the $p \times p$ identity matrix, and K equal to unity in the formula for ρ_j. Thus, in the Corollary, the selected frequencies are $\omega_j = 2\pi j/p$. In the formula for σ_{ab}, the multiplier becomes $2\sigma^2/p^2$ and a, b runs over 1, 2, ..., p.

Remark. When the independent errors are normal, θ_n is the Maximum Likelihood estimate of θ for every n. This is true since the Least Squares estimate φ_n of φ becomes the Maximum Likelihood estimate, and the Maximum Likelihood estimate of any 1-1 vector-valued function is the function of the Maximum Likelihood estimate.

4. PROOF OF THE LEMMAS

Proof of Lemma 1. In the standard fashion we write our p^{th} order equation as a first order vector equation:

$$\vec{x}_t = \Theta \vec{x}_{t-1} + v_t e_1$$

where

$$\vec{x}_{t-1} = \begin{bmatrix} x_{t-1} \\ \cdot \\ \cdot \\ \cdot \\ x_{t-p} \end{bmatrix} \qquad e_1 = \begin{bmatrix} 1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix} \qquad \Theta = \left[\begin{array}{ccc|c} \theta_1 & \cdots & \theta_{p-1} & \theta_p \\ \hline & & & 0 \\ & I & & \cdot \\ & & & \cdot \\ & & & \cdot \\ & & & 0 \end{array}\right]$$

Iterating back to $t=0$, the solution for positive times is

$$x_t = e_1' \vec{x}_t = e_1' \Theta^t \vec{x}_0 + \sum_{n=0}^{t-1} \beta_n v_{t-n}$$

where

$$\beta_n = e_1' \Theta^n e_1 .$$

We first use the assumption of distinct roots to evaluate β_n.

Throughout λ denotes an indeterminate complex scalar. Define the unimodular (i.e. determinant = 1) matrix

$$E(\lambda) = \begin{bmatrix} 1 & \lambda & \lambda^2 & \cdots & \lambda^{p-1} \\ & 1 & \lambda & \cdots & \lambda^{p-2} \\ & & & & \\ & 0 & & & \\ & & & 1 & \lambda \\ & & & & 1 \end{bmatrix}$$

Then

$$E(\lambda) = E_1(\lambda)\, E_2(\lambda) \cdots E_{p-1}(\lambda)$$

where $E_j(\lambda)$ is the identity matrix with λ added into the $(j, j+1)$ position. Right-multiplication of any $p \times p$ matrix $A = [a_1, \ldots, a_p]$ by $E_j(\lambda)$ adds λ times the j^{th} column to the $j+1^{st}$. Consequently,

$$AE(\lambda) = [\, a_1, a_2 + a_1\lambda,\ a_3 + a_2\lambda + a_1\lambda^2, \ldots, a_p + a_{p-1}\lambda + \cdots + \lambda^{p-1} \,]$$

where the successive columns are λ-polynomials with vector coefficients. The inverse of the operator,

$$E(\lambda)^{-1} = E_1(\lambda) + E_2(\lambda) + \cdots + E_{p-1}(\lambda),$$

exists independently of λ. If we replace A by the characteristic matrix of Θ we find

$$(\Theta - \lambda I)\, E(\lambda) = - \left[\begin{array}{cccc|c} g_1(\lambda) & g_2(\lambda) & \cdots & g_{p-1}(\lambda) & g_p(\lambda) \\ \hline & & & & 0 \\ & I & & & \vdots \\ & & & & 0 \end{array} \right] = F(\lambda)$$

wherein

$$g_j(\lambda) = \lambda^j - \theta_1 \lambda^{j-1} - \cdots - \theta_{j-1}\lambda - \theta_j \qquad (j = 1, 2, \ldots, p)$$

and g_p is g in the statement of the lemma. Since $|E(\lambda)| \equiv 1$, $\Theta - \lambda I$ and $F(\lambda)$ are rank equivalent, and the characteristic polynomial of Θ is simply

$$|\Theta - \lambda I| = |F(\lambda)| = (-1)^{p+1} g_p(\lambda) .$$

Regarded as a function of λ, the characteristic matrix has full rank unless λ coincides with one of the roots

$$\lambda_1, \ldots, \lambda_p$$

of $g_p(\lambda) = 0$. In this case, $\Theta - \lambda_j I$ has rank $p-1$. Thus, if λ_j is repeated there is only one nonzero vector solution to the homogeneous equation $(\Theta - \lambda_j I)x = 0$. Consequently, Θ is equivalent (more properly, collinear) to the diagonal matrix Λ of its eigenvalues if and only if the eigenvalues are distinct. This is our assumption.

Since Θ is not normal, i.e. does not commute with its transpose, two sets of eigenvectors are involved. To each λ belonging to the spectrum $\lambda_1, \ldots, \lambda_p$ there are distinct nonzero vectors $\ell = \ell_\lambda$ and $r = r_\lambda$, unique up to a multiplicative constant, satisfying

$$\Theta' \ell = \lambda \ell \qquad \Theta r = \lambda r .$$

We use the "left-right" nomenclature relative to Θ. Let

$$L = [\ell_1, \ldots, \ell_p] \quad R = [r_1, \ldots, r_p]$$

be the matrices of these columns generated as λ assumes the p spectral values. Then

$$\begin{aligned} \Theta R &= \Theta[r_1, \ldots, r_p] = [\Theta r_1, \ldots, \Theta r_p] = [\lambda_1 r_1, \ldots, \lambda_p r_p] \\ &= R\Lambda \end{aligned}$$

and

$$\Theta' L = L \Lambda \qquad \text{or} \qquad L'\Theta = \Lambda L'$$

Thus, (without restriction on the eigenvalues)

$$\Lambda(L'R) = L'\Theta R = (L'R)\,\Lambda$$

where $L'R$ has the inner product $\ell_i' r_j$ for ij^{th} entry. In terms of elements, the statement that $L'R$ commutes with Λ is

$$(\lambda_i - \lambda_j)\,\ell_i' r_j = 0 \,.$$

Thus, for distinct eigenvalues, ℓ_i is perpendicular to r_j for all $i \neq j$ (and the converse is also true). The two vector sets are called bi-orthogonal, and

$$L'R = D = \begin{bmatrix} d_{11} & & & & 0 \\ & d_{22} & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ 0 & & & & d_{pp} \end{bmatrix} \qquad d_{jj} = \ell_j' r_j$$

We then have, since $|D| \neq 0$,

$$\begin{aligned} \Theta &= R\Lambda R^{-1} \\ &= (RD^{-\frac{1}{2}})\,\Lambda(RD^{-\frac{1}{2}})' \,, \end{aligned}$$

and the proper normalization for ℓ_j and r_j is $1/\sqrt{d_{jj}}$.

We now compute L, R and D as functions of the eigenvalues. We have

$$(\Theta' - \lambda_j I)\ell_j = E(\lambda_j)'^{-1} F(\lambda_j)' \ell_j .$$

By definition ℓ_j makes the left side the 0 vector, and hence $F(\lambda_j)'\ell_j = 0$. These equations are

$$\left[\begin{array}{c|cccc} -g_1(\lambda_j) & 1 & & & \\ -g_2(\lambda_j) & & 1 & & \\ \cdot & & & \cdot & \\ \cdot & & & & \cdot \\ \cdot & & & & \\ -g_{p-1}(\lambda_j) & & & & 1 \\ \hline 0 & 0 & \cdots & & 0 \end{array}\right] \begin{bmatrix} 1 \\ g_1(\lambda_j) \\ g_2(\lambda_j) \\ \cdot \\ \cdot \\ \cdot \\ g_{p-1}(\lambda_j) \end{bmatrix} = 0$$

because $g_p(\lambda) = 0$ for $\lambda = \lambda_j$. The vector equation $\Theta r_j = \lambda_j r_j$ is

$$\left[\begin{array}{cccc|c} \theta_1 & \theta_2 & \cdots & \theta_{p-1} & \theta_p \\ \hline 1 & & & & 0 \\ & 1 & & & \cdot \\ & & \cdot & & \cdot \\ & & & \cdot & \cdot \\ & & & \cdot & \\ & & & 1 & 0 \end{array}\right] \begin{bmatrix} \lambda_j^{p-1} \\ \lambda_j^{p-2} \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix} = \begin{bmatrix} \lambda_j^{p} \\ \lambda_j^{p-1} \\ \cdot \\ \cdot \\ \cdot \\ \lambda_j \end{bmatrix}$$

The first of these says λ_j is a root, and the balance are redundancies. Thus,

$$\ell_j = \begin{bmatrix} 1 \\ g_1(\lambda_j) \\ \cdot \\ \cdot \\ \cdot \\ g_{p-1}(\lambda_j) \end{bmatrix} \qquad r_j = \begin{bmatrix} \lambda_j^{p-1} \\ \lambda_j^{p-2} \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix}$$

R is a Vandermonde matrix with

$$|R| = (-1)^{p+1} \prod_{i<j} (\lambda_i - \lambda_j).$$

To compute the normalization

$$d_{jj} = \ell_j' r_j = \lambda_j^{p-1} + \sum_{k=1}^{p-1} g_k(\lambda_j)\lambda_j^{p-k-1}$$

we derive a λ-identity. If we multiply

$$g_k(\lambda) = \lambda^k - \sum_{i=1}^{k} \theta_i \lambda^{k-i}$$

by λ^{p-k-1} and sum over k we have

$$\sum_{k=1}^{p-1} g_k(\lambda)\lambda^{p-k-1} = (p-1)\lambda^{p-1} - \sum_{k=1}^{p-1} \sum_{i=1}^{k} \theta_i \lambda^{p-i-1}$$

$$= p\lambda^{p-1} - \lambda^{p-1} - \sum_{k=1}^{p-1} \theta_k (p-k)\lambda^{p-k-1} .$$

Moving λ^{p-1} to the left side we obtain

$$\lambda^{p-1} + \sum_{k=1}^{p-1} g_k(\lambda)\lambda^{p-k-1} = \frac{d}{d\lambda}\left[\lambda^p - \sum_{k=1}^{p-1} \theta_k \lambda^{p-k} + c\right]$$

We can set the arbitrary constant equal to $-\theta_p$. There results

$$\lambda^{p-1} + \sum_{k=1}^{p-1} g_k(\lambda)\,\lambda^{p-k-1} = \frac{d}{d\lambda} g_p(\lambda)$$

$$= \frac{d}{d\lambda} \prod_{i=1}^{p} (\lambda-\lambda_i) = \sum_{k=1}^{p} \prod_{i \neq k} (\lambda-\lambda_i)$$

In particular, for $\lambda = \lambda_j$,

$$d_{jj} = \prod_{i \neq j} (\lambda_j - \lambda_i) \ .$$

To find the dependence of β_n on the eigenvalues we use $L'\Theta = \Lambda L'$. We have

$$L'\Theta^n = \Lambda^n L' = \begin{bmatrix} \lambda_1^n \ell_1' \\ \lambda_2^n \ell_2' \\ \cdot \\ \cdot \\ \cdot \\ \lambda_p^n \ell_p' \end{bmatrix}$$

Since the leading element of every ℓ_j is 1,

$$L'\Theta^n e_1 = \begin{bmatrix} \lambda_1^n \\ \cdot \\ \cdot \\ \cdot \\ \lambda_p^n \end{bmatrix}$$

But $L'R = D$ implies $L'^{-1} = RD^{-1}$, so

$$\Theta^n e_1 = RD^{-1} \begin{bmatrix} \lambda_1^n \\ \cdot \\ \cdot \\ \cdot \\ \lambda_p^n \end{bmatrix}$$

Since $e_1'R$ is the first row of R,

$$e_1'\Theta^n e_1 = [\lambda_1^{p-1} \cdots \lambda_p^{p-1}]\, D^{-1} \begin{bmatrix} \lambda_1^n \\ \cdot \\ \cdot \\ \cdot \\ \lambda_p^n \end{bmatrix} \qquad ,$$

and therefore

$$\beta_n = \sum_{j=1}^{p} \frac{\lambda_j^{p-1}}{d_{jj}} \lambda_j^n = \sum_{j=1}^{p} \alpha_j \lambda_j^n .$$

The vector α satisfies

$$L\alpha = e_1 .$$

In particular, $\alpha_1 + \cdots + \alpha_p = 1$, as should be since β_0 must be unity.

We now use the assumption that

$$\max_{1 \le j \le p} |\lambda_j| < 1$$

where $|\ |$ denotes modulus. For any $p \times q$ matrix A with elements a_{ij}, define the norm

$$\| A \| = \sqrt{\sum_{i=1}^{p} \sum_{j=1}^{q} a_{ij}^2}$$

which generalizes length. Then

$$\begin{aligned} | e_1' \Theta^n \vec{x}_0 | &\leq \| e_1' \| \, \| \Theta^n \vec{x}_0 \| \\ &\leq \| \Theta^n \| \, \| \vec{x}_0 \| \ . \end{aligned}$$

But

$$\begin{aligned} \| \Theta^n \| \leq \| R \| \, \| \Lambda^n \| \, \| R^{-1} \| &= \text{const.} \ \sqrt{\lambda_1^{2n} + \cdots + \lambda_p^{2n}} \\ &\to 0 \qquad \text{as } n \to \infty \ . \end{aligned}$$

Translating the origin back to $t = -\infty$, this establishes the formula asserted in Lemma 1.

Proof of Lemma 2. For $v_t = K \cos \omega t$ the steady state solution is, according to Lemma 1,

$$x_t = (Ka')\cos \omega t + (Kb')\sin \omega t$$

where

$$a' = \sum_{n \geq 0} \beta_n \cos n\omega = a/K$$

$$b' = \sum_{n \geq 0} \beta_n \sin n\omega = a/K \ .$$

Setting

$$A = 1 - \sum_{k=1}^{p} \theta_k \cos k\omega$$

$$B = \sum_{k=1}^{p} \theta_k \sin k\omega \ ,$$

the assertion to be proved is clearly equivalent to

$$\frac{a'}{a'^2 + b'^2} = A$$

$$\frac{b'}{a'^2 + b'^2} = B \ .$$

This in turn holds if

$$(A + iB)(a' - ib') = 1,$$

since then

$$A + iB = \frac{1}{a' - ib'} = \frac{a' + ib'}{a'^2 + b'^2} = \frac{a'}{a'^2 + b'^2} + i\frac{b'}{a'^2 + b'^2} \ .$$

Putting

$$z = e^{i\omega}$$

we have

$$A + iB = 1 - \sum_{k=1}^{p} \theta_k z^{-k} = z^{-p} g_p(z)$$

and

$$a' - ib' = \sum_{n\geq 0} \beta_n z^{-n}$$

$$= \sum_{n\geq 0} \left(\sum_{k=1}^{p} \alpha_k \lambda_k^n \right) z^{-n} = \sum_{k=1}^{p} \alpha_k \sum_{n\geq 0} (\lambda_k / z)^n \ .$$

But $|\lambda_k / z| \leq |\lambda_k| < 1$ by hypothesis for all k, so

$$a' - ib' = \sum_{k=1}^{p} \alpha_k \frac{z}{z - \lambda_k} \quad \text{with } \alpha_k = \frac{\lambda_k^{p-1}}{\prod_{j \neq k} (\lambda_k - \lambda_j)} \ .$$

It thus suffices to prove

$$g_p(z) \sum_{k=1}^{p} \frac{\alpha_k}{z-\lambda_k} \equiv z^{p-1} \quad .$$

Since $g_p(z) = \prod_{j=1}^{p} (z-\lambda_j)$, the left side is

$$L(z) = \sum_{k=1}^{p} \alpha_k \prod_{j \neq k} (z-\lambda_j) = \sum_{k=1}^{p} \lambda_k^{p-1} \prod_{j \neq k} \frac{z-\lambda_j}{\lambda_k - \lambda_j}$$

$$= \sum_{k=1}^{p} \lambda_k^{p-1} L_k(z) \; .$$

Each of the summands is a (Lagrange) polynomial of degree $p-1$ in z, so we must have

$$L(z) = c_1 z^{p-1} + c_2 z^{p-2} + \cdots + c_p \; .$$

The coefficients are determined by the values of L at any p distinct points. We of course use the characteristic roots, since

$$L_k(\lambda_i) = \begin{cases} 1 & \text{if} \quad k=i \\ 0 & \text{if} \quad k \neq i \; . \end{cases}$$

Thus

$$L(\lambda_i) = \lambda_i^{p-1} \qquad (i = 1, 2, \ldots, p) \; .$$

This shows $c_1 = 1$ and $c_2 = \cdots = c_p = 0$, i.e. $L(z) = z^{p-1}$.

5. PROOF OF THE THEOREMS

Proof of Theorem 1. It clearly suffices to consider either an odd or an even number of parameters, and we choose the latter. Thus, throughout the proof,

$$p+1 = 2q+2 \qquad (q \geq 0).$$

Furthermore, we set

$$\omega_0 = 0 \qquad \omega_{q+1} = \pi .$$

According to Lemma 2, then, the output which results from the input

$$u_t = \frac{1}{\sqrt{2}} \cos \omega_0 t + \sum_{j=1}^{q} \cos \omega_j t + \frac{1}{\sqrt{2}} \cos \omega_{q+1} t$$

is

$$x_t = \frac{1}{\sqrt{2}} (a_0 \cos \omega_0 t + b_0 \sin \omega_0 t) + \sum_{j=1}^{q} (a_j \cos \omega_j t + b_j \sin \omega_j t)$$

$$+ \frac{1}{\sqrt{2}} (a_{q+1} \cos \omega_{q+1} t + b_{q+1} \sin \omega_{q+1} t)$$

$$= h_t' \varphi ,$$

where h_t is as in Step 1 and

$$\varphi' = [a_0, a_1, b_1, \ldots, a_q, b_q, a_{q+1}] .$$

Furthermore,

$$c_j \equiv \frac{a_j}{a_j^2 + b_j^2} = \sum_{k=0}^{p} \gamma_k \cos k\omega_j$$

$$j = 0, 1, \ldots, q, q+1$$

$$d_j \equiv -\frac{b_j}{a_j^2 + b_j^2} = \sum_{k=0}^{p} \gamma_k \sin k\omega_j$$

which imply $b_0 = b_{q+1} = 0$. Putting

$$\psi' = [c_0, c_1, d_1, \ldots, c_q, d_q, c_{q+1}]$$

$$\gamma' = [\gamma_0, \gamma_1, \ldots, \gamma_p],$$

where $\gamma_0 = 1/K$ and $\gamma_k = -\theta_k/K$ $(k=1, 2, \ldots, p)$, these equations are

$$\psi = M\gamma$$

with M as given in Step 3.

Let $\vec{\xi}_n$ be the column vector of the first n noise realizations. We have a data model which is linear in the components of φ,

$$\vec{y}_n = H'_n \varphi + \vec{\xi}_n .$$

The estimate resulting from Step 1 can be written $\varphi_n = \varphi + B_n H_n \vec{\xi}_n$, or

$$A_n(\varphi_n - \varphi) = H_n \vec{\xi}_n \qquad (A_n = B_n^{-1}).$$

For the p+1 by p+1 matrix $A_n = \sum_{t=1}^{n} h_t h'_t$ we have

$$\frac{2}{n}A_n = \frac{2}{n}\sum_{t=1}^{n}\begin{bmatrix} \frac{1}{2} & \frac{1}{\sqrt{2}}\cos\omega_1 t & \frac{1}{\sqrt{2}}\sin\omega_1 t & \frac{1}{\sqrt{2}}\cos\omega_2 t & \cdots & \frac{1}{\sqrt{2}}\sin\omega_q t & \frac{(-1)^t}{\sqrt{2}} \\ & \cos^2\omega_1 t & \cos\omega_1 t\sin\omega_1 t & \cos\omega_1 t\cos\omega_2 t & \cdots & \cos\omega_1 t\sin\omega_q t & \cos\omega_1 t\frac{(-1)^t}{} \\ & & \sin^2\omega_1 t & \sin\omega_1 t\cos\omega_2 t & \cdots & \sin\omega_1 t\sin\omega_q t & \sin\omega_1 \frac{(-1)^t}{\sqrt{2}} \\ & & & \cos^2\omega_2 t & & & \\ & \text{(symmetric)} & & & & & \\ & & & & & \sin^2\omega_q t & \sin\omega_q t\frac{(-1)^t}{\sqrt{2}} \\ & & & & & & \frac{1}{2} \end{bmatrix}$$

From the identities

$$\sum_{t=1}^{n}\cos 2\lambda t = \frac{\sin n\lambda}{\sin\lambda}\cos(n+1)\lambda$$

$$(\lambda \neq \text{a multiple of } \pi)$$

$$\sum_{t=1}^{n}\sin 2\lambda t = \frac{\sin n\lambda}{\sin\lambda}\sin(n+1)\lambda$$

it follows for $n \rightarrow \infty$ that

$$\left.\begin{array}{l} \frac{2}{n}\sum_{t=1}^{n} \cos\lambda_1 t \cos\lambda_2 t \\ \\ \frac{2}{n}\sum_{t=1}^{n} \sin\lambda_1 t \sin\lambda_2 t \end{array}\right\} = O(\frac{1}{n}) \quad \text{for all } 0 \leq \lambda_1 \lambda_2 \leq \pi \text{ with } \lambda_1 \neq \lambda_2$$

$$\left.\begin{array}{l} \frac{2}{n}\sum_{t=1}^{n} \cos^2 \lambda^t \\ \\ \frac{2}{n}\sum_{t=1}^{n} \sin^2 \lambda^t \end{array}\right\} = 1 + O(\frac{1}{n}) \qquad \text{for all } 0 < \lambda < \pi$$

$$\frac{2}{n}\sum_{t=1}^{n} \cos\lambda_1 t \sin\lambda_2 t = O(\frac{1}{n}) \qquad \text{for all } 0 \leq \lambda_1, \lambda_2 \leq \pi .$$

Since $\omega_1, \ldots, \omega_q$ were chosen as interior points of $(0, \pi)$, we therefore have

$$\frac{2}{n} A_n = I + \frac{1}{n} E_n$$

for some matrix E_n whose elements remain uniformly bounded as $n \to \infty$. (To get the identity matrix in the limit was our reason for using the amplitude $1/\sqrt{2}$ for the first and last input.) Thus,

$$(I + \frac{1}{n} E_n)(\varphi_n - \varphi) = \frac{2}{n} H_n \vec{\xi}_n$$

and the estimate φ_n will converge to φ in the same probabilistic sense that the vector

$\frac{2}{n} H_n \vec{\xi}_n$ converges to the zero vector.

The $p+1$ components of $\frac{2}{n} H_n \vec{\xi}_n$ are arithmetic means

$$\frac{\sqrt{2}}{n} \sum_{t=1}^{n} \xi_t$$

$$\frac{2}{n} \sum_{t=1}^{n} \cos \omega_1 t \, \xi_t$$

$$\frac{2}{n} \sum_{t=1}^{n} \sin \omega_1 t \, \xi_t$$

$$\vdots$$

$$\frac{2}{n} \sum_{t=1}^{n} \cos \omega_q t \, \xi_t$$

$$\frac{2}{n} \sum_{t=1}^{n} \sin \omega_q t \, \xi_t$$

$$\frac{\sqrt{2}}{n} \sum_{t=1}^{n} (-1)^t \, \xi_t$$

each of which has zero expectation. For any average of the form

$$S_n = \frac{1}{n} \sum_{t=1}^{n} \cos \omega t \, \xi_t$$

we have

$$\mathcal{E} S_n \cos\omega n \xi_n = \frac{1}{n} \sum_{t=1}^{n} \cos\omega t \cos\omega n\, \sigma_{n-t} .$$

By hypothesis, therefore,

$$|\mathcal{E} S_n \cos\omega n \xi_n| \le \frac{1}{n} \sum_{t=0}^{n-1} |\sigma_t| = O(\frac{1}{n^{\epsilon}}) .$$

It follows from a LLN for dependent random variables that $S_n \to 0$ with probability one as $n \to \infty$ (and, incidentally, also in mean square).

Since strong convergence is preserved through continuous transformations, we successively conclude from $\varphi_n \to \varphi$ that $\psi_n \to \psi$, $\gamma_n \to \gamma$, and the conclusion of the lemma.

Proof of Theorem 2. We continue to take $p+1$ even, and note the modifications of the results needed for $p+1$ odd when we are done. The asymptotic distribution at $\sqrt{n}(\varphi_n - \varphi)$ is the same as that of $\frac{2}{\sqrt{n}} H_n \vec{\xi}_n$, which we express in symbols as

$$\sqrt{n}(\varphi_n - \varphi) \sim \frac{2}{\sqrt{n}} H_n \vec{\xi}_n$$

Under the assumptions of the theorem the latter, by Liapounov's CLT, tends to be distributed in large samples as a $p+1$ dimensional normal random variable with zero mean vector and covariance matrix equal to

$$\lim_n \mathcal{E}\left(\frac{2}{\sqrt{n}} H_n \vec{\xi}_n\right)\left(\frac{2}{\sqrt{n}} H_n \vec{\xi}_n\right)'$$

According to the asymptotic orthogonality relations used in the proof of Theorem 1,

$$\mathcal{E}\frac{4}{n}\sum_{t=1}^{n}\sum_{s=1}^{n}\cos\omega_i t\,\xi_t\cos\omega_j s\,\xi_s=\frac{4\sigma^2}{n}\sum_{t=1}^{n}\cos\omega_i t\cos\omega_j t\to 2\sigma^2\delta_{ij}$$

as $n\to\infty$ for all $i, j = 0, 1, \ldots, q, q+1$, provided i and j are not both 0 nor both $q+1$. The same limit holds when both cosines are replaced by sines. For $i=j=0$ (i.e. $\omega_0=0$) and $i=j=q+1$ (i.e. $\omega_{q+1}=\pi$) we have

$$\mathcal{E}\frac{2}{n}\sum_{t=1}^{n}\sum_{s=1}^{n}\xi_t\xi_s=\mathcal{E}\frac{2}{n}\sum_{t=1}^{n}\sum_{s=1}^{n}(-1)^t\xi_t(-1)^s\xi_s=2\sigma^2$$

for every n. Finally,

$$\mathcal{E}\frac{4}{n}\sum_{t=1}^{n}\sum_{s=1}^{n}\cos\omega_i t\,\xi_t\sin\omega_j s\,\xi_s=\frac{4\sigma^2}{n}\sum_{t=1}^{n}\cos\omega_i t\sin\omega_j t\to 0$$

without restriction on $i, j = 0, 1, \ldots, q, q+1$. Thus,

$$\sqrt{n}(\varphi_n-\varphi)\quad\sim\quad N(0, 2\sigma^2 I)$$

where I is the $p+1$ by $p+1$ identity. This checks with the known result for $N(0,\sigma^2)$ errors ; viz., that φ_n is $N(\varphi, \sigma^2 B_n)$ for every n. (Recall $B_n \cong 2/n$.)

Since the components of ψ_n are functions of those of φ_n we have, by the "delta method,"

$$\sqrt{n}(\psi_n-\psi)\quad\sim\quad N(0, 2\sigma^2 P^2)$$

$$\sqrt{n}(\gamma_n-\gamma)\quad\sim\quad N(0, 2\sigma^2 M^{-1}P^2M^{-1\prime})$$

where

$$P^2 = D'D \quad\text{and}\quad D = \frac{\partial\psi}{\partial\varphi}$$

is the p+1 by p+1 matrix of partial derivatives computed at the true parameter values (induced by K and the θ's). We have

$$\frac{\partial c_j}{\partial a_i} = \frac{\partial d_j}{\partial b_i} = \delta_{ij}\,\alpha_j \qquad \alpha_j = \frac{b_j^2 - a_j^2}{(a_j^2+b_j^2)^2}$$

$$\frac{\partial d_j}{\partial a_i} = -\frac{\partial c_j}{\partial b_i} = \delta_{ij}\,\beta_j \qquad \beta_j = \frac{2a_jb_j}{(a_j^2+b_j^2)^2}$$

for all $i, j = 0, 1, \ldots, q, q+1$, provided we continue to take $b_0 = b_{q+1} = 0$ and ignore the variables not involved. We see that

$$D = \begin{bmatrix} \alpha_0 & & & & & & 0 \\ & A_1 & & & & & \\ & & A_2 & & & & \\ & & & \ddots & & & \\ & & & & A_q & & \\ 0 & & & & & & \alpha_{q+1} \end{bmatrix} \qquad A_i = \begin{bmatrix} \alpha_i & \beta_i \\ -\beta_i & \alpha_i \end{bmatrix}$$

Since the inner product between any two different columns of D vanishes, we have

$$P^2 = \mathrm{diag}[\rho_0^2, \rho_1^2, \rho_1^2, \ldots, \rho_q^2, \rho_q^2, \rho_{q+1}^2] \quad \rho_j^2 = \alpha_j^2 + \beta_j^2 \quad (\beta_0 = \beta_{q+1} = 0)\ .$$

The typical entry is

$$\rho^2 = \frac{(b^2-a^2)^2}{(a^2+b^2)^4} + \frac{4a^2b^2}{(a^2+b^2)^4} = \frac{1}{(a^2+b^2)^2} .$$

But from Lemma 2

$$\frac{1}{a^2+b^2} = \frac{a^2+b^2}{(a^2+b^2)^2} = \left(\sum_{k=0}^{p} \gamma_k \cos k\omega\right)^2 + \left(\sum_{k=0}^{p} \gamma_k \sin k\omega\right)^2 .$$

In other words,

$$\rho_j = \left| \sum_{k=0}^{p} \gamma_k e^{ik\omega_j} \right|^2$$

for $j=0, 1, \ldots, q, q+1$ and $0=\omega_0 < \omega_1 < \cdots < \omega_q < \omega_{q+1} = \pi$.

Finally, we apply the "delta method" once again to get the large sample distribution of the estimates of $K = \frac{1}{\gamma_0}$, $\theta_1 = -\frac{\gamma_1}{\gamma_0}, \ldots, \theta_p = -\frac{\gamma_p}{\gamma_0}$. We have

$$\sqrt{n}(\theta_n - \theta) \sim N(0, 2\sigma^2 \Delta' M^{-1} P^2 M^{-1'} \Delta)$$

where

$$\Delta = \frac{\partial(K, \theta_1, \ldots, \theta_p)}{\partial(\gamma_0, \gamma_1, \ldots, \gamma_p)}$$

$$= -K \left[\begin{array}{c|cccc} K & \theta_1 & \theta_2 & \cdots & \theta_p \\ \hline 0 & & & & \\ \vdots & & & I & \\ 0 & & & & \end{array}\right]$$

where I is the $p \times p$ identity.

We can give an alternate expression for the elements of P in terms of the characteristic polynomial

$$g(z) = z^p - \theta_1 z^{p-1} - \cdots - \theta_p \ .$$

Dividing through by K we have

$$\frac{1}{K} g(z) = z^p \sum_{k=0}^{p} \gamma_k z^{-k} \ .$$

Putting $z = e^{i\omega}$ and taking squared modulii gives

$$\frac{1}{K^2} |g(e^{i\omega})|^2 = |\sum_{k=0}^{p} \gamma_k e^{ik\omega}|^2 \ .$$

We therefore have

$$\rho_j = \frac{1}{K^2} |1 - \sum_{k=1}^{p} \theta_k e^{ik\omega}|^2 = \frac{1}{K^2} |g(e^{i\omega})|^2 \ .$$

One can bound these quantities using the fact that $|g(e^{i\omega})| = \prod_{k=1}^{p} |e^{i\omega} - \lambda_k|$ satisfies

$$\prod_{k=1}^{p} (1 - |\lambda_k|) \leq |g(e^{i\omega})| \leq \prod_{k=1}^{p} (1 + |\lambda_k|) \ .$$

For an odd number of parameters,

$$p + 1 = 2q + 1 \qquad (q \geq 0)$$

we simply delete the last row of M and the last entry of P.

Proof of the Corollary. From the identities used in the proof of Theorem 1 for the sum of the first n cosines and sines we have, with

$$\omega_j = \frac{2\pi j}{p+1} \qquad (j = 0, 1, \ldots, q, q+1 = \frac{p+1}{2}),$$

relations

$$\sum_{k=0}^{p} \cos \omega_i k \cos \omega_j k = \sum_{k=1}^{p+1} \cos \omega_i k \cos \omega_j k + 1 - \cos \omega_i (p+1) \cos \omega_j (p+1)$$

$$= \frac{p+1}{2} \delta_{ij} + 1 - 1$$

for all i, j, provided both are not 0 or q+1. From similar ones involving sines, and cosines and sines, we see that

$$MM' = S^{-1}$$

where S was defined in Sec. 2. It is now a simple matter to solve $M\gamma = \psi$. Setting

$$M_0 = S^{\frac{1}{2}} M,$$

we have $M_0 M_0' = I$, so that

$$\gamma = M_0' S^{\frac{1}{2}} \psi = (SM)' \psi .$$

Premultiplication by the diagonal matrix S multiplies the rows of M. The scalar γ_k is the inner product between row k of $(SM)'$, i.e. column k of SM, and ψ $(k = 0, 1, \ldots, p)$. Thus

$$\gamma_k = \frac{c_0}{p+1} + \frac{2}{p+1} \sum_{j=1}^{q} (c_j \cos k\omega_j + d_j \sin k\omega_j) + \frac{(-1)^k}{p+1} c_{q+1} .$$

For $p+1 = 2q+1$, the last term is deleted.

The covariance matrix Σ under Theorem 2(a) becomes

$$\frac{1}{2\sigma^2}\,\Sigma = M_0' S^{\frac{1}{2}} P^2 S^{\frac{1}{2}} M_0 = M' SP^2 SM = (PSM)'\ (PSM)$$

The entry labeled a, b (a, b = 0, 1, . . . , p) is the inner product between the corresponding columns of PSM. Thus

$$\left(\frac{1}{2\sigma^2}\,\Sigma\right)_{ab} = \left(\frac{\rho_0}{p+1}\right)^2 + \left(\frac{2\rho_1}{p+1}\right)^2 [\cos a\omega_1 \cos b\omega_1 + \sin a\omega_1 \sin b\omega_1] + \cdots$$

$$\cdots + \left(\frac{2\rho_q}{p+1}\right)^2 [\cos a\omega_q \cos b\omega_q + \sin a\omega_q \sin b\omega_q] + \left(\frac{\rho_{q+1}}{p+1}\right)^2 (-1)^{a+b}$$

which is the asserted formula. Again, if $p+1 = 2q+1$, the last term is deleted.

The eigenvalues of Σ are clearly those of $2\sigma^2 S^{\frac{1}{2}} P^2 S^{\frac{1}{2}}$, viz.

$$\frac{2\sigma^2}{p+1}\,\{\rho_0^2,\ 2\rho_1^2,\ 2\rho_1^2, \ldots, 2\rho_q^2,\ 2\rho_q^2,\ \rho_{q+1}^2\}\ .$$

The determinant is their product:

$$\det \Sigma = \left(\frac{2\sigma^2}{p+1}\right)^{p+1} \rho_0^2 \cdot 2\rho_1^2 \cdot 2\rho_1^2 \cdots 2\rho_q^2 \cdot 2\rho_q^2 \cdot \rho_{q+1}^2$$

$$= \tfrac{1}{4}\left(\frac{4\sigma^2}{p+1}\right)^{p+1} \rho_0^2\,\rho_1^4 \cdots \rho_q^4\,\rho_{q+1}^2\ .$$

Since

$$\det \Delta = (-K)^{p+1} K = (-1)^{p+1} K^{p+2}\ ,$$

we have

$$\det(\Delta'\ \Sigma\Delta) = \tfrac{1}{4}\left(\frac{4\sigma^2}{p+1}\right)^{p+1} \frac{1}{K^{2p}} \cdot (K^2\rho_0)^2 (K^2\rho_1)^4 \cdots (K^2\rho_q)^4 (K^2\rho_{q+1})^2$$

The quantities $K^2\rho_j$ depend only on the θ's. Using the formula in terms of the λ's we see that $K^2\rho_j < 4^p$.

DISTRIBUTION LIST

Director's Office

W.H. Radford
C.R. Wieser
D. E. Dustin

Division 2

F.C. Frick
V.A. Nedzel

Group 21

H. L. Kasnitz
G.M. Shannon

Group 22

W. Z. Lemnios
R.J. Bergemann, Jr.

C.R. Arnold
A. Freed
L. A. Gardner, Jr. (10)
A.A. Grometstein
O. A. Z. Leneman
A.J. Morency
C. E. Nielsen, Jr.
J. Rheinstein
M. Rockowitz
F.C. Schweppe
A. F. Smith
P.A. Willmann

Group 28

J.A. Arnow
H.K. Knudsen
J.B. Lewis
H.C. Peterson

Group 35

M. Balser

L.R. Martin
J. Penhune
K.E. Ralston
A.R. Warwas

Division 4

J. Freedman
H.G. Weiss

W.W. Ward

Group 41

J. P. Hawkes
H.M. Jones
H.O. Schneider

Group 42

R.P. Wishner

Group 45

S.J. Miller

M. Axelbank
A. Bertolini
S. F. Catalano
D.F. DeLong
R.W. Straw

Group 62

F. E. Heart

K. Jordan
I. Stiglitz

Group 64

P.E. Green

Group 64 (Continued)

H.W. Briscoe

J. Capon
R.T. Lacoss

WSMR

R.L. Hardy
C.J. McKenney

Kwajalein

G.R. Armstrong
T.H. Einstein
E. Korngold
B. Nanni

Hq. ESD (Lt. J. Egan)
L.G. Hanscom Field
Bedford, Mass.

A. Albert
ARCON
803 Mass. Avenue
Lexington, Mass.

Prof. M. Athans
Rm. 24-306
MIT, Cambridge, Mass.

P.L. Falb
Univ. of Michigan
Information and Control Eng.
Ann Arbor, Michigan

Prof. Larry Ho
Harvard University
Cambridge, Mass.

LAG:smm
11 February 1966

DOCUMENT CONTROL DATA – R&D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY *(Corporate author)* Lincoln Laboratory, M.I.T.	2a. REPORT SECURITY CLASSIFICATION Unclassified 2b. GROUP None

3. REPORT TITLE

A Method for Estimating the Parameters of a Stable Difference Equation Driven by Sinusoids and Observed in the Presence of Nonwhite Noise

4. DESCRIPTIVE NOTES *(Type of report and inclusive dates)*

Technical Note

5. AUTHOR(S) *(Last name, first name, initial)*

Gardner, Leland A., Jr.

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
11 February 1966	40	None

8a. CONTRACT OR GRANT NO. AF 19(628)-5167 b. PROJECT NO. 627A c. d.	9a. ORIGINATOR'S REPORT NUMBER(S) TN 1966-9 9b. OTHER REPORT NO(S) *(Any other numbers that may be assigned this report)* ESD-TDR-66-41

10. AVAILABILITY/LIMITATION NOTICES

Distribution of this document is unlimited.

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY
None	Air Force Systems Command, USAF

13. ABSTRACT

A computationally feasible and strongly consistent method is considered for estimating the coefficients of

$$\gamma_0 x_t + \gamma_1 x_{t-1} + \ldots + \gamma_p x_{t-p} = u_t .$$

The steady-state solution is observed in the presence of nonwhite noise, and the system is driven by a certain selected superposition of sinusoids, u_t. When the noise is white, the estimates are shown to possess an asymptotic normal distribution with covariance matrix which depends on the coefficients only via the values of

$$\left| \sum_{k=0}^{p} \gamma_k e^{ik\omega} \right|^2$$

evaluated at the input frequencies.

14. KEY WORDS

difference equations
sinusoids
nonwhite noise