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William W. White





OPERATIONS RESEARCH CENTER

COLLEGE OF ENGINEERING

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UNIVERSITY OF CALIFORNIA - BERKELEY

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# ABSTRACT

This paper extends some results of Gomory on a group theoretic approach to linear integer programming. An algorithm for solving integer programming problems is presented, and the relation of this work to cuts and appropriate geometric interpretations are given.

The theoretical basis for the algorithm establishes a characterization of integer solutions to linear programs by considering the Gomory column group associated with the optimal linear programming basis. It is shown that if an optimal integer solution exists, it can be obtained by finding the $r^{th}$ optimal solution to an optimization problem over elements in this Gomory group, for some finite r. A dynamic programming procedure for finding this $r^{th}$ optimal solution is given, and shown to be equivalent to finding an $r^{th}$ shortest route through a specially constructed network.

A particular class of cutting planes is discussed. These cuts are shown to be parallel to a subset of Gomory cuts and at least as strong as the Gomory cuts.

Specially structured integer programs are discussed with a view toward speeding computation. Included in this discussion is a specialization to zero-one variables. Some computational results are given. It is seen that computational efficiency depends crucially on the size of the absolute value of the determinant of the optimal linear programming basis. In general, the smaller this absolute value, the more efficient the algorithm.

## PREFACE

Recently, R. E. Gomory published a paper [8] in which an algorithm for solving a certain class of integer linear programming problems was presented. Of particular interest was the approach employed to obtain this algorithm, for it revealed many characteristics of solutions to integer programs.

In this paper we hope to follow the lead which Gomory has presented. In Chapter I we develop the notation used in the rest of the paper and present some of Gomory's results pursuant to the material developed in later sections. In Chapter II we give some characterizations of integer solutions to linear programs and present some generalizations of results already obtained by Gomory. We develop in Chapter III the computational aspects of an algorithm suggested by the results of Chapter II. Chapter IV contains some further properties of integer solutions, and investigates the close connection between results of the previous sections and Gomory's Method of Integer Forms [9]. Chapter V contains results on the application of the material already developed to problems with special properties, such as zero-one restrictions on the variables. In view of what is developed in Chapter IV these results also carry over to the Method of Integer

Forms. Chapter VI contains a discussion of computational results. Both Chapters III and V are concluded with examples.

I would like to thank G. B. Dantzig and E. W. Barankin for their suggestions for improving both the form and the content of this paper. Special acknowledgements are due to F. W. Glover, with whom I have worked closely during this past year. Many of the ideas presented here are a direct outgrowth of my association with Dr. Glover. D. Van Every is responsible for the excellent job of programming the algorithm presented in Chapter III. I would also like to thank J. Haldi for permission to use the series of test problems. No one reading this paper can miss its reliance upon the work of R. E. Gomory, without whom the field of integer programming, and this paper in particular, would not be what it is today. These people I would like to thank for what may be worthwhile in this paper. The shortcomings, of course, are my own.

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# CHAPTER I

## BACKGROUND

Define problem P1 as the ordinary linear programming problem

$$\text{minimize} \quad z_1 = cx$$

$$\text{subject to} \quad Ax = b, \; x \geq 0,$$

and problem P2 as problem P1 with the further restriction that the variables take on integral values. Here b is an integer m-vector, c is a real (m+n)-vector, A is an m x (m+n) matrix of integers, and x is an (m+n)-vector. A solution x will be called feasible if it satisfies the above constraints, and will be called optimal if it also minimizes $z_1$. A will be assumed to be of full rank partitioned into two submatrices A' and I, where I is an mxm identity matrix. Thus $A = [A', I]$ and the equality constraints could have been written in the form $A'x \leq b$, for x appropriately defined.

Let B denote an mxm optimal basis matrix for problem P1, and let N denote the mxn submatrix of A corresponding to the non-basic columns of A. Partition the vector c into two parts, $c_B$ and $c_N$, corresponding to B and N respectively, and let the vector x be partitioned similarly, $x = [x_B, x_N]$. Then problem P1 is equivalent to the following problem P1':

$$\text{minimize} \quad z_1 = -c_B B^{-1} b + (c_N - c_B B^{-1} N) x_N$$

$$\text{subject to} \quad I x_B + B^{-1} N x_N = B^{-1} b, \quad x_B \geq 0, \quad x_N \geq 0.$$

From the foregoing assumptions, $(c_N - c_B B^{-1}) \geq 0$, $B^{-1} b \geq 0$, and an optimal solution to P1 is given by:

$$x_B = B^{-1} b, \quad x_N = 0, \quad \text{and} \quad z_1 = -c_B B^{-1} b.$$

Let problem P2' be the same as problem P1' with the further restriction that the variables take on integral values. Then P2' is equivalent to P2, and the optimal solution to P1' obtained above is also an optimal solution to P2' and P2 if and only if $B^{-1} b$ is a vector of integers.

It is clear that for $x_B$ to be integer, $B^{-1} b - B^{-1} N x_N$ must be a vector with integral components. Thus any solution to P2' must satisfy

$$B^{-1} N x_N \equiv B^{-1} b \pmod 1, \quad x_N \geq 0, \quad x_N \text{ integer.}^{(1)}$$

Since $x_B \geq 0$, it is also necessary that $B^{-1} N x_N \leq B^{-1} b$. An optimal

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(1) $a \equiv b \pmod c$, read "a is congruent to b modulo c", means that $(a - b)$ is an integral multiple of $c$.

solution to P2' must satisfy these two restrictions, and also minimize $(c_N - c_B B^{-1} N)x_N$.

Gomory [8] has shown that for a certain class of vectors b, a minimizing solution to the set of linear congruences also satisfies the inequality relations. To review Gomory's results, denote the colums of B by $a_i$, $1 \le i \le m$, and the columns of N by $a_{i+m}$, $1 \le i \le n$. Let $\ell = \max_{1 \le i \le n} || a_{i+m} ||$, i.e., the maximum of the Euclidean norms, and D be the absolute value of the determinant of B. Finally, let $K^B = \{b \mid B^{-1}b \ge 0\}$, and let $K^B(d)$ be the reduced cone obtained from $K^B$ by removing all points within a distance d of the boundary of $K^B$. Observe that $K^B$ is the cone of vectors b for which B is an optimal basis for P1. Gomory has shown that if $b \in K^B(\ell(D-1))$ then a minimizing solution to the set of linear congruences also satisfies the inequality relations.

If, however, $b \notin K^B(\ell(D-1))$, then it is not necessarily true that a solution which minimizes $(c_N - c_B B^{-1} N)x_N$ subject to the congruence relations also satisfies $B^{-1}Nx_N \le B^{-1}b$. If these inequalities are not violated too "badly," the present minimizing solution may be acceptable, for the magnitude of the violations may not exceed the degree of accuracy imposed by the coefficients of problem P1. This could be true, for example, if an equation of P1 is a technological constraint, with the coefficients subject to measurement error. When it is necessary that the inequalities be satisfied, however, a different approach is necessary. One possibility is to use Gomory's Method of Integer Forms [9]. Another algorithm, which will be developed later in this paper, makes fuller use of the system of linear congruences.

As Gomory has shown [8, 9], a very important property of problem P2 is its underlying group structure. Let M(I) be the module[(1)] of all integer points in m-space, and let M(B) be the module of integer combinations of the columns $\alpha_i$, $1 \leq i \leq m$, of the basis matrix B. Then the factor module $G = M(I)/M(B)$ is a finite additive group of D elements, where, as above, $D = |\det B|$. Consider the columns $B^{-1}\alpha_{i+m}$, $1 \leq i \leq n$, of the matrix $B^{-1}N$, and the column $B^{-1}b$, and replace each entry in each column by its fractional part, i.e., consider the entry modulo 1. These columns then generate a finite additive group F, all operations taken modulo 1. Similarly, the row coefficients of $Ix_B + B^{-1}Nx_N = B^{-1}b$, taken module 1, also generate a finite additive group $F^T$. It can be shown (Gomory, [9]) that F, $F^T$, and G are all isomorphic to each other, under the assumption that $A = [A', I]$ with A' a matrix of integers.

In the remainder of this paper, $\bar{\alpha}_{i+m}$ will denote the element of the Gomory column group F corresponding to the column vector $\alpha_{i+m}$, $1 \leq i \leq n$. Thus $B^{-1}\alpha_{i+m} \equiv \bar{\alpha}_{i+m} \pmod 1$. The column $\bar{b}$ will be the element corresponding to b, $b \equiv \bar{b} \pmod 1$. A typical

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(1) A module is an additive group.

element of F will be $\bar{p}$, and the zero element will be $\bar{0}$. Note that each element $\bar{p}$ of F is a column vector of m non-negative components, each component being less than one. In fact, each component can be expressed as an integer between zero and D-1 (inclusive), divided by D.

# CHAPTER II

## CHARACTERIZATIONS OF INTEGER SOLUTIONS

In Chapter I it was pointed out that any solution to the integer programming problem P2' must satisfy $B^{-1}Nx_N \equiv B^{-1}b \pmod 1$. If each column of $B^{-1}N$ is reduced modulo 1, the congruence must continue to hold. Let $c^* = (c^*_{1+m}, c^*_{2+m}, \ldots, c^*_{n+m})$ be the objective function coefficients of P1', i.e., $c^* = c_N - c_B B^{-1}N$. Define problem P3 as the following minimization problem over the Gomory column group F:

$$\text{minimize} \quad \varphi(b) = \sum_{i=1}^{n} c^*_{i+m} y_i$$

$$\text{subject to} \quad \sum_{i=1}^{n} \bar{a}_{i+m} y_i \equiv \bar{b} \pmod 1$$

with $y \geq 0$, integral.

From Chapter I it is evident that there is a close relation between solutions to problem P3 and solutions to problem P2. This chapter will develop this relation and examine properties of solutions to P3.

Let C denote the set of all solutions to the linear programming problem P1, when the non-negativity restrictions on the optimal basic variables x have been dropped. Then $C = \{(x, y) \mid Bx + Ny = b,\ y \geq 0\}$. Theorem 1 presents a relation between points in C and solutions to P3.

Theorem 1: The vector $\hat{y}$ is a feasible integral solution to P3 if and only if there exists an $\hat{x}$ such that $(\hat{x}, \hat{y})$ is a lattice point of C.

Proof:

Let $(\hat{x}, \hat{y})$ be a lattice point of C. Then $B\hat{x} + N\hat{y} = b$, with $\hat{x}$ and $\hat{y}$ integral, and $\hat{y} \geq 0$. Then $I\hat{x} = B^{-1}b - B^{-1}N\hat{y}$. Since $\hat{x}$ is integral, $B^{-1}b - B^{-1}N\hat{y} \equiv 0 \pmod 1$. Considering the components of $B^{-1}b$ and $B^{-1}N$ modulo 1, it follows that $\sum_{i=1}^{n} \bar{a}_{i+m}\hat{y}_i \equiv \bar{b} \pmod 1$.

Conversely, let $\hat{y}$ be a feasible solution to P3, i. e., $\hat{y}$ satisfies $\sum_{i=1}^{n} \bar{a}_{i+m}\hat{y}_i \equiv \bar{b} \pmod 1$ with $\hat{y} \geq 0$, $\hat{y}$ integral. Then $B^{-1}N\hat{y} \equiv B^{-1}b \pmod 1$, and by definition of "modulo 1" it follows that there exists an integer-valued vector K such that $B^{-1}N\hat{y} = B^{-1}b + K$. Taking $\hat{x} = K$, it follows that $B\hat{x} + N\hat{y} = b$ and thus $(\hat{x}, \hat{y}) \in C$.

From Theorem 1 it follows that if there exists a feasible solution to problem P2, then there is a corresponding feasible solution to problem P3. However, the theorem is more general. If the non-negativity restrictions on certain of the variables are dropped, then this correspondence still holds. As is evident from the proof of Theorem 1, this correspondence is one to one. Thus from the converse section of the theorem, every solution to problem P3 generates a solution to problem P2, provided that the non-negativity restrictions on those variables which are basic in the optimal solution to P1 are dropped.

This dropping of non-negativity restrictions is the crux of the problem. It is an outcome of Gomory's work that, neglecting these non-negativity constraints, an optimal solution to P3 generates an optimal solution to P2. If, in addition, these constraints are satisfied, then the solution generated is actually optimal for P2. However, it is not uncommon that the solution to P2 generated by the optimal solution P3 is not feasible for P2.

Geometrically, the following situation occurs. Let $C_x = \{x | x = B^{-1}(b - Ny),\ y \geq 0\}$. $C_x$ is a cone in m-space with its origin at $B^{-1}b$. The origin $B^{-1}b$ of this cone represents an optimal solution to problem P1, and the hyperplanes forming the boundaries of this cone are just those constraints of problem P1 which are binding for the optimal linear programming solution. Each lattice point of C corresponds to a lattice point of $C_x$, and each lattice point of $C_x$ is generated by a solution of problem P3. In fact, an optimal lattice point of $C_x$. However, the constraints $x \geq 0$ restrict the feasible region of $C_x$, and an optimal lattice point of $C_x$ satisfying these additional restrictions (if one exists) may not coincide with the lattice point obtained from a solution to problem P3 via $x = B^{-1}(b - Ny)$.

It is true that if an optimal integer solution to P2 exists, it can be generated from some solution to P3. Since the solutions of P3 are in integers, one can rank solutions of P3 in terms of their objective function value. Let $\varphi^{(r)}(b)$ be the $r^{th}$ smallest value of the objective function of P3, and let $y^{(r)}$ be a corresponding solution ($y^{(r)}$ is not necessarily unique). Theorem II establishes a connection between solutions to P3 and optimal solutions to P2, if they exist.

<u>Theorem II</u>: If an optimal solution to P2 exists, it is given by $(x, y) = (B^{-1}(b - Ny^{(r*)}), y^{(r*)})$, where $r*$ is the smallest $r$ such that $B^{-1}(b - Ny^{(r)}) \geq 0$.

<u>Proof</u>:

If $(\hat{x}, \hat{y})$ is an optimal solution to P2, then $(\hat{x}, \hat{y})$ is a lattice point of C with $\hat{x} \geq 0$. By Theorem I, $\hat{y}$ is a feasible solution of P3. Thus $\hat{y} = y^{(r*)}$, since all lattice points of C must have their y-components solving P3, and the value of y which gives the smallest value of $\varphi(b)$ such that $B^{-1}(b - Ny) \geq 0$ is $y^{(r*)}$.

Theorem II implies that if there is an optimal solution to P2, it can be determined by looking successively at the "next best" solutions to P3, until we find the first solution $\hat{y}$ for which $B^{-1}(b - N\hat{y}) \geq 0$. A procedure for attaining this end is given in Chapter III.

The following theorems and corollary establish some properties of solutions to problem P3. Two of the proofs are lengthy and will be deferred to the Appendices. It is hoped, however, that these proofs may lend some insight into problem P3.

<u>Theorem III</u>: There is an $r^{th}$ optimal solution[1] to problem P3, unless $c^*_{i+m} = 0$ for all i, $1 \le i \le n$.

<u>Proof</u>:

Since the original matrix A of problem P1 is of the form $[A', I]$ and b is integral, there is a solution to P1 in which the variables take on integral, though not necessarily non-negative, values. The components of this solution corresponding to the components of y satisfy the congruence relations. Since the Gomory group F is of order D, an arbitrary number of feasible solutions to P3 can be generated merely by adding sufficiently large integer multiples of D to the components of y. Thus, if $c^*_{i=m} \ne 0$ for some i, $1 \le i \le n$, an arbitrary (but countable) number of feasible solutions to P3 can be generated, each giving a different value for $\varphi(b)$.

Theorem IV:[2] If $c^*_{i+m} \ne 0$ for some i, $1 \le i \le n$, then there is an

(1) If one solution is $r^{th}$ optimal, all other solutions with the same objective function value are also $r^{th}$ optimal.

(2) In a verbal communication with the author on 27 December 1965, R.E. Gomory asserted the validity of this theorem for $r = 1$ and stated that a proof could be obtained along the lines presented in the Appendix. The proof for general r is a generalization of Gomory's suggestions.

$r^{th}$ optimal solution to P3, say y, with $\prod_{i=1}^{n} (y_i + 1) \le rD$.

Proof: See Appendix I.

Corollary I:(1) If $c^*_{i+m} \ne 0$ for some i, $1 \le i \le n$, then there is an $r^{th}$ optimal solution to P3, say y, with $\sum_{i=1}^{n} y_i \le rD - 1$.

Proof:

Let y be an $r^{th}$ optimal solution for which Theorem IV holds. Then $rD \ge \prod_{i=1}^{n} (y_i + 1) \ge 1 + \sum_{i=1}^{n} y_i$.

Theorem V: Let $I = \{i \mid c^*_{i+m} = 0, 1 \le i \le n\}$. Then any $r^{th}$ optimal solution to problem P3, say y, satisfies $\sum_{i \notin I} y_i \le rD - 1$.

Proof: See Appendix I.

Theorem III assures the existence of a sequence of solutions to problem P3, provided that there is a non-trivial objective function for problem P1. Theorem IV implies that, to find an $r^{th}$ optimal solution to P3, it is sufficient to examine only those solutions y for which $\prod_{i=1}^{n} (y_i + 1) \le rD$. If, however, it is necessary to determine all $r^{th}$ optimal solutions, Theorem V implies that it is necessary to

(1) Again, this was suggested by Gomory for $r = 1$. Corollary I was actually proven for $r = 1$ by Gomory in [8], using the underlying group structure of problem P3. The proof of Theorem V is a modification and generalization of Gomory's proof for Corollary I.

consider a much larger class of solutions. In particular, if $I \neq \Phi$, there will be an infinite number of $r^{th}$ optimal solutions for every r. In this case it is necessary to assume that, for each $i \in I$, there is an upper bound $M_i$, such that $y_i \leq M_i$. For the computational scheme developed in section III, it will be necessary to assume that such bounds exist, either explicitly or implicitly.

netrically, these theorems describe some interesting properties of integer programming problems. Consider the set of solutions $C_y$ to problem P3. $C_y$ is a set of lattice points in the non-negative orthant of n-space, since $C_y = \{y \mid \sum_{i=1}^{n} \bar{\alpha}_{i+m} y_i \equiv \bar{b} \pmod 1$, $y \geq 0$, y integral$\}$. Now the cone $C_x$ (as defined above, $C_x = \{x \mid x = B^{-1}(b - Ny),\ y \geq 0\}$) is a linear transformation of the non-negative orthant of n-space into m-space, and any point in this non-negative orthant is mapped onto a single point in $C_x$. In view of Theorem I, a lattice point of $C_y$ is therefore mapped onto a lattice point of $C_x$. It follows that the convex hull of $C_y$, $\hat{C}_y$, is mapped onto the convex hull $\hat{C}_x$ of the lattice points of $C_x$. Note that the direct sum of $\hat{C}_y$ and $\hat{C}_x$ is the convex hull of the lattice points of C, where $C = \{(x, y) \mid Bx + Ny = b,\ y \geq 0\}$ as defined previously.

Consider an arbitrary objective function for problem P3, subject to the restriction that the coefficients are non-negative. These coefficients can be chosen so that there is a unique minimizing solution to P3. This solution must be an extreme point of $\hat{C}_y$. Furthermore, by an appropriate choice of coefficients in the objective function, each extreme point of $\hat{C}_y$ can be obtained as a unique solution to P3. From Theorem IV it follows that, if y is an extreme point of $\hat{C}_y$, then $\prod_{i=1}^{n} (y_i + 1) \leq D$. In other words, every extreme point of $\hat{C}_y$ lies on a ray from the origin to the hyperbola $\prod_{i=1}^{n} (y_i+1)=D$ in n-space. By virtue of the mapping described in the previous paragraph, these results carry over to $C_x$ and $\hat{C}_x$: every extreme point of $\hat{C}_x$ lies within a "transformed" hyperbola's distance of the center, $B^{-1}b$, of the cone $C_x$.

In Chapter I, a condition was given which Gomory had shown to be sufficient for an optimal solution of P3 to generate an optimal solution to P2. The theorem due to Gomory from which this condition was derived is the same as Corollary I above for r = 1. A similar sufficient condition can now be developed here. Define $\ell'$ to be an m-vector whose $k^{th}$ component is equal to the largest entry in row k of the matrix $B^{-1}N$. In view of Corollary I, it follows that $B^{-1}Ny \leq \ell' \sum_{i=1}^{n} y_i \leq \ell'(D-1)$, for some optimal solution to problem P3. If $B^{-1}b \geq \ell'(D-1)$ then the optimal solution to problem P3 generates an optimal solution to P2.

Geometrically, this condition assures that the non-negativity restrictions, which problem P2 imposes on the cone $C_x$, lie outside the "transformed" hyperbola for the cone $C_x$. Thus in this case every extreme point of the convex hull $\hat{C}_y$ generates a feasible solution to problem P2.

The following theorem, which completes Chapter II, provides for the finiteness of the algorithm proposed in the next Chapter.

Theorem VI: Let $I = \{i \mid c^*_{i+m} = 0,\ 1 \le i \le n\}$. Let $y_i \le M_i < \infty$ for all $i \in I$. Let there be a finite feasible solution to P2. Then an optimal solution to P2 can be obtained in finite number of steps, provided that the algorithm for obtaining $\varphi^{(r)}(b)$ is finite.

Proof:

Since there is a finite feasible solution to P2, there is a finite optimal solution to P2, which, by Theorem II, can be generated by $y^{(r)}$, for some $r$, where $y^{(r)}$ is an $r^{th}$ optimal solution to P3, and $y^{(r)}$ is finite.

Since $y^{(r)}$ is finite, $\varphi^{(r)}(b)$ is finite. For $k < r$, $\varphi^{(k)}(b)$ strictly increases as $k$ increases, and thus can do so only finitely often.

By Theorem V and the fact that $y_i \le M_i < \infty$ for $i \in I$, there are only a finite number of solutions to examine for each value of the objective function.

The theorem follows, provided that the algorithm for obtaining $\varphi^{(r)}(b)$ is finite.

# CHAPTER III

## AN ALGORITHM FOR INTEGER PROGRAMS

As stated in Chapter II, an algorithm for finding $r^{th}$ best solutions to problem P3 could form the basis of a procedure for finding solutions to the integer programming problem P2. This chapter will present an algorithm for accomplishing this end. The approach used is a straightforward extension of the dynamic programming method proposed by Gomory [8], and can be interpreted as finding an $r^{th}$ shortest route through an appropriately constructed network. The recursion relation is fairly simple in form,[1] and although the corresponding network contains cycles, the computation is not mathematically involved. As with all $r^{th}$ shortest route algorithms, the necessary bookkeeping grows quite large as $r$ increases.

For $\bar{p}$ in the Gomory column group $F$, define the following problem as problem P3$(s, \bar{p})$:

$$\text{Minimize} \quad \varphi_s(\bar{p}) = \sum_{i=1}^{s} c^*_{i+m} y_i$$

$$\text{subject to} \quad \sum_{i=1}^{s} \bar{a}_{i+m} y_i \equiv \bar{p} \pmod 1, \quad y \geq 0, \quad y \text{ integer.}$$

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(1) The author is indebted to F. Glover for suggesting the present form of the recursion, and for sketching a proof of its validity.

Let $\varphi_s^{(r)}(\bar{p})$ be the $r^{th}$ smallest value of $\varphi_s(\bar{p})$. Note that P3(n, $\bar{b}$) is the same as problem P3, and that $\varphi_n^{(r)}(\bar{b}) = \varphi^{(r)}(\bar{b})$. Let $\bar{a}_{i+m}$ be of order $\bar{d}_i$ for $1 \le i \le n$, where, of course, $\bar{d}_i | D$(1). Denote by $\min_{(k)}\{\cdot\}$ the $k^{th}$ smallest term within the brackets, and take $\min_{(k)}\{\cdot\} = +\infty$ if there is no such term.

Solutions to problem P3(s, $\bar{p}$) can be obtained recursively. Since $c^*_{i+m} \ge 0$ for $i = 1, \ldots, n$, it is immediately obvious that

$$\varphi_s^{(1)}(\bar{0}) = 0 \qquad \text{for all } s$$

and that

$$\varphi_1^{(r)}(\bar{0}) = (r-1) \cdot c^*_{1+m} \cdot \bar{d}, \qquad \text{for all } r,$$

the latter provided that $c^*_{1+m} > 0$. If $c^*_{1+m} = 0$, define $\varphi_1^{(r)}(\bar{0}) = +\infty$ for $r \ge 2$. Define $\varphi_s^{(0)}(\bar{p}) = 0$. Recursively,

$$\varphi_s^{(r)}(\bar{0}) = \min_{2 \le k \le r} {}_{(r-1)} \{\varphi_{s-1}^{(k)}(\bar{0}),\ \varphi_s^{(k-1)}((\bar{d}_s - 1) \cdot \bar{a}_{s+m}) + c^*_{s+m}\},$$

for $r, s \ge 2$.

The next relation is also obvious: when $t$ is integral,

$$\varphi_1^{(r)}(t \cdot \bar{a}_{1+m}) = t \cdot c^*_{1+m} + \varphi_1^{(r)}(\bar{0}), \quad \text{for } r \ge 1,\ 1 \le t \le \bar{d}_1 - 1.$$

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(1) $a|b$, read "a divides b", means that b is an integral multiple of a. Here $\bar{d}$ is the order of $\bar{p}$ if $\bar{d}$ is the smallest positive integer such that $\bar{d} \cdot \bar{p} = 0$, for $\bar{p} \in F$.

If $\bar{d}_1 \neq D$, there exist elements $\bar{p}' \in F$ for which there is no integral $t$ such that $\bar{p}' = t \cdot \bar{a}_{1+m}$. For such elements $\bar{p}'$, take $\varphi_1^{(r)}(\bar{p}') = +\infty$ for all $r \geq 1$.

The general recursion relations will be stated next. Proofs that the relations are correct will be deferred until later. For $t$ integral,

$$\varphi_s^{(r)}(t \cdot \bar{a}_{s+m}) = \min_{1 \leq k \leq r} {}^{(r)}\{\varphi_{s-1}^{(k)}(t \cdot \bar{a}_{s+m}),\ \varphi_s^{(k)}(t - 1) \cdot \bar{a}_{s+m}) + c^*_{s+m}\}$$

for $r \geq 1$, $s \geq 2$, and $1 \leq t \leq \bar{d}_s - 1$. Again, if $\bar{d}_s \neq D$, there exist elements $\bar{p}' \in F$ for which there is no integral $t$ such that $\bar{p}' = t \cdot \bar{a}_{s+m}$. Let $P_s$ be the set of all such $\bar{p}' \in F$. Define $\bar{p}^* \in F$ by

$$\varphi_s^{(1)}(\bar{p}^*) = \min_{\bar{p} \in P_s} \{\varphi_{s-1}^{(1)}(\bar{p})\} \qquad \text{for} \quad s \geq 2.$$

Then

$$\varphi_s^{(r)}(\bar{p}^*) = \min_{2 \leq k \leq r} {}^{(r-1)}\{\varphi_{s-1}^{(k)}(\bar{p}^*),\ \varphi_s^{(k)}(\bar{p}^* + (t - 1) \cdot \bar{a}_{s+m}) + c^*_{s+m}\}$$

for $r, s \geq 2$. The following relation now holds for integral $t$:

$$\varphi_s^{(r)}(\bar{p}^* + t \cdot \bar{a}_{s+m}) = \min_{1 \leq k \leq r} {}^{(r)}\{\varphi_{s-1}^{(k)}(\bar{p}^* + t \cdot \bar{a}_{s+m}),$$

$$\varphi_s^{(k)}(\bar{p}^* + (t - 1) \cdot \bar{a}_{s+m}) + c^*_{s+m}\}$$

for $r \geq 1$, $s \geq 2$, $1 \leq t \leq \bar{d}_s - 1$. It may still be possible that there exist $\bar{p}' \epsilon P_s$ for which there is no $t$ such that $\bar{p}^* + t \cdot \bar{a}_{s+m} = \bar{p}'$. Then, letting $P_s^*$ be the set of such $\bar{p}' \epsilon P_s$, one can define $p^{**} \epsilon P_s^*$ similarly to $\bar{p}^*$ and continue. From these relations, one can determine $\varphi_s^{(r)}(\bar{p})$ for all $\bar{p} \epsilon F$. Thus, by using these formulas, $\varphi^{(r)}(\bar{b})$ can be obtained, since the formulas determine recursively $\varphi^{(r)}(\bar{p})$ for all $\bar{p} \epsilon F$.

Consider the case $r = 1$. The following discussion should help to illustrate the basic concepts used in obtaining the above relations. Suppose that $\varphi_s^{(1)}(\bar{p})$ is known for all $\bar{p} \epsilon F$ and $s < s^*$. Since $c_{i+m}^* \geq 0$ for all $i$, it is obvious that a minimizing solution for problem $P3(s^*, \bar{0})$ is $y_1 = y_2 = \ldots = y_{s^*} = 0$, and thus $\varphi_{s^*}^{(1)}(\bar{0}) = 0$. Note that if, say, $c_{1+m}^* = 0$, $y_1$ can be any non-negative integral multiple of $\bar{d}_1$ (since $\bar{d}_1$ annihilates $\bar{a}_{i+m}$). Thus in this case there are countably many minimizing solutions to $P3(s^*, \bar{0})$.

Now, assume that problem $P3(s^*, t \cdot \bar{a}_{s^*+m})$ has been solved for all $t < t^*$, where $1 < t^* < \bar{d}_{s^*}$. Thus $\varphi_{s^*}^{(1)}(t \cdot \bar{a}_{s^*+m})$ has been obtained for $t < t^*$. Let a value of $y_{s^*}$ in problem $P3(s^*, t \cdot \bar{a}_{s^*+m})$ be $y_{s^*}(t)$, for $t < t^*$, and consider problem $P3(s^*, t^* \cdot \bar{a}_{s^*+m})$. One way to obtain a feasible solution for this problem is to use an optimal solution to problem $P3(s^* - 1, t^* \cdot \bar{a}_{s^*+m})$ and take $y_{s^*}(t^*) = 0$. The cost associated with this solution is

$$\varphi^{(1)}_{s^*-1}(t^* \cdot \bar{a}_{s^*+m}) + c^*_{s^*+m} \cdot y_{s^*}(t^*) = \varphi^{(1)}_{s^*-1}(t^* \cdot \bar{a}_{s^*+m}) .$$

Another way to obtain a feasible solution for this problem is to use an optimal solution to problem $P3(s^*, (t^*-1) \cdot \bar{a}_{s^*+m})$ and take $y_{s^*} = y_{s^*}(t^*) + 1$. The cost associated with this solution is

$$\varphi^{(1)}_{s^*}((t^*-1) \cdot \bar{a}_{s^*+m}) + c^*_{s^*+m} \cdot (y_{s^*}(t^*) - y_{s^*}(t^*-1))$$

$$= \varphi^{(1)}_{s^*}((t^*-1) \cdot \bar{a}_{s^*+m}) + c^*_{s^*+m} .$$

When it is later shown that the $\varphi$'s as defined in the previous paragraphs are actually the correct values, it will also be shown that any other way of obtaining a feasible solution to problem $P3(s^*, t^* \cdot \bar{a}_{s^*+m})$ will give a corresponding cost which is at least as large as one of the two costs given above (this is the "Principle of Optimality" inherent in dynamic programming algorithms). Thus the solution which gives the optimal $\varphi^{(1)}_{s^*}(t^* \cdot \bar{a}_{s^*+m})$ is just that one corresponding to the minimum of the above two expressions. The calculations for $\varphi^{(r)}_{s}(t \cdot \bar{a}_{s+m})$ are direct generalizations of the above reasoning.

If there is a $\bar{p} \in F$ for which there is no $t$ such that $\bar{p} = t \cdot \bar{a}_{s+m}$, the recursion for problem $P3(s, \bar{p}^*)$ can be interpreted as a "displacement" of problem $P3(s, \bar{0})$ by $\bar{p}^*$ "units". $\bar{p}^*$ is determined so that the "displacement" has minimal cost of all such possible "displacements." Again, there is a direct generalization to the $r^{th}$

optimal problem, which accounts for the similarity between the expressions for $\varphi_s^{(r)}(\bar{0})$ and $\varphi_s^{(r)}(\bar{p}*)$, and between the expressions for $\varphi_s^{(r)}(\bar{0})$ and $\varphi_s^{(r)}(\bar{p}*)$, and between the expressions for $\varphi_s^{(r)}(t \cdot \bar{a}_{s+m})$ and $\varphi_s^{(r)}(\bar{p}* + t \cdot \bar{a}_{s+m})$.

From what has been stated above, one can give problem P3 a network interpretation. Construct a network as follows:

(1) Place the nodes in a rectangular array, D nodes in each column and n nodes in each row.

(2) Identify each node by a number $j_i$, where j corresponds to the $j^{th}$ group element and i corresponds to the $i^{th}$ variable. Let the $j^{th}$ group element be $\bar{p}_j$ and take $\bar{p}_0 = \bar{0}$, $\bar{p}_{j*} = \bar{b}$.

(3) Draw in directed arcs from node $j_{i-1}$ to node $j_i$, and put a "cost" of zero on each such arc.

(4) Draw in directed arcs from node $k_i$ to node $j_i$ if and only if $\bar{p}_k + \bar{a}_{i+m} = \bar{p}_j$. On each such arc put a "cost" equal to $c^*_{i+m}$.

(5) Identify the source as node $0_1$ and the sink as node $j^*_n$.

The resulting network is directed, has cycles, and has non-negative costs on each arc. With the exception of the nodes in the first and last columns, each node is incident to exactly four arcs, two directed into the node and two directed out from the node. The nodes in the first column have only one inner-directed arc, while the nodes in the last column have only one outer-directed arc.

Let the vector $(\varphi_s^{(1)}(\bar{p}_j),\ \varphi_s^{(2)}(\bar{p}_j),\ ---)$ be associated with node $j_s$. Then, assuming the recursion formulas are correct, $\varphi_s^{(r)}(\bar{p}_j)$ is the length of an $r^{th}$ shortest route from the source to node $j_s$. Thus an $r^{th}$ optimal solution to problem P3 corresponds to an $r^{th}$ shortest route through this network. The recursions described are precisely the recursions of the Bellman-Kalaba algorithm for the $r^{th}$ shortest route through a directed network, where the routes may include loops. Assuming that this network interpretation is correct, any $r^{th}$ shortest route algorithm which allows routes to have loops will generate a solution to problem P3.

To determine from the network the values for the variables of problem P3, one looks at the arcs constituting a route and sets $y_i$ equal to the number of arcs of the form $(k_i,\ j_i)$ present on the route. The theorems of Chapter II carry over to the network. In particular, by Corollary I, there is always an $r^{th}$ shortest route, provided that $c^*_{i+m} > 0$ for some $i$, which has at most $r-1$ loops.

Define $N(s, \bar{p}_j)$ to be that network constructed by the rules given above, except that the sink is taken to be node $j_s$. The next theorem, which is the basic theorem of this chapter, will only be stated here, since the proof is lengthy.

Theorem VII: The formulas defining $\varphi$ given above are the correct values for solving problem P3(s, $\bar{p}$) for all s, $1 \le s \le n$, and all $\bar{p} \in F$. Furthermore, problem P3(s, $\bar{p}_j$) is equivalent to problem N(s, $\bar{p}_j$) for all s and j.

Proof: See Appendix II.

There are certain simplifications which can be made in the network, and thus in the corresponding recursions for P3, if the $\bar{\alpha}_{i+m}$'s have special properties. Suppose that $\bar{d}_1 \neq D$. Then there are elements $\bar{p}_j$ in the Gomory column group F for which there are no numbers t for which $\bar{p}_j = t \cdot \bar{\alpha}_{1+m}$. These are the $\bar{p}_j$ for which $\varphi_1^{(1)}(\bar{p}_j) = +\infty$. However, as can be seen from the associated network, these nodes can never be reached starting from the source. Thus all these nodes $j_1$ can be dropped from the network, along with the corresponding arcs. Similarly, if it is also true that $\bar{d}_2 \neq D$, all nodes $j_2$ together with their associated arcs can be dropped from consideration, where $\bar{p}_j$ is any node for which there are no numbers t, t' such that $\bar{p}_j = t \cdot \bar{\alpha}_{1+m} + t' \cdot \bar{\alpha}_{2+m}$. This process can be repeated for all s, $1 \le s \le n$, and can easily be incorporated into the calculations.

It is unnecessary to construct the network when solving problem **P3**, for it is constructed implicitly during the calculations for r = 1. The following is a brief outline of the procedure for finding $\varphi^{(1)}(\bar{b})$:

(1) Set $\varphi_1^{(1)}(t \cdot \bar{a}_{1+m}) = t \cdot c^*_{1+m}$ for $t = 0, 1, \ldots, \bar{d}_1 - 1$, and put $t \cdot \bar{a}_{1+m}$ in the set $P_1$, for each $t$.

(2) Given the values $\varphi_s^{(1)}(\bar{p})$ for all $\bar{p} \epsilon P_s$, set $P_{s+1} = \varphi$.

(3) Find a $\bar{p}^*$ and a $\varphi_{s+1}^{(1)}(\bar{p}^*)$ by $\varphi_{s+1}^{(1)}(\bar{p}^*) = \min_{\bar{p} \epsilon P_s, \bar{p} \notin P_{s+1}} \{\varphi_s^{(1)}(\bar{p})\}$.

(4) Find $\varphi_{s+1}^{(1)}(\bar{p}^* + t \cdot \bar{a}_{s+1+m})$ for $t = 1, 2, \ldots, \bar{d}_{s+1} - 1$ by

$$\varphi_{s+1}^{(1)}(\bar{p}^* + t \cdot \bar{a}_{s+1+m}) = \min\{\varphi_s^{(1)}(\bar{p}^* + t \cdot \bar{a}_{s+1+m}),$$

$$\varphi_{s+1}^{(1)}(\bar{p}^* + (t - 1) \cdot \bar{a}_{s+1+m}) + c^*_{s+1+m}\}$$

where $\varphi_s^{(1)}(\bar{p}^* + t \cdot \bar{a}_{s+1+m}) = +\infty$ if $\bar{p}^* + t \cdot \bar{a}_{s+1+m} \notin P_s$.

Put $\bar{p}^* + t \cdot \bar{a}_{s+1+m}$ in the set $P_{s+1}$.

(5) If $P_{s+1} \supseteq P_s$ and $s + 1 = n$, $\varphi^{(1)}(\bar{b}) = \varphi_{s+1}^{(1)}(\bar{b})$.

If $P_{s+1} \supseteq P_s$ and $s + 1 < n$, increase $s$ by 1 and return to (2). Otherwise return to (3).

During this procedure the network has been implicitly constructed and a set of elements $\bar{p}^*$ has been determined for each $s$. When determining $\varphi^{(r)}(b)$ for arbitrary $r$, these same $\bar{p}^*$ are used. It is also unnecessary to determine the $\bar{d}_s$ ahead of time. As one increases $t$ (in steps (1) and (4)) one eventually finds a $t > 0$ for which $t \cdot \bar{a}_{s+m} = \bar{0}$. This $t$ is $\bar{d}_s$.

Assuming that $\varphi^{(r)}(\bar{b})$ has been determined for all $r < r^*$, the following is a brief outline for finding $\varphi^{(r^*)}(\bar{b})$:

(1) For $\bar{p} \epsilon P_1$, set $\varphi_1^{(r*)}(\bar{p}) = +\infty$ if $c^*_{1+m} = 0$, and set

$$\varphi_1^{(r*)}(\bar{p}*) = \varphi_1^{(r*-1)}(\bar{p}) + c^*_{1+m} \cdot \bar{d}_1 \text{ if } c^*_{1+m} > 0.$$

(2) For each $\bar{p}* \epsilon P_s$, where the $\bar{p}*$ were determined when $r = 1$, set

$$\varphi_s^{(r*)}(\bar{p}*) = \min_{2 \le k \le r*}{}^{(r*-1)}\{\varphi_{s-1}^{(k)}(\bar{p}* - \bar{a}_{s+m}) + c^*_{s+m}\}.$$

(3) For $t = 1, 2, \ldots, \bar{d}_s - 1$, and for each $\bar{p}* \epsilon P_s$, set

$$\varphi_s^{(r*)}(\bar{p}* + t \cdot \bar{a}_{s+m}) = \min_{1 \le k \le r*}{}^{(r*)}\{\varphi_{s-1}^{(k)}(\bar{p}* + t \cdot \bar{a}_{s+m}),$$

$$\varphi_s^{(k)}(\bar{p}* + (t-1) \cdot \bar{a}_{s+m}) + c^*_{s+m}\}.$$

(4) If $s = n$, $\varphi^{(r*)}(\bar{b}) = \varphi_s^{(r*)}(\bar{b})$. Otherwise increase $s$ by 1 and return to (2).

As is evident from these steps, this procedure relies on much of the information generated when $r = 1$.

The main problem with $r^{th}$ shortest route a gorithms is the amount of record keeping required. A vector of $r$ components must be recorded for each node to determine the length of a single $r^{th}$ shortest route. However, the formulas can be simplified for ease in calculating the $\varphi$'s, so that only one comparison is necessary for each $\varphi$ value. Partition the argument of $\min_{(r)}\{\cdot\}$ into two sets, the first of which contains the $r - 1$ smallest terms which have already been determined. Partition the second set into two sets, $s_1$ and $s_2$, with $s_1$ containing terms which arise from $\varphi_{s-1}^{(k)}(\bar{p})$ and $s_2$ containing terms which arise from $\varphi_s^{(k)}(\bar{p} - \bar{a}_{s+m}) + c^*_{s+m}$. Since the $\varphi$'s are monotonic non-decreasing in r at each node, it is sufficient to determine $\min_{(r)} \{\cdot\}$ by finding min{first element in $s_1$, first element in $s_2$}. Thus the computation of the $\varphi$'s should be fairly rapid.

In order to determine the solution y that corresponds to an $r^{th}$ shortest route of length $\varphi^{(r)}(\bar{b})$, one may associate a pair $(\bar{r}, \delta)$ with each value $\varphi_s^{(r)}(\bar{p})$. The number $\bar{r}$ tells which route gave rise to $\varphi_s^{(r)}(\bar{p})$, while $\delta = 1$ if that route was via $\varphi_s^{(\bar{r})}(\bar{p} - \bar{a}_{s+m})$ and $\delta = 0$ if that route was via $\varphi_{s-1}^{(\bar{r})}(\bar{p})$. The y levels are then determined as follows:

(1) Assume all y's are initially zero.

(2) Proceeding recursively from $\varphi_n^{(r)}(\bar{b})$, if the pair associated with $\varphi_s^{(k)}(\bar{p})$ is $(\bar{k}, \delta)$, increase $y_s$ by $\delta$ and consider $\varphi_{s+1-\delta}^{(\bar{k})}(\bar{p} - \delta \cdot \bar{a}_{s+m})$ next.

Appropriate adjustments must of course be made when s = 1. The values of s and $\bar{r}$ are monotonically non-increasing during this backward pass, so that this look-up procedure does not jump around too badly.

The major difficulty in this procedure lies in the existence of ties that occur when there is more than one route with the same length.

As $r$ increases, the number of ties usually increases. Since all $r^{th}$ best solutions to problem P3 must be known when generating solutions to problem P2, enough information must be recorded so that the backward pass recovers all $r^{th}$ best solutions.

Given the results which have been obtained in Chapters II and III, an algorithm for the determination of integer solutions to linear programs can now be stated:

(1) Determine an optimal solution to problem P1.

(2) Find, for each $i \, (1 \le i \le n)$, $\bar{a}_{i+m}$ from $B^{-1} a_{i+m} \equiv \bar{a}_{i+m} \pmod 1$.

(3) Starting initially with $r = 1$, find all $r^{th}$ best solutions to problem P3. If any solution $y$ satisfies $B^{-1}(b - Ny) \ge 0$, the vector $(x, y) = (B^{-1}(b - Ny), y)$ is an optimal solution to problem P2, with an objective function value of $-c_B B^{-1} b + \varphi^{(r)}(\bar{b})$. If there are no such solutions $y$, increase $r$ by 1 and repeat this step.

The conditions under which this algorithm is finite are given in Theorem VI.

Example:

$$
\begin{aligned}
\text{minimize} \quad & z = -x_2 \\
\text{subject to} \quad & -4x_1 + 4x_2 \le 3 \\
& 2x_1 - x_2 \le 2 \\
& x_1 + x_2 \le 6 \\
& x_1, x_2 \text{ non-negative integers.}
\end{aligned}
$$

Upon addition of slack variables, problem P2 is expressed in the required form. The initial tableau is

| z | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | 1 |
|---|---|---|---|---|---|---|
| 0 | -4 | 4 | 1 | 0 | 0 | 3 |
| 0 | 2 | -1 | 0 | 1 | 0 | 2 |
| 0 | 1 | 1 | 0 | 0 | 1 | 6 |
| -1 | 0 | -1 | 0 | 0 | 0 | 0 |

The optimal solution to problem P1 is given by the final tableau:

| z | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | 1 |
|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 1/8 | 0 | 4/8 | 27/8 |
| 0 | 0 | 0 | 3/8 | 1 | -4/8 | 1/8 |
| 0 | 1 | 0 | -1/8 | 0 | 4/8 | 21/8 |
| -1 | 0 | 0 | 1/8 | 0 | 4/8 | 27/8 |

From the final tableau, it may be seen that the optimal basis B has columns corresponding to $x_1$, $x_2$ and $x_4$, and $|\det B| = D = 8$. Letting $y_1 = x_3$ and $y_2 = x_5$, it follows that $\bar{\alpha}_{1+m} = (1/8,\ 3/8,\ 7/8)^T$ and $\bar{\alpha}_{2+m} = (4/8,\ 4/8,\ 4/8)$. The group F in this case has eight elements: $g_0 = (0, 0, 0)^T$, $g_1 = (1/8, 3/8, 7/8)^T$, $g_2 = (2/8, 6/8, 6/8)^T$, $g_3 = (3/8, 1/8, 5/8)^T$, $g_4 = (4/8, 4/8, 4/8)^T$, $g_5 = (5/8, 7/8, 3/8)^T$, $g_6 = (6/8, 2/8, 2/8)^T$, $g_7 = (7/8, 5/8, 1/8)^T$. Note that $\bar{a}_{1+m} = g_1$ and $\bar{a}_{2+m} = g_4$ and that $g_1$ generates the whole group, whereas $g_4$ generates a subgroup of two elements.

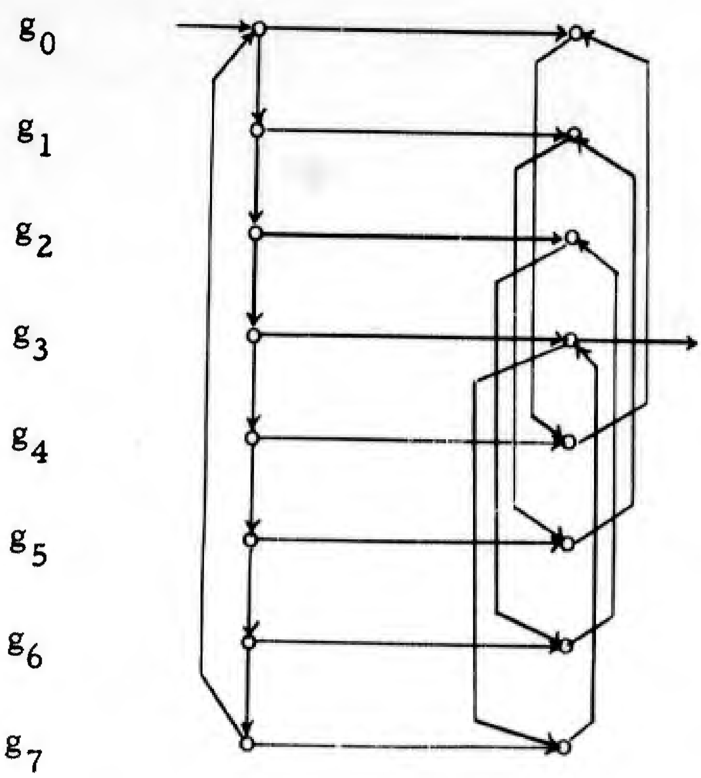
Problem P3 is thus

$$\text{minimize} \quad 1/8\, y_1 + 4/8\, y_2$$

$$\text{subject to} \quad \bar{a}_{1+m} y_1 + \bar{a}_{2+m} y_2 \equiv \bar{b}$$

$$y_1,\ y_2 \text{ non-negative integers}$$

where $\bar{b} = g_3$. For clarity, the corresponding network is shown below.



For $r = 1$, the $\varphi$ values are given by

| | $g_0$ | $g_1$ | $g_2$ | $g_3$ | $g_4$ | $g_5$ | $g_6$ | $g_7$ |
|---|---|---|---|---|---|---|---|---|
| s = 1 | 0 | 1/8 | 2/8 | 3/8 | 4/8 | 5/8 | 6/8 | 7/8 |
| s = 2 | 0 | 1/8 | 2/8 | 3/8 | 4/8 | 5/8 | 6/8 | 7/8 |

and, for $\bar{b} = g_3$, the first optimal solution is $y_1^{(1)} = 3$, $y_2^{(1)} = 0$, $\varphi^{(1)}(\bar{b}) = 3/8$. (Note that if $\bar{b} = g_4$, there would be two optimal solutions). Now, $B^{-1}b - B^{-1}Ny^{(1)} = (3, -1, 3)^T$, so $y^{(1)}$ does not generate a feasible solution to P2.

For $r = 2$, the following tableau gives us the $\varphi$ values:

$\bar{p} =$

| | $g_0$ | $g_1$ | $g_2$ | $g_3$ | $g_4$ | $g_5$ | $g_6$ | $g_7$ |
|---|---|---|---|---|---|---|---|---|
| s = 1 | 8/8 | 9/8 | 10/8 | 11/8 | 12/8 | 13/8 | 14/8 | 15/8 |
| s = 2 | 8/8 | 9/8 | 10/8 | 11/8 | 12/8 | 13/8 | 14/8 | 15/8 |

Here, however, there are three $2^{nd}$ optimal solutions:

$y_1^{(2a)} = 11$, $y_2^{(2a)} = 0$; $y_1^{(2b)} = 7$, $y_2^{(2b)} = 1$; $y_1^{(2c)} = 3$, $y_2^{(2c)} = 2$,

each giving $\varphi^{(2)}(\bar{b}) = 11/8$. Now, examining each solution,

$$B^{-1}b - B^{-1}Ny^{(2a)} = (2, -4, 4)^T$$

$$B^{-1}b - B^{-1}Ny^{(2b)} = (2, -2, 3)^T$$

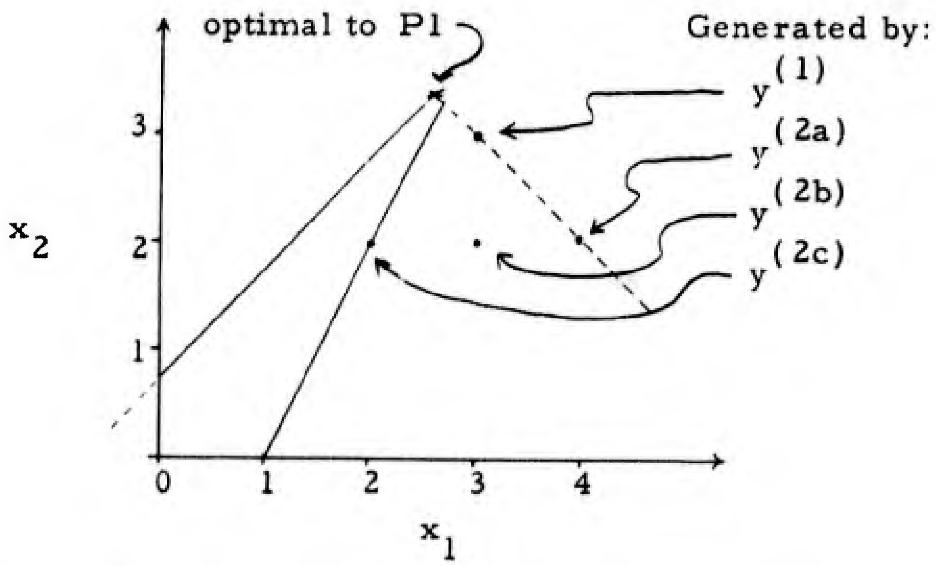
$$B^{-1}b - B^{-1}Ny^{(2c)} = (2, 0, 2)^T$$

Thus solution $y^{(2c)}$ to the $2^{nd}$ optimal of P3 generates the optimal solution to P2:

$$x_1 = 2, \; x_2 = 0, \; x_3 = 3, \; x_4 = 2, \; x_5 = 2$$

with a cost of $z = -27/8 + Q^{(2)}(\bar{b}) = -2$.

Geometrically, the problem is pictured below (note that $y^{(1)}$ generates the optimal solution to P2 in the absence of the constraint $2x_1 - x_2 \le 2$, which lies interior to the cone given by the binding constraints on P1):




The cost function is parallel to the horizontal axis.

# CHAPTER IV

## ON THE RELATION OF PROBLEM P3 TO THE METHOD OF INTEGER FORMS

A geometric interpretation of problem P3 was given in Chapter II. It was pointed out that the set $C_y$ of solutions to problem P3 was a set of lattice points, $C_y = \{y \mid \sum_{i=1}^{n} \bar{a}_{i+m} y_i \equiv \bar{b} \pmod 1),\ y \geq 0,$ y integral$\}$, imbedded in the non-negative orthant of n-space. It was further remarked that, by an appropriate choice of objective function for P3, any extreme point of the convex hull $\hat{C}_y$ of $C_y$ could be obtained as a minimizing solution of problem P3. In this section, the bounding hyperplanes of $\hat{C}_y$ will be described. Due to the close relationship of these hyperplanes to Gomory cuts, a number of results will be obtained that help to characterize the behavior of Gomory's Method of Integer Forms [9].

In the context of this chapter, a <u>supporting hyperplane</u> to $\hat{C}_y$ is hyperplane in n-space such that $\hat{C}_y$ lies in one of its halfspaces and at least one point of $\hat{C}_y$ lies on the hyperplane. A <u>bounding hyperplane</u> is a supporting hyperplane with n linearly independent points of $\hat{C}_y$ incident to the hyperplane. Given any feasible solution to P3, another feasible solution can be constructed by adding a non-negative integral multiple of D to any component of the given solution. Thus $\hat{C}_y$ is unbounded, and attention will be restricted to bounding hyperplanes with finite intercepts.

Any linear form (in n variables) can be minimized with respect to the constraints of problem P3. Any linear form whose minimum is finite, together with its minimum value, is a supporting hyperplane for P3. Let $\sum_{i=1}^{n} \bar{c}_{i+m} y_i = \bar{z}$ be such a supporting hyperplane. Since $\hat{C}_y$ is unbounded, it follows that $\bar{c}_{i+m} \geq 0$ for all $1 \leq i \leq n$, and hence $\bar{z} \geq 0$. In fact, all supporting hyperplanes, and thus all bounding hyperplanes, can be written with non-negative coefficients and a non-negative right hand side. Note that, if there is a feasible solution to problem P3 with some component, say the $k^{th}$ component, equal to zero, then $y_k = 0$ is a bounding hyperplane. Those bounding hyperplanes with a right hand side of zero will be called trivial bounding hyperplanes.

Let $\sum_{i=1}^{n} \bar{c}_{i+m} y_i = \bar{z}$ be a supporting hyperplane. It follows that any point of $C_y$ (any lattice point of $\hat{C}_y$) must satisfy $\sum_{i=1}^{n} \bar{c}_{i+m} y_i \geq \bar{z}$. Furthermore, the y-components of any solution to the integer programming problem P2 must also satisfy this inequality. If, in addition, $\bar{z} > 0$, this inequality is a "cut" which could be added to the linear programming problem P1 to eliminate certain non-integer points from the set of feasible solutions. As in Gomory's Method of Integer Forms, a sequence of such cuts could be used to approach an integer solution to problem P2 (though, of course, the process might not be finite in this case). If the inequality actually corresponded to a bounding hyperplane of $\hat{C}_y$, the cut might be considered as a "best" cut, since it describes a face of the convex hull $\hat{C}_y$. However, a "best" cut may not be very strong, since it does not necessarily form a bounding hyperplane for the convex hull of solutions to problem P2.

At any rate, if all the non-trivial bounding hyperplanes of $\hat{C}_y$ were known, then there would be a set of "best" cuts which could be added to problem P1. The cuts in this set are at least as strong as those obtained by the Method of Integer Forms. As is shown later in this chapter, there is a finite procedure for generating these bounding hyperplanes to P3, and an optimal solution to P2, if it exists, can be obtained in a finite number of steps by adding cuts to P1 and reoptimizing.

Let $\sum_{i=1}^{n} \bar{c}_{i+m} y_i = \bar{z}$ be a non-trivial bounding hyperplane of the convex hull $\hat{C}_y$. Since the hyperplane is incident to $n$ lattice points of $\hat{C}_y$, and since these lattice points are integral, all of the coefficients $\bar{c}_{i+m}$ must be rational numbers. Thus the equation may be multiplied by a positive integer to clear fractions.

The following theorem helps to characterize the bounding hyperplanes of $\hat{C}_y$. As the proof is lengthy, it will be given in Appendix III.

Theorem VIII: Let $\sum_{i=1}^{n} \bar{c}_{i+m} y_i = \bar{z}$ be a non-trivial bounding hyperplane of $\hat{C}_y$ with integral coefficients. Then the coefficients can be made to satisfy $0 \le \bar{c}_{i+m} < D$ for $i = 1, 2, \ldots, n$, and all feasible integer solutions to problem P3 satisfy $\sum_{i=1}^{n} \bar{c}_{i+m} y_i \equiv \bar{z} \pmod{D}$.

Proof: See Appendix III.

Note that it is not necessarily true that $\bar{z} < D$. For example, consider the following problem P3 consisting of only one congruence:

$$2/30\, y_1 + 9/30\, y_2 \equiv 1/30 \pmod{1},$$

where $D = 30$. It can be shown that there are exactly two extreme point solutions, $y^1 = (y_1^1, y_2^1) = (2, 3)$ and $y^2 = (y_1^2, y_2^2) = (11, 1)$. A bounding hyperplane for the corresponding $\hat{C}_y$ which is incident to both of these extreme points is

$$2y_1 + 9y_2 = 31,$$

and in this case, $\bar{z} = 31 > 30 = D$. However, the following corollary can be established:

Corollary II: Let $\sum_{i=1}^{n} \bar{c}_{i+m} y_i = \bar{z}$ be a bounding hyperplane for $\hat{C}_y$ with $n^*$ of the coefficients $\bar{c}_{i+m}$ $(i = 1, 2, \ldots, n)$ strictly positive. Then

$$\text{(i)} \qquad \bar{z} \le n^* \cdot (D - 1)$$

and

$$\text{(ii)} \qquad \bar{z} \le (D - 1) \log_2 D .$$

Proof:

(i) By the same type of argument used in Theorem VIII to show that $\bar{c}_{i+m} \le D - 1$ for all i, it can be shown that for any lattice point $y^*$ incident to the bounding hyperplane, $\bar{c}_{i+m} y_i^* \le D - 1$ for each i. Thus $\sum_{i=1}^{n} \bar{c}_{i+m} y_i^* \le n^*(D - 1)$, since at most $n^*$ terms in this sum are strictly positive.

(ii) For each i, $i = 1, 2, \ldots, n$, there is a lattice point of $\hat{C}_y$ incident to the hyperplane for which the $i^{th}$ component is strictly positive (if this were not true, there would not be n linear independent lattice points incident to the hyperplane). Let $y^*$ be an extreme point of $\hat{C}_y$ incident to the hyperplane. By virtue of the foregoing, if some component of $y^*$, say $y_k^*$, is zero, then $c_{k+m}^* = 0$. Letting $\hat{n}$ be the number of positive components of $y^*$, it follows that $\hat{n} \ge n^*$. By Theorem IV, $\prod_{i=1}^{n} (y_i^* + 1) \le D$, since $y^*$ is an extreme point solution. Since $\prod_{i=1}^{n} (y_i^* + 1) \ge 2^{\hat{n}}$ and $\hat{n} \ge n^*$, $D \ge 2^{n^*}$. Thus (ii) follows.

By Theorem VIII, any bounding hyperplane of $\hat{C}_y$ has a rather restricted form with integral coefficients. Now, divide the equation of the hyperplane through by D. The hyperplane $\sum_{i=1}^{n} \bar{c}_{i+m} y_i = \bar{z}$ can be considered as having fractional coefficients $\bar{c}_{i+m}$, with $0 \le \bar{c}_{i+m} < 1$ for each i, $i = 1, 2, \ldots, n$. Furthermore, all integer solutions to P3 satisfy $\sum_{i=1}^{n} \bar{c}_{i+m} y_i \equiv \bar{z} \pmod 1$. Let $z^0$ be the fractional part of $\bar{z}$, i.e., $\bar{z} \equiv z^0 \pmod 1$ and $0 \le z^0 < 1$. Then any feasible solution to P3 satisfies $\sum_{i=1}^{n} \bar{c}_{i+m} y_i \equiv z^0 \pmod 1$.

Consider problem P3 in matrix form, so that, with m rows, there are m congruence relations. As Gomory has shown in [9], any integer linear combination of these congruences taken modulo 1 produces another congruence which is satisfied by any feasible solution to P3. As mentioned in Chapter I, considering each such possible congruence as a row vector of $m + 1$ components, these vectors form an additive group of exactly D elements, under operations taken modulo 1. This Gomory row group is isomorphic to the Gomory column group F, and forms the basis for the cutting plane approach of [9].

The following theorem completes the characterization of the bounding hyperplanes of $\hat{C}_y$. Again, the proof will be deferred.

Theorem IX: Let $\sum_{i=1}^{n} \bar{c}_{i+m} y_i = \bar{z}$ be a non-trivial bounding hyperplane of $\hat{C}_y$, with the coefficients written as integers over a common denominator D. Then $(\bar{c}_{1+m}, \bar{c}_{2+m}, ---, \bar{c}_{n+m}, z^0)$ belongs to the Gomory row group.

Proof: See Appendix III.

It follows directly from the theorem that every non-trivial bounding hyperplane of the convex hull $\hat{C}_y$ is parallel to an equation whose coefficients are components of some element of the Gomory row group. Since the row group has D - 1 non-trivial elements, there are at most D - 1 non-trivial bounding hyperplanes.

It was remarked earlier that if all the bounding hyperplanes were known, then there would be a set of cuts which could be added to problem P1 to further restrict the solution set. There is now a mechanism by which this end may be achieved.

Corollary III: Let $(\bar{a}_{j1}, \bar{a}_{j2}, \ldots, \bar{a}_{jn}, \bar{b}_j)$ be a non-trivial element of the Gomory row group. Minimize the linear form $\sum_{i=1}^{n} \bar{a}_{ji} y_i$ subject to the constraints of problem P3, and let the minimum value obtained by this function be $\bar{z}_j$ (of course, $\bar{z}_j \geq \bar{b}_j$ and $\bar{z}_j \equiv \bar{b}_j \pmod 1$). Then the inequality $\sum_{i=1}^{n} \bar{a}_{ji} y_i \geq \bar{z}_j$ is a valid cut which could be added to problem P1.

Proof:

Follows directly from Theorem IX and the definition of a bounding hyperplane.

Note that if this procedure is repeated for each of the D - 1 non-trivial elements of the Gomory row group, a set of cuts will be obtained, a subset of which will correspond to the set of non-trivial bounding hyperplanes for the convex hull $\hat{C}_y$. Thus in theory the bounding hyperplanes of $\hat{C}_y$ can be obtained in a finite number of steps.

This procedure, of course, involves D - 1 minimizations. Usually a large number of cuts which are not bounding hyperplanes will be obtained. These cuts are extraneous, since the cuts corresponding to the bounding hyperplanes are stronger. Unfortunately, there seems to be at present no way of knowing *a priori* which elements of the row group generate the bounding hyperplanes. Furthermore, even after generating the cuts, one does not know which of the cuts generated corresponds to a bounding hyperplane, unless one attempts to find n linear independent solutions for each minimization.

There is another difficulty which arises when one would like to determine the cuts to add to problem P1. Ideally, the best cuts of the described class are those which are binding on the new optimal solution to problem P1 when it is augmented to include all cuts of this class (as Gomory [2] has shown, those cuts which are not binding on the new optimal solution to P1 can be dropped from the tableau). One might think that those cuts derived from the bounding hyperplanes which define the optimal solution to problem P3 would be best in this sense. However, it is easy to construct non-pathological problems for which the hyperplanes of $\hat{C}_y$ defining the optimal solution for problem P3 are not binding on the reoptimized solution to P1. Thus knowledge of the optimal extreme point of $\hat{C}_y$ does not necessarily imply knowledge of the "best" cuts to add.

It was mentioned above that the Gomory row group forms the basis for the cutting plane approach developed by Gomory in his Method of Integer Forms [9]. In particular, if $(\bar{a}_{j1}, \bar{a}_{j2}, \ldots, \bar{a}_{jn}, \bar{b}_j)$ is an element of the Gomory row group, then $\sum_{i=1}^{n} \bar{a}_{ji} y_i \geq \bar{b}_j$ is a Gomory cut. From Theorem IX and the above comments, it follows that of the set of all possible Gomory cuts, there is a subset of cuts which are parallel to those obtained from the bounding hyperplanes. However, the example following Theorem VIII above serves to show that the Gomory cuts may not be the same as the cuts derived from the bounding hyperplanes. For the problem given there, the bounding hyperplane generates the cut $2/30\ y_1 + 9/30\ y_2 \geq 31/30$, while the corresponding Gomory cut is $2/30\ y_1 + 9/30\ y_2 \geq 1/30$. Here the first cut is uniformly stronger than the second.(1)

Since the Gomory cuts are no stronger than the cuts derived from the bounding hyperplanes, it follows that the Gomory cuts also do not exclude any lattice point of $\hat{C}_y$ from consideration, even though there may be many such lattice points that do not generate feasible

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(1) The first cut is one of a class given by Glover [5]. It can be directly inferred from the second cut, since the fact that $y_1$ and $y_2$ must be integral implies that $2/30\ y_1 + 9/30\ y_2 > 1/30$, and the fact that $2/30\ y_1 + 9/30 y_2 \equiv 1/30 \pmod 1$ implies that $2/30\ y_1 + 9/30\ y_2 \geq 1 + 1/30$.

solutions for problem P2. In order to exclude lattice points which solve P3 but do not generate feasible solutions to P2 it is necessary to develop cuts having negative coefficients.[1]

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[1] F. Glover in [4] has derived a class of cuts which can exhibit this property.

# CHAPTER V

## APPLICATIONS TO PROBLEMS WITH SPECIAL STRUCTURES AND RESTRICTIONS

Many integer programming problems have special structures and restrictions which can be exploited to advantage by the methods given earlier. An important example of this occurs when some of the variables are restricted to be either zero or one. Another occurs when the matrix A can be partitioned into submatrices, some of which are unimodular. In this chapter some results will be presented which make it possible to handle these structures and restrictions in an efficient manner.

There are two types of simplifications that can be derived from structural considerations: those which can be used to advantage in the formulation of problem P3, and those which can be used to modify the algorithm for solving problem P3. The former will be considered first.

In the earlier part of this paper, the columns of problem P3 were denoted by $\bar{\alpha}_{i+m}$, where $B^{-1}\alpha_{i+m} \equiv \bar{\alpha}_{i+m} \pmod 1$. The $\bar{\alpha}_{i+m}$, of course, were assumed to have as many components as the number of rows in the initial tableau. Gomory [8] has shown that this is usually unnecessary. The Gomory group F of columns (taken modulo 1) can be written as the direct sum of cyclic subgroups: if there are D* cyclic subgroups, F can be generated by exactly D* elements. Since the column group is isomorphic to the row group, it is sufficient to consider D* rows and hence a column group (isomorphic to F) F*, in which there are exactly D* entries in each column. In particular, if some row of the matrix of coefficients of problem P3 is a linear integer combination of the other rows, it can be dropped from consideration. In [8], Gomory has constructed a minimal set of rows to use for problem P3, based on a proof of the Elementary Divisor Theorem.(1)

In this section, consideration will be limited to determining how to condense problem P3 by direct observation of the structure of the coefficient matrix of problem P1. To accomplish this end, we state the following theorem.

Theorem X: Let B be an optimal basis matrix for P1. Let $B^0$ be a square submatrix of B, and let $D^0$ be the absolute value of the determinant of $B^0$. If $D^0 = 1$, the rows of the optimal tableau corresponding to the basic variables of $B^0$ can be dropped from consideration when constructing problem P3.

Proof: See Appendix IV.

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(1) For this proof, see Van der Waerden [13].

If there are many submatrices of B for which the corresponding determinant has an absolute value equal to one, it is not necessarily true that all the corresponding rows may be dropped. It is permissible to drop all such rows only if the matrix consisting of these submatrices plus their linking elements has a determinant with an absolute value equal to one. A sufficient condition for this to be true is, of course, that all the submatrices be disjoint, i.e., there are no non-zero linking elements. From these arguments, the following two corollaries are immediate:

Corollary IV:[1] If a slack variable from the initial tableau is basic in the optimal tableau, then its corresponding equation in the optimal tableau can be dropped from consideration when constructing problem P3, without affecting the validity of Theorem X.

Corollary V: If there is a set of rows in the original problem whose coefficients form the incidence matrix of a network, then those rows corresponding to a subset of optimal basic variables, which variables also form a basis for the incidence matrix, can be dropped from

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[1] Note that the objective function variable can be interpreted as a slack. Thus if the coefficients of the original objective function for P1 are integers, requiring the objective function variable to be integral adds no information to problem P3. In this case, of course, any cut obtained from the objective function using the Method of Integer Forms can also be obtained from some integer combination of the constraints.

consideration when constructing problem P3, without affecting the validity of Theorem X.

Using the foregoing results, one can quite often cut down on the number of components of the columns of problem P3 by direct observation of the structure of the original problem.

In order to construct problem P3, it is still necessary to have at least a subset of the coefficients in the optimal tableau for each non-basic variable. There are, however, certain linear programming techniques which can be used to accomplish this end. In this section, only two of these will be covered, the concepts of which can be extended in a straightforward manner to the more general techniques.

The first technique, given by Dantzig in [2], is applicable when the variables have upper bounds. By using Dantzig's Upper Bounding Algorithm, one can solve the linear programming problem P1 using only the initial constraints, while implicitly taking into account the upper bound restrictions. In this fashion, a much smaller matrix is operated upon. The optimal tableau obtained by this method, however, is not quite of the form necessary to construct problem P3. This can happen because some of the variables may be "non-basic at their upper bound," and for which the relative optimal cost $c^*_{i+m}$ is non-positive. Suppose such a non-basic variable is $y_k$, and its upper bound is $M_k$. Replacing $y_k$ in the optimal tableau by

$y'_k = M_k - y_k$ and adjusting the coefficients (and the right hand side) of the tableau accordingly will bring the tableau into the required form. One can construct problem P3 directly from this altered tableau. In this fashion, problem P3 is much smaller than it would be if the Simplex Method were applied directly to solve problem P1.

One word of caution, however, is necessary. Because the bounds are being considered implicitly, it may happen that the solution to problem P2 generated by a solution to problem P3 may be non-negative without being feasible. It is therefore also necessary to check that no variable violates its upper bound.

Another technique, developed by Dantzig and Van Slyke in [3], utilizes the concept of a working basis. In many integer programming problems there are often constraints of the form $\sum_{i \in S} x_i + \sum_{i \in T} y_i = 1$ for some disjoint sets of variables S and T. Here the Dantzig- Van Slyke Generalized Upper Bounding Technique can be used to obtain a linear programming solution for problems with such constraints. If, for example, problem P1 contains M general constraints and L non-overlapping constraints of this bounding type, then their technique utilizes a working basis of rank M (assuming the problem is of full rank). One might hope that there is a way to construct problem P3 using columns which have at most M components, and, in fact, this is the case.

For linear programs, the technique uses a working basis $\bar{B}$, its inverse $\bar{B}^{-1}$, and a set of L "key variables", one for each bounding constraint. Each row of $\bar{B}$ corresponds to one of the M general constraints. Let $A^{k_\ell}$ denote that column of M components of the initial tableau associated with the M general constraints. These M components of $A^{k_\ell}$ are the coefficients of the key variable $x_{k_\ell}$ associated with the $\ell^{th}$ bounding constraint. At optimality, create an expanded matrix of M rows and M + L columns by entering $\bar{B}^{-1}$ in the first $M \times M$ places, and by entering $-\bar{B}^{-1}A^{k_\ell}$ in the $(M + \ell)$ column. Call this matrix $\bar{B}^*$. Letting N refer to the original matrix of coefficients of the variables which are non-basic at optimality (each column of N having M + L components), use $\bar{B}^*N$ and $\bar{B}^*b$ to obtain problem P3.[1] Thus each column in P3 has M components.

That it is sufficient to use these coefficients for constructing problem P3 follows from the fact that each key variable is basic for a bounding constraint, and that the bounding constraints are non-overlapping and have coefficients equal to one. For problem P2, one again must check that the upper bounds are satisfied.

The above results, of course, hold also for Gomory's Method of Integer Forms [9]. Chapter IV has indicated a relationship between problem P3 and this cutting plane approach. It follows that it is

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(1) Computationally speaking, it is quicker to use a different ordering of these operations to obtain $\bar{B}^*N$ and $\bar{B}^*b$.

sufficient to consider the rows of problem P3 to generate cutting planes for this method.

For problems exhibiting the special structures discussed above, the theorems of the earlier chapters are valid even though the original A-matrix of problem P1 may not be partitioned into two parts, $A = [A', I]$. Slack variables are not required for rows that lie in a sub-matrix whose determinant has an absolute value equal to one.

In some cases, the algorithm for solving problem P3 can be adjusted to take certain side restrictions into account. Perhaps the easiest way to see these adjustments is to consider the network underlying problem P3. Suppose that some set of S variables, say $y_s, y_{s+1}, \ldots, y_{s+S-1}$, must satisfy certain conditions, i.e., it is known that the S-tuple $(y_s, \ldots, y_{s+S-1})$ must lie in some set Q of lattice points. If the set Q is not too large, then it can be enumerated.

Let $(y_s^q, \ldots, y_{s+S-1}^q)$ be an element of the set Q. Define $\bar{a}_s^q$ by $\bar{a}_s^q = \sum_{i=s}^{s+S-1} \bar{a}_{i+m} y_i^q$ for each q in Q, and let $c^{*q}_{s+m} = \sum_{i=s}^{s+S-1} c^*_{i+m} y_i^q$. Now, construct the original network as before, and drop all nodes $j_i$ for $i = s, \ldots, s+S-1$ and all arcs incident to these nodes. If there are $|Q|$ elements in Q, construct $|Q|$ directed arcs incident to each node $K_{s-1}$ as follows (recall that k referred to the $k^{th}$ group element $\bar{p}_k$ and $s - 1$ referred to the $(s - 1)^{st}$ variable):

arc $(k_{s-1}, j_{s+S})$ is added to the network if and only if there exists a q such that $\bar{p}_k + \bar{a}_s^q = \bar{p}_j$. The cost on this arc is $c^{*q}_{s+m}$. There can be more than one arc incident to any pair of nodes. It is evident that any route through this altered network can be translated into a solution for problem P3, and that the side restrictions defining the set Q are always satisfied for each such solution to P3.(1)

While such restrictions limit the class of solutions to problem P3, they are usually paid for in terms of computational inefficiency. Since this method amounts to partial enumeration of solutions to problem P3, the drawbacks inherent in this method are obvious, particularly since the algorithm for solving problem P3 requires the construction of the complete network underlying P3. There are, however, two instances in which it may be profitable to actually perform such alterations.

The first instance occurs if there are very few $\bar{a}_s^q$ compared to the size of D. If these restricted variables are then listed first, this would be roughly equivalent to reducing the problem to one in which the first variable has a small order, thus easing computation.

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(1) Chronologically, the specialization for zero-one restrictions was developed first. F. Glover then pointed out the similarity between this work and some of his recent results, which inspired the presentation given here. For a wider discussion of side conditions in a slightly different context, see Glover [7]. One might also consider the possibility of combining the procedures given here with some of the existing partial (or truncated) enumeration procedures (see, for example, Balas [1], Glover [6].

The second instance occurs if the form of the restrictions is such that calculations for finding the $\bar{a}_s^q$ are simple (in particular, this implies that $|Q|$ is not very large). This is the case, for example if the side restriction is of the form $\sum_{i=s}^{s+S-1} a_i y_i \le U$, where $U > 0$, and $a_i > 0$, for all i. If U is small, then it is possible that a considerable saving of computation may be realized.

Consider, for example, the situation when the side restriction is of the form $\sum_{i=s}^{s+S-1} y_i \le 1$. By taking $S = 1$, this corresponds to restricting the variable $y_s$ to either zero or one. In this case, the set Q has precisely S + 1 elements, and one can take $\bar{a}_s^q = \bar{a}_{s+m+q}$ for $q = 0, \ldots, S-1$, and $\bar{a}_s^S = \bar{0}$. The formulas for $\varphi$ and the bookkeeping also simplify. If $s = 1$, then $\varphi_0^{(1)}(\bar{p}) = 0$ and one can take $\varphi_0^{(r)}(\bar{0}) = +\infty$ for $r \ge 2$ (obviously one need never consider $\varphi_0^{(r)}(\bar{p})$ for $p \ne 0$). One calculates $\varphi_{s+S}^{(r)}(\bar{p})$ by:

$$\varphi_{s+S}^{(1)}(\bar{0}) = 0$$

$$\varphi_{s+S}^{(r)}(\bar{0}) = \min_{\substack{2\le k\le r \\ 0\le i\le S-1}} {}^{(r-1)}\{\varphi_{s-1}^{(k)}(\bar{0}),\ \varphi_{s+S}^{(k-1)}((\bar{d}_{s+S} - 1)\bar{a}_{s+S+m}) + c^*_{s+S+m},$$

$$\varphi_{s-1}^{(k-1)}((\bar{d}_{s+i} - 1)\bar{a}_{s+i+m}) + c^*_{s+i+m}\} \quad \text{for} \quad r \ge 2$$

$$\varphi_{s+S}^{(1)}(\bar{p}^*) = \min_{\substack{\bar{p}\in P_{s-1},\ \bar{p}\notin P_{s+S} \\ 0\le i\le S-1}} \{\varphi_{s-1}^{(1)}(\bar{p}),\ \varphi_{s-1}^{(1)}(\bar{p} - \bar{a}_{s+i+m}) + c^*_{s+i+m}\}$$

$$\varphi^{(r)}_{s+S}(\bar{p}^*) = \min_{\substack{2 \le k \le r \\ 0 \le i \le s-1}} {}^{(r-1)} \{ \varphi^{(k)}_{s-1}(\bar{p}^*), \ \varphi^{(k-1)}_{s+S}(\bar{p}^* - \bar{a}_{s+S+m}) + c^*_{s+S+m},$$

$$\varphi^{(k-1)}_{s-1}(\bar{p}^* - \bar{a}_{s+i+m}) + c^*_{s+i+m} \} \quad \text{for} \quad r \ge 2$$

$$\varphi^{(r)}_{s+S}(\bar{p}) = \min_{\substack{2 \le k \le r \\ 0 \le i \le s-1}} {}^{(r)} \{ \varphi^{(k)}_{s-1}(\bar{p}), \ \varphi^{(k)}_{s+S}(\bar{p} - \bar{a}_{s+S+m}) + c^*_{s+S+m},$$

$$\varphi^{(k)}_{s-1}(\bar{p} - \bar{a}_{s+i+m}) + c^*_{s+i+m} \}$$

for $\bar{p} = \bar{p}^* + t \cdot \bar{a}_{s+S+m}$ for some integral $t$,

where $P_{s-1}$, $P_{s+S}$ are defined in the same manner as $P_s$ in Chapter II, page 22. Note that it is unnecessary to calculate $\varphi^{(r)}_{s+i}(\bar{p})$ for $0 \le i \le S - 1$. Thus, although each $\varphi$ value is determined by reference to a larger number of terms than previously, there are fewer values of $\varphi$ to determine.

Because zero-one restrictions are taken explicitly into account in the above formulas, one might expect the solutions obtained by these formulas to have a somewhat better chance of providing an optimal solution to P2 than the recursions developed earlier, in particular in conjunction with techniques such as the Dantzig-Van Slyke Generalized Upper Bounding Technique given above.

Example:

$$\begin{array}{llllllll}
\text{maximize} & 9x_1 + x_2 + 7x_3 + 8x_4 + 5x_5 + 7x_6 \\
\text{subject to} & 7x_1 + 8x_4 \leq 6 \\
& 4x_2 + 6x_5 \leq 6 \\
& 5x_3 + 4x_6 \leq 5 \\
& x_1 + x_2 + x_3 = 1 \\
& x_4 + x_5 + x_6 = 1
\end{array}$$

and $x_1, \ldots, x_6$ are non-negative integers.

Upon the addition of slack variables, the initial tableau is

| $x_0$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | $x_7$ | $x_8$ | $x_9$ | 1 |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | -9 | -1 | -7 | -8 | -5 | -7 | 0 | 0 | 0 | 0 |
| 0 | 7 | 0 | 0 | 8 | 0 | 0 | 1 | 0 | 0 | 6 |
| 0 | 0 | 4 | 0 | 0 | 6 | 0 | 0 | 1 | 0 | 6 |
| 0 | 0 | 0 | 5 | 0 | 0 | 4 | 0 | 0 | 1 | 6 |
| 0 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| 0 | 0 | 0 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 1 |

One can obtain an optimal linear programming solution to this problem by applying the Dantzig-Van Slyke Generalized Upper Bounding Technique [3]. Initially, one can start with a feasible basis using columns 0, 2, 6, 7, 8, and 9, for which columns 2 and 6 are key. Letting $A^j$ denote the first four components of column j, the initial working basis is $B = [A^0, A^7, A^8, A^9]$. Continuing as in [3], at optimality one has a basis consisting of columns 0, 1, 3, 6, 8, and 9, for which columns 3 and 6 are key. The optimal working basis is $\bar{B} = [A^0, A^1 - A^3, A^8, A^9]$. The non-basic columns are 2, 4, 5, and 7. In particular, $D = |\det \bar{B}| = 7$, and

$$\bar{B}^{-1} = \begin{bmatrix} 1 & 2/7 & 0 & 0 \\ 0 & 1/7 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 5/7 & 0 & 1 \end{bmatrix}$$

Following the preceding chapter, $\bar{B}^{-1}$ is expanded to $\bar{B}* = [\bar{B}^{-1}, -\bar{B}^{-1}A^3, -\bar{B}^{-1}A^6]$, or

$$\bar{B}* = \begin{bmatrix} 1 & 2/7 & 0 & 0 & 7 & 7 \\ 0 & 1/7 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 5/7 & 0 & 1 & -5 & -4 \end{bmatrix}$$

Now,

$$\overline{B}^*N = \begin{bmatrix} 6 & 9/7 & 2 & 2/7 \\ 0 & 8/7 & 0 & 1/7 \\ 4 & 0 & 6 & 0 \\ -5 & 12/7 & -4 & 5/7 \end{bmatrix} \qquad \overline{B}^*b = \begin{bmatrix} 110/7 \\ 6/7 \\ 6 \\ 2/7 \end{bmatrix}$$

The optimal linear programming solution is

non-basic variables: $x_2 = x_4 = x_5 = x_7 = 0$

Working basic variables: $x_0 = 110/7$, $x_1 = 6/7$, $x_8 = 6$, $x_9 = 2/7$

key basic variables: $x_3 = 1 - x_1 - x_2 = 1/7$, $x_6 = 1 - x_4 - x_5 = 1$

Since the solution is not integral, one constructs problem P3. The first row of $\overline{B}^*N$ gives the coefficients of the objective function. Note now that the optimal working basis variables $x_8$ and $x_9$ were originally slack variables. Since they now correspond to rows three and four of $\overline{B}^*$, one can, by corollary III of the preceding chapter, drop rows three and four when constructing problem P3. Hence one has

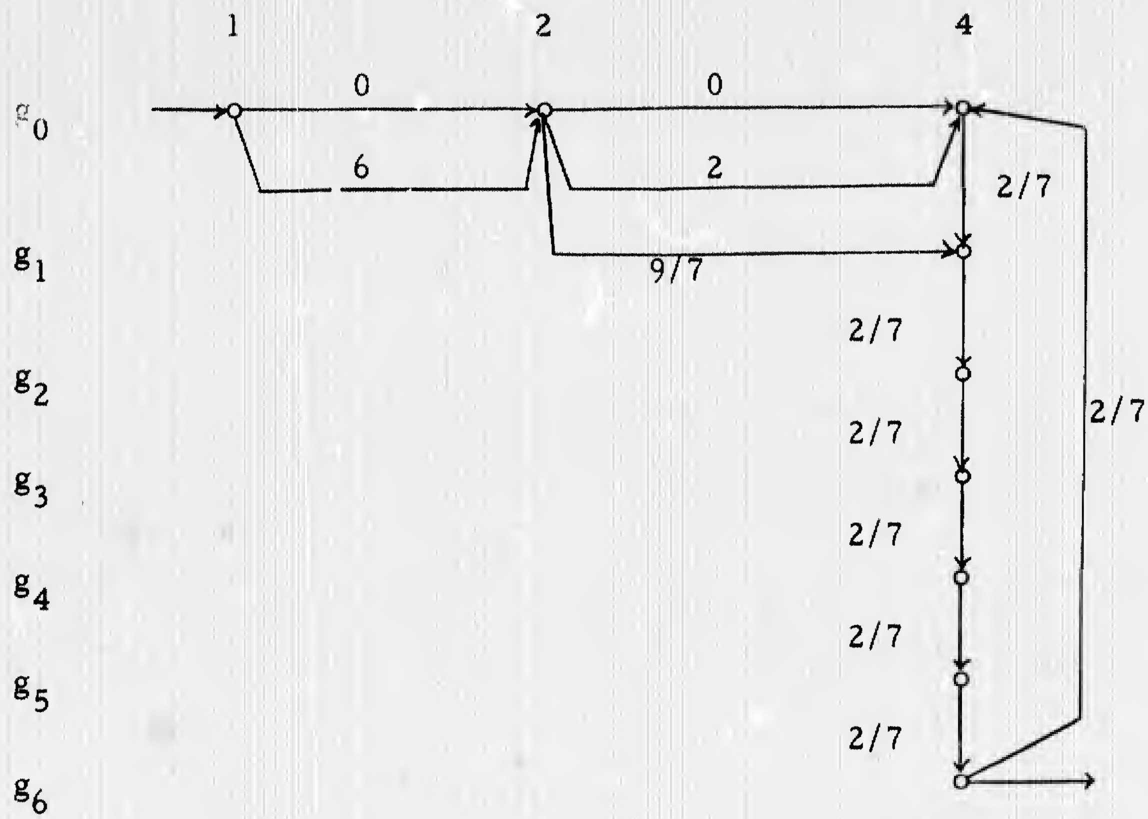
Problem P3:

minimize $6y_1 + 9/7y_2 + 2y_3 + 2/7y_4$

subject to $0y_1 + 1/7y_2 + 0y_3 + 1/7y_4 \equiv 6/7 \pmod 1$

with $y_1, \ldots, y_4$ non-negative integers.

Here $y_1$ corresponds to $x_2$, $y_2$ to $x_4$, $y_3$ to $x_5$, and $y_4$ to $x_7$. The group has $D = 7$ elements and is cyclic: the group elements are $g = (0)$, $g_1 = (1/7)$, $g_2 = (2/7)$, $g_3 = (3/7)$, $g_4 = (4/7)$, $g_5 = (5/7)$, $g_6 = (6/7)$. For problem P3, $\bar{a}_{1+m} = \bar{a}_{s+m} = g_0$, $\bar{a}_{2+m} = \bar{a}_{4+m} = g_1$, and $\bar{b} = g_6$. Note now that there are bounds on some of the variables; as one can see from the initial tableau, $0 \leq y_1 \leq 1$ and $0 \leq y_2 + y_3 \leq 1$. Taking these bounds into account, the underlying network of P3 is




The costs are indicated on each arc. The duplication of certain arcs is due to the fact that both $\bar{a}_{1+m}$ and $\bar{a}_{3+m}$ equal $g_0$. There is no "column" 3 because $y_2 + y_3 \leq 1$. For $r = 1$, the following $\varphi$ values are obtained:

$\bar{p} =$

| | | $g_0$ | $g_1$ | $g_2$ | $g_3$ | $g_4$ | $g_5$ | $g_6$ |
|---|---|---|---|---|---|---|---|---|
| | 1 | 0 | X | X | X | X | X | X |
| s = | 2 | 0 | X | X | X | X | X | X |
| | 4 | 0 | 2/7 | 4/7 | 6/7 | 8/7 | 10/7 | 12/7 |

Backtracking for $\bar{b} = g_0$, the solution to P3 is $y_1^{(1)} = y_2^{(2)} = y_3^{(1)} = 0$, $y_4^{(1)} = 6$, with $\phi^{(1)}(\bar{b}) = 12/7$. But $\bar{B}{*}b - \bar{B}{*}Ny^{(1)} = (14, 0, 6, -4)^T$, which is not feasible for problem P2. For $r = 2$, the $\phi$ values are

$\bar{p} =$

| | | $g_0$ | $g_1$ | $g_2$ | $g_3$ | $g_4$ | $g_5$ | $g_6$ |
|---|---|---|---|---|---|---|---|---|
| | 1 | $+\infty$ | X | X | X | X | X | X |
| s = | 2 | 6 | X | X | X | X | X | X |
| | 4 | 2 | 9/7 | 11/7 | 13/7 | 15/7 | 17/7 | 19/7 |

and, for $\bar{b} = g_0$, the 2nd optimal solution is $y_1^{(2)} = y_3^{(2)} = 0$, $y_2^{(2)} = 1$, $y_4^{(2)} = 5$ with $\phi^{(2)}(\bar{b}) = 19/7$. Again, $\bar{B}{*}b - \bar{B}{*}Ny^{(2)} = (13, -1, 6, -5)^T$ which is not feasible for problem P2. For $r = 3$, the $\phi$ values are

| | | $g_0$ | $g_1$ | $g_2$ | $g_3$ | $g_4$ | $g_5$ | $g_6$ |
|---|---|---|---|---|---|---|---|---|
| | 1 | $+\infty$ | X | X | X | X | X | X |
| s = | 2 | $+\infty$ | X | X | X | X | X | X |
| | 4 | 3 | 16/7 | 18/7 | 20/7 | 22/7 | 24/7 | 26/7 |

and, for $\bar{b} = g_6$, there are two 3rd optimal solutions: $y_1^{(3a)} = y_2^{(3a)} = y_3^{(3a)} = 0$, $y_4^{(3a)} = 13$; $y_1^{(3b)} = y_2^{(3b)} = 0$, $y_3^{(3b)} = 1$, $y_4^{(3b)} = 6$. In both cases, of course, $\phi^{(3)}(\bar{b}) = 26/7$. Now, $\bar{B}^*b - \bar{B}^*Ny^{(3a)} = (12, -1, 6, -9)^T$, which is infeasible. However, $\bar{B}^*b - \bar{B}^*Ny^{(3b)} = (12, 0, 0, 0)^T$ which is feasible. The optimal solution to problem P2 is

non-basic variables: $x_2 = x_4 = 0$, $x_5 = 1$, $x_7 = 6$

working basic variables: $x_0 = 12$, $x_1 = 0$, $x_8 = 0$, $x_9 = 0$

key basic variables: $x_3 = 1 - x_1 - x_2 = 1$, $x_6 = 1 - x_4 - x_5 = 0$.

# CHAPTER VI

## COMPUTATIONAL RESULTS

The algorithm of Chapter III[1] has been programmed for the IBM 7094 DCS system at the Computer Center of the University of California in Berkeley. This chapter describes some of the features of the code and gives the results obtained for a number of sample problems. It concludes with a discussion of the computational aspects of the algorithm, and suggestions for future research.

The code itself is in Fortran IV and is single precision. The linear programming subroutine[2] for finding the optimal linear programming solution was therefore altered for Fortran IV. For ease in coding, the underlying network is constructed before proceeding to find the shortest route. In the process of constructing the network, a disk file is used for temporary storage of the non-basic columns. As the disk is used to simulate tape, including REWIND, the construction will take longer if physical tape is used. The program, however, can be modified to do these operations in core, without too great a loss of memory.

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(1) The algorithm has been slightly modified to take advantage of the appearance of slack variables in the optimal linear programming basis, as indicated in Chapter V.

(2) Share Library Subroutine HO RS MSUB (Version 1).

To handle the case when $c^*_{i+m} = 0$ for $i$ belonging in some set $I$, a special subroutine is employed which creates a table of multiples of $\bar{d}_i$ for each $i$ in $I$. It is presently assumed that $y_i \leq D \cdot \bar{d}_i$ for each $i$ in $I$, and the table can hold at most 400 entries.[1] The routes are calculated as usual, except that no cycles are allowed on the zero cost arcs. The entries from the table are then combined with the solution obtained from the route to generate the multiple solutions for P3.

The code assumes that the data is entered in integers, including the objective function. It allows for four decimal digits plus a sign. After constructing the network but before finding the first shortest route, the relative costs are converted into integers by multiplying through by D. When taking the non-basic columns modulo one and finding the group structure, round-off occurs at four places after the decimal point. The code terminates the problem if an optimal integer solution has been reached or if one of the parameters restricting the size has been violated.

The code is written to handle variable dimensions. Dimension parameters determined at the start are

(i) M, the number of rows of P1 including the objective function,

(ii) NR, the number of variables including slacks.

At the end of the linear program, the following parameters are

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(1) If the table is exceeded, multiple solutions corresponding to the overflow will not be examined.

determined:

(i) MNS, the number of basic activities in the optimal linear programming basis, excluding those activities having a unit submatrix (e.g., slacks)

(ii) N, the number of non-basic activities in the optimal linear programming solution.

In the remainder of the code, the following dimension parameters are used:

(i) DMAX, the order of the Gomory group,

(ii) RMAX, the number r of route lengths required for an optimal solution,

(iii) KMAX, the number of routes of a given length,

(iv) SUMD, where SUMD = $\sum_{i=1}^{n} \bar{d}_i$ (of course, SUMD ≤ N*DMAX).

The restrictions on problem size are as follows:

(i) M ≤ 100, n ≤ 100,

(ii) 2*M*NR + M*M + 3*M ≤ CORE,

(iii) N*MNS + (4 + MNS)*DMAX ≤ CORE,

(iv) 4*DMAX * (RMAX + 1) ≤ CORE,

(v) 2*N + 2*(RMAX + 1)*SUMD + KMAX*(3 + N) ≤ CORE,

where CORE is a variable. The parameters DMAX, RMAX, and KMAX are checked successively from the last three constraints.[1]

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[1] For a more detailed discussion of the code, see [14]. The current program allows for a KMAX of at least N. It does not, at present, allow for multiple right hand sides (see [8]).

Initially, a set of seven test problems were developed to test various aspects of the algorithm. These problems were chosen so that solutions could be obtained by hand calculation and checked by inspection. Indeed, some can be solved graphically. These problems are given in Appendix V. Table I gives the breakdown of the computer results. The times for various operations are given to establish some basis for determining the behavior of the coded algorithm.

The code was then tried on some of the integer programming test problems collected by Haldi [10]. The results are given in Table II. These problems have also been "solved" using three different Share cutting plane codes (LIP1, IP02, IP03), and the results are given in both [10] and [11]. Ready comparison between the Share code results and the present is not easy since the Share code results are given in terms of iterations (pivot steps) while the present code has no directly analogous iterative steps.

From Tables I and II, it is seen that the construction of the underlying network is the principle factor in governing the speed of the code, particularly in the larger problems. This is mainly due to the large number of comparisons necessary, which are roughly on the order of $1/2\ mnD^2$. A more efficient way of constructing the network is clearly necessary as a first step toward shortening the time for solution.

## TABLE I. RESULTS[(a)] OF SAMPLE PROBLEMS

| Problem No. | No. Rows | No. Vars.[(b)] | D[(c)] | No. Optimal Solutions | Time[(d)] To L. P. Solution | Time to Construct Network | Order Of Route | No. Routes of Same Length | Time To Compute Length | Time To Backtrack | Time to Check Feasibility |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 5 | 9 | 42 | 1 | 0.1 | 1.1 | 1 | 1 | 2.2 | 0.6 | 0.5 |
| 2 | 2 | 4 | 4 | 1 | 0.1 | 0.1 | 1 | 1 | 1.5 | 0.4 | 0.8 |
| 3 | 3 | 5 | 8 | 1 | 0.1 | 0.2 | 1 | 1 | 1.5 | 0.4 | 1.2 |
| | | | | | | | 2 | 3 | 0.5 | 0.6 | 0.8 |
| 4 | 3 | 5 | 5 | 1 | 0.1 | 0.1 | 1 | 1 | 1.0 | 0.4 | 1.1 |
| | | | | | | | 2 | 2 | 0.3 | 0.3 | 0.8 |
| 5 | 4 | 7 | 8 | 2 | 0.1 | 0.2 | 1 | 1 | 0.9 | 0.6 | 0.6 |
| | | | | | | | 2 | 3 | 0.5 | 0.3 | 0.7 |
| 6 | 2 | 4 | 29 | 1 | 0.1 | 0.6 | 1 | 1 | 1.5 | 0.3 | 0.6 |
| 7 | 2 | 7 | 6 | 1 | 0.1 | 0.3 | 1 | 1 | 1.5 | 0.9 | 0.9 |

(a) All times are in seconds.

(b) Includes slacks.

(c) A ">k" means that D > k and that k is the bound imposed by the dimensional constraints.

(d) Does not include I/0 operations or a matrix inversion.

(e) The times do not add up to the total since some operations were not timed individually. On small problems I/0 operations can take 50% of the total time.

TABLE II. RESULTS[a] OF SELECTED HALDI PROBLEMS[b]

| Problem No. | No. Rows[c] | No. Vars.[d] | D[e] | No. Optimal Solutions | Time[f] To L.P. Solution | Time to Construct Network | Order Of Route | No. Routes of Same Length | Time to Compute Length | Time To Backtrack | Time to Check Feasibility | Total Time[g] |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| F. C. 1 | 6 | 11 | 183 | 14 | 0. 1 | 23. 0 | 1 | 35 | 7. 5 | 2. 6 | 2. 7 | 36. 0 |
| F. C. 2 | 6 | 11 | 258 | 4 | 0. 1 | 35. 0 | 1 | 9 | 10. 0 | 2. 6 | 1. 1 | 59. 0 |
| F. C. 3 | 6 | 11 | 320 | 7 | 0. 1 | 47. 0 | 1 | 10 | 11. 0 | 3. 0 | 1. 1 | 68. 0 |
| F. C. 4 | 6 | 11 | 18 | 2 | 0. 1 | 1. 0 | 1 | 2 | 1. 6 | 0. 6 | 0. 7 | 9. 0 |
| F. C. 6 | 6 | 11 | >1798 | - | 0. 1 | - | - | - | - | - | - | 273. 0 |
| F. C. 9 | 9 | 15 | >1497 | - | 0. 2 | - | - | - | - | - | - | 239. 0 |
| F. C. 10 | 16 | 28 | >1192 | - | 0. 8 | - | - | - | - | - | - | 124. 0 |
| Mach Sch 1 | 21 | 56 | >639 | - | 3. 7 | - | - | - | - | - | - | 73. 0 |
| IBM Test 1 | 7 | 14 | 32 | 7 | 0. 2 | 2. 3 | 1 | 7 | 3. 0 | 0. 7 | 1. 0 | 12. 0 |
| IBM Test 2 | 7 | 14 | 32 | 40 | 0. 2 | 2. 3 | 1 | 49 | 2. 0 | 1. 1 | 4. 3 | 11. 0 |
| IBM Test 3 | 3 | 7 | 72 | 1 | 0. 1 | 2. 7 | 1 | 1 | 2. 5 | 0. 7 | 0. 6 | |
| | | | | | | | 2 | 1 | 2. 1 | 1. 0 | 0. 5 | |
| | | | | | | | 3 | 1 | 2. 6 | 1. 0 | 0. 5 | |
| | | | | | | | 4 | 1 | 3. 2 | 1. 3 | 0. 5 | |
| | | | | | | | 5 | 2 | 3. 9 | 2. 0 | 0. 8 | 27. 0 |
| IBM Test 4 | 15 | 30 | >989 | - | 1. 0 | - | - | - | - | - | - | 197. 0 |
| IBM Test 5 | 15 | 30 | >846 | - | 1. 0 | - | - | - | - | - | - | 88. 0 |
| IBM Test 8 | 12 | 49 | >1034 | - | 0. 9 | - | - | - | - | - | - | 209. 0 |

(a) All times are in seconds.
(b) See Haldi [10].
(c) F. C. problems 1, 2, 3, 4, 9, and 10 include rows corresponding to zero-one restrictions.
(d) Includes slacks.
(e) A ">k" means that $D > k$ and that $k$ is the bound imposed by the dimensional constraints.
(f) Does not include I/0 operations or a matrix inversion.
(g) The times do not add up to the total since some operations were not timed individually.

It is evident that the efficiency of the present algorithm depends crucially on the size of D. Thus, as is pointed out in [11], any rescaling of rows or columns in the constraint matrix could be a great help. As a first step in this direction, one should try to ensure that no row has a common factor.

One can also take out a common factor in a column provided that care is taken not to allow solutions to the modified problem to generate non-integer solutions to the original problem. Suppose, for example, the $i^{th}$ column of the original problem P1 has a common factor k, where k is a positive integer. If, at optimality in the modified problem, this column is non-basic, then using $k \cdot \bar{\alpha}_{i+m}$ instead of $\bar{\alpha}_{i+m}$ in the construction of the network and in the formulas for $\varphi$ guarantees that the corresponding variable is integral in the original problem. On the other hand, if the $i^{th}$ column is in the optimal basis to the modified problem, then one can add a subroutine to the feasibility check to disallow solutions for which the modified $x_i$ is not a multiple of k. In this latter case, D would be reduced by a factor of 1/k in the modified problem, but the number of routes necessary to find a solution to the original problem could increase by a factor of k.

It is possible to construct perverse problems for which the algorithm may be quite inefficient, even with D small. Such a problem can arise, for example, if one of the non-binding linear programming constraints is very close and almost parallel to one of the constraints which is binding at optimality. The problem will be still more resistant to solution if, in addition, the objective function is almost parallel to some other binding constraint. One suspects, however, that in practice such problems are not too common, and that, practically speaking, it is the size of D which determines the efficiency of the algorithm.

There are some modifications which might be made to this algorithm to aid in computation. As suggested in Chapter I, an infeasible solution might be acceptable if it is not "too" infeasible. For instance, one might search for a feasible solution until the objective function reaches a certain value, and after that point look for any solution which is within a certain degree of being feasible.

Another possibility is to proceed to a certain stage in the computation and then to add a cut (or set of cuts) to the linear program, reoptimize, and start over again. One could, of course, use the present value of the objective function as a cut. This, however, would drive all of the $c^*_{i+m}$ to zero upon reoptimizing (except for the one corresponding to the added cut), a situation which could lead to difficulties in the reformulated problem P3. A third possiblity would be to try to find cuts to add which result in a small value of D thereby easing the construction of the new problem P3. A fourth possibility would be to look for cuts which, upon reoptimizing, make the new non-binding linear programming constraints as "relaxed" as possible, so that they intersect the new cone $C_y$ far from the origin (see Section II).

One may also consider starting from a non-optimal but dual feasible solution to the linear program P1 (either before or after adding cuts). The theory developed in the previous chapters certainly allows for this possibility, and the algorithm goes through unchanged.[1] Here one would expect to have to examine more routes than if one started from an optimal solution. However, one can construct problems that require an arbitrary but finite number of routes from an optimal solution to P1, whereas, backing slightly away from an optimal solution requires only one route. In theory, one could even start from a dual feasible integer solution to problem P1. Here each column of P3 could have order one, and the algorithm could degenerate into a strictly combinatorial search based on the objective function to problem P3. In this context, the use of side restrictions would be valuable (see Section V).

It appears, then, that the efficiency of the algorithm presented here depends highly on the particular nature of the problem to be solved. The problem size itself is apparently less critical than the size of D and the nature of the non-binding constraints of the optimal linear program. Unfortunately, these factors are difficult to determine for any

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(1) Consider, for example, combining this dual feasible property with the third possibility in the preceding paragraph. Instead of reoptimizing after adding a cut (or cuts), just pivot the cut(s) in. As is mentioned in [9], pp. 289-291, this procedure reduces the value of D. G. T. Martin also employs this type of device in his Accelerated Algorithm (see [12]).

given problem without first attempting to solve it. Some of the modifications suggested above may help to improve the performance of the method for problems that are structured unfavorably.

# APPENDIX I

## PROOFS OF SELECTED THEOREMS OF CHAPTER II

Theorem IV: If $c^*_{i+m} \neq 0$ for some $i$, $1 \leq i \leq n$, then there is an $r^{th}$ optimal solution to P3, say $y$, with $\prod_{i=1}^{n} (y_i + 1) \leq rD$.

Proof:

Take $y$ as an $r^{th}$ optimal solution with minimal product $\prod_{i=1}^{n} (y_i + 1)$. Form the n-dimensional array with entries $\bar{I}(p_1, p_2, \ldots, p_n) \equiv \sum_{i=1}^{n} \bar{a}_{i+m} p_i \pmod 1$ where $p_i$ is integral, $0 \leq p_i \leq y_i$. Note that $\bar{I}(y_1, y_2, \ldots, y_n) = \bar{b}$ and that $\bar{I}(p_1, p_2, \ldots, p_n)$ belongs in the Gomory column group F for all n-tuples.

Suppose that $\prod_{i=1}^{n} (y_i + 1) \geq rD + 1$, so that there are at least $rD + 1$ entries in the array. Then, since at most D entries are distinct (since F has D elements), there must be at least $r + 1$ entries in the array which are the same, say $\bar{I}(p_1^{(1)}, \ldots, p_n^{(1)}) = \bar{I}(p_1^{(2)}, \ldots, p_n^{(2)}) = \ldots = \bar{I}(p_1^{(r+1)}, \ldots, p_n^{(r+1)})$. Without loss of generality, one can assume that $\sum_{i=1}^{n} c^*_{i+m} p_i^{(j)} \leq \sum_{i=1}^{n} c^*_{i+m} p_i^{(j+1)}$ for $j = 1, \ldots, r$. Define

$y^{(j+)} = y - p^{(1)} + p^{(j)}$. Since $0 \leq p_i^{(j)} \leq y_i$ for all $i$ and $j$, $y^{(j+)}$ is a feasible solution to problem P3. Similarly, $y^{(j-)} = y + p^{(1)} - p^{(j)}$

is feasible for all j. Since $p^{(j_1)} \neq p^{(j_2)}$ for $j_1 \neq j_2$, these solutions are distinct. Furthermore, $y^{(j+)} + y^{(j-)} = 2y$ for all j, and

$$\sum_{i=1}^{n} c^*_{i+m} y_i^{(r+1-)} \leq \ldots \leq \sum_{i=1}^{n} c^*_{i+m} y_i^{(2-)} \leq \sum_{i=1}^{n} c^*_{i+m} y_i \leq \sum_{i=1}^{n} c^*_{i+m} y_i^{(2+)}$$

$$\leq \ldots \leq \sum_{i=1}^{n} c^*_{i+m} y_i^{(r+1+)} .$$

Since there are r solutions, $y^{(r+1-)}$ to $y^{(2-)}$, whose objective function values do not exceed that of y, there must be at least two of these solutions which give the same objective function value, in order to fulfill the assumption that y is $r^{th}$ optimal. Thus either $\sum_{i=1}^{n} c^*_{i=m} y_i^{(j-)} = \sum_{i=1}^{n} c^*_{i+m} y_i$ for some j, or

$$\sum_{i=1}^{n} c^*_{i+m} y_i^{(j_1-)} = \sum_{i=1}^{n} c^*_{i+m} y_i^{(j_2-)} < \sum_{i=1}^{n} c^*_{i+m} y_i \quad \text{for some } j_1 > j_2 .$$

Suppose the first possibility obtains. Then, since $y^{(j-)} + y^{(j+)} = 2y$, it must be true that $\sum_{i=1}^{n} c^*_{i+m} y_i^{(j+)} = \sum_{i=1}^{n} c^*_{i+m} y_i$. However, by the following lemma, $\prod_{i=1}^{n} (y_i^{(j-)} + 1) \geq \prod_{i=1}^{n} (y_i + 1)$ then $\prod_{i=1}^{n} (y_i + 1) > \prod_{i=1}^{n} (y_i^{(j+)} + 1)$ and conversely. Both implications contradict the assumption that y is $r^{th}$ optimal with minimal product $\prod_{i=1}^{n} (y_i + 1)$.

Next, suppose the second possibility obtains. Form $\hat{y} = y + y^{(j_1-)} - y^{(j_2-)}$ and $\hat{\hat{y}} = y - y^{(j_1-)} + y^{(j_2-)}$. Both $\hat{y}$ and $\hat{\hat{y}}$ are feasible solutions to P3 and $y \neq \hat{y}$, $y \neq \hat{\hat{y}}$, $\hat{y} \neq \hat{\hat{y}}$. Furthermore, $\sum_{i=1}^{n} c^*_{i+m} \hat{y}_i = \sum_{i=1}^{n} c^*_{i+m} \hat{\hat{y}}_i$. By the same argument as in the previous paragraph, either $\prod_{i=1}^{n} (\hat{y}_i + 1) < \prod_{i=1}^{n} (y_i + 1)$ or $\prod_{i=1}^{n} (\hat{\hat{y}}_i + 1) < \prod_{i=1}^{n} (y_i + 1)$, thus giving the contradiction which establishes the theorem.

<u>Lemma</u>: Let $y'$, $y''$, and $y$ be three n-tuples of nonnegative numbers with $y' + y'' = 2y$. Let $y_i \neq y'_i$, $y_i \neq y''_i$, and $y'_i \neq y''_i$ for each $i = 1, \ldots, n$. Then it cannot be that both $\prod_{i=1}^{n} (y'_i + 1) \geq \prod_{i=1}^{n} (y_i + 1)$ and $\prod_{i=1}^{n} (y''_i + 1) \geq \prod_{i=1}^{n} (y_i + 1)$.

<u>Proof</u>:

Suppose the assumptions of the lemma are satisfied, but that both inequalities hold. Then

$$\prod_{i=1}^{n} (y'_i + 1) \cdot \prod_{i=1}^{n} (y''_i + 1) \geq \left[\prod_{i=1}^{n} (y_i + 1)\right]^2$$

$$= \left[\prod_{i=1}^{n} \frac{y'_i + 1 + y''_i + 1}{2}\right]^2$$

$$= \left[\prod_{i=1}^{n} \tfrac{1}{2}\left((\sqrt{y'_i + 1})^2 + (\sqrt{y''_i + 1})^2\right)\right]^2$$

$$\geq \left[ \prod_{i=1}^{n} (\sqrt{y_i' + 1})(\sqrt{y_i'' + 1}) \right]^2$$

$$= \prod_{i=1}^{n} (y_i' + 1) \cdot \prod_{i=1}^{n} (y_i'' + 1)$$

where the middle inequality follows from the fact that $1/2(a_i^2 + b_i^2) \geq a_i b_i$, for each i. Since the first and last terms are the same, there must be equality throughout. In particular, it must be true that, for each i,

$$\frac{(y_i' + 1) + (y_i'' + 1)}{2} = \sqrt{y_i' + 1} \cdot \sqrt{y_i'' + 1}.$$

Squaring and substituting for $y_i + 1$, it must be true that $(y_i + 1)^2 = (y_i' + 1)(y_i'' + 1)$ for each i.

Now,

$$\begin{aligned} 0 &= -(y_i + 1)^2 + (y_i' + 1)(y_i'' + 1) \\ &= (y_i + 1)^2 + (y_i' + 1)(y_i'' + 1) - (y_i + 1) \cdot 2(y_i + 1) \\ &= (y_i + 1)^2 + (y_i' + 1)(y_i'' + 1) - (y_i' + 1)(y_i + 1) - (y_i + 1)(y_i'' + 1) \\ &= (y_i - y_i')(y_i - y_i''). \end{aligned}$$

Thus for each i, either $y_i = y_i'$; or $y_i = y_i''$, a contradiction which proves the lemma.

Theorem V: Let $I = \{i \mid c^*_{i+m} = 0,\ 1 \le i \le n\}$. Then any $r^{th}$ optimal solution to problem P3, say $y$, satisfies $\sum_{i \notin I} y_i \le rD - 1$.

Proof:

In a similar vein to the proof of Theorem VI, suppose that there is an $r^{th}$ optimal solution, say $y$, for which $\sum_{i \notin I} y_i \ge rD$. Form a vector $\bar{T}$ containing $\sum_{i \notin I} y_i$ entries, each one an element of the Gomory column group $F$, where element $\bar{a}_{i+m}$ occurs exactly $y_i$ times, if $i \notin I$. Form another vector $\bar{S}$, of $1 + \sum_{i \notin I} y_i$ entries, where the $p^{th}$ component, $\bar{S}_p$, is the group sum of the first $p$ components of $\bar{T}$, and $\bar{S}_0$ is taken as $\bar{0}$.

Since $F$ has $D$ elements, and $\bar{S}$ has at least $rD+1$ components, there must be some element of $F$ which occurs at least $r+1$ times in the vector $\bar{S}$. Thus, for some $p_1 < p_2 < \dots < p_{r+1}$, $\bar{S}_{p_1} = \bar{S}_{p_2} = \dots = \bar{S}_{p_{r+1}}$. It follows that, for any $j$, $\bar{S}_{p_{r+1}} - \bar{S}_{p_j} = \bar{0}$, $1 \le j \le r$.[1]

For each $j$, $1 \le j \le r$, delete successively the elements $\bar{T}_{p_j+1}, \bar{T}_{p_j+2}, \dots, \bar{T}_{p_{r+1}}$ from the $\bar{T}$ vector. For each $j$, this constructs a corresponding solution, $y^j$, which is feasible for P3. Furthermore, $y_i^j \le y_i^{j+1} \le y_i$ for each $i$ and $j$, and there is an $i$ such that $y_i^j < y_i^{j+1}$ for each $j$, and also an $i$ such that $y_i^j < y_i$ for each $j$. Thus $\sum_{i \notin I} y_i^1 < \sum_{i \notin I} y_i^2 < \dots < \sum_{i \notin I} y_i^r < \sum_{i \notin I} y_i$.

However, since $c^*_{i+m} > 0$ for all $i \notin I$, it must therefore be true that $\sum_{i \notin I} c^*_{i+m} y_i^1 < \dots < \sum_{i \notin I} c^*_{i+m} y_i^r < \sum_{i \notin I} c^*_{i+m} y_i$. Thus $y$ can not be $r^{th}$ optimal, which contradicts our initial assumption.

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[1] i.e., $\bar{T}_{p_j+1} + \bar{T}_{p_j+2} + \dots + \bar{T}_{p_{r+1}} = \bar{0}$.

# APPENDIX II

## PROOF OF A THEOREM OF CHAPTER III

Theorem VII: The formulas defining $\varphi$ given in the text are the correct values for solving problem $P3(s, \bar{p})$ for all $s$ $(1 \leq s \leq n)$ and all $\bar{p} \in \bar{F}$. Furthermore, problem $P3(s, \bar{p}_j)$ is equivalent to problem $N(s, \bar{p}_j)$ for all $s$ and $j$.

Proof:

As was remarked in the body of the text, to establish the equivalence of $P3(s, \bar{p}_j)$ and $N(s, \bar{p}_j)$, it suffices to show that the formulas defining $\varphi$ are correct.

It is obvious that $\varphi_s^{(1)}(\bar{0}) = 0$ is an optimal solution to $P3(s, \bar{0})$ for all $s$. It is further obvious that $\varphi_1^{(1)}(\bar{p}) = t \cdot c^*_{1+m}$ or $+\infty$, depending upon whether $\bar{p} = t \cdot \bar{a}_{i+m}$ for some integral $t$ or not, since in the second case there is no feasible solution to $P3(1, \bar{p})$ and in the first case a cheapest solution to $P3(1, \bar{p})$ is to take $y_1 \equiv t \pmod{\bar{d}_1}$, $0 \leq y_1 < \bar{d}_1$.

Now, fix $r = 1$, and proceed by induction: Assume that $\varphi_i^{(1)}(\bar{p})$ is correct for problem $P3(i, \bar{p})$ for all $i < s$ and all $\bar{p}$. Consider $P3(s, \bar{p})$, where $\bar{p} \not\equiv t \cdot a_{s+m}$ for some integral $t$, and let this set of

$\bar{p}$'s be $\tilde{P}_s$. Any feasible solution to P3(s-1, $\bar{p}$) can be extended to a feasible solution to P3(s, $\bar{p}$) merely by taking $y_s = 0$. Thus it must be true that $\varphi_s^{(1)}(\bar{p}) \le \varphi_{s-1}^{(1)}(\bar{p})$ for all $\bar{p} \in P_s$. However, since $\bar{p} \in P_s$, no feasible solution to P3(s - 1, $\bar{p}'$) for $\bar{p}' \notin P_s$ can be extended to a feasible solution to P3(s, $\bar{p}$) without first extending it to a feasible solution to P3(s - 1, $\bar{p}''$) for some $\bar{p}'' \in P_s$. Therefore, since this must be true of each optimal solution, there must be at least one $\bar{p}^* \in P_s$ such that $\varphi_s^{(1)}(\bar{p}) \ge \varphi_{s-1}^{(1)}(\bar{p}^*)$ for all $\bar{p} \in P_s$. It follows that

$$\varphi_{s-1}^{(1)}(\bar{p}) \ge \varphi_{s-1}^{(1)}(\bar{p}^*) \text{ for all } \bar{p} \in P_s .$$

Hence there is a $\bar{p}^*$ with

$$\varphi_s^{(1)}(\bar{p}^*) = \varphi_{s-1}^{(1)}(\bar{p}^*) = \min_{\bar{p} \in P_s} \{\varphi_{s-1}^{(1)}(\bar{p})\}.$$

To complete the proof for $r = 1$, assume that $\varphi_s^{(1)}(\bar{p}^* + j \cdot \bar{a}_{s+m})$ is given correctly for $0 \le j < t < \bar{d}_s - 1$. Include the case when $\bar{p}^* \equiv 0$. Note that

$$\varphi_s^{(1)}(\bar{p}^* + t \cdot \bar{a}_{s+m}) \le \min \{\varphi_{s-1}^{(1)}(\bar{p}^* + t \cdot \bar{a}_{s+m}),$$
$$\varphi_s^{(1)}(\bar{p}^* + (t-1)\bar{a}_{s+m}) + c^*_{s+m}\},$$

since both expressions in the min $\{\cdot\}$ arise from feasible solutions. But by the inductive assumption, the fact that any feasible solution to P3(s, $\bar{p}^* + t \cdot \bar{a}_{s+m}$) can be extended from a solution to a previously obtained problem via either $P3(s-1, \bar{p}^* + t \cdot \bar{a}_{s+m})$ or $P3(s, \bar{p}^* + (t-1) \cdot \bar{a}_{s+m})$ implies that it is best to extend a solution from either $P3(s-1, \bar{p}^* + t \cdot \bar{a}_{s+m})$ or $P3((s, \bar{p}^* + (t-1) \cdot \bar{a}_{s+m})$. Thus there must be equality.

Now, assume the formulas are correct for all $r < R$. Obviously, $\varphi_1^{(R)}(\bar{0}) = (R-1) \cdot c^*_{i+m} \cdot \bar{d}_1$ if $c^*_{i+m} \neq 0$, and $\varphi_1^{(R)}(\bar{p}) = t \cdot c^*_{1+m} + \varphi_1^{(R)}(\bar{0})$ if $\bar{p} \equiv t \cdot \bar{a}_{1+m}$, with $\varphi_1^{(R)}(\bar{p}) = +\infty$ otherwise. Hence, assume the formulas are correct for $i \leq s$. Consider problem $P3(s, \bar{p}^*)$ where $\bar{p}^*$ has been defined when $r = 1$. Following the argument presented above, any $r^{th}$ feasible solution to $P3(s-1, \bar{p}^*)$ may be extended to an $R^{th}$ feasible solution to $P3(s, \bar{p}^*)$ for $r \leq R$ by taking $y_s = 0$ provided that the $r^{th}$ feasible solution is not $k^{th}$ optimal for some $k$, $k \leq R-1$. Thus $\varphi_s^{(R)}(\bar{p}^*) \leq \min_{2 \leq r \leq R}{}^{(R-1)} \{\varphi_{s-1}^{(r)}(\bar{p}^*)\}$. However, there is another possibility: any $r-1^{st}$ feasible solution to $P3(s, \bar{p}^* - \bar{a}_{s+m})$ may be extended to an $R^{th}$ feasible solution to $P3(s, \bar{p}^*)$ for $r \leq R$ by increasing $y_s$ by 1, again provided that the $r-1^{st}$ feasible solution is not $k^{th}$ optimal for some $k$, $k \leq R-1$. Thus also $\varphi_s^{(R)}(\bar{p}^*) \leq \min_{2 \leq r \leq R}{}^{(R-1)} \{\varphi_s^{(r-1)}(\bar{p}^* - \bar{a}_{s+m}) + c^*_{s+m}\}$. Hence $\varphi_s^{(R)}(\bar{p}^*) \leq \min_{2 \leq r \leq R}{}^{(R-1)} \{\varphi_{s-1}^{(r)}(\bar{p}^*), \varphi_s^{(r-1)}(\bar{p}^* - \bar{a}_{s+m}\}$. As previously argued, however, since $\bar{p} \in P_s$ no $r^{th}$ feasible solution to $P3(s, \bar{p}')$ or $P3(s-1, \bar{p}')$ can be extended to an $R^{th}$ feasible solution to $P3(s, \bar{p})$ if $\bar{p}' \notin P_s$ unless it is first extended to a solution to either $P3(s-1, \bar{p})$ or $P3(s, \bar{p})$ for $\bar{p} \in P_s$. Hence, by the inductive assumption, one needs only to consider extensions from $r^{th}$ feasible solutions to either $P3(s-1, \bar{p})$ or $P3(s, \bar{p})$ for $\bar{p} \in P_s$. But, since this must be true of any optimal solution, there must be equality in the expression for $\varphi_s^{(R)}(\bar{p}^*)$, unless $\varphi_s^{(R)}(\bar{p}^*)$ was obtained via an $R^{th}$ best solution to $P3(s, \bar{p}^* - \bar{a}_{s+m})$. But this is impossible, since $\varphi_s^{(R)}(\bar{p}^* - \bar{a}_{s+m}) + c^*_{s+m} > \varphi_s^{(R-1)}(\bar{p}^* - \bar{a}_{s+m}) + c^*_{s+m}$, and the quantity on the right represents the cost for an $R^{th}$ feasible extension from $P3(s, \bar{p}^* - \bar{a}_{s+m})$, i.e., there is always a feasible extension which costs less. Thus the formula for $\varphi_s^{(R)}(\bar{p}^*)$ is correct. It is obvious that if no $R^{th}$ optimal solution has been obtained for $P3(s, \bar{p})$ for any $\bar{p} \in F$, then one can always take $\bar{p}^* \equiv 0$.

To finish the general case, assume $\varphi_s^{(r)}(\bar{p}^* + j \cdot \bar{a}_{s+m})$ is correct for all $r \le R$ and all $j < t \le \bar{d}_s - 1$. As above, it is evident that

$$\varphi_s^{(R)}(\bar{p}^* + t \cdot \bar{a}_{s+m}) \le \min_{1 \le r \le R}{}^{(R)} \{\varphi_{s-1}^{(r)}(\bar{p}^* + t \cdot \bar{a}_{s+m}),$$

$$\varphi_s^{(r)}(\bar{p}^* + (t-1) \cdot \bar{a}_{s+m}) + c^*_{s+m}\},$$

since all expressions in the brackets correspond to costs for $R^{th}$ feasible solutions, which were obtained either by extending those $r^{th}$ feasible solutions $(r \le R)$ for $P3(s-1, \bar{p}^* + t \cdot \bar{a}_{s+m})$ which were not $k^{th}$ optimal $(k < R)$ for $P3(s, \bar{p}^* + t \cdot \bar{a}_{s+m})$, or by extending those $r^{th}$ feasible solutions $(r \le R)$ for $P3(s, \bar{p}^* + (t-1) \cdot \bar{a}_{s+m})$ whose extensions were not $k^{th}$ optimal $(k < R)$ for $P3(s, \bar{p}^* + t \cdot \bar{a}_{s+m})$. Hence the $R^{th}$ minimum is taken. Now again, by the inductive assumption, the fact that any $R^{th}$ feasible solution to $P3(s, \bar{p}^* + t \cdot \bar{a}_{s+m})$ can be extended in either of these two ways implies that it is best to do so. Thus there are no other ways of extension which give a lower cost, so there must be equality.

# APPENDIX III

## PROOFS OF SELECTED THEOREMS OF CHAPTER V

<u>Theorem VIII</u>: Let $\sum_{i=1}^{n} \bar{c}_{i+m} y_i = \bar{z}$ be a non-trivial bounding hyperplane of $\hat{C}_y$ with integral coefficients. Then the coefficients can be made to satisfy $0 \le \bar{c}_{i+m} < D$ for $i = 1, 2, \ldots, n$, and all feasible solutions to problem P3 satisfy $\sum_{i=1}^{n} \bar{c}_{i+m} y_i \equiv \bar{z} \pmod{D}$.

<u>Proof</u>:

The proof will utilize induction over $\bar{n}$, the number of positive coefficients $\bar{c}_{i+m}$.

(i) For $\bar{n} = 1$. Assume it is the first coefficient which is positive. Let $y^*$ be a solution which is incident to the hyperplane, i. e., $\sum_{i=1}^{n} \bar{c}_{i+m} y_i^* = \bar{c}_{1+m} y_1^* = \bar{z}$. Thus $y^*$ minimizes $\sum_{i=1}^{n} \bar{c}_{i+m} y_i$ subject to the constraints of P3. It is known that $\bar{z} > 0$ and $y_1^* < \bar{d}_1$ where $\bar{d}_1$ is the order of $\bar{\alpha}_{i+m}$. Since $\bar{c}_{1+m} y_1^* = \bar{z}$ and $y_1^* < \bar{d}_1$, $\bar{z}/\bar{c}_{1+m}$ is a positive integer less than $\bar{d}_1$. Letting $\tilde{z} = D/\bar{d}_1 \cdot \bar{z}/\bar{c}_{1+m}$, since $\bar{d}_1 | D$ it follows that $\tilde{z}$ is a positive integer less than D. Let $\tilde{c}_{1+m} = D/\bar{d}_1$, so that $\tilde{c}_{1+m}$ is a positive integer less than D. Then $\tilde{c}_{1+m} y_1^* = \tilde{z}$ has the coefficients of the required form.

Let $\hat{y}$ be any feasible solution to P3, $\hat{y} = (\hat{y}_1, y_2, \ldots, \hat{y}_n)$. If $\hat{y}_1 = y_1^* + k_1\bar{d}_1$ for $k_1$ a non-negative integer, then $\tilde{c}_{1+m}\hat{y}_1 = \tilde{c}_{1+m} \cdot k_1\bar{d}_1$. But $\tilde{c}_{1+m} \cdot k_1\bar{d}_1 = k_1 D \frac{\bar{z}}{\bar{c}_{1+m}}$ with $\bar{z}/\bar{c}_{1+m}$ integral. Thus $\tilde{c}_{1+m}\hat{y}_1 \equiv \tilde{c}_{1+m}y_1^* \equiv \tilde{z} \pmod{D}$.

Suppose now that $\hat{y}$ is a feasible solution to P3, but that $\hat{y}_1 \neq y_1^* + k_1\bar{d}_1$ for all integral $k_1$. Here $\hat{y}_1 > y_1^*$ since $y_1^*$ is optimal. It can be assumed that $\hat{y}_1 < \bar{d}_1$ (if this is not the case, then $\hat{y}_1$ could be reduced by multiples of $\bar{d}_1$ until this is true, without loss of generality). Thus $0 < \hat{y}_1 - y_1^* < \bar{d}_1$. Similarly, without loss of generality, it can be assumed that $\hat{y}_i > y_i^*$ for all $i$, $i = 1, 2, \ldots, n$. Letting $y^k = y^* + k(\hat{y} - y^*)$ for $k$ a non-negative integer, it follows that $y^k$ is a feasible solution for P3, for all such $k$. Obviously $y^0 = y^*$, $y^1 = \hat{y}$.

Consider the set of feasible solutions $y^{-k}$, where $y^{-k}$ is obtained from $y^k$ be taking the $i^{th}$ component modulo $\bar{d}_i$, for each $i$ (i.e., $y_i^{-k} \equiv y_i^k \pmod{\bar{d}_i}$ and $0 \leq y_i^{-k} < \bar{d}_i$). Since $\bar{a}_{1+m}$ is of order $\bar{d}_1$, there are at most $\bar{d}_1$ distinct values for the first component of solutions in this set of feasible solutions. Attention can now be restricted to those $\bar{d}_1$ solutions for which $0 \leq k < \bar{d}_1$ (note that $y_1^{-k+r\bar{d}_1} \equiv y_1^{-k} \pmod{\bar{d}_1}$ for all non-negative integral $r$).

Suppose that the first component of each of these $\bar{d}_1$ solutions is distinct. Then, since $0 \leq y_1^{-k} < \bar{d}_1$ for each $k$, there must be some

$k^*$ for which $\bar{y}_1^{-k^*} = 0$. But then $\bar{y}^{-k^*}$ is a feasible solution for P3 which contradicts the assumption that $y^*$ is optimal. Thus there must be less than $\bar{d}_1$ solutions with distinct first components. Since $k < \bar{d}_1$, there must be two different values of $k$, say $k'$ and $k''$, for which $\bar{y}_1^{-k'} = \bar{y}_1^{-k''}$. Assuming $k' > k''$, we have

$$0 \equiv \bar{y}_1^{-k'} - \bar{y}_1^{k''} \equiv y_1^* + k'(\hat{y}_1 - y_1^*) - y_1^* - k''(\hat{y}_1 - y_1^*)$$

$$\equiv (k' - k'')(\hat{y}_1 - y_1^*) \quad (\text{mod } \bar{d}_1).$$

Letting $\bar{k} = k' - k''$, it follows that $\bar{k}(\hat{y}_1 - y_1^*) \equiv 0 \pmod{\bar{d}_1}$ with $0 < \bar{k} < \bar{d}_1$. Since for each $k < \bar{d}_1$, $\bar{y}_1^{-k+\bar{k}} = \bar{y}_1^{-k}$, there can be at most $\bar{k}$ solutions with distinct first components, so that attention can be further restricted to those $\bar{k}$ solutions $\bar{y}_1^{-k}$ for which $0 \leq k < \bar{k}$. Without loss of generality, it can be assumed that the first components of these $\bar{k}$ solutions are distinct (if not, the above argument can be iterated to find a smaller $\bar{k}$ for which this holds; such a $\bar{k}$ must exist since $0 < \bar{y}_1^{-0} = y_1^* < \hat{y}_1 = \bar{y}_1^{-1} < \bar{d}_1$). Now, for $k < \bar{k}$, $\bar{y}_1^{-k+1} - \bar{y}_1^{-k} \equiv \hat{y}_1 - y_1^* \pmod{\bar{d}_1}$, so that one can conclude that these $\bar{k}$ distinct first components are equally spaced on the interval $(0, \bar{d}_1)$. Let $\hat{d}_1$ be the spacing. Then $\bar{k}\hat{d}_1 = \bar{d}_1$, and $y_1 + r\hat{d}_1$ is the first component of some feasible solution for each non-negative integral $r$.

Let $\hat{c}_{1+m} = D/\hat{d}_1$ and $\hat{z} = D/\hat{d}_1 \cdot \bar{z}/\bar{c}_{1+m}$. Then, since $\hat{d}_1 | \bar{d}_1$ and $\bar{d}_1 | D$, $\hat{c}_{1+m}$ is integral and $0 < \hat{c}_{1+m} < D$. Now,

$$\hat{c}_{1+m}(y_1^* + r\hat{d}_1) = D/\hat{d}_1(\bar{z}/\bar{c}_{1+m} + r\hat{d}_1) = \hat{z} + rD \equiv \hat{z} \pmod{D}.$$

If $r = \bar{k} \cdot k_1$, then $y_1^* + r\hat{d}_1 = y_1^* + k_1\bar{d}_1$ satisfies this congruence, and since $\hat{y}_1 = y_1^* + r_1\hat{d}_1$ for some $r_1$, $1 \le r_1 < \bar{k}$, then $\hat{y}_1$ satisfies this congruence. Thus any feasible solution to P3 satisfies the congruence $\hat{c}_{1+m}y_1 \equiv \hat{z} \pmod{D}$, and $\hat{c}_{1+m}y_1^* = \hat{z}$. Thus $\hat{c}_{1+m}y_1 = \hat{z}$ can be used as the bounding hyperplane.

(ii) Assume the theorem is true for all $k < \bar{n}$, and assume the hyperplane now has $\bar{c}_{i+m} > 0$ for $i = 1, 2, \ldots, \bar{n}$, with $\bar{n} \le n$. Let $y^*$ be a solution with $\sum_{i=1}^{n} \bar{c}_{i+m}y_i^* = \bar{z}$ with, of course, $\bar{z} > 0$. Since $\bar{c}_{i+m} \ge 0$ and $y_i^* \ge 0$ for all $i$, $\bar{c}_{i+m}y_i^* \le \bar{z}$ for each $i$.

Let $i_0$ refer to some single index, $1 \le i_0 \le \bar{n}$. Then $\sum_{\substack{i=1 \\ i \ne i_0}}^{n} \bar{c}_{i+m}y_i^* = \bar{z} - \bar{c}_{i_0+m}y_{i_0}^*$ and $\sum_{\substack{i=1 \\ i \ne i_0}}^{n} \bar{a}_{i+m}y_i^* \equiv \bar{b} - \bar{a}_{i_0+m}y_{i_0}^* \pmod{1}$. By the algorithm for obtaining $y^*$, $y^*$ with the $i_0^{th}$ component removed is an optimal solution for the altered problem:

$$\text{minimize} \quad \sum_{\substack{i=1 \\ i \ne i_0}}^{n} \bar{c}_{i+m}y_i$$

$$\text{subject to} \quad \sum_{\substack{i=1 \\ i \ne i_0}}^{n} \bar{a}_{i+m}y_i \equiv \bar{b} - \bar{a}_{i_0+m}y_{i_0}^* \pmod{1}$$

with $y_i$ integral and non-negative.

Thus unless $\bar{z} = \bar{c}_{i_0+m} y^*_{i_0}$, this is a problem with $\bar{n} - 1$ positive coefficients for which the inductive assumption holds. If $\bar{z} = \bar{c}_{i_0+m} y^*_{i_0}$ then $y^*_i = 0$ for $i = 1, 2, \ldots, \bar{n}$ and $i \neq i_0$. But since there are n linearly independent solutions which satisfy $\sum_{i=1}^{n} \bar{c}_{i+m} y_i = \bar{z}$, there must be at least one for which an $i^{th}$ component, $i \neq i_0$, is strictly positive, with $i \leq \bar{n}$. One can always take this solution as $y^*$, so that it can be assumed that $\bar{c}_{i_0+m} y^*_{i_0} < \bar{z}$.

Thus, for fixed $i_0$, the inductive assumption gives us that $0 \leq \bar{c}_{i+m} < D$ for $i \neq i_0$, and all feasible solutions to P3 with the $i_0^{th}$ component fixed at $y^*_{i_0}$ satisfy $\sum_{i=1}^{n} \bar{c}_{i+m} y_i \equiv \bar{z} \pmod{D}$. Since $i_0$ was arbitrary, $0 \leq \bar{c}_{i+m} < D$ for all i.

The only thing left to show is that the statement "all feasible solutions to P3 with the $i_0^{th}$ component fixed at $y^*_{i_0}$ satisfy $\sum_{i=1}^{n} \bar{c}_{i+m} y_i \equiv \bar{z} \pmod{D}$" implies the same statement without the fixed restriction. This, however, is evident since, for any given feasible solution with the restriction, all components other than the $i_0^{th}$ can be fixed at their present levels, and $i_0^{th}$ can be varied, thus reducing the problem to one with a single variable. The inductive hypothesis gives the result in this case. By doing this for each such feasible solution, all feasible solutions for P3 are obtained, each of which satisfies $\sum_{i=1}^{n} \bar{c}_{i+m} y_i \equiv \bar{z} \pmod{D}$.

Theorem IX: Let $\sum_{i=1}^{n} \bar{c}_{i+m} y_i = \bar{z}$ be a non-trivial bounding hyperplane of $\hat{C}_y$, with the coefficients written as integers over a common denominator D. Then $(\bar{c}_{1+m}, \bar{c}_{2+m}, \ldots, \bar{c}_{n+m}, z^0)$ belongs to the Gomory row group, where $\bar{z} \equiv z^0 \pmod 1$ and $0 \le z^0 < 1$.

Proof:

Consider the congruences of problem P3. Since any feasible solution to P3 must satisfy these congruences, any feasible solution must satisfy any linear integer combination of these congruences. There are D distinct such linear integer combinations, including the original congruences themselves. The vectors of coefficients of these congruences are the elements of the Gomory row group. In particular, to define the solution set to problem P3, it is sufficient to consider any set of these congruences whose vectors of coefficients generate the Gomory row group.

There is always a set of columns of problem P3 which generate the Gomory column group. Consider any set of these columns which are independent generators of the Gomory column group. Let these columns be the first $\bar{m}$ columns of problem P3. Since the column group is isomorphic to the row group, there is a corresponding set of $\bar{m}$ congruences which suffice to define the solution set of problem P3.

Consider the $\bar{m} \times n$ matrix of coefficients of these congruences.

Lemma: The submatrix consisting of the first $\bar{m}$ columns of this matrix of coefficients can be diagonalized. The $\bar{m}$ rows still generate the row group and the $\bar{m}$ columns still generate the column group.

Proof:[1]

It can be assumed that under the isomorphism, row i corresponds to column i, $1 \leq i \leq \bar{m}$. The allowable row operations consist of replacing one row by that row plus an integral multiple of another row. Under row operations of this sort, the resulting isomorphism still matches row i with column i. This follows since a row operation only changes one row, and the row group is still generated by the $\bar{m}$ rows. Since all unchanged rows still correspond with their respective columns, the changed row must still correspond with its column.

Now, one can always find an entry in the first column which has the same order as the first column. If there is no such entry in the present matrix, one can proceed by row operations to obtain such an entry as follows. Let the order of the first column be $\bar{d}_1$. Then all entries in the first column can be written as non-negative integers less than $\bar{d}_1$ divided by $\bar{d}_1$. If there is no entry in the first column of order $\bar{d}_1$, then no numerator of any entry can be relatively prime to $\bar{d}_1$. But in this case, in order to have the order of column

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(1) This proof is similar in spirit to Van der Waerden's proof of the Elementary Divisor Theorem [13]. Here, however, column operations are not permitted.

one equal to $\bar{d}_1$, the numerators of two entries in the column must be relatively prime. By the Euclidean algorithm applied to the group generated by the rows of column one, there is a sequence of row operations by which one can obtain the numerator of some entry equal to one. This entry then has order $\bar{d}_1$.

Such an entry can be assumed to occur in row one. If this is not already true, one can add multiples of such a row to the first row until the desired result is obtained. Note that at this point, an isomorphism exists under which row i still corresponds to column i for each i, $1 \le i \le \bar{m}$.

Next, proceed by induction: if $\bar{m} = 1$, the foregoing was unnecessary, for the matrix is trivially diagonal. So, assume the lemma is true for Gomory groups generated by $\bar{m} - 1$ elements. By the foregoing, multiples of the first row can be added to each of the following rows to obtain the last $\bar{m} - 1$ components in the first column equal to zero. By the inductive hypothesis, the $(\bar{m} - 1) \times (\bar{m} - 1)$ sub-matrix obtained by dropping the first row and first column can be diagonalized. By the isomorphism row i corresponds to column i for each i. Since for $i \ge 2$ there is only one element in row i, its order must be equal to that of row i, and therefore that of column i. Thus one can add multiples of row i to row 1 to eliminate the entry in row 1 and column i. Thus the lemma holds.

Returning to the proof of the theorem, it is clear that, after possibly re-ordering the columns, it is sufficient to consider the following set of congruences as constraints of P3:

$$\bar{a}_{1,1+m}y_1 \qquad\qquad + \bar{a}_{1,\bar{m}+m+1}y_{\bar{m}+1} + \dots + \bar{a}_{1,n+m}y_n \equiv \bar{b}_1$$

$$\bar{a}_{2,2+m}y_2 \qquad + \bar{a}_{2,\bar{m}+m+1}y_{\bar{m}+1} + \dots + \bar{a}_{2,n+m}y_n \equiv \bar{b}_2$$

(mod 1)

$$\bar{a}_{\bar{m},\bar{m}+m}y_{\bar{m}} + \bar{a}_{\bar{m},\bar{m}+m+1}y_{\bar{m}+1} + \dots + \bar{a}_{\bar{m},n+m}y_n \equiv \bar{b}_{\bar{m}}$$

It has been shown in Theorem VIII that all feasible solutions to P3 satisfy $\sum_{i=1}^{n} \bar{c}_{i+m}y_i \equiv z^0$ (mod 1). Let $\bar{d}_i$ be the order of column i of P3. For any $i \le \bar{m}$ for which $\bar{c}_{i+m} > 0$, suppose that $\bar{c}_{i+m}$ is not of order $\bar{d}_i$, say for $i = i_0$. Then, unless $\bar{c}_{i_0+m}$ is of an order which divides $\bar{d}_{i_0}$, one can construct a feasible solution to P3 which does not satisfy $\sum_{i=1}^{n} \bar{c}_{i+m}y_i \equiv z^0$ (mod 1) merely by taking some feasible solution to P3 and increasing the $i_0$ component by $\bar{d}_{i_0}$. Thus $\bar{c}_{i+m}$ must be of an order which divides $\bar{d}_i$ (possibly equal to $\bar{d}_i$ itself) for $i \le \bar{m}$, $\bar{c}_{i+m} > 0$.

Now, to prove the theorem, it must be shown that there exist integers $\lambda_1, \lambda_2, \dots, \lambda_{\bar{m}}$ such that, multiplying each row j by $\lambda_j$ and summing (modulo 1), the vector $(\bar{c}_{1+m}, \bar{c}_{2+m}, \dots, \bar{c}_{n+m}, z^0)$ is obtained. By the preceding paragraph, it is evident that for $j \le \bar{m}$, there exist $\lambda_j$ such that $\lambda_j\bar{a}_{j,j+m} \equiv \bar{c}_{j+m}$ (mod 1). Thus it remains to be shown that $\sum_{j=1}^{\bar{m}} \lambda_j a_{j,i+m} \equiv \bar{c}_{i+m}$ (mod 1) for $i \geq \bar{m} + 1$.

Since all feasible solutions to P3 satisfy $\sum_{i=1}^{n} \bar{c}_{i+m} y_i \equiv \bar{z}^0$ (mod 1), it follows that, by multiplying row j by $\lambda_j$ and "subtracting" the sum from $\sum_{i=1}^{n} \bar{c}_{i+m} y_i \equiv z^0$ (mod 1), every feasible solution to P3 must satisfy

$$\sum_{i=\bar{m}+1}^{n} (\bar{c}_{i+m} - \sum_{j=1}^{\bar{m}} \lambda_j \bar{a}_{j,i+m}) y_i \equiv (z^0 - \sum_{j=1}^{\bar{m}} \lambda_j \bar{b}_j) \quad (\text{mod } 1),$$

where it is understood that the coefficients are considered modulo 1 (and the subtraction is group subtraction).

Since columns 1 to $\bar{m}$ generate the column group, there is a feasible solution for which $y_i = 0$ for $i \geq \bar{m} + 1$. Using this solution in the above congruence, it must be true that

$$z^0 - \sum_{j=1}^{\bar{m}} \lambda_j \bar{b}_j \equiv 0 \quad (\text{mod } 1).$$

Thus all feasible solutions to P3 must satisfy

$$\sum_{i=\bar{m}+1}^{n} (\bar{c}_{i+m} - \sum_{j=1}^{\bar{m}} \lambda_j \bar{a}_{j,i+m}) y_i \equiv 0 \quad (\text{mod } 1).$$

Again, since columns 1 to $\bar{m}$ generate the column group, for each column i, $i \geq \bar{m} + 1$, there is an integer linear combination of the first $\bar{m}$ columns which equals column i. Thus, there must also be an integer linear combination of the first $\bar{m}$ columns which is the inverse of column i, $i \geq \bar{m} + 1$, that is, there exist non-negative integers $\beta_1(i), \beta_2(i), \ldots, \beta_{\bar{m}}(i)$ such that, for $i \geq \bar{m} + 1$, $\bar{a}_{j, j+m} \cdot \beta_j(i) + \bar{a}_{j, i+m} \equiv 0 \pmod 1$ for each j, $j \leq \bar{m}$.

Let $y^0$ be a feasible solution for problem P3 in which $y_i^0 = 0$ for $i \geq \bar{m} + 1$. Let a vector $y(i_0)$ be defined by

$$y_j(i_0) = y_j^0 + \beta_j(i_0) \qquad \text{for } j \leq \bar{m}$$

$$y_{i_0}(i_0) = 1$$

$$y_i(i_0) = 0 \qquad \text{for } i \geq \bar{m} + 1, \quad i \neq i_0.$$

Then, for each j,

$$\bar{a}_{j, j+m} y_j(i_0) + \sum_{i=\bar{m}+1}^{n} \bar{a}_{j, i+m} y_j(i_0)$$

$$= \bar{a}_{j, j+m} y_j^0 + \bar{a}_{j, j+m} \beta_j(i_0) + \bar{a}_{j, i_0 + m}$$

$$\equiv \bar{b}_j \pmod 1.$$

Thus $y(i_0)$ is a feasible solution for problem P3. Thus $y(i_0)$ must satisfy

$$\sum_{i=\bar{m}+1}^{n} (\bar{c}_{i+m} - \sum_{j=1}^{\bar{m}} \lambda_j \bar{a}_{j,\, i+m}) y_i(i_0) \equiv 0 \quad (\text{mod } 1).$$

By definition of $y(i_0)$, this means that

$$\bar{c}_{i_0+m} - \sum_{j=1}^{\bar{m}} \lambda_j \bar{a}_{j,\, i_0+m} \equiv 0 \quad (\text{mod } 1)$$

Since $i_0$ was arbitrary, the theorem follows.

# APPENDIX IV

## PROOF OF A THEOREM OF CHAPTER V

Theorem X: Let B be an optimal basis matrix for P1. Let $B^0$ be a square submatrix of B, and let $D^0$ be the absolute value of the determinant of $B^0$. If $D^0 = 1$, the rows of the optimal tableau corresponding to the basic variables of $B^0$ can be dropped from consideration when constructing problem P3.

Proof:

Consider problem P1 in the form $Bx + Ny = b$, where N is the submatrix corresponding to the non-basic variables. By assumption all coefficients are integral.

Assume that $B^0$ is $m^0 \times m^0$ and that B can be partitioned into four submatrices, $B = \begin{bmatrix} B_1 & B_2 \\ B_3 & B^0 \end{bmatrix}$. Similarly, N is partitioned into $N = \begin{bmatrix} N_1 \\ N_2 \end{bmatrix}$, b into $\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ and the cost function $c_B x + c_N y$ into $c_1 x_1 + c_2 x_2 + c_N y$, the appropriate dimensions following from the fact that $B^0$ is $m^0 \times m^0$.

Let B* be defined by

$$B^* = \begin{bmatrix} 1 & -B_2B^{0-1} \\ 0 & B^{0-1} \end{bmatrix}.$$

Then multiplying problem P1 through by $B^*$ corresponds to a block pivot, since

$$B^*B = \begin{bmatrix} B_1 - B_2B^{0-1}B_3 & 0 \\ B^{0-1}B_3 & 1 \end{bmatrix}.$$

If $D^0 = 1$, then all entries of $B^{0-1}$ are integral. Therefore, in the following system of equations, all coefficients are integral:

$$\begin{aligned} (B_1 - B_2B^{0-1}B_3)x_1 & & + (N_1 - B_2B^{0-1}N_2)y &= b_1 - B_2B^{0-1}b_2 \\ (B^{0-1}B^3)x_1 &+ Ix_2 + & (B^{0-1}N_2)y &= B^{0-1}b_2. \end{aligned}$$

Thus any solution to the first set of equations completely determines the values of $x_2$. In particular, any integer solution to the first set of equations generates integer values of $x_2$.

Solving problem P2, then, amounts to finding a solution to the following problem:

$$\text{minimize} \quad (c_1 - c_2 B^{0-1} B_3) x_1 + (c_N - c_2 B^{0-1} N_2) y$$

$$\text{subject to} \quad (B_1 - B_2 B^{0-1} B_3) x_1 + (N_1 - B_2 B^{0-1} N_2) y = b_1 - B_2 B^{0-1} b_2$$

$$\text{and} \quad x_1 \geq 0, \quad y \geq 0, \quad \text{and} \quad x_1, y \text{ integral.}$$

The values for $x_2$ are obtained by back substitution, and must be checked for feasibility.

However, in defining problem P3, it suffices to consider the above problem, since any integer solution to it generates integer values for $x_2$.

# APPENDIX V

## SELECTED SAMPLE PROBLEMS

The following sample problems are all of the form:

$$\begin{aligned} &\text{Maximize} && cx \\ &\text{Subject to} && Ax \le b \\ &\text{and} && x \ge 0, \text{ integer.} \end{aligned}$$

Problem 1.

$$c = (8 \quad 6 \quad 1 \quad 1)$$

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 7 & 0 & 8 & 0 \\ -4 & -4 & 0 & 6 \\ 0 & 5 & -4 & -4 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 1 \\ 6 \\ 6 \\ 5 \end{bmatrix}.$$

Problem 2.

$$c = (0 \quad 1)$$

$$A = \begin{bmatrix} 2 & 0 \\ -2 & 2 \end{bmatrix}, \quad b = \begin{bmatrix} 3 \\ -1 \end{bmatrix}.$$

Problem 3.

$$c = (0 \quad 1)$$

$$A = \begin{bmatrix} -4 & 4 \\ 2 & -1 \\ 1 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 3 \\ 2 \\ 6 \end{bmatrix}.$$

Problem 4. $c = (5 \quad 4)$

$$A = \begin{bmatrix} 2 & -2 \\ 1 & 2 \\ 4 & 3 \end{bmatrix}, \qquad b = \begin{bmatrix} 7 \\ 9 \\ 20 \end{bmatrix}.$$

Problem 5. $c = (1 \quad 1 \quad 1)$

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \\ 1 & 1 & -4 \end{bmatrix}, \qquad b = \begin{bmatrix} 3 \\ 3 \\ 3 \\ -3 \end{bmatrix}.$$

Problem 6. $c = (6 \quad 5)$

$$A = \begin{bmatrix} -3 & 2 \\ 7 & 5 \end{bmatrix}, \qquad b = \begin{bmatrix} 3 \\ 15 \end{bmatrix}.$$

Problem 7.[(1)] $c = (1 \quad 2 \quad 3 \quad 1 \quad 1)$

$$A = \begin{bmatrix} 1 & 0 & 4 & 2 & 1 \\ 4 & 3 & 1 & -4 & -1 \end{bmatrix}, \qquad b = \begin{bmatrix} 41 \\ 47 \end{bmatrix}.$$

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(1) Due to R. E. Gomory.

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13. ABSTRACT This paper extends some results of Gomory on a group theoretic approach to linear integer programming. An algorithm for solving integer programming problems is presented, and the relation of this work to cuts and appropriate geometric interpretations are given. The theoretical basis for the algorithm establishes a characterization of integer solutions to linear programs by considering the Gomory column group associated with the optimal linear programming basis. It is shown that if an optimal integer solution exists, it can be obtained by finding the $r^{th}$ optimal solution to an optimization problem over elements in this Gomory group, for some finite r. A dynamic programming procedure for finding this $r^{th}$ optimal solution is given, and shown to be equivalent to finding an $r^{th}$ shortest route through a specially constructed network. A particular class of cutting planes is discussed. These cuts are shown to be parallel to a subset of Gomory cuts and at least as strong as the Gomory cuts. Specially structured integer programs are discussed with a view toward speeding computation. Included in this discussion is a specialization to zero-one variables. Some computational results are given. It is seen that computational efficiency depends crucially on the size of the absolute value of the determinant of the optimal linear programming basis. In general, the smaller this absolute value, the more efficient the algorithm.

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