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DRL-376 CM-860

STUDIES ON LONGITUDINAL AND BENDING WAVES IN LONG ELASTIC RODS

by

H. J. Plass, Jr. and C. C. Steyer

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Contract NOrd-16498
Task UTX-1
Problem UTX-1-B-1

DRL-376, CM-837
24 April 1956

STUDIES ON LONGITUDINAL AND BENDING WAVES IN LONG ELASTIC RODS

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H. J. Plass, Jr. and C. C. Steyer

Defense Research Laboratory
The University of Texas

Austin, Texas

24 April 1956
HJP:CCS:bgh

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SYMBOLS FOR MAIN BODY OF REPORT

Section 2: Theoretical Studies -- Longitudinal Waves

$\bar{a}$ = radius of bar (inches)

B = constant

c_D = dilatational wave velocity $= \sqrt{\frac{(1-\nu)E}{(1-2\nu)(1+\nu)\rho}}$ (in./sec.)

c_S = shear wave velocity $= \sqrt{G/\rho}$ (in./sec)

c_o = velocity of long waves in a bar $= \sqrt{E/\rho}$

$\bar{c}_o$ $= \frac{c_D^2}{c_S^2} = \frac{2(1-\nu)}{1-2\nu}$

E = modulus of elasticity (lbs./in.^{2})

$E_{\bar{r}\bar{r}}$ = bar strain as defined

$E_{\bar{x}\bar{x}}$ = bar strain as defined

f = arbitrary function of $\bar{x}$ and $\bar{t}$

F = arbitrary function of $\bar{x}$ and $\bar{t}$

G = Lame's constant

$\bar{H}$ $= \frac{\nu K}{c_o}$

K = radius of gyration of the cross-sectional area of the bar

P = dimensionless normal axial force $= \frac{P_{\bar{x}}}{E\bar{a}^2}$

$\bar{P}(\bar{t})$ = force input function

$P_{\bar{r}}$ = radial "bar stress" in cylindrical coordinates

$P_{\bar{x}}$ = axial "bar stress" in cylindrical coordinates

P_θ = tangential "bar stress" in cylindrical coordinates

Q = dimensionless shear $= \frac{\bar{Q}}{G\bar{a}^2}$

$\bar{Q}$ = "bar stress" in cylindrical coordinates

$\bar{r}$ = cylindrical coordinate

R = dimensionless normal radial force = $\frac{P_{\bar{r}}}{E\bar{a}^2}$

t = dimensionless time = $\frac{c_o\bar{t}}{\bar{a}}$

$\bar{t}$ = time (sec.)

u = axial displacement (in.)

u_r = radial displacement in cylindrical coordinates

u_x = axial displacement in cylindrical coordinates

u_θ = tangential displacement in cylindrical coordinates

$\bar{U}$ = displacement function in cylindrical coordinates

V = dimensionless axial velocity of section = $\frac{V_o}{c_o}$

V_o = axial velocity (ft./sec.)

V_1 = radial velocity (ft./sec.)

W = dimensionless radial velocity of section = $\frac{V_1}{c_S}$

$\bar{W}$ = displacement function in cylindrical coordinates

x = dimensionless distance along axis = $\frac{\bar{x}}{\bar{a}}$

$\bar{x}$ = axial coordinate (in.)

$\gamma_{\bar{x}\bar{r}}$ = shear strain in cylindrical coordinates

$\Gamma_{\bar{x}\bar{r}}$ = "bar strain" as defined

$\epsilon_{\bar{r}\bar{r}}$ = radial strain in cylindrical coordinates

$\epsilon_{\bar{x}\bar{x}}$ = axial strain in cylindrical coordinates

θ = cylindrical coordinate

λ = Lamé's constant

ν = Poisson's ratio

ρ = density (lb.sec.^{2}/in.^{4})

$\bar{\rho}$ = variable of integration

$\sigma_{\bar{x}\bar{x}}$ = axial stress

$\bar{\tau}$ = duration of impact (sec.)

Section 3: Theoretical Studies -- Bending Waves

$a = \frac{2}{k(C - 1)}$

A = dimensionless cross-section area = $\frac{\text{area}(\text{in.}^2)}{h^2(\text{in.}^2)}$

$b = \frac{1}{k\sqrt{C}}$

$c_o = \sqrt{\frac{E}{\rho}}$ = bar velocity (in./sec.)

$c_Q = \sqrt{K' \quad \frac{G}{\rho}}$ = shear wave velocity

$C = \frac{c_o^2}{c_Q^2}$ = $\frac{\text{square of bending wave velocity}}{\text{square of shear wave velocity}}$

E = modulus of elasticity (lb./in.^{2})

G = shear modulus (lb./in.^{2})

h = diameter (in.)

H = constant

k = dimensionless section radius of gyration = $\frac{\text{radius of gyration(in.)}}{h(\text{in.})}$

K' = a correction factor = $\frac{\text{average shear stress in section}}{\text{maximum shear stress in section}}$

$\mathcal{M} = \sqrt{\frac{1 + C}{2}}$

M = dimensionless moment = $\frac{\text{moment (in. - lb.)}}{Eh^3(\text{in. - lb.})}$

$\mathcal{N} = \frac{C - 1}{C + 1}$

Q = dimensionless shear force = $\frac{\text{shear force (lb.)}}{Eh^2(\text{lb.})}$

r = integration variable

R = $\sqrt{r^2 + \eta^2(a^2 - r^2)}$

S = $\eta\sqrt{r}\left[\eta\sqrt{a^2 + r^2} - r\right]^{1/2}$

S_o = $(S)_{r = \frac{\pi}{\tau}}$

t = dimensionless time = $\dfrac{\text{time(sec.)}}{\frac{h}{c_o}\text{ (sec.)}}$

T = $\eta\sqrt{r}\left[\eta\sqrt{a^2 + r^2} + r\right]^{1/2}$

T_o = $(T)\left(r = \frac{\pi}{\tau}\right)$

v = dimensionless transverse velocity of element

= $\dfrac{\text{physical velocity (in./sec.)}}{c_o\text{(in./sec.)}}$

w = dimensionless transverse displacement = $\dfrac{\text{displacement(in.)}}{h\text{(in.)}}$

x = dimensionless axial coordinate = $\dfrac{\text{axial distance (in.)}}{h\text{(in.)}}$

ζ = $\sqrt{\dfrac{r(R + r)}{2}}$

ρ = density (lb. sec.^{2}/in.^{4})

τ = dimensionless duration of applied pulse

φ = variable of integration

ψ = $\sqrt{\dfrac{r(R - r)}{2}}$

ω = dimensionless angular velocity of section

= $\dfrac{\text{physical angular velocity (sec.}^{-1}\text{)}}{\left(\frac{c_o}{h}\right)\text{(sec.}^{-1}\text{)}}$

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STUDIES ON LONGITUDINAL AND BENDING WAVES IN LONG ELASTIC RODS

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1. General Introduction

The effect on structural elements of rapidly varying, suddenly applied disturbances has become more significant in recent engineering work. Such impacts produce traveling waves in the members upon which they act. The prediction of the stresses and displacements associated with these waves is too difficult a task if the theory used is the exact theory of elasticity. As a result of this, approximate theories have been sought and devised, which lead to results of sufficient accuracy as long as certain limitations exist. The simplest approximate traveling wave theory is that which describes the longitudinal waves in a long rod resulting from an axial disturbance applied to the end of the rod. It is assumed that the stress on a cross section of the rod is uniform and that the lateral motion, e.g., the expansion associated with compression, is negligible. It predicts that the time variation of stress at some station along the rod is the same as that which was applied to the end, except that it is delayed by the time required for the wave to travel from the end to that station. If the impacts supplied to the bar are sufficiently long in duration, and change gradually with time, then this approximate theory gives fairly accurate results. However, when the impact has short duration, or does not change gradually, the theory is no longer adequate. The reasons for its failure are that the quantities neglected are no longer negligible. There is considerable inertia associated with lateral motion, and there are shear strains arising in the neighborhood of the rapidly changing axial stress. Of course, there is an exact theory which can be used, but in most cases the mathematical complications in employing it for practical problems are prohibitive. Some compromises must be made. A similar set of circumstances exists for the flexural, or bending waves in rods. The simplest theory, known as the elementary bending wave theory, ignores rotatory inertia of longitudinal elements of the bar, and shear deformation. Exact elasticity theory leads to cumbersome mathematics. Again, compromises are in order.

These theories of intermediate difficulty have already been proposed in the literature. A theory for longitudinal waves in which a correction for lateral inertia is included, is found in Love's [Ref. (1)] treatise on elasticity. Timoshenko [Refs. (2)(3)] has published corrections for shear and rotatory inertia in the vibratory beam differential equation. Further longitudinal wave corrections are those given by Mindlin and Herrmann [Ref.(4)] and Bishop [Ref.(5)]. Though these theories have existed for some time, there has been very little activity in obtaining their solutions under sudden load conditions. Some efforts have been made by Davies [Ref.(6)] using the Love equation for longitudinal waves (in which an error has been discovered) and by Uflyand [Ref.(7)], Dengler and Goland [Ref.(8)],Leonard and Budiansky [Ref.(9)], Miklowitz [Ref.(10)] and Boley and Chao [Ref.(11)], using the Timoshenko equation. Unfortunately, errors have been pointed out in the papers of Uflyand and Dengler and Goland. In a recent paper by Goland, Wickersham and Dengler [Ref.(12)], this error is corrected.

It was felt that more work was needed along the line of obtaining solutions to these intermediate equations. It was also thought that more experimental results for waves in rods were needed. These two tasks were undertaken about a year ago by a small group at DRL working mostly on a part-time basis. The two avenues of approach were coordinated by seeking only those solutions whose conditions agree closely with those imposed experimentally, and by choosing experimental techniques which give rise to boundary and initial conditions easily approximated mathematically.

Ripperger [Ref.(13)] has investigated experimentally the propagation of longitudinal waves in elastic rods, and in a more recent report [Ref.(14)], has extended his experimental studies to bending waves. The present report covers primarily the theoretical phase of the project, although some experimental results are included for comparison purposes. Except for a relatively small proportion of desk computations, used as check values for electronic computer data, the bulk of computed results was done on the NORC computer at the U. S. Naval Proving Ground at Dahlgren, Virginia.

2. Theoretical Studies - Longitudinal Waves

a. Simple Theory

The simple theory [Ref.(15)] for the propagation of longitudinal waves in rods is derived by assuming that plane transverse sections of the rod remain plane during the passage of stress waves, and that the stress acts uniformly over each section. This simple theory results in the differential equation commonly called the wave equation:

$$\rho \frac{\partial^2 u}{\partial \bar{t}^2} = E \frac{\partial^2 u}{\partial \bar{x}^2} \qquad (2\text{-}1)$$

where

$\bar{x}$ = axial coordinate (in.)
$\bar{t}$ = time (sec.)
u = axial displacement (in.)
E = modulus of elasticity (lbs/in.^{2})
ρ = density (lb.sec.^{2}/in.^{4})

This equation shows that longitudinal waves are propagated along the bar with velocity $\sqrt{E/\rho}$. The solution of Eq. (2-1) may be written in the form:

$$u = f(c_o\bar{t} - \bar{x}) + F(c_o\bar{t} + \bar{x}) \qquad (2\text{-}2)$$

where

$$c_o = \sqrt{\frac{E}{\rho}}$$

and F and f are arbitrary functions depending on initial conditions. The function f corresponds to a wave traveling in the direction of increasing $\bar{x}$, while the function F corresponds to a wave traveling in the opposite direction.

Now the ratio between axial stress $\sigma_{\bar{x}\bar{x}}$ and the axial strain $\frac{\partial u}{\partial \bar{x}}$ in a transverse element is Young's modulus so that

$$\frac{\partial u}{\partial \bar{x}} = \frac{\sigma_{\bar{x}\bar{x}}}{E} .$$

By differentiating Eq.(2-2) with respect to $\bar{x}$ and with respect to $\bar{t}$ it can be shown that

$$\frac{\partial u}{\partial \bar{t}} = c_0 \frac{\partial u}{\partial \bar{x}} ,$$

so that

$$\sigma_{\bar{x}\bar{x}} = \rho c_0 \frac{\partial u}{\partial \bar{t}} \tag{2-3}$$

Thus, Eq. (2-3) shows that a linear relationship exists between the stress at any point and the particle velocity.

Since the velocity of propagation c_0 is independent of the frequency of the stress waves, a stress pulse, according to the above simple theory, will travel down the bar without changing form.

Experiments show, however, that the length of the pulse continually increases as it travels along the rod. The longitudinal expansion and contraction of a section of the rod will necessarily result in lateral contraction and expansion. This lateral motion will result in a non-uniform distribution of stress across the sections of the rod, and plane transverse sections will become distorted. The effect of this lateral motion in rods becomes of importance when the wave lengths are of the same order as the diameter of the rods.

Two theories have been developed which consider the effect of this lateral motion in rods. Love had developed a differential equation which considers the effect of radial inertia, while Mindlin and Hermann [Ref.(4)] developed a differential equation which includes both radial inertia and shear corrections. Experimental results that may be used as a basis for comparison are available in a report by Ripperger [Ref.(13)].

b. Love's Equation

Love [Ref.(1)] developed the following equation:

$$\rho\left(\frac{\partial^2 u}{\partial t^2} - \nu K^2 \frac{\partial^4 u}{\partial x^2 \partial t}\right) = E \frac{\partial^2 u}{\partial x^2}, \qquad (2-4)$$

where

ν = Poisson's ratio

K = radius of gyration of the cross-sectional area of the bar about the $\tilde{x}$-axis.

This was developed using the energy method. Davies [Ref.(6)] attempted to solve this equation using certain geometric boundary conditions. One of these boundary conditions is not compatible with this problem. Davies ignored the fact that certain boundary conditions, not necessarily permitting physical interpretation, also result from the use of the energy method. These boundary conditions must be used in connection with Eq. (2-4) in order to give compatible results. A detailed development of this is given in Appendix A. The Love equation was solved with these boundary conditions by the use of the Laplace transform method parallel to the method used by Davies.

The physical conditions consist of a semi-infinite bar struck at the end $\bar{x} = 0$ with a certain force $\bar{P}(\bar{t})$. The initial conditions for this case are,

$$u(\bar{x},0) = 0 \tag{2-5}$$

$$u_{\bar{t}}(\bar{x},0) = 0 \tag{2-6}$$

and the boundary conditions derived from energy considerations are,

$$\bar{H}^2 u_{\bar{x}\bar{t}\bar{t}}(0,\bar{t}) + u_{\bar{x}}(0,t) = \frac{\bar{P}(\bar{t})}{E} \tag{2-7}$$

$$\bar{H}^2 u_{\bar{x}\bar{t}\bar{t}}(\infty,\bar{t}) + u_{\bar{x}}(\infty,\bar{t}) = 0 \tag{2-8}$$

where

$$\bar{H} = \frac{\nu K}{c_o}$$

and the subscripts $\bar{x}$ and $\bar{t}$ denote differentiation with respect to that variable.

In order to provide a comparison with experimental results, the following assumption was made concerning the force input.

$$\bar{P}(\bar{t}) = \frac{B\bar{\tau}}{\pi} \sin \frac{\pi\bar{t}}{\bar{\tau}}$$

where

$\bar{\tau}$ = duration of impact

B = constant

The solution obtained by the use of the Laplace transform for this condition is:

$$u(\bar{x},\bar{t}) = \frac{Bc_o}{\bar{E}} \left\{ \frac{\sin\frac{\pi\bar{t}}{\bar{\tau}} \sin\frac{\pi\bar{x}}{c_o\,\bar{\tau}\sqrt{1-\bar{H}^2\,\pi^2/\bar{\tau}^2}}}{\frac{\pi^2}{\bar{\tau}^2}\; 1-\frac{\bar{H}^2\pi^2}{\bar{\tau}^2}} - \frac{2}{\pi}\int_0^{1/\bar{H}} \frac{\sin\bar{\rho}\bar{t}\,\cos\frac{\bar{\rho}\bar{x}}{c_o\sqrt{1-\bar{H}^2\bar{\rho}^2}}}{\bar{\rho}\sqrt{1-\bar{H}^2\bar{\rho}^2}\;\left(\frac{\pi^2}{\bar{\tau}^2}-\bar{\rho}^2\right)}\, d\bar{\rho} \right\} \begin{array}{l} t \le \bar{\tau} \\ \frac{\pi}{\bar{\tau}} < \frac{1}{\bar{H}} \end{array} \tag{2-9}$$

and

$$u(\bar{x},\bar{t}) = \frac{Bc_o}{E}\left\{ \frac{\sin\frac{\pi\bar{t}}{\bar{\tau}}\ \sin\frac{\pi\bar{x}}{c_o\bar{\tau}\sqrt{1-\bar{H}^2\pi^2/\bar{\tau}^2}}}{\frac{\pi^2}{\bar{\tau}^2}\sqrt{1-\frac{\bar{H}^2\tau^2}{\bar{\tau}^2}}} - \frac{2}{\pi}\int_0^{1/\bar{H}} \frac{\sin\bar{\rho}\bar{t}\ \cos\frac{\bar{\rho}\bar{x}}{c_o\sqrt{1-\bar{H}^2\bar{\rho}^2}}}{\bar{\rho}\sqrt{1-\bar{H}^2\bar{\rho}^2}\left(\frac{\pi^2}{\bar{\tau}^2}-\bar{\rho}^2\right)}\,d\bar{\rho} \right. \tag{2-10}$$

$$\left. + \frac{\sin\frac{\pi(\bar{t}-\bar{\tau})}{\bar{\tau}}\ \sin\frac{\pi\bar{x}}{c_o\bar{\tau}\sqrt{1-\bar{H}^2\pi^2/\bar{\tau}^2}}}{\frac{\pi^2}{\bar{\tau}^2}\sqrt{1-\bar{H}^2\frac{\pi^2}{\bar{\tau}^2}}} - \frac{2}{\pi}\int_0^{1/\bar{H}} \frac{\sin\bar{\rho}(\bar{t}-\bar{\tau})\ \cos\frac{\bar{\rho}\bar{x}}{c_o\sqrt{1-\bar{H}^2\bar{\rho}^2}}}{\bar{\rho}\sqrt{1-\bar{H}^2\bar{\rho}^2}\left(\frac{\pi^2}{\bar{\tau}^2}-\bar{\rho}^2\right)}\,d\bar{\rho} \right\} \begin{array}{l} \bar{t} \geq \bar{\tau} \\ \frac{\pi}{\bar{\tau}} \leq \frac{1}{\bar{H}} \end{array}$$

In order to obtain results compatible with the boundary conditions the expression $\bar{H}^2 u_{\bar{x}\bar{t}\bar{t}} + u_{\bar{x}}$ was used to represent the strain in the axial direction. This expression is:

$$\bar{H}^2 u_{\bar{x}\bar{t}\bar{t}}(\bar{x},\bar{t}) + u_x(\bar{x},\bar{t}) = \frac{2B}{\pi\bar{E}} \int_0^{1/\bar{H}} \frac{\sin \dfrac{\bar{\rho}\bar{x}}{c_0\sqrt{1 - \bar{H}^2\bar{\rho}^{2}}}}{\dfrac{\pi^2}{\bar{\tau}^2} - \bar{\rho}^2} [\sin\bar{\rho}\bar{t} + \sin \rho(\bar{t} - \bar{\tau})]\, d\bar{\rho} \qquad (2\text{-}11)$$

$$t > \bar{\tau}$$

$$\frac{\pi}{\bar{\tau}} < \frac{1}{\bar{H}}$$

Since the evaluation of this integral is very laborious, computations were made for only one particular case, namely a sinusoidal force input with a duration of 4.4. x 10^{-6} seconds acting on a rod 1/8" in diameter. Simpson's rule was used to evaluate the integral of Eq. (2-11). It is believed that a sufficient number of intervals was used to obtain results of sufficient accuracy to permit comparison with the experimental results obtained by Ripperger. These comparisons are shown in Fig. 2-1.

c. Mindlin-Herrmann Theory

Pochhammer [Ref.(16)] developed an exact solution of the equations of elasticity for waves in a semi-infinite bar of uniform circular cross-section. However, the complexity of the Pochhammer solution reveals the impracticability of attempting to employ the general equations of elasticity in attacking problems of vibrations in bars and rods. The equation developed by Love included a correction for radial inertia but no correction for shear. However, the effect of shear may be equally as important as the effect of radial inertia. Mindlin and Herrmann took both of these contributions into consideration by making certain displacement assumptions and then using Hamilton's principle.

In their paper, Mindlin and Herrmann make the following assumptions concerning displacements using cylindrical coordinates $(\bar{r}, \theta, \bar{x}, \bar{t})$ for a circular rod of radius $\bar{a}$:

$$u_{\bar{r}} = \frac{\bar{r}}{\bar{a}}\,\bar{U}(\bar{x},\bar{t}) \qquad u_{\theta} = 0 \qquad u_{\bar{x}} = \bar{W}(\bar{x},\bar{t})$$

Four "bar stress components" are defined as follows:

$$P_{\bar{r}} = \int_0^{\bar{a}} \sigma_{\bar{r}\bar{r}}\,\bar{r}d\bar{r} \qquad P_{\theta} = \int_0^{\bar{a}} \sigma_{\theta\theta}\,\bar{r}d\bar{r}$$

$$P_{\bar{x}} = \int_0^{\bar{a}} \sigma_{\bar{x}\bar{x}}\,\bar{r}d\bar{r} \qquad \bar{Q} = \int_0^{\bar{a}} \frac{\sigma_{\bar{x}\bar{r}}}{\bar{a}}\,\bar{r}^2 d\bar{r}$$

Assuming surface traction on the cylinder to be zero, then the use of Hamilton's principle gives the following differential equation:

$$\frac{\partial\bar{Q}}{\partial\bar{x}} - \frac{P_{\bar{r}} + P_{\theta}}{\bar{a}} = \frac{\rho\bar{a}^2}{4}\,\frac{\partial^2 U}{\partial\bar{t}^2} \tag{2-12}$$

$$\frac{\partial P_{\bar{x}}}{\partial\bar{x}} = \frac{\rho\bar{a}^2}{2}\,\frac{\partial^2\bar{W}}{\partial\bar{t}^2} \tag{2-13}$$

In this particular case the strains in terms of the stresses become,

$$\epsilon_{\bar{r}\bar{r}} = \frac{\lambda + 2G}{2G(3\lambda + 2G)}\,\sigma_{\bar{r}\bar{r}} - \frac{\lambda}{2G(3\lambda + 2G)}\,\sigma_{\bar{r}\bar{r}} \tag{2-14}$$

$$\epsilon_{\bar{x}\bar{x}} = \frac{\lambda + G}{G(3\lambda + 2G)}\,\sigma_{\bar{r}\bar{r}} - \frac{\lambda}{G(3\lambda + 2G)}\,\sigma_{\bar{r}\bar{r}} \tag{2-15}$$

$$\gamma_{\bar{x}\bar{r}} = \frac{\sigma_{\bar{x}\bar{r}}}{G} \tag{2-16}$$

where λ and G are Lamé's constants.

The "bar strains" are defined as follows and are expressed in terms of the displacements:

$$E_{\bar r \bar r} = \int_0^{\bar a} \bar r \; \epsilon_{\bar r \bar r} \; d\bar r = \frac{U\bar a}{2}$$

$$E_{\bar x \bar x} = \int_0^{\bar a} \bar r \; \epsilon_{\bar x \bar x} \; d\bar r = \frac{\bar a^2}{2} \; \frac{\partial \bar W}{\partial \bar x} \qquad (2\text{-}17)$$

$$\Gamma_{\bar x \bar r} = \int_0^{\bar a} \frac{G\bar r^2}{\bar a} \; \gamma_{\bar x \bar r} \; dr = \frac{G\bar a^2}{4} \; \frac{\partial \bar U}{\partial \bar x}$$

Multiplying Eqs. (2-14) and (2-15) by $\bar r d\bar r$ and integrating, differentiating Eqs. (2-17), and using the definitions for the "bar stresses and strains", the following results are obtained:

$$\frac{\bar a}{2} V_1 = \left[(1 - \nu) \frac{\partial P_{\bar r}}{\partial \bar t} - \frac{\partial P_{\bar x}}{\partial \bar t} \right] \qquad (2\text{-}18)$$

$$\frac{\bar a^2}{2} \; \frac{\partial V_o}{\partial \bar x} = \frac{1}{E} \left[\frac{\partial P_{\bar x}}{\partial \bar t} - 2 \; \frac{\partial P_{\bar r}}{\partial \bar t} \right] \qquad (2\text{-}19)$$

$$\frac{\partial \bar Q}{\partial \bar t} = \frac{G\bar a^2}{4} \; \frac{\partial V_1}{\partial \bar x} \qquad (2\text{-}20)$$

$$\frac{\partial \bar{Q}}{\partial \bar{x}} - 2\,\frac{P_{\bar{r}}}{\bar{a}} = \frac{\rho \bar{a}}{4}\,\frac{\partial V_1}{\partial \bar{t}} \qquad (2\text{-}21)$$

$$\frac{\partial P_{\bar{x}}}{\partial \bar{x}} = \frac{\rho \bar{a}^2}{2}\,\frac{\partial V_o}{\partial \bar{t}} \qquad (2\text{-}22)$$

where

$$V_1 = \frac{\partial U}{\partial \bar{t}}, \quad V_o = \frac{\partial \bar{W}}{\partial \bar{t}}$$

These five equations are put into dimensionless form by the following substitutions:

P = dimensionless normal axial force $= \dfrac{\text{physical force (lb)}}{E\bar{a}^2\text{(lb)}} = \dfrac{P_{\bar{x}}}{E\bar{a}^2}$

R = dimensionless normal radial force $= \dfrac{\text{physical force (lb)}}{E\bar{a}^2\text{(lb)}} = \dfrac{P_{\bar{r}}}{E\bar{a}^2}$

Q = dimensionless shear $= \dfrac{\text{physical shear (lb)}}{G\bar{a}^2\text{(lb)}} = \dfrac{\bar{Q}}{G\bar{a}^2}$

V = dimensionless axial velocity of section $= \dfrac{\text{physical velocity(in/sec)}}{c_D\text{(in/sec)}} = \dfrac{V_o}{c_D}$

W = dimensionless radial velocity of section $= \dfrac{\text{physical velocity(in/sec)}}{c_S\text{(in/sec)}} = \dfrac{V_1}{c_S}$

x = dimensionless distance along axis $= \dfrac{\text{physical distance(in.)}}{\bar{a}\text{(in.)}} = \dfrac{\bar{x}}{\bar{a}}$

t = dimensionless time $= \dfrac{\text{physical time(sec)}}{\bar{a}/c_D\text{ (sec)}} = \dfrac{c_o\bar{t}}{\bar{a}}$

where

$$c_D = \sqrt{\left[\frac{1 - \nu}{(1 - 2\nu)(1 + \nu)}\right] \left[\frac{E}{\rho}\right]} = \text{dilatation wave velocity (in/sec)}$$

$$c_S = \sqrt{\frac{G}{\rho}} = \text{shear wave velocity (in/sec)}$$

also we let

$$\bar{c}_o = \frac{c_D^2}{c_S^2} = \frac{2(1 - \nu)}{(1 - 2\nu)}$$

Then Eqs. (2-18), (2-19), (2-20), (2-21), and (2-22) become respectively:

$$W - 2(1 - \nu)\sqrt{\bar{c}_o}\,\frac{\partial R}{\partial t} + 2\nu\sqrt{\bar{c}_o}\,\frac{\partial P}{\partial t} = 0 \qquad (2\text{-}24)$$

$$\frac{\partial V}{\partial x} - 2\frac{\partial P}{\partial t} + 4\nu\frac{\partial R}{\partial t} = 0 \qquad (2\text{-}25)$$

$$\frac{\partial W}{\partial x} - 4\sqrt{\bar{c}_o}\,\frac{\partial a}{\partial t} = 0 \qquad (2\text{-}26)$$

$$\frac{\partial W}{\partial t} - \frac{4}{\bar{c}_o}\frac{\partial Q}{\partial x} + \frac{16}{\sqrt{\bar{c}_o}}(1 + \nu)\,R = 0 \qquad (2\text{-}27)$$

$$\frac{\partial V}{\partial t} - \frac{4(1 + \nu)}{\bar{c}_o}\frac{\partial P}{\partial x} = 0 \qquad (2\text{-}28)$$

The characteristics in this problem are found as shown in Appendix B-1. The characteristic curves that satisfy the differential equations are:

$$dx = 0$$

$$\left.\begin{aligned} dt &= dx \\ dt &= -dx \end{aligned}\right\} \text{Dilatation}$$

$$\left.\begin{aligned} dt &= \sqrt{\bar{c}_o}\, dx \\ dt &= -\sqrt{\bar{c}_o}\, dx \end{aligned}\right\} \text{Shear}$$

The partial differential equations used in conjunction with the characteristic differential equation reduce to the following:

$$dP - \frac{1 - \nu}{\nu}\, dR + \frac{W}{2\sqrt{\bar{c}_o}}\, dt = 0 \qquad \text{along } dx = 0$$

$$dP - \frac{\bar{c}_o}{4(1 + \nu)}\, dV - \frac{\nu\sqrt{\bar{c}_o}\, W}{2(1 - \nu)(1 + \nu)}\, dt = 0 \qquad \text{along } dt = dx$$

$$dP + \frac{\bar{c}_o}{4(1 + \nu)}\, dV - \frac{\nu\sqrt{\bar{c}_o}\, W}{2(1 - \nu)(1 + \nu)}\, dt = 0 \qquad \text{along } dt = -dx$$

$$dQ - \frac{dW}{4} - \frac{4(1 - \nu)R}{\sqrt{\bar{c}_o}}\, dt = 0 \qquad \text{along } dt = \sqrt{\bar{c}_o}\, dx$$

$$dQ + \frac{dW}{4} + \frac{4(1 + \nu)R}{\sqrt{\bar{c}_o}}\, dt = 0 \qquad \text{along } dt = -\sqrt{\bar{c}_o}\, dx$$

These last five equations were converted to difference equations by the method shown in Appendix B-1. The NORC Digital Computer was not available to perform the vast amount of computations necessary for a complete study of this problem.

3. Theoretical Studies - Bending Waves

In the study of bending waves, two theories have been investigated. One theory is the so-called elementary theory, in which no corrections for shear deformation or rotatory inertia are included. The other is the Timoshenko theory, in which both of these corrections are included in a simple manner. For each of these theories a set of results is obtained for a semi-infinite bar whose end is either simply supported or unsupported.

a. Elementary Bending Theory

Consider an element of a long beam, as shown in Fig. 3-1. It is convenient to write the equations governing the motion of the beam in dimensionless form. The following dimensionless quantities are introduced:

x = dimensionless axial coordinate = $\frac{\text{axial distance (in.)}}{h\text{(in.)}}$
where h = diameter (in.)

w = dimensionless transverse displacement = $\frac{\text{displacement(in.)}}{h\text{(in.)}}$

t = dimensionless time = $\frac{\text{time (sec.)}}{h/c_o \text{ (sec.)}}$

Q = dimensionless shear force = $\frac{\text{shear force (lb.)}}{Eh^2 \text{ (lb)}}$

M = dimensionless moment = $\frac{\text{moment (in. - lb)}}{Eh^3 \text{ (in. - lb)}}$

A = dimensionless cross section area = $\frac{\text{area (in.}^2\text{)}}{h^2 \text{ (in.}^2\text{)}}$

k = dimensionless section radius of gyration = $\frac{\text{radius of gyration(in.)}}{h\text{(in.)}}$

E = modulus of elasticity (lb/in.^{2})

$c_o = \sqrt{\frac{E}{\rho}}$ = bar velocity (in./sec.)

ρ = density (lb sec^2/in.^{4})

Using these dimensionless variables, the following relations for the motion of the beam are obtained, ignoring rotatory inertia and shear deformation.

$$Q = \frac{\partial M}{\partial x} \quad \text{(Equilibrium of moments)} \tag{3-1}$$

$$\frac{\partial Q}{\partial x} + A\frac{\partial^2 w}{\partial t^2} = 0 \quad \text{(Transverse motion)} \tag{3-2}$$

$$M = Ak^2 \frac{\partial^2 w}{\partial x^2} \quad \text{(Elasticity law)} \tag{3-3}$$

On combining these into one differential equation for $w(x,t)$ by eliminating M and Q, the following is obtained:

$$k^2 \frac{\partial^4 w}{\partial x^4} + \frac{\partial^2 w}{\partial t^2} = 0 \tag{3-4}$$

This is the dimensionless form of the equation most often used for the study of vibration of beams. In this report, Eq. (3-4) is solved for a few cases of a semi-infinite beam loaded at its end by means of a pure bending couple having a time variation in the form of a half sine wave. Solutions are obtained for both the simply supported and the unsupported end.

The details of the solution of Eq. (3-4) for both kinds of boundary conditions are given in Appendix C. The assumed moment variation is:

$$\begin{aligned} M(0,t) &= H \sin \frac{\pi t}{\tau}, \quad (t \geq 0) \\ &= 0, \qquad\qquad (t > 0) \end{aligned} \tag{3-5}$$

If the impact end is simply supported, or pinned, the following expression for M(x,t) is obtained:

$$M(x,t) = H \left\{ -\frac{\tau}{\pi^2} \int_0^\infty e^{-\varphi t} \frac{\varphi^2}{\varphi^2 + \frac{\pi^2}{2}} \sin\left(x\sqrt{\frac{\varphi}{2k}}\right) \cosh\left(x\sqrt{\frac{\varphi}{2k}}\right) d\varphi \right.$$

$$+ \frac{1}{2} e^{-x\sqrt{\frac{\pi}{k}}} \sin\frac{\pi t}{\tau} + \frac{1}{2} \sin\left(\frac{\pi t}{\tau} - x\sqrt{\frac{\pi}{k\tau}}\right)$$

$$\left. + \frac{\pi t}{2\sqrt{k}\,(\pi t)^{3/2}} \cos\left(\frac{x^2}{4kt} + \frac{\pi}{4}\right) \right\} \qquad (3\text{-}6)$$

It is possible to obtain the value of the integral in Eq. (3-6) in the form of an infinite series. This is shown in Appendix C. If the end is unsupported, the following expression for M(x,t) is obtained:

$$M(x,t) = H \left\{ \frac{1}{\tau} \int_0^\infty \frac{e^{-\varphi t}}{\varphi^2 + \frac{\pi^2}{\tau^2}} \sinh\left(x\sqrt{\frac{\varphi}{2k}}\right) \left[\sin\left(x\sqrt{\frac{\varphi}{2k}}\right) \right. \right.$$

$$\left. - \cos\left(x\sqrt{\frac{\varphi}{2k}}\right) \right] d\varphi$$

$$+ \frac{1}{\sqrt{2}} \left[\cos\left(x\sqrt{\frac{\pi}{k}}\right) \sin\left(\frac{\pi t}{\tau} + \frac{\pi}{4}\right) \right. \qquad (3\text{-}7)$$

$$\left. + \sin\left(\frac{\pi t}{\tau} - \frac{\pi}{4}\right) e^{-x\sqrt{\frac{\pi}{k\tau}}} + \sin x\sqrt{\frac{\pi}{k\tau}} \right]$$

As is done for Eq.(3-6), the integral in Eq. (3-7) is evaluated by means of series expansions. These details are included in Appendix C.

Computations have been made, using Eqs. (3-6) and (3-7) for M vs t with $x = 2$, and $x = 4$. The results of these calculations are presented in Fig. 3-5(b) and (d), labeled "Elementary Theory".

b. Timoshenko Bending Theory

Consider a beam element similar to that used in developing the partial differential equations of the previous article. Whereas in the previous derivation, shear deformation and rotatory inertia are ignored, they are now considered in a simple manner. Fig. 3-2 shows an element having shear deformation. That is, the angle ψ and the slope $\frac{\partial w}{\partial x}$ are not equal. In addition to those dimensionless variables introduced in Article 3a, the following will also be used:

$$\omega = \text{dimensionless angular velocity of section}$$

$$= \frac{\text{physical angular velocity } (\text{sec}^{-1})}{(\frac{c_o}{h})\ (\text{sec}^{-1})}$$

$$v = \text{dimensionless transverse velocity of element}$$

$$= \frac{\text{physical velocity (in/sec)}}{c_o(\text{in/sec})}$$

$$C = \frac{c_o^2}{c_Q^2} = \frac{\text{square of bending wave velocity}}{\text{square of shear wave velocity}}$$

where

$$c_o = \sqrt{\frac{E}{\rho}} = \text{bending wave velocity}$$

$$c_o = \sqrt{K' \frac{G}{\rho}} = \text{shear wave velocity}$$

and as before,

E = modulus of elasticity (lb/in^2)
ρ = mass density (lb sec^2/in^4)

and

G = shear modulus (lb/in^2)
K' = a correction factor = $\dfrac{\text{average shear stress in section}}{\text{maximum shear stress in section}}$

The equation governing the motion and stresses in the beam are as follows:

$$\frac{\partial M}{\partial x} - Q = Ak^2 \frac{\partial \omega}{\partial t} \quad \text{(Rotatory motion)} \qquad (3\text{-}8)$$

$$\frac{\partial Q}{\partial x} = A \frac{\partial v}{\partial t} \quad \text{(Transverse motion)} \qquad (3\text{-}9)$$

$$\frac{\partial M}{\partial t} = Ak^2 \frac{\partial \omega}{\partial x} \quad \text{(Time derivative of moment-curvature relation)} \qquad (3\text{-}10)$$

$$C \frac{\partial Q}{\partial t} = A\left(\frac{\partial v}{\partial x} + \omega\right) \quad \text{(Time derivative of shear deformation relation)} \qquad (3\text{-}11)$$

When all but one variable, say M, are eliminated from this set of four first-order equations, a single fourth-order equation, known as the Timoshenko equation, is obtained. It is:

$$k^2 \frac{\partial^4 M}{\partial x^4} + \frac{\partial^2 M}{\partial t^2} - k^2(1 + C) \frac{\partial^4 M}{\partial x^2 \partial t} + k^2 C \frac{\partial^4 M}{\partial t^4} = 0 \qquad (3\text{-}12)$$

In this report, solutions to Eqs. (3-8), (3-9). (3-10), and (3-11) are found for a variety of boundary conditions involving moment, angular velocity, shear force, and transverse velocity. In each case, the time variation assumed is sinusoidal and similar to that given for $M(0,t)$ in Eq. (3-5). For the moment and angular velocity impacts, two boundary conditions are assumed for the loaded end, namely simply supported and unsupported. For the shear force and transverse velocity impacts the end is assumed unsupported and free of any moment. In nearly all cases the solutions were obtained by means of the method of characteristics, the details of which are given in Appendix E. For one case only the solution was obtained by Laplace transform for check purposes. Except for a few results obtained on desk machines, computations using the method of characteristics were done on the NORC (Naval Ordance Research Computer) at the U. S. Naval Proving Ground in Dahlgren, Virginia.

The solution obtained by Laplace transform (see Appendix D for details) for the case of the semi-infinite bar with its end simply supported and subjected to a suddenly applied sinusoidal moment

$$\left[M(0,t) = 0 \text{ for } t \leq 0; \;\; H \sin\frac{\pi t}{\tau} \text{ for } t \geq 0\right]$$ is as follows:

for $t \leq \sqrt{C}\,x$,

$$M(x,t) = \frac{H}{2}\left[1 - \frac{\frac{\pi}{\tau}}{\sqrt{a^2 + (\frac{\pi}{\tau})^2}}\right]\left[-\sin T_o x \cos\frac{\pi t}{\tau} + \cos T_o x \sin\frac{\pi t}{\tau}\right]$$

$$+ \frac{H}{2}\left[1 + \frac{\frac{\pi}{\tau}}{a^2 + (\frac{\pi}{\tau})^2}\right]\left[-\sin S_o x \cos\frac{\pi t}{\tau} + \cos S_o x \sin\frac{\pi t}{\tau}\right]$$

$$- \frac{H}{\tau}\int_0^b \left[1 + \frac{r}{\sqrt{a^2 + r^2}}\right]\frac{1}{(\frac{\pi}{\tau})^2 - r^2}\cos rt \sinh Sx \, dr, \qquad (3\text{-}13)$$

if

$$\frac{\pi}{\tau} \geq b$$

For $x \le \sqrt{C}\; x$

$$M(x,t) = \frac{H}{2}\left[1 + \frac{\frac{\pi}{\tau}}{\sqrt{a^2 + (\frac{\pi}{\tau})^2}}\right]\left[- \sin S_o x \cos \frac{\pi t}{\tau} + \cos S_o x \sin \frac{\pi t}{\tau}\right]$$

$$- \frac{H}{\tau}\int_0^a \frac{1}{r^2 + (\frac{\pi}{\tau})^2}\left[\frac{r}{\sqrt{a^2 - r^2}} \cos M\psi x \sinh (M\zeta x - rt) + \sin M\psi x \cosh (M\zeta x - rt)\right] dr \qquad (3\text{-}14)$$

$$- \frac{H}{\tau}\int_0^b \left[1 + \frac{r}{a^2 + r^2}\right]\frac{1}{(\frac{\pi}{\tau})^2 - r^2} \cos \; rt \sinh Sx \, dr$$

if

$$\frac{\pi}{\tau} > b$$

also for $t \ge \sqrt{C}\; x$

$$M(x,t) = \frac{H}{2}\left[1 - \frac{\frac{\pi}{\tau}}{\sqrt{a^2 + (\frac{\pi}{\tau})^2}}\right] - \sin T_o x \cos \frac{\pi t}{\tau} + \cos T_o x \sin \frac{\pi t}{\tau}\Big]$$

$$+ \frac{H}{2}\left[1 + \frac{\frac{\pi}{\tau}}{\sqrt{a^2 + (\frac{\pi}{\tau})^2}}\right] \cosh S_o x \sin \frac{\pi t}{\tau} \qquad (3\text{-}15)$$

$$- \frac{H}{\tau}\int_0^b \left[1 + \frac{}{\sqrt{a^2 + r^2}}\right]\frac{1}{(\frac{\pi}{\tau})^2 - r^2} \cos rt \sinh Sx \, dr$$

if

$$\frac{\pi}{\tau} < b$$

For $x \leq t \leq \sqrt{C}\, x$

$$M(x,t) = \frac{H}{2}\left[1 + \frac{\frac{\pi}{\tau}}{\sqrt{a^2 + (\frac{\pi}{\tau})^2}}\right] \cosh S_o x \sin \frac{\pi t}{\tau}$$

$$- \frac{H}{\tau}\int_0^a \frac{1}{r^2 + (\frac{\pi}{\tau})^2}\left[\frac{r}{\sqrt{a^2 - r^2}} \cos \mathcal{M}\psi x \sinh (\mathcal{M}\zeta x - rt)\right.$$

$$\left. + \sin \mathcal{M}\psi x \cosh (\mathcal{M}\zeta x - rt)\right] dr \qquad (3-\text{[illegible]})$$

$$- \frac{H}{\tau}\int_0^b \left[1 + \frac{r}{\sqrt{a^2 - r^2}}\right] \frac{1}{(\frac{\pi}{\tau})^2 - r} \cosh rt \sinh Sx\, dr$$

if $\frac{\pi}{\tau} < b$

where

$$\mathcal{M} = \sqrt{\frac{1 + C}{2}}, \quad \mathcal{N} = \frac{C - 1}{C + 1}, \quad a = \frac{2}{k(C - 1)}, \quad b = \frac{1}{k\sqrt{C}}$$

$$R = \sqrt{r^2 + \mathcal{N}^2(a^2 - r^2)}$$

$$\zeta = \sqrt{\frac{r(R + r)}{2}}, \quad \psi = \sqrt{\frac{r(R - r)}{2}}$$

$$S = \mathcal{M}\sqrt{r}\left[\mathcal{N}\sqrt{a^2 + r^2} - r\right]^{1/2}$$

$$T = \mathcal{M}\sqrt{r}\left[\mathcal{N}\sqrt{a^2 + r^2} + r\right]^{1/2}$$

$$S_o = (S)_{r = \frac{\pi}{\tau}}$$

$$T_o = (T)_{r = \frac{\pi}{\tau}}$$

A comparison between the Laplace transform solution and the solution obtained by characteristics is given in Fig. 3-3. It is evident from this figure that the solution by characteristics is free from major errors.

By means of superposition, the solutions for a single half sine wave input can be found. This has been done in every case studied. The results for a variety of impacts and boundary conditions are presented in Figs. 3-4, 3-5, . . ., 3-11. In some of the above mentioned figures comparisons are made between the computed (Timoshenko theory) and the experimental results for those cases where the assumed conditions are most nearly like those of the experiment. Perfect agreement does not occur, and ought not be expected, since the actual time variation of input moment is not exactly in the form of a half-sine wave, as is assumed for computational purposes. Also, it should be pointed out that the Timoshenko theory is, of course, an approximate theory. The determination of the degree of closeness to reality of the predictions of the approximate theories in question is one of the purposes of writing this report. Comments concerning the general conclusions to be drawn from these comparisons are given in Section 4.

4. General Conclusions

The conclusions to be drawn from this study are necessarily limited to bending waves, as too few computed results for longitudinal waves are available. The findings concerning bending waves are arranged as follows!

a. Comparisons between theoretical results, both from elementary theory and the Timoshenko Equation, with experimental results.
b. Comparison of theoretical results (Timoshenko Equation) for same type of impact, but different method of support.
c. Comparison of theoretical results (Timoshenko Equation) for similar types of support, but varied impact conditions.
d. Comparison of theoretical results (Timoshenko Equation) for similar impact and support conditions, but different material properties.

a. Comparison of Theory and Experiment

The experimental results of Ripperger [Ref. (14).] for moment impacts having timewise variations very nearly in the form of a half sine wave are selected for comparison with theoretical results. The moments in every case were generated by means of the eccentric impact of a small steel ball on the end of the rod. The longitudinal wave, also generated by such an impact, was eliminated from the strain measurements by properly connecting diametrically opposed strain gages. For such a connection only the bending wave strains are recorded. In Fig. 3-5(a), (b), (e), and (f), are shown some strain records converted to dimensionless moment records by means of multiplication by proper constants. It is seen upon examining these figures that, except for the shortest pulse, the agreement between the theoretical (Timoshenko Equation) results and the experimental results is quite good. The theoretical results are those for $C = 4$, that is, for the bending velocity c_o equal to twice the shear velocity c_Q. This is the nearest value to that for circular steel rods, according to Timoshenko's assumptions. Hence, the bulk of the calculations were made with $C = 4$. A reduction in the value of C, with fixed $c_o = \frac{E}{\rho}$, is thus equivalent to an increase in the shear velocity c_Q. Such a change would result in somewhat the same form for the moment-time curves at the various stations but a more rapid propagation of that portion of the energy which is primarily in the shear mode of transmission. The experimental data agree, as far as time relationship is concerned, most closely with the theoretical results for $C = 4$. On comparing the curves of Fig. 3-5(b), ($C = 4$, $\tau = 2.4$) with those of Fig. 3-10(a), ($C = 3$, $\tau = 2.4$), it is seen that the large peaks for $C = 3$ occur earlier in time than those for $C = 4$, as anticipated. The fact that experimental results are predicted as accurately as they are by the Timoshenko theory for $C = 4$, is encouraging. Because of the frequently good agreement, it can be said with a fair degree of confidence that the results presented in the other figures in which experimental data are not presented are reliable. For impact loads, the Timoshenko theory is quite satisfactory; however, in no case was it found that the elementary theory predicts results in agreement with experiment, except for a general agreement in amplitude.

b. Comparison of Theoretical Results (Timoshenko Equation) for Similar Impacts, but Different Methods of Support

The moment-time relations at various stations resulting from half-sine moment impacts are shown in Fig. 3-4, (a) through (c), and Fig. 3-5, (a) through (f). The curves for Fig. 3-4 are those resulting from assuming the end sumply supported, while those of Fig. 3-5 are for an unsupported end. Certain similarities as well as differences are noted in comparing figures for equal duration impacts [e.g., Figs. 3-4(a) and 3-5(b)]. When the impact end is simply supported the moments do not build up anywhere to values as large as those occurring when the impact end is unsupported. During the earliest stages of the development of the M - t wave there are, however, only small differences. It is also noticed that for the simply supported impact end the largest peaks occur later in time than for the unsupported impact end. This seems to indicate that large shear waves are formed more slowly when the end is supported.

c. Comparison of Theoretical Results (Timoshenko Equation)for Similar Types of Impact, but Different Methods of Support

On comparing Fig. 3-5 with Fig. 3-9, for unsupported beams with moment and angular velocity half sine impacts, it is seen that the main difference between the M - t trace at the station under study is in the amount of spreading out of the wave. For moment impacts the M - t curve is less dispersed than for angular velocity impacts. It should also be pointed out that the two sets of curves have opposite polarity, since the definition of positive ω and positive M are opposite for the section on which they are assumed to act.

Similar comparisons for Fig. 3-6 and 3-7, for unsupported beams with end shear force and end transverse half sine impacts, yield similar results. The transverse velocity impact in this case is the less dispersed of the two.

For simply supported bars, moment and angular velocity impacts are compared in Fig. 3-4 and 3-8. Similar conclusions are obtained to those for unsupported bars. The dispersion for moment impacts is not as great as that for angular velocity impacts.

No attempt is made here to explain on physical grounds, in terms of shear and bending wave velocities, the differences noted between the M - t curves under closely related end impacts.

d. Comparison of Theoretical Results (Timoshenko Equation) for Similar Impacts and different Material Properties

Comparisons are made between the curves of Figs. 3-5 and 3-11 for unsupported beams with equal duration half sine moment impacts. In one case $C = 4$, and in the other $C = 3$, where $C = \frac{c_o^2}{c_Q^2}$, the square of the ratio of velocities of bending waves to shear waves. In defining dimensionless quantities, c_o was used a reference. Therefore a change in the value of C from 4 to 3 corresponds to an increase of the shear velocity c_Q. This change could be brought about by a change in section shape or a change in material. The earliest arrivals of the M - t waves have very little difference in shape for these two cases. However, the bulk of the pulse for $C = 3$ arrives earlier than the bulk for $C = 4$, indicating that shear type waves, traveling at the lower velocity c_Q, are primarily responsible for the large part of the M - t waves. It can also be inferred that the early arrival waves are transmitted at the bending velocity $c_o = \frac{E}{\rho}$. There are not available any experimental results reliable enough in the early region to be compared with these conclusions. There is some doubt about the speed of propagation c_o. Certain conclusions based on exact theory lead to a conclusion that this speed should be the dilatational velocity, c_D, somewhat larger than c_o.

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APPENDIX A

SOLUTION OF LOVE'S EQUATION

A-1. Love [Ref. (1)] used Hamilton's principle to derive a wave equation which takes into consideration the effect of the lateral inertia of the bar. This equation is:

$$u_{\bar{t}\bar{t}} - c_o^2 \left(u_{\bar{x}\bar{x}} + \bar{H}^2 u_{\bar{x}\bar{x}\bar{t}\bar{t}}\right) = 0 \qquad \text{(A-1)}$$

where $c_o^2 = \frac{E}{\rho}$, $\bar{H} = \frac{\nu k}{c_o}$ and the subscripts $\bar{x}$ and $\bar{t}$ denote differentiation with respect to that variable.

Davies [Ref. (6)] attempted to obtain a solution of this equation by the Laplace transform method for a bar of finite length (ℓ). Davies assumed the following boundary conditions:

$$u(\bar{x},0) = 0 \qquad u_{\bar{x}}(0,\bar{t}) = 0$$

$$u_{\bar{t}}(\bar{x},0) = 0 \qquad u_{\bar{x}}(\ell,\bar{t}) = \frac{\bar{P}}{E}$$

The solution as given by Davies is:

$$u = \frac{\bar{\bar{P}}\bar{\bar{t}}^2}{2\rho\ell} + \frac{\bar{\bar{P}}\bar{\bar{H}}^2}{\rho\ell} + \frac{2P\ell}{\pi^2 c_o^2 \rho} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \left(\cos\frac{n\pi\bar{x}}{\ell}\right) \left(1 - \frac{1}{1+n^2\psi^2} \cos\frac{n\pi c_o \bar{t}}{\ell\sqrt{1+n^2\psi^2}}\right.$$

where (A-2)

$$\psi = \frac{\pi\bar{H}c_o}{\ell}$$

An inspection of Eq. (A-2) shows that the initial condition $u(\bar{x},0) = 0$, assumed to be true in arriving at the solution, is really not satisfied. This indicates a fundamental error in the original formulation of the problem. That this condition cannot be satisfied is also demonstrated by applying a theorem from Laplace transform theory [Ref. (17)]. This theorem states that $\lim_{s\to\infty} s\bar{u}(\bar{x},s) = 0$ for the initial condition $\lim_{t\to 0} u(\bar{x},\bar{t}) = 0$ to be satisfied, where $\bar{u}$ and $\bar{s}$ denote the variables in the transform space.

Davies has for $\bar{u}$,

$$\bar{u} = \frac{Pc_o\sqrt{1+\bar{H}^2s^2}}{Es^2}\;\frac{\cosh\dfrac{s\bar{x}}{c_o\sqrt{1+\bar{H}^2s^2}}}{\sinh\dfrac{s\ell}{c_o\sqrt{1+\bar{H}^2s^2}}} \qquad \text{(A-3)}$$

so that

$$\lim_{s\to\infty} s\bar{u}(\bar{x},s) = \lim_{s\to\infty} \frac{Pc_o\sqrt{1+\bar{H}^2s^2}}{Es}\;\frac{\cosh\dfrac{s\bar{x}}{c_o\sqrt{1+\bar{H}^2s^2}}}{\sinh\dfrac{s\ell}{c_o\sqrt{1+\bar{H}^2s^2}}} \neq 0$$

This shows that the initial condition $\lim_{t\to 0} u(\bar{x},\bar{t}) = 0$ cannot be satisfied.

The reason for this discrepancy is that the boundary conditions assumed in Davies' analysis are incompatible with the stress-displacement relations associated with Love's equation. An analysis based on Hamilton's principle now follows which corrects this error.

Using the equation given by Love [Ref.(1)] the following expression gives the total energy of the rod ($\bar{V}$).

$$\bar{V} = \int_0^{T_1}\int_0^{\ell} \left[\frac{1}{2}\rho\bar{A} \left\{ (u_{\bar{t}})^2 + \nu^2K^2(u_{\bar{x}\bar{t}})^2 \right\} - \frac{1}{2} E\bar{A} \; (u_{\bar{x}})^2 \right] d\bar{x}\,d\bar{t} + Z$$

where

Z = Energy due to the load at the end.

$\bar{A}$ = Cross-sectional area.

T_1 = Final time value.

Then using Hamilton's principle:

$$\delta\bar{V} = \int_0^{T_1}\int_0^{\ell} \left[\frac{\rho\bar{A}}{2} \left\{ 2u_{\bar{t}}\delta u_{\bar{t}} + 2\nu K^2 u_{\bar{x}}\delta u_{\bar{x}\bar{t}} \right\} - E\bar{A}u_{\bar{x}}\delta u_{\bar{x}} \right] d\bar{x}\,d\bar{t} + \delta Z = 0$$

Then integrating by parts:

$$\delta\bar{V} = \int_0^{T_1}\int_0^{\ell} (-\rho A u_{\bar{t}\bar{t}} + E\bar{A}u_{\bar{x}\bar{x}}) \; \delta u\,d\bar{x}\,d\bar{t} + \int_0^{T_1}\int_0^{\ell} \rho\bar{A}(\nu^2K^2u_{\bar{x}\bar{x}\bar{t}\bar{t}}\delta u)\; d\bar{x}\,d\bar{t}$$

$$- \int_0^{\ell} \rho\bar{A}\nu^2K^2u_{\bar{x}\bar{x}\bar{t}}\delta u \Big]_0^{T_1} d\bar{x} + \rho\bar{A}\nu^2K^2u_{\bar{x}\bar{t}}\;\delta u \Big]_0^{\ell}\Big]_0^{T_1}$$

$$- \int_0^{T_1} \rho\bar{A}\nu^2K^2u_{\bar{x}\bar{t}\bar{t}}\delta u \Big]_0^{\ell} d\bar{t} - \int_0^{T_1} E\bar{A}u_{\bar{x}}\delta u \Big]_0^{\ell} d\bar{t} + \delta Z$$

In varying the integral it is assumed that the displacement alone is subject to variation so that the variation is zero at $\bar{t} = 0$ and $\bar{t} = T_1$

Then

$$\delta\bar{V} = \int_0^{T_1}\int_0^{\ell} \Big[(-\rho\bar{A}u_{\bar{t}\bar{t}} + E\bar{A}u_{\bar{x}\bar{x}} + \rho\bar{A}v^2K^2u_{\bar{x}\bar{x}\bar{t}\bar{t}})\delta u\Big]\, d\bar{x}d\bar{t}$$

$$- \int_0^{T_1} \Big[\Big[(\rho\bar{A}v^2K^2u_{\bar{x}\bar{t}\bar{t}} + E\bar{A}u_{\bar{x}})\Big]_0^{\ell} \delta u\Big]\, d\bar{t} + \delta Z = 0$$

The equation given by Love is obtained by equating the coefficient of δu in the first integral to zero:

$$\rho(u_{\bar{t}\bar{t}} - v^2K^2u_{\bar{x}\bar{x}\bar{t}\bar{t}}) = Eu_{\bar{x}\bar{x}}$$

or

$$u_{\bar{t}\bar{t}} - c_o\,(u_{\bar{x}\bar{x}} + \bar{H}^2u_{\bar{x}\bar{x}\bar{t}\bar{t}}) = 0 \qquad (A\text{-}4)$$

The following related boundary conditions are obtained by equating the coefficient of δu in the **second** integral to zero:

$$\rho v^2K^2u_{\bar{x}\bar{t}\bar{t}}(0,\bar{t}) + Eu_{\bar{x}}(0,t) = 0 \qquad (A\text{-}5)$$

$$\rho v^2K^2u_{\bar{x}\bar{t}\bar{t}}(\ell,\bar{t}) + Eu_{\bar{x}}(\ell,t) = \bar{P}(\bar{t}) \qquad (A\text{-}6)$$

or

$$\bar{H}^2 u_{\bar{x}\bar{t}\bar{t}}(0,\bar{t}) + u_{\bar{x}}(0,\bar{t}) = 0 \quad \text{at } \bar{x} = 0 \tag{A-7}$$

$$\bar{H}^2 u_{\bar{x}\bar{t}\bar{t}}(\ell,\bar{t}) + u_{\bar{x}}(\ell,\bar{t}) = \frac{\bar{P}(t)}{E} \quad \text{at } \bar{x} = \ell \tag{A-8}$$

where $\bar{P}(t)$ is the unit stress at the impact end.

The initial conditions are

$$u(\bar{x},0) = 0 \tag{A-9}$$

$$u_{\bar{t}}(\bar{x},0) = 0 \tag{A-10}$$

The transform equation resulting from Eq. (A-1) is

$$s^2\,\bar{u}(\bar{x},s) - su(\bar{x},0) - u_{\bar{t}}(\bar{x},0) - c_o^2 \frac{\partial^2}{\partial \bar{x}^2}\left\{\bar{u}(\bar{x},s) + \bar{H}^2\left[s^2\,u(\bar{x},s) - su(\bar{x},0) - u_{\bar{t}}(\bar{x},0)\right]\right\} = 0 \tag{A-11}$$

The use of the initial conditions, Eqs. (A-9) and (A-10), in Eq. (A-11) results in the following expression:

$$\bar{u}_{\bar{x}\bar{x}} - \frac{s^2}{c_o^2(1 + \bar{H}^2 s^2)}\,\bar{u} = 0 \tag{A-12}$$

Considering $\bar{P}$ constant, the use of the boundary conditions, Eqs. (A-7) and (A-8), results in the following expressions:

$$u_{\bar{x}}(0,s) = 0 \tag{A-13}$$

$$u_{\bar{x}}(\ell,s) = \frac{\bar{P}}{Es(1 + \bar{H}^2 s^2)} \tag{A-14}$$

The solution of Eq. (A-12) is

$$\bar{u}(x,s) = B_1 \sinh \frac{s\bar{x}}{c_o\sqrt{1+\bar{H}^2 s^2}} + B_2 \cosh \frac{s\bar{x}}{c_o\sqrt{1+\bar{H}^2 s^2}}$$

From Eqs. (A-13) and (A-14) it follows that:

$$B_1 = 0$$

and

$$B_2 = \frac{\bar{P}c_o}{Es^2\sqrt{1+\bar{H}^2 s^2}\,\sinh \dfrac{s\ell}{c_o\sqrt{1+\bar{H}^2 s^2}}}$$

or the final result is:

$$\bar{u}(\bar{x},s) = \frac{\bar{P}c_o \cosh \dfrac{s\bar{x}}{c_o\sqrt{1+\bar{H}^2 s^2}}}{Es^2\sqrt{1+\bar{H}^2 s^2}\,\sinh \dfrac{s\ell}{c_o\sqrt{1+\bar{H}^2 s^2}}} \tag{A-15}$$

In this case it can be shown that the initial condition u(x,0) = 0 is satisfied, since:

$$\lim_{s\to\infty} s\bar{u}(\bar{x},s) = \lim_{s\to\infty} \frac{\bar{P}c_o \cosh \dfrac{s\bar{x}}{c_o\sqrt{1+\bar{H}^2 s^2}}}{Es\sqrt{1+\bar{H}^2 s^2}\,\sinh \dfrac{s\ell}{c_o\sqrt{1+\bar{H}^2 s^2}}} = 0$$

The inversion theorem is used to determine u:

$$u(\bar{x},\bar{t}) = \frac{\bar{P}c_o}{2\pi iE} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{\dfrac{\bar{\rho}\bar{x}}{c_o\sqrt{1 + \bar{H}^2\rho^2}}}{\bar{\rho}^2 \sqrt{1 + \bar{H}^2\bar{\rho}^2} \sinh \dfrac{\bar{\rho}\ell}{c_o\sqrt{1 + \bar{H}^2\bar{\rho}^2}}} d\bar{\rho} \qquad \text{(A-16)}$$

The integrand is a single valued function of s with a triple pole at s = 0 and simple poles at the points

$$\frac{\bar{\rho}\ell}{c_o \sqrt{1 + \bar{H}^2\bar{\rho}^2}} = in\pi$$

where n is a positive integer. In addition, as $n \to \infty$, $\rho \to \pm \frac{i}{\bar{H}}$: i.e., the set of simple poles has two limit points which are essential singularities of the integrand.

Considering a suitable closed circuit in the complex plane in the usual way, the value of the integral in Eq. (A-16) can be shown by Cauchy's theorem to be equal to $2\pi i$ times the sum of the residues of the integrand at its poles. By excluding the two limit points by small circles and proceeding to the limit when these circles are infinitely small, it can be proved that the limit points make no contribution to the integral.

The value of the residue at the triple pole s = 0 is:

$$\frac{c_o\bar{t}^2}{2} + \frac{\bar{x}^2}{2c_o\ell} - \frac{\ell}{6c_o}$$

and the residue at the poles $s = \frac{\pm i n\pi c_o}{\sqrt{\ell^2 + n^2\pi^2 c_o^2 \bar{H}^2}}$ is:

$$\frac{-2\ell \cos \frac{n\pi\bar{x}}{\ell}}{(-1)^n n^2\pi^2 c_o} \cos \frac{n\pi c_o \bar{t}}{\sqrt{\ell^2 + n^2\pi^2 c_o^2 \bar{H}^2}}$$

The following expression for u is then obtained from Eq. (A-16):

$$u(\bar{x},\bar{t}) = \frac{\bar{P}c_o}{E}\left\{\frac{c_o\bar{t}^2}{2} + \frac{\bar{x}^2}{2c_o\ell} - \frac{\ell}{6c_o} - \frac{2\ell}{c_o\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos\frac{n\pi\bar{x}}{\ell}\right.$$

$$\cos\frac{n\pi c_o\bar{t}}{\sqrt{\ell^2 + n^2\pi^2 c_o^2\bar{H}^2}} \qquad \text{(A-17)}$$

The expression under the $\sum$ is the same as that found by Davies except for the term, $\frac{1}{1+\frac{n^2\pi^2\bar{H}^2c_o^2}{\ell^2}}$, found in the solution of Davies.

Equation (A-17) can be used to obtain values of $\bar{u}$ and $\bar{u}_{\bar{x}}$. Because the series converges very slowly this process becomes, however, extremely laborious. Also the infinite velocities inherent in this solution cause an immediate reflection which contradicts the known physical phenomena.

A-2. It is believed that a solution for a semi-infinite bar might compare more closely with experimental results since this would eliminate the reflection of waves with infinite velocities. Also the solution might be more adaptable to quicker computations.

The following discussion will be an attempt to solve the Love equation for a particular force impact that approximates the impact in certain experimental work.

Again writing the Love equation,

$$u_{\bar t\bar t} - c_o^2 \left(u_{\bar x\bar x} + \bar H^2 u_{\bar x\bar x\bar t\bar t}\right) = 0 \tag{A-18}$$

The end $\bar x = 0$ will be chosen as the end at which the impact is applied. The force function that is applied is of the half-wave sinusoidal type and is described as follows:

$$\bar P(t) = \frac{B\bar\tau}{\pi} \sin \frac{\pi t}{\bar\tau} \qquad 0 \leq t \leq \bar\tau$$

$$\bar P(t) = 0 \qquad \begin{cases} \tau \leq t \\ t < 0 \end{cases} \tag{A-19}$$

where B is proportional to the impact stress, and $\bar\tau$ is the duration of the pulse.

The boundary conditions that result from Hamilton's principle are:

$$\bar H^2 u_{\bar x\bar t\bar t}(0,\bar t) + u_{\bar x}(0,\bar t) = \frac{\bar P(\bar t)}{E} \tag{A-20}$$

$$\bar H^2 u_{\bar x\bar t\bar t}(\infty, \bar t) + u_{\bar x}(\infty, \bar t) = 0 \tag{A-21}$$

The initial conditions are,

$$u(\bar x,0) = 0 \tag{A-22}$$

$$u_{\bar t}(\bar x,0) = 0 \tag{A-23}$$

The required solution is obtained by the use of the Laplace transform. As before, the symbols $\bar u$ and s will be used to denote the variables in the transform space. The use of initial conditions (A-22) and (A-23) in the transform of Eq. (A-18) lead to

$$\bar u_{\bar x \bar x} - \frac{s^2 \bar u}{c_o^2(1 + \bar H^2 s^2)} = 0 \qquad \text{(A-24)}$$

The solution of this equation is

$$\bar u(\bar x, s) = B_3 e^{\frac{s\bar x}{c_o\sqrt{1 + \bar H^2 s^2}}} + B_4 e^{-\frac{s\bar x}{c_o\sqrt{1 + \bar H^2 s^2}}}$$

Boundary condition (A-21) in the transform space becomes,

$$(1 + \bar H^2 s^2)\, u_{\bar x}(\infty, s) = 0$$

or

$$\bar u_{\bar x}(\infty, s) = 0 \qquad \text{(A-25)}$$

Boundary condition (A-20) in the transform space becomes:

$$(1 + \bar H^2 s^2) \quad \bar u_x(0, s) = \frac{B}{E(s^2 + \frac{\pi^2}{\bar T^2})}$$

or

$$\bar u_{\bar x}(0, s) = \frac{B}{E(1 + \bar H^2 s^2)(s^2 + \frac{\pi^2}{\bar T^2})} \qquad \text{(A-26)}$$

The use of conditions (A-25) and (A-26) result in the following values of B_3 and B_4.

$$B_3 = 0 \qquad B_4 = \frac{-Bc_0}{Es\sqrt{1 + \bar{H}^2 s^2}\;(s^2 + \frac{\pi^2}{\bar{\tau}^2})}$$

Then

$$u(\bar{x},s) = \frac{-Bc_0}{Es\sqrt{1 + \bar{H}^2 s^2}(s^2 + \frac{\pi^2}{\bar{\tau}^2})}\, e^{-\frac{s\bar{x}}{c_0\sqrt{1 + \bar{H}^2 s^2}}} \tag{A-27}$$

Since $\lim\limits_{s\to\infty} s\bar{u}(x,s) = 0$, then the initial condition $(\bar{x},\bar{t})_{t\to 0} = 0$ is satisfied.

Then the use of the inversion theorem gives the following formula for the longitudinal displacement.

$$u(\bar{x},\bar{t}) = \frac{-Bc_0}{2\pi Ei}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{\left(\bar{\rho}\bar{t} - \frac{\bar{\rho}\bar{x}}{c_0\sqrt{1 + \bar{H}^2\bar{\rho}^2}}\right)}}{\bar{\rho}\sqrt{1 + \bar{H}^2\bar{\rho}^2}(\bar{\rho}^2 + \frac{\pi^2}{\bar{\tau}^2})}\, d\bar{\rho} \tag{A-28}$$

The integrand is a multivalued function of $\bar{\rho}$ with poles at $\bar{\rho} = 0$, $\bar{\rho} = \pm \frac{i\pi}{\bar{\tau}}$, $\bar{\rho} = \pm \frac{i}{\bar{H}}$ and branch points at $\bar{\rho} = \pm \frac{i}{\bar{H}}$ The contour shown in Fig. A-1 will be used to evaluate the integral for the particular case $\frac{\pi}{\bar{\tau}} = \frac{1}{\bar{H}}$ Using the notation as shown in Fig. A-1 the following is true by Cauchy's theorem, since all poles and essential singularities are excluded.

$$\begin{aligned} &I_B + I_{C_1} + I_D + I_{A_4} + I_{A_1} + I_{A_2} + I_{A_3} \\ &+ I_{A_5} + I_{a_3} + I_{a_2} + I_{a_1} + I_{a_4} + I_d \\ &+ I_{C_2} = 0 \end{aligned}$$

It can be shown that

$$I_{A_1} = I_{a_1} = I_{A_5} = 0$$

$$I_0 + I_D = 0$$

and $I_{C_1} + I_{C_2} = 0$ as $R \to \infty$

Also

$$I_{a_2} + I_{A_2} = \frac{-2\pi i\,(Bc_0)\sin\frac{\pi\bar{t}}{\bar{\tau}}\sin\dfrac{\pi\bar{x}}{c_0\,\bar{\tau}\sqrt{1-\frac{\bar{H}^2\pi^2}{2}}}}{E\left(\frac{\pi^2}{\bar{\tau}^2}\right)\sqrt{1-\frac{\bar{H}^2\pi^2}{\bar{\tau}^2}}}$$

and

$$I_{A_3} + I_{a_3} + I_{A_4} + I_{a_4} = \frac{2\pi i(2Bc_0)}{\pi E}\int_0^{1/\bar{H}} \frac{\sin\bar{\rho}\bar{t}\cos\dfrac{\bar{\rho}\bar{x}}{c_0\sqrt{1-\bar{H}^2\bar{\rho}^2}}}{\bar{\rho}\sqrt{1-\bar{H}^2\bar{\rho}^2\left(\frac{\pi^2}{\bar{\tau}^2}-\bar{\rho}^2\right)}}\,d\bar{\rho}$$

But

$$u(\bar{x},\bar{t}) = \frac{1}{2\pi i}\,I_3$$

so that

$$u(\bar{x},\bar{t}) = \frac{Bc_o}{E}\left\{ \frac{\sin\frac{\pi\bar{t}}{\tau}\ \sin\dfrac{\pi\bar{x}}{c_o\bar{\tau}\sqrt{1-\frac{\bar{H}^2\pi^2}{\bar{\tau}^2}}}}{\frac{\pi^2}{\bar{\tau}^2}\sqrt{1-\frac{\bar{H}^2\pi^2}{\bar{\tau}^2}}} - \frac{2}{\pi}\int_o^{1/\bar{H}} \frac{\sin\bar{\rho}\bar{t}\ \cos\dfrac{\bar{\rho}\bar{x}}{c_o\sqrt{1-\bar{H}^2\bar{\rho}^2}}}{\bar{\rho}(1-\bar{H}^2\bar{\rho}^2)\left(\frac{\pi^2}{\bar{\tau}^2}-\bar{\rho}^2\right)}\,d\bar{\rho}\right\} \qquad (A\text{-}29)$$

and

$$
\begin{aligned}
u(\bar{x},\bar{t}) = \frac{Bc_o}{E} \Bigg\{ & \frac{\sin\frac{\pi\bar{t}}{\bar{\tau}}\ \sin\frac{\pi\bar{x}}{c_o\sqrt{1-\frac{\bar{H}^2\pi^2}{2}}}}{\frac{\pi^2}{\bar{\tau}^2}\sqrt{1-\frac{\bar{H}^2\pi^2}{2}}} - \frac{2}{\pi}\int_0^{1/\bar{H}} \frac{\sin\bar{\rho}\bar{t}\ \cos\frac{\bar{\rho}\bar{x}}{c_o\sqrt{1-\bar{H}^2\bar{\rho}^2}}}{\bar{\rho}\sqrt{1-\bar{H}^2\bar{\rho}^2}\left(\frac{\pi^2}{\bar{\tau}^2}-\bar{\rho}^2\right)} \\
& + \frac{\sin\frac{(\bar{t}-\bar{\tau})}{}\ \sin\frac{\pi\bar{x}}{c_o\bar{\tau}\sqrt{1-\frac{\bar{H}^2\pi^2}{\bar{\tau}^2}}}}{\frac{\pi^2}{\bar{\tau}^2}\sqrt{1-\frac{\bar{H}^2\pi^2}{\bar{\tau}^2}}} - \frac{2}{\pi}\int_0^{1/\bar{H}} \frac{\sin\bar{\rho}(t-\bar{\tau})\ \cos\frac{\bar{\rho}\bar{x}}{c_o\sqrt{1-\bar{H}^2\bar{\rho}^2}}}{\bar{\rho}\sqrt{1-\bar{H}^2\bar{\rho}^2}\left(\frac{\pi^2}{\bar{\tau}^2}-\bar{\rho}^2\right)}\,d\bar{\rho} \Bigg\} \quad \begin{array}{l} t \geq \bar{\tau} \\ \frac{\pi}{\bar{\tau}} \geq \frac{1}{\bar{H}} \end{array} \qquad \text{(A-30)}
\end{aligned}
$$

The solution for the case $\frac{\pi}{\bar{\tau}} < \frac{1}{\bar{H}}$ was derived because this enables comparison with certain experimental results. The expression $u_{\bar{x}}(\bar{x},\bar{t}) + \bar{H}^2 u_{\bar{x}\bar{t}\bar{t}}(\bar{x},\bar{t})$ will be used for the strain in the longitudinal direction, in order to obtain results compatible with the boundary conditions resulting from the use of Hamilton's principle.

Then,

$$u_{\bar{x}}(\bar{x},\bar{t}) + \bar{H}^2 u_{\bar{x}\bar{t}\bar{t}}(\bar{x},\bar{t}) = \frac{B}{\pi E}\left\{ \sin\frac{\pi E}{\bar{\tau}}\cos\frac{\pi\bar{x}}{c_o\bar{\tau}\sqrt{1-\frac{\bar{H}^2\pi^2}{\bar{\tau}^2}}} + 2\int_0^{1/\bar{H}}\frac{\sin\bar{\rho}\bar{t}\,\sin\frac{\bar{\rho}\bar{x}}{c_o\sqrt{1-\bar{H}^2\bar{\rho}^2}}}{\frac{\pi^2}{\bar{\tau}^2}-\bar{\rho}^2}\,d\bar{\rho}\right\} \quad \begin{matrix} t \leq \bar{\tau} \\ \frac{\pi}{\bar{\tau}} \leq \frac{1}{\bar{H}} \end{matrix} \qquad \text{(A-31)}$$

$$u_{\bar{x}}(\bar{x},\bar{t}) + \bar{H}^2 u_{\bar{x}\bar{t}\bar{t}}(\bar{x},\bar{t}) = \frac{2B}{\pi E}\int_0^{1/\bar{H}}\frac{\sin\frac{\bar{\rho}\bar{x}}{c_o\sqrt{1-\bar{H}^2\bar{\rho}^2}}}{\frac{\pi^2}{\bar{\tau}^2}-\bar{\rho}^2}\left[\sin\bar{\rho}\bar{t} + \sin\bar{\rho}\,(\bar{t}-\bar{\tau})\right]d\bar{\rho}, \quad \begin{matrix} \bar{t} \leq \bar{\tau} \\ \frac{\pi}{\bar{\tau}} \leq \frac{1}{\bar{H}} \end{matrix} \qquad \text{(A-32)}$$

In order to compare theoretical calculations with experimental results obtained by Ripperger [Ref. (13)], the above integral was evaluated for the following conditions:

$\bar{\tau} = 4.4 \times 10^{-6}$ sec.

$\bar{H} = .06499 \times 10^{-6}$ sec.

$c_o = 205{,}000$ in/sec.

bar dia. = 1/8".

In order to detect a change in the wave shape, if possible, results were calculated at points one inch and eighteen inches from the impact end. The integral in Eq. (A-32) was evaluated by using Simpson's rule. For $\bar{x} = 1$, twelve intervals were used and for $\bar{x} = 18$, ninety-six intervals were used between 0 and $\frac{2\pi}{\bar{\tau}}$.

The value of the integrand at $\rho = \frac{\pi}{\bar{\tau}}$ can be found by use of L'Hospitals rule. It can also be shown that

$$\int_{0}^{\frac{2\pi}{\bar{\tau}}} \frac{\sin \frac{\bar{\rho}\bar{x}}{c_o\sqrt{1 - \bar{H}^2\bar{\rho}^2}}}{\frac{\pi^2}{\bar{\tau}^2} - \bar{\rho}^2} [\sin \bar{\rho}\bar{t} + \sin \bar{\rho}\,(\bar{t} - \bar{\tau})]\, d\bar{\rho} \gg \int_{\frac{2\pi}{\bar{\tau}}}^{1/\bar{H}} \frac{\sin \frac{\bar{\rho}\bar{x}}{c_o\sqrt{1 - \bar{H}^2\bar{\rho}^2}}}{\frac{\pi^2}{\bar{\tau}^2} - \bar{\rho}^2} [\sin \bar{\rho}\bar{t} + \sin \bar{\rho}\,(\bar{t} - \bar{\tau})]\, d\bar{\rho}.$$

so the second term above was neglected in the evaluation of the integral of Eq. (A-32).

APPENDIX B

SOLUTION OF MINDLIN EQUATION BY METHOD OF CHARACTERISTICS

The development of a solution to the differential equation which takes into consideration both radial and shear corrections is based on the assumptions made and the equations developed by Mindlin and Hermann [Ref.(4)]. The development of the solution by the method of characteristics parallels that used by H. J. Plass [Ref. (18)].

Derivation and notation of Mindlin and Hermann will be used. This is based on a cylindrical coordinate system ($\bar{r}$, θ, $\bar{x}$, $\bar{t}$) with $\bar{a}$ = radius of rod, where

$u_{\bar{r}}$ = displacement in radial direction

u_{θ} = displacement in angular direction

$u_{\bar{x}}$ = displacement in axial direction

As in common practice the symbols σ, ϵ, and γ, with the proper subscripts will be used to denote the stresses and the strains.

The following assumptions were made by Mindlin and Hermann concerning the displacements

$$u_r = \frac{\bar{r}U(\bar{x},\bar{t})}{\bar{a}} \qquad u_\theta = 0 \qquad u_x = \bar{W}(\bar{x},\bar{t}) \qquad \text{(B-1)}$$

and the following are defined:

$$P_{\bar{r}} = \int_0^{\bar{a}} \sigma_{rr} \, \bar{r} d\bar{r} \qquad \text{(B-2)}$$

$$P_{\bar{x}} = \int_0^{a} \sigma_{\bar{x}\bar{x}} \, \bar{r} d\bar{r} \qquad \text{(B-3)}$$

$$P_{\theta} = \int_0^{\bar{a}} \sigma_{\theta\theta} \, \bar{r} d\bar{r} \qquad \text{(B-4)}$$

$$\bar{Q} = \int_0^{\bar{a}} \frac{\sigma_{\bar{x}\bar{r}}}{\bar{a}} \, \bar{r}^2 d\bar{r} \qquad \text{(B-5)}$$

By the use of Hamilton's principle and the above definitions the two following equations are obtained:

$$\frac{\partial \bar{Q}}{\partial \bar{x}} - \frac{P_{\bar{r}} + P_{\theta}}{\bar{a}} + \bar{R} = \frac{\rho \bar{a}^2}{4} \frac{\partial^2 W}{\partial \bar{t}^2} \qquad \text{(B-6)}$$

$$\frac{\partial P_{\bar{x}}}{\partial \bar{x}} + a\bar{Z} = \frac{\rho \bar{a}^2}{2} \frac{\partial^2 \bar{W}}{\partial \bar{t}^2} \qquad \text{(B-7)}$$

where $\bar{R} = \sigma_{\bar{r}\bar{r}} \Big]_{\bar{r}=\bar{a}}$ $\quad Z = \sigma_{\bar{x}\bar{r}} \Big]_{\bar{r}=\bar{a}}$

For the case of principal interest, namely, the case where no forces are applied to the surface of the cylinder, it follows that $\tilde{R} = \tilde{Z} = 0$. Thus, the equations of motion become;

$$\frac{\partial Q}{\partial \bar{x}} - \frac{P_{\bar{r}} + P_{\theta}}{\bar{a}} = \frac{\rho \bar{a}^2}{4} \frac{\partial^2 U}{\partial \bar{t}^2} \qquad \text{(B-8)}$$

$$\frac{\partial P_{\bar{x}}}{\partial \bar{x}} = \frac{\rho \bar{a}^2}{2} \frac{\partial^2 \bar{W}}{\partial \bar{t}^2} \qquad \text{(B-9)}$$

From assumption (B-1) the strains in cylindrical coordinates become

$$\begin{aligned} &\epsilon_{rr} = \frac{U}{\bar{a}} \qquad \epsilon_{\theta\theta} = \frac{U}{\bar{a}} \qquad \epsilon_{\bar{x}\bar{x}} = \frac{\partial \bar{W}}{\partial \bar{x}} \\ &\bar{r}\theta = 0 \qquad \gamma_{\bar{r}\bar{x}} = \frac{r}{\bar{a}} \frac{\partial U}{\partial \bar{x}} \qquad \gamma_{\bar{x}\theta} = 0 \end{aligned} \qquad \text{(B-10)}$$

Then in the present case:

$$\begin{aligned} &\sigma_{\bar{r}\bar{r}} = \lambda\Delta + 2G\epsilon_{\bar{r}\bar{r}} \qquad &&\sigma_{\theta\theta} = \lambda\Delta + 2G\epsilon_{\theta\theta} \\ &\sigma_{\bar{x}\bar{x}} = \lambda\Delta + 2G\epsilon_{\bar{x}\bar{x}} \qquad &&\sigma_{\bar{x}\bar{r}} = G\gamma_{\bar{x}\bar{r}} \end{aligned} \qquad \text{(B-11)}$$

or

$$\sigma_{\bar{r}\bar{r}} = \sigma_{\theta\theta} = 2(\lambda + G)\epsilon_{\bar{r}\bar{r}} + \lambda\epsilon_{\bar{x}\bar{x}} \quad \text{and} \quad P_{\bar{r}} = P_{\theta}$$

$$\sigma_{\bar{r}\bar{r}} = 2\lambda\epsilon_{\bar{r}\bar{r}} + (\lambda + 2G)\epsilon_{\bar{x}\bar{x}}$$

where λ and G are Lame's constants [Ref. (13)] and Δ is the dilatation.

These two equations result in the following relationships:

$$\epsilon_{\bar{r}\bar{r}} = \frac{\lambda + 2G}{2G(3\lambda + 2G)} \quad \sigma_{\bar{r}\bar{r}} - \frac{\lambda}{2G(3\lambda + 2G)} \quad \sigma_{\bar{x}\bar{x}} \tag{B-12}$$

$$\epsilon_{\bar{x}\bar{x}} = \frac{\lambda + G}{G(3\lambda + 2G)} \quad \sigma_{\bar{x}\bar{x}} \quad \frac{\lambda}{G(3\lambda + 2G)} \quad \sigma_{\bar{r}\bar{r}} \tag{B-13}$$

also from (B-11)

$$\gamma_{\bar{x}\bar{r}} = \frac{\sigma_{\bar{x}\bar{r}}}{G}$$

Defining

$$\begin{aligned} E_{\bar{r}\bar{r}} &= \int_0^a \bar{r}\epsilon_{\bar{r}\bar{r}} d\bar{r} \\ E_{\bar{x}\bar{x}} &= \int_0^a \bar{r}\epsilon_{\bar{x}\bar{x}} d\bar{r} \\ \Gamma_{\bar{x}\bar{r}} &= \int_0^a \frac{G\bar{r}^2}{\bar{a}} \gamma_{\bar{x}\bar{r}} d\bar{r} \end{aligned} \tag{B-14}$$

Then using the equations of (B-10) in Eqs. (B-14), it follows that:

$$\begin{aligned} E_{\bar{r}\bar{r}} &= \frac{\bar{u}\bar{a}}{2} \\ E_{\bar{x}\bar{x}} &= \frac{\bar{a}^2}{2} \frac{\partial \bar{W}}{\partial \bar{x}} \\ \bar{Q} &= \frac{G\bar{a}^2}{4} \frac{\partial U}{\partial \bar{x}} \end{aligned} \tag{B-15}$$

Multiplying Eqs. (B-11) and (B-12) by $\bar r d\bar r$ and integrating, using Definitions (B-2), (B-3), and (B-5), results in the following:

$$E_{\bar r\bar r} = \frac{\lambda + 2G}{2G(3 \quad + 2G)} P_{\bar r} - \frac{\lambda}{2G(3 \quad + 2G)} P_{\bar x} \qquad \text{(B-16)}$$

$$E_{\bar x\bar x} = \frac{\lambda + G}{G(3\lambda + 2G)} P_{\bar x} \quad \frac{\lambda}{2G(3\lambda + 2G)} P_{\bar r} \qquad \text{(B-17)}$$

And the use of $\gamma_{\bar x\bar r} = \frac{\sigma_{\bar x\bar r}}{G}$ in the last equation (B-13) results in:

$$\Gamma_{\bar x\bar r} = \bar Q \qquad \text{(B-18)}$$

Then using the relationship $E = \frac{G(3\lambda + 2G)}{\lambda + G}$, $G = \frac{E}{2(1 + \nu)}$ it follows that:

$$E_{\bar r\bar r} = \frac{1}{E}\left[(1 - \nu)P_{\bar r} - \nu P_{\bar x}\right] \qquad \text{(B-19)}$$

$$E_{\bar x\bar x} = \frac{1}{E}\left[P_{\bar x} - 2\nu P_{r}\right] \qquad \text{(B-20)}$$

From Eqs. (B-15) the equations of continuity become:

$$\frac{\partial E_{\bar r\bar r}}{\partial \bar t} = \frac{\bar a}{2}\frac{\partial U}{\partial \bar t} \qquad \text{(B-21)}$$

$$\frac{\partial E_{\bar x\bar x}}{\partial \bar t} = \frac{\bar a^2}{2}\frac{\partial^2 \bar W}{\partial \bar x d\bar t} \qquad \text{(B-22)}$$

$$\partial \Gamma_{\bar x\bar r} = \frac{G\bar a^2}{\gamma}\frac{\partial^2 U}{\partial \bar x d\bar t} = \frac{\partial \bar Q}{\partial \bar t} \qquad \text{(B-23)}$$

Also

$$\frac{\partial E_{\bar{r}\bar{r}}}{\partial \bar{t}} = \frac{1}{E} \left[(1 - \nu) \frac{\partial P_{\bar{x}}}{\partial \bar{t}} - \nu \frac{\partial P_{\bar{x}}}{\partial \bar{t}} \right] \qquad \text{(B-24)}$$

$$\frac{\partial E_{\bar{x}\bar{x}}}{\partial \bar{t}} = \frac{1}{E} \left[\frac{\partial P_{\bar{x}}}{\partial \bar{t}} - 2\nu \frac{\partial P_{\bar{r}}}{\partial \bar{t}} \right] \qquad \text{(B-25)}$$

The use of (B-21) and (B-22) in Eqs. (B-24) and (B-25), together with Eqs. (B-23), (B-8), and (B-9), results in the five differential equations in the five unknowns, $P_{\bar{x}}$, $P_{\bar{r}}$, $\bar{Q}$, V_1, and V_o.

$$\frac{\bar{a}}{2} V_1 = \frac{1}{E} (1 - \nu) \frac{\partial P_{\bar{r}}}{\partial \bar{t}} - \nu \frac{\partial P_{\bar{x}}}{\partial \bar{t}} \qquad \text{(B-26)}$$

$$\frac{\bar{a}^2}{2} \frac{\partial V_o}{\partial \bar{x}} = \left[\frac{1}{E} \frac{\partial P_{\bar{x}}}{\partial \bar{t}} - 2\nu \frac{\partial P_{\bar{r}}}{\partial \bar{t}} \right] \qquad \text{(B-27)}$$

$$\frac{\partial \bar{Q}}{\partial \bar{t}} = \frac{G\bar{a}^2}{4} \frac{\partial V_1}{\partial \bar{x}} \qquad \text{(B-28)}$$

$$\frac{\partial \bar{Q}}{\partial \bar{x}} - 2 \frac{P_{\bar{r}}}{\bar{a}} = \frac{\rho \bar{a}^2}{4} \frac{\partial V_1}{\partial \bar{t}} \qquad \text{(B-29)}$$

$$\frac{\partial P_{\bar{x}}}{\partial \bar{x}} = \frac{\rho \bar{a}^2}{2} \frac{\partial V_o}{\partial \bar{t}}$$

where

$$V_1 = \frac{\partial U}{\partial \bar{t}} \qquad V_o = \frac{\partial \bar{W}}{\partial \bar{t}} \qquad \text{(B-30)}$$

These five equations are put into dimensionless form by the use of the following substitutions:

P = dimensionless normal axial force $= \dfrac{\text{physical force (lb)}}{E\bar{a}^2\ \text{(lb)}} = \dfrac{\bar{P}_x}{E\bar{a}^2}$

R = dimensionless normal radial force $= \dfrac{\text{physical force (lb)}}{E\bar{a}^2\ \text{(lb)}} = \dfrac{\bar{P}_r}{E\bar{a}^2}$

Q = dimensionless shear $= \dfrac{\text{physical shear (lb)}}{G\bar{a}^2\ \text{(lb)}} = \dfrac{\bar{Q}}{G\bar{a}^2}$

V = dimensionless axial velocity of section $= \dfrac{\text{physical velocity (in/sec)}}{c_D\text{(in/sec)}} = \dfrac{V_o}{c_D}$

W = dimensionless radial velocity of section $= \dfrac{\text{physical velocity (in/sec)}}{c_S\ \text{(in/sec)}} = \dfrac{V_1}{c_S}$

x = dimensionless distance along axis $= \dfrac{\text{physical distance (in.)}}{\bar{a}/\text{(in.)}} = \dfrac{\bar{x}}{\bar{a}}$

t = dimensionless time $= \dfrac{\text{physical time (sec)}}{\bar{a}/c_D\text{(sec)}} = \dfrac{c_D t}{\bar{a}}$

where

$$c_D = \sqrt{\left[\frac{1-\nu}{(1-2\nu)(1+\nu)}\right]\left[\frac{E}{\rho}\right]} = \text{dilatation wave velocity (in/sec)}$$

$$c_S = \sqrt{\frac{G}{\rho}} = \text{shear wave velocity (in/sec)}$$

also we let

$$\bar{c}_o = \frac{c_D^2}{c_S^2} = \frac{2(1-\nu)}{(1-2\nu)}$$

Then

$$\frac{\partial P_r}{\partial \bar{t}} = E\bar{a}c_D \frac{\partial R}{\partial t} \; ; \qquad \frac{\partial P_{\bar{x}}}{\partial \bar{x}} = E\bar{a}c_D \frac{\partial P}{\partial t}$$

$$\frac{\partial \bar{Q}}{\partial \bar{t}} = G\bar{a}c_D \frac{\partial Q}{\partial t} \; ; \qquad \frac{\partial V_1}{\partial \bar{x}} = \frac{c_s}{\bar{a}} \frac{\partial W}{\partial x}$$

$$\frac{\partial \bar{Q}}{\partial \bar{x}} = G\bar{a} \frac{\partial Q}{\partial x} \; ; \qquad \frac{\partial V_1}{\partial \bar{t}} = \frac{c_s c_D}{\bar{a}} \frac{\partial W}{\partial t}$$

$$\frac{\partial P_{\bar{x}}}{\partial \bar{x}} = E\bar{a} \frac{\partial P}{\partial x} \; ; \qquad \frac{\partial V_o}{\partial \bar{t}} = \frac{c_D^2}{\bar{a}} \frac{\partial V}{\partial t}$$

Then Eqs. (B-26), (B-27), (B-28), (B-29), and (B-30) become respectively:

$$W - 2(1 - \nu)\sqrt{\bar{c}_o \frac{\partial R}{\partial t}} + 2\nu\sqrt{\bar{c}_o \frac{\partial P}{\partial t}} = 0 \qquad \text{(B-31)}$$

$$\frac{\partial V}{\partial x} - 2\frac{\partial P}{\partial t} + 4\nu \frac{\partial R}{\partial t} = 0 \qquad \text{(B-32)}$$

$$\frac{\partial W}{\partial x} - 4\sqrt{\bar{c}_o}\, \frac{\partial a}{\partial t} = 0 \qquad \text{(B-33)}$$

$$\frac{\partial W}{\partial t} - \frac{4}{\sqrt{\bar{c}_o}} \frac{\partial Q}{\partial x} + \frac{16}{\sqrt{\bar{c}_o}} (1 + \nu) R = 0 \qquad \text{(B-34)}$$

$$\frac{\partial V}{\partial t} - \frac{4(1 + \nu)}{\bar{c}_o} \frac{\partial P}{\partial x} = 0 \qquad \text{(B-35)}$$

To find the characteristics in this problem the following procedure was used. Let $x(\theta)$, $t(\theta)$, where θ is a parameter, be the equations of a curve in the x - t plane. Assume P, R, Q, V, W to be given along this curve; then P_θ, R_θ, etc., (where the subscript indicates differentiation) can be found for all points of the curve. That is,

$$P_x x_\theta + P_t t_\theta = P_\theta$$

$$R\, x_\theta + R_t\, t_\theta = R_\theta, \text{ etc,}$$

There are five such equations possible. The five other relations among the x and t derivatives at the five variables are Eqs. (B-31), (B-32), (B-33), (B-34), (B-35). The result is then ten equations for the ten unknowns P_x, P_t, R_x, R_t, etc.

The condition that there be no unique solution for all ten is that the determinant of these unknowns vanish, that is, the characteristics are found from the condition that:

$$\begin{vmatrix} x_\theta & t_\theta & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & x_\theta & t_\theta & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & x_\theta & t_\theta & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & x_\theta & t_\theta & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_\theta & t_\theta \\ 0 & 0 & 0 & 0 & 0 & 1-\nu & 0 & -\nu & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 4\nu & 0 & -2 & 0 & 0 \\ 0 & 0 & 1/4 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{\bar{c}_o} \\ 0 & 0 & -\frac{\sqrt{c_o}}{4} & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -\frac{\bar{c}_o}{4(1+\nu)} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{vmatrix} = 0 \qquad \text{(B-36)}$$

It can be shown that (B-36) reduces to

$$x_\theta\left(x_\theta^2 - \frac{t_\theta^2}{c_o}\right)\left(x_\theta^2 - t_\theta^2\right) = 0 \qquad \text{(B-37)}$$

Thus, the characteristic curves satisfy the differential equations:

$$\left.\begin{array}{l} dx = 0 \\ \left.\begin{array}{l} dt = dx \\ dt = -\,dx \end{array}\right\} \text{Dilatation} \\ \left.\begin{array}{l} dt = \sqrt{\bar{c}_o}\,dx \\ dt = -\sqrt{\bar{c}_o}\,dx \end{array}\right\} \text{Shear} \end{array}\right. \qquad \text{(B-38)}$$

These curves are plotted in Fig. (B-1).

Multiplying Eq. (B-33) by t_θ , and Eq. (B-34) by x_θ and adding the two equation results in the following:

$$Q_t t_\theta + Q_x x_\theta + \; = \frac{dQ}{d\theta} = \frac{1}{\sqrt{4\;\bar{c}_o}} \quad \frac{\partial W}{\partial x}\;\frac{dt}{d\theta} + 4(1+\nu)\,R\,\frac{dx}{d\theta} + \frac{\sqrt{\bar{c}_o}}{4}\;\frac{\partial W}{\partial t}\;\frac{dx}{d\theta} \qquad \text{(B-39)}$$

Along the characteristic $dt = \sqrt{\bar{c}_o}\,dx$,

Let $x = \theta$ so that $t = \sqrt{c_o}\,\theta$

and $dt = \sqrt{c_o}\,d\theta$, $dx = d\theta$

also $\dfrac{dW}{d\theta} = \sqrt{\bar{c}_o}\;\;\dfrac{\partial W}{\partial t} + \dfrac{\partial W}{\partial x}$

Then Eq. (B-39) becomes

$$\frac{dQ}{d\theta} = \frac{1}{4}\;\frac{\partial W}{\partial x} + 4(1+\nu)\,R + \frac{\sqrt{\bar{c}_o}}{4}\;\;\frac{\partial W}{\partial t}$$

$$= \frac{1}{4}\;\frac{dW}{d\theta} + 4(1+\nu)\,R$$

or

$$dQ = \frac{1}{4}\;dW + 4(1+\nu)\,R\,d\theta$$

or

$$dQ = \frac{dW}{4} + \frac{4(1 + \nu)}{\sqrt{\bar{c}_o}} \, Rdt \quad \text{along } dt = \sqrt{\bar{c}_o}\, dx$$

In a similar manner all the partial differential equations used in conjunction with the characteristic differential equations reduce to the following:

$$\text{Along} \quad dx = 0, \quad dP - \frac{1 - \nu}{\nu} \, dR + \frac{W}{2\nu\sqrt{\bar{c}_o}} \, dt = 0$$

$$\text{Along} \quad dt = dx, \quad dP - \frac{\bar{c}_o}{4(1 + \nu)} \, dV - \frac{\nu\sqrt{\bar{c}_o} W}{2(1 - \nu)(1 + \nu)} \, dt = 0$$

$$\text{Along} \quad dt = -dx, \quad dP + \frac{\bar{c}_o}{4(1 + \nu)} \, dV - \frac{\nu\sqrt{\bar{c}_o} W}{2(1 - \nu)(1 + \nu)} \, dt = 0 \qquad \text{(B-40)}$$

$$\text{Along} \quad dt = \bar{c}_o \, dx, \quad dQ - \frac{dW}{4} - \frac{4(1 + \nu)}{\sqrt{\bar{c}_o}} \, dt = 0$$

$$\text{Along} \quad dt = -\sqrt{c_o} \, dx, \quad dQ + \frac{dW}{4} + \frac{4(1 + \nu)R}{\sqrt{\bar{c}_o}} \, dt = 0$$

These can be converted to difference equations. An element is chosen that is bounded by two pairs of characteristics of the dilatational wave families inside of which are two characteristics of the shear families. See Fig. (B-2).

The difference equations used for obtaining the values of P_P, R_P, Q_P, V_P, W_P, at the Point P in terms of values at A, B, and C are:

$$
\begin{aligned}
&(P_P - P_B) - \frac{(1-\nu)}{\nu}(R_P - R_B) = -\frac{1}{2\nu\sqrt{\bar{c}_o}}(W_P + W_B)\ \Delta t \\
&(P_P - P_A) - \frac{\bar{c}_o}{4(1+\nu)}(V_P - V_A) = \frac{\nu\sqrt{\bar{c}_o}}{2(1-\nu)(1+\nu)}\left(\frac{W_P + W_A}{2}\right)\Delta t \\
&(P_P - P_C) + \frac{\bar{c}_o}{4(1+\nu)}(V_P - V_C) = \frac{\sqrt{\bar{c}_o}}{2(1-\nu)(1+\nu)}\left(\frac{W_P + W_C}{2}\right)\Delta t \\
&(Q_P - Q_{A'}) - \frac{1}{4}(W_P - W_{A'}) = \frac{4(1+\nu)}{\sqrt{\bar{c}_o}}\ \frac{(R_P + R_{A'})}{2}\Delta t' \\
&(Q_P - Q_{C'}) + \frac{1}{4}(W_P - W_{C'}) = -\frac{4(1+\nu)}{\sqrt{\bar{c}_o}}\left(\frac{R_P + R_{C'}}{2}\right)\Delta t'
\end{aligned}
\qquad \text{(B-41)}
$$

where

$$\Delta t' = \frac{2c_D}{c_S + c_D}\ \Delta t = \frac{2\sqrt{\bar{c}_o}}{1 + \sqrt{\bar{c}_o}}\ \Delta t$$

Using linear interpolation between B and A or B and G, the following values are obtained:

$$Q_{A'} = Q_B + \frac{2c_S}{c_S + c_D}(Q_A - Q_B) = Q_B + \frac{2}{1 + \sqrt{\bar{c}_o}}(Q_A - Q_B)$$

$$Q_{C'} = Q_B + \frac{2c_S}{c_S + c_D}(Q_C - Q_B) = Q_B + \frac{2}{1 + \sqrt{\bar{c}_o}}(Q_C - Q_B)$$

and similarly for $W_{A'}$, $W_{C'}$, $R_{A'}$, $R_{C'}$.

If we rewrite these equations, with all the terms having P as a subscript on the left side, we get,

$$\left.\begin{aligned}
&P_P - \frac{1-\nu}{\nu} R_B + \frac{W_P}{2\nu\sqrt{\bar{c}_o}}\,\Delta t = T_1 \\
&P_P - \frac{\bar{c}_o}{4(1+\nu)} V_P - \frac{\nu\sqrt{\bar{c}_o}\,W_P}{4(1+\nu)(1-\nu)}\,\Delta t = T_2 \\
&P_P + \frac{\bar{c}_o}{4(1+\nu)} VP - \frac{\nu\sqrt{\bar{c}_o}\,W_P}{4(1+\nu)(1-\nu)}\,\Delta t = T_3 \\
&Q_P - \frac{W_P}{4} - \frac{4(1+\nu)R_P}{1+\sqrt{\bar{c}_o}}\,\Delta t = T_4 \\
&Q_P + \frac{W_P}{4} + \frac{4(1+\nu)R_P}{1+\sqrt{\bar{c}_o}}\,\Delta t = T_5
\end{aligned}\right\} \qquad \text{(B-42)}$$

where:

$$\left.\begin{aligned}
&T_1 = P_B - \frac{1-\nu}{\nu} R_B - \frac{W_B}{2\nu\sqrt{\bar{c}_o}}\,\Delta t \\
&T_2 = P_A - \frac{\bar{c}_o V_A}{4(1+\nu)} + \frac{\nu\sqrt{\bar{c}_o}\,W_A}{4(1+\nu)(1-\nu)}\,\Delta t \\
&T_3 = P_C + \frac{\bar{c}_o V_o}{4(1+\nu)} + \frac{\nu\sqrt{\bar{c}_o}\,W_C}{4(1+\nu)(1-\nu)}\,R_A \cdot \Delta t \\
&T_4 = \frac{2}{1+\sqrt{\bar{c}_o}}\left[Q_A - \frac{W_A}{4} + \frac{4(1+\nu)}{1+\bar{c}_o} R_A \cdot \Delta t\right] \\
&\qquad + \left(1 - \frac{2}{1+\bar{c}_o}\right)\left[Q_B - \frac{W_B}{4} + \frac{4(1+\nu)}{1+\bar{c}_o} R_B \cdot \Delta t\right] \\
&T_5 = \frac{2}{1+\bar{c}_o}\quad Q_C + \frac{W_C}{4} - \frac{4(1+\nu)}{1+\bar{c}_o} R_C\,\Delta t \\
&\qquad + \left(1 - \frac{2}{1+\bar{c}_o}\right)\left[Q_B + \frac{W_B}{4} - \frac{4(1+\nu)}{1+\bar{c}_o} R_B \cdot \Delta t\right]
\end{aligned}\right\} \qquad \text{(B-43)}$$

Equations (B-42) must be solved simultaneously for P_P, R_P, Q_P, V_P, W_P. Their solution depends upon having previously obtained values at A, B, and C by means of these same equations. If values are also known at D, we then can repeat the process above for the Point R in Fig. (B-3), etc. Since new values depend upon previously attained values, there has to be a starting point somewhere.

In this problem, starting values will also be specified along the leading wave front $t = x$, and along the t-axis (boundary conditions). It is assumed that the time variation of the impact force at the end of the bar is not continuous but sinusoidal. The values at the leading wave front of both P and V are zero. Since the shear wave is a more slowly moving wave, it is assumed that R, Q, and W are also zero along the leading wave front. No effect of these waves is felt until later.
Therefore,

$$\text{along } t = x, \quad P = Q = R = V = W = 0$$

Along the t-axis the boundary conditions (two can be specified) to be considered must be satisfied. For example, if the end $x = 0$ were free and the end is subjected to no shear force but only to an axial force $P(0,t)$ whose time variation is specified by some function $f(t)$, $t \geq 0$,

$$\begin{array}{lll} \text{Along t-axis} & (x = 0) & \\ P = f(t) & t \geq 0 & \\ Q = 0 & t \geq 0 & \text{(B-44)} \end{array}$$

Also, another condition that might have a physical counterpart is the condition that the end $x = 0$ be free, subjected to no shear force but only subjected to an impressed axial velocity $V(0,t)$ whose time variation is specified by some function $g(t)$, $t \geq 0$. In this case along t-axis ($x = 0$)

$$\begin{aligned} V &= g(t) & t &\geq 0 \\ Q &= 0 & t &\geq 0 \end{aligned} \qquad \text{(B-45)}$$

In order to illustrate the calculation setup, Fig. (B-4) is included to illustrate condition (B-44). In this case, at mesh points along the t-axis P and Q are specified, and R, V, and W are to be determined. To attain them only the first, third and fifth of Eqs. (B-42) are to be used, with $Q_P = 0$, and P_P given its proper values. Then the three equations for use along the t-axis are:

$$P_P + \frac{W_P}{2\nu\sqrt{\bar{c}_o}} \Delta t - \frac{1 - \nu}{\nu} R_P = T_1$$

$$P_P + \frac{\bar{c}_o}{4(1 + \nu)} V_P - \frac{\nu \bar{c}_o W_P}{4(1 + \nu)(1 - \nu)} \Delta t = T_\beta \qquad \text{(B-46)}$$

$$\frac{W_P}{4} + \frac{4(1 + \nu)R_P}{1 + \sqrt{\bar{c}_o}} \Delta t = T_5$$

The above equations are then solved simultaneously for R_P, V_P, W_P for their values along $x = 0$.

APPENDIX C

ELEMENTARY BENDING WAVE THEORY FOR A SEMI-INFINITE BEAM WITH SINUSOIDAL MOMENT IMPACT

In Section 3 the solutions were given for the moment $M(x,t)$ in a semi-infinite beam with its end subjected to a suddenly applied sinusoidal impact $M(0,t) = H \sin \frac{\pi t}{\tau}$, $t \geq 0$. Two conditions, simply supported and unsupported, were assumed for the impact end. The details of the solutions for the two cases are given here.

C-1. Simply Supported End

Consider first the case of the simply supported end. For this the following partial differential equation for the transverse deflection $w(x,t)$, [Eq. (3-4)] must be satisfied:

$$k^2 \frac{\partial^4 w}{\partial x^4} + \frac{\partial^2 w}{\partial t^2} = 0 \tag{C-1}$$

The boundary and initial conditions are:

$$w(0,t) = 0 \tag{C-2}$$

$$M(0,t) = Ak^2 w_{xx}(0,t) = H \sin \frac{\pi t}{\tau} \tag{C-3}$$

$$w(\infty,t) < \infty \tag{C-4}$$

$$w(x,0) = 0 \tag{C-5}$$

$$w_t(x,0) = 0 \tag{C-6}$$

In the above equations, and in what follows, x and t subscripts indicate partial differentiation.

Application of the Laplace transformation, with respect to t, in Eq. (C-1) yields:

$$s^2\bar{w}(x,s) - s\bar{w}(x,0) - \bar{w}_t(x,0) + k^2\bar{w}_{xxxx}(x,s) = 0 \qquad \text{(C-7)}$$

where

$$\bar{w}(x,s) = \int_0^\infty e^{-st}\, w(x,t)dt.$$

By virtue of the initial conditions (C-5) and (C-6), Eq. (C-7) becomes:

$$s^2\bar{w}(x,s) + k^2\bar{w}_{xxxx}(x,s) = 0 \qquad \text{(C-8)}$$

The boundary conditions (C-2), (C-3), (C-4) transform to the following:

$$\bar{w}(0,s) = 0 \qquad \text{(C-9)}$$

$$Ak^2\bar{w}_{xx}(0,s) = H\,\frac{\lambda}{s^2 + \lambda^2} \qquad \text{(C-10)}$$

$$\bar{w}\,(\infty,s) < \infty \qquad \text{(C-11)}$$

where $\lambda = \frac{\pi}{\tau}$. The general solution to Eq. (C-8) is:

$$\begin{aligned}\bar{w}(x,s) = {} & e^{x\sqrt{\frac{s}{2k}}}\left[A(s)\cos\left(x\sqrt{\frac{s}{2k}}\right) + B(s)\sin\left(x\sqrt{\frac{s}{2k}}\right)\right] \\ & + e^{-x\sqrt{\frac{s}{2k}}}\left[C(s)\cos\left(x\sqrt{\frac{s}{2k}}\right) + D(s)\sin\left(x\sqrt{\frac{s}{2k}}\right)\right] \qquad \text{(C-12)}\end{aligned}$$

The boundary conditions (C-9) and (C-11) reduce Eq. (C-12) to the following:

$$\bar{w}(x,s) = D(s)\ {}^{-x\sqrt{\frac{s}{2k}}}\ \sin\left(x\sqrt{\frac{s}{2k}}\right) \qquad \text{(C-13)}$$

In order to apply Eq. (C-10), it is necessary to have $\bar{w}_{xx}$, which is:

$$\bar{w}_{xx}(x,s) = - D(s)\left(\frac{s}{k}\right) e^{-x\sqrt{\frac{s}{2k}}} \cos\left(x\sqrt{\frac{s}{2k}}\right) \tag{C-14}$$

The following is then obtained:

$$D(s) = - \frac{H}{Ak^2}\left(\frac{k}{s}\right)\left(\frac{\lambda}{s^2 + \lambda^2}\right) = \frac{H}{Ak\lambda}\left(\frac{s}{s^2 + \lambda^2} - \frac{1}{s}\right) \tag{C-15}$$

Therefore,

$$\bar{w}(x,s) = \frac{H}{Ak\lambda}\left(\frac{s}{s^2 + \lambda^2} - \frac{1}{s}\right) e^{-x\sqrt{\frac{s}{2k}}} \sin\left(x\sqrt{\frac{s}{2k}}\right) \tag{C-16}$$

It is convenient to change the form of this expression to the following, in order that tables of transforms can be used.

$$\bar{w}(x,s) = Re\left\{ i \frac{H}{Ak\lambda}\left(\frac{s}{s^2 + \lambda^2} - \frac{1}{s}\right) e^{-(1+i)\, x \sqrt{s/2k}} \right\} \tag{C-17}$$

This form is like that in Tables of Integral Transforms, Vol. 1, edited by A. Erdelyi, McGraw-Hill Book Company, 1954, Section 5.6, No. 3 and No. 11. On using the inversion formulas listed there, the following is obtained:

$$w(x,t) = Re\left\{ i \frac{H}{Ak\lambda}\left[e^{-(1+i)\; x/2 \; \sqrt{\lambda/k}} \cos\left(\lambda t - [\,1 + i\,]\, \frac{x}{2} \sqrt{\frac{\lambda}{k}}\right) - \frac{1}{\pi}\int_0^\infty e^{-\varphi t} \sin\left[(1 + i)\; x\sqrt{\frac{\varphi}{2k}}\,\right] d\varphi - 1 + \frac{2}{\sqrt{\pi}} \int_0^{\frac{1+i}{2}\frac{x}{\sqrt{2kt}}} e^{-\varphi^2}\, d\varphi \right]\right\} \tag{C-18}$$

To obtain the expression for M(x,t), it is necessary merely to use the relation, Eq. (3-3), which is repeated here for convenience:

$$M = Ak^2 \frac{\partial^2 w}{\partial x^2} \qquad \text{(C-19)}$$

After some manipulation M(x,t) is found to be:

$$\begin{aligned} M(x,t) = H\,\mathrm{Re}\Bigg\{ & \frac{1}{2} e^{-x\sqrt{\lambda/k}} \cos \lambda t + \frac{1}{2} e^{-x\sqrt{\lambda/k}} \sin \lambda t \\ & - \frac{1}{2} \cos\left(\lambda t - x\sqrt{\frac{\lambda}{k}}\right) + \frac{1}{2} \sin\left(\lambda t - x\sqrt{\frac{\lambda}{k}}\right) \\ & - \frac{1}{\pi\lambda} \int_0^\infty \Bigg[e^{-\varphi t} \frac{\varphi^2}{\varphi^2 + \lambda^2} \sin x\sqrt{\frac{\varphi}{2k}} \cosh x\sqrt{\frac{\varphi}{2k}} \\ & \qquad + i \cos x\sqrt{\frac{\varphi}{2k}} \sinh x\sqrt{\frac{\varphi}{2k}} \Bigg] d\varphi \\ & + \frac{x}{2\lambda\sqrt{\pi k}\, t^{3/2}} \left[\cos\left(\frac{x^2}{4kt} + \frac{\pi}{4}\right) - i \sin\left(\frac{x^2}{4kt} + \frac{\pi}{4}\right) \right] \Bigg\} \end{aligned} \qquad \text{(C-}$$

Eq. (C-20) is now reduced further using $\lambda = \frac{\pi}{\tau}$, to obtain the form stated in Section 3, Eq. (3-6), namely:

$$\begin{aligned} M(x,t) = H \Bigg\{ & - \frac{\tau}{\pi^2} \int_0^\infty e^{-\varphi t} \frac{\varphi^2}{\varphi^2 + \frac{\pi^2}{2}} \sin\left(x\sqrt{\frac{\varphi}{2k}}\right) \cosh\left(x\sqrt{\frac{\varphi}{2k}}\right) d\varphi \\ & + \frac{1}{2} e^{-x\sqrt{\pi/k\tau}} \sin \frac{\pi t}{\tau} + \frac{1}{2} \sin\left(\frac{\pi t}{\tau} - x \frac{\pi}{k\tau}\right) \\ & + \frac{x\tau}{2\sqrt{k}(\pi t)^{3/2}} \cos\left(\frac{x^2}{4kt} + \frac{\pi}{4}\right) \end{aligned} \qquad \text{(C-21)}$$

The infinite integral in Eq. (C-21) is evaluated in a special way. The methods used are discussed below. To simply the manipulations, certain changes of variable are made as follows:

$$X = x\sqrt{\frac{\pi}{k\tau}}\ ,\ T = \frac{\pi t}{\tau}\ ,\ \Phi = \frac{\tau}{\pi}\ \varphi \qquad \text{(C-22)}$$

With these changes, the integral (including the prefactor, $-\frac{\tau}{\pi^2}$) in Eq. (C-21) becomes:

$$I = -\frac{1}{\pi}\int_0^{\infty} e^{-\Phi T}\ \frac{\Phi^2}{\Phi^2 + 1}\ \sin\left(X\sqrt{\frac{\Phi}{2}}\right)\cosh\left(X\sqrt{\frac{\Phi}{2}}\right) d\Phi \qquad \text{(C-23)}$$

The product $\sin\left(X\sqrt{\frac{\Phi}{2}}\right) \cosh\left(X\sqrt{\frac{\Phi}{2}}\right)$ is expanded in a power series in $(X\sqrt{\Phi})$ as follows:

$$\sin\left(X\sqrt{\frac{\Phi}{2}}\right)\cosh\left(X\sqrt{\frac{\Phi}{2}}\right) = \frac{1}{\sqrt{2}}\left[X\sqrt{\Phi} + \frac{X^3\Phi^{3/2}}{3!} - \frac{X^7\Phi^{5/2}}{5!} - \frac{X^7\Phi^{7/2}}{7!} + \ldots\right] \qquad \text{(C-24)}$$

Therefore,

$$I = -\frac{1}{\pi\sqrt{2}}\int_0^{\infty} e^{-\Phi T}\ \frac{1}{\Phi^2 + 1}\left[X\Phi^{5/2} + \frac{X^3\Phi^{7/2}}{3!} - \frac{X^5\Phi^{9/2}}{5!} - \frac{X^7\Phi^{11/2}}{7!} + \ldots\right] d\Phi \qquad \text{(C-25)}$$

This can be written as follows:

$$I = - \frac{1}{\pi\sqrt{2}} \left\{ XI_o + \frac{X^3}{3!} I_1 - \frac{X^5}{5!} I_2 - \frac{X^7}{7!} I_3 + \dots \right\} \qquad \text{(C-26)}$$

where

$$I_n = \int_0^\infty e^{-\Phi T} \frac{\Phi^{n + \frac{5}{2}}}{\Phi^2 + 1} \, d\Phi, \quad n = 0, 1, 2, 3, \dots . \qquad \text{(C-27)}$$

The integral I_n, Eq. (C-27), can be considered as a Laplace transform with respect to the variable Φ. Its value is given in Tables of Integral Transforms, Vol. 1, Erdelyi, Sec. 4.3, No. 9. Using this formula, the following is obtained:

$$I_n = \pi(-1)^{n+1} \left\{ \cos \left[T + \left(\frac{n}{2} + \frac{7}{4}\right) \pi \right] + U_{-n-\frac{3}{2}}(2T,0) \right\} \qquad \text{(C-28)}$$

where

$$U_{-n-\frac{3}{2}}(2T,0) = \lim_{z \to 0} \sum_{m=0}^{\infty} (-1)^m \left(\frac{2T}{z}\right)^{2m-n-\frac{3}{2}} J_{2m-n-3/2}(z) \qquad \text{(C-29)}$$

After some manipulation the above becomes:

$$U_{-n-\frac{3}{2}}(2T,0) = \frac{T^{-n-\frac{3}{2}}}{\Gamma(-n-\frac{1}{2})} - \frac{T^{-n+\frac{1}{2}}}{\Gamma(-n+\frac{3}{2})} + \frac{T^{-n+\frac{5}{2}}}{\Gamma(-n+\frac{7}{2})} \dots \qquad \text{(C-30)}$$

Thus I_n can be evaluated for any T , and substituted into Eq. (C-26) to obtain I, the integral in Eq. (C-21). The remaining terms in Eq. (C-21) for M(x,t) can be found from tables.

C-2. Unsupported End

In the above discussion the solution and method of computation were presented for the case of a sinusoidal impact on the end of a simply supported beam, according to the elementary theory of beam vibrations. In this section are presented the corresponding solution and the computation method for the case of the unsupported end. As for the simply supported end, the differential equation to be solved here is the same, namely Eq. (C-1). The difference is only in one boundary condition; for the unsupported end zero shear force Q is assumed, instead of zero deflection. That is, Eq. (C-2) is replaced by:

$$Q(0,t) = Ak^2 w_{xxx}(0,t) = 0 \qquad \text{(C-31)}$$

The other boundary conditions and the initial conditions are exactly the same as those given for the simply supported case. They are Eqs. (C-3), (C-4), (C-5), and (C-6). The Laplace transform of Eq. (C-31) is:

$$Ak^2 \bar{w}_{xxx}(0,s) = 0 \qquad \text{(C-32)}$$

This replaces Eq. (C-9) in the previous discussion. The use of the general solution for the transform, Eq. (C-12), and the present boundary conditions, results in the following expression for $\bar{w}(x,s)$:

$$\bar{w}(x,s) = \frac{H}{\lambda A k}\left(\frac{1}{s} - \frac{s}{s^2 + \lambda^2}\right) e^{-x\sqrt{s/2k}} \sin\left(x\sqrt{\frac{s}{2k}}\right) - \sin \; x \frac{s}{2k} \qquad \text{(C-33)}$$

The transformed moment $\bar{M}(x,s)$ is obtained from Eq. (C-33) by twice differentiating with respect to x, and multiplying by Ak^2. The result is:

$$\bar{M}(x,s) = H \frac{\lambda}{(s^2 + \lambda^2)} e^{-x\sqrt{s/2k}} \left[\sin\left(x\sqrt{\frac{s}{2k}}\right) + \cos\left(x\sqrt{\frac{s}{2k}}\right) \right] \qquad \text{(C-34)}$$

For the purposes of inversion, the form of Eq. (C-34) is changed to the following:

$$\bar{M}(x,s) = \frac{\lambda}{2i} \left\{ (1 + i)\ \bar{m}_1(x,s) - (1 - i)\ \bar{m}_2(x,s) \right\} \qquad \text{(C-35)}$$

where

$$\bar{m}_1(x,s) = \frac{1}{s^2 + \lambda^2} e^{-(1 - i)\,x\sqrt{s/2k}} \qquad \text{(C-36)}$$

$$\bar{m}_2(x,s) = \frac{1}{s^2 + \lambda^2} e^{-(1 + i)\,x\sqrt{s/2k}} \qquad \text{(C-37)}$$

To invert Eq. (C-35), use is made of the inversion integral. Each part is inverted separately. The formulas for the inverses are:

$$m_1(x,t) = \frac{1}{2\pi i} \int_{\gamma-\infty}^{\gamma+\infty} e^{st}\, \bar{m}_1(x,s)\,ds \qquad \text{(C-38)}$$

$$m_2(x,t) = \frac{1}{2\pi i} \int_{\gamma-\infty}^{\gamma+\infty} e^{st}\, \bar{m}_2(x,s)\,ds \qquad \text{(C-39)}$$

where γ is greater than the real part of any singularities of $\bar{m}_1$ or $\bar{m}_2$ in the complex s plane. The singularities of $\bar{m}_1$ and $\bar{m}_2$ are:

a pole at $s = +i\lambda$
a pole at $s = -i\lambda$
a branch point at $s = 0$

Thus γ can be any positive real number.

The path of integration for the evaluation of Eqs. (C-38) and (C-39) is the path AB in Fig. C-1, (at end of the report), for which A and B are assumed to recede farther and farther from the real axis. In order to use Cauchy's theorem, a closed contour is formed by adding the parts shown in Fig. C-1 to AB. Since the entire closed contour does not enclose any singularities, the value of the integral around the complete path is zero. Another way of expressing this conclusion is to say that the integral from A to B along the crooked path is equal to the integral from A to B along the straight direct path. It can be shown that the integral along AG + CB, the large circular arc, approaches zero when $s \to \infty$. Therefore, the integral from A to B is equivalent to the sum of the integrals along GFE and EDC. In each of these paths, one circulation in the counterclockwise direction occurs around the poles at $s = -i\lambda$ and $+i\lambda$; also one circulation around the branch point at s=0 occurs. Since $\pm i\lambda$ are not branch points, the total contributions from the vertical paths is zero. Thus,

$$\int_{\underset{\circlearrowleft}{A}}^{B} = \int_{\underset{\rightarrow}{GE}} + \int_{\underset{\leftarrow}{EC}} + \int_{\underset{\circlearrowleft}{F}} + \int_{\underset{\circlearrowleft}{D}} + \int_{\underset{\circlearrowleft}{E}}$$

The theorem of residues can be used for $\int_{\underset{\circlearrowleft}{F}}$ and $\int_{\underset{\circlearrowleft}{D}}$. It is possible to show that $\int_{\underset{\circlearrowleft}{E}} \to 0$ as the radius of the loop around the branch point at the origin approaches zero. The residue at $s = i\lambda$ for m_1 or m_2 is given by:

$$\lim_{s \to i\omega} (s - i\omega) e^{st} \bar{m}_{1,2}$$

The remainder of the integral for m_1 or m_2 is expressed as an infinite integral along the negative real axis. The results, after a certain amount of manipulation are:

$$m_1(x,t) = \frac{e^{[i\lambda t - (1 - i)x\sqrt{i\lambda/2k}]} - e^{[-i\lambda t - (1 - i)\, x\sqrt{-i\lambda/2k}]}}{2i\omega}$$

$$+ \frac{1}{2\pi i}\int_0^\infty \frac{e^{-\varphi t}}{\varphi^2 + \lambda^2}\left[e^{i(1 - i)x\sqrt{\varphi/2k}} - e^{-i(1 - i)x\sqrt{\varphi/2k}}\right] d\varphi \qquad \text{(C-40)}$$

$$m_2(x,t) = \frac{e^{[i\lambda t - (1 + i)x\sqrt{i\lambda/2k}} - e^{[-i\lambda t - (1 + i)x\sqrt{-i\lambda/2k}}}{2i\omega}\, d\varphi$$

$$+ \frac{1}{2\pi i}\int_0^\infty \frac{e^{-\varphi t}}{\varphi^2 + \lambda^2}\left[e^{i(1 + i)x\sqrt{\varphi/2k}} - e^{-i(1 + i)x\sqrt{\varphi/2k}}\right] d\varphi \qquad \text{(C-41)}$$

From Eq. (C-35),

$$M(x,t) = \frac{\lambda}{2i}\left\{(1 + i)\, m_1(x,t) - (1 - i)\, m_2(x,t)\right\} \qquad \text{(C-42)}$$

On using Eqs. (C-40) and (C-41) in Eq. (C-42), the following is obtained:

$$M(x,t) = H\left\{\frac{1}{\sqrt{2}}\left[\cos x\sqrt{\frac{\pi}{k\tau}}\ \sin\left(\frac{\pi t}{\tau} + \frac{\pi}{4}\right)\right.\right.$$

$$\left. + \sin\left(\frac{\pi t}{\tau} - \frac{\pi}{4}\right) e^{-x\sqrt{\pi/k\tau}} + \sin x\,\frac{\pi}{k\tau}\right]$$

$$+ \frac{1}{\tau}\int_0^\infty \frac{e^{-\varphi t}}{\varphi^2 + \lambda^2}\ \sinh\left[\left(x\sqrt{\frac{\varphi}{2k}}\right)\cos\left(x\sqrt{\frac{\varphi}{2k}}\right) - \cosh\left(x\sqrt{\frac{\varphi}{2k}}\right)\sin\left(x\sqrt{\frac{\varphi}{2k}}\right)\right] d\varphi$$

----- (C-43)

To evaluate the infinite integral in Eq. (C-43), methods are used similar to those used in evaluating the integral in Eq. (C-21). The same change of variable, as given in (C-22), is used to simplify the manipulations in evaluating the integral of Eq. (C-43). The integral, including the factor $\frac{1}{T}$ when rewritten in terms of the new variables X, T, and Φ, becomes

$$I' = \frac{1}{\pi} \int_0^\infty \frac{e^{-\Phi T}}{\Phi^2 + 1} \left[\sinh\left(X\sqrt{\frac{\Phi}{2}}\right) \cos\left(X\sqrt{\frac{\Phi}{2}}\right) - \cosh\left(X\sqrt{\frac{\Phi}{2}}\right) \sin\left(X\sqrt{\frac{\Phi}{2}}\right)\right] d\Phi \qquad \text{------(C-44)}$$

The series expansion for the bracket containing the hyperbolic function and the circular functions is:

$$\left[\sinh\left(X\sqrt{\frac{\Phi}{2}}\right) \cos\left(X\sqrt{\frac{\Phi}{2}}\right) - \cosh\left(X\sqrt{\frac{\Phi}{2}}\right) \sin\left(X\sqrt{\frac{\Phi}{2}}\right)\right]$$

$$= -\sqrt{2}\left[\frac{1}{3!} X^3 \Phi^{3/2} - \frac{1}{7!} X^7 \Phi^{7/2} + \frac{1}{11!} X^{11} \Phi^{11/2} -- \ldots\right] \qquad \text{(C-45)}$$

Thus, the integral I', as defined in Eq. (C-44) is given by the series

$$I' = -\frac{\sqrt{2}}{\pi}\left\{\frac{X^3}{3!} I'_1 - \frac{X^7}{7!} I'_3 + \frac{X^{11}}{11!} I'_5 -- \ldots\right\} \qquad \text{(C-46)}$$

where

$$I'_n = \int_0^\infty e^{-\Phi T} \frac{\Phi^{n+\frac{1}{2}}}{\Phi^2 + 1} d\Phi, \; n = 1, 3, 5, \ldots \qquad \text{(C-47)}$$

Using the same formula as referred to after Eq. (C-27), after certain manipulations, I'_n becomes:

$$I'_n = \pi \left\{ \cos \left[T + \left(\frac{n}{2} + \frac{3}{4}\right)\pi \right] + \left[\frac{T^{-n+\frac{1}{2}}}{\Gamma(-n+\frac{3}{2})} - \frac{T^{-n+\frac{5}{2}}}{\Gamma(-n+\frac{7}{2})} + \frac{T^{-n+\frac{9}{2}}}{\Gamma(-n+\frac{9}{2})} \; -- \; . \; . \; . \right] \right\} \qquad \text{(C-48)}$$

Thus I'_n can be evaluated, and substituted into Eqs. (C-46), and (C-43) to obtain $M(x,t)$.

APPENDIX D

LAPLACE TRANSFORM SOLUTION OF THE TIMOSHENKO WAVE EQUATION FOR A SEMI-INFINITE SIMPLY SUPPORTED BEAM WITH SINUSOIDAL MOMENT IMPACT

In Section 3, the equations describing the motion of a beam in bending, including the effects of shear and rotatory inertia, are given. See Eqs. (3-8), (3-9), (3-10), (3-11). They are repeated here for convenience.

$$\frac{\partial M}{\partial x} - Q = Ak^2 \frac{\partial \omega}{\partial t} \tag{D-1}$$

$$\frac{\partial Q}{\partial x} = A \frac{\partial v}{\partial t} \tag{D-2}$$

$$\frac{\partial M}{\partial t} = Ak^2 \frac{\partial \omega}{\partial x} \tag{D-3}$$

$$C \frac{\partial Q}{\partial t} = A \left(\frac{\partial v}{\partial x} + \omega\right) \tag{D-4}$$

In order to have an independent check on the numerical solutions described in Section 3, the above equations have been solved for one particular case, using the Laplace transform technique. The boundary conditions assumed are those corresponding to a simply supported end subjected to a suddenly applied sinusoidal moment impact. That is:

$$\left.\begin{aligned} M(0,t) &= H \sin \frac{\pi t}{\tau}, \quad t \leq 0 \qquad &\text{(D-5)} \\ &= 0 \qquad\qquad\quad, \quad t \leq 0 & \\ v(0,t) &= 0 &\text{(D-6)} \end{aligned}\right\}$$

The initial conditions assumed are that all four variables M, ω, Q, v are zero prior to the impact. That is:

$$M(x,0) = \omega(x,0) = Q(x,0) = v(x,0) = 0 \tag{D-7}$$

Application of the Laplace transform to the four linear equation, (D-1), (D-2), (D-3), (D-4), and use of the initial conditions (D-7) yields:

$$\bar{M}_x - \bar{Q} = Ak^2 s\bar{\omega} \qquad \text{(D-8)}$$

$$\bar{Q}_x = As\bar{v} \qquad \text{(D-9)}$$

$$\bar{\omega}_x = \frac{1}{Ak^2}\, s\, \bar{M} \qquad \text{(D-10)}$$

$$\bar{v}_x = \frac{C}{A}\; s\bar{Q} - \bar{\omega} \qquad \text{(D-11)}$$

$$\text{where } \bar{M}(x,s) = \int_0^\infty e^{st}\; M(x,t)dt, \text{ etc.} \qquad \text{(D-12)}$$

The transformed boundary conditions (D-5) and (D-6) are as follows:

$$\bar{M}(0,s) = \frac{\pi H}{\tau}\; \frac{1}{s^2 + (\frac{\pi}{\tau})^2} \qquad \text{(D-13)}$$

$$\bar{v}(0,s) = 0 \qquad \text{(D-14)}$$

Substitution of Eqs. (D-13) and (D-14) into the transformed differential equations, (D-8), (D-9), (D-10), and (D-11) with subsequent elimination of all but $\bar{M}$ and its derivatives, the equivalent of Eq. (D-14) is obtained. That is:

$$\bar{M}_{xx}(0,s) = \frac{\pi H}{\tau}\; \frac{s^2}{s^2 + (\frac{\pi}{\tau})^2} \qquad \text{(D-15)}$$

Also, on eliminating all but $\bar{M}$ and its derivatives in Eqs. (D-8), (D-9), (D-10), and (D-11), the differential equation for $\bar{M}$ is obtained. It is:

$$k^2\bar{M}_{xxxx}(x,s) - k^2 s^2 (1 + C)\bar{M}_{xx}(x,s) + s^2(1 + k^2 Cs^2)\bar{M}(x,s) = 0 \qquad \text{(D-16)}$$

The solution of Eq. (D-16), satisfying the boundary conditions (D-13) and (D-15), is:

$$\bar{M}(x,s) = \bar{M}_1(x,s) + \bar{M}_2(x,s) \tag{D-17}$$

where

$$\bar{M}_1(x,s) = \frac{\pi H}{2\tau} \frac{(\lambda_1^2 - s^2C)\;{}^{-\lambda_1 x}}{\mathcal{M}^2\mathcal{N}s(s^2 - a^2)^{1/2}(s^2 + \frac{\pi^2}{\tau^2})}, \tag{D-18}$$

$$\bar{M}_2(x,s) = -\frac{\pi H}{2\tau} \frac{(\lambda_2^2 - s^2C)\;{}^{-\lambda_2 x}}{\mathcal{M}^2\mathcal{N}s(s^2 - a^2)^{1/2}(s^2 + \frac{\pi^2}{\tau^2})} \tag{D-19}$$

and where

$$\lambda_1 = \left[\mathcal{M}^2 s^2 + \mathcal{M}^2\mathcal{N}s\,(s^2 - a^2)^{1/2}\right]^{1/2} \tag{D-20}$$

$$\lambda_2 = \left[\mathcal{M}^2\mathcal{N}s^2 - \mathcal{M}^2\mathcal{N}s(s^2 - a^2)^{1/2}\right]^{1/2} \tag{D-21}$$

The definitions of the symbols used in the above are given in Section 3, but are repeated here also.

$C = \frac{c_o^2}{c_Q^2}$

$c_o = \sqrt{\frac{E}{\rho}}$ = bending wave velocity

$c_Q = \sqrt{K' \frac{G}{\rho}}$ = shear wave velocity

K' = Timoshenko shear correction factor

$\mathcal{M} = \sqrt{\frac{1 + C}{2}}$

$\mathcal{N} = \frac{C - 1}{C + 1}$

$a = \frac{2}{k(C - 1)}$

$b = \frac{1}{k\sqrt{C}}$

The formula for $M(x,t)$ is obtained by means of the inversion integral. That is,

$$M(x,t) = M_1(x,t) + M_2(x,t) \tag{D-22}$$

where

$$M_{1,2}(x,t) = \frac{1}{2\pi i} \int_{\gamma+\infty}^{\gamma+\infty} \bar{M}_{1,2}(x,s) e^{st} \, ds \tag{D-23}$$

In the above integral, γ is greater than the real part of any singularity of $\bar{M}_1$ and $\bar{M}_2$. The singularities of $\bar{M}_1$ and $\bar{M}_2$ are:

Branch point at	$s = 0$
Branch points at	$s = \pm a$
Branch points at	$s = \pm ib$
Simple poles at	$s = \pm i\frac{\pi}{T}$

In Fig. D-1 is shown the closed contour for which Cauchy's theorem can be used. The path of integration is along the vertical line from $\gamma - i\beta$ to $\gamma + i\beta$, as $\beta \to \infty$. Because of Cauchy's theorem the crooked contour from $\gamma - i\beta$, around all the singularities, to $\gamma + i\beta$ is equivalent to the straight one.

For the functions $\bar{M}_1(x,s)$ and $\bar{M}_2(x,s)$ it can be shown that the portions of the integral contributed by integrating along ℓ_1, L_1, ℓ_6, L_6, ℓ_7, L_7, r_3, R_3, r_1 and r_2 are all zero. This leaves only ℓ_2, L_2, ℓ_3, L_3, ℓ_4, L_4, ℓ_5, L_5, r_4 and R_4 to produce non-zero values. In Fig. D-2 are shown the values of s, and certain functions of s needed for evaluation of the integrals, along the portions of the contour just listed.

Using the results shown on Fig. D-2, it can be shown that for $x \leq t \leq \sqrt{C}\, x$, the integral for $M_1(x,t)$ vanishes. Hence,

$$\left.\begin{aligned} M(x,t) &= M_2(x,t) \text{ for } x \leq t \leq \sqrt{C}\, x \\ M(x,t) &= M_1(x,t) + M_2(x,t) \text{ for } t \geq \sqrt{C}\, x \end{aligned}\right\} \qquad \text{(D-24)}$$

After some manipulation, the following results are obtained:

$$\Big[M_1(x,t)\Big]_{L_2+\ell_2+L_5+\ell_5} = \frac{H}{\tau}\int_0^a \frac{1}{r^2+(\frac{\pi}{\tau})^2}\left\{\frac{r}{\sqrt{a^2-r^2}}\cos M\psi x \sinh (M\zeta x - rt) + \sin M\psi x \cosh (M\zeta x - rt)\right\} dr \qquad \text{(D-25)}$$

$$\Big[M_1(x,t)\Big]_{L_3+\ell_3+L_4+\ell_4} = 0 \qquad \text{(D-26)}$$

$$\Big[M_2(x,t)\Big]_{L_2+\ell_2+L_5+\ell_5} = -\frac{H}{\tau}\int_0^a \frac{1}{r^2+(\frac{\pi}{\tau})^2}\left\{\frac{r}{\sqrt{a^2-r^2}}\cos M\psi x \sinh (M\zeta x - rt) + \sin M\psi x \cosh (M\zeta x - rt)\right\} dr \qquad \text{(D-27)}$$

$$\Big[M_2(x,t)\Big]_{L_3+\ell_3+L_4+\ell_4} = -\frac{H}{\tau}\int_0^b \left[1+\frac{r}{\sqrt{a^2+r^2}}\right]\frac{1}{(\frac{\pi}{\tau})^2-r^2}\cos rt \sinh Sx \, dr \qquad \text{(D-28)}$$

The poles at $s = \pm i \frac{\pi}{\tau}$ contribute to M_1 and M_2 also. Using the fact that M_1 or M_2 is equal to the sum of the residues of $\bar{M}_1 e^{st}$ or $\bar{M}_2 e^{st}$ at the two poles, the following are obtained for $\underline{\frac{\pi}{\tau} > b}$,

$$(M_1)_{poles} = \frac{H}{2}\left[1 - \frac{\frac{\pi}{\tau}}{\sqrt{a^2 + \left(\frac{\pi}{\tau}\right)^2}}\right]\left\{- \sin T_o x \cos \frac{\pi t}{\tau} + \cos T_o x \sin \frac{\pi t}{\tau}\right\} \qquad \text{(D-29)}$$

$$(M_2)_{poles} = \frac{H}{2}\left[1 + \frac{\frac{\pi}{\tau}}{\sqrt{a^2 + \frac{\pi}{\tau}^2}}\right]\left\{- \sin S_o x \cos \frac{\pi t}{\tau} + \cos S_o x \sin \frac{\pi t}{\tau}\right\} \qquad \text{(D-30)}$$

Thus, on using Eqs. (D-24), (D-25), (D-26), (D-27), (D-28), (D-29), (D-30), there results:

For $t \geq Cx$

$$\begin{aligned} M(x,t) = {} & \frac{H}{2}\left[1 - \frac{\frac{\pi}{\tau}}{\sqrt{a^2 + (\frac{\pi}{\tau})}}\right]\left[- \sin T_o x \cos \frac{\pi t}{\tau} + \cos T_o x \sin \frac{\pi t}{\tau}\right] \\ & + \frac{H}{2}\left[1 + \frac{\frac{\pi}{\tau}}{\sqrt{a^2 + (\frac{\pi}{\tau})^2}}\right]\left[- \sin S_o x \cos \frac{\pi t}{\tau} + \cos S_o x \sin \frac{\pi t}{\tau}\right] \qquad \text{(D-31)} \\ & - \frac{H}{\tau}\int_0^b \left[1 + \frac{r}{\sqrt{a^2 + r^2}}\right]\frac{1}{(\frac{\pi}{\tau})^2 - r^2} \cos rt \sinh Sx \, dr, \end{aligned}$$

if $\frac{\pi}{\tau} > b$

For $x \leq t \leq \sqrt{C}\, x$,

$$M(x,t) = \frac{H}{2} \left[1 + \frac{\frac{\pi}{\tau}}{\sqrt{a^2 + (\frac{\pi}{\tau})^2}} \right] \left[-\sin S_o x \cos \frac{\pi t}{\tau} + \cos S_o x \sin \frac{\pi t}{\tau} \right]$$

$$- \frac{H}{\tau} \int_0^a \frac{1}{r^2 + (\frac{\pi}{\tau})^2} \left[\frac{r}{\sqrt{a^2 - r^2}} \cos \mathcal{M}\psi x \sinh (\mathcal{M}\zeta x - rt) + \sin \mathcal{M}\psi x \cosh (\mathcal{M}\zeta x - rt) \right] dr \qquad \text{(D-32)}$$

$$- \frac{H}{\tau} \int_0^b \left[1 + \frac{r}{\sqrt{a^2 + r^2}} \right] \frac{1}{(\frac{\pi}{\tau})^2 - r^2} \cos rt \sinh Sx \, dr$$

$$\text{if } \frac{\pi}{\tau} > b$$

If $\underline{\frac{\pi}{\tau} < b}$, the corresponding results are:

For $t \geq \sqrt{C}\, x$,

$$M(x,t) = \frac{H}{2} \left[1 - \frac{\frac{\pi}{\tau}}{\sqrt{a^2 + (\frac{\pi}{\tau})^2}} \right] \left[-\sin T_o x \cos \frac{\pi t}{\tau} + \cos T_o x \sin \frac{\pi t}{\tau} \right]$$

$$+ \frac{H}{2} \left[1 + \frac{\frac{\pi}{\tau}}{\sqrt{a^2 + (\frac{\pi}{\tau})^2}} \right] \cosh S_o x \sin \frac{\pi t}{\tau} \qquad \text{(D-33)}$$

$$- \frac{H}{\tau} \int_0^b \left[1 + \frac{r}{\sqrt{a^2 + r^2}} \right] \frac{1}{(\frac{\pi}{\tau})^2 - r^2} \cos rt \sinh Sx \, dr$$

$$\text{if } \frac{\pi}{\tau} < b$$

For $x \leq t \leq \sqrt{C}\, x$,

$$
\begin{aligned}
M(x,t) = {} & \frac{H}{2}\left[1 + \frac{\frac{\pi}{\tau}}{\sqrt{a^2 + (\frac{\pi}{\tau})^2}}\right]\cosh S_o x \sin\frac{\pi t}{\tau} \\
& - \frac{H}{\tau}\int_0^a \frac{1}{r^2 + (\frac{\pi}{\tau})^2}\left[\frac{r}{\sqrt{a^2 - r^2}}\cos \mathcal{M}\psi x \sinh(\mathcal{M}\zeta x - rt) + \sin \mathcal{M}\psi x \cosh(\mathcal{M}\zeta x - rt)\right] dr \\
& - \frac{H}{\tau}\int_0^b \left[1 + \frac{r}{\sqrt{a^2 + r^2}}\right]\frac{1}{(\frac{\pi}{\tau})^2 - r^2}\cos rt \sinh Sx\, dr
\end{aligned}
\qquad \text{(D-34)}
$$

if $\frac{\pi}{\tau} < b$

where in the foregoing,

$\mathcal{M} = \sqrt{\frac{1 + C}{2}}$, $\mathcal{N} = \frac{C - 1}{C + 1}$, $a = \frac{2}{k(C - 1)}$, $b = \frac{1}{k\sqrt{C}}$

$R = \sqrt{r^2 + \mathcal{N}^2 (a^2 - r^2)}$

$\zeta = \sqrt{\frac{r(R + r)}{2}}$, $\psi = \sqrt{\frac{r(R - r)}{2}}$

$S = \mathcal{M}\sqrt{r}\left[\mathcal{N}\sqrt{a^2 + r^2} - r\right]^{1/2}$

$T = \mathcal{M}\sqrt{r}\left[\mathcal{N}\sqrt{a^2 + r^2} + r\right]^{1/2}$

$S_o = (S)_{r = \frac{\pi}{\tau}}$

$T_o = (T)_{r = \frac{\pi}{\tau}}$

The integrals appearing in the formulas for $M_1(x,t)$ and $M_2(x,t)$ have to be evaluated numerically. First the expressions are rewritten as follows: (Only the case $\frac{\pi}{\tau} < b$ is computed) For $t \geq \sqrt{C}\, x$,

$$M(x,t) = \frac{H}{2} \left[1 - \frac{\frac{\pi}{\tau}}{\sqrt{a^2 + (\frac{\pi}{\tau})^2}} \right] \left[- \sin T_o x \cos \frac{\pi t}{\tau} + \cos T_o x \sin \frac{\pi t}{\tau} \right] \tag{D-35}$$

$$+ \frac{H}{2} \left[1 + \frac{\frac{\pi}{\tau}}{a^2 + (\frac{\pi}{\tau})^2} \right] \cosh S_o x \sin \frac{\pi t}{\tau}$$

$$- \frac{H}{\tau} K_3$$

For $x \leq t \leq \sqrt{C}\, x$,

$$M(x,t) = \frac{H}{2} \left[1 + \frac{\frac{\pi}{\tau}}{\sqrt{a^2 + (\frac{\pi}{\tau})^2}} \right] \cosh S_o x \sin \frac{\pi t}{\tau} \tag{D-36}$$

$$- \frac{H}{\tau} \left[K_1 + K_2 + K_3 \right]$$

where

$$K_1 = \int_0^a \frac{1}{r^2 + (\frac{\pi}{\tau})^2} \; \frac{r}{\sqrt{a^2 - r^2}} \; \cos M\psi x \sinh (M\zeta x - rt)dr \tag{D-37}$$

$$K_2 = \int_0^a \frac{1}{r^2 + (\frac{\pi}{\tau})^2} \; \sin M\psi x \cosh (M\zeta x - rt)dr \tag{D-38}$$

$$K_3 = \int_0^b \left[1 + \frac{r}{\sqrt{a^2 + r^2}} \right] \frac{1}{(\frac{\pi}{\tau})^2 - r^2} \; \cos rt \sinh Sx \, dr \tag{D-39}$$

The integrals K_1 and K_3 are improper; the integrand of K_1 is singular at $r = a$, and the integrand of K_3 is singular at $r = \frac{\pi}{\tau}$ (it is assumed that $\frac{\pi}{\tau} < b$; hence, this singularity is between the limits of the integral). The integrand of K_2 is regular at all points between 0 and a, and hence can be calculated directly by means of a numerical integration scheme. The integrals K_1 and K_3 are split into two parts, one of which contains the singularity in a convenient integrable form, and the other part regular. Consider K_1 , first:

Let,

$$f_1(r;x,t) = \frac{r}{r^2 + (\frac{\pi}{\tau})^2} \cos M\psi x \sinh(M\zeta x - rt) \qquad \text{(D-40)}$$

Then

$$K_1 = \int_0^a \frac{f_1(r;x,t)}{\sqrt{a^2 - r^2}} dr \qquad \text{(D-41)}$$

Write $f_1(r;x,t)$ as follows:

$$f_1(r;x,t) = f_1(a;x,t) + \left[f_1(r;x,t) - f_1(a;x,t) \right] \qquad \text{(D-42)}$$

Then K_1 becomes

$$K_1 = \int_0^a \frac{f_1(a;x,t)}{\sqrt{a^2 - r^2}} dr + \int_0^a \frac{f_1(r;x,t) - f_1(a;x,t)}{\sqrt{a^2 - r^2}} dr \qquad \text{(D-43)}$$

Now consider the following limit:

$$\lim_{r \to a} \left[\frac{f_1(r;x,t) - f_1(a;x,t)}{\sqrt{a^2 - r^2}} \right] \text{(of form } \tfrac{0}{0}\text{)}$$

$$= \lim_{r \to a} \frac{\left[\frac{\partial f_1(r;x,t)}{\partial r}\right]}{\frac{\partial}{\partial r}\left(\sqrt{a^2 - r^2}\right)} = \lim_{r \to a} \frac{\sqrt{a^2 - r^2}}{r} \quad \frac{\partial f_1(r;x,t)}{\partial r}$$

$$= 0 \text{ since } \lim_{r \to a} \frac{\partial f_1(r;x,t)}{\partial r} \text{ is finite.}$$

Therefore the integral,

$$L_1 = \int_0^a \frac{f_1(r;x,t) - f_1(a;x_2t)}{\sqrt{a^2 - r^2}} \, dr \qquad \text{(D-44)}$$

is regular and can be evaluated with standard numerical techniques. The following is obtained after integrating the first term of Eq. (D-43)

$$K_1 = f_1(a;x,t) \frac{\pi}{2} + L_1 \qquad \text{(D-45)}$$

where L_1 is to be evaluated numerically.

A similar procedure is followed to obtain a convenient expression for evaluating K_3. The intermediate steps are omitted, and only the final result is presented.

$$K_3 = f_3\left(\frac{\pi}{\tau};x,t\right) \frac{1}{2\left(\frac{\pi}{\tau}\right)} \ln\left(\frac{b + \frac{\pi}{\tau}}{b - \frac{\pi}{\tau}}\right) + L_3 \qquad \text{(D-46)}$$

where

$$f_3(r;x,t) = \left[1 + \frac{r}{\sqrt{a^2 + r^2}}\right] \cos rt \sinh Sx \qquad \text{(D-47)}$$

$$L_3 = \int_0^b \frac{\left[f_3(r;x,t) - f_3\left(\frac{\pi}{\tau};x,t\right)\right]}{\left(\frac{\pi}{\tau}\right)^2 - r^2} \, dr \qquad \text{(D-48)}$$

The integrand of L_3 is regular and hence L_3 can be found numerically. The case for which computations have been carried out is the following:

Bar: circular, diameter 0.516 inches; material, steel

$E = 30 \times 10^6$ lb/in.^{2}

$\nu = 0.30$

$\rho = 7.187 \times 10^{-4}$ lb. sec^2/in.^{4}

shear correction factor $K' = \frac{3}{4}$

ratio of wave velocities $\frac{c_M}{c_Q} = 2$; thus $C = \frac{c_o^2}{c_Q^2} = 4$.

Impact half period = 4 microseconds, hence $\tau = 2.37558$.

Derived constants:

$\mathfrak{M} = 1.58115$, $\mathfrak{N} = 0.6000$, $a = 2.66667$, $b = 2.00000$

$S_o = 1.23789$

$T_o = 3.20577$

Assumed value of $H = 1$.

Values were obtained for $x = 2.0$, $t = 2, 2.5, 3, 3.5, 4, \ldots 12$. These are plotted on Fig. 3-3, labeled "Laplace Transform".

APPENDIX E

CHARACTERISTICS SOLUTION FOR TIMOSHENKO EQUATION

In Section 3b it is stated that the differential equations of the variables M, ω, Q, v, along the four families of characteristics are:

$$\left.\begin{aligned}
dM - Ak^2 d\omega &= Q dt , && \text{along } dt = dx \\
dM + Ak^2 d\omega &= - Q dt , && \text{along } dt = - dx \\
dQ - \frac{A}{\sqrt{C}}\, dv &= \frac{A}{C}\, \omega dt , && \text{along } dt = \sqrt{C}\ dx \\
dQ + \frac{A}{\sqrt{C}}\, dv &= \frac{A}{C}\, \omega dt , && \text{along } dt = \sqrt{C}\ dx
\end{aligned}\right\} \qquad \text{(E-1)}$$

These are converted to difference equations by making use of the element in Fig. E-1, bounded by characteristics of the bending wave family.

Within this element there are two members of the shear wave family, shown dotted. The fifference equations, obtained directly from the differential equations, Eqs. (E-1), are:

$$\left.\begin{aligned}
(M_P - M_A) - Ak^2(\omega_P - \omega_A) &= \frac{Q_P + Q_A}{2}\ \Delta t \\
(M_P - M_C) + Ak^2(\omega_P - \omega_C) &= - \frac{Q_P + Q_C}{2}\ \Delta t \\
(Q_P - Q_{A'}) - \frac{A}{\sqrt{C}}\ (v_P - v_{A'}) &= \frac{A}{C}\ \frac{\omega_P + \omega_{A'}}{2}\ \Delta t' \\
(Q_P - Q_{C'}) + \frac{A}{\sqrt{C}}\ (v_P - v_{C'}) &= \frac{A}{C}\ \frac{\omega_P + \omega_{C'}}{2}\ \Delta t'
\end{aligned}\right\} \qquad \text{(E-2)}$$

In the foregoing,

$$\left.\begin{aligned}
\Delta t' &= \frac{2\sqrt{C}}{1+\sqrt{C}}\,\Delta t \\
Q_{A'} &= Q_B + \frac{2}{1+\sqrt{C}}\,(Q_A - Q_B) \\
Q_{C'} &= Q_B + \frac{2}{1+\sqrt{C}}\,(Q_C - Q_B) \\
&\text{etc., for } v_{A'},\ v_{C'},\ \omega_{A'},\ \omega_{C'}
\end{aligned}\right\} \tag{E-3}$$

If these equations are rewritten with all the terms having P as a subscript on the left side, the following are obtained:

$$\begin{aligned}
&M_P - Ak^2\omega_P - \tfrac{1}{2}\Delta t Q_P = S_1 \\
&M_P + Ak^2\omega_P + \tfrac{1}{2}\Delta t Q_P = S_2 \\
&Q_P - \frac{A}{\sqrt{C}}\left(v_P + \frac{1}{1+\sqrt{C}}\,\Delta t\omega_P\right) = S_3 \\
&Q_P + \frac{A}{\sqrt{C}}\left(v_P - \frac{1}{1+\sqrt{C}}\,\Delta t\omega_P\right) = S_4
\end{aligned} \tag{E-4}$$

where

$$S_1 = M_A - Ak^2\omega_A + \frac{1}{2}\Delta t Q_A$$

$$S_2 = M_C + Ak^2\omega_C - \frac{1}{2}\Delta t Q_C$$

$$\begin{aligned} S_3 &= \left(\frac{2}{1+\sqrt{C}}\right)\left[Q_A - \frac{A}{\sqrt{C}}\left(v_A - \frac{1}{1+\sqrt{C}}\Delta t\omega_A\right)\right] \\ &\quad + \left(1 - \frac{2}{1+\sqrt{C}}\right)\left[Q_B - \frac{A}{\sqrt{C}}\left(v_B - \frac{1}{1+\sqrt{C}}\Delta t\omega_B\right)\right] \qquad \text{(E-5)} \\ S_4 &= \left(\frac{2}{1+\sqrt{C}}\right)\left[Q_C + \frac{A}{\sqrt{C}}\left(v_C + \frac{1}{1+\sqrt{C}}\Delta t\omega_C\right)\right] \\ &\quad + \left(1 - \frac{2}{1+\sqrt{C}}\right)\left[Q_B + \frac{A}{\sqrt{C}}\left(v_B + \frac{1}{1+\sqrt{C}}\Delta t\omega_B\right)\right] \end{aligned}$$

Equations (E-4) are solved simultaneously for M_P, ω_P, Q_P, v_P. Their solution depends upon having previously obtained values of all four variables at A, B and C by means of the same equations. If the values are also known at D, in Fig. E-2, then the same process can be used to obtain the values at R.

Since new values depend upon previously obtained values, there has to be a starting point somewhere. In the problem solved in this report, the end of the bar is subject to various types of impacts in the form of half sine waves. In all cases, the values along the leading wave front, $x = t$, are zero. That is, $M = \omega = Q = v = 0$ along this line. Along the t-axis two of the variables are specified, say M and v; the others are determined from the two equations associated with the right half of the element of Fig. E-1. For example, if M and v are specified along the t-axis at the mesh points, then the second and fourth equations of Eqs. (E-4) are used to determine ω and Q at these points.

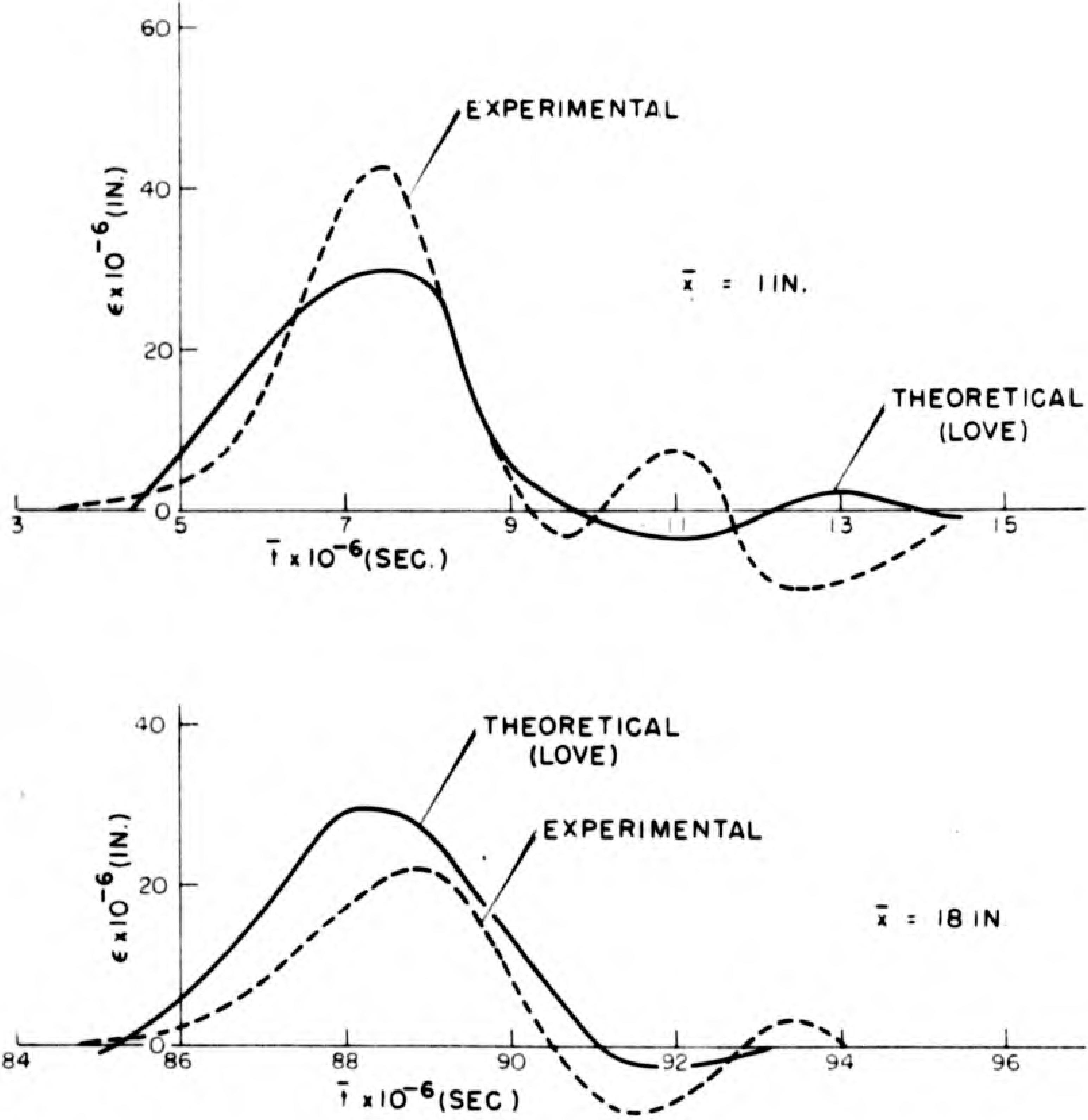

FIG. 2-1: COMPARISON OF THEORETICAL CALCULATIONS AND EXPERIMENTAL RESULTS FOR THE CASE GIVEN IN FIG. 9 OF RIPPERGER'S REPORT (REF. 13) WITH $\bar{a}$ = 1/16 IN. AND $\bar{\tau} = 4.4 \times 10^{-6}$ SEC

DRL – UT
DWG AA 1707
HJP – CLW
4 – 5 – 56

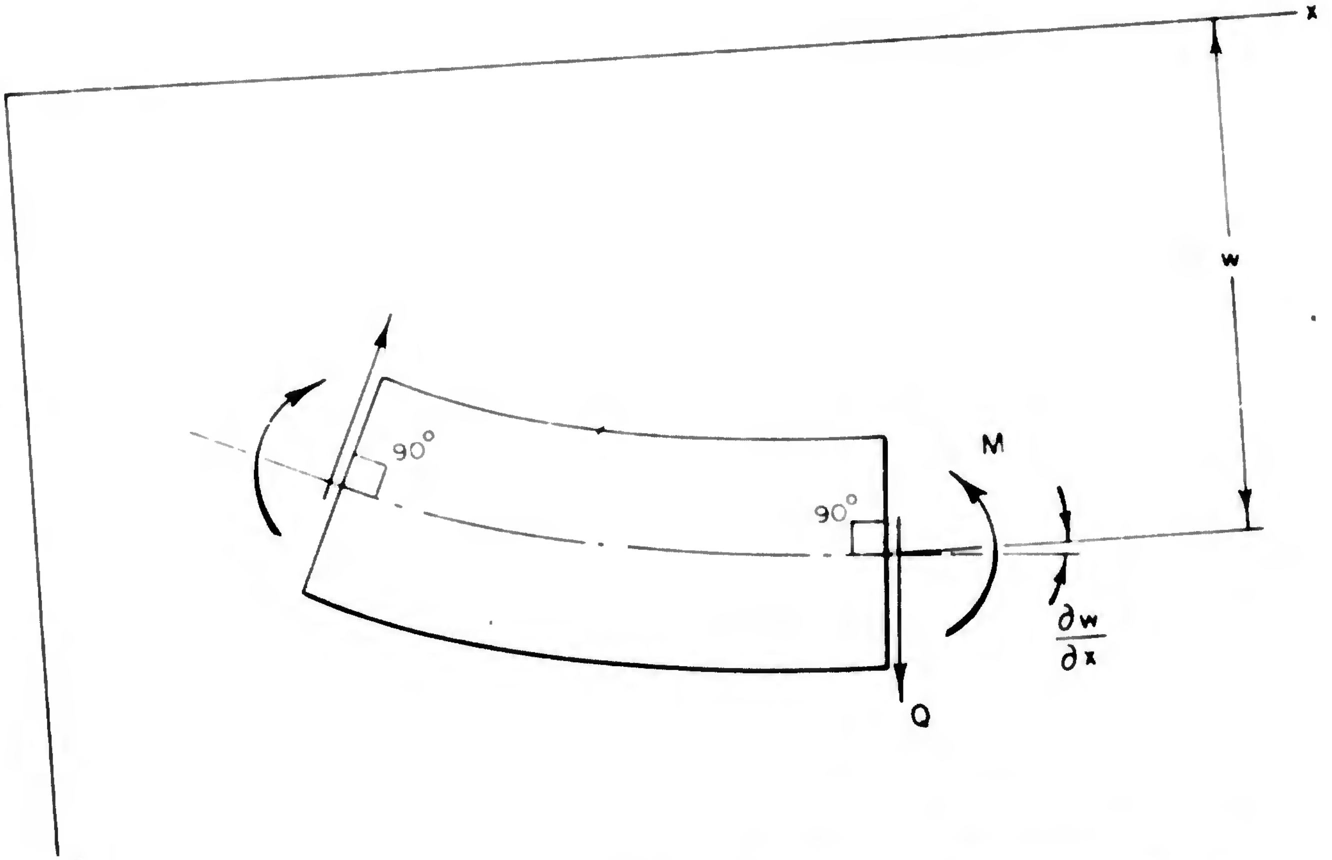

FIG. 3-1 ELEMENT OF BEAM AS ASSUMED IN THE ELEMENTARY THEORY

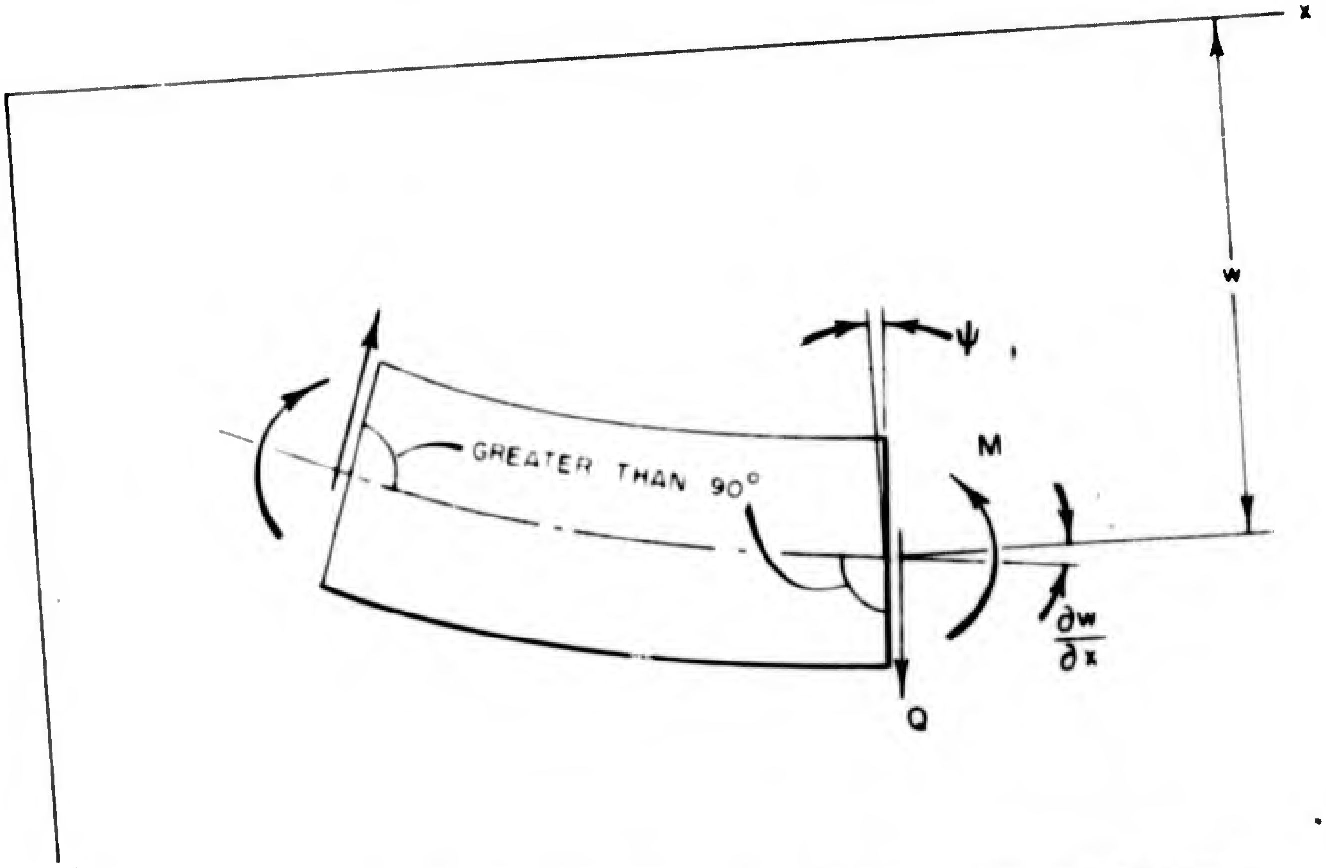

FIG. 3-2 ELEMENT OF BEAM AS ASSUMED IN THE TIMOSHENKO THEORY

DRL - UT
DWG AA 1645
HJP - CLW
3 - 5 - 56

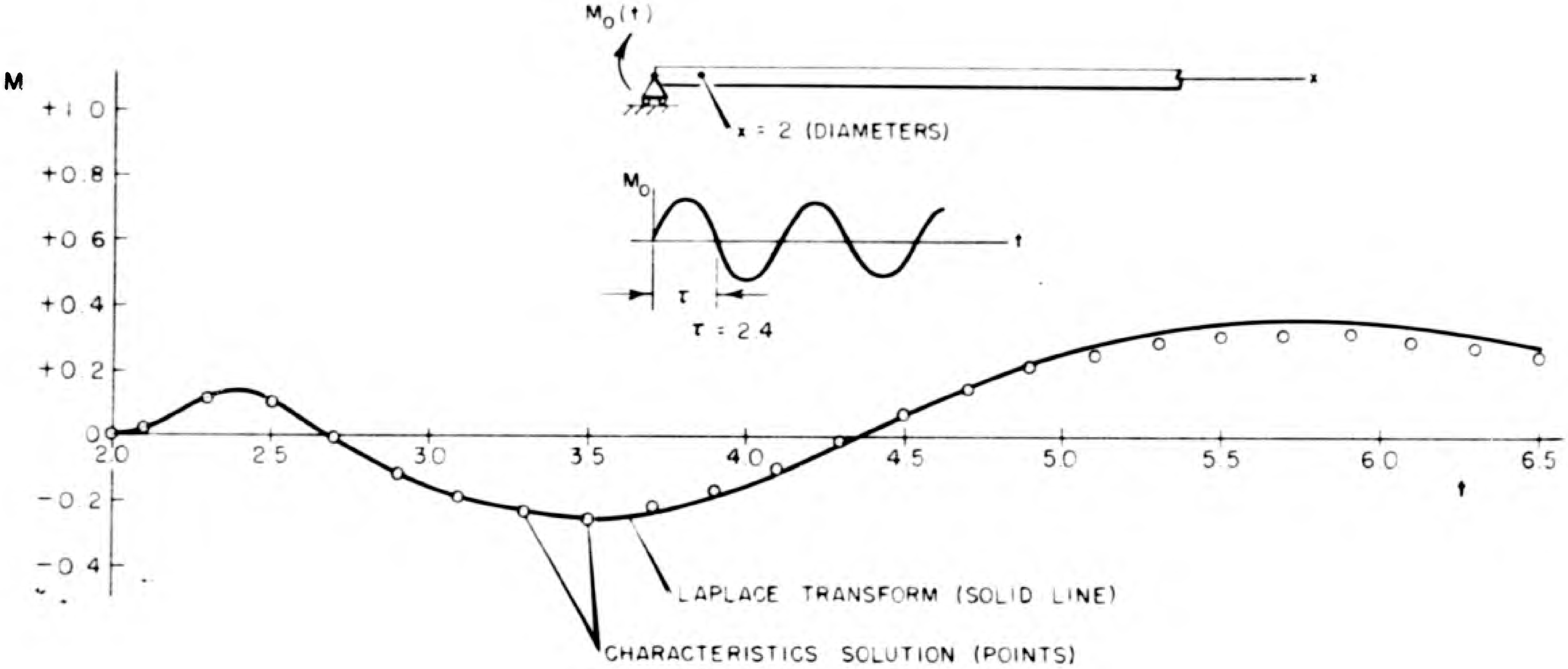

FIG. 3-3 COMPARISON OF RESULTS COMPILED FOR x = 2 FROM LAPLACE TRANSFORM AND CHARACTERISTICS

M AND t ARE DIMENSIONLESS.

DRL – UT
DWG AA 1646
HJP – CLW
3 – 12 – 56

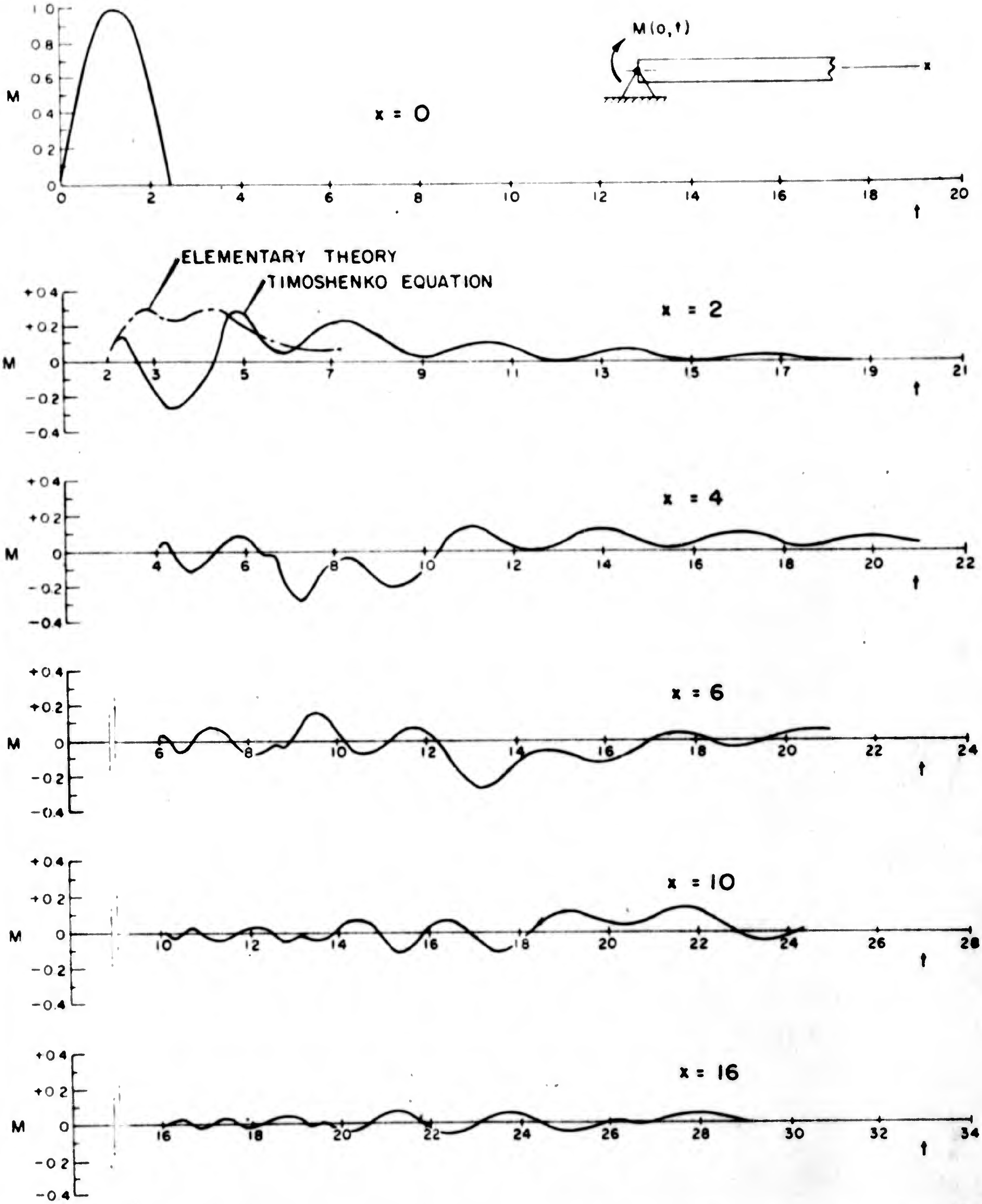

FIG. 3-4 (a): M–t CURVES AT VARIOUS STATIONS ALONG BAR WITH PINNED END, RESULTING FROM HALF SINE MOMENT IMPACT; $\tau = 2.4$, $C = 4$

DRL – UT
DWG AA 1647
HJP – CLW
2 – 20 – 56

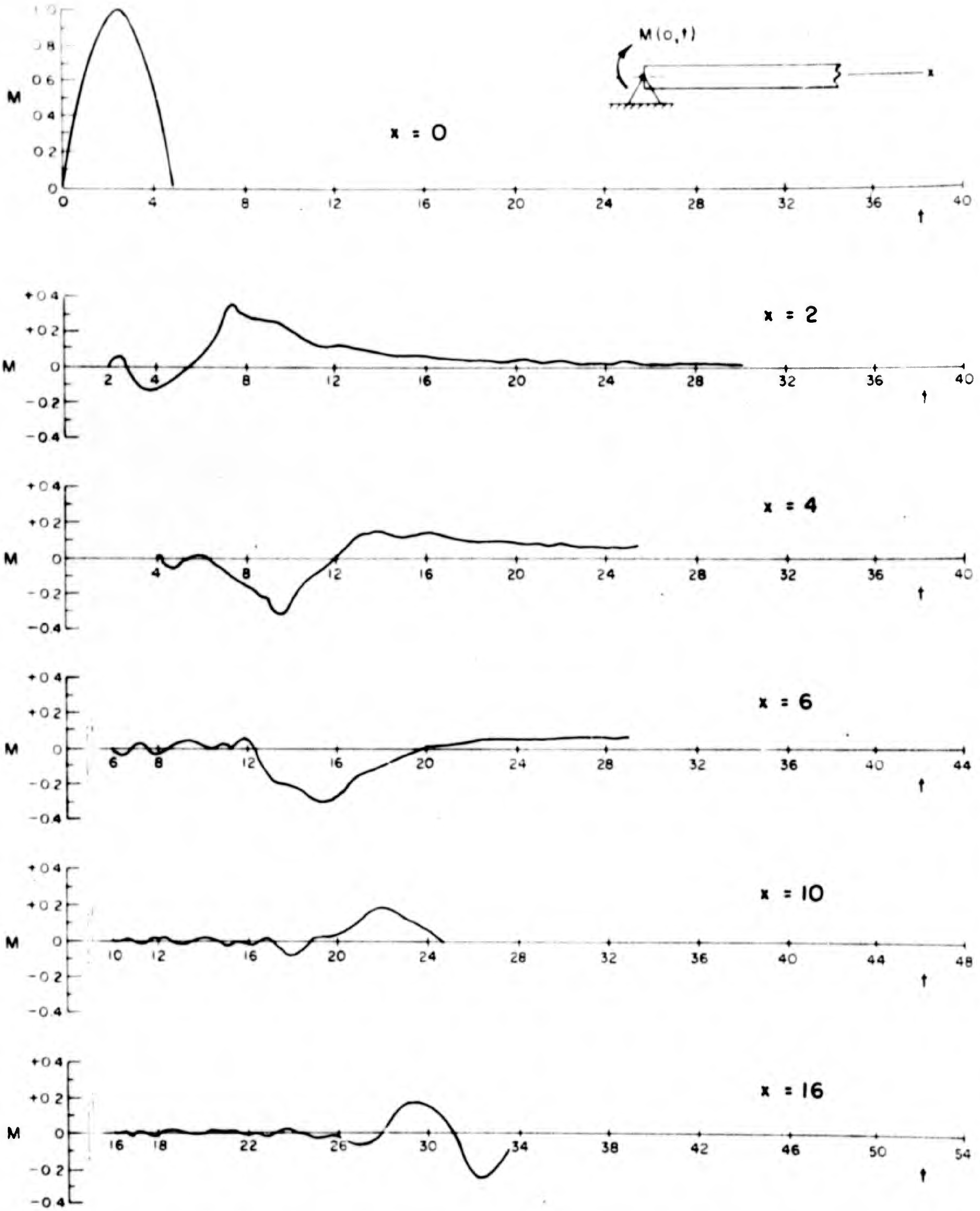

FIG. 3-4 (b): M-t CURVES AT VARIOUS STATIONS ALONG BAR WITH PINNED END, RESULTING FROM HALF SINE MOMENT IMPACT; $\tau = 4.8$, $C = 4$

DRL - UT
DWG AA 1648
HJP - CLW
2-20-56

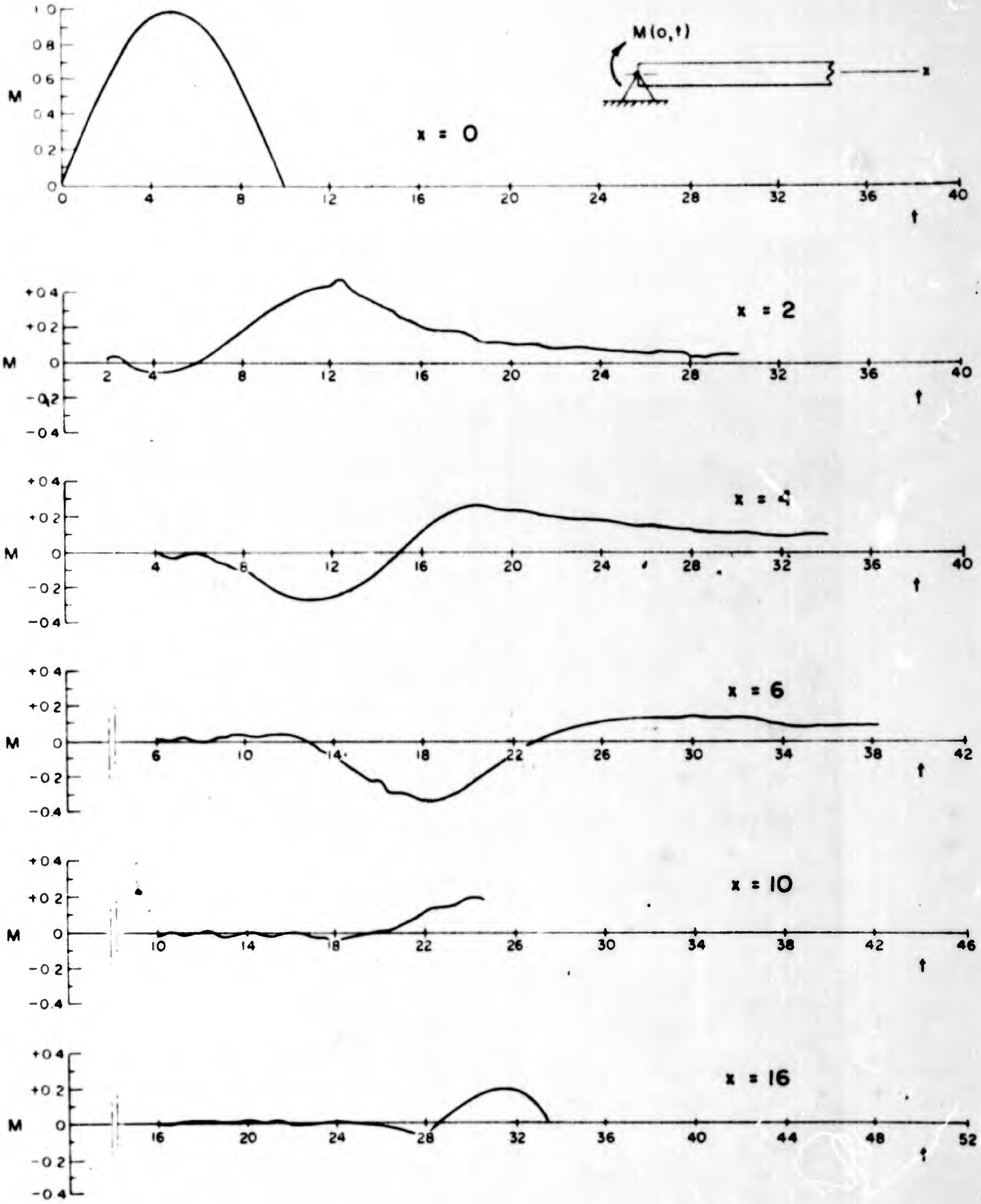

FIG. 3-4 (c): M-t CURVES AT VARIOUS STATIONS ALONG BAR WITH PINNED END, RESULTING FROM HALF SINE MOMENT IMPACT; $\tau = 10$, C = 4

DRL – UT
DWG AA 1649
HJP – CLW
2 - 20 - 56

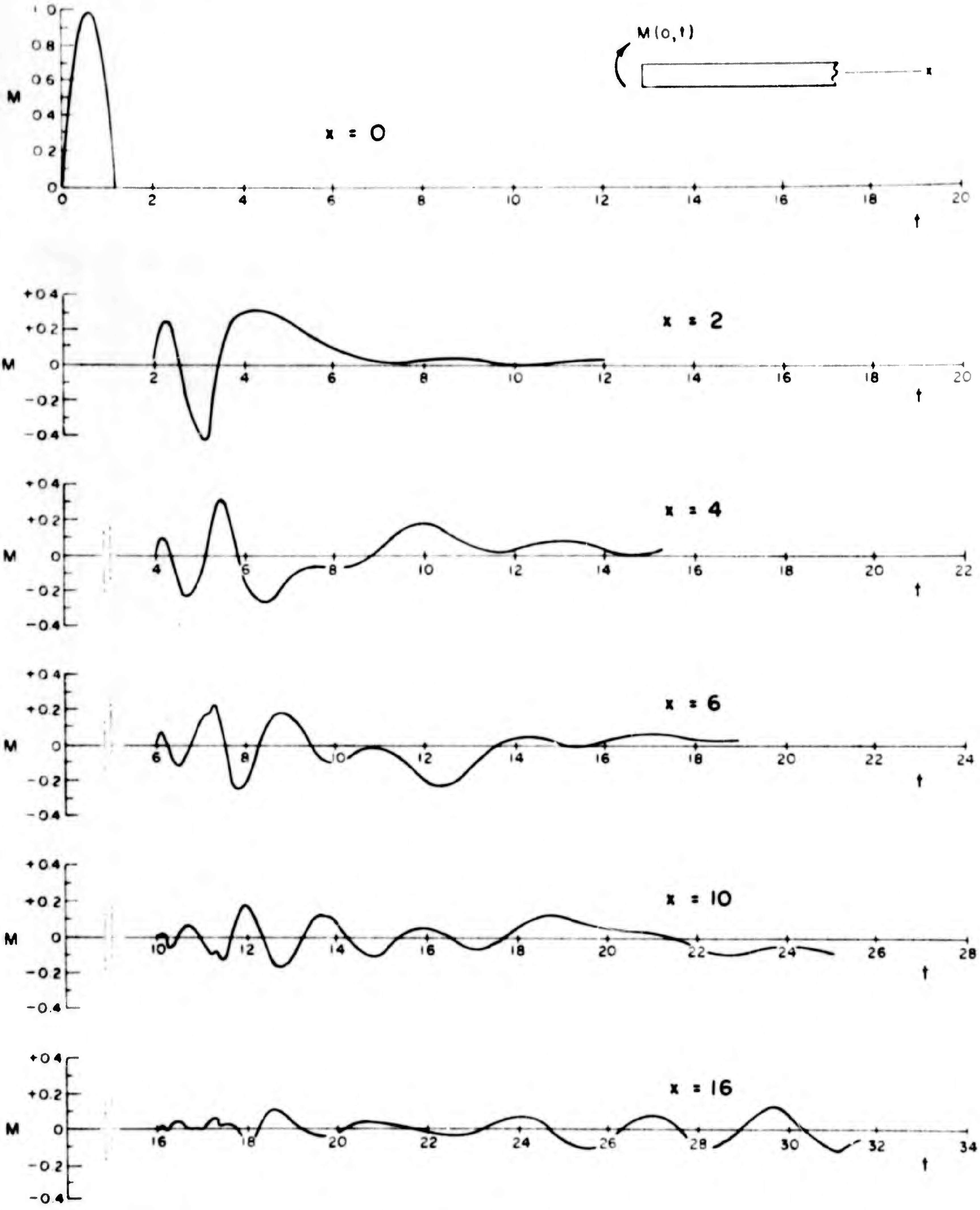

FIG. 3-5 (a): M-t CURVES AT VARIOUS STATIONS ALONG BAR WITH UNSUPPORTED END, RESULTING FROM HALF SINE MOMENT IMPACT; $\tau = 1.2$, $C = 4$

DRL - UT
DWG AA 1650
HJP - CLW
2 - 20 - 56

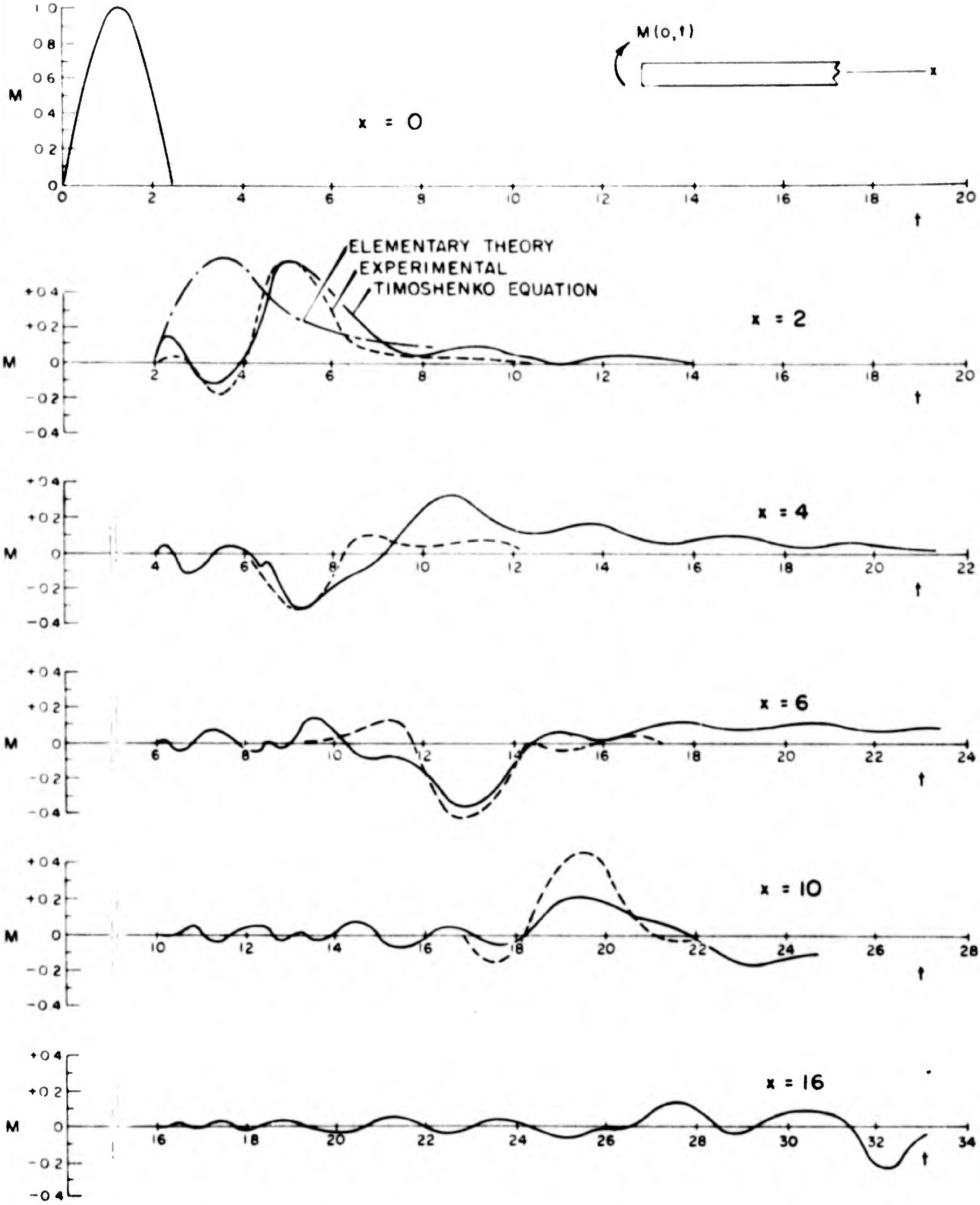

FIG. 3-5 (b): M-t CURVES AT VARIOUS STATIONS ALONG BAR WITH UNSUPPORTED END, RESULTING FROM HALF SINE MOMENT IMPACT; τ = 2.4 , C = 4

DRL - UT
DWG AA 1651
HJP - CLW
2-20-56

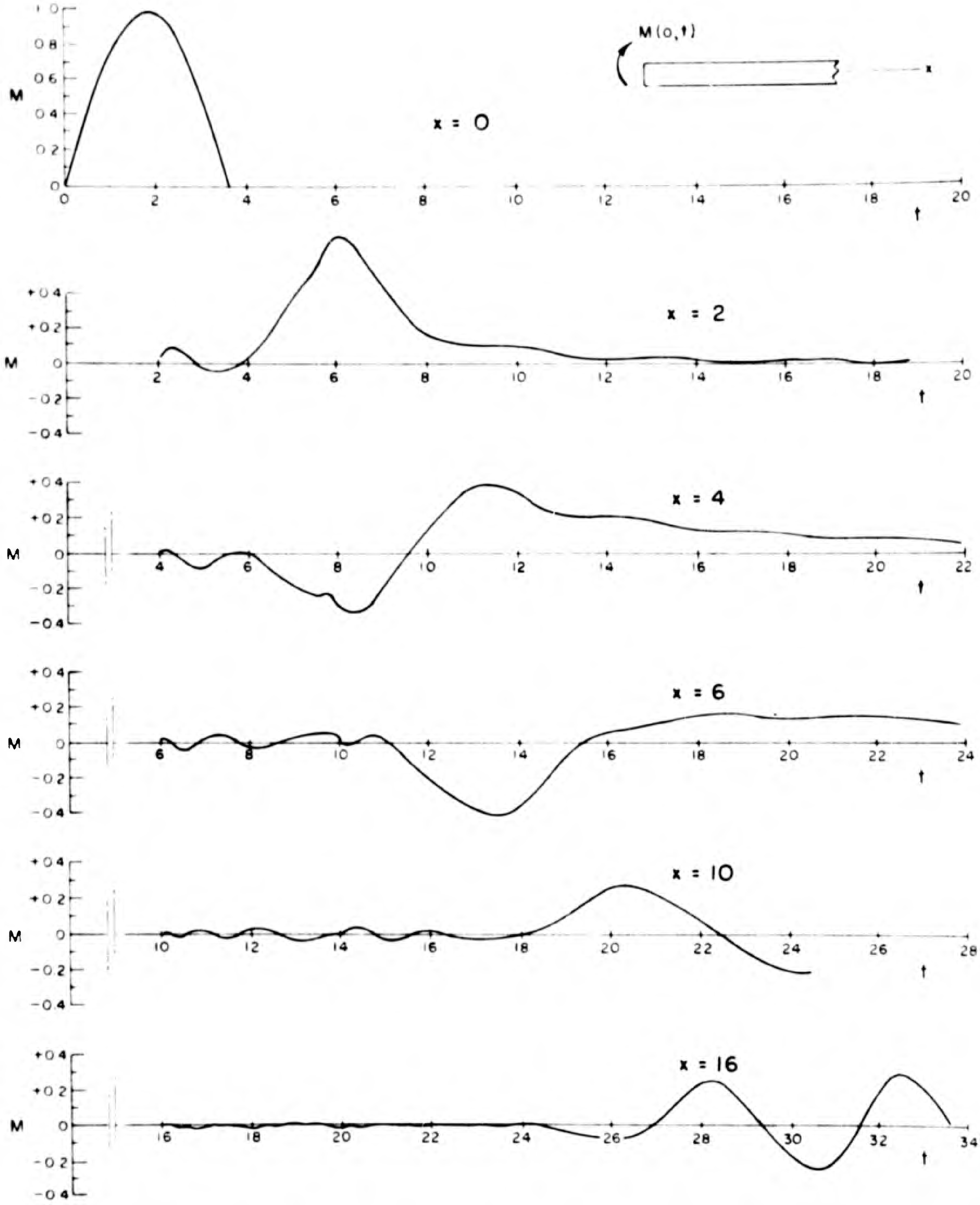

FIG. 3-5 (c): M-t CURVES AT VARIOUS STATIONS ALONG BAR WITH UNSUPPORTED END, RESULTING FROM HALF SINE MOMENT IMPACT; $\tau = 3.6$, $C = 4$

DRL - UT
DWG AA 1652
HJP - CLW
2 - 20 - 56

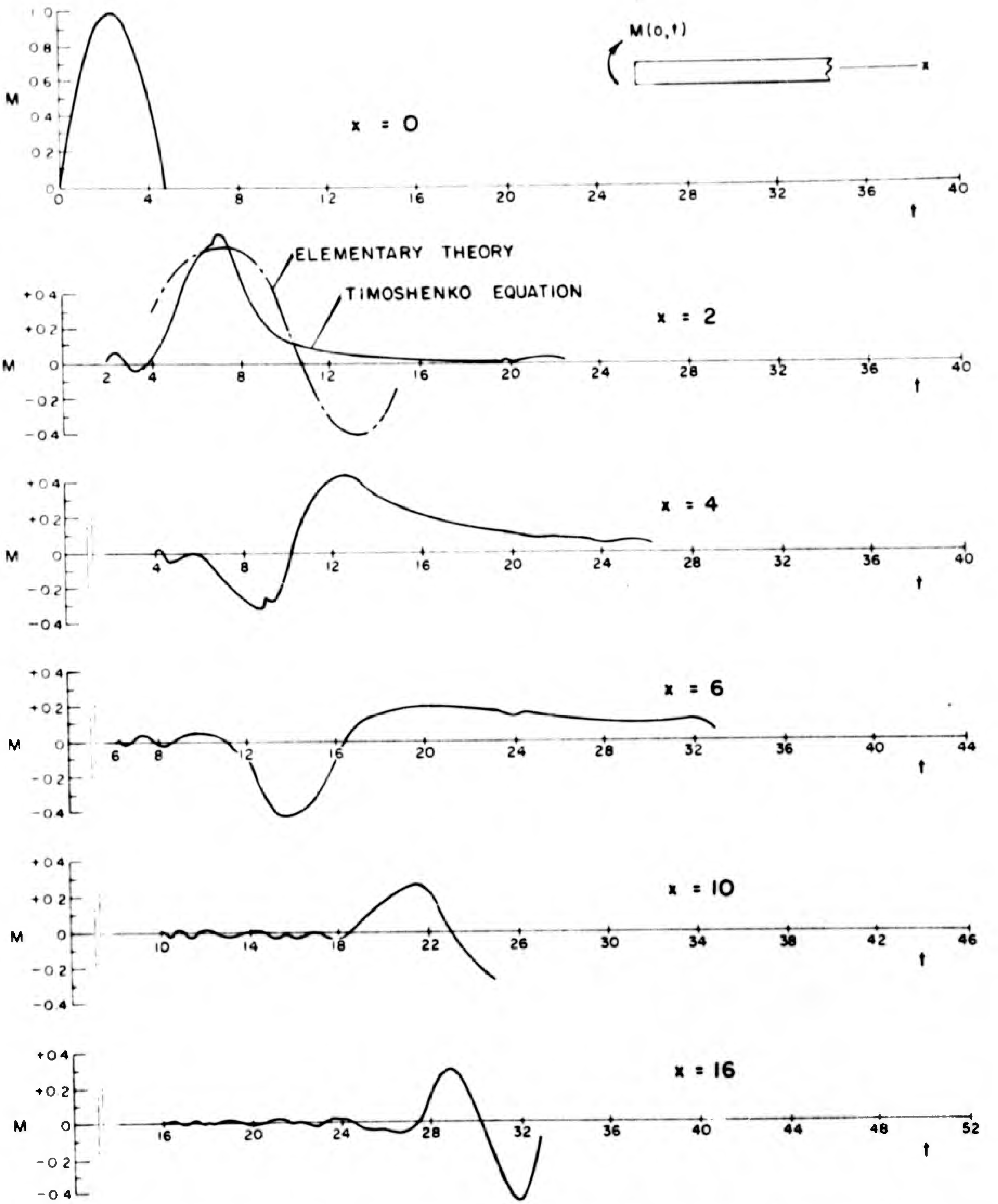

FIG. 3-5 (d): M-t CURVES AT VARIOUS STATIONS ALONG BAR WITH UNSUPPORTED END, RESULTING FROM HALF SINE MOMENT IMPACT; $\tau = 4.8$, $C = 4$

DRL - UT
DWG AA 1653
HJP - CLW
2 - 20 - 56

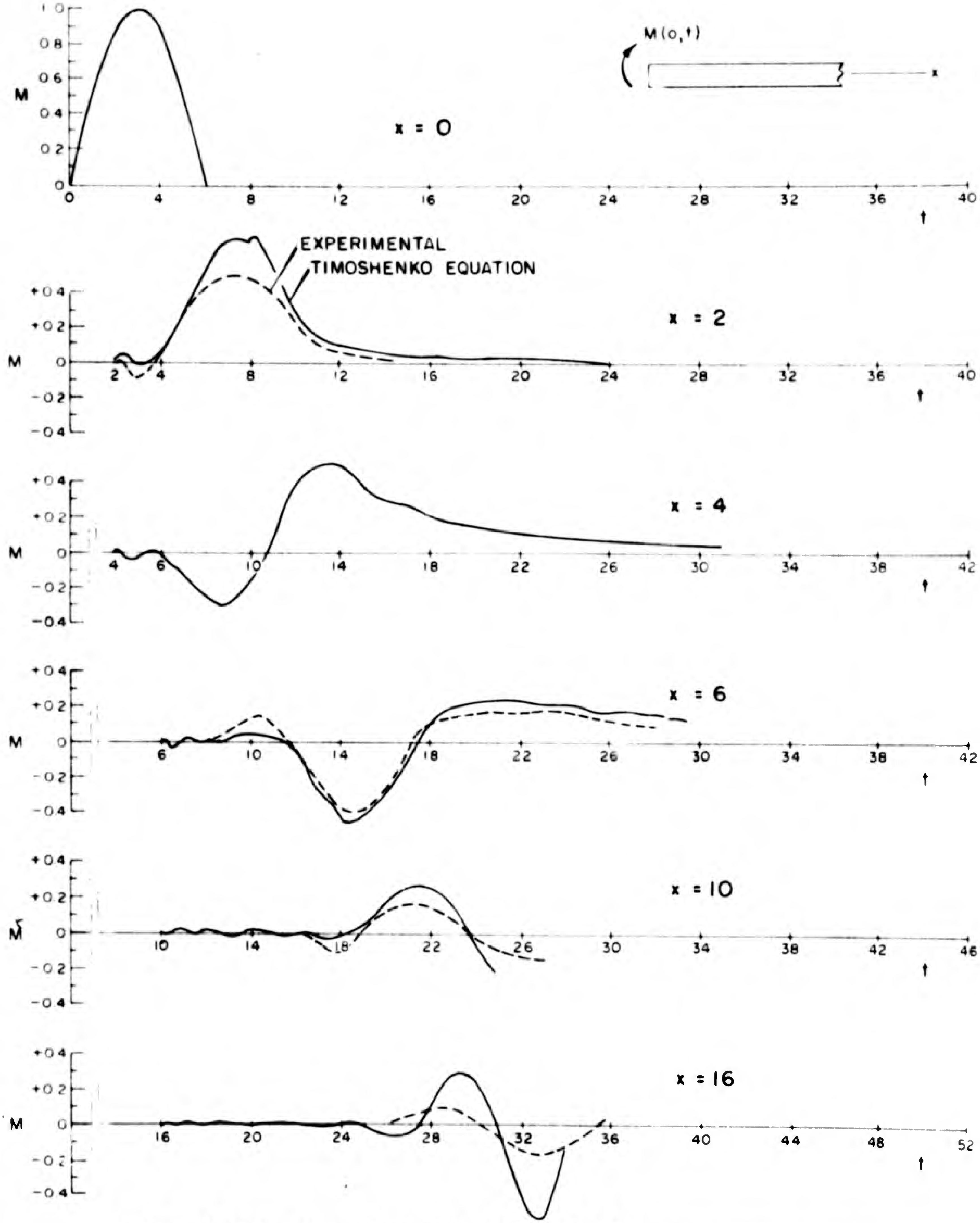

FIG. 3-5 (e): M-t CURVES AT VARIOUS STATIONS ALONG BAR WITH UNSUPPORTED END, RESULTING FROM HALF SINE MOMENT IMPACT; $\tau = 6.0$, $C = 4$

DRL - UT
DWG AA 1654
HJP - CLW
2-20-56

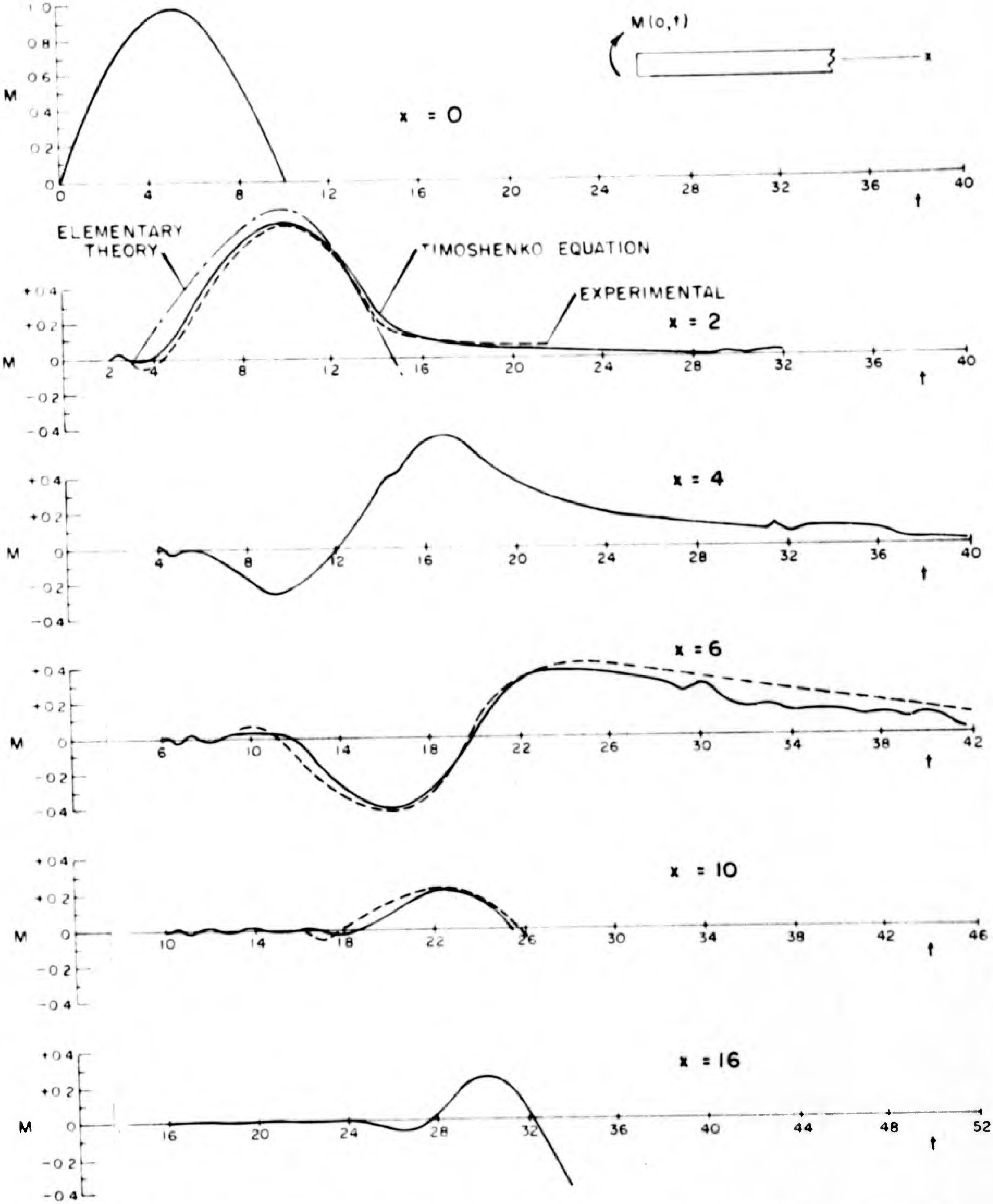

FIG.3-5 (f): M-t CURVES AT VARIOUS STATIONS ALONG BAR WITH UNSUPPORTED END, RESULTING FROM HALF SINE MOMENT IMPACT; $\tau = 10.0$, $C = 4$

DRL - UT
DWG AA 1655
HJP - CLW
2 - 20 - 56

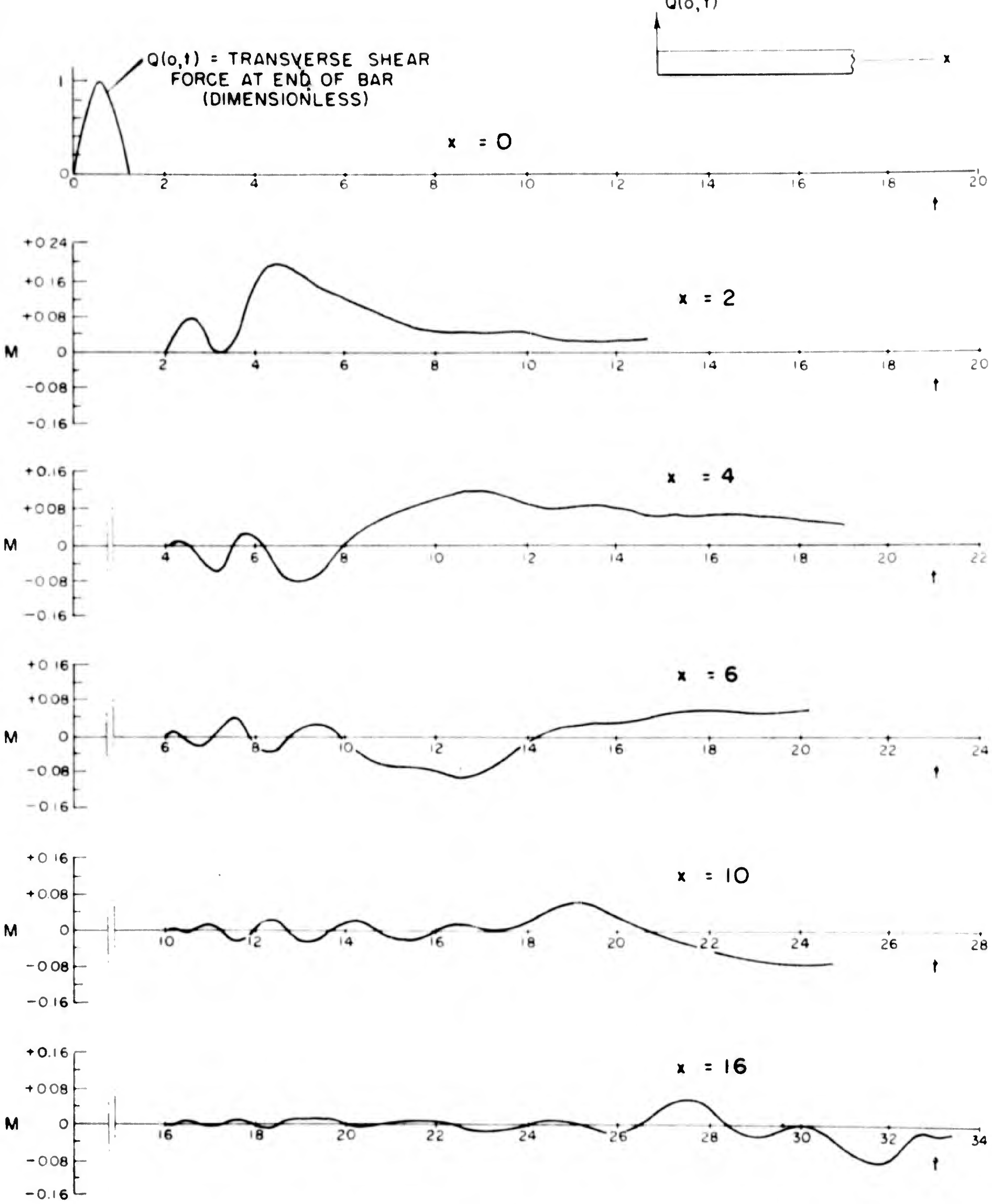

FIG.3-6 (a): M-t CURVES AT VARIOUS STATIONS ALONG BAR RESULTING FROM TRANSVERSE SHEAR FORCE HALF-SINE IMPACT, $\tau = 1.2$, $C = 4$.

DRL - UT
DWG AA 1681
HJP - CLW
3 - 20 - 56

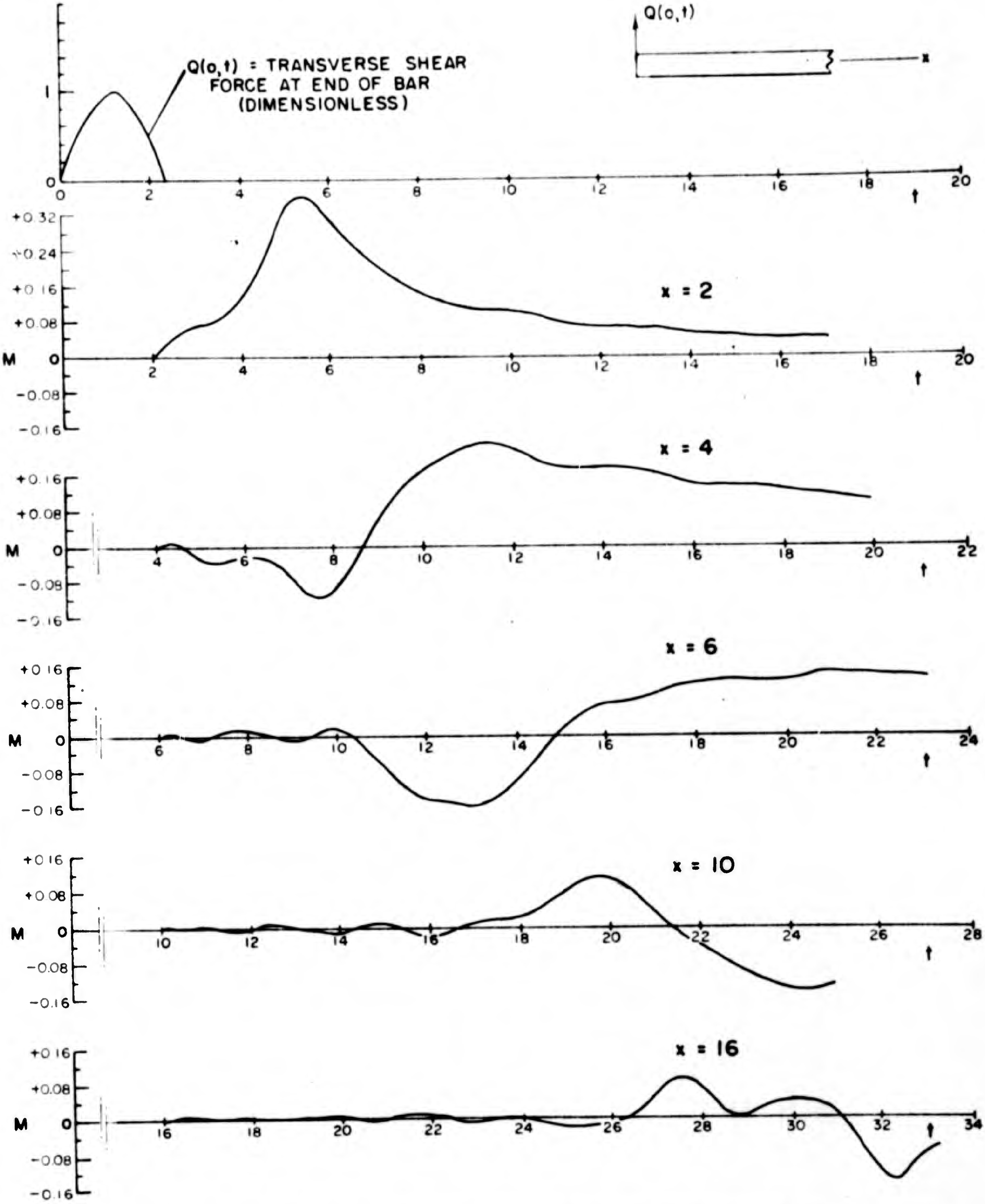

FIG. 3-6 (b): M-t CURVES AT VARIOUS STATIONS ALONG BAR RESULTING FROM TRANSVERSE SHEAR FORCE HALF-SINE IMPACT; $\tau = 2.4$, $C = 4$.

DRL-UT
DWG AA 1682
MJP-CLW
2-20-56

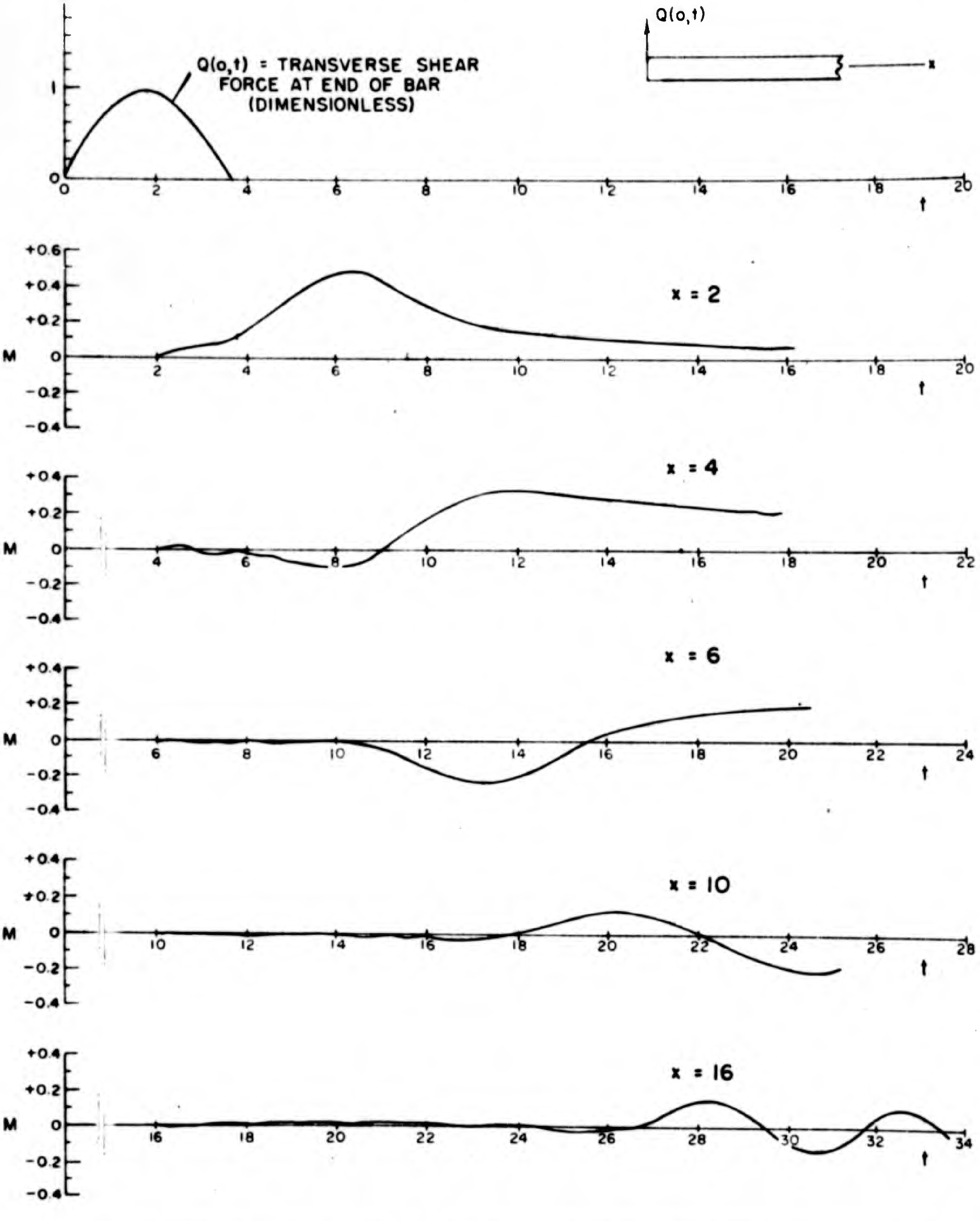

FIG.3-6 (c). M-t CURVES AT VARIOUS STATIONS ALONG BAR RESULTING FROM TRANSVERSE SHEAR FORCE HALF-SINE IMPACT; $\tau = 3.6$, $C = 4$.

DRL-UT
DWG AA 1684
HJP - CLW
2-20-56

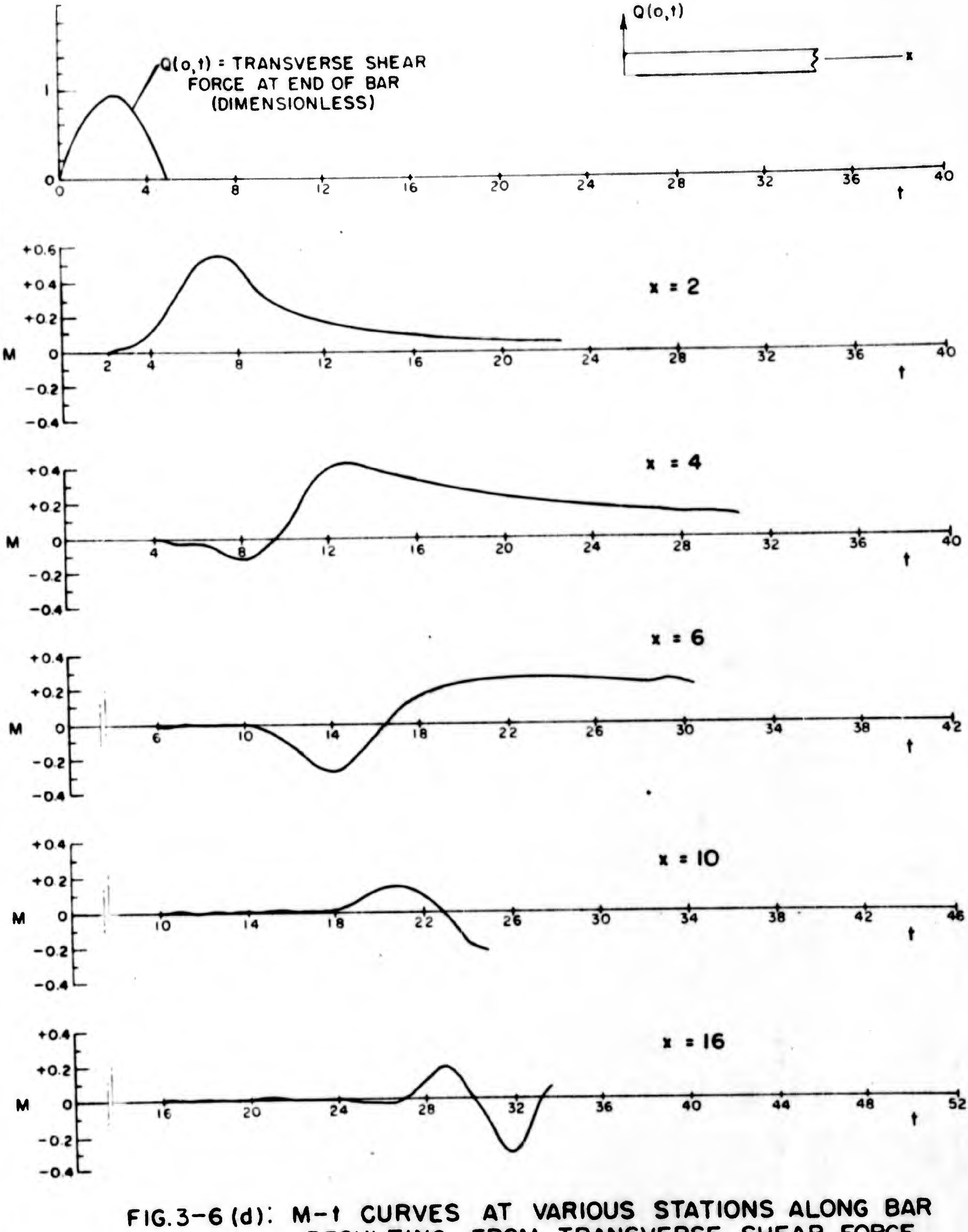

FIG. 3-6 (d): M-t CURVES AT VARIOUS STATIONS ALONG BAR RESULTING FROM TRANSVERSE SHEAR FORCE HALF-SINE IMPACT; $\tau = 4.8$ $C = 4$.

DRL - UT
DWG AA 1685
HJP - CLW
2 - 20 - 56

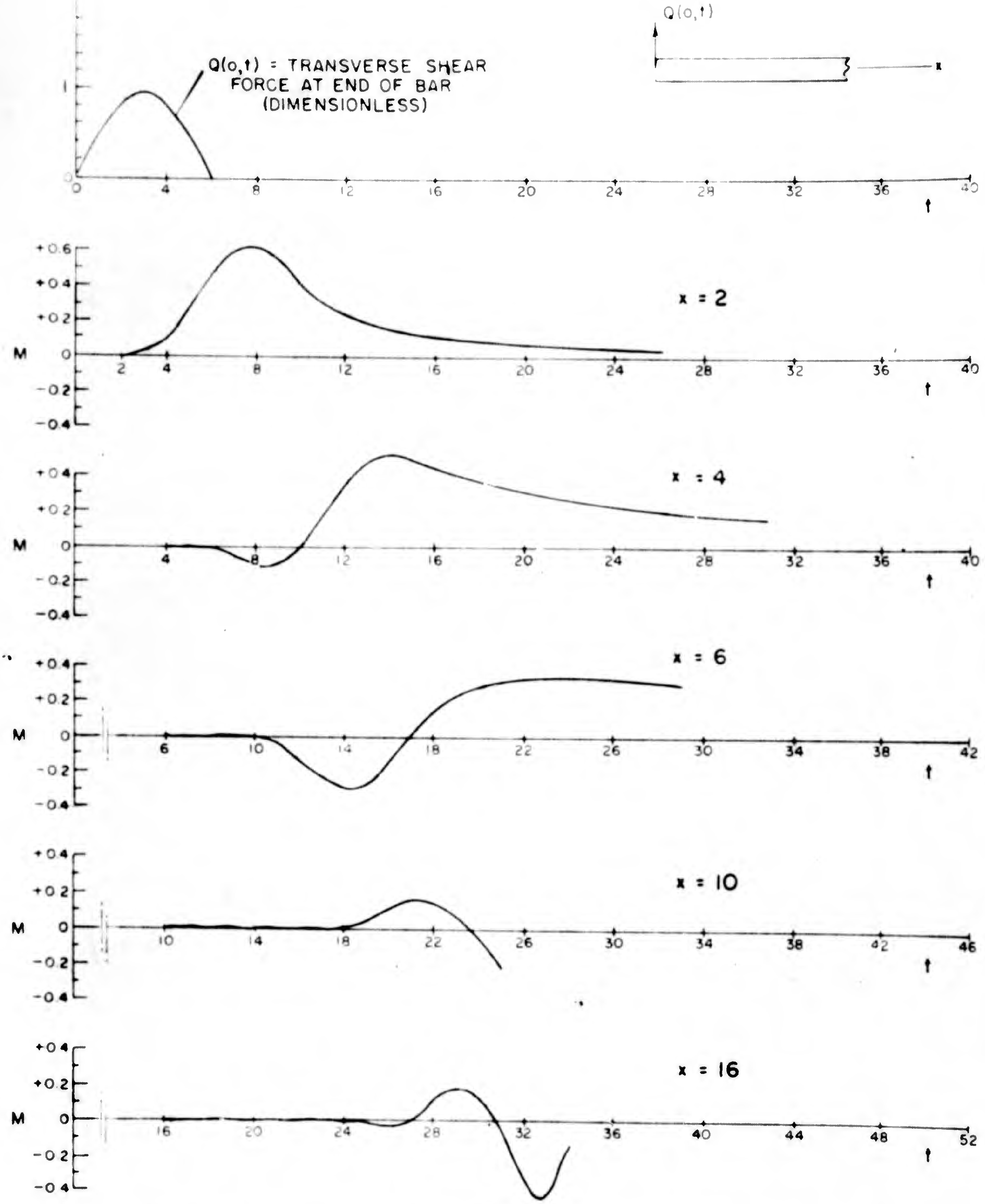

FIG.3-6 (e): M-t CURVES AT VARIOUS STATIONS ALONG BAR RESULTING FROM TRANSVERSE SHEAR FORCE HALF-SINE IMPACT; $\tau = 6.0$, C = 4.

DRL-UT
DWG AA 1686
HJP-CLW
2-20-56

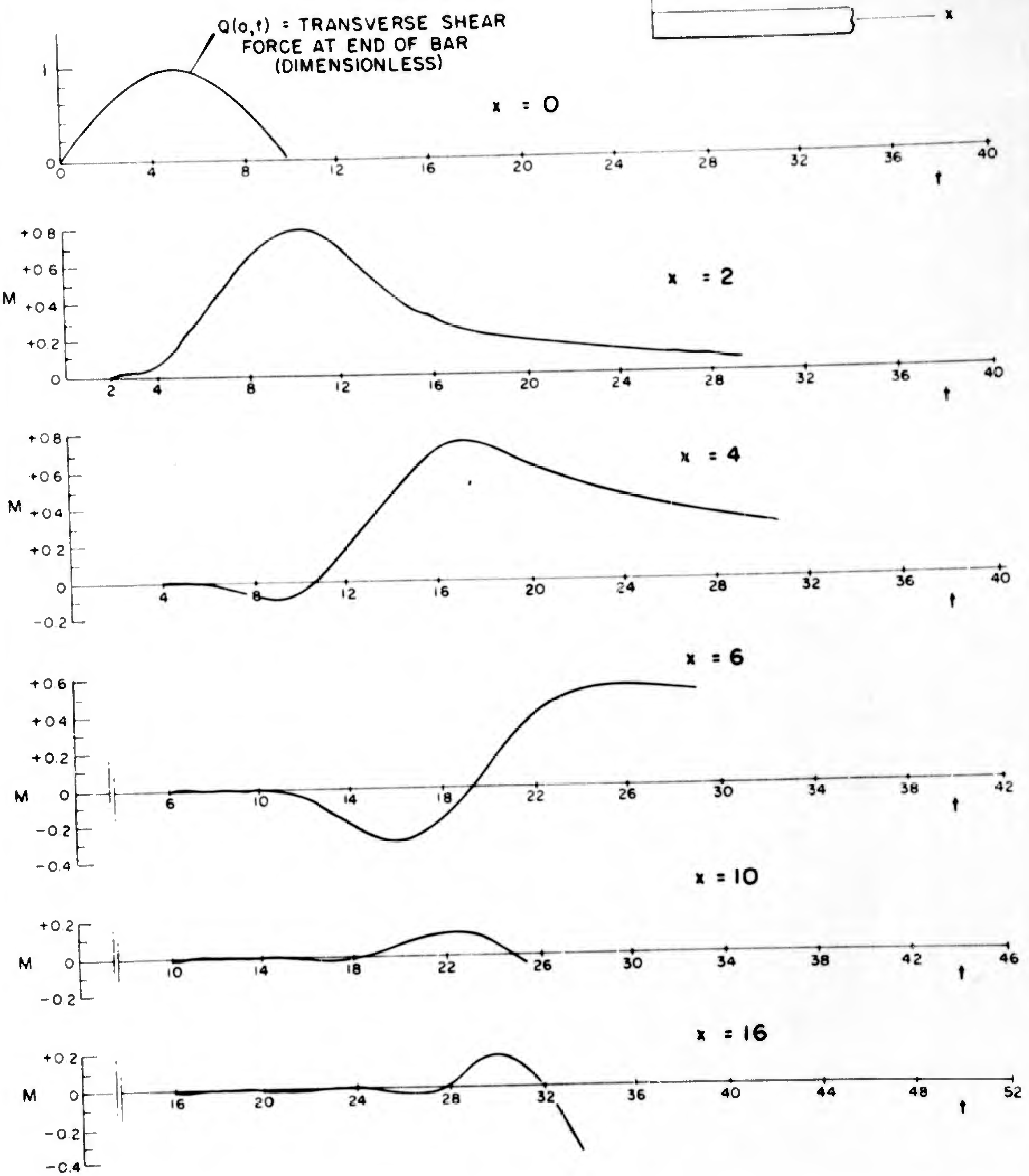

FIG. 3-6 (f): M-t CURVES AT VARIOUS STATIONS ALONG BAR RESULTING FROM TRANSVERSE SHEAR FORCE HALF-SINE IMPACT; $\tau = 10.0$, $C = 4$.

DRL - UT
DWG AA 1687
HJP—CLW
3-20-56

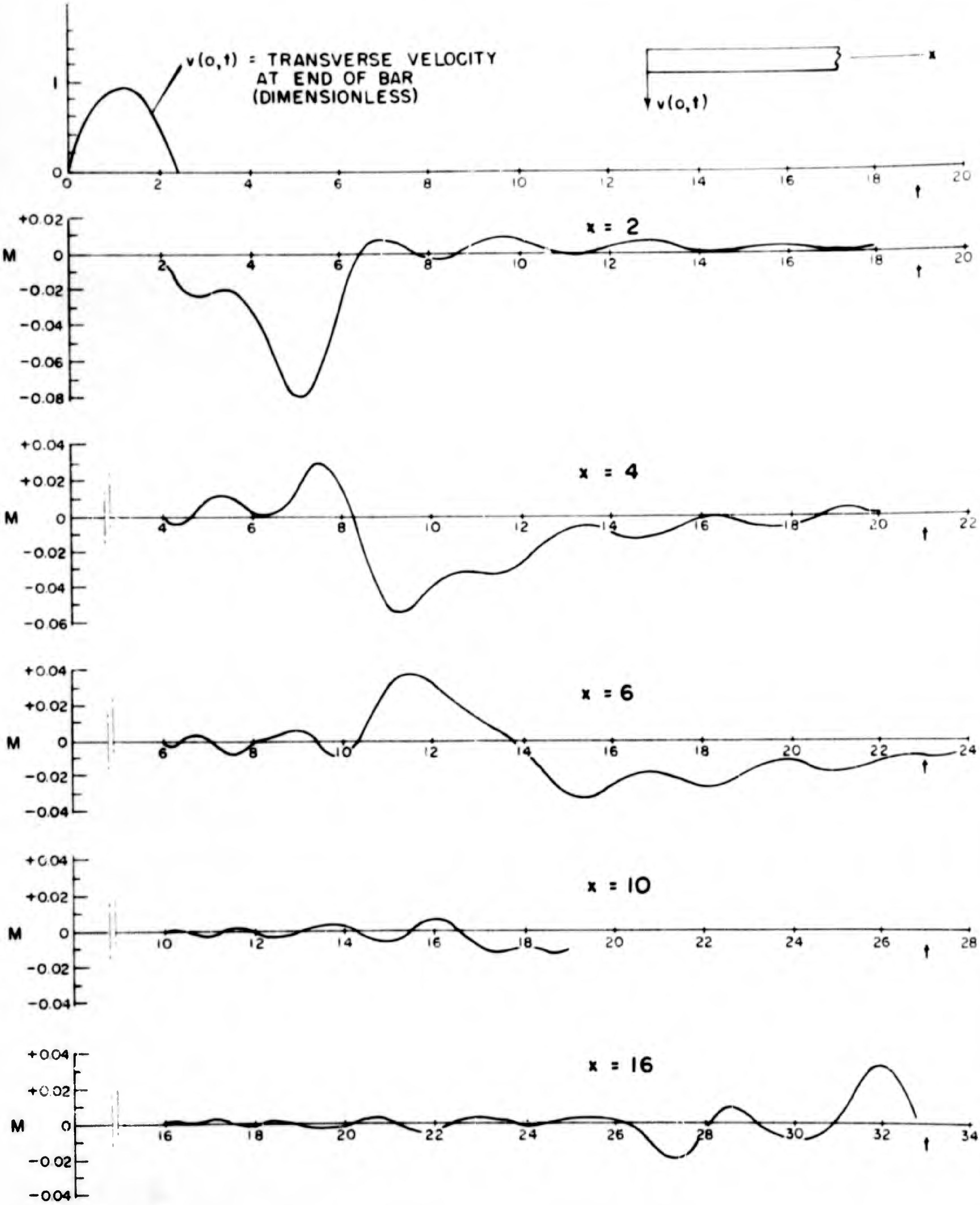

FIG. 3-7(a): M-t CURVES AT VARIOUS STATIONS ALONG BAR RESULTING FROM TRANSVERSE VELOCITY HALF-SINE IMPACT; τ = 2.4, C = 4.

DRL - UT
DWG AA 1688
HJP - CLW
3 - 15 - 56

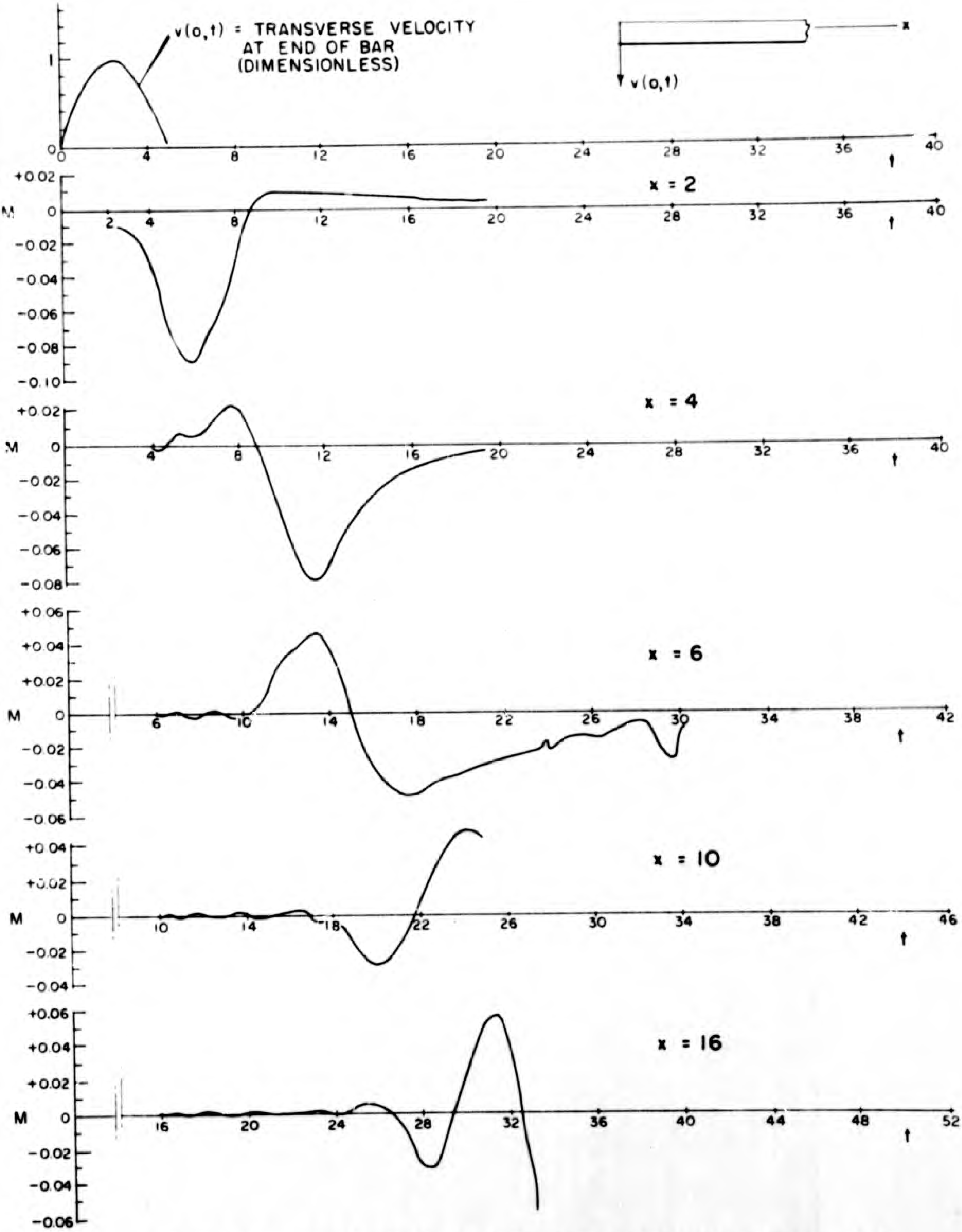

FIG. 3-7(b): M-t CURVES AT VARIOUS STATIONS ALONG BAR RESULTING FROM TRANSVERSE VELOCITY HALF-SINE IMPACT; $\tau = 4.8$, $C = 4$.

DRL - UT
DWG AA 1689
HJP - CLW
3 - 15 - 55

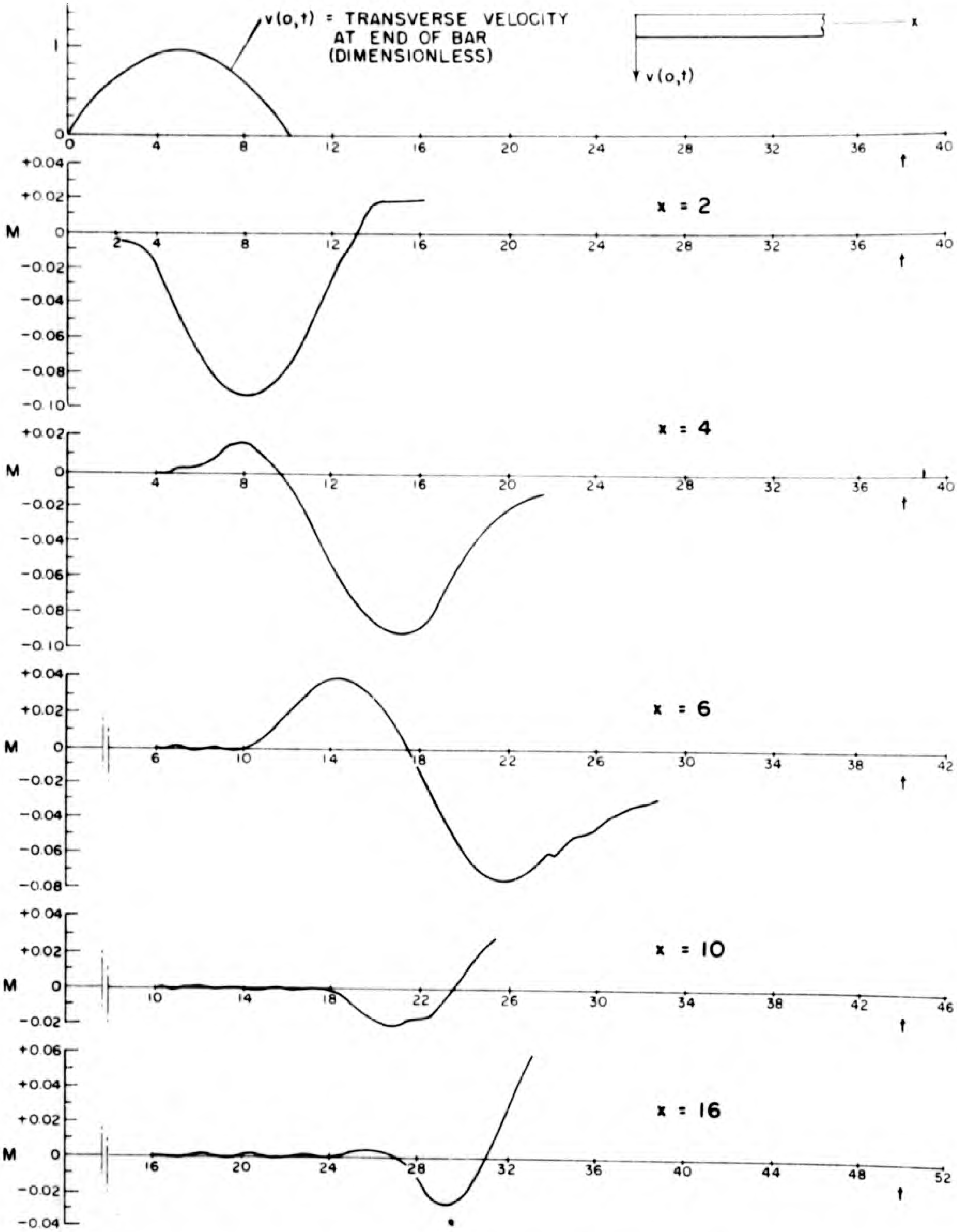

FIG. 3-7(c): M-t CURVES AT VARIOUS STATIONS ALONG BAR RESULTING FROM TRANSVERSE VELOCITY HALF-SINE IMPACT; $\tau = 10.0$, C = 4

DRL - UT
DWG AA 1690
HJP - CLW
3 - 15 - 55

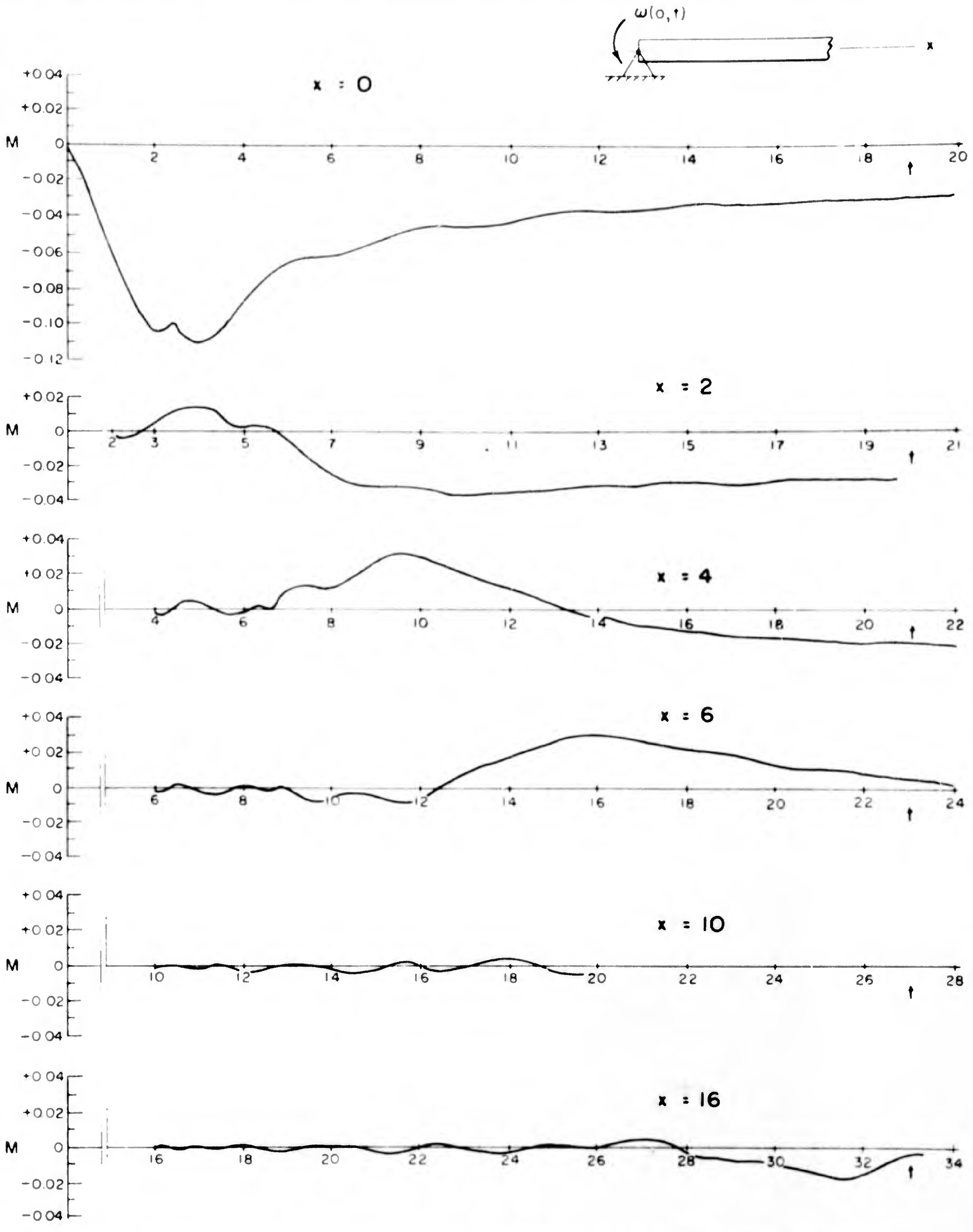

FIG 3-8(a). M-t CURVES AT VARIOUS STATIONS ALONG BAR RESULTING FROM ANGULAR VELOCITY HALF-SINE IMPACT. END SIMPLY SUPPORTED. $\tau = 2.4$, $C = 4$.

DRL - UT
DWG AA 1691
HJP - CLW
3 - 16 - 56

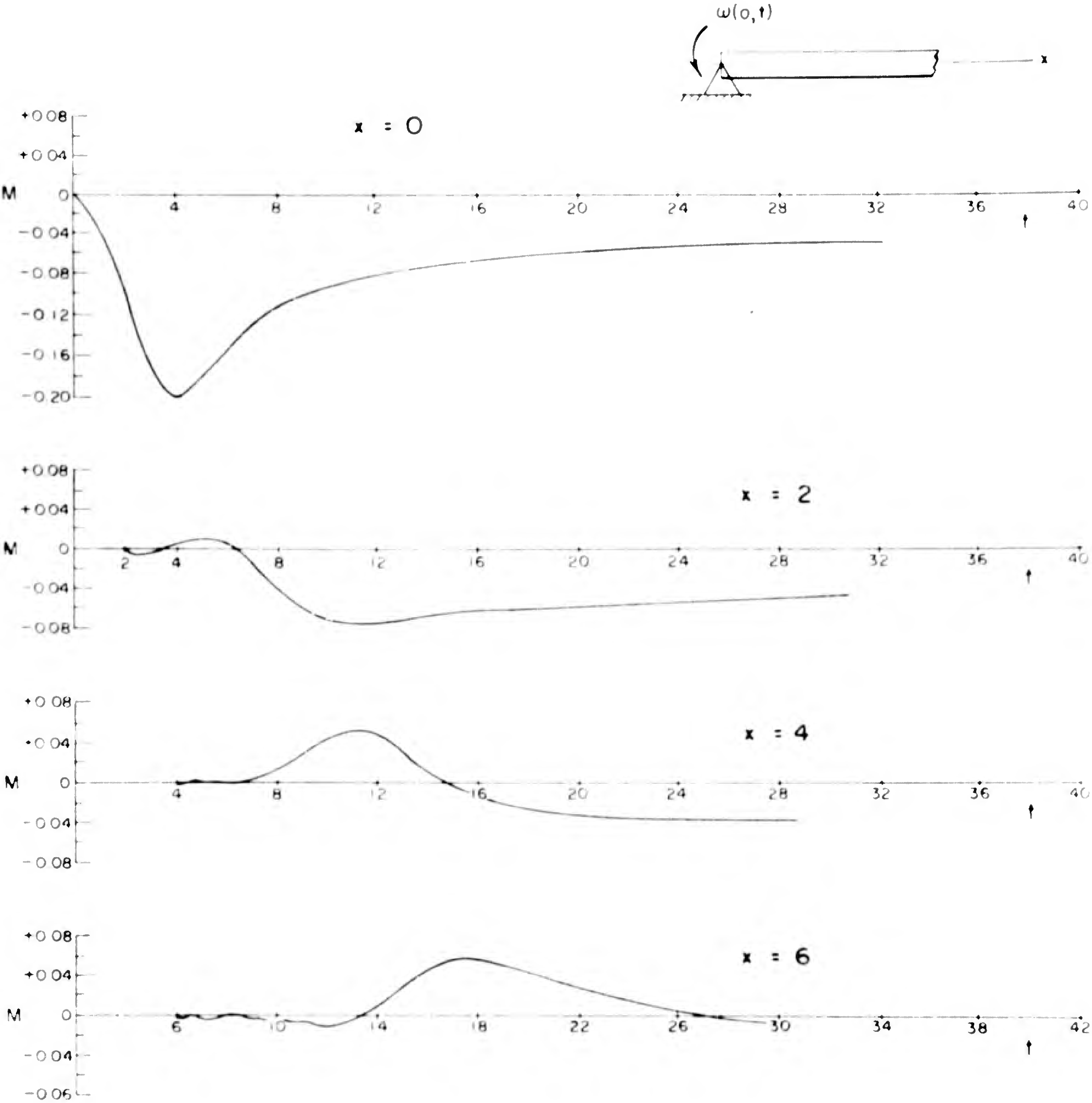

FIG 3-8(b). M-t CURVES AT VARIOUS STATIONS ALONG BAR
RESULTING FROM ANGULAR VELOCITY
HALF-SINE IMPACT.
END SIMPLY SUPPORTED.
$\tau = 48$, $C = 4$.

DRL - UT
DWG AA 1692
HJP - CLW
3 - 16 - 56

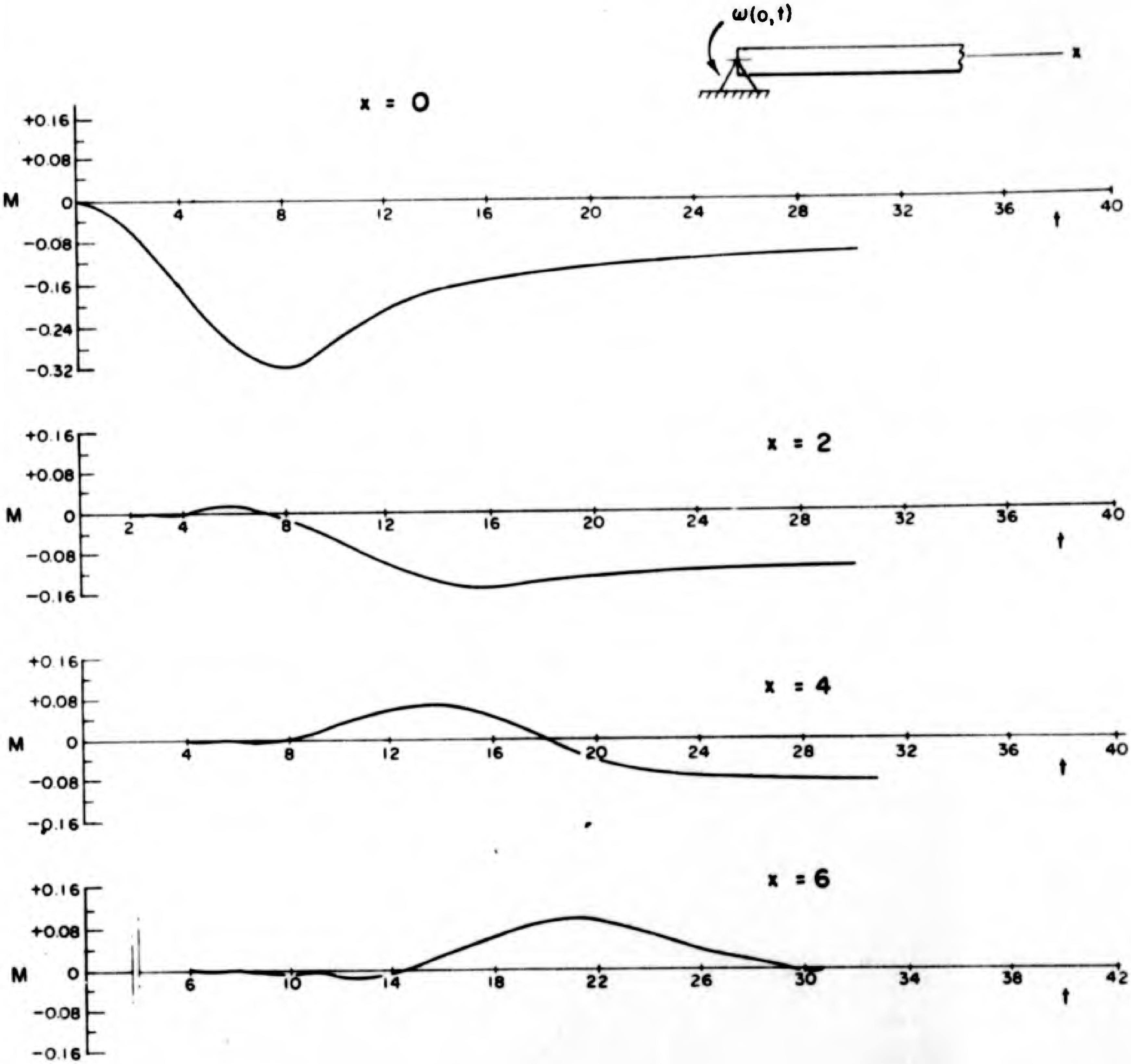

FIG. 3-8(c): M-t CURVES AT VARIOUS STATIONS ALONG BAR RESULTING FROM ANGULAR VELOCITY HALF-SINE IMPACT. END SIMPLY SUPPORTED. $\tau = 10.0$, $C = 4$.

DRL - UT
DWG AA 1693
HJP - CLW
3 - 19 - 56

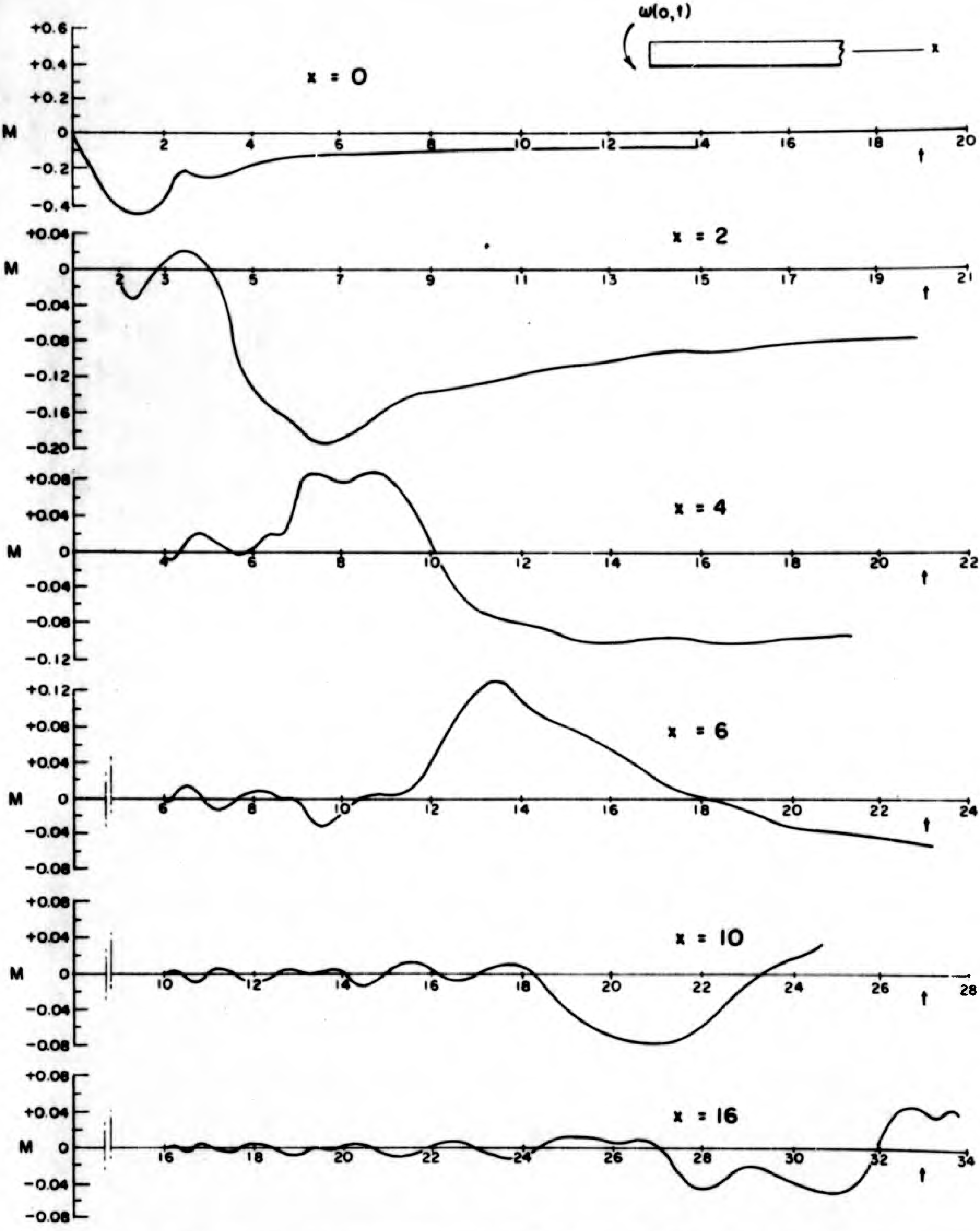

FIG. 3-9(a): M-t CURVES AT VARIOUS STATIONS ALONG BAR RESULTING FROM ANGULAR VELOCITY HALF-SINE IMPACT. END UNSUPPORTED. $\tau = 2.4$, $C = 4$.

DRL - UT
DWG AA 1694
HJP - CLW
3 - 16 - 56

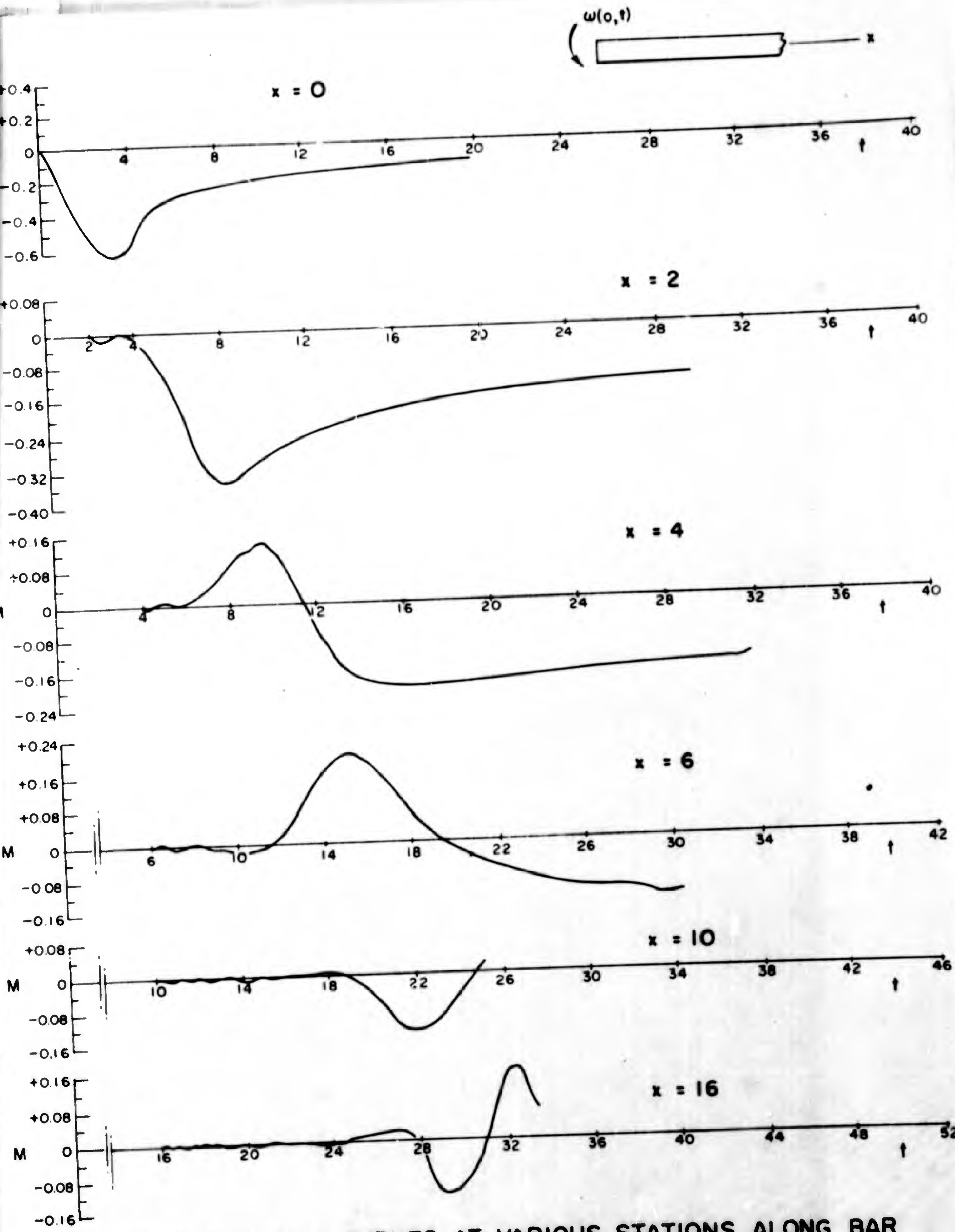

FIG. 3-9 (b): M-t CURVES AT VARIOUS STATIONS ALONG BAR RESULTING FROM ANGULAR VELOCITY HALF-SINE IMPACT. END UNSUPPORTED. $\tau = 4.8$, $C = 4$.

DRL - UT
DWG AA 1695
HJP - CLW
3 - 19 - 56

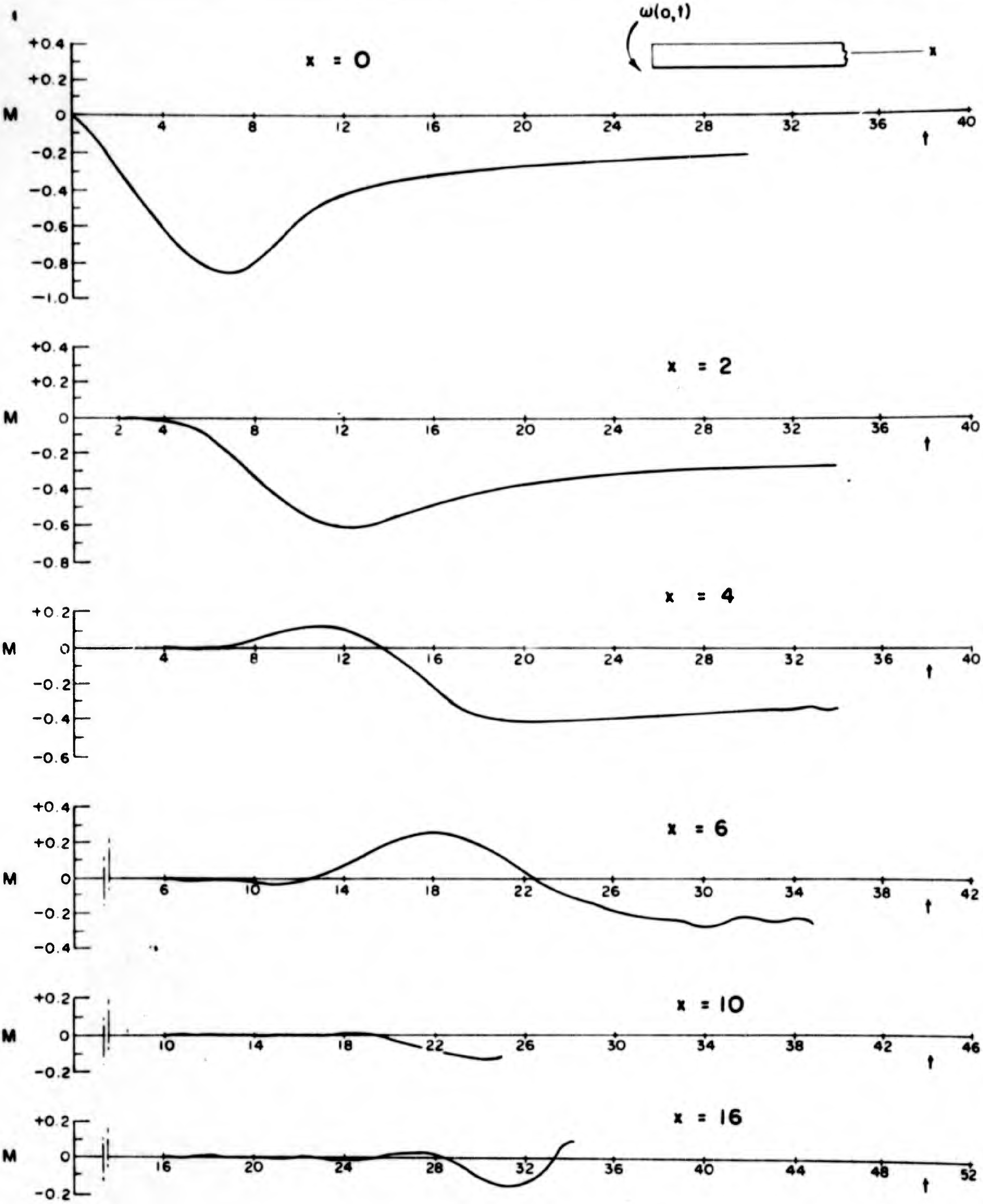

FIG. 3-9 (c): M-t CURVES AT VARIOUS STATIONS ALONG BAR
RESULTING FROM ANGULAR VELOCITY
HALF-SINE IMPACT.
END UNSUPPORTED.
τ = 10.0, C = 4.

DRL - UT
DWG AA 1696
HJP - CLW
3-19-56

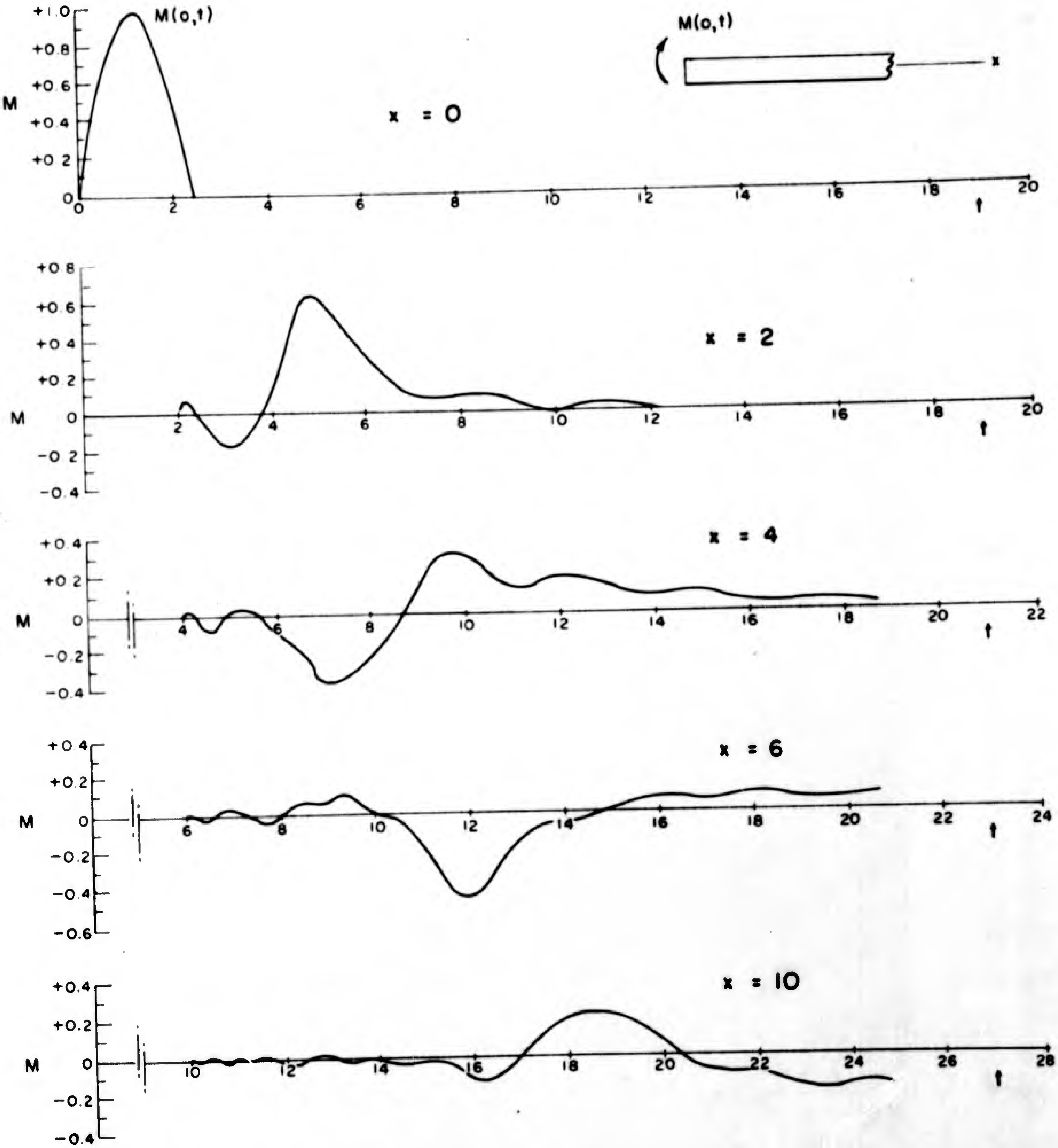

FIG. 3-10(a): M-t CURVES AT VARIOUS STATIONS ALONG BAR RESULTING FROM MOMENT HALF-SINE IMPACT. END UNSUPPORTED. τ = 2.4, C = 3.

DRL – UT
DWG AA 1697
HJP – CLW
3 – 20 – 56

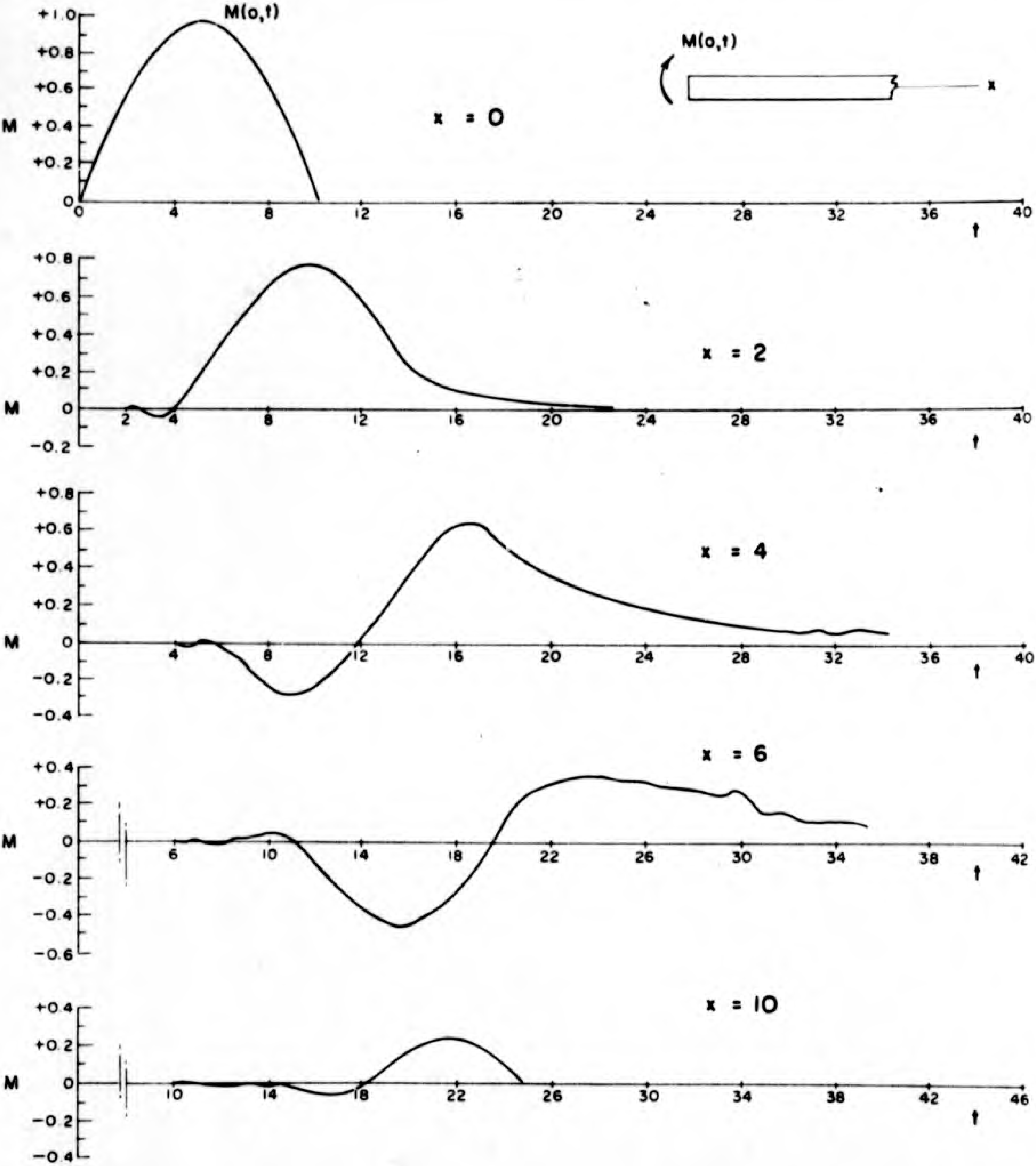

FIG. 3-10 (b): M-t CURVES AT VARIOUS STATIONS ALONG BAR RESULTING FROM MOMENT HALF-SINE IMPACT. END UNSUPPORTED. $\tau = 10.0$, C = 3.

DRL - UT
DWG AA1698
HJP - CLW
3-20-56

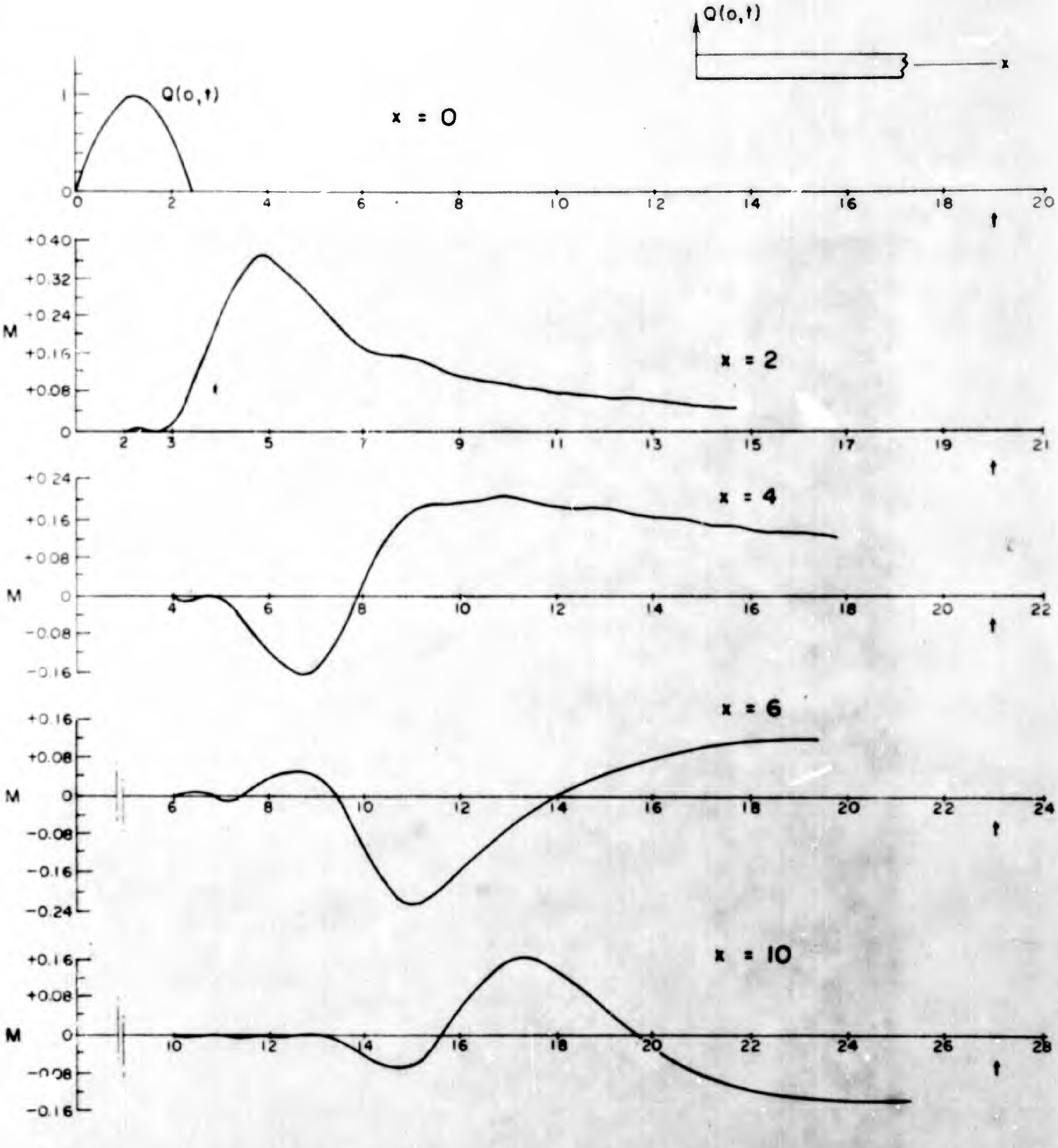

FIG. 3-11(a): M-t CURVES AT VARIOUS STATIONS ALONG BAR RESULTING FROM TRANSVERSE SHEAR FORCE HALF-SINE IMPACT; $\tau = 2.4$, $C = 3$.

DRL - UT
DWG AA 1683
HJP - CLW
3 - 19 - 56

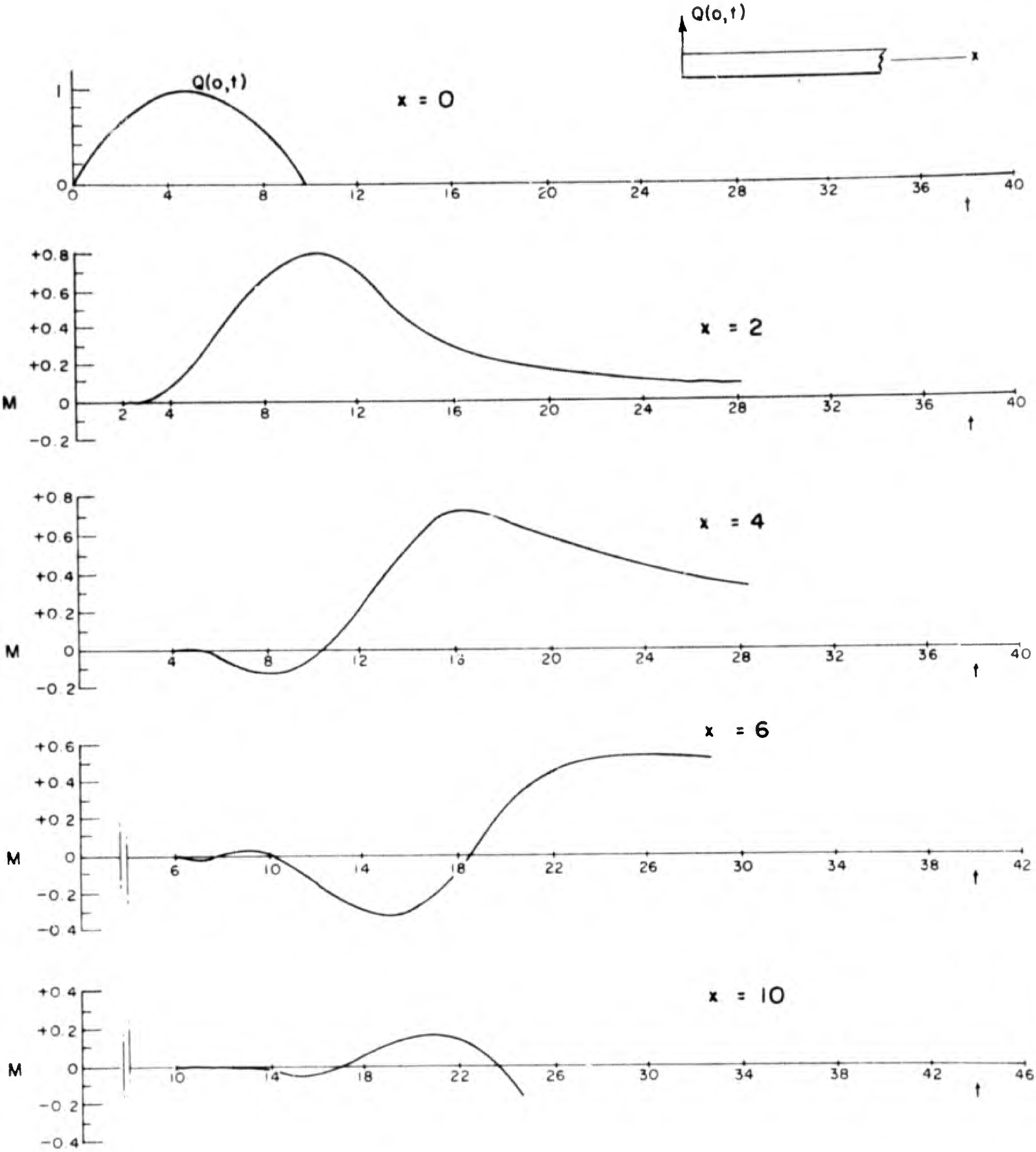

FIG. 3-11(b): M-t CURVES AT VARIOUS STATIONS ALONG BAR RESULTING FROM TRANSVERSE SHEAR FORCE HALF-SINE IMPACT; $\tau = 10.0$, $C = 3$.

DRL - UT
DWG AA 1699
HJP - CLW
3 - 19 - 56

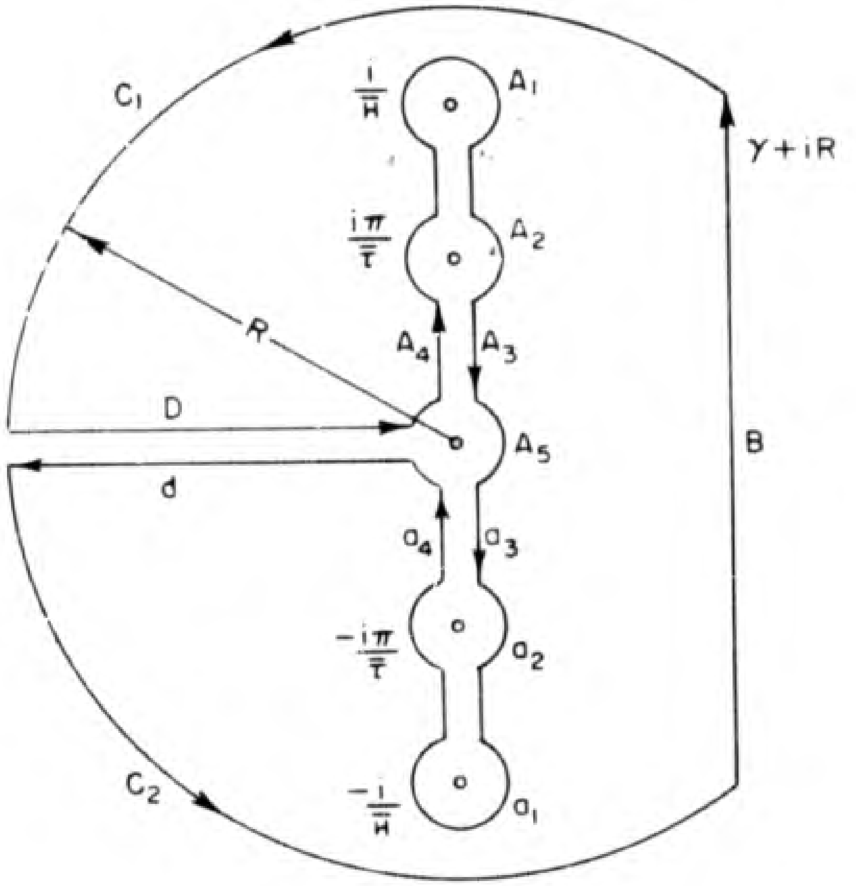

FIG. A-1: INVERSION CONTOUR FOR THE SOLUTION OF THE LOVE EQUATION FOR THE SEMI-INFINITE BAR

DRL — UT
DWG AA 1704
HJP — CLW
4 - 5 - 56

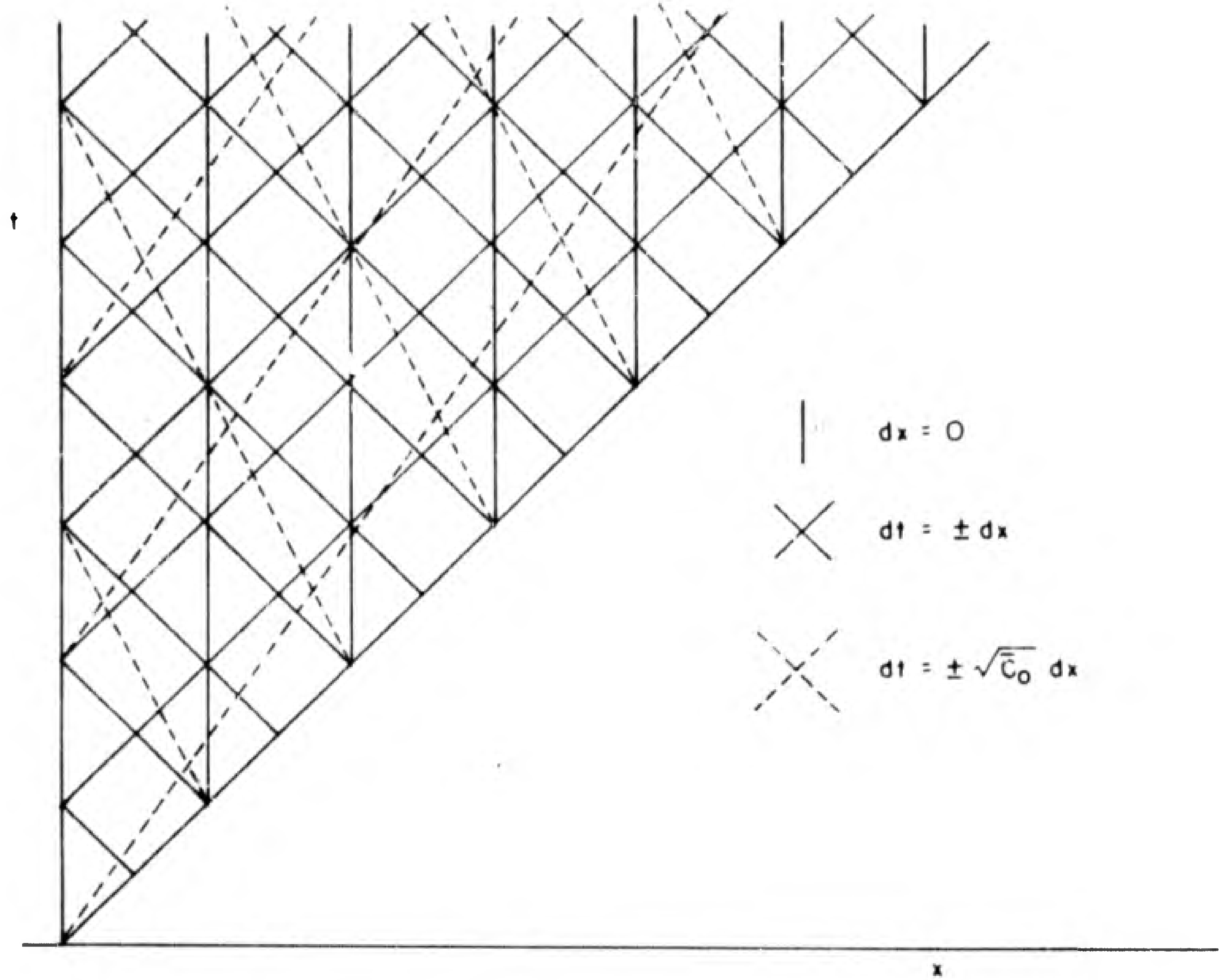

FIG. B-1: CURVES SHOWING THE FAMILIES OF CHARACTERISTICS

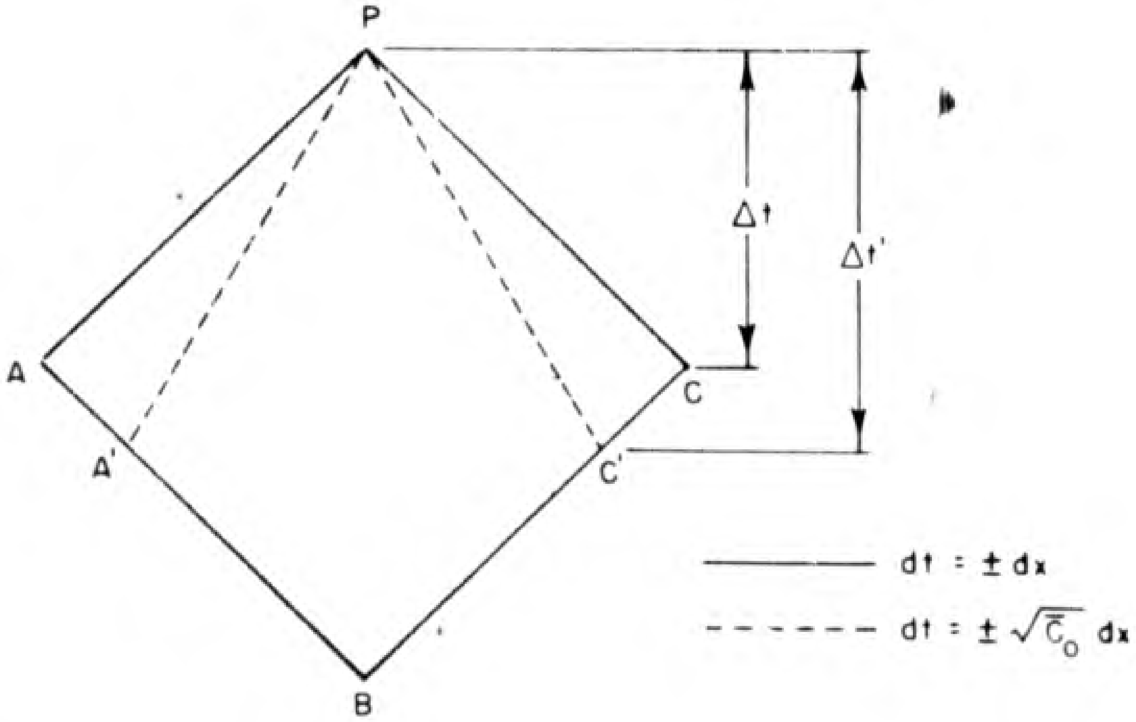

FIG. B-2: CURVES SHOWING FOUR ADJACENT POINTS IN THE FAMILIES OF CHARACTERISTICS

DRL – UT
DWG AA 1705
HJP – CLW
4 – 5 – 56

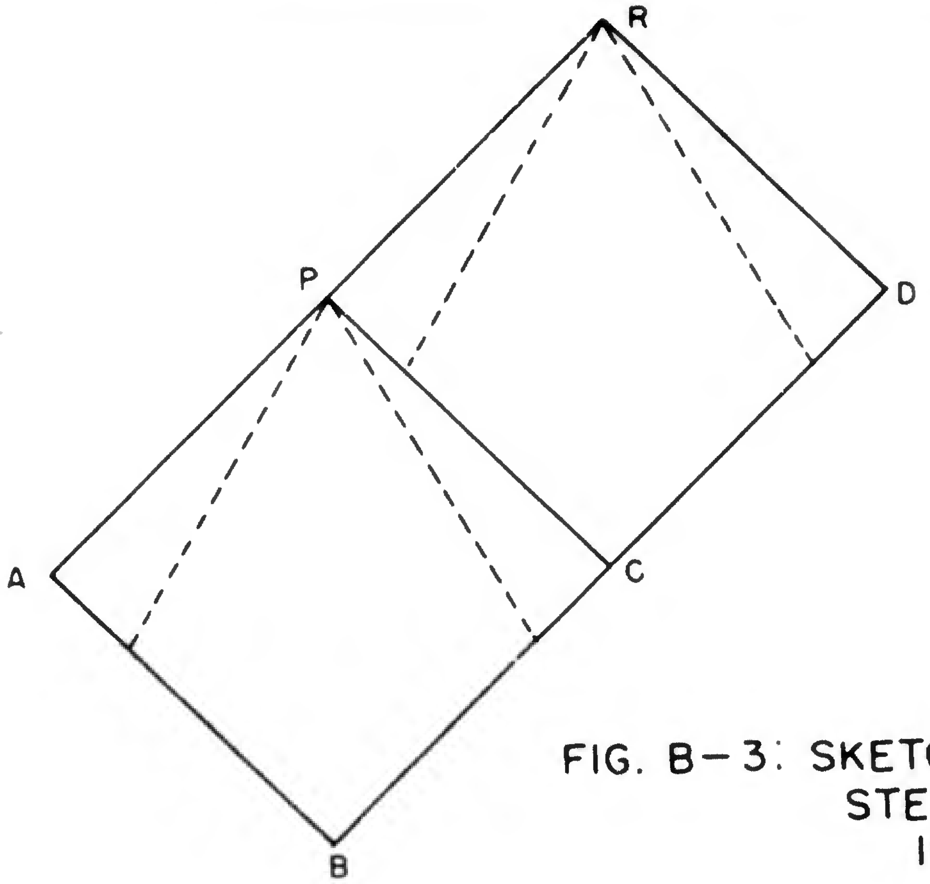

FIG. B-3: SKETCH SHOWING STEP BY STEP PROCEDURE USED IN CALCULATIONS

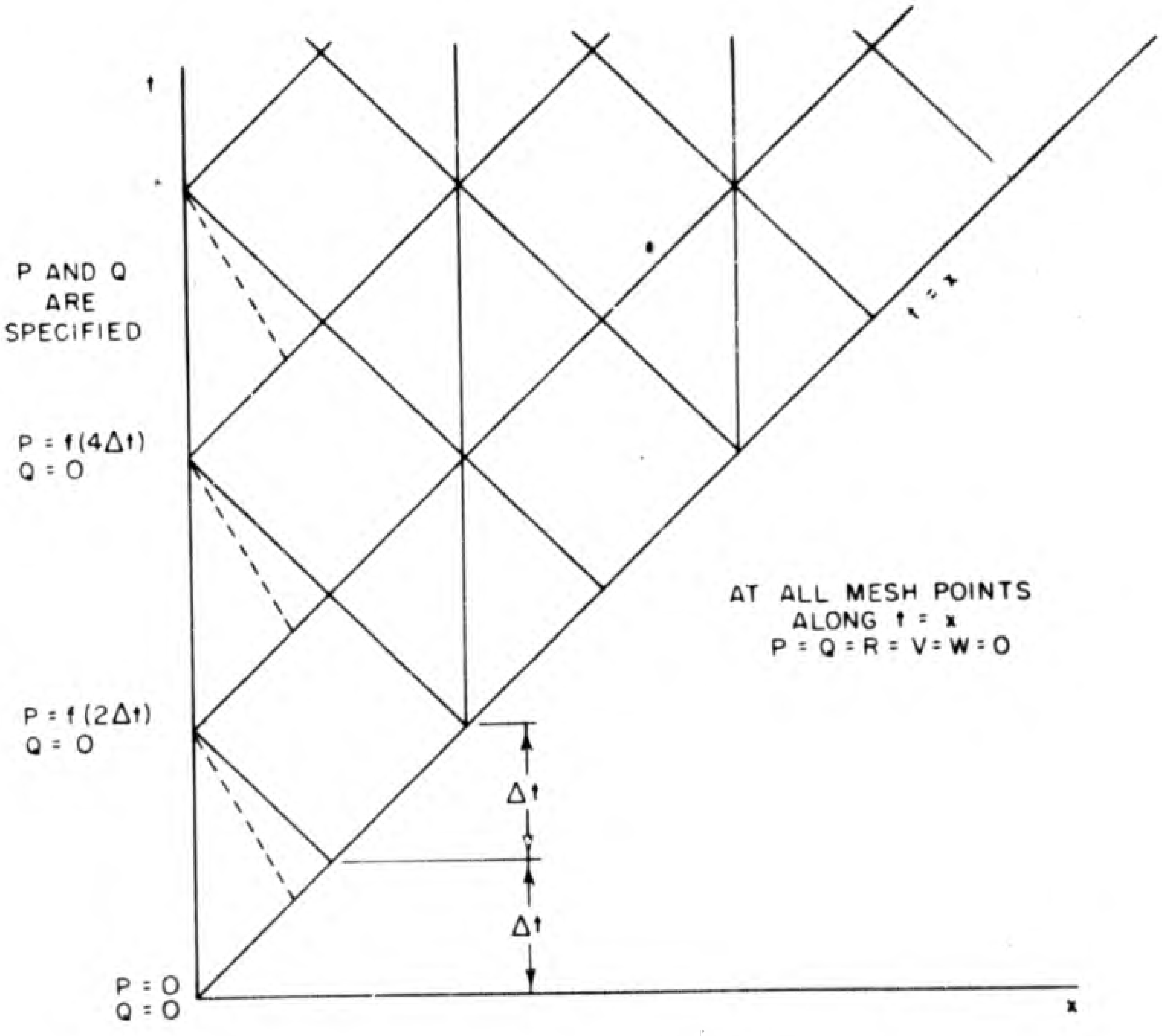

FIG. B-4: CURVES ILLUSTRATING PROCEDURE USED IN CALCULATIONS

DRL - UT
DWG AA 1706
HJP - CLW
4 - 5 - 56

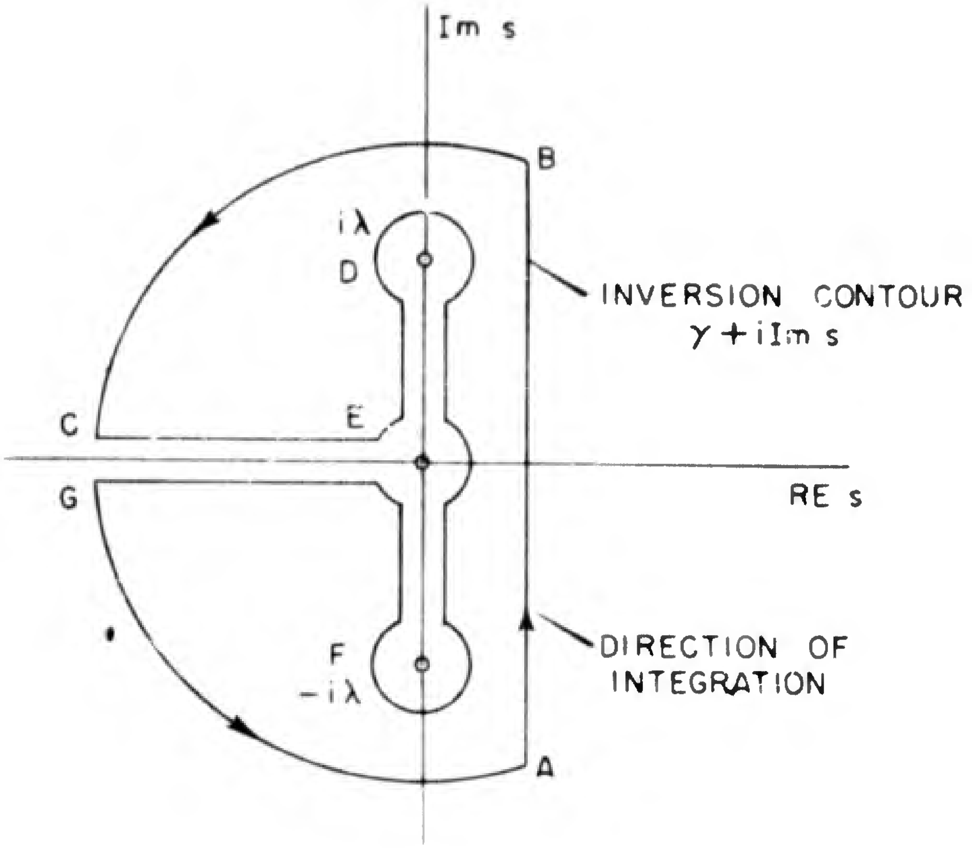

FIG. C−1: CONTOUR FOR EVALUATION OF EQS. (C−38) AND (C−39)

DRL − UT
DWG AA 1700
HJP − CLW
4 − 4 − 56

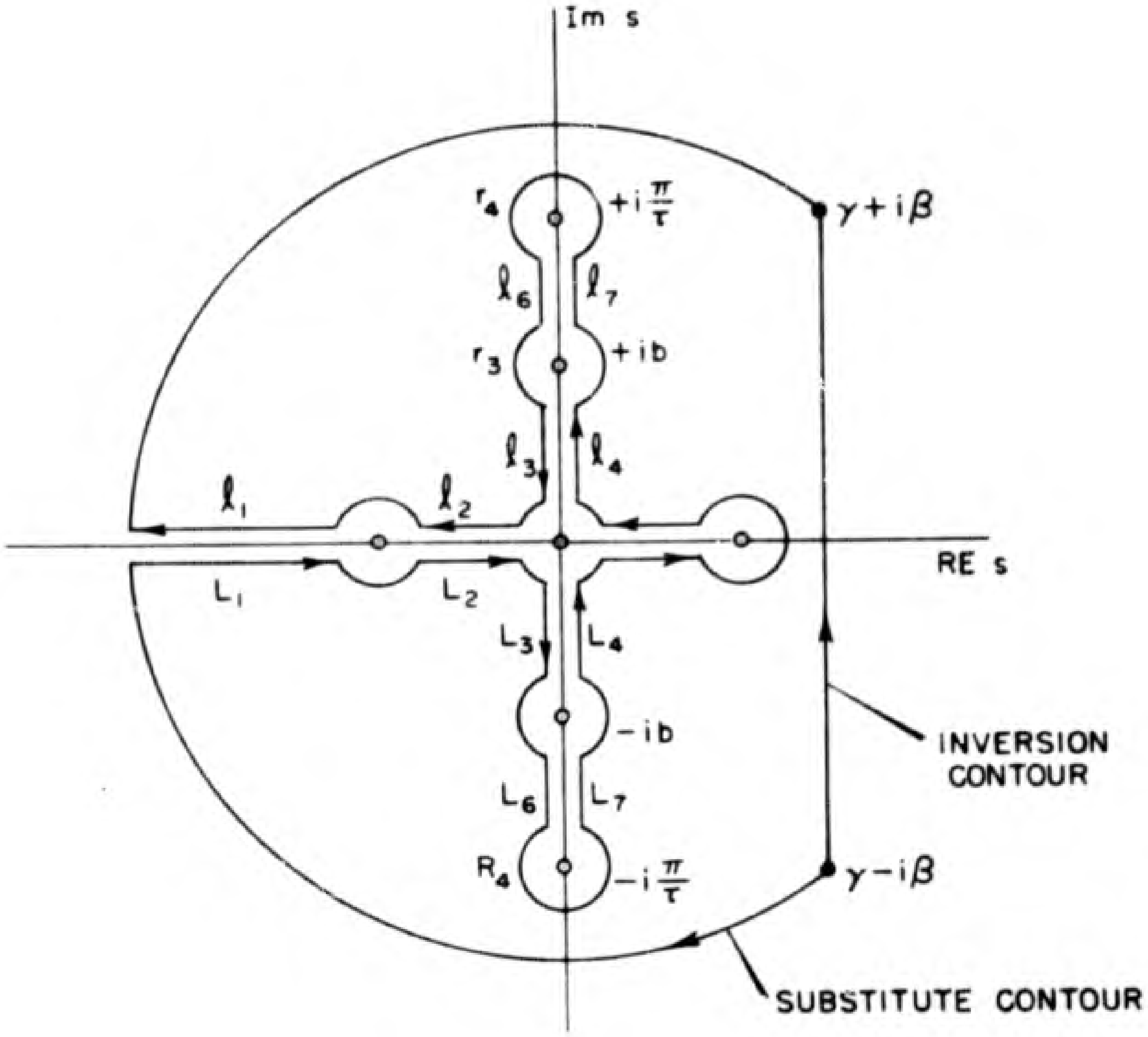

FIG. D-1: CONTOUR FOR EVALUATION OF EQ. (D-23)

DRL - UT
DWG AA 1701
HJP - CLW
4 - 4 - 56

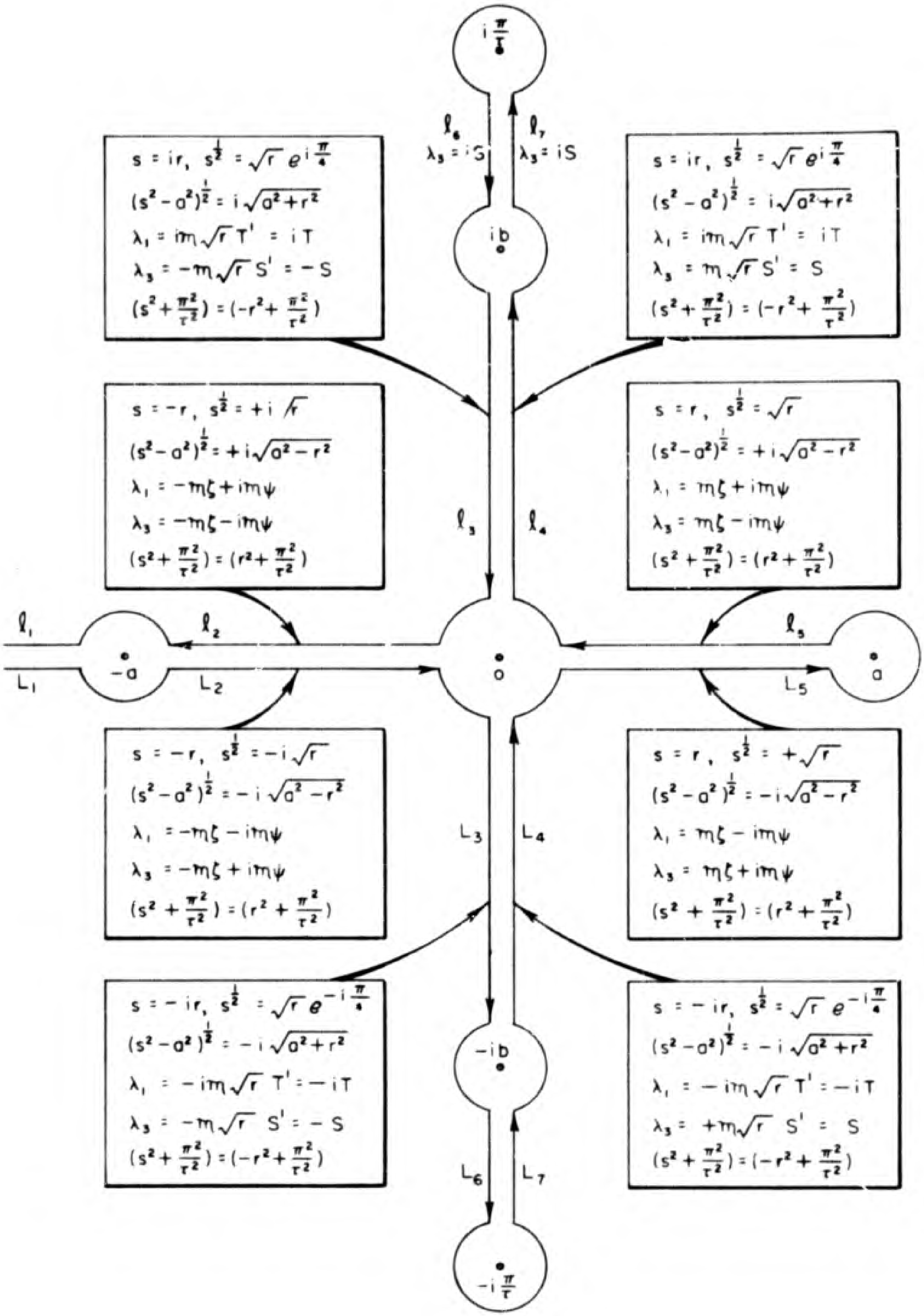

FIG. D-2: SIGNIFICANT EXPRESSIONS USEFUL IN INTEGRATION OF EQ. (D-23)

DRL - UT
DWG AA 1702

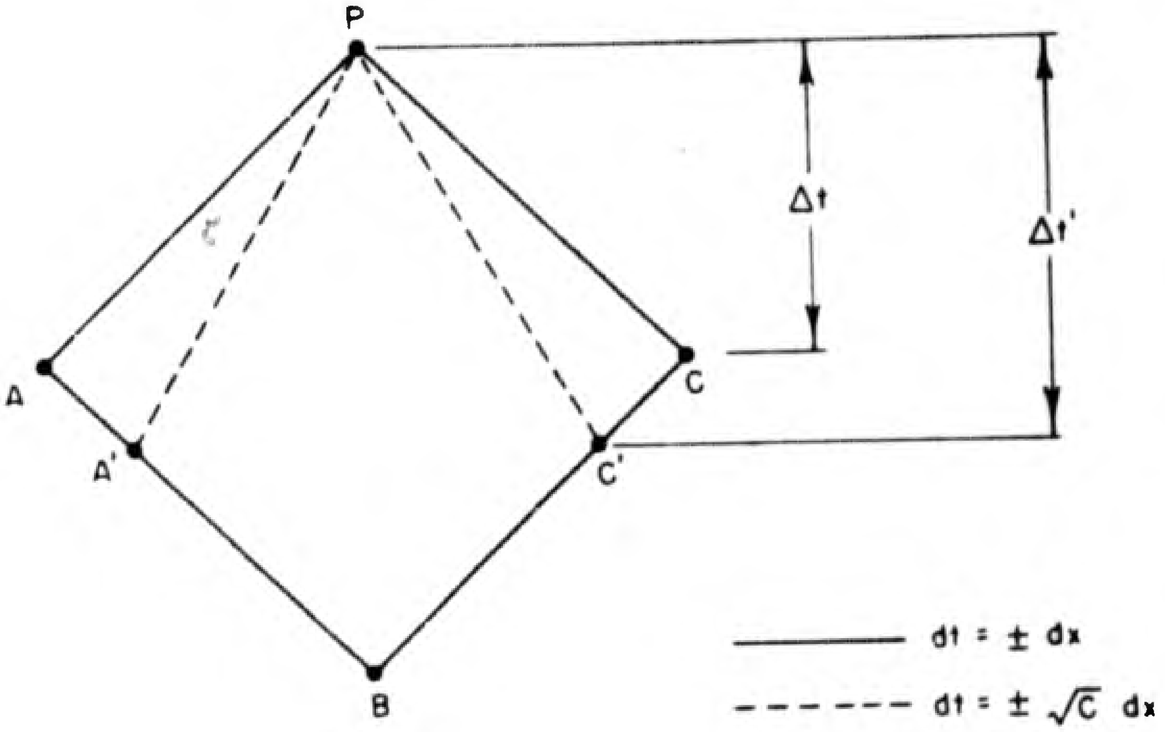

FIG. E-1: ELEMENT USED IN NUMERICAL COMPUTATION BY METHOD OF CHARACTERISTICS

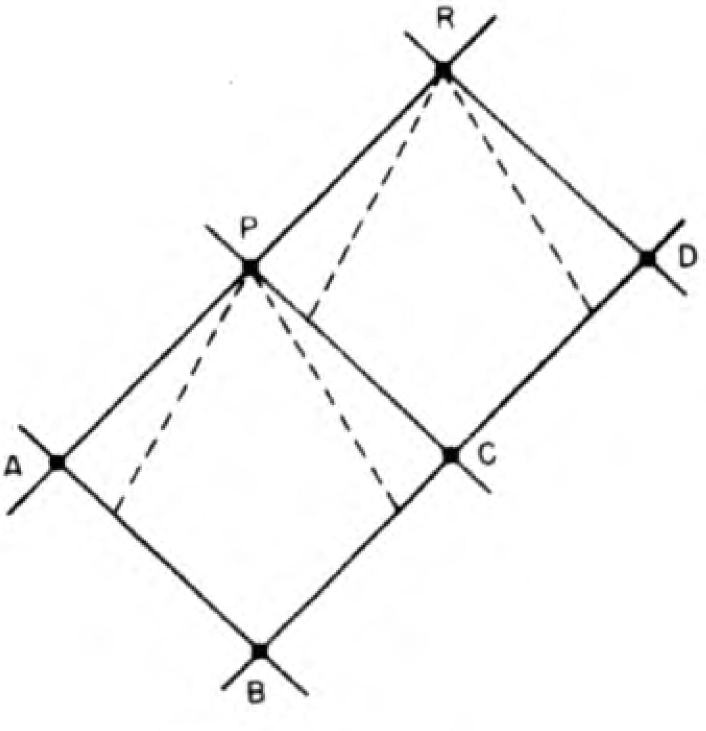

FIG. E-2: TWO CONSECUTIVE ELEMENTS

DRL — UT
DWG AA 1703
HJP — CLW
4 – 5 – 56