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ANECHOIC COATING PROJECT

REPORT ON THE DETERMINATION OF ACOUSTIC PROPERTIES OF MATERIALS BY PROPAGATION OF WAVES IN WATER FILLED TUBES

by
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SUMMARY

Sound is basically the vibration of continuous elastic substances. The vibration takes the form of an acoustic wave which can be mathematically analyzed in the same manner as an electric wave can be analyzed. Sound waves generated in an acoustic medium and confined by the rigid boundaries of a tube may be described in much the same manner as one-dimensional transverse waves in a vibrating string of finite length or by two-dimensional waves in a membrane fixed along its edge.

The first part of this report concerns itself with the development of the wave equation in its general form. For this purpose, there is a discussion of the Equation of State, the Newtonian Mechanical Equations as applied to an ideal fluid, and the Law of Conservation of Mass. These relationships lead to a second order partial differential equation known as the three-dimensional wave equation. It essentially describes a wave being propagated in a deformable medium in three mutually perpendicular directions. In order to apply the wave equation to the analysis of sound waves in water filled tubes, the equation is converted from Cartesian to cylindrical coordinates.

It is then shown that the wave equation is separable and that its solution may be represented as the product of four distinct functions, each related to only one variable. Each of these product functions is solved by successively reducing the partial differential equation to several ordinary differential equations. The general solution to the wave equation is then shown to be a complex waveform being simultaneously propagated along the longitudinal and radial axes of the tube as well as in the Θ-space. The net result is a wave which is propagated with a screw-like motion along the longitudinal axis of the tube. The wave may be either progressive or stationary, depending upon given boundary conditions. Assuming that the walls of the tube in which the sound wave is being propagated form a perfect reflector and that the waves must be finite at the origin of the tube, it is shown that the wave equation reduces to a plane wave, that is, one which is propagated along the longitudinal axis of the tube only. It is further shown that, as the radial waves form a discontinuous series of frequencies, a certain minimum frequency must not be exceeded in order to maintain plane monochromatic waves.

The final part of this report deals with the application of theory to sound waves in water filled tubes. The acoustic impedance of a material placed at one end of the tube is shown to be a complex quantity which is a function of known constants of the water, sound wave and tube. It is also a function of the phase angle between incident and reflected rays at the interface of the water and material and of the amplitude ratio of reflected to incident pressure waves (pulse tube method); or the ratio of minimum to maximum pressure and the distance of the pressure nodes from the interface (standing wave method). Pressure ratios and phase angles are measureable quantities. Once determined, the acoustic impedance may be calculated or graphically determined from Smith Charts.

1. Definition of a Fluid. Variables Describing Continuous Deformations[1,2]

Acoustic waves are generally generated by continuous and oscillatory changes in the physical properties of matter (pressure, density, etc.). They give rise to audible sounds when their frequencies vary from about 20 cps to 15,000 cps. There are also ultrasonic frequencies above 15,000 cps and subsonic frequencies below 20 cps. The manner in which a material transmits, absorbs or reflects acoustic waves is closely connected with its internal structure. Yet the relationships between the elemental structure of materials and the values of their empirical constants has not been calculated to date.

The way continuous deformations give rise to acoustic waves is a consequence of the Newtonian mechanical equations for fluids, plus certain supplementary conditions and approximations.

Let us assume that we are dealing with a quasi-continuous medium like air, water or steel. The first two media are called "fluids" because of their propensity to flow. But the theory of relativity taught us that between the three substances there is only a difference of degree, a solid under extreme conditions of pressure and temperature being allowed to creep like a liquid. Thus, any substance can be called a "fluid". Fluid properties are defined by pressure P, mass density σ, specific weight ρ, viscosity μ, surface tension δ, and modulus of elasticity E. For a low Mach number $M = \frac{v}{c}$ (ratio of velocity of a fluid v with respect to a given coordinate system to sound velocity c in the fluid), both hydro-dynamics and aerodynamics may be treated in the same manner. Compressibility becomes important and must be taken into account only when the Mach number is over about $\frac{1}{2}$.

In a fluid we can define at each point P at time t a pressure P, a density σ and a velocity with components u, v, w with respect to a rectangular coordinate system, which is the local velocity of a particle of the fluid passing a point P (x, y, z) at time t and which could be very different from the average velocity of the whole fluid. The five quantities P, σ, u, v and w are in general functions of the coordinates x, y, z and time t. In order to determine the behavior of each element or particle of the fluid when t varies, we must know the values of these five quantities at any time. Therefore, we must have five relations between these five quantities. The usual way to get them is to write five differential equations determining them in terms of five arbitrary initial constants P_o, σ_o, u_o, v_o and w_o representing the values of P, σ, u, v and w at time $t = t_o$.

2. Equation of State[1,2]

The first relation we need may be derived from the equation of state. In any fluid there is a relationship between pressure P, density σ and absolute temperature T:

1) $\psi(P, \sigma, T) = 0$

For an ideal gas at constant temperature (1) takes the form

2) $Pv = RT$

where P is the absolute pressure, v is the specific volume, T the absolute temperature, and R the universal gas constant. On setting $v = 1/\sigma$ we get $P = \sigma RT$. In the adiabatic case equation (2) is non-linear and takes the form Pv^{α} = constant or

3) $P = g(\sigma)$ or $\sigma = f(P)$

where α is the gas constant.

For perturbations of small amplitudes (which is usually the case for acoustic waves if we disregard shock waves) equation (3) can be placed by a linear relation, obtained by retaining the first two terms of a Taylor expansion of (3) and disregarding higher order terms.

4) $$P = Po + \left(\frac{\partial P}{\partial \sigma}\right)_{\sigma=\sigma_0} \Delta\sigma + \cdot \cdot \cdot \cdot$$

so that $\Delta P = P - Po = \left(\frac{\partial P}{\partial \sigma}\right) \Delta\sigma$. Since the velocity of sound $c = \sqrt{\partial P/\partial \sigma}$ we have from equation (4)

5) $P = Po + c^2 \Delta\sigma$

This equation may be regarded as the equation of state for perturbations of small amplitude in ideal gases. The equation of state for real gases at constant temperature would closely approximate Van der Waal's equation

6) $(P + a/V^2)(v-b) = RT$

where P and V are the pressure and volume and a and b are molecular constants. Real gas equations of state can also represent the behavior of liquids within a limited range of pressures and temperatures. But, for a liquid in bulk, the equation of state does not contain P as any liquid is very nearly incompressible. Density σ depends only on temperature T and we have between certain limits $\sigma = \sigma_0 (1 + KT)^{-1}$, K being the coefficient of dilation of the liquid due to heat and σ_0 its density at T = 0. For most cases, water may be regarded as incompressible, but not as far as sound waves are concerned. In other words, water appears incompressible for static pressures (between certain limits). The passage of a sound wave through water is really a pressure wave with a small variation of density approximately expressed by $\Delta P = c^2 \Delta\sigma$, the large value of c^2 associating a small variation of density $\Delta\sigma$ to a large variation of pressure ΔP.

3. Definition of External Forces

The four remaining equations are derived from the three equations of motion (Newtonian mechanical equations applied to fluids) and the law of mass conservation.

Suppose that at time t a fluid F occupies a volume V bounded by a surface S, satisfying the usual conditions of smoothness and uniqueness. The external forces acting on the elements of this fluid are of two kinds: those acting on the volume elements occupied by F and those acting on the surface elements immersed in or limiting F.

a. The external forces in a volume, like gravity, electric field, hydrostatic pressure, etc., . . . are defined in terms of the element of mass dm of the fluid F contained in the volume dV, so that $dm = \sigma dV$, σ being the density of the fluid F at a point P in the volumetric element dV. These elementary forces f_v are considered to be proportional to dm and therefore are defined by the equation

7) $$df_v = K_v dm = K_v \sigma dV$$

K_v being a proportionality constant equal to the force per unit mass.

b. There are also forces acting through areas immersed in or limiting F, such as pressure exerted by the walls of the container confining F. Let dS be an elemental area. The surface forces f_s are considered to be proportional to the area upon which they act and thus we may write

8) $$df_s = K_s \, dS$$

where K_s is a proportionality constant equal to the force per unit area.

4. Mechanical Equations[3]

We may now write the mechanical equations of motion for a fluid F corresponding to the Newtonian equations for any material system. This is simply an application of Newtonian mechanics to continuous media. Consider a point P in the fluid with mass m, velocity (u, v, w) submitted to a force (F_x, F_y, F_z) with respect to the (x, y, z) coordinate system. The Newtonian equations of motion are:

9) $$F_x = m\frac{du}{dt}, \quad F_y = m\frac{dv}{dt}, \quad F_z = m\frac{dw}{dt}$$

If X, Y, Z are components of the force per unit mass acting on the fluid F at P then

10) $$X\sigma\, dV = Y\sigma\, dV = Z\sigma\, dV$$

are components of the total force acting on the mass dm. If J is the total acceleration of a particle with coordinates x, y, z and mass dm, then the components of J are

$J_x = \frac{d^2x}{dt^2}$, $J_y = \frac{d^2y}{dt^2}$, and $J_z = \frac{d^2z}{dt^2}$ The elementary inertial force dm J applied to the particle has the components dm J_x, dm J_y and dm J_z. Integration of (10) yields the components of the total force acting on the volume V occupied by fluid F at time t.

11) $$\iiint_V \sigma X\, dV, \quad \iiint_V \sigma Y\, dV, \quad \iiint_V \sigma Z\, dV$$

Similarly the total acceleration for the same volume V at time t has the components

12) $$\iiint_V J_x\, dV, \quad \iiint_V J_y\, dV, \quad \iiint_V J_z\, dV$$

giving rise to inertial forces $\iiint_V J_x \sigma\, dV$, $\iiint_V J_y \sigma\, dV$, $\iiint_V J_z \sigma\, dV$ Since the fluid is in a state of equilibrium, these volume forces must be compensated by the total force through the surface S limiting the volume V. This surface force has the components

13) $$\iint_S T_x\, dS, \quad \iint_S T_y\, dS, \quad \iint_S T_z\, dS$$

so that the mechanical equations of motion become

14) $$\iiint_V \sigma(X - J_x)\, dV - \iint_S T_x\, dS = 0$$
$$\iiint_V \sigma(Y - J_y)\, dV - \iint_S T_y\, dS = 0$$
$$\iiint_V \sigma(Z - J_z)\, dV - \iint_S T_z\, dS = 0$$

We may also say that the sum of all external forces must be equal to the inertial force.

5. Euler's Equations for an Ideal Fluid

Equations (14) become simpler when we consider the case of an ideal (or perfect) fluid for which the viscosity is negligible and the shear modulus is zero. Usual fluids like water as far as acoustic waves are concerned approach ideality sufficiently to make use of this simplification. It is in agreement with Prandtl's hypothesis according to which, with fluids of low viscosity, the effects of viscosity are limited to a narrow region along the boundaries. In many real cases, fluid friction is so small that the hypothesis of ideality is sufficiently accurate. For an ideal fluid, the surface forces through dS are all normal to dS and directed in the positive direction. Denoting as α, β, γ the direction cosines of the outer normal to S at dS, we get for the components of the forces T_x, T_y and T_z

15)
$$T_x = P_x\alpha = \int_{P_0}^{P_1} dP_x$$
$$T_y = P_y\beta = \int_{P_0}^{P_1} dP_y$$
$$T_z = P_z\gamma = \int_{P_0}^{P_1} dP_z$$

where P_x, P_y and P_z are the components of pressure in the x, y and z directions. Since $dP_x = \left(\frac{\partial P}{\partial x}\right)dx$, $dP_y = \left(\frac{\partial P}{\partial y}\right)dy$, $dP_z = \left(\frac{\partial P}{\partial z}\right)dz$, we get $\int dP_x = \int\left(\frac{\partial P}{\partial x}\right)dx = P_x\alpha$, etc. . . . consequently,

16)
$$\iint_S T_x\, dS = \iint_S P_x\alpha\, dS = \iint_S \left(\int dP_x\right) dS = \iint_S\int\left(\frac{\partial P}{\partial x}dx\right) dS = \iiint_V \left(\frac{\partial P}{\partial x}\right) dV$$

Substituting in equations (14) we obtain

17)
$$\iiint_V \sigma(X-J_x)\, dV - \iiint_V \left(\frac{\partial P}{\partial x}\right) dV = 0$$
$$\iiint_V \sigma(Y-J_y)\, dV - \iiint_V \left(\frac{\partial P}{\partial y}\right) dV = 0$$
$$\iiint_V \sigma(Z-J_z)\, dV - \iiint_V \left(\frac{\partial P}{\partial z}\right) dV = 0$$

These equations must hold for any volume whatsoever, in particular for an arbitrary infinitesimal volume dV. This requires that the integrand vanish. Therefore

18)
$$\sigma(X-J_x) - \frac{\partial P}{\partial x} = 0$$
$$\sigma(Y-J_y) - \frac{\partial P}{\partial y} = 0$$
$$\sigma(Z-J_z) - \frac{\partial P}{\partial z} = 0$$

Now, let us follow the motion of a certain particle of the fluid in terms of time t. When passing the point P at time t it has the velocity (u, v, w) and obviously the values of these components depend on coordinates x, y, z of the particle. In turn, the latter depend on time t, since the coordinates of the particle vary when time increases. Thus, u, v and w are functions of x, y, z and t

19) $$dx/dt = u(x, y, z, t)$$

$$dy/dt = v(x, y, z, t)$$

$$dz/dt = w(x, y, z, t)$$

and x = x(t), y = y(t), z = z(t). Consequently the acceleration becomes

20) $$J_x = \frac{d^2x}{dt^2} = \frac{du}{dt} = \frac{\partial u}{\partial t} + \left(\frac{\partial u}{\partial x}\right)u + \left(\frac{\partial u}{\partial y}\right)v + \left(\frac{\partial u}{\partial z}\right)w$$

$$J_y = \frac{d^2y}{dt^2} = \frac{dv}{dt} = \frac{\partial v}{\partial t} + \left(\frac{\partial v}{\partial x}\right)u + \left(\frac{\partial v}{\partial y}\right)v + \left(\frac{\partial v}{\partial z}\right)w$$

$$J_z = \frac{d^2z}{dt^2} = \frac{dw}{dt} = \frac{\partial w}{\partial t} + \left(\frac{\partial w}{\partial x}\right)u + \left(\frac{\partial w}{\partial y}\right)v + \left(\frac{\partial w}{\partial z}\right)w$$

using the well known formulas of partial derivatives and the chain rule. Replacing in (18) J_x, J_y, J_z by their values (20), we obtain the Euler equations

21) $$\frac{1}{\sigma}\frac{\partial P}{\partial x} = X - \frac{\partial u}{\partial t} - u\left(\frac{\partial u}{\partial x}\right) - v\left(\frac{\partial u}{\partial y}\right) - w\left(\frac{\partial u}{\partial z}\right)$$

$$\frac{1}{\sigma}\frac{\partial P}{\partial y} = Y - \frac{\partial v}{\partial t} - u\left(\frac{\partial v}{\partial x}\right) - v\left(\frac{\partial v}{\partial y}\right) - w\left(\frac{\partial v}{\partial z}\right)$$

$$\frac{1}{\sigma}\frac{\partial P}{\partial z} = Z - \frac{\partial w}{\partial t} - u\left(\frac{\partial w}{\partial x}\right) - v\left(\frac{\partial w}{\partial y}\right) - w\left(\frac{\partial w}{\partial z}\right)$$

which, when integrated give three conditions to be satisfied by the variables u, v, w. Together with the equation of state (1) taken in the adiabatic form not involving temperature T, i.e., pv^{α} = constant, since temperature has no time to vary during the processes we are dealing with, we obtain four conditions. We still need another equation to determine the variables P, σ, u, v and w. It is given by the Law of Conservation of Mass, and the Continuity Equation.

6. Continuity Equation

Consider a particle of the perfect fluid (ideal frictionless medium) with mass dm occupying the volume dV at time t and having the coordinates x, y, z. We have $dm = \sigma dV$ where σ is the density at the point P(x, y, z) at time t. When t becomes $t' = t + dt$ is at $Q(x', y', z')$ with coordinates

22) $$x' = x + u\,dt \qquad y' = y + v\,dt \qquad z' = z + w\,dt$$

The volume occupied by the same particle becomes dV', its density σ' at time t but the mass remains the same so that

23) $$dm = \sigma dV = \sigma' dV'$$

Since σ is a function of x, y, z and t we may write $\sigma = f(x, y, z, t)$ so that

24) $$\sigma' = f(x', y', z', t') = f(x + udt, y + vdt, z + wdt, t + dt)$$

Using a Taylor series expansion limited to the second term we have

25) $$\sigma' = \sigma + \left(\left(\frac{\partial\sigma}{\partial x}\right)u + \left(\frac{\partial\sigma}{\partial y}\right)v + \left(\frac{\partial\sigma}{\partial z}\right)w + \frac{\partial\sigma}{\partial t}\right)dt$$

On the other hand $dV' = dx'\,dy'\,dz'$. The term $dx'\,dy'\,dz'$ may be computed by differentiating formulas (22) which yield

26) $$dx' = dx + du\,dt \qquad dy' = dy + dv\,dt \qquad dz' = dz + dw\,dt$$

Partial differentiation of (22) or division of (26) by dx, dy, dz gives

27) $$\frac{\partial x'}{\partial x} = 1+\left(\frac{\partial u}{\partial x}\right)dt \qquad \frac{\partial x'}{\partial y} = \left(\frac{\partial u}{\partial y}\right)dt \qquad \frac{\partial x'}{\partial z} = \left(\frac{\partial u}{\partial z}\right)dt$$

$$\frac{\partial y'}{\partial x} = \left(\frac{\partial v}{\partial x}\right)dt \qquad \frac{\partial y'}{\partial y} = 1+\left(\frac{\partial v}{\partial y}\right)dt \qquad \frac{\partial y'}{\partial z} = \left(\frac{\partial v}{\partial z}\right)dt$$

$$\frac{\partial z'}{\partial x} = \left(\frac{\partial w}{\partial x}\right)dx \qquad \frac{\partial z'}{\partial y} = \left(\frac{\partial w}{\partial y}\right)dt \qquad \frac{\partial z'}{\partial z} = 1+\left(\frac{\partial w}{\partial z}\right)dt$$

Substituting in $dV' = dx' \, dy' \, dz'$ and neglecting terms containing $(dt)^2$ and $(dt)^3$ we have

$$28) \qquad dV = dV\left[1 + \left(\frac{\partial u}{\partial \psi} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) dt\right].$$

From (23) we deduce, in the same approximation, using (25) and (28)

$$29) \qquad \left[\sigma + \left(\left(\frac{\partial \sigma}{\partial x}\right)u + \left(\frac{\partial \sigma}{\partial y}\right)v + \left(\frac{\partial \sigma}{\partial z}\right)w + \left(\frac{\partial \sigma}{\partial t}\right)\right) dt\right]$$

$$dV\left[1 + \left(\frac{\partial u}{\partial \psi} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)dt\right] = \sigma dV$$

Neglecting the $(dt)^2$ term (29) becomes

$$\sigma dV + dt\, dV\left[\sigma\left[\frac{\partial u}{\partial \psi} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right] + \left[\left(\frac{\partial \sigma}{\partial \psi}\right)u + \left(\frac{\partial \sigma}{\partial y}\right)v + \left(\frac{\partial \sigma}{\partial z}\right)w + \frac{\partial \sigma}{\partial t}\right]\right]$$

which reduces to

$$30) \qquad \sigma\left[\frac{\partial u}{\partial \psi} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right] + \left(\frac{\partial \sigma}{\partial \psi}\right)u + \left(\frac{\partial \sigma}{\partial y}\right)v + \left(\frac{\partial \sigma}{\partial z}\right)w + \frac{\partial \sigma}{\partial t} = 0$$

(30) can be written

$$31) \qquad \frac{\partial \sigma}{\partial t} + \frac{\partial \sigma u}{\partial \psi} + \frac{\partial \sigma v}{\partial y} + \frac{\partial \sigma w}{\partial z} = 0$$

which is the continuity equation. It represents the conservation of mass in a differential form. Indeed, integrating (31) over volume V we obtain

$$32) \qquad \iiint_V \left(\frac{\partial \sigma}{\partial t}\right) dV + \iiint_V \left(\frac{\partial \sigma u}{\partial \psi} + \frac{\partial \sigma v}{\partial y} + \frac{\partial \sigma w}{\partial z}\right) dV = 0$$

By Gauss' theorem for a vector $\overline{K}$ where

$\overline{K} = (K_x, K_y, K_z)$ we have $\iint_S K_n \, dS = \iiint_V \operatorname{div} \overline{K} \, dV$, which means that the surface integral of the normal component K_n of K over the surface S equals the volume integral over the volume V bounded by S where $\operatorname{div} \overline{K} = \frac{\partial K_\psi}{\partial \psi} + \frac{\partial K_y}{\partial y} + \frac{\partial K_z}{\partial z}$.

The integral $\iiint_V \left(\frac{\partial \sigma u}{\partial \psi} + \frac{\partial \sigma v}{\partial y} + \frac{\partial \sigma w}{\partial z}\right) dV$ of (32) can therefore be written in the form of a surface integral, surface S bounding volume V. Denoting as $\overline{K}$ the vector with components $\sigma u, \sigma v$, and σw, $\iiint_V \text{div } \overline{K}\, dV = \iint_S K_n\, dS$. If S is the surface bounding the fluid, no amount of fluid flows through this surface and as $\overline{K}$ is the vector through this surface then $\iint_S K_n\, dS$ must be zero, where K_n is the component of $\overline{K}$ normal to the surface. The integral

$$\iiint_V \left(\frac{\partial \sigma}{\partial t}\right) dV \text{ of } (32)$$

may be written as

$$\frac{\partial}{\partial t} \iiint_V \sigma\, dV = \frac{\partial}{\partial t} \iiint_V dm = \frac{\partial m}{\partial t}.$$

Since the second integral of (3) must be zero then (32) reduces to $\frac{\partial m}{\partial t} = 0$ and the mass is conserved. This implies that no fluid is created or destroyed inside V.

7. Force Potential and Velocity Potential

The Euler equations (21) may be put into a simpler form when three supplementary restrictions are valid.

First, assume that the external volume force X, Y, Z derives from an energy potential V, so that

33) $$X = \frac{\partial V}{\partial \psi}, \quad Y = \frac{\partial V}{\partial y}, \quad Z = \frac{\partial V}{\partial z}$$

Second, there is no rotational motion inside the fluid so that the fluid velocity must derive from a potential Φ

34) $$u = -\frac{\partial \Phi}{\partial \psi} \qquad V = -\frac{\partial \Phi}{\partial y} \qquad W = -\frac{\partial \Phi}{\partial z}$$

The negative sign indicates that an increasing energy potential V causes the velocity of the fluid to decrease.

Third, the equation of state is valid in the adiabatic form $\sigma = f(P)$ solved with respect to σ.

When these three conditions are satisfied we may define a new function

35) $$Q = V - \int \frac{dP}{\sigma}$$

Introducing this equation into the Euler equations (21) and replacing X, Y, Z by (33) and u, v, w by (34) respectively, we obtain for the x component

36) $$\frac{\partial Q}{\partial \psi} = \frac{\partial V}{\partial \psi} - \frac{1}{\sigma}\frac{\partial P}{\partial \psi}$$
$$= -\frac{\partial^2 \Phi}{\partial t \partial \psi} + \frac{\partial \Phi}{\partial \psi}\frac{\partial^2 \Phi}{\partial \psi^2} + \frac{\partial \Phi}{\partial y}\frac{\partial^2 \Phi}{\partial y \partial \psi} + \frac{\partial \Phi}{\partial z}\frac{\partial^2 \Phi}{\partial z \partial \psi}$$

and we get similar formulas for $\partial Q/\partial y$ and $\partial Q/\partial z$, replacing x in (36) by y and z sucessively. As Φ is a uniform function of x, y, z and t, we have

$$\frac{\partial^2 \Phi}{\partial \psi \partial y} = \frac{\partial^2 \Phi}{\partial y \partial \psi}, \quad \frac{\partial^2 \Phi}{\partial t \partial \psi} = \frac{\partial^2 \Phi}{\partial \psi \partial t}, \text{ etc.} \ldots \text{ for } x, y, z, t.$$

Thus, the right hand side of (36) can be written in the form

37) $$\frac{\partial}{\partial \psi}\left[-\frac{\partial \Phi}{\partial t} + \frac{1}{2}\left[\left(\frac{\partial \Phi}{\partial \psi}\right)^2 + \left(\frac{\partial \Phi}{\partial y}\right)^2 + \left(\frac{\partial \Phi}{\partial z}\right)^2\right]\right]$$

since $\frac{\partial\phi}{\partial x} \cdot \frac{\partial^2\phi}{\partial x^2} = \frac{1}{2}\frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial x}\right)^2$

$$\frac{\partial\phi}{\partial y}\,\frac{\partial^2\phi}{\partial x\partial y} = \frac{1}{2}\,\frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial y}\right)^2$$

$$\frac{\partial\phi}{\partial z}\,\frac{\partial^2\phi}{\partial x\partial z} = \frac{1}{2}\,\frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial z}\right)^2$$

consequently, we may write (36) in the form

38) $$\frac{\partial}{\partial x}\left[Q + \frac{\partial\phi}{\partial t} - \frac{1}{2}\left\{\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2\right\}\right] = 0$$

The quantity within the outer bracket,

$$Q + \frac{\partial\phi}{\partial t} - \frac{1}{2}\left\{\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2\right\}$$

must therefore be independent of x as its partial derivative with respect to x is zero. Similary, replacing x by y and z successively, we can demonstrate that the same quantity is independent of y and z. Thus it depends only on t and we can call it H (t). If it is assumed that H (t) is continuous in the domain of variation of t, then there exists an integral $\eta(t) = \int H(t)\,dt$ so that

$$H(t) = \frac{d\,\eta(t)}{dt}$$

and we may write

39) $$Q + \frac{\partial\phi}{dt} - \frac{1}{2}\left\{\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2\right\} - \frac{d\,\eta(t)}{dt} = 0$$

or

40) $$Q + \frac{\partial(\phi - \eta(t))}{\partial t} - \frac{1}{2}\left\{\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2\right\} = 0$$

Now (40) we can replace everywhere ϕ by $\phi' = \phi - \eta(t)$ since $\eta(t)$ depends only on t and not on x, y, z. We obtain

41) $$Q + \frac{\partial\phi'}{\partial t} - \frac{1}{2}\left\{\left(\frac{\partial\phi'}{\partial x}\right)^2 + \left(\frac{\partial\phi'}{\partial y}\right)^2 + \left(\frac{\partial\phi'}{\partial z}\right)^2\right\} = 0$$

or, replacing $\partial\Phi'/\partial y$ etc. by u, v, w, we have

41')

$$Q + \frac{\partial\Phi'}{\partial t} - \frac{1}{2}\left[u^2+v^2+w^2\right] = 0$$

This equation, together with the continuity equation (31) and the equation of state in the form (3) gives three equations to determine the three unknowns Φ', P and σ in terms of x, y, z, t and arbitrary constants.

8. Wave Equation [4]

If the velocity of the particles of the continuous fluid is small (a condition always satisfied if we put aside shock waves of great intensity or explosive impulses) we can neglect the quadratic terms $u^2 + v^2 + w^2$ in (41'). The latter reduces to

42) $$Q + \frac{\partial \Phi'}{\partial t} = 0 \text{ or } V - \int \frac{dP}{\sigma} + \frac{\partial \Phi'}{\partial t} = 0$$

owing to the definition (35) of Q. Suppose now that there are no external volume forces; i.e., V = 0. Then (42) becomes

43) $$\int \frac{dP}{\sigma} = \frac{\partial \Phi'}{\partial t}$$

Now, for waves of weak amplitude the mass density σ is very nearly a constant and for a variation of pressure from P_o to P we may write

44) $$\int_{P_o}^{P} \frac{dP}{\sigma} = \frac{1}{\sigma_o} (P - P_o)$$

where σ_o is the average constant density. Equation (43) then becomes

45) $$P - P_o = \sigma_o \frac{\partial \Phi'}{\partial t}$$

consider now the continuity equation (31) where u, v and w have been replaced by their expression (34) in terms of Φ. Since

$$\frac{\partial \Phi}{\partial x} = \frac{\partial \Phi'}{\partial x}, \frac{\partial \Phi}{\partial y} = \frac{\partial \Phi'}{\partial y}, \frac{\partial \Phi}{\partial z} = \frac{\partial \Phi'}{\partial z}$$

we may substitute Φ for Φ'. We obtain

46) $$u\left(\frac{\partial \sigma}{\partial x}\right) + v\left(\frac{\partial \sigma}{\partial y}\right) + w\left(\frac{\partial \sigma}{\partial z}\right) + \frac{\partial \sigma}{\partial t} - \sigma\left(\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2}\right) = 0$$

Since $\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} = \nabla^2 \Phi$ where ∇ is the Laplacian operator, we may write (46) in the form $\frac{\partial \sigma}{\partial t} + u\left(\frac{\partial \sigma}{\partial x}\right) + v\left(\frac{\partial \sigma}{\partial y}\right) + w\left(\frac{\partial \sigma}{\partial z}\right) = \sigma \nabla^2 \Phi$ or, since $\sigma = \sigma_o$ independent of x, y and z

47) $$\frac{\partial \sigma_o}{\partial t} = \sigma_o \nabla^2 \Phi.$$

According to the equation of state (3), σ_o depends explicity on P. Consequently, we can write

48) $$\frac{\partial \sigma_o}{\partial t} = \frac{\partial \sigma_o}{\partial P}\frac{\partial P}{\partial t}$$

Differentiating (45) with respect to t and noting that σ_o and P_o are constant with respect to t and that Φ' may be written Φ:

49) $$\frac{\partial P}{\partial t} = \sigma_o \frac{\partial^2 \Phi}{\partial t^2}$$

Substituting this value for $\partial P/\partial t$ in (48) we get

50) $$\frac{\partial \sigma_o}{\partial t} = \frac{\partial \sigma_o}{\partial P}\sigma_o \frac{\partial^2 \Phi}{\partial t^2}$$

Substituting (50) in (47) we have

51) $$\frac{\partial \sigma_o}{\partial P}\sigma_o \frac{\partial^2 \Phi}{\partial t^2} = \sigma_o \nabla^2 \Phi$$

On setting $c^2 = \frac{\partial P}{\partial \sigma_o}$ equation (51) becomes

52) $$\nabla^2 \Phi = \frac{1}{c^2}\frac{\partial^2 \Phi}{\partial t^2}$$

Equation (52) is called the wave equation. It represents waves propagating in the (x, y, z) space with velocity c. It can also be deduced in the following way: suppose that the volume V containing a certain fixed mass of fluid varies and becomes V'. This defines the cubic dilation δ as the ratio

53) $$\delta = \frac{V' - V}{V}$$

So that $V' = V(1 + \delta)$. On the other hand, if the density of the part of fluid contained in a fixed volume V changes so that the density σ becomes σ', the condensation ϵ is defined by

54) $$\epsilon = \frac{\sigma' - \sigma}{\sigma}$$

So that $\sigma' = \sigma(1+\epsilon)$. In general, changes in density and volume in fluids result from pressure changes. The excess pressure $P-P_o$ over the static pressure P_o is called the acoustic pressure. The bulk modulus B is defined as

55) $$B = -\frac{P}{\delta}$$

As $V''\sigma = V\sigma = m$ because of the conservation of mass, using (53) and (54) we get $V'\sigma' = V\sigma = m = V\sigma(1+\delta)(1+\epsilon) = V\sigma(1+\delta+\epsilon+\delta\epsilon)$. Neglecting the cross-product $\delta\epsilon$ we get $1 + \delta + \epsilon = 1$ or $\epsilon = -\delta$ and therefore $B = \frac{P}{\epsilon}$. Next.

consider a surface S and a volume element Sdz and suppose for simplicity that the fluid is deformed only in the z direction. A particle of the fluid which was at z at time t undergoes a displacement dz = dl. The displacement of the particle occupying the position z + dz at some time t will be l + dl where dl = $\left(\frac{\partial l}{\partial z}\right)$ dz. If the pressure at z is P, the pressure at z + dz is P + dP = P + $\left(\frac{\partial P}{\partial z}\right)$dz, where P is the force per surface unit. Consequently, the net force in the z direction is equal to SP - S $\left(P + \left(\frac{\partial P}{\partial z}\right)dz\right) = -S\left(\frac{\partial P}{\partial z}\right)dz$. The mass of fluid contained in the volume element Sdz is $\sigma S dz$. Thus the Newtonian dynamic equation in the z- direction is

$$55) \qquad -S\left(\frac{\partial P}{\partial z}\right)dz = S\sigma dz \frac{\partial^2 l}{\partial t^2}$$

hence $-\frac{\partial P}{\partial z} = \sigma \frac{\partial^2 l}{\partial t^2}$. Now, since $P = B\epsilon$, we deduce by differentiation, B being a constant:

$$\frac{\partial P}{\partial z} = B\frac{\partial \epsilon}{\partial z} = -B\frac{\partial \delta}{\partial z}.$$

But

$$= \frac{v' - v}{v} = \frac{S\left(\frac{\partial l}{\partial z}\right)dz}{Sdz} = \frac{\partial l}{\partial z}, \text{ So that}$$

$$\frac{\partial P}{\partial z} = -B\frac{\partial \delta}{\partial z} = -B\frac{\partial^2 l}{\partial z^2} \quad \text{and we have}$$

$$56) \qquad \frac{\partial^2 l}{\partial t^2} = \left(\frac{B}{\sigma}\right)\frac{\partial^2 l}{\partial t^2}$$

which is a wave equation in the z direction, emphasizing the role of the coefficient B. For adiabatic processes in ideal gases we have Pv^{α} = constant so that we can define the adiabatic bulk modulus by differentiation:

$$v^{\alpha} dP + \alpha v^{\alpha-1} P dv = 0.$$

Thus $dP = -\alpha \frac{dv}{v} P = -\alpha \delta P$ so that $\alpha P = -\frac{dP}{\delta} = B$ and $c = \sqrt{\frac{B}{\sigma}} = \sqrt{\frac{\alpha P}{\sigma}}$. We can verify that (56) or (52) represents a wave equation in the z direction. In that case, the general solution to this equation is h = f (z + ct) + g (z-ct) = f (u) + g(v) on setting z + ct = u, z - ct = v. Differentiating h with respect to t and z we obtain:

$$\frac{\partial h}{\partial t} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial g}{\partial v}\frac{\partial v}{\partial t} = c\left(\frac{\partial f}{\partial u} - \frac{\partial g}{\partial v}\right)$$

$$\frac{\partial^2 h}{\partial t^2} = c\left[\frac{\partial^2 f}{\partial u^2}\frac{\partial u}{\partial t} - \frac{\partial^2 g}{\partial v^2}\frac{\partial v}{\partial t}\right]$$

$$= c^2 \left(\frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 g}{\partial v^2} \right)$$

$$\frac{\partial h}{\partial z} = \left(\frac{\partial f}{\partial u} \frac{\partial u}{\partial z} + \frac{\partial g}{\partial v} \frac{\partial v}{\partial z} \right) = \frac{\partial f}{\partial u} + \frac{\partial g}{\partial v}$$

$$\frac{\partial^2 h}{\partial z^2} = \frac{\partial^2 f}{\partial u^2} \frac{\partial u}{\partial z} + \frac{\partial^2 g}{\partial v^2} \frac{\partial v}{\partial z} = \frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 g}{\partial z^2}$$

$$= \frac{1}{c^2} \frac{\partial^2 h}{\partial t^2}$$

The functions f and g represent any differentiable function. According to the equation $h = f(z + ct)$ the deformation $f(z)$ which at time $t = 0$ was at $z = z_0$ is at time t at $z + ct$ with the same functional form $f(z)$. Thus, the wave propagated from z to $z + ct$ with velocity c. In the same way a wave with the functional form $g(z)$ is propagated from $z = z_0$ to $z - ct$ with velocity c in time t. In the general case, consider the wave equation (52) where c is a constant and we take as a single independent variable $\mu = \alpha x + \beta y + \gamma z + ct$ where α, β and γ are constants. The wave equation becomes

$$c^2(\alpha^2 + \beta^2 + \gamma^2 - 1) \frac{\partial^2 \Phi}{\partial \mu^2} = 0$$

Thus, when $\alpha^2 + \beta^2 + \gamma^2 = 1$, any differentiable function Φ is a solution of (52). Also, $\eta = \alpha x + \beta y + \gamma z - ct$ leads to a solution of (52). Thus, $\Phi = f_1(\mu) + f_2(\eta)$ is a solution where f_1, f_2 are arbitrary differentiable functions.

9. Wave Equation in Cylindrical Coordinates

For our aim, which is to analyse the behavior of sound waves in tubes, we have to express the wave equation (52) in a cylindrical coordinate system instead of the usual rectangular system. The cylindrical system is more adapted to our problem and has the supplementary advantage that we can define in it separated solutions-i.e., solutions formed by a product of functions, each factor of which depends only on one coordinate. These separated solutions satisfy a system of ordinary differential equations - which are much easier to handle then partial differential equations - and all solutions of the partial differential equation can be determined as linear combinations of each term of the set of separated solutions. Thus, to integrate equation (52) is equivalent to determining the separated solutions. The coordinates for which such solutions exist are called separable coordinates.

We choose as coordinates a directed radius r starting from the origin of the rectangular coordinate system and situated in the (x, y) plane and angle Θ which is the positive angle determined by r with the x-axis. The (x, y) plane can be regarded as defining a cross-section of a certain cylindrical tube. The z axis in this case is along the longitudinal axis of the tube.

To express (52) in cylindrical coordinates we notice that r, Θ are simply polar coordinates in the (x, y) plane so that the definition of the cylindrical coordinates is given by

57) $$x = r\cos\Theta \qquad y = r\sin\Theta \qquad z = z$$

or, in terms of x and y

58) $$r = \sqrt{x^2+y^2} \qquad \Theta = \tan^{-1}\frac{y}{x}$$

Differentiating (58) we get

59) $$\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2+y^2}} = \cos\Theta \qquad \frac{\partial\Theta}{\partial y} = \frac{x}{x^2+y^2} = \frac{1}{r}\cos\Theta$$

$$\frac{\partial r}{\partial y} = \frac{y}{\sqrt{x^2+y^2}} = \sin\Theta$$

$$\frac{\partial\Theta}{\partial x} = -\frac{y}{x^2+y^2} = -\frac{1}{r}\sin\Theta$$

The wave function $\Phi(x, y, z, t)$ when expressed by means of the variables (r, Θ, z, t) becomes another function $\Phi'(r, \Theta, z, t)$. In order not to needlessly complicate the notations we shall retain Φ in any case. Applying the formulas of partial differentiation we get

60)
$$\frac{\partial \Phi}{\partial x} = \frac{\partial \Phi}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial \Phi}{\partial \Theta}\frac{\partial \Theta}{\partial x}$$
$$\frac{\partial \Phi}{\partial y} = \frac{\partial \Phi}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial \Phi}{\partial \Theta}\frac{\partial \Theta}{\partial y}$$

Substituting (59) into (60) we get

61)
$$\frac{\partial \Phi}{\partial x} = \frac{\partial \Phi}{\partial r}\cos\Theta - \frac{\partial \Phi}{\partial \Theta}\frac{\sin\Theta}{r}$$
$$\frac{\partial \Phi}{\partial y} = \frac{\partial \Phi}{\partial r}\sin\Theta + \frac{\partial \Phi}{\partial \Theta}\frac{\cos\Theta}{r}$$

Consequently
$$\frac{\partial^2 \Phi}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{\partial \Phi}{\partial r}\cos\Theta - \frac{\partial \Phi}{\partial \Theta}\frac{\sin\Theta}{r}\right]$$

63)
$$\frac{\partial^2 \Phi}{\partial y^2} = \frac{\partial}{\partial y}\left[\frac{\partial \Phi}{\partial r}\sin\Theta + \frac{\partial \Phi}{\partial \Theta}\frac{\cos\Theta}{r}\right]$$

differentiating (63) we have

64)
$$\frac{\partial^2 \Phi}{\partial x^2} = \frac{\partial}{\partial r}\left[\frac{\partial \Phi}{\partial r}\cos\Theta - \frac{\partial \Phi}{\partial \Theta}\frac{\sin\Theta}{r}\right]\frac{\partial r}{\partial x} + \frac{\partial}{\partial \Theta}\left[\frac{\partial \Phi}{\partial r}\cos\Theta - \frac{\partial \Phi}{\partial \Theta}\frac{\sin\Theta}{r}\right]\frac{\partial \Theta}{\partial x}$$
$$= \left[\frac{\partial^2 \Phi}{\partial r^2}\cos\Theta + \frac{\partial \Phi}{\partial \Theta}\frac{\sin\Theta}{r^2} - \frac{\partial^2 \Phi}{\partial r \partial \Theta}\frac{\sin\Theta}{r}\right]\cos\Theta$$
$$- \frac{\sin\Theta}{r}\left[\frac{\partial^2 \Phi}{\partial r \partial \Theta}\cos\Theta - \frac{\partial \Phi}{\partial r}\sin\Theta - \frac{\partial^2 \Phi}{\partial \Theta^2}\frac{\sin\Theta}{r} - \frac{\partial \Phi}{\partial \Theta}\frac{\cos\Theta}{r}\right]$$

$$\frac{\partial^2 \Phi}{\partial y^2} = \frac{\partial}{\partial r}\left[\frac{\partial \Phi}{\partial r}\sin\Theta + \frac{\partial \Phi}{\partial \Theta}\frac{\cos\Theta}{r}\right]\frac{\partial r}{\partial y} + \frac{\partial}{\partial \Theta}\left[\frac{\partial \Phi}{\partial r}\sin\Theta + \frac{\partial \Phi}{\partial \Theta}\frac{\cos\Theta}{r}\right]\frac{\partial \Theta}{\partial y}$$

$$= \left[\frac{\partial^2 \Phi}{\partial r^2} \sin\theta + \frac{\partial^2 \Phi}{\partial\theta\partial r} \frac{\cos\theta}{r} - \frac{\partial \Phi}{\partial\theta} \frac{\cos\theta}{r^2} \right] \sin\theta +$$

$$\frac{\cos\theta}{r} \left[\frac{\partial^2 \Phi}{\partial r\partial\theta} \sin\theta + \frac{\partial \Phi}{\partial r} \cos\theta + \frac{\partial^2 \Phi}{\partial\theta^2} \frac{\cos\theta}{r} - \frac{\partial \Phi}{\partial\theta} \frac{\sin\theta}{r} \right]$$

Adding $\frac{\partial^2 \Phi}{\partial x^2}$ and $\frac{\partial^2 \Phi}{\partial y^2}$ we get

65) $$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = \frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r} \frac{\partial \Phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \Phi}{\partial\theta^2}$$

so that the wave equation (52) takes the form

66) $$\frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r} \frac{\partial \Phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \Phi}{\partial\theta^2} + \frac{\partial^2 \Phi}{\partial z^2} = \frac{1}{c^2} \frac{\partial^2 \Phi}{\partial t^2}$$

10. Separated Solution of the Wave Equation[5]

We have already seen that in cylindrical coordinates equation (66) is separable. consequently, a solution to this equation is given by

67) $$\Phi\ (r, \Theta, z, t) = A(r)\ B(\Theta)\ C(z)\ D(t)$$

Each factor of the product being related to one variable. Substituting in (66) the product (67) we obtain

68) $$BCD \frac{\partial^2 A}{\partial r^2} + BCD \frac{\partial A}{r \partial r} + ACD \frac{\partial^2 B}{r^2 \partial \theta^2} + ABD \frac{\partial^2 C}{\partial z^2} = ABC \frac{\partial^2 D}{c^2 \partial t^2}$$

Dividing every term by ABCD (68) becomes

69) $$\frac{1}{A} \frac{\partial^2 A}{\partial r^2} + \frac{1}{Ar} \frac{\partial A}{\partial r} + \frac{1}{Br^2} \frac{\partial^2 B}{\partial \theta^2} + \frac{1}{C} \frac{\partial^2 C}{\partial z^2} = \frac{1}{Dc^2} \frac{\partial^2 D}{\partial t^2}$$

The left-hand side of (69) does not depend on t since it is only formed by the functions r, Θ, z and their derivatives. On the contrary, the right hand side of (69) is only formed by a function of t and its second derivative. Thus, when t varies, the left hand side remains constant and when r, Θ, z vary the right hand side remains constant. As the right hand side is equal to the left hand side we conclude that it is equal to the same constant, that we can term k^2. Hence

70) $$\frac{1}{Dc^2} \frac{\partial^2 D}{\partial t^2} = k^2 \quad \text{or} \quad \frac{\partial^2 D}{\partial t^2} = k^2 c^2 D$$

The general solution of this ordinary differential equatior of the second order with constant coefficients is of the form

71) $$D = h_1 e^{kct} + h_2 e^{-kct}$$

where h_1 and h_2 are arbitrary constants. We see that (71) represents harmonic waves only when k is purely imaginary. Hence we may set

72) $$kc = 2\pi j \nu \quad \text{or} \quad k = 2\pi j \nu / c = 2\pi j / \lambda$$

where λ is the wave length of the periodic phenomenon and ν is the frequency. In general, k is complex, k = a+jb, and involves an attenuation factor "a" so that (71) becomes

$$D = h_1 e^{(a+jb)ct} + h_2 e^{-(a+jb)ct}$$

a and b being real numbers. The arbitrary constants h_1 and h_2 are in general complex numbers. Putting the right-hand side of (69) equal to K^2, we can write

$$73)\qquad \frac{1}{A}\frac{\partial^2 A}{\partial r^2} + \frac{1}{Ar}\frac{\partial A}{\partial r} + \frac{1}{Br^2}\frac{\partial^2 B}{\partial \theta^2} = k^2 - \frac{1}{C}\frac{\partial^2 C}{\partial z^2}$$

We find ourselves in the same situation as previously. The right-hand side of (73) is only a function of z and the left-hand side only a function of r and θ. As they are equal to each other, they can be equated to the same constant $(k')^2$. Thus

$$74)\qquad \frac{\partial^2 C}{\partial z^2} = (k^2 - (k')^2)C$$

The general solution of this ordinary differential equation of the second order is

$$75)\qquad C = C_1 e^{\sqrt{k^2-(k')^2}\,z} + C_2 e^{-\sqrt{k^2-(k')^2}\,z}$$

Replacing the right-hand side of (73) by $(k')^2$ and subtracting $\frac{\partial^2 B}{Br^2\partial\theta^2}$, we obtain

$$76)\qquad \frac{1}{A}\frac{\partial^2 A}{\partial r^2} + \frac{1}{Ar}\frac{\partial A}{\partial r} = (k')^2 - \frac{1}{Br^2}\frac{\partial^2 B}{\partial \theta^2}$$

multiplying by r^2

$$77)\qquad r^2\left(\frac{1}{A}\frac{\partial^2 A}{\partial r^2} + \frac{1}{Ar}\frac{\partial A}{\partial r} - (k')^2\right) = -\frac{1}{B}\frac{\partial^2 B}{\partial \theta^2}$$

Again, the right-hand side of (77) is only a function of θ and the left-hand side is only a function of r. Thus, they are both equal to the same constant that is customarily called m^2. Accordingly, we have

$$78)\qquad \frac{\partial^2 B}{\partial \theta^2} = -m^2 B$$

The general solution of which is

$$79)\qquad B = B_1 e^{jm\theta} + B_2 e^{-jm\theta}$$

B_1 and B_2 being arbitrary constants. m must be a real number, otherwise (79) would give vanishing or unbounded solutions. If m is not an integer, the wave $B_1 e^{jm\theta} + B_2 e^{-jm\theta}$ represents a progressive wave in the θ-space. For, B is changed to $B' \neq B$ by a 2π rotation. When m is an integer, for $\theta = 0$ or $n\,2\pi$, n an integer, $B' = B_1 + B_2 = B$ remains the same. Physically, this means that the wave front after a complete rotation of n 2π degrees comes back to the same phase. The wave is thus stationary in the Θ-space.

Next, replacing in (77) the right-hand side by m^2, substituting for $(k')^2$, $h^2 = (k')^2$ and conveniently arranging the terms we get

80) $$\frac{d^2A}{dr^2} + \frac{1}{r}\frac{dA}{dr} + \left(h^2 - \frac{m^2}{r^2}\right) A = 0$$

Finally, replacing r by $s = hr$ we obtain the general Bessel equation

81) $$\frac{d^2A}{ds^2} + \frac{1}{s}\frac{dA}{ds} + \left(1 - \frac{m^2}{s^2}\right) A = 0$$

The partial derivative $\partial/\partial s$ has been replaced by the total derivative d/ds since s is the only variable. The general solution of (81) can be given in terms of the Hankel functions of order m of the first and second kind $H_m^{(1)}$, $H_m^{(2)}$:

82) $$A = \alpha_1 H_m^{(1)}(s) + \alpha_2 H_m^{(2)}(s).$$

In practical applications the Hankel Functions are used to represent progressing cylindrical and spherical waves for the following reason. If we assume that s is very large so that we can neglect $1/s^2$, (81) is simplified. On setting $A\sqrt{s} = f$, it becomes in this approximation

83) $$\frac{d^2f}{ds^2} + f = 0$$

This equation has the particular solutions $\alpha e^{\pm js}$ representing progressive waves in the s-direction. In fact, for large s, the Hankel functions have the asymptotic form

$$H_m^{(1)} = \sqrt{\frac{2}{\pi s}}\; e^{j\left(s - \frac{1}{2}m\pi - \frac{1}{4}\pi\right)} \qquad -\pi < \arg s < 2\pi$$

$$H_m^{(2)} = \sqrt{\frac{2}{\pi s}}\; e^{-j\left(s - \frac{1}{2}m\pi - \frac{1}{4}\pi\right)} \qquad -2\pi < \arg s < \pi$$

Hence, they differ from $f/\sqrt{s}$ only by a constant. Now, we can define the Bessel functions $J_m(s)$ by the relation

84) $$J_m(s) = \frac{1}{2}\left[H_m^{(1)}(s) + H_m^{(2)}(s)\right]$$

Because of the linearity of equation (81) they are also particular solutions of this equation. According to (84) their asymptotic form for large s becomes

85) $$J_m(s) = \frac{1}{2}\sqrt{\frac{2}{\pi s}}\left[e^{j\left(s-\frac{m\pi}{2}-\frac{\pi}{4}\right)} + e^{-j\left(s-\frac{m\pi}{2}-\frac{\pi}{4}\right)}\right]$$
$$= \sqrt{\frac{2}{\pi s}}\cos\left(s-\frac{m\pi}{2}-\frac{\pi}{4}\right)$$

(85) is more adapted to the description of stationary waves as it involves waves in the $\pm$ s directions. When m is not an integer, the general solution of (81) takes the form

86) $$A = \beta_1 J_m(s) + \beta_2 J_{-m}(s)$$

where

$$J_m(s) = \left(\frac{s}{2}\right)^m \sum_{n=0}^{\infty} \frac{\left(\frac{s}{2}\right)^{2n}(-1)^n}{n!\,\Gamma(m+n+1)}$$

$$J_{-m}(s) = \left(\frac{s}{2}\right)^{-m} \sum_{n=0}^{\infty} \frac{\left(\frac{s}{2}\right)^{2n}(-1)^n}{n!\,\Gamma(-m+n+1)}$$

where Γ is the gamma function.

When m is an integer, $J_{-m}(s)$ degenerates to the form $(-1)^m J_m(s)$, making $J_m(s)$ and $J_{-m}(s)$ linearly dependent. As we must have two independent solutions, the second solution is now given by the Neumann function $N_m(s)$, defined in terms of the Hankel function by

$$N_m(s) = \frac{1}{2j}\left(H_m^{(1)}(s) - H_m^{(2)}(s)\right)$$
$$= \frac{2}{\pi}\left(\gamma + \ln\frac{s}{2}\right) J_m(s)$$
$$= \frac{1}{\pi}\sum_{n=0}^{n=m-1} \frac{(m-n-1)!}{n!}\left(\frac{s}{2}\right)^{2n-m}$$
$$- \frac{1}{\pi}\sum_{n=0}^{\infty} (-1)^n \frac{(h_{n+m} + h_n)}{n!\,(n+m)!}\left(\frac{s}{2}\right)^{m+2n} + \cdots$$

$m = 1, 2, 3 \ldots \qquad \gamma = 0.57721$

$$h_k = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{k} \; ; \; h_0 = 0$$

so that

87) $A = \alpha_1 J_m(s) + \alpha_2 N_m(s)$

The Asymptotic form of $N_m(s)$ is proportional to $\sin\left(s - \frac{m\pi}{2} - \frac{\pi}{4}\right)$ so that the Neumann function represents a generalization of the sine function. When h^2 is negative nothing important is changed in the form of the solution, although the Bessel function depends on the imaginary argument of s, $hr = jk^1r$. For, a solution of (81) is also given by

88) $I_m(js) = e^{-j\frac{m\pi}{2}} J_m(jk^1r)$

where $I_m(js)$ is a real function of k^1r which, when m is an integer, degenerates to $(\pm 1, \pm j)\, J_m(s)$.

The Bessel equation (81) is solved by a series expansion. On setting $A = s^a C_1 + s^{a+1}C_2 + \ldots + s^{a+n}C_{n+1} + \ldots$, differentiating A and equating the coefficients of the same power of s we obtain for the Bessel function of m order (m an integer) the series

89) $$J_m(s) = \frac{1}{m!}\left(\frac{s}{2}\right)^m - \frac{1}{(m+1)!}\left(\frac{s}{2}\right)^{m+2} + \ldots$$

$$\ldots + (-1)^k \frac{1}{k!\,(m+k)!}\left(\frac{s}{2}\right)^{m+2k} + \ldots$$

Series (89) converges for finite values of s, oscillating as the trigonometric functions do, but with an amplitude which decreases as s increases. Between the first two Bessel functions J_0 and J_1 we have the relation

90) $$\int J_1 \, ds = - J_0(s)$$

Remembering that the fundamental particular solution of (66) is (67), we have to substitute for B, C, D the values (71), (75), (79). For the radial function, if we consider the progressing waves we can choose (82).

As we need only two constants we write the particular solution in the form

(91) $$\Phi_0 = \left(\alpha_1 H_m^{(1)} + \alpha_2 H_m^{(2)}\right) e^{\pm jm\theta} e^{\pm jbct} e^{\pm\sqrt{k^2-(k')^2}\, z}$$

If m is not an integer this describes a progressing wave that propagates in the z direction in a screw-like manner. When m is an integer we may also represent Φ by a superposition of stationary waves. In this case we choose for the radial wave function the solution (87), so that

$$92)\qquad \Phi_0 = (\alpha_1 J_m + \alpha_2 H_m)\, e^{\pm jm\theta}\, e^{\pm jbct}\, e^{\pm j\sqrt{(k')^2 - k^2}\, z}$$

The general solution is a linear combination of (91) or (92) and may be written for instance using (92)

$$93)\qquad \Phi = \sum_{m,k,k'} (\alpha_1 J_m + \alpha_2 N_m)\, e^{\pm jm\theta}\, e^{\pm jbct}\, e^{\pm j\sqrt{(k')^2 - k^2}\, z}$$

m, k', k being variable parameters.

11. Digression about Function Spaces [6]

It would appear surprising that the solutions of certain differential equations could be expressed in several different ways, for instance that the general wave solution of (81) could be expressed either in terms of elementary progressive waves or in terms of stationary waves. Such a property follows from the fact that the elementary functions, for instance J_m or N_m or in the case of a Fourier series a_o, $a_n \cos nx$, $b_n \sin nx$, form what is called a complete system of functions. This is readily understood by appealing to geometrical intuition. Consider a linear vector space of n dimensions, R_n. This means that there exists a fundamental system of unit vectors $\bar{e}_1, \bar{e}_2, \ldots, \bar{e}_n$ such that all other vectors of R_n, for instance $\bar{\mu}$ can be represented in terms of the $\bar{\mu}_k$ or more precisely can be regarded as linear functions of $\bar{e}_k$:

$$94) \qquad \bar{\mu} = \overline{\mu_1 e_1} + \overline{\mu_2 e_2} + \ldots + \overline{\mu_n e_n} = \sum_{k=1}^{n} \overline{\mu_k e_k}$$

We say that the $\bar{e}_k$ form a complete system of basic vectors. Of course, we can choose another set of unit vectors $\bar{e}_k'$ which will also be complete. This means that the $\bar{\mu}_k'$ can be expressed as linear functions of $\bar{e}_j$:

$$95) \qquad \bar{\mu} = \overline{\mu_1' e_1'} + \overline{\mu_2' e_2'} + \ldots + \overline{\mu_n' e_n'} = \sum_{k=1}^{n} \overline{\mu_k' e_k'}$$

We can also say that the set $\bar{\mu}$ has the components $\bar{\mu}'_k$ in the representation associated with the $\bar{e}'_k$ and $\bar{\mu}_k$ in the representation associated with $\bar{e}_k$ or that $\bar{e}_k$, $\bar{e}'_k$ define two different coordinate systems. The $\bar{\mu}$, $\bar{e}_k$ and $\bar{e}'_k$ are quantities that are defined by a magnitude and a direction. There are, however, mathematical objects of a more general character which satisfy the same conditions of completeness and linearity and consequently define a linear space. The elements of this linear space instead of being vectors are functions. For instance, we can speak of the space of continuous functions f(x) for $0<x<1$. If we take as basic vectors the quantities 1, cos x, cos 2x, ... , cos nx . . . , sin x, sin 2x, ... , sin nx, . . . , this system of functions is complete with respect to uniform convergence in the space of continuous periodic functions. This means that any continuous periodic function f(x) being given in a certain interval, it will always be possible to represent such a function in the following way:

$$96) \qquad f(x) = a_o + \sum_{k=1}^{\infty} a_k \cos kx + \sum_{k=1}^{\infty} b_k \sin kx$$

We recognize the expansion (96) as a fourier series. But, on the other hand, the system of functions 1, x, x^2, . . . , x^n, . . . is complete with respect to ordinary point-wise convergence in the space of continuous functions. Thus, if in example (96) f(x) represents a continuous periodic function, it can also be represented in terms of the basic functions 1, x, x^2, ...

These examples are sufficient to show that under very general conditions a continuous function can be linearly represented in terms of a sequence of definite other functions and that the choice of these functions could not be unique.

For instance, a function could be represented in terms of the Legendre polynomials or in terms of the Hermite polynomials. More particularly, the Bessel function $J_m(s)$ where m is an integer form a complete system able to represent under general conditions a continous function f(s). A general wave solution of (81) can thus be represented in terms of progressive waves or in terms of stationary waves.

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12. Proper Frequencies of Radial Waves[7]

No simple result could be obtained from the study of waves as complicated as those described by (93) that have variable frequencies and amplitudes. Experiments to test the acoustic properties of a sample at the termination of a tube are necessary using monochromatic plane waves like

$$97) \qquad \Phi(z, t) = Ae^{jkz}\,e^{jkct}$$

with constant amplitude and frequency. In this way, we are able to unequivocally compare the amplitude and phase after being reflected by the sample with the incident ones. Thus the experimental problem is to produce only waves of the form (97) and not of the general form (93). This problem can be discussed mathematically because (97) is a degenerate form of (93). Thus, we have to determine under which conditions (93) reduces to (97). Physically, production of waves like (93) means that it is not sufficient for producing a monochromatic wave to give a piston at the end of a tube a sinusoidal motion $u = u_o \sin 2\pi\nu t$ because Huyghen's principal states that each point of the piston surface can be regarded as the source of a wavelet expanding in the three dimensions of space and thus giving rise in general not only to waves along the z axis (which is the tube axis) but to waves in the perpendicular (x, y) plane that are described by Bessel's function and consequently do not have constant amplitude and frequency. This part of the general wave is just what is described by $\alpha_{1m} J_m + \alpha_{2m} N_m$ in (93). As s = hr it is in the direction of the polar axis $r = \sqrt{x^2 + y^2}$ which is perpendicular to the z axis.

Now these radial waves are associated with a discontinuous series of frequencies, because they have to satisfy the two following conditions that are very restrictive:

1. Their amplitude cannot be infinite at the origin r = 0, which would be contrary to experimental results.

2. They must be reflected by the walls of the tube, regarded as a perfect reflector.

The first condition eliminates the Neumann functions $N_m(s)$ which are not finite at $r = 0$.

The second condition requires that the components of velocity of the radial waves at the wall of the tube vanish so that the waves are entirely reflected.

Thus, if the radius of the tube is defined by r = a, we have, as the velocity components derive from a potential Φ according to (34)

$$u_a = -\left(\frac{\partial\Phi}{\partial x}\right)_a = v_a = -\left(\frac{\partial\Phi}{\partial y}\right)_a = 0.$$

Hence, by solving (61) we obtain

$$\left(\frac{\partial\Phi}{\partial r}\right)_a = \left(\cos\theta\,\frac{\partial\Phi}{\partial x} + \sin\theta\,\frac{\partial\Phi}{\partial y}\right)_a = 0$$

According to the first condition we have to put in equation (93) $N_m \leq 0$ for all m. Further, taking the derivative of (93) with respect to r and letting

$$\left(\frac{\partial \Phi}{\partial r}\right)_a = 0, \text{ we obtain}$$

98) $$\left(\frac{\partial J_m}{\partial r}\right)_{r = a} = 0 \text{ for } r = a \text{ or for } s = ha$$

a being given, condition (98) cannot be satisfied for any arbitrary value of h or s, but only for a sequence of discontinuous values h_n or s_n. Thus, if we term h_m the parameter associated with the function of m order J_m, the different values of h_m for which (98) is satisfied can be denoted h_{mn}, corresponding to s_{mn}. In the following table we give the values of s_{mn} corresponding to the zeros of

$$\left(\frac{\partial J_m}{\partial r}\right) = J_m' \text{ for } m = 0, 1, 2, 3$$
$$\text{and } n = 0, 1, 2, 3.$$

TABLE 1

n	s_{1n} for $J_0'=0$	s_{2n} for $J_1'=0$	s_{3n} for $J_2'=0$	s_{4n} for $J_3'=0$
0	0	1.84	3.05	4.20
1	3.83	5.33	6.70	8.02
2	7.02	8.54	9.97	11.34
3	10.17	11.71	13.17	14.59

Consider the general solution

93') $$\Phi' = \sum_{m,k,k'} \alpha_{1m} J_m (h_{mn}r)\, e^{\pm jm\theta}\, e^{\pm jbct}\, e^{\pm j\sqrt{(k')^2 - k^2}\, z}$$

In order that Φ' be independent of r, which is the first condition for a plane wave along the z axis, $h_{mn}r$ must reduce to a constant, otherwise the function $J_m (h_{mn}r)$ cannot reduce to a constant. The term $h_{mn}r = C$ is impossible since h_{mn} and r vary independently of each other, unless C = 0, which implies that $h_{mn} = 0$ since r always varies from r = 0 to r = a. Thus $s_{mn} = 0$. Referring to the definition of J_m given by (89) we see that this solution is satisfied only by $J_m = 0$ when $m \neq 0$, so that in this case wave (93') vanishes. For m = 0 we have the solution $J_0 = 1$ for any value of r and also

$$\frac{\partial J_0}{\partial r} = J_0' = 0.$$

Thus all conditions are satisfied. As $h^2 = -(k')^2$, $k' = 0$, (93) reduces to

99) $$\Phi = \alpha_{10}\, e^{jkct}\, e^{jkz}$$

i.e., to the monochromatic plane wave (97). This lasts until we meet with the least value of s_{mn} which is s_{10} corresponding to $J_1' = 0$ according to Table 1. To explain this, consider pure radial waves along the r axis so that there is no propagation along the z axis. According to (75) this means that $k^2 = (k')^2 = -h^2$. Taking account of (72) and (71) we see that h corresponds to the frequency of a periodic phenomenon ν such that

100) $$\nu = \frac{hc}{2\pi}$$

so that $D = h_1 e^{jhct} + h_2 e^{-jhct}$

Now Table 1 shows us that the smallest value of $s_{mn} = h_{mn}a$ is at $s_{10} = 1.84$. Thus, the smallest value of h_{mn} is $h_{10} = 1.84/a$ and the smallest radial frequency is

101) $$\nu_{10} = \frac{1.84c}{2\pi a}$$

Similarly the second possible frequency ν corresponds to s_{20} in Table 1 so that

$$\nu_{20} = \frac{3.05c}{2\pi a}$$

is the following frequency. Similarly, the next admissable frequency is $\nu_{01} = \frac{3.83c}{2\pi a}$

In general, the different radial frequencies are obtained by

$$\nu_{mn} = h_{mn}c/2\pi = s_{mn}c/2\pi a.$$

Thus, we have to stay away from these frequencies if we want to operate on a pure monochromatic plane wave.

13. Definition of Specific Acoustic Impedance - Comparison Between Water and Air Filled Tubes[8]

The wave equation (52) is satisfied by potential Φ and also, as equation (56) has shown, by displacement h. Morever, the same equation is satisfied by pressure P, local velocity (u, v, w) and density σ. To show that it is approximately satisfied by pressure P, we only have to use equation (45), where σ_0 is a constant and Φ' is simply called Φ. For, differentiating (52) with respect to time t, multiplying by the constant σ_0 and taking account of the commutativity of the differential operators

$$\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \text{ and } \frac{\partial}{\partial t} \text{ since } \Phi$$

is a uniform function:

102) $$\sigma_0 \frac{\partial}{\partial t}(\nabla^2 \Phi) = \frac{\sigma_0}{c^2}\frac{\partial}{\partial t}\left(\frac{\partial^2 \Phi}{\partial t^2}\right) \text{ or}$$

$$\nabla^2\left(\sigma_0 \frac{\partial \Phi}{\partial t}\right) = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\left(\frac{\partial \Phi}{\partial t}\right) \text{ or}$$

$$\nabla^2 P = \frac{1}{c^2}\frac{\partial^2 P}{\partial t^2}$$

Similarly the velocity depending on the potential Φ we get for u

103) $$-\frac{\partial}{\partial x}(\nabla^2 \Phi) = -\frac{\partial}{\partial x}\left(\frac{1}{c^2}\frac{\partial^2 \Phi}{\partial t^2}\right) \text{ or}$$

$$\nabla^2\left(-\frac{\partial \Phi}{\partial x}\right) = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\left(-\frac{\partial \Phi}{\partial x}\right) \text{ or}$$

$$\nabla^2 u = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

and similarly for v and w:

$$\nabla^2 w = \frac{1}{c^2}\frac{\partial^2 w}{\partial t^2}$$

$$\nabla^2 v = \frac{1}{c^2}\frac{\partial^2 v}{\partial t^2}$$

Consider the simple case where a monochromatic wave propagates in a certain indefinite medium in the positive z direction so that

104) $\Phi(z, t) = Ae^{jkz} e^{jkct}$

from (45), the acoustic pressure is given by

105) $P - Po = \sigma_0 \frac{\partial \Phi}{\partial t} = \sigma_0 jAkc\, e^{jkz} e^{jkct}$

and the velocity w of elements of the medium in the z direction is given by

106) $w = -\frac{\partial \Phi}{\partial z} = -jAk \; e^{jkz} e^{jkct}$ or

$$w = -\frac{1}{jk\sigma_0 c} \frac{\partial P}{\partial z}$$

The quantity e^{jkct} appears everywhere. To avoid using variable quantities of this kind, it is convenient to define the new concept of specific acoustic impedance of the medium transmitting the plane wave as the ratio of the external pressure $P - Po$ to the local velocity w. It represents the reaction of the medium to the external pressure and is a certain measure characteristic of this medium. For instance, if no velocity is communicated by external pressure to the particles of the medium the specific acoustic impedance is infinite. It is defined by

107) $Z = \frac{P-Po}{w} = \sigma_0 c$

according to (105) and (106) for a plane wave.

We see that finally, despite the intervention of the macroscopic quantity w, it can be measured only by determining macroscopic quantities like σ_0 and c.

As the subject of this report is concerned with the application of theory to water filled tubes, it is suitable to compare the acoustic properties of air and water as far as their behavior in tubes is concerned.

a. In air at rest the sound velocity is c = 1130 ft/sec. In water, c = 5000 ft/sec. Thus, the wavelength $\lambda = \frac{c}{\nu}$ for a given frequency is 4.42 times larger in water than it is in air.

b. The specific impedance for air at 20°C is $\sigma_0 c = 41.264 \frac{gm}{cm^2\text{-}sec}$ and for sea water at 20°C is $\sigma_0 c = 1.5 \times 10^5 \frac{gm}{cm^2\text{-}sec}$. For iron and soft steel, $\sigma_0 c = 3.99 \times 10^6, \frac{gm}{cm^2\text{-}sec}$ more than 10^5 times the value for air, but only 26.6 times larger than the specific impedance of water. On the other hand, $Z_{water}/Z_{air} = 3640$. Thus, for a given sound velocity, the amplitude of a pressure wave in water will be 3640 times greater than that in air.

c. If the acoustic impedance Z_1 of medium 1 with local velocity w_1 is much larger than the acoustic impedance Z_2 of medium 2 with local velocity w_2 for equal pressures in mediums 1 and 2 we have
$Z_1 = \frac{P-Po}{w_1}$, $Z_2 = \frac{P-Po}{w_2}$, $\frac{Z_1}{Z_2} = \frac{w_2}{w_1}$. Since $Z_1 >> Z_2$ we have $w_2 >> w_1$. Thus, acoustic

waves from medium 2 will not penetrate medium 1 due to the smallness of w_1. Since the acoustic impedance of steel is not much greater than that of water, we can expect that acoustic waves traveling in steel tubes filled with water will penetrate the walls of the tube to some degree. The standing waves actually induce vibrations in the tube which in turn transmits these vibrations to the water, disturbing the plane wave propagation. This necessitates a study of the characteristic frequencies of the tube. Difficulties encountered by tube vibrations can be partially eliminated by increasing the tube thickness*, by damping the vibrations by surrounding the tube with sand or by avoiding the range of characteristic frequencies. The Germans performed an experimental study of their tubes before using them.

d. The attenuation of waves is much less in water than in air, which simplifies the equations.

e. The phenomenon of refraction is important in water.

f. Noises and reverberation are obstacles to reception as in air. They are particularly important in pulse transmission and reception.

In general, the length, inside diameter and tube thickness cannot be arbitrarily chosen but are connected with the possibility of definite oscillations. The length of the tube, in the case of standing waves, is related to the maximum wave-length by the formula

108) $$L = \frac{\lambda}{4}$$

for a tube closed at one end and open at the other or by $L = \lambda/2$ for a tube closed at both ends. A minimum frequency of v_{min} = 100 cps will correspond to a maximum wavelength λ = 1525 cm, assuming that the tube is infinitely rigid and that the velocity of sound c = 152,500 cm/sec. Thus, according to (108) the length of the tube will be equal to 3.8 meters.

For a pulse tube with a wave train formed by 10 wavelengths and a minimum frequency v_{min} = 1000 cps, we have λ = 152.5 cm for each wave and λ = 1525 cm for the entire wave train. The length of the tube must be at least equal to $\lambda/2$ in order for the pulse termination not to interfere with its front at the initial section of the tube after a complete cycle. Other elements intervene against lengthening the tube too much. As the acoustic impedance of water is close to that of steel, proper oscillations are excited in the tube which cannot be avoided by clamping the tube. Their study is related to that of elasticity in solids.

*Professor Skudrzyk of Pennsylvania State University has noted that, as the walls of the tube become thicker, more of the acoustic energy of the sound wave travels in the wall, thus complicating the mechanism of plane wave propagation.

14. Reduction in Velocity[9]

The sound velocity in water filled tubes is not quite the same as that in free water, due to the flexibility of the tube. The sound velocity in a tube with ideally rigid walls being denoted as c_0 (same as in free water) and c' being the actual velocity in the tube, the ratio c'/c_0 is given by Korteweg's formula

$$109)\quad \frac{c'}{c_0} = \left(1 + \frac{2aK}{hE}\right)^{-\frac{1}{2}}$$

where a is the internal radius of the tube, h is the thickness of its wall, K is the Bulk Modulus of Elasticity of water, E is Young's modulus for the material of the tube. An improved form of the equation has been given by Hutte:

$$110)\quad \frac{c'}{c_0} = \left[1 + 2\,\frac{K\left[(r_2/r_1)^2 + 1\right]}{E\left[(r_2/r_1)^2 - 1\right]}\right]^{-\frac{1}{2}}$$

where r_1 is the internal radius and r_2 the external radius of the tube. Even in the case of infinitely large wall thickness, c'/c_0 is less than 1, due to the compressibility of the wall material. For steel, there is a limiting value of $c'/c_0 = 0.9892$. We can see that reduction in velocity can be corrected by an increase of the tube thickness. For instance, if $r_2 = 2r_1$, formula (110) yields

$$111)\quad \frac{c'}{c_0} = 0.982$$

If $r_2 = \frac{3}{2} r_1$, $\frac{c'}{c_0} = 0.972$.

Thus, the thicker* the tube, the better the experiment, as the reduction in velocity becomes negligible. One must also consider the fact that some acoustic energy travels in the walls of the tube and hence tube thickness will influence proper radiations in the tube.

As far as the internal diameter is concerned, it should be less than the minimum wavelength for two reasons:

First, the tube through which the wave-train propagates can be regarded as an obstacle to this train. When the wavelength is very small compared to the linear dimensions of the obstacle there is a shadow. When the linear dimensions of the obstacle are of the order of the wavelength of the vibration, diffraction occurs, so that propagation of a train of plane waves along rays is not possible. When

*See footnote, page ~~40~~ 38.

the wavelength is several times larger than the obstacle (in this case the internal radius of the tube), the obstacle does not appreciably perturb the wave motion.

Second, we have already seen that the first frequency corresponding to a radial vibration is given by (101) so that only a plane wave is propagated below the lower frequency limit ν_{10} given by (101). For standing wave tubes, there are strong characteristic vibrations of the tube walls excited by the radial oscillations and measurements are very difficult above the limit frequency ν_{10}. The characteristic vibrations of the tube include longitudinal vibrations, bending oscillations symmetrical to the axis and two and three-dimensional bending oscillations. The resonance frequencies of the longitudinal vibrations are again associated with Bessel's functions. They are given by the formula

112) $$\nu_{Lm} = \frac{c}{2mL}$$

where L is the length of the tube. For waves of the first kind $m \geq 1$ for the number of nodes on the total length L. For waves of the second kind, $m \geq 2$. The third kind of waves have 2n nodes on the circumference of any circular ring ($n \geq 2$) and $m \geq 1$ nodal circles in the entire length, the characteristic frequencies being higher than those of the second kind of vibrations. This limits the length that can be given to the tube as, according to (112), the longer the tube, the lower the first characteristic frequency. According to equation (112), it is impossible to avoid the characteristic frequencies of the tube. For L = 100 cm, ν_{L1} = 762.5 cps, for L = 200 cm, ν_{L1} = 381.3 cps, for L = 300 cm, ν_{L1} = 254.2 cps. To avoid such a nuisance, it is recommended that the thickness of the tube (r_2-r_1) be such that the ratio $h = (r_2-r_1)/r_1$ be in the range $0.2 \leq h \leq 3$. Moreover, the amplitudes of the tube vibration should be reduced by rubber rings and a filter formed by two rings fastened together by a brazed bronze sheet.

Next, consider formula (101) which gives the lowest Bessel frequency above which no pure plane wave propagates through the water of the tube. From (101) we have

$$2a = \frac{1.84}{\pi}\lambda = 0.586\lambda \text{ or } \lambda = 3.42a,$$

where a is the radius of the tube. For a = 2.5 cm and c = 152,500 cm/sec, we have $\lambda_{min} = 3.42 \times 2.5 = 8.55$ cm and $\nu_{max} = \frac{c}{\lambda} = \frac{152500}{8.55} = 17{,}850$ cps. Above ν_{max} = 17,850 cps we have radial waves superimposed upon the transverse waves and the wave may no longer be considered a plane wave.

15. Propagation of Plane Waves in Water Filled Tubes[9,10]

We begin by a little digression about complex numbers. Any complex number may be written in the form $w = u + jv$ where u and v are real numbers and $j = \sqrt{-1}$. The numbers u and v may be regarded as perpendicular coordinates in the w plane. Thus a complex number is utilized when we are dealing with a quantity that depends on two other independent quantities. Passing to polar coordinates we may write $w = r \cos \Theta$, $v = r \sin \Theta$ so that w takes the form $w = re^{j\Theta}$ where r plays the rcle of an amplitude and Θ that of a phase angle, which is very convenient for the study of periodic motions. As sample materials in water filled tubes modify both amplitude and phase of plane waves, their action is most conveniently represented by complex variables. By definition, $|w| = r = \sqrt{u^2 + v^2}$, and $\Theta = \tan^{-1}\frac{v}{u}$.

Now, returning to the case of a water filled tube with uniform cross-sectional area S, starting at the origin of coordinates z = 0, and extending to the right along the z axis. The harmonic wave produced at the origin by an audio oscillator gives rise to a series of plane waves like (104) and (106) whose characteristic impedance is given by (107).

When the tube is of finite length L and closed at z = L, the plane wave generated at z = 0 will be reflected at the termination z = L of the tube and there will be a mixture of incident and reflected waves. The total pressure at any point z at any time t will be given by

113) $$P = (P_+ e^{-j\varepsilon z/c} + P_- e^{+j\varepsilon z/c})\, e^{\pm j\varepsilon t}$$

where $\varepsilon = 2\pi\nu$ is the circular frequency. P_+ is the amplitude of the incident wave and P_- the amplitude of the reflected wave. P_- includes any change of phase at the termination and consequently the amplitudes are complex. Using the definition of pressure

$$P = \delta_o \frac{\partial \Phi}{\partial t}$$

given by (45) (where P is written instead of P-Po and Φ instead of Φ'*) and for Φ the plane wave equation

114) $$\Phi = (\Phi_+ e^{-j\varepsilon z/c} + \Phi_- e^{+j\varepsilon z/c})\, e^{\pm j\varepsilon t}$$

we can write P in the form

115) $$P = \pm (\delta_o j\varepsilon)(\Phi_+ e^{-j\varepsilon z/c} + \Phi_- e^{+j\varepsilon z/c})\, e^{\pm j\varepsilon t}$$

so that

116) $$\delta_o j\varepsilon\Phi_+ = P_+ \text{ and}$$

$$\delta_o j\varepsilon\Phi_- = P_-$$

*Recall that $\frac{\partial\Phi}{\partial\psi} = \frac{\partial\Phi'}{\partial\psi}$, $\frac{\partial\Phi}{\partial y} = \frac{\partial\Phi'}{\partial y}$, $\frac{\partial\Phi}{\partial z} = \frac{\partial\Phi'}{\partial z}$ and that for waves of weak amplitude we may let $\frac{\partial}{\partial t} H(t) \approx 0$ so that Φ may replace Φ' everywhere in our equations.

In the same way, because of (34), the velocity reduces to $w = -\frac{\partial \Phi}{\partial z}$. Thus, according to (114) and (106) we may write

117) $$w = (j\varepsilon/c)\,(\Phi_+ e^{-j\varepsilon z/c} - \Phi_- e^{j\varepsilon z/c})\, e^{\pm j\varepsilon t}$$

so that

118) $$w - \frac{1}{j\varepsilon\sigma_o}\frac{\partial P}{\partial z}$$

The specific acoustic impedance may now be given by

119) $$Z = \frac{P}{w} = \pm\sigma_o c\left[\frac{\Phi_+ e^{-j\varepsilon z/c} + \Phi_- e^{+j\varepsilon z/c}}{\Phi_+ e^{-j\varepsilon z/c} - \Phi_- e^{+j\varepsilon z/c}}\right]$$

From (116) we have

120) $$Z = \pm\sigma_o c\left[\frac{P_+ \; e^{-j\varepsilon z/c} + P_- \; e^{+j\varepsilon z/c}}{P_+ \; e^{-j\varepsilon z/c} - P_- \; e^{+j\varepsilon z/c}}\right]$$

At z = L, the impedance has the value

121) $$Z = \pm\sigma_o c\left[\frac{P_+ \; e^{-j\varepsilon L/c} + P_- \; e^{+j\varepsilon L/c}}{P_+ \; e^{-j\varepsilon L/c} - P_- \; e^{+j\varepsilon L/c}}\right].$$

Obviously, we could have located the termination at z = 0 and the initial section at z = - L which would give for (120)

122) $$Z = \pm\sigma_o c\left[\frac{P_+ + P_-}{P_+ - P_-}\right]$$

We can put formulas (121) or (122) in a more convenient form to determine Z_L from the experimental data. We define the pressure ratio

123) $$\frac{P_-}{P_+} = e^{-2\mu} \text{ so that}$$

$\mu = \frac{\log P_+ - \log P_-}{2}$. Dividing the numerator and denominator of (121) by P_+ and multiplying them by e^{μ} we get

124) $$Z = \pm \sigma_0 c \frac{e^{\mu - j\varepsilon L/c} + e^{-(\mu - j\varepsilon L/c)}}{e^{\mu + j\varepsilon L/C} - e^{-(\mu - j\varepsilon L/c)}}$$

The numerator of (124) is simply 2 cosh $(\mu - j\varepsilon L/c)$ and the denominator is 2 sinh $(\mu - j\varepsilon L/c)$ so that (124) reduces to

125) $$Z = \pm \sigma_0 c \coth(\mu - j\varepsilon L/c)$$

The parameters σ_0, $\varepsilon = 2\pi\nu$, L and c are known. μ is determined experimentally. There are two methods. First, the method of the pulse tube and second, the method of the standing wave tube. We shall first discuss the pulse tube method.

16. Measurements by the Pulse Tube Method[9,10,12]

Let $\mu = \mu_1 + j\mu_2$ and $e^{-2\mu_1} = r$, $2\mu_2 = \theta$ so that (123) becomes

126) $$\frac{F_-}{F_+} = e^{-2\mu_1} e^{-2j\mu_2} = re^{-j\theta}$$

Thus, r represents the ratio of the real amplitudes of the incident and reflected waves at z = L where the material sample has been set. It is an experimental number that can be measured by the methods indicated in the usual works. In the same way, θ represents the real change of phase at the interface and can be directly measured. Consequently μ and Z_L are known.

In (124) we can divide numerator and denominator by $e^{\mu - j\varepsilon L/c}$ and denote $e^{-2(\mu - j\varepsilon L/c)}$ as η. We obtain

127) $$Z = \pm \sigma_0 c \frac{1+\eta}{1-\eta} \text{ or, setting } \frac{Z_L}{\sigma_0 c} = \gamma$$

we have

128) $$\gamma = \frac{Z}{\sigma_0 c} = \frac{1+\eta}{1-\eta}$$

Equation (128) represents a well-known conformal transformation of the theory of functions of a complex variable. On setting $\gamma = u + jv$, $\eta = x+jy$, the lines $u = u_0$ = const., $v = v_0$ = constant generate a rectangular coordinate system in the u, v plane. In the x, y plane these lines are represented by two families of tangential circles. For, replacing in (128) η by x+jy, multiplying and dividing the right-hand side of (128) by (1-x)+jy we obtain

129) $$\gamma = u+jv = \frac{(1+jy)^2 - x^2}{(1-x)^2 + y^2}$$

so that

130) $$\gamma = u+jv = \frac{1-y^2-x^2}{(1-x)^2+y^2} + j\frac{2y}{(1-x)^2+y^2}$$

On setting u = constant = u_0, v = constant = v_0, we obtain from (130)

131) $$\left(x - \frac{u_0}{u_0+1}\right)^2 + y^2 = \frac{1}{(1+u_0)^2} \qquad \text{(a)}$$

$$(x-1)^2 + \left(y - \frac{1}{v_0}\right)^2 = \frac{1}{v_c^2} \qquad \text{(b)}$$

Replacing u_o in (131a) with numerical values we obtain

TABLE II

u_o	Equation of Circle	Center at	Radius
-4	$(x + \frac{4}{3})^2 + y^2 = \frac{1}{9}$	$(-\frac{4}{3}, 0)$	$\frac{1}{3}$
-3	$(x + \frac{3}{2})^2 + y^2 = \frac{1}{4}$	$(-\frac{3}{2}, 0)$	$\frac{1}{2}$
-2	$(x + 2)^2 + y^2 = 1$	$(-2, 0)$	1
-1	$(x - \infty)^2 + y^2 = \infty$	$(\infty, 0)$	∞
0	$x^2 + y^2 = 1$	$(0, 0)$	1
1	$(x - \frac{1}{2})^2 + y^2 = \frac{1}{4}$	$(\frac{1}{2}, 0)$	$\frac{1}{2}$
2	$(x - \frac{2}{3})^2 + y^2 = \frac{1}{9}$	$(\frac{2}{3}, 0)$	$\frac{1}{3}$
3	$(x - \frac{3}{4})^2 + y^2 = \frac{1}{16}$	$(\frac{3}{4}, 0)$	$\frac{1}{4}$
4	$(x - \frac{4}{5})^2 + y^2 = \frac{1}{25}$	$(\frac{4}{5}, 0)$	$\frac{1}{5}$

Thus, lines of constant $\mu = u_o > -1$ in the u, v plane transform into circles with centers

$$\left(\frac{u_o}{u_o+1}, 0 \right) \text{ and radii } \left| \frac{1}{1+u_o} \right|$$

tangent to the line x = 1 in the x, y plane. Lines of constant $u = u_o < -1$ in the u, v plane transform into circles with centers

$$\left(\frac{u_o}{u_o+1}, 0 \right) \text{ and radii } \left| \frac{1}{1+u_o} \right|$$

tangent to the line x = -1 in the x, y plane. The line $u_o = -1$ in the u, v plane in undefined in the x, y plane. Similarly, replacing v_o in (131b) with numerical values we obtain

TABLE III

v_o	Equation of Circle	Center at	Radius
-4	$(x - 1)^2 + (y + \frac{1}{4})^2 = 1/16$	$(1, -\frac{1}{4})$	$\frac{1}{4}$
-3	$(x - 1)^2 + (y + 1/3)^2 = 1/9$	$(1, - 1/3)$	1/3
-2	$(x - 1)^2 + (y + \frac{1}{2})^2 = \frac{1}{4}$	$(1, -\frac{1}{2})$	$\frac{1}{2}$
-1	$(x - 1)^2 + (y + 1)^2 = 1$	$(1, - 1)$	1
0	$(x - 1)^2 + (y -\infty)^2 = \infty$	$(1,\infty)$	∞
1	$(x - 1)^2 + (y - 1)^2 = 1$	$(1, 1)$	1
2	$(x - 1)^2 + (y - \frac{1}{2})^2 = \frac{1}{4}$	$(1, \frac{1}{2})$	$\frac{1}{2}$
3	$(x - 1)^2 + (y - 1/3)^2 = 1/9$	$(1, 1/3)$	1/3
4	$(x - 1)^2 + (y - \frac{1}{4})^2 = 1/16$	$(1, \frac{1}{4})$	$\frac{1}{4}$

Thus, lines of constant $v = v_o \neq 0$ in the u, v plane transform into circles with centers

$$(1, \frac{1}{v_o}) \text{ and radii } \left|\frac{1}{v_o}\right|$$

tangent to the line y = 0 in the x, y plane. The line $v_o = 0$ in the u, v plane is undefined in the x, y plane. These relationships are shown in Figure 1. Charts giving values of u, v corresponding to values of x, y are available commercially and are known as Smith charts.

Returning to (125) and substituting cosh ix = cos x, sinh ix = i sin x we may write

$$132) \quad \frac{Z}{\sigma_o c} = \gamma = \frac{\tan h\, \mu\, (1 + \tan^2 \frac{\varepsilon L}{c}) + j \tan \frac{\varepsilon L}{c} (1 - \tan h^2 \mu)}{\tan h^2 \mu + \tan^2 \frac{\varepsilon L}{c}}$$

which determines u and v in terms of μ and $\frac{\varepsilon L}{c}$. When μ^2 and $\frac{\varepsilon^2 L^2}{c^2}$ are $< \frac{\pi^2}{4}$ we can also use the series

$$133) \quad \tan h\, \mu = \mu - \frac{\mu^3}{3} + 2\frac{\mu^5}{15} - 17\frac{\mu^7}{315} + \ldots$$

$$\tan \frac{\varepsilon L}{c} = \frac{\varepsilon L}{c} + \frac{\varepsilon^3 L^3}{3c^3} + 2\frac{\varepsilon^5 L^5}{15c^5} + \frac{17\varepsilon^7 L^7}{315c^7} + \ldots$$

Equation (132) in a first approximation for high values of the impedance with μ, $\frac{\varepsilon L}{c}$ less than 0.1 reduces to

$$134) \quad \gamma = \frac{\mu\left(1 + \frac{\varepsilon^2 L^2}{c^2} + j\,\frac{\varepsilon L}{c}\,(1-\mu^2)\right)}{\mu^2 + \frac{\varepsilon^2 L^2}{c^2}}$$

These formulas give the characteristic impedance of the water at the interface $z = L$ separating the material under investigation and water. It is also of great interest to know the value of the acoustic impedance inside a layer formed by this material which is supposedly homogeneous.

This has been done as follows. Consider that the test substance forms a layer in the tube between $z = 0$ and $z = L$. (Figure 1) Above the test substance we have a brass disk of thickness U and below the test substance we have a water-filled section S extending from $z = -h$ to $z = 0$. Let σ_0 = density of the water, c_0 = velocity of sound in the tube between $z = -h$ and $z = 0$, Z_0 = specific acoustic impedance of the water-filled section of the tube. $P_+{}^o$ and $P_-{}^o$ are defined as the pressure amplitudes of the incident and reflected waves in S, respectively. The total pressure in S is then given by:

$$135) \quad P_0 = \left(P_+{}^o \cdot e^{-\tau_0 z} + P_-{}^o \; e^{+\tau_0 z}\right) e^{\pm j\varepsilon t}$$

where $\tau_0 = \alpha_0 + j\,/c_0$. The constant α_0 is an attenuation constant which is now being introduced. We then have a corresponding equation for the local velocity W_0 in S so that the specific acoustic impedance Z_0 becomes:

$$136) \quad Z_0 = \frac{P_0}{W_0} = \pm\,(j\varepsilon\sigma_0/\tau_0)\,\frac{(P_+{}^o\; e^{-\tau_0 z} + P_-{}^o\; e^{+\tau_0 z})}{(P_+{}^o\; e^{-\tau_0 z} - P_-{}^o\; e^{-\tau_0 z})}$$

To determine the value of Z_0 at $z = -h$ or at the interface $z = 0$ we need only replace z in (136) by $-h$ or 0. At $z = 0$ (the interface) we have

$$137) \quad Z_{00} = \left(\frac{j\varepsilon\sigma_0}{\tau_0}\right)\frac{\left(1 + \frac{P_-{}^o}{P_+{}^o}\right)}{\left(1 - \frac{P_-{}^o}{P_+{}^o}\right)}$$

as usual.

In water, we can neglect the attenuation coefficient α_0 so that τ_0 reduces to $j\varepsilon/c_0$ as in our former equations. Replacing P_- and P_+ in (123) by $P_-{}^o$ and $P_+{}^o$

we may write $P_-^0/P_+^0 = re^{-j\theta}$ where $\mu = \mu_1 + j\mu_2$, $r = e^{-2\mu_1}$ and $\theta = -2\mu_2$ as before. Taking the value of r in decibels we get

138) $$db = 20 \log r.$$

Numerical values of $Z_0/\sigma_0 c$ may be obtained by Smith's charts as in the foregoing formulas.

Now, let us call σ_1 the density of the sample and c_1 the sound velocity in the sample. P_+^1 and P_-^1 are defined as the incident and reflected waves in the sample and τ_1 the value of τ in the sample. The pressure inside the sample becomes

139) $$P_1 = (P_+^1 e^{-\tau_1 z} + P_-^1 e^{+\tau_1 z})\, e^{\pm j\varepsilon t}$$

We have a corresponding formula for the local velocity w_1 in the sample so that the specific acoustic impedance Z_1 of the sample becomes

140) $$Z_1 = (j\varepsilon\sigma_1/\tau_1)\,\frac{(P_+^1 e^{-\tau_1 z} + P_-^1 e^{+\tau_1 z})}{(P_+^1 e^{-\tau_1 z} - P_-^1 e^{+\tau_1 z})}$$

where z varies from z = 0 to z = L. The impedance Z_{10} at the interface is obtained by setting z = 0 in (140). We have

141) $$Z_{10} = \left(\frac{j\varepsilon\sigma_1}{\tau_1}\right)\frac{1+\dfrac{P_-^1}{P_+^1}}{1-\dfrac{P_-^1}{P_+^1}}$$

The impedance at the brass washer may be obtained by setting z = L in (140). We have for Z_{1L}

142) $$Z_{1L} = (j\varepsilon\sigma_1/\tau_1)\,\frac{1+\dfrac{P_-^1}{P_+^1}\,e^{+2\tau_1 L}}{1-\dfrac{P_-^1}{P_+^1}\,e^{+2\tau_1 L}}$$

Solving (142) for P_-^1/P_+^1 we have

143) $$\frac{P_-^1}{P_+^1} = e^{-2\tau_1 L}\,\frac{\tau_1 Z_{1L} - j\varepsilon\sigma_1}{j\varepsilon\sigma_1 + \tau_1 Z_{1L}}$$

From (139) we see that the pressure at $z = L$, P_{1L} is given by

144) $$P_{1L} = (P_+^1 e^{-\tau_1 L} + P_-^1 e^{+\tau_1 L}) e^{\pm j\varepsilon t}$$

and the corresponding velocity at $z = L$, w_{1L} is obtained by application of (118) to (139) so that

145) $$w_{1L} = -\frac{1}{j\varepsilon\sigma_1}\frac{dP_1}{dz}$$

$$= \frac{\tau_1}{j\varepsilon\sigma_1}(P_+^1 e^{-\tau_1 L} - P_-^1 e^{+\tau_1 L}) e^{\pm j\varepsilon t}$$

The brass interface may be regarded as a perfect reflector so that at $z = L$

146) $$w_{1L} = 0$$

Thus, from (145)

147) $$P_+^1 e^{-\tau_1 L} - P_-^1 e^{+\tau_1 L} = 0 \text{ or}$$

$$\frac{P_-^1}{P_+^1} = e^{-2\tau_1 L}$$

Returning to equation (141) we may write

148) $$Z_{10} = (j\varepsilon\sigma_1/\tau_1)\frac{1+ e^{-2\tau_1 L}}{1- e^{-2\tau_1 L}}$$

Dividing both sides of (148) by $\varepsilon\sigma_1 L$, where L is the length of the sample, and multiplying numerator and denominator by $e^{\tau_1 L}$ we obtain

149) $$Z_{10}/\varepsilon\sigma_1 L = \frac{j}{\tau_1 L}\left[\frac{e^{\tau_1 L} + e^{-\tau_1 L}}{e^{\tau_1 L} - e^{-\tau_1 L}}\right]$$

which reduces to

150) $$Z_{10}/\varepsilon\sigma_1 L = \frac{j}{\tau_1 L}\coth \tau_1 L$$

On setting $Z_{10} = R_{10} + jX_{10}$ and $\tau_1 = \mu_1 + j\mu_2$ and substituting $\coth jz = -j \cot z$ we have

151) $R_{10}/\varepsilon\sigma_1 L$

$$= \frac{\mu_2 L \tanh \mu_1 L\,(1+\tan^2 \mu_2 L) + \mu_1 L \tan \mu_2 L\,(1-\tanh^2 \mu_1 L)}{(\mu_1^2 L^2 + \mu_2^2 L^2)(\tanh^2 \mu_1 L + \tan^2 \mu_2 L)}$$

$X_{10}/\varepsilon\sigma_1 L$

$$= \frac{\mu_1 L \tanh \mu_1 L\,(1+\tan^2 \mu_2 L) - \mu_2 L \tan \mu_2 L\,(1-\tanh^2 \mu_1 L)}{(\mu_1^2 L^2 + \mu_2^2 L^2)(\tanh^2 \mu_1 L + \tan^2 \mu_2 L)}$$

Since μ_1 and μ_2 are known, we may calculate $R_{10}/\varepsilon\sigma_1 L$ and $X_{10}/\varepsilon\sigma_1 L$ from (151). Special charts are available which plot the functions (151), facilitating computations. Once (151) are known, the components of the impedance of the sample may be determined.

17. The Standing Wave Method

The measurements of Z up to now have been considered as performed by means of the pulse tube method. We had to determine the complex ratio

$$\frac{P_-}{P_+} = re^{j\theta}$$

and the physical quantities accessible to measurements were the amplitude r and the phase angle θ at certain interfaces z = 0 and z = L. The relationship between these numbers and the numbers $R_{10}/\varepsilon\sigma_1 L$ and $X_{10}/\varepsilon\sigma_1 L$ of $Z_{10}/\varepsilon\sigma_1 L = R_{10}/\varepsilon\sigma_1 L + jX_{10}/\varepsilon\sigma_1 L$ was then determined from equations (151) or from special charts of equations (151).

When measurements by standing waves are performed, the fundamental quantities to be measured are the ratio P_{min}/P_{max} of the minimum to maximum pressure and the phase difference as obtained by measurement of the distance of the pressure nodes from the sample interface. The equation for pressure (113) and the relationship given by (123) still applies. Then $P_+e^{-\mu} = P_0 = P_- e^{\mu}$ and (113) can be rewritten

152) $$P = P_0 \left(e^{\mu - j\varepsilon z/c} + e^{-(\mu - j\varepsilon z/c)}\right) e^{\pm j\varepsilon t}$$

P_0 is a complex quantity whereas the pressure measured at different points in the tube is real. To establish a relationship between these two quantities, we must deal with the absolute value of P, which is a real number defined by

$$|P| = \sqrt{PP^*}$$

where P* is the complex conjugate of P. Then, from (152)

153) $$|P| = |P_0| \left|e^{\mu - j\varepsilon z/c} + e^{-(\mu - j\varepsilon z/c)}\right|$$

Letting $\mu = \mu_1 + j\mu_2$ and noting that $2 \cosh x = e^x + e^{-x}$ we can write (153) in the form

154) $$|P| = |2P_0| \left|\cosh \left(\mu_1 - j(\varepsilon z/c - \mu_2)\right)\right|$$
$$= |2P_0| \left|\cosh \mu_1 \cos(\varepsilon z/c - \mu_2) - j \sinh \mu_1 \sin(\varepsilon z/c - \mu_2)\right|$$

Now, the absolute value of P is

155) $$|P| = |2P_0| \sqrt{\cosh^2 \mu_1 \cos^2 (\varepsilon z/c - \mu_2) + \sinh^2 \mu_1 \sin^2 (\varepsilon z/c - \mu_2)}$$

which can be written ($\cosh^2 \mu_1 = \sinh^2 \mu_1 + 1$)

156) $$|P| = |2P_0| \sqrt{\sinh^2 \mu_1 + \cos^2 (\varepsilon z/c - \mu_2)}$$

We see that $|P|$ is a minimum if $\cos^2(\varepsilon z/c - \mu_2) = 0$ which is satisfied by

157) $\mu_2 - \varepsilon z/c = (n + \frac{1}{2})\pi, \; n = 0, 1, 2, 3 \ldots$

The maximum of $|P|$ occurs if $\cos^2(\varepsilon z/c + \mu_2) = 1$ which corresponds to

158) $\mu_2 - \varepsilon z/c = n\pi, \; n = 0, 1, 2, 3 \ldots$

Thus we have

159) $|P|\text{maximum} = 2|P_0|\cosh \mu_1$

$|P|\text{minimum} = 2|P_0|\sinh \mu_1$

For a perfect reflector, $\mu_1 = 0$, $|P|\text{minimum} = 0$, $|P|\text{maximum} = 2|P_0|$. In other cases, the ratio

$$|P \text{ min}| / |P \text{ max}|$$

is experimentally known and we have

160) $|P \text{ min}| / |P \text{ max}| = \tanh \mu_1$ or

$\mu_1 = \tanh^{-1} |P \text{ min}| / |P \text{ max}|$ or

if $|P \text{ min}| / |P \text{ max}| = \beta$ we have

$\mu_1 = \tanh^{-1} \beta$. and thus

161) $\mu_1 = \beta + \beta^3/3 + \beta^5/5 + \ldots + \beta^n/n + \ldots$

which gives μ_1 by successive convergent approximations for $\beta^2 < 1$, a condition always satisfied.

The problem can also be treated without using complex numbers. Consider the local displacement caused by the acoustic waves in the tube. It is given by

162) $h = a \cos k(ct - z) - r \cos k(ct + z)$

$= (a + r) \sin k ct \sin k z + (a - r) \cos k ct \cos kz$

where a is the amplitude of the incident wave, r the amplitude of the reflected wave. The motion can be regarded as due to the superposition of two stationary waves of amplitudes (a + r) and (a-r) and, according to the definition of the pressure P in terms of the displacement h

163) $P_i = \sigma_o c \frac{\partial h_i}{\partial t}$, $P_r = -\sigma_o c \frac{\partial h_r}{\partial t}$

we have

164) $$P = P_i + P_r = \sigma_o kc^2 \left[(a + r) \cos kct \sin kz - (a - r) \sin kct \cos kz \right]$$

where $k = 2\pi\nu/c$, ν being the frequency of the sound velocity c, σ_o the average water density. We have

165) $$\frac{P_{min}}{P_{max}} = \frac{a - r}{a + r} = \frac{1}{\text{Standing Wave Ratio}}$$

The reflection coefficient r/a of an imperfect reflector is expressed as

166) $$\frac{r}{a} = \frac{SWR - 1}{SWR + 1}$$

For example, if SWR = 2, $\frac{r}{a} = \frac{1}{3}$

18. Illustrative Example[11, 12]

Consider a set of measurements to be made by the pulse tube method. The frequency range is from 2,000 cps to 20,000 cps and there are a minimum of 10 waves per pulse train. The characteristic impedance of fresh water is $\sigma_o c_o = 1.5 \times 10^5$ gm/cm^2 sec; the thickness of the sample is 1.055 cm and its density is $\sigma_1 = 1.18$ gm/cm^3. The operating frequency is 15,000 cps. We would like to determine the characteristic impedance of the sample, its attenuation coefficient and the velocity of sound in the sample. Measurements indicate that the reflection coefficient is 0.945 and the phase change - 31.1°.

At a minimum frequency of 2000 cps the wavelength per train of ten waves is

$$\lambda = \frac{c}{\nu} n = \frac{5000}{2000} \times 10 = 25 \text{ feet}$$

The minimum theoretical length of the tube is therefore

$$L = \frac{\lambda}{2} = 12.5 \text{ feet}$$

Choice of this length implies that there is no time delay between the instant the wave train leaves the oscillator and the instant it returns. Under actual conditions there is a finite time lapse between projected and reflected wave trains, and the tube is generally made longer than theoretical length to accommodate this time difference. We shall choose a tube factor F = 0.75 so that

$$L = F\lambda = 0.75 \times 25 = 19 \text{ feet}$$

is the actual length of the tube.

If we choose our axes at the sample interface we have, from (122)

$$\frac{Z_{oo}}{\sigma_o c_o} = \frac{1 + \frac{P_-}{P_+}}{1 - \frac{P_-}{P_+}}$$

where $\frac{Z_{oo}}{\sigma_o c_o} = \gamma_{oo} = u + jv$ and $\frac{P_-}{P_+} = x + jy = \eta$.

The complex ratio $\frac{P_-}{P_+}$ is termed the reflection coefficient and consists of an amplitude and a phase change, which, for this problem are 0.945 and - 31.1° respectively. Hence, we may rewrite (122) as

$$\gamma_{oo} = \frac{1 + \eta}{1 - \eta}$$

where $\eta = x + j\,y = 0.945\, e^{-j\,(31.1°)}$. Thus,

167) $$x^2 + y^2 = (0.945)^2 = 0.893$$
and
$$\tan(-31.1°) = \frac{y}{x} = -0.604$$

solving equations (167) we have x = .808 and y = - .488 so that $\eta = 0.808 - j\,0.488$.
From (130) we have

$$u = \frac{1-y^2-x^2}{(1-x)^2 + y^2} = .397$$

$$v = \frac{2y}{(1-x)^2 + y^2} = -3.55$$

and $\gamma_{oo} = u + jv = 0.397 - j\,3.55$.

The same result can be quickly obtained from a Smith Chart (Figure 3). Equations (131) represent circles in the (x, y) plane with centers at (.284, 0) and (1, - .282) respectively with radii of 0.716 and 0.282. To determine u and v from a Smith Chart, draw a vector of length r = 0.945 at an angle of - 31.1° with the x axis. The head of the vector r lies at the point of intersection of the two circles u and v. Values of u = 0.40 and v = -3.50 may be read directly from the chart without further calculation. Thus

$$\gamma_{oo} = \frac{Z_{oo}}{\sigma_o c_o} = 0.397 - j\,3.55 = \frac{Z_{10}}{\sigma_o c_o}.$$

Since

$$\frac{Z_{10}}{\sigma_1 \varepsilon L} = \frac{Z_{10}}{\sigma_o c_o} \times \frac{\sigma_o c_o}{\sigma_1 \varepsilon L}$$

we have

$$\frac{Z_{10}}{\sigma_1 \varepsilon L} = \frac{1.5 \times 10^5}{1.18 \times 2\pi \times 15000 \times 1.055}\left(0.397 - j\,3.55\right)$$

$$\frac{Z_{10}}{\sigma_1 \varepsilon L} = \frac{Z_{10}}{\sigma_o c_o} \times 1.285 = 0.51 - j\,4.56$$

The function

(151) $$\frac{Z_{10}}{\sigma_1 \varepsilon L} = 0.51 - j\,4.56 = \frac{R_{10}}{\sigma_1 \varepsilon L} + j\,\frac{X_{10}}{\sigma_1 \varepsilon L}$$

is a function of the parameters $\mu_1 L$ and $\mu_2 L$ and is available in chart form (U.S. Navy Underwater Sound Laboratory Chart). A typical chart is shown in Figure 4. Entering the chart with values of

$$\frac{R_{10}}{\sigma_1 \varepsilon L} = 0.51 \text{ and } \frac{X_{10}}{\sigma_1 \varepsilon L} = -4.56$$

we find that $\mu_1 L = .025$ and $\mu_2 L = .455$.

Values of $\mu_1 L$ and $\mu_2 L$ obtained from the NAVUWTRSOUNDLAB chart may be verified by substituting $\mu_1 L = 0.025$ and $\mu_2 L = 0.455$ in (151) and calculating $R_{10}/\sigma_1 \varepsilon L$ and $X_{10}/\sigma_1 \varepsilon L$. Thus

$$R_{10}/\sigma_1 \varepsilon L =$$

$$\frac{.455 \tanh .025 (1+\tan^2 .455) + .025 \tan .455 (1-\tanh^2 .025)}{(.025^2 + .455^2)(\tanh^2 .025 + \tan^2 .455)}$$

$$R_{10}/\sigma_1 \varepsilon L = 0.528$$

$$X_{10}/\sigma_1 \varepsilon L =$$

$$\frac{.025 \tanh .025 (1+\tan^2 .455) - .450 \tan .455 (1-\tanh^2 .025)}{(.025^2 + .455^2)(\tanh^2 .025 + \tan^2 .455)}$$

$$X_{10}/\sigma_1 \varepsilon L = -4.449$$

These values agree within 4% of the chart values of $\frac{R_{10}}{\sigma_1 \varepsilon L} = 0.51$ and $\frac{X_{10}}{\sigma_1 \varepsilon L} = -4.56$.

Thus

$$\mu_1 L = 0.025; \ \mu_1 = \frac{.025}{1.055} = .0237 \text{ cm}^{-1}$$

$$\mu_2 L = 0.455; \ \mu_2 = \frac{.455}{1.055} = .431 \text{ cm}^{-1}$$

and, since

$$\gamma_{10} = \frac{Z_{10}}{\sigma_1 c_1} = \alpha_1 + j \frac{\varepsilon}{c_1} = \mu_1 + j \mu_2$$

we have

$$\alpha_1 = \mu_1 = .0237 = \text{attenuation coefficient}$$

and

$\frac{\varepsilon}{c_1} = \mu_2 = .431$ so that the velocity of sound c_1 in the material becomes

$$c_1 = \frac{\varepsilon}{.431} = \frac{2\pi \times 15000}{.431} = 2.18 \times 10^5 \text{ cm/sec}$$

and the characteristic impedance of the sample is therefore

$$\sigma_1 c_1 = 1.18 \times 2.18 \times 10^5 = 2.57 \times 10^5 \frac{\text{gm}}{\text{cm}^2 - \text{sec}}.$$

CONFORMAL MAPPING OF $\mu = \frac{1+\eta}{1-\eta}$

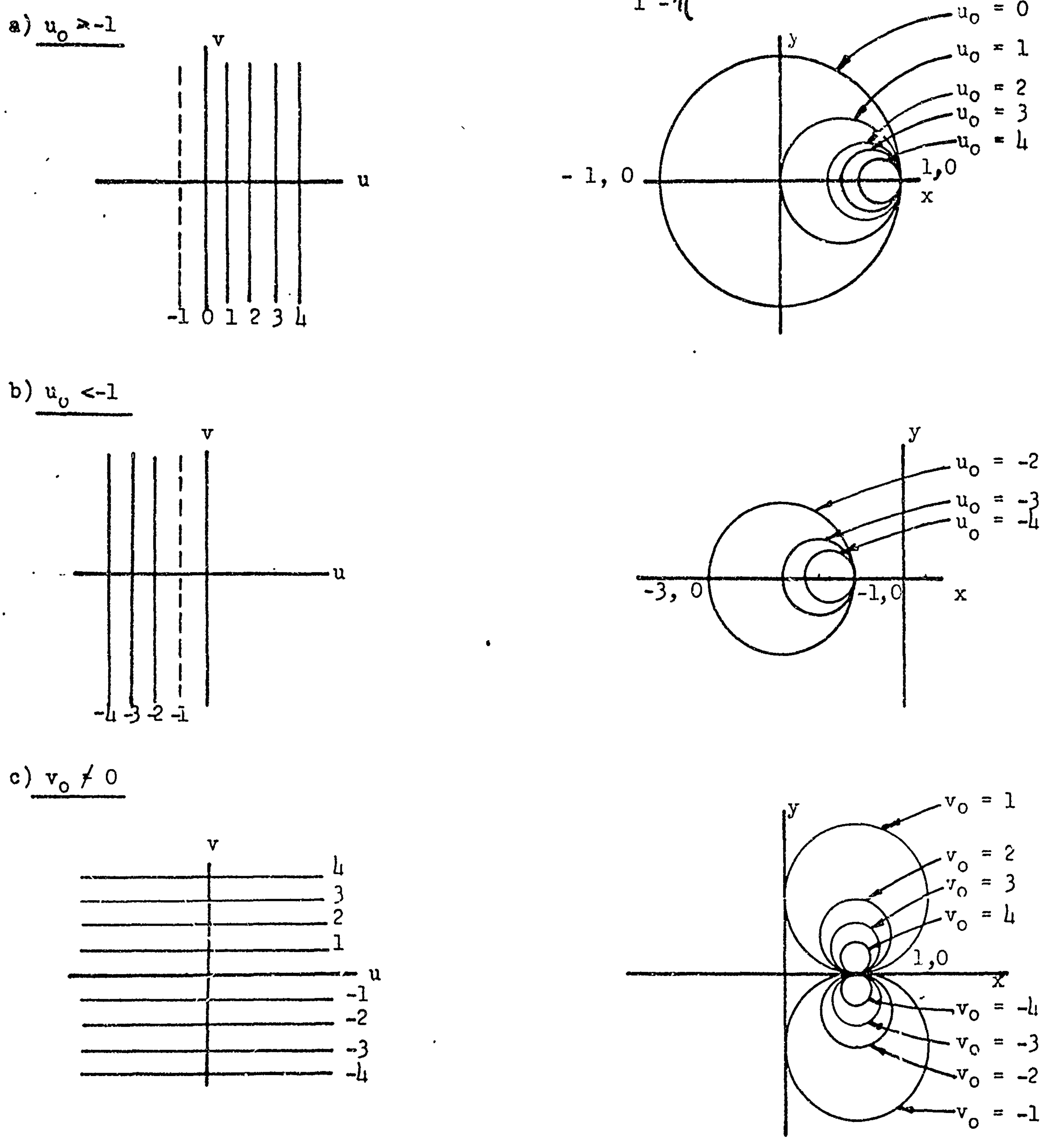


FIGURE 1

PULSE TUBE SCHEMATIC FOR THE MEASUREMENT OF THE SPECIFIC IMPEDANCE OF A BRASS DISC IN A WATER FILLED TUBE

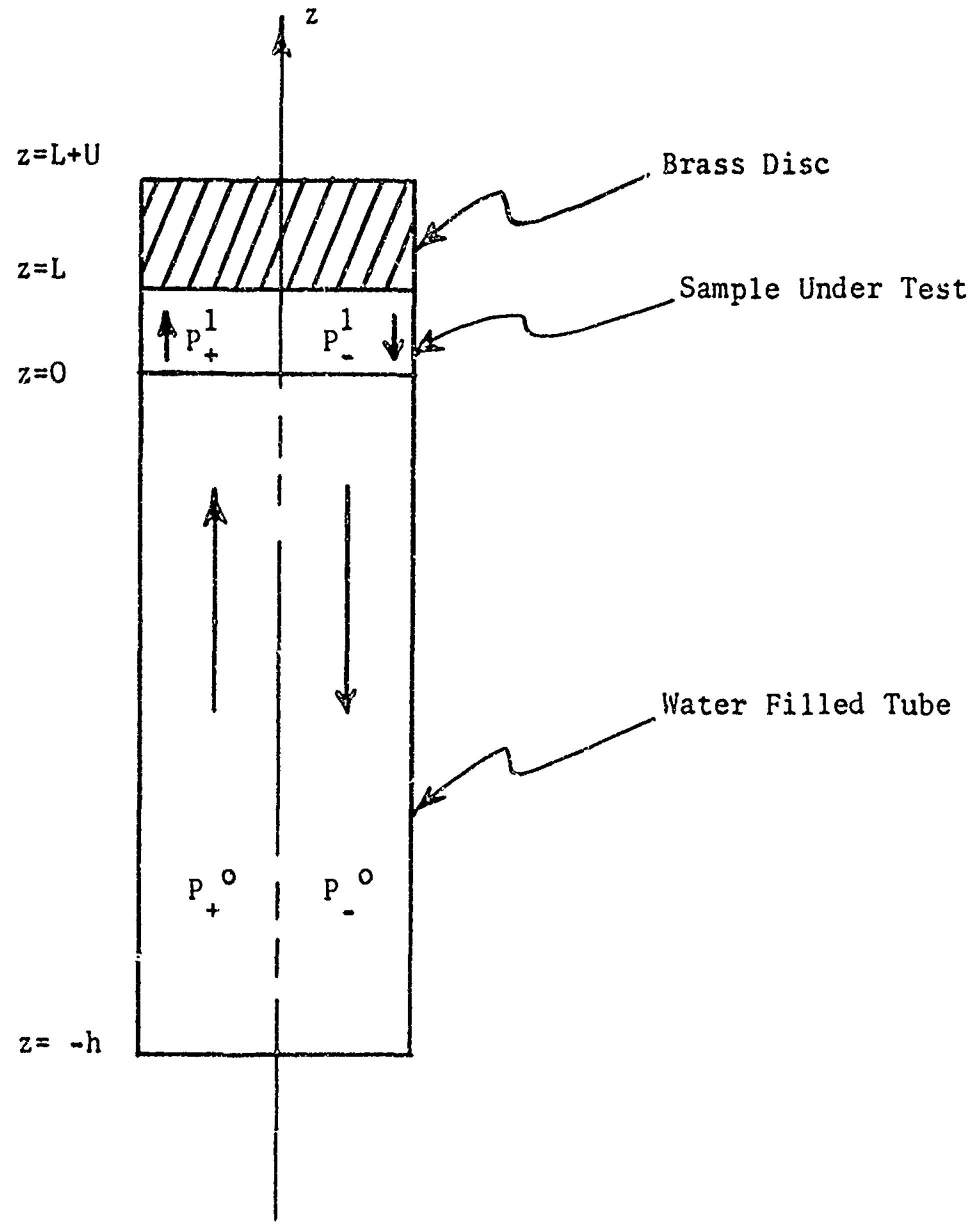


FIGURE 2

SMITH CHART REPRESENTATION OF THE FUNCTION $\gamma = (1+\eta)/(1-\eta)$

IMPEDANCE OR ADMITTANCE COORDINATES

$\gamma = u + jv$
$\eta = x + jy$

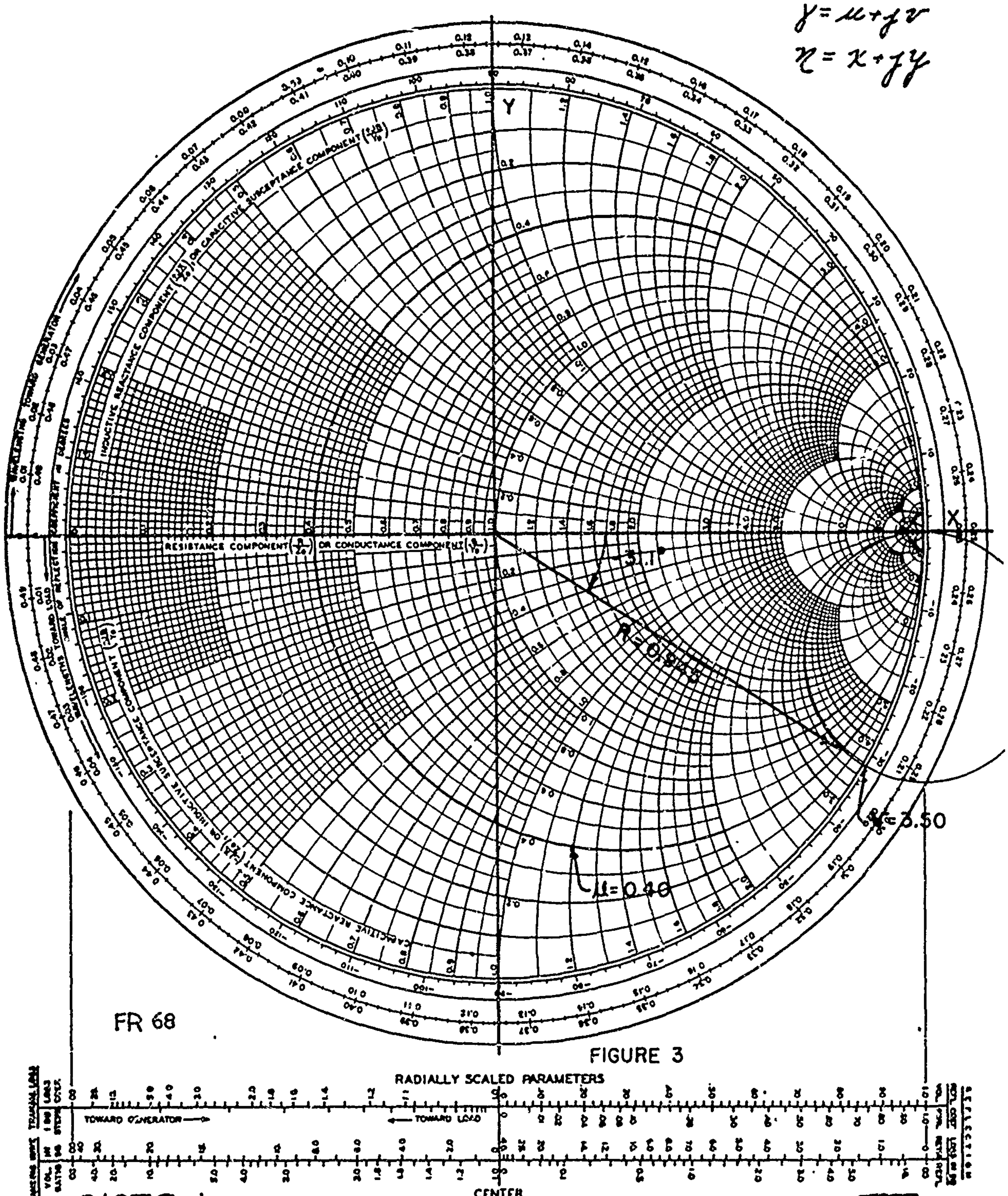
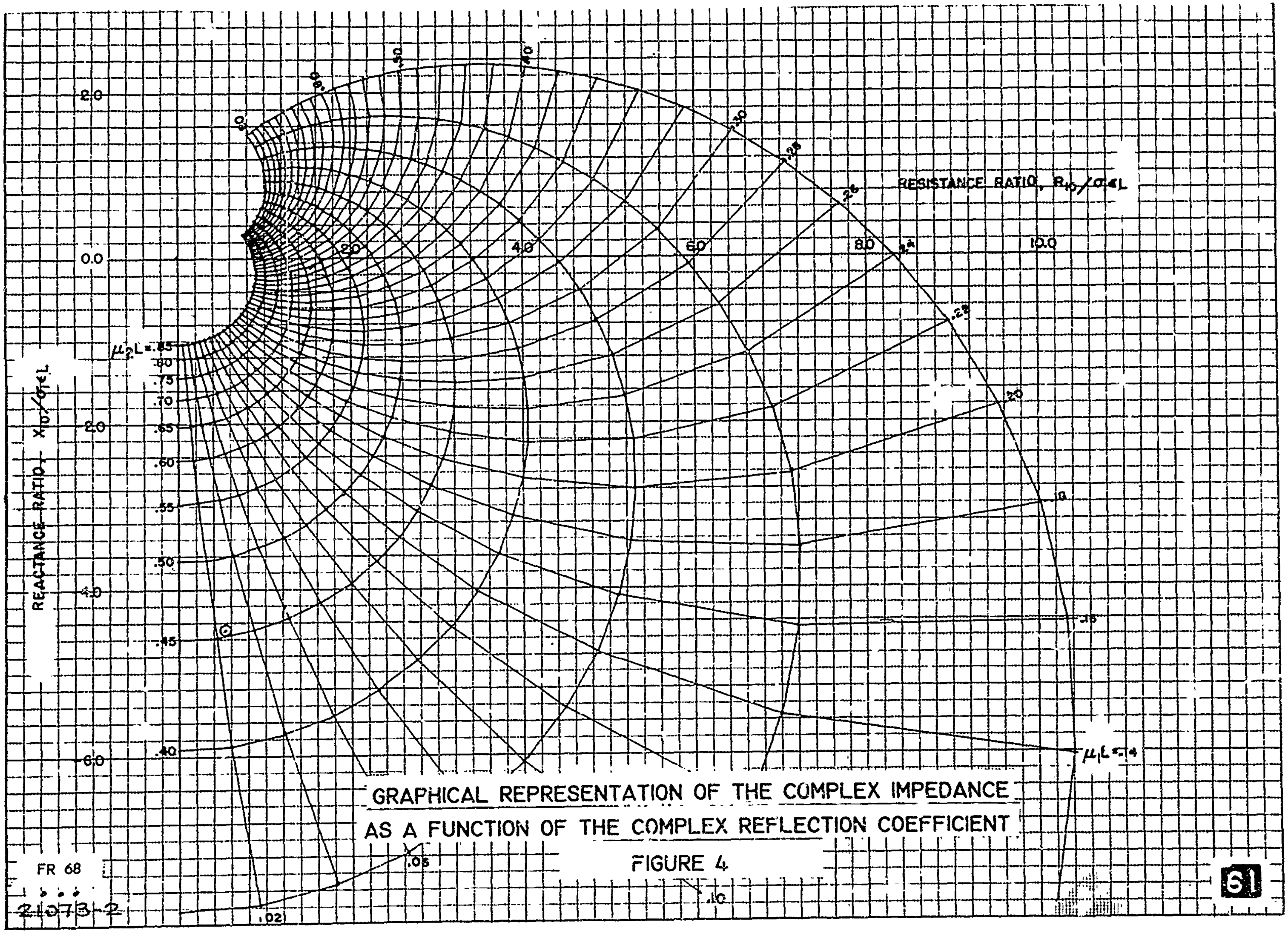


FIGURE 3

21073-1



GRAPHICAL REPRESENTATION OF THE COMPLEX IMPEDANCE
AS A FUNCTION OF THE COMPLEX REFLECTION COEFFICIENT

FIGURE 4

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