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THREE-DIMENSIONAL, ANALYTIC SOLUTIONS TO THE PROBLEMS  
OF DIFFUSION OF WIND-DRIVEN CONTAMINATION

Continuation of a Study of Diffusion of Contamination  
From a Source of Finite Extent  
(Technical Memorandum No. PMR-TM-65-4)

By

H. F. LUDLOFF, Ph.D  
Operations Analysis

14 April 1966

PACIFIC MISSILE RANGE

Point Mugu, California

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# PACIFIC MISSILE RANGE

POINT MUGU, CALIFORNIA

R. N. SHARP, RADM USN  
*Commander*

DR. W. M. SIMPSON  
*Chief Scientist, Acting*

This report describes work completed on 7 December 1965.

Dr. C. J. Thorne, Operations Research Analyst, has reviewed this report for publication.

Approved by:

DR. W. M. SIMPSON  
*Chief Scientist, Acting*

Technical Memorandum PMR-TM-66-4

Published by ..... Editorial Division  
Technical Information Department  
First printing ..... 150 copies  
Security classification ..... UNCLASSIFIED

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## SUMMARY

Estimates of critical distances, up to which dust, aerosols, and (toxic) fumes may be driven, under the influence of various wind and diffusivity conditions, require that three-dimensional solutions to the problem of wind-driven contamination be derived, for application to range safety problems. The so called "conservative," two-dimensional solutions (which might be of some value in certain meteorological situations), are in general, by orders of magnitude, too large to be of practical value (see part III, page 5). After all, nature is three dimensional; this feature determines the rate at which the concentration falls off in the downwind direction.

In part III, analytic solutions for constant wind and constant diffusivity have been derived. By using the method of separation of variables, the three-dimensional concentration functions resulting from two-dimensional as well as three-dimensional sources are deduced. In either case, the intensity  $q(x, 0)$  falls off with the first power of the downwind distance  $x$ , as expected; but the strength factor, which depends on the dimensions of the source in the  $x$ -,  $y$ -, and  $z$ -directions, offers some new insight (see figure 1). Even in three dimensions, the resulting critical distances are too large to be useful, so long as the assumption of constant  $U$  and  $K$  is retained (see part III, page 5).

In part IV, three-dimensional analytic solutions have been derived, under the assumption that wind speed  $U$  and eddy diffusivity  $K$  vary, either in accordance with the conjugate power laws, or in a more general fashion. The results, following therefrom, can be made to agree with published observational data; in particular with the general observation that the rate of decline of concentration with the downwind distance  $x$  and crosswind distance  $y$  along the ground surface obey a certain power law.

In section one of part IV, a new method of solution is suggested which may be used for solving diffusion problems of a more general nature: a "method of superposition of parabolic sources, sinks, etc."

In the remainder, five particular, three-dimensional parabolic source solutions have been derived.

The five mentioned solutions are characterized by:

$$1.) \quad U = \bar{U} \cdot z^{\frac{1}{7}}, \quad K_y = a \cdot z^{\frac{1}{7}}, \quad K_z = b \cdot z^{\frac{6}{7}}, \quad p = \frac{25}{18}$$

$$2.) \quad U = \bar{U} \cdot z^{\frac{6}{7}}, \quad K_y = a \cdot z^{\frac{6}{7}}, \quad K_z = b \cdot z^{\frac{6}{7}}, \quad p = \frac{20}{17}$$

$$3.) \quad U = \bar{U} \cdot z, \quad K_y = a \cdot z, \quad K_z = b \cdot z, \quad p = \frac{3}{2}$$

$$4.) \quad U = \bar{U} \cdot z, \quad K_y = a \cdot z \cdot y^{\frac{1}{2}}, \quad K_z = b \cdot z, \quad p = \frac{5}{2}$$

$$5.) \quad U = \bar{U} \cdot z, \quad K_y = a \cdot z \cdot y, \quad K_z = b \cdot z, \quad p = 2$$

$K_y$  and  $K_z$  are the diffusivities in the y- and z-directions.  $p$  is the exponent, which indicates the rate of decline of concentration, with downwind distance  $x$ . The examples are ordered after increasing  $p$ .

### PART III

#### THE CASE OF CONSTANT WIND VELOCITY AND CONSTANT DIFFUSIVITY

##### THREE-DIMENSIONAL DIFFUSION FOR A TWO-DIMENSIONAL SOURCE

In part I, the writer has derived an analytic solution of the problem of diffusion of wind-driven contamination, in two dimensions; wind velocity and diffusivity were assumed to be constant. In part III, an analytic solution of the analogous problem in three dimensions will be derived.

As in figure 1-1 of part I, the wind velocity  $U$  is in the direction  $x$ , and the source is located in the  $y$ - $z$  plane ( $x = 0$ ) and may extend between  $0 \leq z \leq \ell$ ,  $-m \leq y \leq +m$ . (Later on, we shall assume a three-dimensional source.) We desire the three-dimensional distribution of the concentration,  $q(x; y, z)$ , which has to satisfy the differential equation:

$$\frac{\partial \bar{q}}{\partial x} = \frac{K}{U} \left( \frac{\partial^2 \bar{q}}{\partial y^2} + \frac{\partial^2 \bar{q}}{\partial z^2} \right) \quad (1)$$

Again, we stipulate the boundary conditions:

$$\begin{aligned} \bar{q} = \bar{q}_0(y, z) = 1 & \text{ for } -\ell \leq z \leq +\ell, \text{ and } -m \leq y \leq +m \\ & = 0 \text{ outside of this rectangle in the } y\text{-}z \text{ plane} \end{aligned} \quad (2)$$

The symmetrical form of equation (2), about the  $x$ - $y$  plane, guarantees that the second boundary condition:

$$K \frac{\partial \bar{q}}{\partial z} = 0, \text{ for } z = 0, \text{ any } x, \text{ any } y \quad (3)$$

will be fulfilled, which signifies the impenetrability of the ground surface.

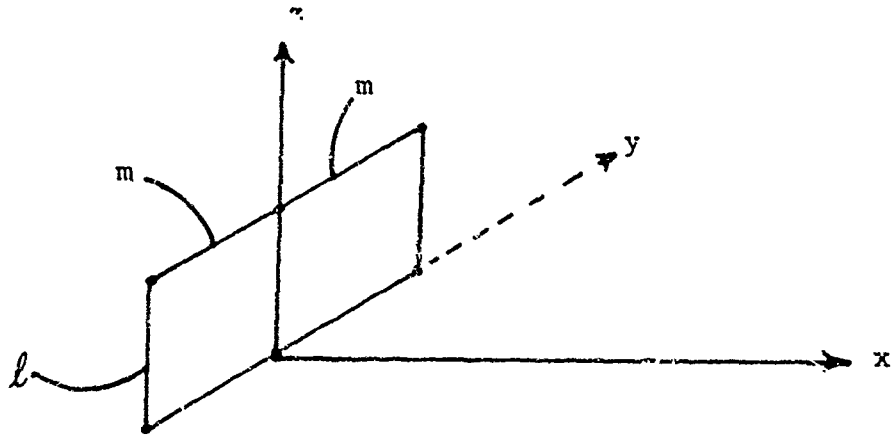


Figure 1. Location of Two-Dimensional Source.

We assume a solution

$$\bar{q}(x; y, z) = q_1(x, y) \cdot q_2(x, z) \quad (4)$$

then

$$\frac{\partial q_1}{\partial x} \cdot q_2 + \frac{\partial q_2}{\partial x} \cdot q_1 = \frac{K}{U} \left[ \frac{\partial^2 q_1}{\partial y^2} \cdot q_2 + \frac{\partial^2 q_2}{\partial z^2} \cdot q_1 \right]$$

Hence

$$\frac{1}{q_1} \left\{ \frac{\partial q_1}{\partial x} - \frac{K}{U} \cdot \frac{\partial^2 q_1}{\partial y^2} \right\} = \frac{1}{-q_2} \left\{ \frac{\partial q_2}{\partial x} - \frac{K}{U} \cdot \frac{\partial^2 q_2}{\partial z^2} \right\} = C$$

If we set  $C = 0$ , then

$$\frac{\partial q_1}{\partial x} = \frac{K}{U} \cdot \frac{\partial^2 q_1}{\partial y^2} \quad \text{and} \quad \frac{\partial q_2}{\partial x} = \frac{K}{U} \cdot \frac{\partial^2 q_2}{\partial z^2} \quad (5)$$

Thus, the form of  $q_1$  and  $q_2$  is as in two dimensions; as a consequence, not only differential equations (1), but also boundary conditions (2) and (3) are fulfilled.

The solution of the problem can be written in the form

$$\begin{aligned} \bar{q}(x; y, z) &= \frac{U}{4\pi K \cdot x} \cdot \int_{-l}^{+l} 1 \cdot d\zeta \int_{-m}^{+m} d\eta \cdot \exp \left\{ -\frac{U}{4K} \frac{[(z - \zeta)^2 + (y - \eta)^2]}{x} \right\} \\ &= \left( \frac{U}{4\pi K \cdot x} \right)^{\frac{1}{2}} \cdot \int_{-l}^{+l} d\zeta \cdot \exp \left\{ -\frac{U}{4K} \frac{[z - \zeta]^2}{x} \right\} \cdot \\ &\quad \left( \frac{U}{4\pi K \cdot x} \right)^{\frac{1}{2}} \cdot \int_{-m}^{+m} d\eta \cdot \exp \left\{ -\frac{U}{4K} \frac{[y - \eta]^2}{x} \right\} \end{aligned} \quad (5)$$

Evaluating either factor of equation (6) as in part I, we get

$$\bar{q}(x; y, z) = \frac{1}{2} \left[ \pm I(u_1) + I(u_2) \right] \cdot \frac{1}{2} \left[ \pm I(w_1) + I(w_2) \right] \quad (7)$$

Here

$$u_{1(2)} = (z(\mp)l) \left( \frac{U}{4K \cdot x} \right)^{\frac{1}{2}}; \quad w_{1(2)} = (y(\mp)m) \left( \frac{U}{4K \cdot x} \right)^{\frac{1}{2}} \quad (8)$$

By using the first term in the asymptotic expansion of the error integral, we may write; in the vicinity of the x-axis, and for  $\left(\frac{l}{x}\right) \ll 1$ :

$$\begin{aligned} q_i &\approx \frac{1}{2} \left[ \pm \frac{2}{\sqrt{\pi}} \left( |u_1| \right) + \frac{2}{\sqrt{\pi}} \left( u_2 \right) \right] = \frac{1}{\sqrt{\pi}} \left[ \frac{l}{x^{\frac{1}{2}}} + \frac{l}{x^{\frac{1}{2}}} \right] \cdot \left( \frac{U}{4K} \right)^{\frac{1}{2}} \\ &= \left( \frac{l}{x} \right)^{\frac{1}{2}} \cdot \left( \frac{lU}{\pi K} \right)^{\frac{1}{2}} \end{aligned} \quad (9)$$

In three dimensions:

$$\bar{q} \approx \left(\frac{l}{x}\right)^{\frac{1}{2}} \cdot \left(\frac{m}{x}\right)^{\frac{1}{2}} \cdot \left(\frac{lU}{\pi K}\right)^{\frac{1}{2}} \cdot \left(\frac{mU}{\pi K}\right)^{\frac{1}{2}} \quad (10)$$

and for  $l = m$ :

$$\bar{q} \approx \frac{1}{\pi} (l \cdot m) \cdot \frac{U}{K} \cdot \frac{1}{x} = \frac{1}{\pi} \text{area} \frac{U}{K} \frac{1}{x} \quad (10a)$$

The concentration at a given distance  $x$ , is proportional to the area of the source and the strength of the wind but inverse to the diffusivity  $K$ , which diffuses the contamination in the  $y$ - and  $z$ -directions.

#### APPLICATION TO RANGE SAFETY CONDITIONS

Let us assume that the critical contamination value  $q_{cr}$  is considered to be 1 per cent of the "initial" contamination  $q_0 = 1$ , at the source. We would like to know, at what distance,  $x_{cr}$ , the value  $q_{cr}$  is reached, particularly downwind on the ground. Depending on, whether we use the two- or three-dimensional model, we get:

(a) Strip source:

$$q_{tw} = \left(\frac{l}{x_{tw}}\right)^{\frac{1}{2}} \cdot \left(\frac{lU}{\pi K}\right)^{\frac{1}{2}} = 0.01 \quad (11)$$

(b) Area source:

$$q_{th} = \left(\frac{l}{x_{th}}\right)^{\frac{1}{2}} \cdot \left(\frac{m}{x_{th}}\right)^{\frac{1}{2}} \cdot \left(\frac{lU}{\pi K}\right)^{\frac{1}{2}} \cdot \left(\frac{mU}{\pi K}\right)^{\frac{1}{2}} = 0.01 \quad (11a)$$

The data given to us, were  $l = 30 \text{ m}$ ,  $\frac{U}{K} \approx 15 \text{ m}^{-1}$ , so that  $\frac{x_{tw}}{l} \sim 1.5 \cdot 10^6$ .

In contrast to this two-dimensional result, we compare the result, obtained in three dimensions; equalizing equations (11) and (11a)

$$\frac{\left(\frac{l \cdot m}{x_{th}}\right)^{\frac{1}{2}}}{x_{th}} = \left(\frac{l}{x_{tw}}\right)^{\frac{1}{2}} \cdot \left(\frac{mU}{\pi K}\right)^{\frac{1}{2}}$$

or

$$\frac{x_{th}}{\left(\frac{l \cdot m}{x_{th}}\right)^{\frac{1}{2}}} = \left(\frac{x_{tw}}{l}\right)^{\frac{1}{2}} \cdot \left(\frac{mU}{\pi K}\right)^{\frac{1}{2}} \quad (12)$$

If, according to the above, we assume  $l = m = 30$ ,  $\frac{U}{K} = 15 \text{ m}^{-1}$ , we get:

$$\frac{x_{th}}{(l_m)^{\frac{1}{2}}} \sim (1.2 \cdot 10^3) \cdot 12.5$$

The three-dimensional distance is obtained, by taking the square root of the two-dimensional distance, and in addition, multiplying with the strength width in the y-direction (ratio of wind width and diffusivity).

### THREE-DIMENSIONAL DIFFUSION FOR A THREE-DIMENSIONAL SOURCE

If the source has also an extension in the x-direction,  $-d \leq x \leq +d$ , (see figure 2) then, if we skip, for the present, the y-direction we get

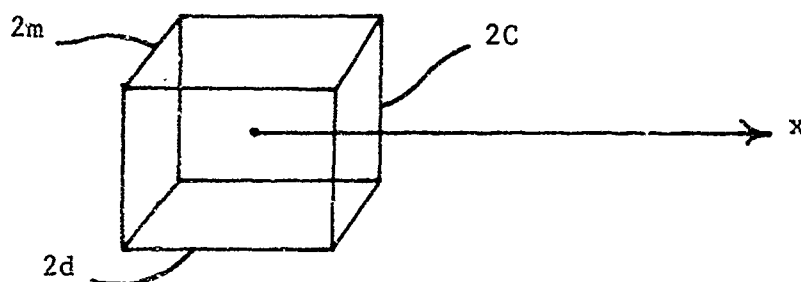


Figure 2. Location of Three-Dimensional Source.

$$q = \left( \frac{U}{4\pi K} \right)^{\frac{1}{2}} \cdot \int_{-d}^{+d} \frac{d\theta}{x+\theta} \cdot \int_{-l}^{+l} \xi \cdot q_0 \cdot \exp \left\{ -\frac{U}{4K} \left[ \frac{(z-\xi)^2}{x+\theta} \right] \right\} \quad (13)$$

$$= \frac{1}{2} \left\{ \int_{-d}^{+d} [u_1(\theta)] d\theta + \int_{-d}^{+d} [u_2(\theta)] d\theta \right\} \quad (13a)$$

Near the x-axis, and for  $\frac{l}{x} \ll 1$ , we may set:

$$q \approx \int \left( \frac{l}{x+\theta} \right)^{\frac{1}{2}} \left( \frac{lU}{\pi K} \right)^{\frac{1}{2}} \cdot d\theta; \quad (14)$$

Set  $x + \theta = \bar{x}$ , then

$$\bar{q} \approx \left(\frac{\ell U}{\pi K}\right)^{\frac{1}{2}} \cdot \int_{x-d}^{x+d} \frac{\ell}{x} d\bar{x} = \left(\frac{\ell U}{\pi K}\right)^{\frac{1}{2}} \cdot 2 \cdot \left(\frac{\ell}{x}\right)^{\frac{1}{2}} \left[ \left(1 + \frac{d}{x}\right)^{\frac{1}{2}} - \left(1 - \frac{d}{x}\right)^{\frac{1}{2}} \right] \quad (15)$$

Expanding the square roots into polynomial series, we have

$$\bar{q} \approx 2 d \cdot \left(\frac{\ell U}{\pi K}\right)^{\frac{1}{2}} \cdot \frac{1}{x^{\frac{1}{2}}} \left\{ 1 + \frac{1}{8} \frac{d^2}{x^2} + \dots \right\} \quad (16)$$

To get consistent results, we have to use also the asymptotic expansion of the error integral up to the second term:

$$I(u_i) = \frac{2}{\sqrt{\pi}} \left\{ u_i - \frac{u_i^3}{3} + \frac{u_i^5}{10} \dots \right\}$$

so that, near the x-axis, and for  $\left(\frac{\ell}{x}\right) \ll 1$ :

$$\frac{1}{2} \left[ I(u_1) + I(u_2) \right] \approx \frac{1}{\sqrt{\pi}} \left\{ \left(\frac{\ell}{x}\right)^{\frac{1}{2}} \left(\frac{\ell U}{K}\right)^{\frac{1}{2}} - \frac{1}{3 \cdot 4} \left(\frac{\ell}{x}\right)^{\frac{3}{2}} \cdot \left(\frac{\ell U}{K}\right)^{\frac{3}{2}} \dots \right\}$$

Hence

$$\bar{q} \approx \int_{-d}^{+d} \frac{d\theta}{\sqrt{\pi}} \left\{ \left(\frac{\ell U}{K}\right)^{\frac{1}{2}} \left(\frac{\ell}{x+\theta}\right)^{\frac{1}{2}} - \frac{1}{3 \cdot 4} \left(\frac{\ell U}{K}\right)^{\frac{3}{2}} \cdot \left(\frac{\ell}{x+\theta}\right)^{\frac{3}{2}} + \dots \right\} \quad (17)$$

Now, evaluating the integral, and observing that

$$(x+d)^{\frac{1}{2}} - (x-d)^{\frac{1}{2}} \approx x^{\frac{1}{2}} \left[ \frac{d}{x} + 0 \cdot \left(\frac{d}{x}\right)^2 + \frac{2}{16} \left(\frac{d}{x}\right)^3 + \dots \right]$$

and

$$(x+d)^{-\frac{1}{2}} - (x-d)^{-\frac{1}{2}} \approx -\frac{1}{x^{\frac{3}{2}}} \cdot \frac{d}{x} + \dots$$

we obtain finally:

$$\bar{q} \approx \frac{1}{\sqrt{\pi}} \cdot 2 d \left\{ \left( \frac{\ell U}{K} \right)^{\frac{1}{2}} \cdot \left( \frac{\ell}{x} \right)^{\frac{1}{2}} - \frac{1}{3 \cdot 4} \left( \frac{\ell U}{K} \right)^{\frac{3}{2}} \cdot \left( \frac{\ell}{x} \right)^{\frac{3}{2}} + \dots \right\} + 0 \left\{ \left( \frac{\ell}{x} \right)^{\frac{5}{2}} \right\} \quad (18)$$

The first term of the series indicates, that the source has gained strength by a factor  $2d$ . The second term takes into account that the initial contamination is distributed in the  $x$ -direction (multipole effect).

Analogous results are obtained, if we add to equation (18) the results obtained for the  $y$ -direction:

$$q \quad \frac{2 d}{\pi} \left\{ \frac{(\ell_m)}{x} \cdot \frac{U}{K} + \dots \right\}$$

## PART IV

### WIND SPEED AND DIFFUSIVITY VARYING ACCORDING TO THE CONJUGATE POWER LAWS

#### INTRODUCTION

In the following, the idealization that wind speed  $U$  and diffusivity  $K$  are constant, will be discarded. Since it is well known that at the earth's (or ocean) surface the wind speed must decrease to zero, according to the physical laws of viscous flow, and, since the air in the atmosphere is in turbulent motion, a natural assumption is that the variation of wind speed with altitude obey the  $\frac{1}{7}$  power law of turbulent flow, as derived by von Karman and confirmed in numerous laboratory tests. At high altitudes, wind conditions will vary according to orographic conditions, and finally there may be an inversion layer. Such complicated conditions can be best dealt with by numerical integration of the differential equation, for which an example is given in part II. Representative, simple conditions should be treated analytically, when possible. This is done in the subsequent part IV.

In marked contrast to the case of constant  $U$  and  $K$ , where we obtained a range of concentration characterized by

$$q(x, 0) \propto \left(\frac{\ell}{x}\right)^{\frac{1}{2}} \text{ in two, and } q(x, 0) \propto \left(\frac{\ell}{x}\right) \text{ in three dimensions, we}$$

shall now expect a decrease in  $q(x, 0)$  with higher negative powers of  $x$ . In the case of constant  $U$ , the wind velocity at the ground is larger than zero, and hence, the emanating contamination is carried farther. For a speed distribution following the  $\frac{1}{7}$  power law, the ground layer is dragged by turbulent mixing with the upper layers, which are moving with increasingly greater speeds. Therefore, the highest concentration is close to the source but falls off more rapidly with distance downwind than the concentration for the constant wind.

#### SUGGESTION OF A NEW METHOD OF SOLUTION

In part III, the writer has derived three-dimensional solutions to the problem of wind-driven contamination, whereby it was assumed that wind speed and diffusivity are constants. "The failure of these solutions to conform to the meteorological observations, produces unassailable evidence that eddy diffusion in the atmosphere cannot be represented by the 'Fickian' equation," (i.e., the diffusion equation with constant  $U$  and  $K$ ). Quotation is taken from O.G. Sutton, Micro-meteorology, page 138.

It is suggested that Schmidt's conjugate power laws may be more accurate. These laws assume that the wind speed varies as the  $\frac{1}{7}$  power of a dimensionless altitude coordinate as it should in idealized turbulent flow, and the diffusivities vary as the  $\frac{6}{7}$  power of the

altitude coordinate. In two dimensions, Sutton (page 281), gives a simple closed form solution for these conditions and a source in the form of a singularity in the x-z plane. This idealization of the source form is a frequent artifice, used in several fields of physics, e.g., electro-dynamics and fluid dynamics, whereby the singular vicinity of the source, which produces the flow, is subsequently excluded along a streamline, equipotential line, curve of constant contamination, or any appropriate straight line, and only the exterior field is considered as real. Furthermore, it is apparent that, if the governing equations are linear, sums or integrals over such sources, can be added up, to give a composite field, with the desired properties (boundaries).

This writer proposes that this "method of superposition of sources, etc." be used to obtain the desired distribution of any given contamination field. One may note that the sources of incompressible flow satisfy the Laplace equation, which is elliptic. The "pseudo sources" of supersonic flow obey a hyperbolic equation (of the type of the wave equation). Since our present sources obey the diffusion equation, they may be called "parabolic or diffusion sources."

The prerequisite for such a procedure is that several parabolic source solutions be known. It is the main objective in this part to derive several elementary three-dimensional parabolic source solutions. These solutions will answer the question about the distribution of concentration, when the actual U and K variations are the assumed ones. Beyond that, a superposition of such elementary solutions may enable us in the future, to deal with problems of a more general nature.

THE CONCENTRATION, UNDER CONDITIONS IN WHICH  $U = \bar{U} \cdot z^{\frac{1}{7}}$ ,  $K_y = a \cdot z^{\frac{1}{7}}$ ,

$$\text{AND } K_z = b \cdot z^{\frac{6}{7}}$$

The problem of our three-dimensional singularity source can be formulated as follows:

We are looking for a solution of the parabolic differential equation

$$\bar{U} \cdot z^{\frac{1}{7}} \cdot \frac{\partial q}{\partial x} = \frac{\partial}{\partial y} \left( a \cdot z^{\frac{1}{7}} \frac{\partial q}{\partial y} \right) + \frac{\partial}{\partial z} \left( b \cdot z^{\frac{6}{7}} \cdot \frac{\partial q}{\partial z} \right) \quad (1)$$

with the following boundary conditions:

$$K \frac{\partial q}{\partial z} = 0, \text{ for } z = 0, \text{ any } x \text{ and any } y \quad (2)$$

$$q = 0, \text{ as } x, y, \text{ or } z \rightarrow \infty \quad (3)$$

$$q \rightarrow \infty, \text{ as } x = z = y \rightarrow 0$$

$$\int_0^{\infty} dz \int_{-\infty}^{+\infty} dy \cdot \bar{U} \cdot z^{\frac{1}{7}} \cdot q(x; y, z) = Q, \text{ independent of } x \quad (4)$$

The last condition represents the conservation of mass flow through any plane  $x = \text{constant}$ .

For a two-dimensional singularity source, i.e., if in equation (1)  $a$  is set equal to zero, a solution has been stated by Sutton, page 281, which is of the form:

$$q_2(x, z) = \frac{C}{x^{\frac{9}{8}}} \cdot \exp\left(-A \frac{z^{\frac{9}{7}}}{x}\right) \quad (5)$$

It is readily verified by differentiation, that this solution with

$$A = \frac{49}{81} \frac{\bar{U}}{b}$$

satisfies differential equation (1) and boundary conditions (2) and (3). Condition (4) can be written, for two dimensions only, as:

$$M_2 = \int_0^{\infty} \frac{1}{x^{\frac{9}{8}}} \cdot \exp\left[-A \cdot \frac{z^{\frac{9}{7}}}{x}\right] dz \sim \int_0^{\infty} \frac{d\left(\frac{z^{\frac{9}{7}}}{x}\right)}{x^{\frac{9}{8}}} \cdot \exp\left(-A \cdot \frac{z^{\frac{9}{7}}}{x}\right) \quad (6)$$

and if

$$\frac{z^{\frac{9}{7}}}{x^{\frac{9}{8}}} = v, \quad M_2 \sim \int_0^{\infty} d v \cdot \exp\left(-A \cdot v^{\frac{9}{8}}\right), \text{ for any fixed } x.$$

We see from the above, that  $M_2$  is independent of  $v$  and  $x$ , i.e., the diffusing mass flow is constant.

Sutton's above two-dimensional diffusion source is capable of generalization to three dimensions, in several ways. To prove this, let us now consider the complete differential equation (1) and complete conservation equation (4).

Generalizing "Ansatz" (5), set

$$q_3 = \frac{C}{x^p} \cdot \exp\left[-A \cdot \frac{z^m}{x} - B \cdot \frac{y^n}{x}\right] \quad (7)$$

whereby the coefficients  $p$ ,  $m$ , and  $n$  will be so determined, that they satisfy the differential equation and boundary conditions.

Inserting equation (7) into equation (1), we obtain:

$$\bar{U} \cdot z^{\frac{1}{7}} \cdot \frac{\partial q}{\partial x} = \bar{U} \cdot z^{\frac{1}{7}} \left\{ -\frac{p \cdot C}{x^{p+1}} + \frac{A \cdot C \cdot z^m}{x^{p+2}} + \frac{B \cdot C \cdot y^n}{x^{p+2}} \right\} \cdot \exp \left[ -A \cdot \frac{z^m}{x} - B \cdot \frac{y^n}{x} \right] \quad (8)$$

$$a \cdot z^{\frac{1}{7}} \cdot \frac{\partial^2 q}{\partial y^2} = a \cdot z^{\frac{1}{7}} \left\{ -\frac{B \cdot C}{x^{p+1}} n(n-1) \cdot y^{n-2} + \frac{B^2 \cdot C}{x^{p+2}} n^2 y^{2n-2} \right\} \cdot \exp \left[ -A \cdot \frac{z^m}{x} - B \cdot \frac{y^n}{x} \right] \quad (9)$$

$$b \cdot z^{\frac{6}{7}} \cdot \frac{\partial q}{\partial z} = -\frac{b A \cdot C}{x^{p+1}} \cdot m \cdot z^{\left(m-1 + \frac{6}{7}\right)} \cdot \exp \left[ -A \cdot \frac{z^m}{x} - B \cdot \frac{y^n}{x} \right]$$

$$\frac{\partial}{\partial z} \left[ b \cdot z^{\frac{6}{7}} \frac{\partial q}{\partial z} \right] = \left[ -\frac{b A \cdot C}{x^{p+1}} m \left( m - 1 + \frac{6}{7} \right) \cdot z^{m-2 + \frac{6}{7}} + \frac{b A^2 \cdot C}{x^{p+2}} m^2 \cdot z^{\left(2m-2 + \frac{6}{7}\right)} \right] \cdot \exp \left[ -A \cdot \frac{z^m}{x} - B \cdot \frac{y^n}{x} \right] \quad (10)$$

Next we have to determine the exponents in equations (8), (9), and (10), so that they match. After this, we shall match the coefficients of equal powers in  $z$ ,  $y$ , and  $x$ .

Equation (8) contains terms with the following powers:

$$\frac{1}{z^{\frac{1}{7}}} / x^{p+1}; \quad \frac{1}{z^{\frac{1}{7}+m}} / x^{p+2}; \quad \frac{1}{z^{\frac{1}{7}}} \cdot y^n / x^{p+2}$$

Equation (9) contains terms with the following powers:

$$y^{n-2} \cdot \frac{1}{z^{\frac{1}{7}}} / x^{p+1}; \quad \frac{1}{z^{\frac{1}{7}}} \cdot y^{2n-2} / x^{p+2}$$

Equation (10) contains terms with the following powers:

$$z^{m-2+\frac{6}{7}} / x^{p+1}; \quad z^{2m-2+\frac{6}{7}} / x^{p+2}$$

We see that all exponents match, if we set  $n = 2$ ; and  $m = \frac{9}{7}$ , as in two dimensions.

Then:

Equation (8) contains the powers

$$\frac{1}{z^7} / x^{p+1}; \frac{10}{z^7} / x^{p+2}; y^2 \frac{1}{z^7} / x^{p+2}$$

Equation (9) contains the powers

$$\frac{1}{z^7} / x^{p+1}; y^2 \frac{1}{z^7} / x^{p+2}$$

Equation (10) contains the powers

$$\frac{1}{z^7} / x^{p+1}; \frac{10}{z^7} / x^{p+2}$$

As we see, p remains open.

Since we have now matched all the exponents, we compare the coefficients of equal powers, which provides the following four equations:

$$\frac{1}{z^7} \frac{1}{x^{p+1}} \left[ -p \cdot \bar{U} = -a \cdot B \cdot 2 - b \cdot A \cdot \frac{9}{7} \cdot \frac{8}{7} \right] \quad (11)$$

$$\frac{1}{z^7} \frac{y^2}{x^{p+2}} \left[ \bar{U} \cdot B = a \cdot B^2 \cdot 4 \right] \quad (12)$$

$$\frac{10}{z^7} \frac{1}{x^{p+2}} \left[ \bar{U} \cdot A = b \cdot A^2 \cdot \left(\frac{9}{7}\right)^2 \right] \quad (13)$$

To this add the information that in two dimensions, if  $a = B = 0$ , we know that  $p = p_0 = \frac{8}{9}$ , so that we have

$$-\frac{8}{9} \bar{U} = -b A \cdot \frac{72}{49} \quad (14)$$

Now subtracting equation (14) from equation (11) and setting  $p = \frac{8}{9} + p_1$ , we get

$$-p_1 \bar{U} = -a B \cdot 2 \quad (11) - (14)$$

$$\bar{U} = a B \cdot 4 \quad (12)$$

$$\text{which yields } p_1 = \frac{1}{2} \text{ and } p = p_0 + p_1 = \frac{25}{18} \quad (15)$$

It is noteworthy that equation (15) holds true for any values  $\bar{U}$ , a, b.

## DISCUSSION OF RESULTS

Actually, these values are determined by the altitude variations of wind and diffusivity. In lieu of observational data, we set  $a = b = \bar{U}$ , and obtain:

$$\frac{8}{9} = A \cdot \frac{72}{49} \quad (14a)$$

$$1 = 4B \quad (12a)$$

$$1 = \frac{81}{49} \cdot A \quad (13a)$$

Equations (13a) and (14a) give the same A value, and equation (12a) gives  $B = \frac{1}{4}$ .

Therefore, the three-dimensional distribution of contamination, for the indicated variations of U and K, which are in agreement with the physical laws (of turbulence) and with observational data, assumes the following simple form:

$$q(x; y, z) = \frac{C}{x^{18}} \cdot \exp \left[ -\frac{49}{81} \cdot \frac{z^9}{x} - \frac{1}{4} \frac{y^2}{x} \right] \quad (16)$$

This is the contamination issuing from one elementary "parabolic source," mentioned earlier.

It is obvious that solution (16) fulfills not only differential equation (1), but also conditions (2) and (3). But we still have to show that it fulfills the condition of conservation of mass flow for three dimensions (4). We can write (4), after separating p into  $P_0 + P_1$ :

$$M_3 = \int_0^{\infty} \frac{\bar{U}}{x^9} \cdot \frac{1}{z^7} \cdot \exp \left[ -A \cdot \frac{z^9}{x} \right] dz \cdot \int_{-\infty}^{+\infty} \frac{1}{x^{\frac{1}{2}}} \cdot \exp \left[ -B \cdot \frac{y^2}{x} \right] dy = Q \quad (17)$$

Now, it has been shown before, that the first factor is independent of x and constant; it remains to show that the second factor is also independent of x.

We see that

$$M_3 = M_2 \cdot \int_{-\infty}^{+\infty} d \left( \frac{y}{x^2} \right) \cdot \exp \left[ - B \left( \frac{y}{x^2} \right)^2 \right] \quad (18)$$

Setting  $\frac{y}{x^2} = r$

$$M_3 = M_2 \cdot \int_{-\infty}^{+\infty} dr \cdot e^{-Br^2} = M_2 \cdot \sqrt{\pi} \cdot (\sqrt{B})^{-1}, \text{ q.e.d.} \quad (19)$$

The diffusing mass flow  $M_3$  is the same at any plane  $x = c$ , also in three dimensions.

THE SPREAD OF CONTAMINATION  $q(x; y, z)$ , FOR  $U = \bar{U} \cdot z^{\frac{6}{7}}$ ,

$$K_y = a \cdot z^{\frac{6}{7}}, \quad K_z = b \cdot z^{\frac{6}{7}}$$

Another three-dimensional singularity source, which has been suggested as representing actual meteorological conditions, is the one indicated above, so that we have:

$$\bar{U} \cdot z^{\frac{6}{7}} \cdot \frac{\partial q}{\partial x} = \frac{\partial}{\partial y} \left( a \cdot z^{\frac{6}{7}} \cdot \frac{\partial q}{\partial y} \right) + \frac{\partial}{\partial z} \left( b \cdot z^{\frac{6}{7}} \cdot \frac{\partial q}{\partial z} \right)$$

Setting as before:

$$q = \frac{C}{x^p} \cdot \exp \left[ - A \cdot \frac{z^m}{x} - B \cdot \frac{y^n}{x} \right] = \frac{C}{x^p} \cdot \exp \left[ - L(x, y, z) \right]$$

we get this time:

$$1. \quad \bar{U} \cdot z^{\frac{6}{7}} \cdot \frac{\partial q}{\partial x} = \bar{U} \cdot z^{\frac{6}{7}} \left[ - \frac{p \cdot C}{x^{p+1}} + \frac{A \cdot C \cdot z^m}{x^{p+2}} + \frac{B \cdot C \cdot y^n}{x^{p+2}} \right] \cdot \exp \left[ - L(x, y, z) \right]$$

$$2. \quad a \cdot z^{\frac{6}{7}} \cdot \frac{\partial^2 q}{\partial y^2} = a \cdot z^{\frac{6}{7}} \left[ - \frac{B \cdot C}{x^{p+1}} n(n-1) \cdot y^{n-2} + \frac{B^2 \cdot C}{x^{p+2}} n^2 \cdot y^{2n-2} \right] \cdot \exp \left[ -L(x, y, z) \right]$$

$$3. \quad \frac{\partial}{\partial z} \left[ b \cdot z^{\frac{6}{7}} \cdot \frac{\partial q}{\partial z} \right] = \left[ - \frac{bA \cdot C \cdot m \left( m-1 + \frac{6}{7} \right)}{x^{p+1}} z^{\left( m-2 + \frac{6}{7} \right)} + \frac{bA^2 \cdot C \cdot m^2}{x^{p+2}} \cdot z^{\left( 2m-2 + \frac{6}{7} \right)} \right] \cdot \exp \left[ -L(x, y, z) \right]$$

1. Contains the powers  $z^{\frac{6}{7}}/x^{p+1}$ ;  $z^{\frac{6+m}{7}}/x^{p+2}$ ;  $z^{\frac{6}{7}} \cdot y^n/x^{p+2}$
2. Contains the powers  $z^{\frac{6}{7}}/x^{p+1}$ ;  $z^{\frac{6}{7}} \cdot y^{2n-2}/x^{p+2}$
3. Contains the powers  $z^{(m-2)+\frac{6}{7}}/x^{p+1}$ ;  $z^{(2m-2)+\frac{5}{7}}/x^{p+2}$

All powers will be matched, if we take  $n = 2$ ,  $m = 2$ , and leave  $p$  open, at first.

By comparing the coefficients of equal powers we get the equations:

$$\frac{z^{\frac{6}{7}}}{x^{p+1}} \left[ -\bar{U} \cdot p = -a \cdot B \cdot 2 - b \cdot A \cdot 2 \frac{13}{7} \right] \quad (\text{I})$$

$$\frac{z^{\frac{6}{7}+2}}{x^{p+1}} \left[ \bar{U} = b A \cdot 4 \right] \quad (\text{II})$$

$$\frac{z^{\frac{6}{7}} \cdot y^2}{x^{p+2}} \left[ \bar{U} = a B \cdot 4 \right] \quad (\text{III})$$

In two dimensions, where  $a = B = 0$ , we get

$$\bar{U} \cdot p_0 = b A \frac{26}{7} \quad (\text{IV})$$

$$\bar{U} = b A \cdot 4 \quad (\text{II})$$

Hence:

$$4p_0 = \frac{26}{7}; \quad p_0 = \frac{13}{14}, \quad \text{for any } \bar{U} \text{ and } b \text{ value.}$$

Set again,  $p = p_0$  and  $p_1$ , then

$$-U \cdot p_1 = -a B \cdot 2 \quad (\text{I}) - (\text{IV})$$

$$U = a B \cdot 4 \quad (\text{III})$$

$$4p_1 = 2, \quad p_1 = \frac{1}{2} \text{ again.}$$

If now we set  $\bar{U} = b = a$ , then

$$B = \frac{1}{4}, \quad A = \frac{1}{4}$$

THE CASE:  $U = \bar{U} \cdot z$ ;  $K_y = a \cdot z$ ;  $K_z = b \cdot z$

$$1. \quad \bar{U} \cdot z \frac{\partial q}{\partial x} = \bar{U} \cdot z \left[ -\frac{p \cdot C}{x^{p+1}} + \frac{A \cdot C \cdot z^m}{x^{p+2}} + \frac{B \cdot C \cdot y^n}{x^{p+2}} \right] \cdot \exp \left[ -L(x, y, z) \right]$$

$$2. \quad a z \cdot \frac{\partial^2 q}{\partial y^2} = a \cdot z \left[ -\frac{B \cdot C}{x^{p+1}} n(n-1) \cdot y^{n-2} + \frac{B^2 \cdot C}{x^{p+2}} n^2 \cdot y^{2n-2} \right] \cdot \exp \left[ -L(x, y, z) \right]$$

$$3. \quad \frac{\partial}{\partial z} \left[ b z \cdot \frac{\partial q}{\partial z} \right] = \left[ -\frac{bA \cdot C \cdot m^2}{x^{p+1}} \cdot z^{m-1} + \frac{bA^2 \cdot C \cdot m^2}{x^{p+2}} \cdot z^{2m-1} \right] \cdot \exp \left[ -L(x, y, z) \right]$$

1. Contains the powers  $z/x^{p+1}$ ;  $z^{m+1}/x^{p+2}$ ;  $z \cdot y^n/x^{p+2}$

2. Contains the powers  $z \cdot y^{n-2}/x^{p+1}$ ;  $z \cdot y^{2n-2}/x^{p+2}$

3. Contains the powers  $z^{m-1}/x^{p+1}$ ;  $z^{2m-1}/x^{p+2}$

We can match the powers, if we set  $m = 2$  and  $n = 2$ , for three dimensions. For the special, two-dimensional case we set  $a = B = 0$  and  $n = 0$ . Then

$$-\bar{U} \cdot p_0 = -b A \cdot 4$$

$$\bar{U} = b A \cdot 4$$

Therefore  $p_0 = 1$  with  $n = 0$ ,  $m = 2$ . For three dimensions

$$-\bar{U} \cdot p_1 = -a B \cdot 2; \quad U = b A \cdot 4; \quad U = a B \cdot 4$$

Hence  $4p_1 = 2$ , and  $p_1 = \frac{1}{2}$ , again.

The addition of the third dimension, appears always to increase the negative power of  $x$  by one half, so long as the crosswind distribution of  $q$ , i.e.,  $K_y$  is not made to depend on  $y$ .

THE CASE:  $U(z) = \bar{U} \cdot z$ ;  $K_y = a z \cdot y$ ;  $K_z = b \cdot z$

The meteorological data seem to require that the fall-off of concentration in the downwind direction is somewhere between  $x^{-2}$  and  $x^{-1.5}$ . Since the preceding cases seem to indicate that any  $z$ -dependence of  $K_y$  does not lead beyond  $p = 1.5$ , we try to gain new insight, by making  $K_y = a \cdot z \cdot y$ .

While equations (1) and (3) of case 5 remain the same, we have now to replace equation (2) by

$$\frac{\partial}{\partial y} \left[ a z \cdot y \cdot \frac{\partial q}{\partial y} \right] = a z \cdot \frac{\partial q}{\partial y} + a z \cdot y \cdot \frac{\partial^2 q}{\partial y^2} \quad (2a)$$

With the general assumed form (7)

$$q_3 = \frac{C}{x^p} \cdot \exp \left[ -L(x, y, z) \right] \quad (7)$$

we have now:

$$\begin{aligned} a z \left[ \frac{\partial q}{\partial y} + y \cdot \frac{\partial^2 q}{\partial y^2} \right] &= a z \left\{ -\frac{B \cdot C \cdot n \cdot y^{n-1}}{x^{p+1}} + \frac{B^2 \cdot C \cdot n^2 \cdot y^{2n-1}}{x^{p+2}} \right. \\ &\quad \left. - \frac{B' \cdot C \cdot n(n-1) \cdot y^{n-1}}{x^{p+1}} \right\} \cdot \exp \left[ -L(x, y, z) \right] \\ &= a z \left\{ -\frac{B \cdot C \cdot n^2}{x^{p+1}} \cdot y^{n-1} + \frac{B^2 \cdot C \cdot n^2}{x^{p+2}} y^{2n-1} \right\} \cdot \exp \left[ -L(x, y, z) \right] \end{aligned}$$

Therefore the terms (1), (2a), and (3) of our differential equation contain the following powers of x, y, and z:

1.  $z/x^{p+1}$ ;  $z^{m+1}/x^{p+2}$ ;  $z \cdot y^n/x^{p+2}$
- 2a.  $z \cdot y^{n-1}/x^{p+1}$ ;  $z \cdot y^{2n-1}/x^{p+2}$
3.  $z^{m-1}/x^{p+1}$ ;  $z^{2m-1}/x^{p+2}$

We see that we can match all powers by setting  $n = 1$ ,  $m = 2$ . First, solve the two-dimensional case, by setting  $a = B = 0$ ,  $m = 2$ . Then, as in case 5

$$-U \cdot p_0 = -b A \cdot 4; \quad U = b A \cdot 4. \quad \text{Hence } p_0 = 1, \text{ again.}$$

For three dimensions:

$$-U \cdot p = -a B \cdot 1 - b A \cdot 4; \quad U = b A \cdot 4; \quad U = a B \cdot 1$$

Hence  $p - p_0 = p_1 = +1$  and  $p = 2$ . The fall-off of  $q(x, 0)$  is proportional to  $x^{-2}$ .

THE CASE:  $U(z) = \bar{U} \cdot z$ ;  $K_y = a \cdot z \cdot y^{\frac{1}{2}}$ ;  $K_z = b \cdot z$

Since the fall-off in case 6 was too large, we try  $K_y = a \cdot z \cdot y^{\frac{1}{2}}$ . Again the terms (1) and (3) remain the same as in cases 5 and 6. But term (2) must be replaced by the following:

$$\frac{\partial}{\partial y} \left[ azy^{\frac{1}{2}} \frac{\partial q}{\partial y} \right] = \frac{1}{2} a z \cdot y^{-\frac{1}{2}} \cdot \frac{\partial q}{\partial y} + a z \cdot y^{\frac{1}{2}} \cdot \frac{\partial^2 q}{\partial y^2} \quad (2b)$$

Substituting expression (7) for q, we get:

$$\begin{aligned} \frac{1}{2} a z \cdot y^{-\frac{1}{2}} \cdot \frac{\partial q}{\partial y} &= \frac{1}{2} a z \cdot y^{-\frac{1}{2}} \left\{ -\frac{B C \cdot n \cdot y^{n-1}}{x^{p+1}} \right\} \cdot \exp \left[ -L(x,y,z) \right] \\ + a z y^{\frac{1}{2}} \cdot \frac{\partial^2 q}{\partial y^2} &= a z y^{\frac{1}{2}} \left\{ \frac{B^2 C \cdot n^2 \cdot y^{2n-2}}{x^{p+2}} - \frac{B C \cdot n(n-1) \cdot y^{n-2}}{x^{p+1}} \right\} \cdot \exp \left[ -L \right] \end{aligned}$$

Hence:

$$(2b) = \left\{ -\frac{B \cdot C \cdot n \cdot y^{n-2}}{x^{p+1}} \left[ \frac{1}{2} + n-1 \right] + \frac{B^2 \cdot C \cdot n^2 \cdot y^{2n-2}}{x^{p+2}} \right\} \cdot \exp \left[ -L(x,y,z) \right]$$

The terms (1), (2b), and (3) of our differential equation contain the following powers of x, y, and z:

1.  $z/x^{p+1}$ ;  $z^{m+1}/x^{p+2}$ ;  $z \cdot y^n/x^{p+2}$
- 2b.  $z \cdot y^{n-\frac{3}{2}}/x^{p+1}$ ;  $z \cdot y^{2n-\frac{3}{2}}/x^{p+2}$
3.  $z^{m-1}/x^{p+1}$ ;  $z^{2m-1}/x^{p+2}$

All terms can be matched, if we take  $m = 2$ ,  $n = \frac{3}{2}$ . From the two-dimensional case, ( $a = B = 0$ ), we get as before:

$$p_0 = 1$$

For three dimensions, we get the equations:

$$-U \cdot p = -a B \cdot \frac{3}{2} - b A \cdot 4; \quad U = b A \cdot 4; \quad U = a B \cdot \frac{9}{4}$$

Subtracting from the first equation:  $-U \cdot p_0 = -b A \cdot 4$ , we obtain now

$$p - p_0 = p_1 = \frac{2}{3}, \text{ so that } p = 1.66$$

This fall-off of  $q(x,0)$  seems to agree fairly well with the observational data.

If necessary, new parabolic source solutions can be readily generated.

## CONCLUSIONS

We have shown that, depending on the variation of the diffusivities  $K_y$  and  $K_z$  with the altitude coordinate  $z$  and crosswind coordinate  $y$ , the rate of decline of concentration  $q$ , with downwind distance,  $x^p$  may follow any power law between  $p \sim 1.5$  and  $p \sim 2.0$ . In the crosswind direction, a Gaussian distribution results. This general behavior of the distribution of concentration, at ground level, is well substantiated by observations in all kinds of flat lands, as reported in the literature. An empirical exponent which is favored is  $p \sim 1.75^*$ .

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\* Pasquill, F. Atmospheric Diffusion, London, Van Nostrand, Ltd., 1962. Page 83.