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APPLIED PHYSICS LABORATORY
THE JOHNS HOPKINS UNIVERSITY
SILVER SPRING MARYLAND

BBD-346
October 31, 1957

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TO: J. W. Follin, Jr.

FROM: J. E. Hanson

SUBJECT: Some Notes on the Application of Calculus of Variations to Smoothing for Finite Time with Acceleration Restrictions and the Exact Solution of Dr. Follin's Variance Equations for the Case of White Noise and White Target Acceleration Spectrum.

REFERENCE

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J. W. Follin, Jr.
", June 19, 1956.

rom J. E. Hanson,
Scan Smoothing
-State Cases",

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REFERENCES:

(1) APL/JHU Memorandum BBD-076, to J. W. Follin, Jr. from C. F. Black, "The Continuous Correction Computer", March 16, 1956.

(2) APL/JHU Memorandum BBD-117, to A. Kossiakoff from J. W. Follin, Jr. "Speech for Agard Meeting September 24 - 26, 1956", June 19, 1956.

(3) APL/JHU Memorandum BBD-285 to J. W. Follin, Jr. from J. E. Hanson, "Some Theoretical Aspects of Optimum Track-While-Scan Smoothing and Prediction, for both the Transient and Steady-State Cases", July 1, 1957.

The purpose of this paper is to present some useful techniques for solving finite-time filtering problems. In the course of the discussion, we shall solve explicitly the three variance equations presented in references (1) and (2).

We begin at time zero, and wind up at a fixed time T. The output of the filter is X(t), and the input is $X_s(t) + N(t)$, where $X_s(t)$ is signal and N(t) is noise. t is time which varies from 0 to T. We wish to minimize the variance of $X(T) - X_s(T)$ subject to certain restrictions on $\ddot{X}(t)$. All random variables are assumed to have mean zero.

We shall assume a specific statistical model, although it will be clear how to modify the analysis if some other model is used. We assume that N(t) has a white power spectral density φ, and $\dddot{X}_s(t)$ has a white power spectral density Θ.

We assume that the filter is linear in the sense that we have

(1). $$\ddot{X}(t) = \int_0^t f(t,\tau)\left[X_s(\tau) + N(\tau)\right] d\tau$$

Note that we do not assume f(t,τ) to take the usual form f(t - τ).

At each time t, $\ddot{X}(t)$ is a linear combination of past information, and it is not a priori clear how this linear combination will change if we go to a different value of t. What we wish to do is solve for $f(t, \tau)$.

Because of our statistical assumptions, it is desirable to express $X(T) - X_s(T)$ in terms of $\ddot{X}_s(t)$ and $N(t)$. To this end we have

(2).
$$X_s(T) = X_{s_o} + \dot{X}_{s_o} T + \int_o^T \left[\int_o^t \ddot{X}_s(\tau)\, d\tau \right] dt$$
$$= X_{s_o} + \dot{X}_{s_o} T + \int_o^T (T - \tau)\, \ddot{X}_s(\tau)\, d\tau$$

obtained by interchanging order of integration.

Similarly

(3).
$$X_s(\tau) = X_{s_o} + \dot{X}_{s_o} \tau + \int_o^\tau (\tau - u) \ddot{X}_s(u)\, du$$

and

(4).
$$X(T) = \int_o^T (T - t)\, \ddot{X}(t)\, dt,$$

since we choose our coordinate system so that initially, $X(0) = \dot{X}(0) = 0$.

Next, we substitute (3) into (1), and then (1) into (4), and after a fair amount of interchanging orders of integration, we have, using (2),

(5).
$$X(T) - X_s(T)$$
$$= \int_o^T N(\tau) g(\tau) d\tau + X_{s_o} \left[\int_o^T g(\tau) d\tau - 1 \right]$$
$$+ \dot{X}_{s_o} \left[\int_o^T \tau g(\tau) d\tau - T \right]$$
$$+ \int_o^T \ddot{X}_s(\tau) \left[\int_\tau^T (u - \tau) g(u) du - T + \tau \right] d\tau$$

where

(6). $$g(\tau) = \int_{\tau}^{T} (T - t) f(t, \tau)\, dt$$

The significance of $g(\tau)$ is that it is the weighting function for $X(T)$. That is,

(7). $$X(T) = \int_{0}^{T} g(\tau) \left[X_s(\tau) + N(\tau)\right] d\tau$$

Let σ^2 denote the variance of (5). We assume that X_{s_o} and $\dot{X}_{s_o}$ are random variables, not necessarily independent. All other pairs of random variables are assumed independent. We denote the mathematical expectation of $X_{s_o}^{\;2}$, $\dot{X}_{s_o}^{\;2}$, $X_{s_o}\dot{X}_{s_o}$ by $\sigma_{X_o, X_o}^{\;2}$, $\sigma_{V_o, V_o}^{\;2}$, and $\sigma_{X_o, V_o}^{\;2}$, respectively.

Next, we square (5), and take mathematical expectations. We get

(8). $$\sigma^2 = \varphi \int_{0}^{T} \left[g(\tau)\right]^2 d\tau + \sigma_{X_o, X_o}^{\;2} \left[\int_{0}^{T} g(\tau)\, d\tau - 1\right]^2 + 2\,\sigma_{X_o, V_o}^{\;2} \left[\int_{0}^{T} g(\tau)\, d\tau - 1\right]\left[\int_{0}^{T} \tau g(\tau) - T\right] + \sigma_{V_o, V_o}^{\;2} \left[\int_{0}^{T} \tau g(\tau)\, d\tau - T\right]^2 + \theta \int_{0}^{T} \left[\int_{\tau}^{T} (u - \tau) g(u)\, du - T + \tau\right]^2 d\tau$$

Since we are interested in placing a restriction on $\ddot{X}(t)$, we need an expression similar to (5) for $\ddot{X}(t)$.

This is

(9).
$$\ddot{X}(t) = X_{s_0}\int_0^t f(t,\tau)\,d\tau + \dot{X}_{s_0}\int_0^t \tau f(t,\tau)\,d\tau + \int_0^t \ddot{X}_s(\tau)\left[\int_\tau^t (u-\tau)f(t,u)\,du\right]d\tau + \int_0^t N(\tau)f(t,\tau)\,d\tau$$

Squaring and taking expectations, we have

(10).
$$E\left[\ddot{X}^2(t)\right] = \sigma^2_{X_0,X_0}\left[\int_0^t f(t,\tau)\,d\tau\right]^2 + 2\,\sigma^2_{X_0,V_0}\left[\int_0^t f(t,\tau)\,d\tau\right]\left[\int_0^t \tau f(t,\tau)\,d\tau\right] + \sigma^2_{V_0,V_0}\left[\int_0^t \tau f(t,\tau)\,d\tau\right]^2 + \varphi\int_0^t\left[f(t,\tau)\right]^2 d\tau + \theta\int_0^t\left[\int_\tau^t (u-\tau)f(t,u)\,du\right]^2 d\tau$$

The limitation we shall place on $\ddot{X}(t)$ is that

$D = \int_0^T E\left[\ddot{X}^2(t)\right]dt$ is given. From (10), we have

(11).
$$D = \sigma^2_{X_0,X_0}\int_0^T\left[\int_0^t f(t,\tau)\,d\tau\right]^2 dt + 2\,\sigma^2_{X_0,V_0}\int_0^T\left[\int_0^t f(t,\tau)\,d\tau\right]\left[\int_0^t \tau f(t,\tau)\,d\tau\right]dt + \sigma^2_{V_0,V_0}\int_0^T\left[\int_0^t \tau f(t,\tau)\,d\tau\right]^2 dt + \varphi\int_0^T\left\{\int_0^t\left[f(t,\tau)\right]^2 d\tau\right\}dt + \theta\int_0^T\left\{\int_0^t\left[\int_\tau^t (u-\tau)f(t,u)\,du\right]^2 d\tau\right\}dt$$

We wish to minimize (8) for a given value of D. Applying calculus of variations, we set $V_f(\sigma^2) + \lambda V_f(D)$ equal to zero, where V_f stands for the variation with respect to $f(t,\tau)$, and λ is a Lagrange multiplier.

Now $(T - t)$ factors out of $V_f(\sigma^2)$, and we compute

(12). $$\frac{V_f(\sigma^2)}{2(T-t)} = \varphi\, g(\tau) + \sigma^2_{X_o,X_o}\left[\int_o^T g(u)\,du - 1\right]$$
$$+ \sigma^2_{X_o,V_o}\left[\int_o^T (\tau + u) g(u)\,du - T - \tau\right]$$
$$+ \sigma^2_{V_o,V_o}\,\tau\left[\int_o^T u g(u)\,du - T\right]$$
$$+ \theta\int_o^\tau\left[\int_u^T (\tau - u)(V-u) g(V)\,dV\right]du$$
$$- \theta\int_o^\tau (T-u)(\tau-u)\,du$$

Also

(13). $$\frac{1}{2} V_f(D) = \sigma^2_{X_o,X_o}\int_o^t f(t,u)\,du$$
$$+ \sigma^2_{X_o,V_o}\int_o^t (\tau + u) f(t,u)\,du$$
$$+ \sigma^2_{V_o,V_o}\,\tau\int_o^t u f(t,u)\,du$$
$$+ \varphi f(t,\tau)$$
$$+ \theta\int_o^\tau\left[\int_u^t (\tau - u)(V-u) f(t,V)\,dV\right]du$$

The solution of $V_f(\sigma^2) + \lambda V_f(D) = o$ for $f(t,\tau)$ appears quite difficult, so we shall content ourselves with some special cases.

THE CASE OF NO ACCELERATION RESTRICTION

Here we set $\lambda = o$, and we set (12) equal to zero and attempt to solve for $f(t,\tau)$. However, we notice from (12) that the only thing we can hope to solve for is $g(\tau)$, and from (6), there are an infinity of functions $f(t,\tau)$ which produce the same $g(\tau)$. Upon reflection, we see from (7) that this should be the case. All that is required is that the output of the filter wind up at the right place when time = T. This can be accomplished by an infinity of paths, since we have placed no restriction on the path.

$g(\tau)$ can be solved for explicitly. If we set (12) equal to zero, differentiate four times, we obtain

$$(14). \qquad \varphi \frac{d^4}{d\tau^4}\left[g(\tau)\right] + \theta\, g(\tau) = o.$$

We solve (14), with four undetermined coefficients, and substitution into $V_f(\sigma^2) = o$ determines the coefficients. We then substitute $g(\tau)$ into (8) to obtain the optimum σ^2. All this is extremely tedious, and the expression for $g(\tau)$ is quite messy. The optimum σ^2 is not too bad, so we shall write it down.

$$(15). \quad \sigma^2_{opt.} = 2K\varphi \frac{A \cosh 2KT + B \sinh 2KT + C \cos 2KT + D \sin 2KT}{8K^4\varphi^2 + 2\left[\sigma^4_{X_o,V_o} - \sigma^2_{X_o,X_o}\sigma^2_{V_o,V_o}\right] + B \cosh 2KT + A \sinh 2KT - D \cos 2KT + C \sin 2KT}$$

where

$$(16). \qquad K = \frac{1}{\sqrt{2}}\left(\frac{\theta}{\varphi}\right)^{1/4}$$

$$(17). \qquad A = 2K\varphi\sigma^2_{V_o,V_o} + 4K^3\varphi\,\sigma^2_{X_o,X_o}.$$

$$(18). \qquad B = 4K^4\varphi^2 + 4K^2\varphi\,\sigma^2_{X_o,V_o} - \sigma^4_{X_o,V_o} + \sigma^2_{X_o,X_o}\sigma^2_{V_o,V_o}$$

(19). $$C = -2K\varphi\sigma^2_{V_o,V_o} + 4K^3\varphi\,\sigma^2_{X_o,X_o}$$

(20). $$D = -4K^4\varphi^2 + 4K^2\varphi\sigma^2_{X_o,V_o} + \sigma^4_{X_o,V_o} - \sigma^2_{X_o,X_o}\,\sigma^2_{V_o,V_o}$$

Since we have a degree of freedom in choosing a filter which will produce this optimum at time T, let us pin the filter down by requiring that at any time, not necessarily T, the output produces the optimum σ^2 at that time. This filter will produce infinite r.m.s. acceleration, but if we have no acceleration restriction, it is obviously a desirable filter. (It is in fact the case that any filter producing the optimum σ^2 at time T will have infinite acceleration.)

To denote the functional dependence of $g(\tau)$ on T, we replace T by t, and $g(\tau)$ by $g(t,\tau)$.

(12) now becomes

(21). $$\varphi g(t,\tau) + \sigma^2_{X_o,X_o}\left[\int_0^t g(t,u)\,du - 1\right]$$
$$+ \sigma^2_{X_o,V_o}\left[\int_0^t (\tau + u)g(t,u)\,du - t - \tau\right]$$
$$+ \sigma^2_{V_o,V_o}\,\tau\left[\int_0^t ug(t,u)\,du - t\right]$$
$$+ \theta\int_0^\tau\left[\int_u^t (\tau - u)(V - u)g(t,V)\,dV\right]du$$
$$- \theta\int_0^\tau (t - u)(\tau - u)\,du = 0.$$

an identity in t and τ, and (7) becomes

(22). $$X(t) = \int_0^t g(t,\tau)\,I(\tau)\,d\tau$$

where $I(\tau) = X_s(\tau) + N(\tau)$ is the total input.

(21) and (22) define the output of the filter as a function of time.

We shall now prove that there exists functions of time which when used as gains in the basic second order filter of references (1) and (2) produces exactly the above filter. To this end, let us define the two functions $h(t,\tau)$ and $V(t)$ as follows:

(23). $$h(t,\tau) = \frac{\partial}{\partial t}\left[g(t,\tau)\right] + g(t,t)\, g(t,\tau)$$

and

(24). $$V(t) = \int_0^t h(t,\tau)\, I(\tau)\, d\tau.$$

Taking the $\frac{\partial}{\partial t}$ of (21), substituting (23), we see that $h(t,\tau)$ satisfies the equation

(25). $$\varphi h(t,\tau) + \sigma^2_{X_o,X_o} \int_0^t h(t,u)du + \sigma^2_{X_o,V_o}\left[\int_0^t (\tau + u)h(t,u)\, du - 1\right]$$
$$+ \sigma^2_{V_o,V_o}\, \tau \left[\int_0^t u\, h(t,u)du - 1\right]$$
$$+ \theta \int_0^\tau \left[\int_u^t (\tau - u)(V - u)\, h(t, V)\, dV\right] du$$
$$- \theta \int_0^\tau (\tau - u)\, du = 0$$

Another relation exists between $g(t,\tau)$ and $h(t,\tau)$, namely

(26). $$\frac{\partial}{\partial t}\left[h(t,\tau)\right] = - h(t,t)g(t,\tau).$$

To prove this, we take the $\frac{\partial}{\partial t}$ of (25), substitute (26) in the result, and we wind up with the identity (21).

Finally, we show

(27). $$\dot{X}(t) = V(t) + g(t,t)\left[I(t) - X(t)\right]$$

(28). $$\dot{V}(t) = h(t,t)\left[I(t) - X(t)\right]$$

These reduce to identities if we substitute (22) and (24) in them, and use (23) and (26). This proves (27) and (28).

In references (1) and (2), the above two equations are assumed, with g(t,t) and h(t,t) replaced with unknown functions of time, and the functions were chosen to optimize the system by a different method. The statistical assumptions were the same as in this paper. In these references, five basic equations are derived, which are

(29). $$\dot{X} = V + \frac{\sigma^2_{X,X}}{\varphi}(I - X)$$

(30). $$\dot{V} = \frac{\sigma^2_{X,V}}{\varphi}(I - X)$$

(31). $$\frac{d}{dt}(\sigma^2_{X,X}) = -\frac{\sigma^4_{X,X}}{\varphi} + 2\sigma^2_{X,V}$$

(32). $$\frac{d}{dt}(\sigma^2_{V,V}) = -\frac{\sigma^4_{X,V}}{\varphi} + \theta$$

(33). $$\frac{d}{dt}(\sigma^2_{X,V}) = -\frac{\sigma^2_{X,X}\,\sigma^2_{X,V}}{\varphi} + \sigma^2_{V,V}$$

The solution to the last three equations determines the optimum gains $\frac{\sigma^2_{X,X}}{\varphi}$ and $\frac{\sigma^2_{X,V}}{\varphi}$, and determines the correlation matrix of errors in position and velocity estimates.

We can now prove that (27) and (28) are identical to (29)-(33) by substituting all our known functions of time into (29)-(33), and showing that they reduce to identities. (This is indeed the case, but we omit the proof because it is tedious.)

As a corollary, we have the explicit solutions to (31)-(33), namely

(34). $\sigma^2_{X,X} = \varphi g(t,t)$

$$= 2K\varphi \frac{A\cosh 2Kt + B\sinh 2Kt + C\cos 2Kt + D\sin 2Kt}{8K^4\varphi^2 + 2\left[\sigma^4_{X_o,V_o} - \sigma^2_{X_o,X_o}\sigma^2_{V_o,V_o}\right] + B\cosh 2Kt + A\sinh 2Kt - D\cos 2Kt + C\sin 2Kt}$$

(35). $\sigma^2_{X,V} = \varphi h(t,t)$

$$= 2K^2\varphi \frac{B\cosh 2Kt + A\sinh 2Kt + D\cos 2Kt - C\sin 2Kt}{8K^4\varphi^2 + 2\left[\sigma^4_{X_o,V_o} - \sigma^2_{X_o,X_o}\sigma^2_{V_o,V_o}\right] + B\cosh 2Kt + A\sinh 2Kt - D\cos 2Kt + C\sin 2Kt}$$

(36).

$$\sigma^2_{V,V} = 4K^3\varphi \frac{A\cosh 2Kt + B\sinh 2Kt - C\cos 2Kt - D\sin 2Kt}{8K^4\varphi^2 + 2\left[\sigma^4_{X_o,V_o} - \sigma^2_{X_o,X_o}\sigma^2_{V_o,V_o}\right] + B\cosh 2Kt + A\sinh 2Kt - D\cos 2Kt + C\sin 2Kt}$$

where K,A,B,C,D are given by (16)-(20), and where initially $\sigma^2_{X,X}$, $\sigma^2_{V,V}$, $\sigma^2_{X,V}$ reduce to $\sigma^2_{X_o,X_o}$, $\sigma^2_{V_o,V_o}$, $\sigma^2_{X_o,V_o}$, respectively.

As another corollary, we have proved that (29)-(33) is the best linear system for optimizing the error under the present statistical assumptions. Reference (3) indicates that it is also optimum for prediction errors.

An interesting special case is when $\theta = o$. In this case we have

$$(37).\quad \sigma^2_{X,X} = \frac{\varphi^2\left[\sigma^2_{X_o,X_o} + 2t\,\sigma^2_{X_o,V_o} + t^2\sigma^2_{V_o,V_o}\right] - \frac{\varphi t^3}{3}\left[\sigma^4_{X_o,V_o} - \sigma^2_{X_o,X_o}\sigma^2_{V_o,V_o}\right]}{\varphi\left[\varphi + t\,\sigma^2_{X_o,X_o} + t^2\sigma^2_{X_o,V_o} + \frac{t^3}{3}\sigma^2_{V_o,V_o}\right] - \frac{t^4}{12}\left[\sigma^4_{X_o,V_o} - \sigma^2_{X_o,X_o}\sigma^2_{V_o,V_o}\right]}$$

$$(38).\quad \sigma^2_{X,V} = \frac{\varphi^2\left[\sigma^2_{X_o,V_o} + t\,\sigma^2_{V_o,V_o}\right] - \frac{\varphi t^2}{2}\left[\sigma^4_{X_o,V_o} - \sigma^2_{X_o,X_o}\,\sigma^2_{V_o,V_o}\right]}{\varphi\left[\varphi + t\,\sigma^2_{X_o,X_o} + t^2\,\sigma^2_{X_o,V_o} + \frac{t^3}{3}\,\sigma^2_{V_o,V_o}\right] - \frac{t^4}{12}\left[\sigma^4_{X_o,V_o} - \sigma^2_{X_o,X_o}\,\sigma^2_{V_o,V_o}\right]}$$

$$(39).\quad \sigma^2_{V,V} = \frac{\varphi^2\sigma^2_{V_o,V_o} - \varphi t\left[\sigma^4_{X_o,V_o} - \sigma^2_{X_o,X_o}\,\sigma^2_{V_o,V_o}\right]}{\varphi\left[\varphi + t\,\sigma^2_{X_o,X_o} + t^2\,\sigma^2_{X_o,V_o} + \frac{t^3}{3}\,\sigma^2_{V_o,V_o}\right] - \frac{t^4}{12}\left[\sigma^4_{X_o,V_o} - \sigma^2_{X_o,X_o}\,\sigma^2_{V_o,V_o}\right]}$$

ANOTHER APPROACH TO THE NO ACCELERATION RESTRICTION CASE

In the discussion following equation (20), we removed the degree of freedom by requiring the filter to produce the optimum error at any time, and we were led to a set of differential equations with time varying coefficients which defined the filter.

We can also remove the degree of freedom in other ways, for example by requiring the filter to be defined by differential equations with constant coefficients. Now we require the existence of a function $\psi(u)$ such that

$$(40).\quad X(t) = \int_0^t \psi(t-\tau)\, I(\tau)\, d\tau.$$

Comparing with (7), we have

$$(41).\quad \psi(T-\tau) = g(\tau)$$ (T is a fixed number in this discussion).

and hence, from (14),

$$(42).\quad \varphi \frac{d^4}{du^4}\left[\psi(u)\right] + \theta\psi(u) = 0$$

if we differentiate (40) four times, we have

$$(43).\quad \ddddot{X}(t) = \psi(0)\,\dddot{I}(t) + \dot{\psi}(0)\,\ddot{I}(t) + \ddot{\psi}(0)\,\dot{I}(t) + \dddot{\psi}(0)\,I(t) + \int_0^t \ddddot{\psi}(t-\tau)\,I(\tau)\,d\tau$$

From (40), (42), (43),

(44). $\varphi \dddot{X}(t) + \theta X(t) = \varphi\psi(o)\,\dddot{I}(t) + \varphi\dot{\psi}(o)\,\ddot{I}(t) + \varphi\ddot{\psi}(o)\,\dot{I}(t) + \varphi\dddot{\psi}(o)\,I(t)$

We compute from the explicit expression for $g(\tau)$ (which we have not presented here)

(45). $$\psi(o) = g(T) = \frac{\sigma^2_{X,X}(T)}{\varphi}$$

(46). $$\dot{\psi}(o) = -\dot{g}(T) = -\frac{\sigma^2_{X,V}(T)}{\varphi}$$

(47). $$\ddot{\psi}(o) = \ddot{g}(T) = o$$

(48). $$\dddot{\psi}(o) = -\dddot{g}(T) = \frac{\theta}{\varphi}.$$

where $\sigma^2_{X,X}(T)$ and $\sigma^2_{X,V}(T)$ are given by (34) and (35), with t replaced by T.

So the optimum transfer function is

(49). $$\frac{X}{I} = \frac{\sigma^2_{X,X}(T)\,p^3 - \sigma^2_{X,V}(T)\,p^2 + \theta}{\varphi\,p^4 + \theta},$$

obviously unstable, but producing the optimum error at the specific instant when time = T.

One could also remove the degree of freedom by looking for homing equations, but the author does not at present know how to do this.

AN ACCELERATION RESTRICTION

Once an acceleration restriction has been added, the degree of freedom disappears, and $f(t,\tau)$ is uniquely determined.

Rather than attempt to solve $V_f(\sigma^2) + \lambda V_f(D) = o$ we shall consider the simpler problem where only the drag due to noise is a restriction.

The variational equation to solve then becomes

(50).
$$\lambda\varphi f(t,\tau) + (T-t)\varphi g(\tau) + (T-t)\,\sigma^2_{X_o,X_o}\left[\int_o^T g(u)\,du - 1\right]$$
$$+ (T-t)\,\sigma^2_{X_o,V_o}\left[\int_o^T(\tau + u)g(u)\,du - T - \tau\right]$$
$$+ (T-t)\,\sigma^2_{V_o,V_o}\,\tau\left[\int_o^T ug(u)\,du - T\right]$$
$$+ (T-t)\Theta\int_o^\tau\left[\int_u^T(\tau-u)(V-u)g(V)\,dV\right]du$$
$$- (T-t)\Theta\int_o^\tau(T-u)(\tau-u)\,du = o.$$

Inspection of (50) shows that $\frac{f(t,\tau)}{T-t}$ is a function of τ alone, and we set

(51).
$$f(t,\tau) = (T-t)F(\tau).$$

Substituting (51) into (6) we have

(52).
$$g(\tau) = \frac{(T-\tau)^3}{3}F(\tau)$$

Substituting (51) and (52) into (50), we have

(53).
$$\lambda\varphi F(\tau) + \varphi \frac{(T-\tau)^3}{3} F(\tau) + \sigma_{X_o,X_o}^2 \left[\int_0^T \frac{(T-u)^3}{3} F(u)du - 1\right]$$
$$+ \sigma_{X_o,V_o}^2 \left[\int_0^T (\tau+u)\frac{(T-u)^3}{3} F(u)du - T - \tau\right]$$
$$+ \sigma_{V_o,V_o}^2 \tau \left[\int_0^T u \frac{(T-u)^3}{3} F(u)du - T\right]$$
$$+ \theta \int_0^\tau \left[\int_u^T (\tau-u)(V-u)\frac{(T-V)^3}{3} F(V)dV\right] du$$
$$- \theta \int_0^\tau (T-u)(\tau-u)du = 0.$$

If we differentiate (53) four times with respect to τ, we have

(54).
$$\frac{d^4}{d\tau^4}\left[\lambda\varphi F(\tau) + \varphi \frac{(T-\tau)^3}{3} F(\tau)\right] + \theta \frac{(T-\tau)^3}{3} F(\tau) = 0.$$

We shall not attempt to solve (54) or (53), but we notice that $F(\tau)$ is a uniquely determined function. The differential equation satisfied by $X(t)$ is easily determined. From (1) and (51) we have

(55).
$$\ddot{X}(t) = (T-t)\int_0^t F(\tau)I(\tau)d\tau$$

Hence

(56).
$$\frac{d}{dt}\left(\frac{\ddot{X}}{T-t}\right) = F(t)I(t).$$

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