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REPORT No. 718

Procedures for Obtaining Binomial Probabilities Within Three Decimal Accuracy Universally.

ED S. SMITH

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REPORT NO. 718

May 1950

PROCEDURES FOR OBTAINING BINOMIAL PROBABILITIES
WITHIN THREE DECIMAL ACCURACY UNIVERSALLY

Ed S. Smith

Project No. TB3-5238 of the Research and
Development Division, Ordnance Department

ABERDEEN PROVING GROUND, MARYLAND

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TABLE OF CONTENTS

	<u>Page No.</u>
Abstract	5
Summary of Recommended Procedures for Different n, p Regions . . .	7
Preface	11
Nomenclature	12
Introduction	15
Need for this work, problems solved, no general method likely, related introductory material, aim to provide a working field reference, and large-number graphs of p^n and $a=np$.	
Binomial Probabilities	22
individual and cumulative: Computation methods and tools (tables C1-C3): Incomplete Beta Function, factorials and Stirling's formulas.	
Normal Probabilities	23
individual and cumulative: Individual term $N_i = \phi(t)/\sigma$ as function of simple deviate t . Cumulative Normal term $N(t_c)$ as function of Normal deviate including .5 continuity-correction. Gram-Charlier series, Type A, with 2 terms for stated accuracy within given limits. The GCAr remainder method for $2 \leq np \leq 22$ and $.1 \leq p \leq .5$.	
Poisson Probabilities	32
individual and cumulative: Poisson cumulative table C7. GC series, Type B, 2 terms for 3-decimal accuracy in stated region.	
Map of Procedures for Obtaining Cumulative Binomial Probabilities. . .	36
Fig. 12 maps recommended methods for 3-decimal accuracy. Regions for using Binomial table C5 appended; Normal table C6, for Normal, Gram-Charlier series, Type A, (GCA), and its remainder GCAr; and Poisson table C7 for Poisson and GC series, Type B (GCB).	
General	39
Limits of significant values, i.e., $.001 \leq B \leq .999$. Methods of checking cumulative Binomial values: Percentage Points, and p^n .	
Conclusion	42
Summary of foregoing. Alternative methods in Appendix A. Systematic reconnaissance in present work, future work suggested, with emphasis on improvements in interpolation and curve-fitting for future work. No general method found.	
Acknowledgments and background	43

TABLE OF CONTENTS

Page No.

APPENDICES:

A. Alternative Methods	45
and reasons for superseding in body of this work: Theoretical formulas. Ferris, with Poisson deviate t_p including fitting constant of 1. GGBr remainder method. Mosteller and Tukey approximations. Other approximations than Normal and Poisson, e.g., $(\sin x)/x$.	
B. Examples of Probability Computations	49
including notes on interpolation.	
C. Tables	57
log n (10-pl.); n! and log n ! (10-pl.); $(\frac{n}{x})$ and log $(\frac{n}{x})$, (5-pl.); e^{-x} (10-pl.); B(c,n,p), (5-pl.); $N, N_1, N_1^{(2)}$, (5-pl.); P(c,a), (5-pl.).	
D. References	77
E. One-page summary: map and context	81

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Ed S. Smith/med
Aberdeen Proving Ground, Md.
May 1950

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ABSTRACT

This self-contained report includes methods, graphs and tables by which binomial probabilities can be evaluated with errors that are always less than substantially 0.001.

SUMMARY OF RECOMMENDED PROCEDURES FOR OBTAINING
VALUES OF THE CUMULATIVE BINOMIAL PROBABILITY
WITHIN 3-DECIMAL ACCURACY UNIVERSALLY

In evaluating the cumulative Binomial probability B or $B(c,n,p) =$

$$\sum_{x=c}^n \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x} \quad \text{for any point } (c,n,p) \text{ in the domain: } 0 \leq p \leq 1,$$

$1 \leq n < \infty$, $0 \leq c \leq n$, the whole domain is divided (see Fig. 1) into six regions in which respective recommended procedures give values of B within .001.

In region 1, values of B can be found directly from a table (C5) of cumulative Binomial probabilities for $1 \leq n \leq 20$. If a table of B is available for other values of n and p , it will of course be used; otherwise the following approximations to B are available for use in the other regions as stated below. Before computing any values of these approximations, one can refer to graphs of percentage points for .001 and .999, see Figs. 14 and 13 of the report, to see whether it is necessary to compute such values.

In region 2, one can use the Poisson approximation $P(c,a) = \sum_{x=c}^{\infty} \frac{a^x e^{-a}}{x!}$ by entering a cumulative Poisson term table (C7) with values of the pair (c,a) . Molina has published convenient tables of Poisson terms for $a=np \leq 100$ which is accordingly taken as the upper limit of region 2. For a given n , the maximum error decreases as p approaches zero, from .001 at the righthand boundary of this region at $p \approx .008$ for $n > 20$.

In region 3, one can use the approximation

$$P_B(c,a) = P(c,a) - \frac{np^2}{2} [P(c,a) - 2P(c-1,a) + P(c-2,a)] \quad \text{where } P(0,a) = P(-1,a) =$$

$P(-2,a) = 1$, by entering the cumulative Poisson table with (c,a) , $(c-1,a)$ and $(c-2,a)$. This approximation is a 2-term modification of the Gram-Charlier series, type B. The maximum error of this approximation decreases from about .001 at $p=.1$, for $n > 20$, to a much lower value at the stated righthand boundary of region 2. While $P_B(c,a)$ can be used to the left of the last named boundary with less than .001 error, this is not necessary since the first term, $P(c,a)$, alone provides this accuracy there.

In region 4, one can use the Normal approximation

$$N(t_c) = \int_{t_c}^{\infty} \phi(t) dt = .5 - \int_0^{t_c} \phi(t) dt \quad \text{where } t_c = (c-a-.5)/\sigma, \quad a=np, \quad \sigma = \sqrt{npq},$$

$q=1-p$ and $\phi(t) = \frac{1}{\sqrt{2\pi}} e^{-t^2/2}$, by entering a Normal integral table (C6) of values of $\int_0^t \phi(t) dt$ with values of t_c . The maximum error of this approximation decreases as n increases and as p approaches .5, being about .001 at $p=.5$ and $n=28$ at the lower end of the lefthand boundary of region 4.

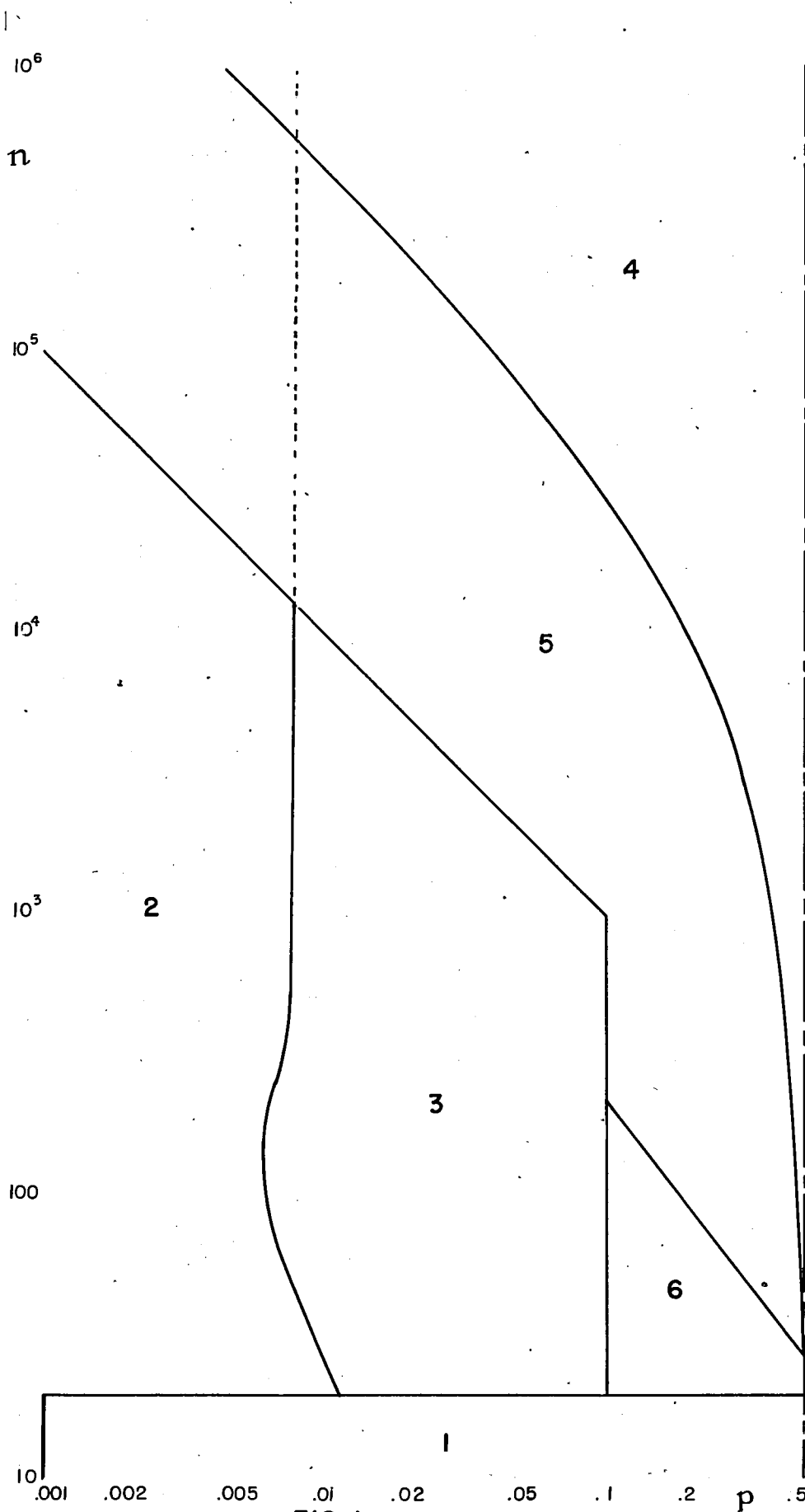


FIG. 1

In region 5, one can use the following approximation which comprises the Normal Approximation, $N(t_c)$, and the second term of the Gram-Charlier series, type A:

$$N_A(t_c) = N(t_c) - \frac{p-q}{6\sigma} \phi^{(2)}(t_c) \quad \text{where the second derivative}$$

$\phi^{(2)}(t_c) = (t_c^2 - 1) \phi(t_c)$. One uses t_c in entering tables (C6) of the Normal integral, density and/or second derivative of the density. The error of the approximation $N_A(t_c)$ does not exceed substantially .001 at the lefthand boundary of region 5. This error decreases as n increases, for a given p , and as p approaches .5, for a given n . While this approximation can be used in region 4 with much less than .001 error, the second term is of course not needed there to have the error less than .001.

In region 6, one can use the following "remainder" modification of the $N_A(t_c)$ approximation with less than .001 error for plural values of c :

$$N_{Ar} = N(t_c) + \alpha \phi^{(2)}(t_c) + r(t_c)/np \quad \text{where } \alpha \approx \frac{.351 (.5-p)^{.87}}{(np)^{.53}}$$

and $r(t_c)$ can be obtained from Fig. 9 of the report. Alternatively, α can be obtained from Fig. 8. As long as $.1 \leq p \leq .5$ and $a=np \geq 2$, this approximation (N_{Ar}) can also be used with less than .001 error for values of n outside region 6, but this is not recommended since it is simpler to use tables of B for lower n and the respective approximation N_A or N for higher n . The approximation N_{Ar} is the only one recommended for cumulative Binomial probabilities in the report, which involves empirical coefficients or curve-fitting.

*For $c=0$, use $B(0,n,p)=1$ and, for $c=1$ and $2 < a < 2.5$, use $B(1,n,p)=1-q^n$.

PREFACE

In many fields utilizing probability theory or mathematical statistics, both individual and cumulative Binomial probabilities must be readily available with up to three-decimal accuracy for increasingly large numbers of trials. This report systematically treats a number of practical procedures for obtaining such probabilities, including an indication of respective Normal or Poisson approximations used in the various mapped regions and the accuracy attained.

The report contains graphs and formulas for readily obtaining cumulative Binomial probabilities* within three-decimal accuracy everywhere. For instance, Gram-Charlier Series of types A and B are found to be useful in the regions in which the Normal and Poisson cumulative approximations, respectively, are the more accurate. For convenient reference by one already familiar with the recommended procedures, a summary of these is provided, including a map (Fig. E-1), of the respective regions in which their error is less than .001. Since for large numbers of trials, the direct computation of an individual Binomial probability is much less tedious than for a cumulative value which involves the computation of many individual terms, no corresponding effort has been made toward developing like means for obtaining individual probabilities.

Appended are alternative methods, typical examples of commonly useful procedures, tables used, and a list of references. Other points of related interest are also covered in the appendices, including interpolation procedures. This report is intended to include enough background and introductory material for its field use with a minimum of other material needed.

*

$$\sum_{x=0}^n \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}$$

is the cumulative Binomial probability or chance of obtaining at least c successes in n trials for probability p of success in a single trial.

NOMENCLATURE

$B(c, n, p)$	cumulative Binomial probability, or B, is the chance of obtaining
c	or more successes ($0 \leq c \leq n$) in
n	trials, for the probability
p	of success in a single trial.
$q = 1 - p$	probability of failure in a single trial.
$\binom{n}{x}$	number of combinations of n things taken x at a time
$n! = n \cdot (n-1) \cdot \dots \cdot 3 \cdot 2 \cdot 1$	"n factorial"
$a = np$	expected number of successes in n trials.
$\sigma = \sqrt{npq}$	"sigma" or standard deviation of the number of successes.
$t = \frac{c-a}{\sigma}$	standard deviate, or deviation from the expected number in units of the standard deviation.
$t_c = \frac{c-a-.5}{\sigma}$	standard deviate including continuity correction of .5
$t_b = \frac{c-a-1}{\sigma}$	Poisson deviate including fitting constant of 1.
$N(t_c)$ or N	cumulative Normal probability $\approx B$.
$\phi(t) = \frac{1}{\sqrt{2\pi}} e^{-t^2/2}$	standard Normal density function, tabulated for $\sigma=1$
$\phi^{(2)}(t_c)$	second derivative of Normal density, used in Gram-Charlier Series, Type A, evaluated at t_c .
$N_A(t_c)$	cumulative probability for GCA series for t_c .
N_{Ar}	cumulative probability for GCA series including a "remainder" correction term.
$\alpha = f(np, .5-p)$	coefficient for $\phi^{(2)}$ term of GCAr series, eq. 22.
$r(t_c)$	coefficient of GCAr remainder, $r(t_c)/np$.
$P(c, a)$	cumulative Poisson probability $\approx B$.
$P_B(c, a)$	cumulative Gram-Charlier series, Type B, probability

$N_i \approx \phi(t)/\sigma$ Normal approximation to B_i , $N_i = N(c=x) - N(c=x+1)$
 $P_i(x, a)$ individual Poisson term.
 $P_{Br}(c, a)$ cumulative probability of GCB series, eq. A2.
 t_{MT} deviate for a recent approximation [16].

INTRODUCTION

Many fields of endeavor need to have reasonably accurate values of cumulative Binomial probabilities $B(c,n,p)$ or B readily available for currently large numbers n of trials. $B(c,n,p)$ is the chance of obtaining at least c successes in n trials, where p is the probability of success in a single trial. Tables ^[1,2]* of cumulative Binomial probabilities are available for n through 150. Since for larger n , the direct computation of $B(c,n,p)$ is rather involved and tedious, various approximations are used in practice. Different approximations are required for a given accuracy in different n,p regions.

Maps of these regions are especially needed by only occasional users since the respective regions of applicability of the several methods are too numerous to be kept readily in mind. A systematic mapping treatment was needed so that, for a given n,p point, or combination of sample size n and chance p of a single success, one can select an approximation giving the necessary accuracy. Since such a treatment was not found in the literature, it is a main purpose of this work to fill that need.

It is also intended that this report complement tables of B for large n since such tables are so extensive that they are not likely to be available to one having only occasional need for values of B . Hence there appears to be a need for a treatment which is brief enough for field or occasional use and yet sufficiently accurate and nearly enough complete to serve many purposes.

In addition to the material required for field use, enough introductory material has been included to facilitate general use of this report, with only occasional reference to sources. The aim is to make it useful to engineers, mathematicians, or others, without requiring previous training in statistics. Since the present treatment may also serve as an introduction to the subject of probabilities for many readers, a partial, cursory review is included of some of the basic or elementary concepts to facilitate a grasp of the notation of probability by those previously unfamiliar with it. This is desirable because such concepts enable many short cuts to be taken in the computation of cumulative Binomial probabilities.

An effort has been made to permit such a mathematician or engineer to handle the simpler cases in a routine manner. But complicated or difficult cases are more readily and efficiently handled by one who is familiar with this specialized field and its conventions, definitions and terminology. The following four paragraphs, A-D, illustrate elementary relations occurring in the field of probability.

* Reference numbers are in brackets, and the references listed in Appendix D.

A. It is well known in this field that the number of combinations of n things taken x at a time is

$$\binom{n}{x} = \frac{n!}{x!(n-x)!} \quad (1)$$

where $n! = n(n-1)(n-2)\dots 1$, and $1! = 0! = 1$.

B. Three basic rules of probability may be noted: (I) If $P(A)$ is the probability that event A will occur and $P(B)$ is the probability that event B will occur, then the probability that either A or B will occur is

$$P = P(A) + P(B) \quad (2)$$

provided that A and B are mutually exclusive events, e.g., A = success and B = failure. (II) If $P(A, B)$ denotes the probability that both A and B will occur and $P_A(B)$ denotes the conditional probability that event B will occur when A is known to have occurred, then the probability that both A and B will occur is

$$P(A, B) = P(A) P_A(B) \quad (3)$$

(III) If the events A and B are independent, eq. 3 reduces to

$$P(A, B) = P(A) P(B) \quad (4)$$

These three rules are powerful tools, with many applications.

C. The probability of obtaining n successes in n trials is

$$B(c=n, n, p) = p^n \quad (5)$$

The chances of failure and success are complementary, or

$$q = 1-p \quad (6)*$$

Hence q^n is the probability of n failures (or the chance of 0 success) in n trials, and the probability of at least one success in n trials is 1 minus the chance of 0 successes or

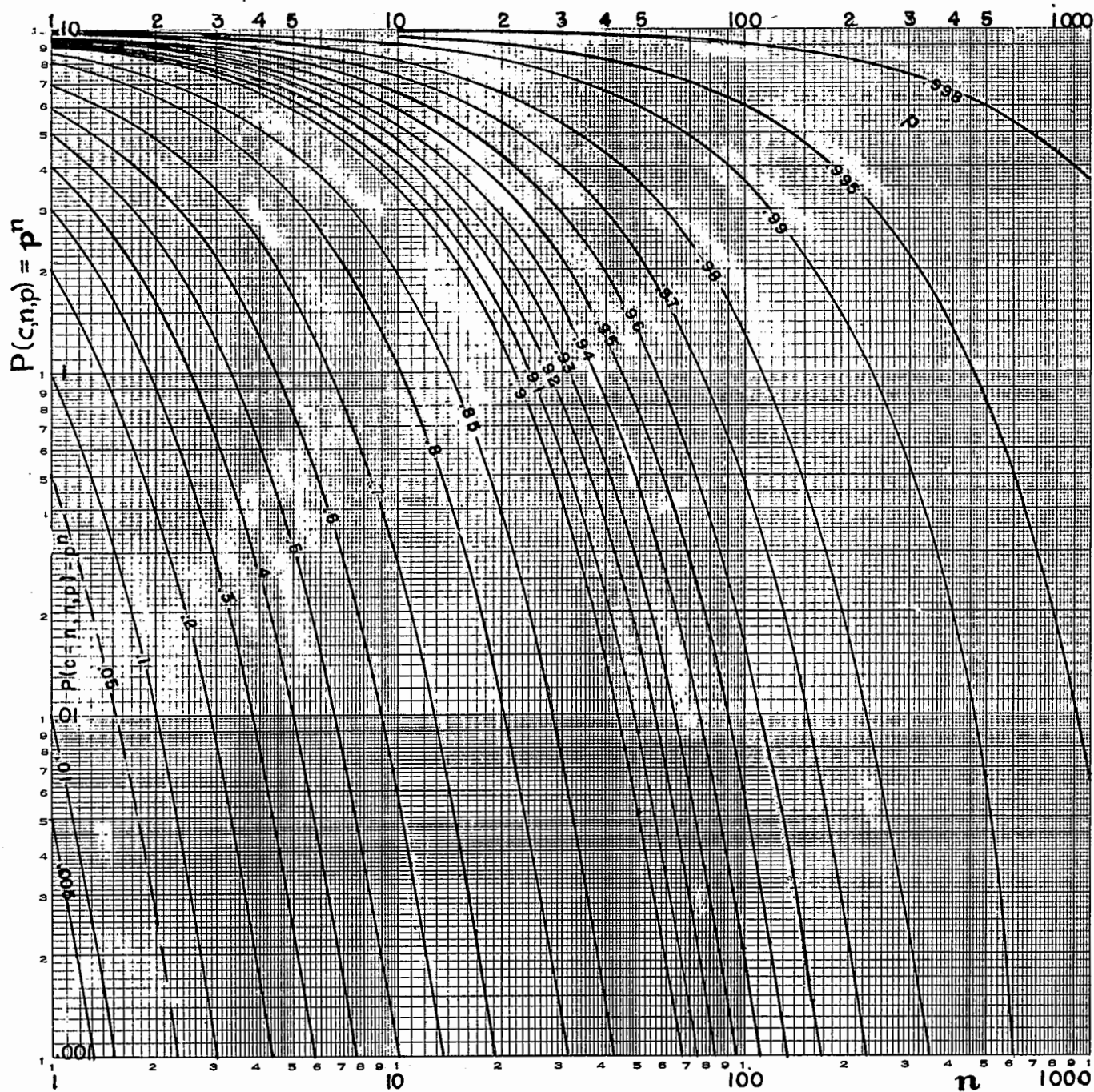
$$B(c=1, n, p) = 1-q^n \quad (7)$$

Hence Fig. 2 may also be used to find $B(c=1, n, p)$. Fig. 2 shows the usefulness of 3-decimal accuracy, i.e., no error larger than .001, in dealing with large n .

*F.N.: It may be parenthetically noted that

$$q = 1-p = e^{-(p + \frac{p^2}{2} + \frac{p^3}{3} + \dots)} \quad \text{and} \quad \ln q = -(p + \frac{p^2}{2} + \frac{p^3}{3} + \dots).$$

p^n = PROBABILITY OF n SUCSESSES IN n TRIALS
 WHERE p = CHANCE OF SUCCESS IN SINGLE TRIAL.



GRAPH OF $P(c, n, p) = p^n$

FIG. 2

D. The expected number of successes in n trials is $a = np$. On Fig. 3 values of a are plotted as contours on an isogram having p and n , respectively, as abscissa and ordinate. Significantly large values of B_i , i.e., individual Binomial probabilities, occur for c 's in the vicinity of a . Likewise* the maximum individual Poisson probabilities occur at $x=a$ and $x=a-1$ for $x \geq 1$, and at $x=0$ for $a < 1$.

The Normal and Poisson distributions** can be used to approximate both Binomial probabilities, comparisons being made for the cumulative case on Figs. 4 and 5.*** Fig. 4 shows that the approximation to the cumulative Binomial by the Normal is the better for p near .5 and by the Poisson for p near 0. Fig. 5 shows that the accuracy of the approximation is much better for the Normal as n alone increases from 50 to 100, and that this is not true for the Poisson. Fig. 5 also shows the difficulty of using the Normal at small n as an approximation to the Binomial. From the comparisons on Figs. 4 and 5, it appears that no single, general method of usefully approximating the Binomial is likely to be found, and that the raw Normal and Poisson distributions can be only a start toward the attainment of three-decimal accuracy in many regions.

Difficulties involved.

Several noteworthy difficulties are involved in attaining a compact treatment of Binomial probabilities of adequate accuracy. One arises in compressing the rather bulky tables into compact isograms, or contour graphs. This compression depends upon success in finding a basis for correlation good enough to reduce the number of parameters from four (the number in the B, c, n, p tables) to three which can of course be mapped on a single sheet.

A second difficulty arises from the stubbornness of integers when the approximation is bound to a continuous relation, or vice versa. A third is that the aid of keeping the maximum error within, e.g., .001 necessitates that a fairly large number of points must be checked in various ranges for each approximation finally used. A fourth is that the carrying of this accuracy down to low n , i.e., of the order of less than $n = 10$, involves the loss of direct help from relations theoretically obtained from the assumption that either n or a product including n approaches infinity, in other words, the approximations may have considerable error for low n .

* See Table II later herein.

** Defined later for those who are not already acquainted with them.

*** Figs. 4 and 5 are on "probability" paper, i.e., graph paper having the ordinate spacing for the Normal cumulative probability with the result that such a distribution function gives a straight line when graphed on this paper.

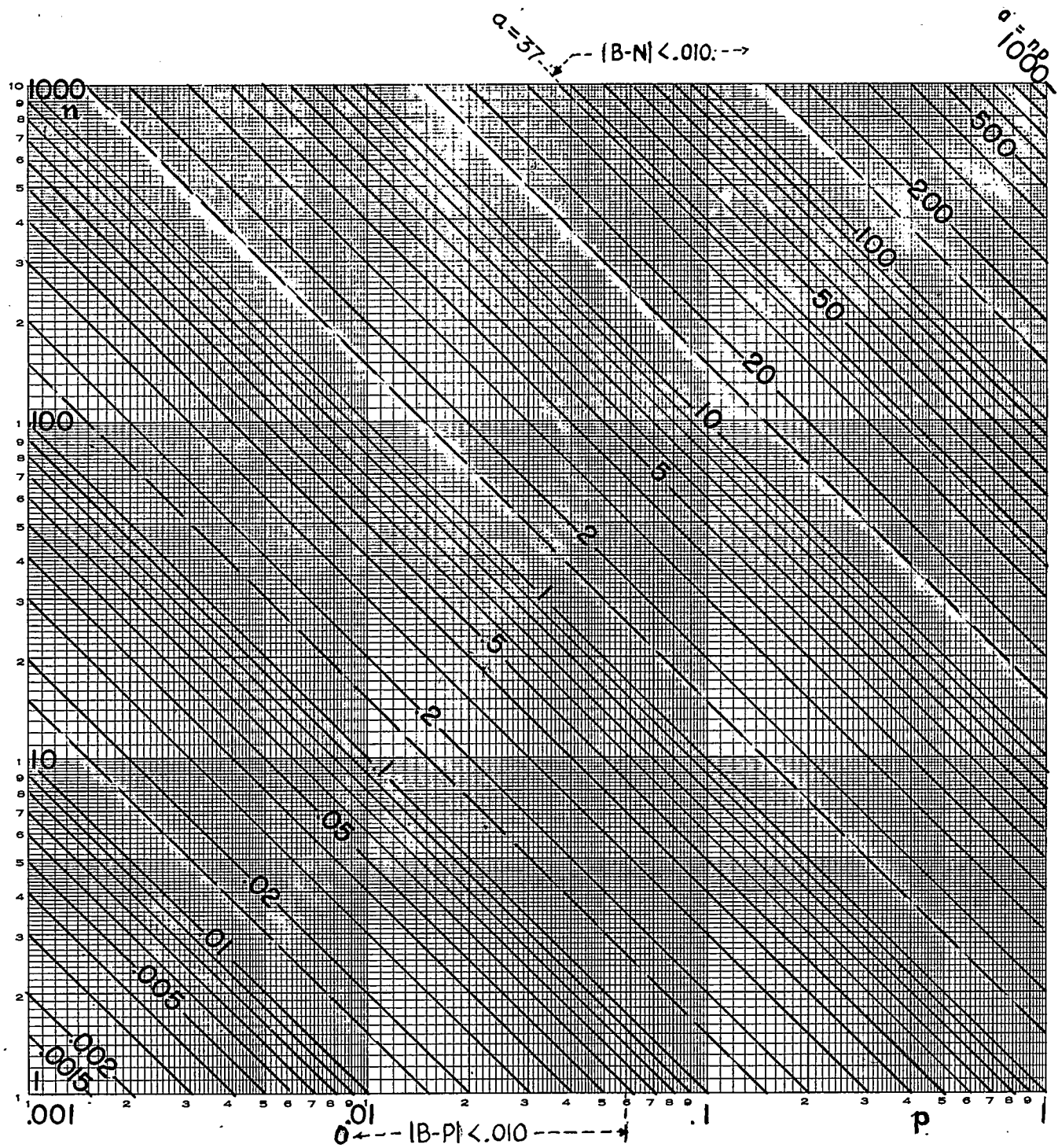
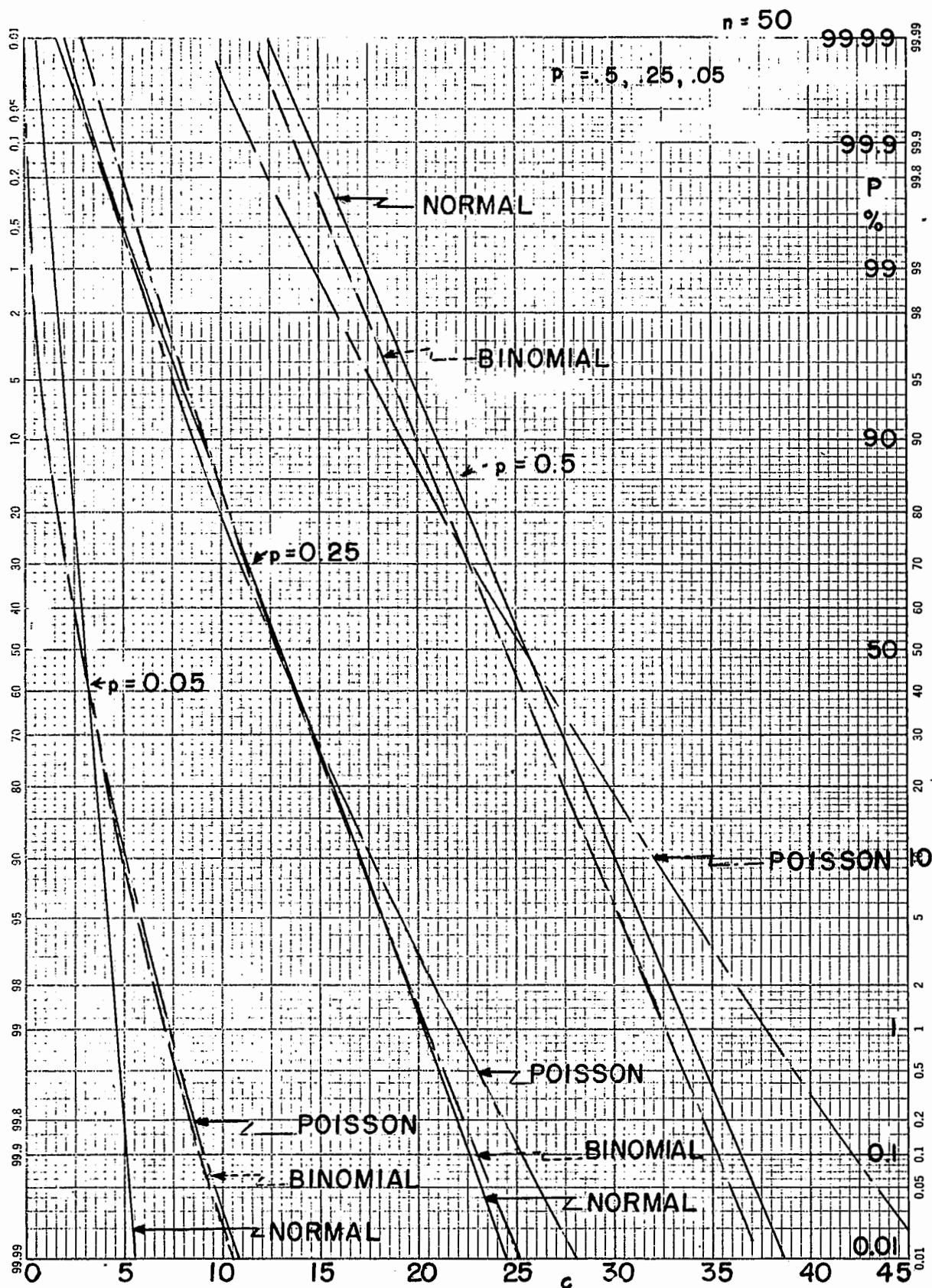
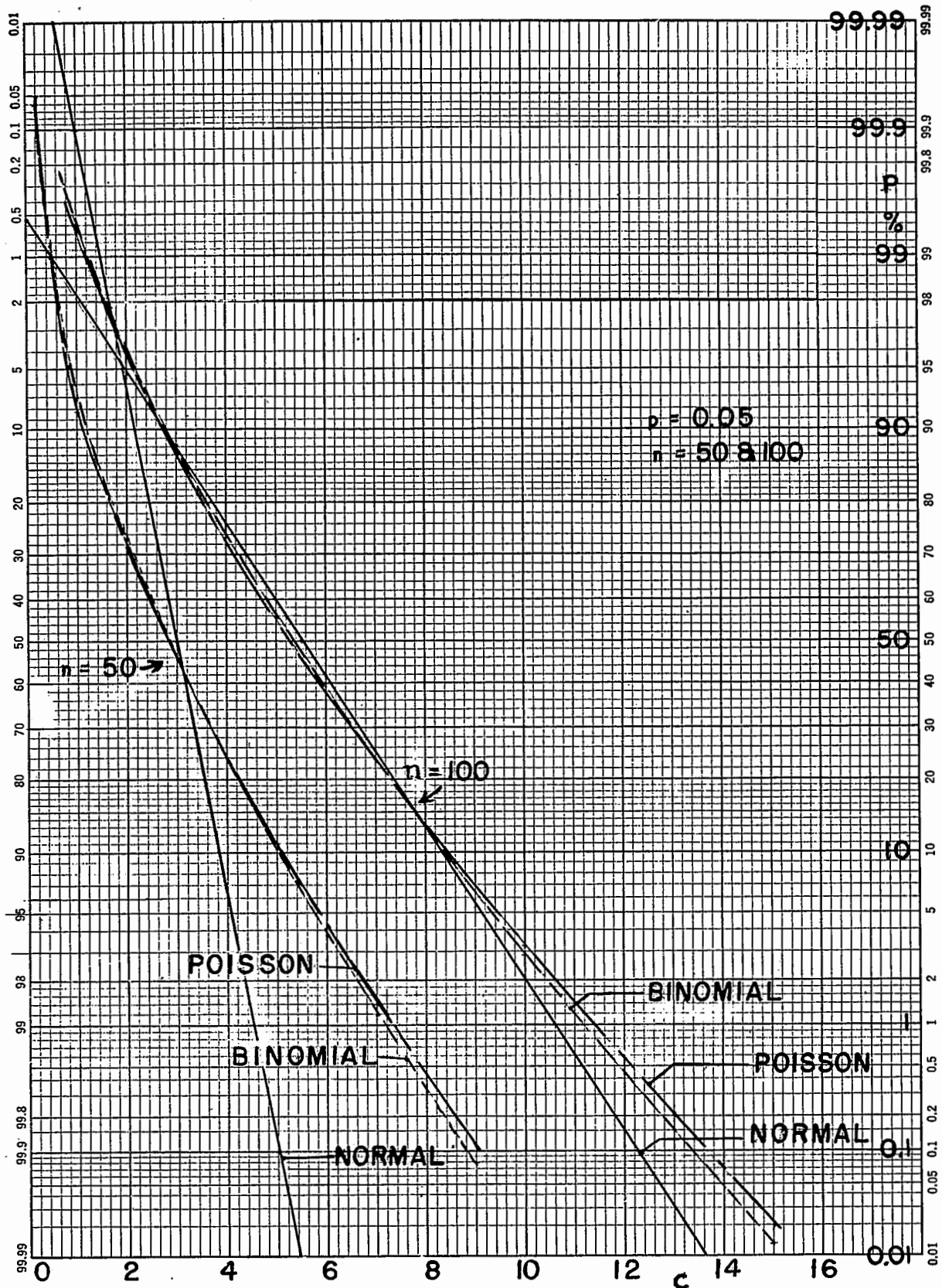


FIG. 3. GRAPH OF $a = np$ = EXPECTED NUMBER OF SUCCESSES.



COMPARISON OF THE NORMAL AND
POISSON TO THE BINOMIAL.

FIG. 4



COMPARISON OF THE NORMAL AND POISSON TO THE BINOMIAL.

FIG. 5

To obtain high accuracy at such low n involved extensive curve fitting as a basis for making useful modifications of such theoretically derived relations. These modifications include the insertion of empirical values into the theoretical relations and the graphing of any remainder (or "error") term against an appropriate parameter, thus taking full advantage of the flexibility of isograms.

Needless to say, no general method was found to apply to all regions; instead, different regions require different approximations. Even the present reconnaissance required considerable work which can only be justified by a considerable saving of time of others who, if this work had not been done, would have had to attack problems piecemeal in the different regions.

BINOMIAL PROBABILITIES

Individual Binomial probabilities are given by the various terms of the Binomial or Bernoulli distribution, and Binomial cumulative probabilities by the sum of such terms. For purposes of the present work, the Normal and Poisson distributions are used in obtaining closely approximate values of the Binomial probabilities, especially the cumulative. However, Normal and Poisson distributions are the correct ones, instead of the Binomial, to use in certain cases not treated herein. Maps provide an indication of where the unmodified Normal and Poisson distributions are useful approximations to the Binomial. The fact that modifications of these basic approximations enable one to obtain substantially 3-decimal accuracy everywhere (for $n > 20$) of values of Binomial Probabilities, is not to be taken as an indication that lower accuracy is not often adequate. Strictly 3-decimal accuracy is not guaranteed everywhere since the attainment of this accuracy at each point would have required a thorough survey with the expenditure of much more time than for the present reconnaissance.

With the Binomial distribution, the individual probability, or general term, is given by

$$B_i \equiv B_i(x, n, p) \equiv \binom{n}{x} p^x q^{n-x} = \frac{n!}{x! (n-x)!} p^x q^{n-x} \quad (8)*$$

and represents the probability of an event's happening exactly x times in n trials if the probability of the event's happening in a single trial is p . This follows since p , the probability of a single trial "success" is complementary to that, q , of a single trial "failure", or $q = 1 - p$ (6), success and failure being mutually exclusive as is necessary for eq. 8's evaluation of a Binomial or Bernoulli individual probability. While the same result can be obtained by counting the success probabilities taken in the different possible ways, eq. 8 is the more convenient basis, especially with a large n . [3, pp. 36-39] An individual Binomial probability

* See Table C1 for a ten-place table of logarithms handy for obtaining the power terms of eq. 8.

is readily computed by use of eq. 8 for any given n , p and x . Hence there is less need, than in the corresponding cumulative case, of devoting much space or effort to its approximations.

For $0 \leq n \leq 100$ and $0 \leq x \leq n$, the values, or their logarithms, of $\binom{n}{x}$ or $\frac{n!}{x!(n-x)!}$ are tabulated [4,5] for convenient use in eq. 8, see

table C3. Tables [6,7] of factorials or their logarithms are available for values of n from 1 through 1200, see table C2. For larger values of n , it is convenient to use either Stirling's formula for factorials:

$$n! = n^n e^{-n} (2\pi n)^{.5} \left(1 + \frac{1}{12n} + \frac{1}{288n^2} + \dots\right) \quad (9)$$

or Stirling's formula for logarithms of factorials:

$$\log n! \cong (n+.5)(\log n) - n(\log e) + \log(2\pi)^{.5} \quad (10)*$$

For the Binomial distribution, the cumulative probability, or that of obtaining at least c success in n trials, is

$$B \equiv P(c, n, p) = \sum_{x=c}^n \frac{n!}{x!(n-x)!} p^x q^{n-x} \quad (11)$$

Eq. 11 can be conveniently used only for small values of n with manual computation. But the results of using Eq. 11 can also be obtained, for $n-c$ from 1 through 50 and p from .01 through .99, from 7-decimal tables of the Incomplete Beta Function.[1] Likewise 6-decimal tables[8] exist for cumulative Binomial probabilities for n from 50 in intervals of $n=5$ through 100 for p from .01 through .99, using the relation $p + q = 1$. A similar 7-decimal table[2] is in preparation for n from 1 by integers through 150 and $.001 \leq p(.001) \leq .010$ and $.01 \leq p(.01) \leq .50$ and hence for $.50(.01).99(.001).999$ because of the nature of the Binomial function. Outside of the n, p regions covered by these tables, one can use suitable approximations including, notably, the Normal and Poisson distributions and the Gram-Charlier Series derived therefrom.

NORMAL PROBABILITIES

The Normal approximation applies adequately for present purposes throughout region "N" of Fig. 1. That region extends from large n and $p = .5$ to the bounding line which has a straight portion for which $np \cong 4000$. The Normal is symmetrical but the Binomial is increasingly skewed as p departs from .5, as is shown on Fig. 4.

* For $n \cong 170$, the error in the factorial approximation is of the order of .0025 and has a rough variation or dispersion with n of at least $\frac{1}{4}$.001.

A Normal approximation to the individual Binomial probability is given by

$$N_i = N_i(x, n, p) = \frac{1}{\sigma \sqrt{2\pi}} e^{-(x-a)^2/2\sigma^2} \quad (12)$$

where $\sigma = \sqrt{npq}$ and $a = np$. The Normal distribution is a function of a continuous variable. More strictly, the approximation should be obtained by integrating Eq. 12 between limits $x - .5$ and $x + .5$; however, this extra work does not seem to the writer to be justified since the individual Binomial probability term itself can be generally obtained directly with less bother. Eq. 12 is for what is commonly called the Normal density distribution.

The Normal probability integral, used as an approximation to the cumulative Binomial probability, is

$$N = N(c, n, p) = \frac{1}{\sigma \sqrt{2\pi}} \int_c^\infty e^{-(x-a)^2/2\sigma^2} dx \quad (13)$$

which is not directly integrable but which may be found from tables of the Normal distribution, which were generally prepared from series expansions of Eq. 13.

These tables [9, 10]* are commonly made up on the basis of a zero mean ($a = np = 0$) and unit standard deviation ($\sigma = 1$). They are entered with the deviate

$$t = \frac{x-a}{\sigma} \quad (14)$$

For this procedure, eqs. 12 and 13 become, respectively,

$$N_i = \frac{1}{\sigma} \phi(t) = \frac{1}{\sigma} \left[\frac{1}{\sqrt{2\pi}} e^{-t^2/2} \right] \quad (15)$$

$$\text{and } N = \int_{t_c}^\infty \phi(t) dt = .5 - \int_0^{t_c} \phi(t) dt \quad (16)$$

where t_c is given by Eq. 19 below.

* A short table of $\int \phi dt$, ϕ and ϕ^2 is appended as table C6.

The first and second derivatives of eq. 15 are

$$N_i^{(1)}(t) = -\frac{t}{\sigma} \phi(t) = -tN_i(t) \quad (17)$$

$$\text{and } N_i^{(2)}(t) = -\frac{1}{\sigma} (1 - t^2) \phi(t) = (t^2 - 1)N_i \quad (18)^*$$

which respectively indicate that the maximum value of the density N_i occurs at $t = 0$ and that the points of inflection are at $x = a \pm \sigma$.

The Normal N generally gives a closer approximation to the cumulative Binomial probability B when a continuity correction of .5 is used in computing the Normal deviate

$$t_c = \frac{c - np - .5}{\sigma} \quad (19)$$

At very large n (> 1000), the effect of the .5 adjustment becomes negligible. Figs. 6 and 7 respectively show the maximum errors of the Normal approximations, on this t_c basis, to the individual and cumulative Binomial probabilities.

The individual approximation term N_i can be taken as the difference between consecutive values of the Normal integral term, or

$$N_i = N(c=x) - N(c=x+1) \quad (20)$$

For example, for $x = 0$, $n = 10$ and $p = .1$:

$$\begin{aligned} N_i(x=0) &= N(c=0) - N(c=1) = N(t_c = -1.581135) - N(t_c = -.527045) \\ &= .44419 - .20090 = .24329, \end{aligned}$$

the $N(t_c)$ values being obtained from tables of $\int_0^{t_c} \phi(t) dt$ for the stated values of t_c .

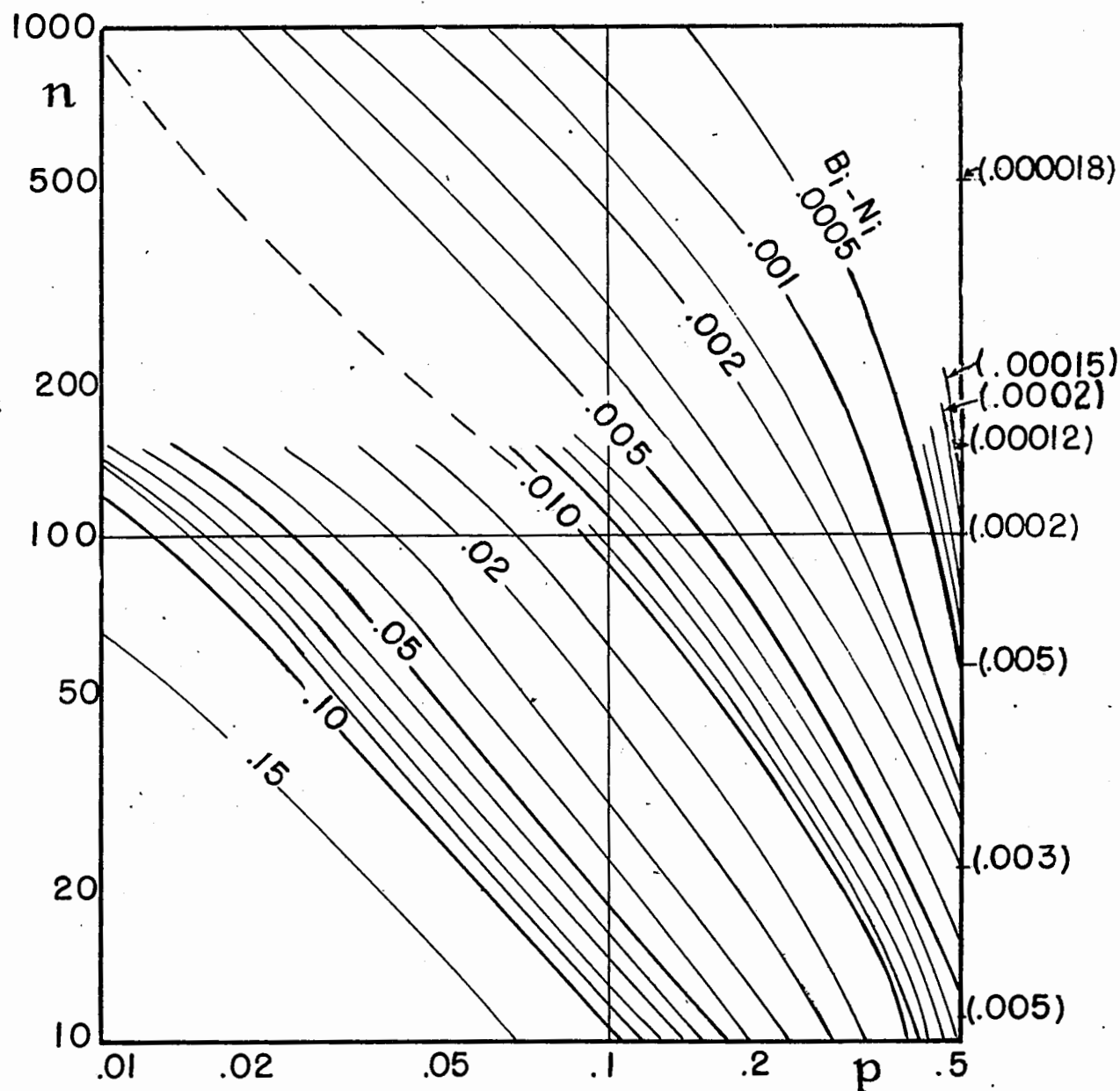
However this difference can be more conveniently, though more roughly, approximated simply by taking the decremental area as the product of the central ordinate ϕ by $\Delta t = \frac{1}{\sigma}$, the central ordinate for the simple deviate t being obtained from a table of $\phi(t)$. Thus, for $x = 0$ in the immediately preceding example, $t = 0 - 1 = -1.05409$,

$$\phi = .22889, \sigma = \sqrt{.9} \text{ and } \frac{\phi}{\sigma} = .24127^{**}, \text{ which is .00202 smaller than}$$

the earlier obtained value of .24329. Since the value of the individual Binomial probability for $x = 0$ is .34868, the Normal approximation is .10539 too low and the corresponding value of $\frac{\phi}{\sigma}$ is .10741 too low.

$$* \text{ Similarly, } N_i^{(3)}(t) = (3t - t^3) \frac{\phi(t)}{\sigma} = (3t - t^3) N_i$$

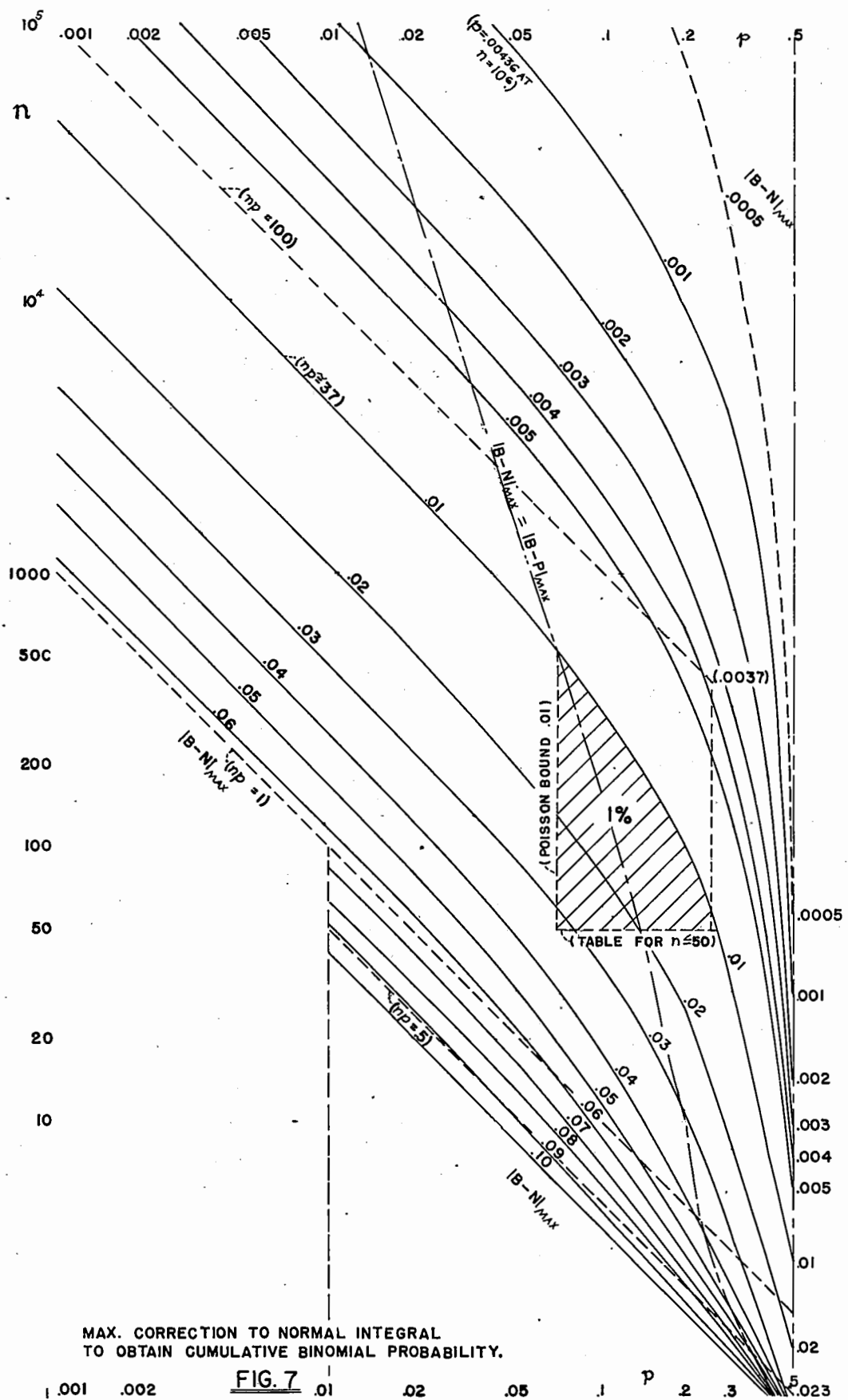
** An identical value is obtained σ by using eq. 15.



MAP OF MAX. CORRECTION OF NORMAL $\phi(t)/\sigma$ TO OBTAIN INDIVIDUAL BINOMIAL PROBABILITY.

$$B_i - N_i \cong B_i - \frac{\phi(t)}{\sigma}$$

FIG. 6



It may be noted that, if one were to mistakenly use the .5 continuity correction in obtaining $\frac{\phi}{\sigma}$, then $t_c = \frac{0-1.5}{\sqrt{.9}} = -1.5811$ and $\frac{\phi}{\sigma} = \frac{.11430}{\sqrt{.9}} = .12048$

which is .22819 too low, or still further off. Also it may be noted that the maximum difference between the Normal and Binomial individual "terms" occurs between $t \approx \pm 1$. The simple deviate was used in computing the maximum error of individual Normal values $\frac{\phi}{\sigma}$ which are

plotted on Fig. 6. Since the individual Binomial B_i is readily calculated, its Normal approximation is calculated as the simpler $\frac{\phi}{\sigma}$ instead

of by the difference N_i , the expedient $\frac{\phi}{\sigma}$ becoming more accurate as n increases.

GRAM-CHARLIER SERIES, TYPE A

For at least 3-decimal accuracy throughout the region in which $np^{1.24} > 12.7$, one can use the first two terms of the Gram-Charlier Series, Type A, for approximating cumulative Binomial probabilities:

$$N_A = \int_{t_c}^{\infty} \phi(t) dt - \frac{p-q}{6\sigma} \phi^{(2)}(t_c) - \frac{1-6pq}{24\sigma^2} \phi^{(3)}(t_c) - \dots \quad (21)$$

$$\text{where } \phi^{(2)}(t_c) = (t_c^2 - 1) \phi(t_c) \quad (22)$$

and $\sigma = \sqrt{npq}$.

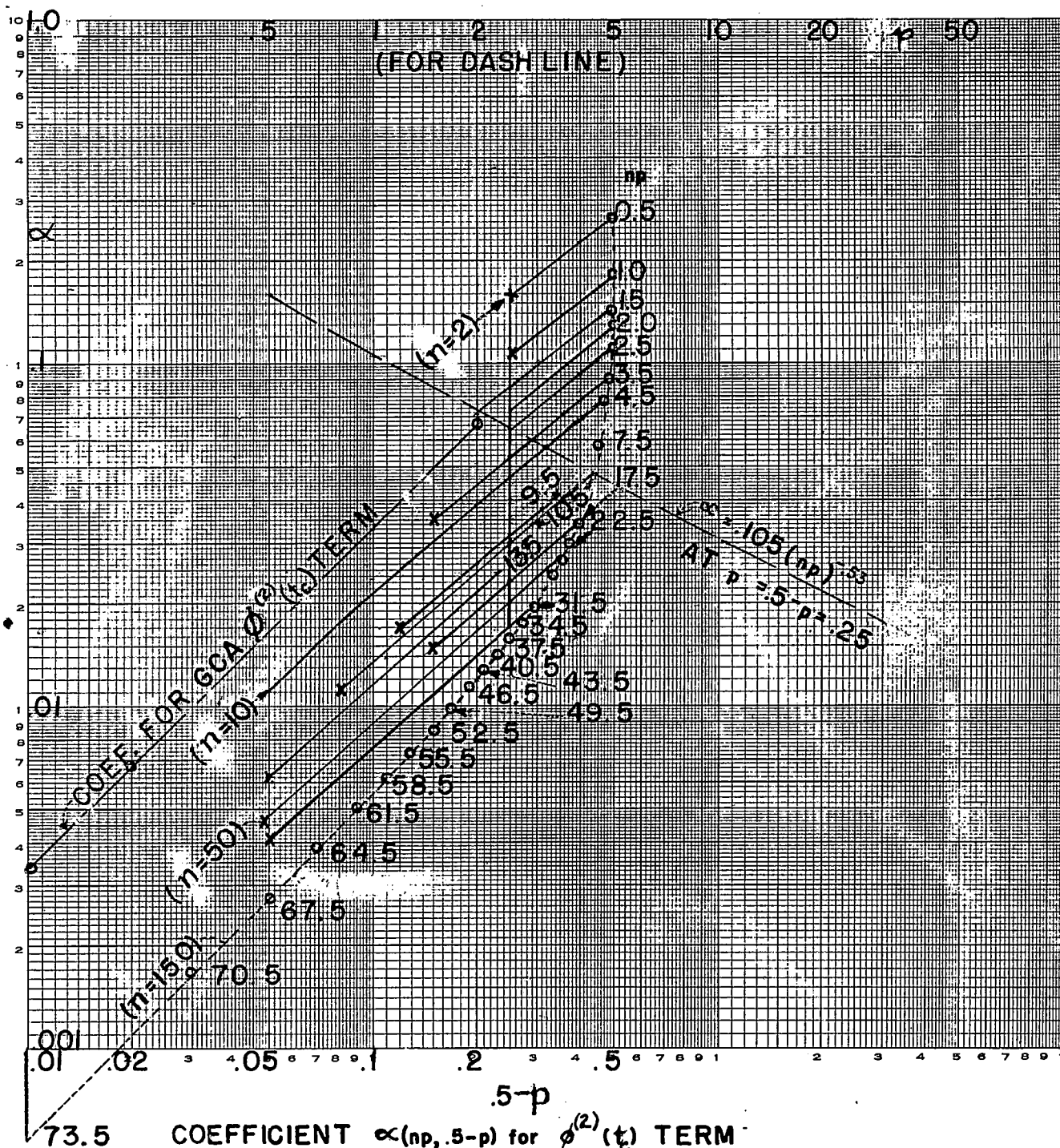
For reasons earlier discussed, the addition of the third and higher terms does not always lead to increased accuracy over the two-term series when the .5 adjustment is used in computing values of t_c . However, the addition of the second term materially increases the accuracy over that of the first term which is of course the Normal cumulative probability itself.

The GCAr "remainder" method.

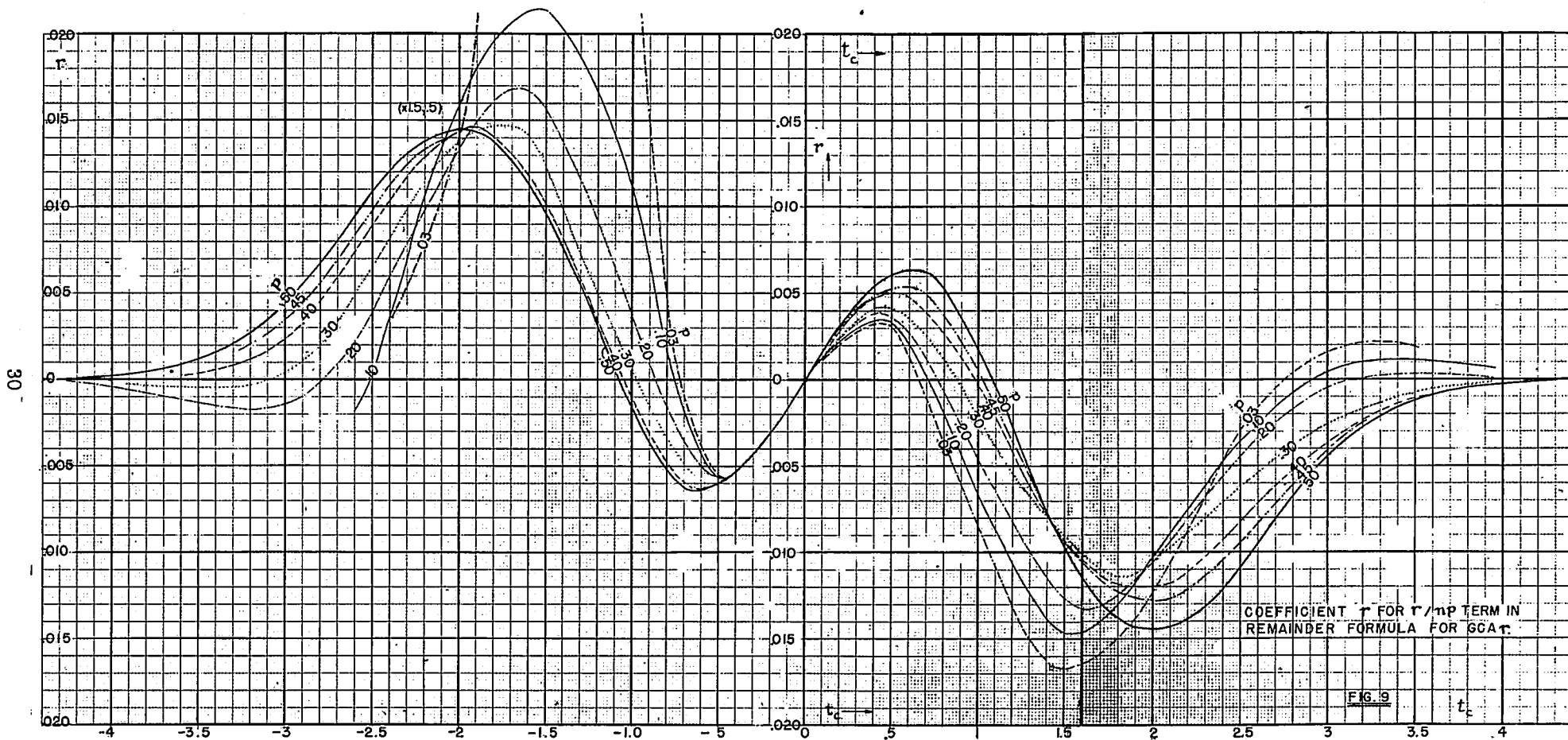
The Gram-Charlier type A series does not give values within the .001 limit for low values of both p and n . The following, related method extends this limit down to $np = 2$ for $.1 \leq p \leq .5$ and plural c . This modification uses the approximation

$$N_{Ar} = N(t_c) + \alpha \phi^{(2)}(t_c) + \frac{r(t_c)}{np} \quad (23)$$

$$\text{in which the coefficient } \alpha \approx .351 \frac{(.5-p)^{.87}}{(np)^{.53}} \quad (23a)$$



COEFFICIENT $\alpha(np, .5-p)$ for $\phi^{(2)}(t)$ TERM
IN REMAINDER FORMULA FOR GCAr. FIG. 8



and is graphed against $.5-p$ on log-log paper in Fig. 8*, while the remainder coefficient r is plotted against the deviate t_c for different values of p on Fig. 9. In effect, this use of the r, t_c diagram amounts to determining the difference $B-N-\alpha\phi^{(2)}$ and applying this as a correction, with a resultant error that is of the order of errors resulting from the graphical interpolation generally involved.

α is seen to correspond loosely with

$$A_1 = \frac{1}{3} \frac{(.5-p)}{(np)^{.5}(1-p)^{.5}} \quad (24)$$

the coefficient for $\phi^{(2)}$ in the GCA series, values of $A_1\phi^{(2)}$ and $\alpha\phi^{(2)}$ for $n = 50$ and $\phi_{\max}^{(2)} = .39894$ being as follows:

TABLE I

p	.10	.25	.40	.49	.50
$A_1\phi_{\max}^{(2)}$	.02689**	.01086	.003839	.0003762	0
$\alpha\phi_{\max}^{(2)}$	<u>.02508</u>	<u>.01099</u>	<u>.003861</u>	<u>.0004677</u>	<u>0</u>
Δ	.00181	.00013	.000022	.0000915	0
A_1/α	1.0722	.9878	.9943	.8044	0

* The values of np shown on Fig. 8 are those which were used in computing the values of α shown by the solid lines. The righthand ends of these lines are shown for $n = 150$. In using this graph, these solid lines give one the slope of the line one sketches in for the pertinent np , while the last-named line is put through a value of α at $p = .5-p = .25$ which is found from the dash line and the np scale at the top edge of this graph. Example 9 in Appendix B illustrates the use of this graph which is both more accurate and handier than eq. 23a for anyone who computes many values of α . However, eq. 23a can be used instead by anyone who prefers formulas to graphs or considers Fig. 8 complicated.

Fig. 8 also has A_1 plotted on the same scale as α against $.5-p$ as a dot-dash line for comparison of this theoretical coefficient with the actual α .

** The n and p for this point are far below the respective n and p recommended herein for the GCA series itself.

Since for a given p , the difference Δ decreases as n increases, there is no need to use the remainder method for $np > 22$. There the GCA series is preferable as it enables one more directly to obtain reliable values for $p < .1$.

r , the remainder.

The GCA $\phi^{(3)}$ term, like that of the other odd derivatives of this series, has true odd* symmetry only for $p = .5$. However, the r, t_c graph flexibly takes care of this lack of true odd symmetry for all other values of p , due to the excellent correlation, as to np for $np > 2$, of an r, t_c curve for a given p . For example, the values of r for $np = 1.5$ do not depart much from the curve for $p = .5$. Fig. 9 shows that the "inboard" swings are smaller than the "outboard" swings for any given p , whereas the opposite is true for the third derivative of ϕ .

For the even symmetry components (mostly $\phi^{(2)}_{\max}$) less than .001; it is found from the empirical formula (23a) for α that $np > 11,200(.5-p)^{1.64}$ or that $n_1 > 11,200(.5-p)^{1.64}/p$. This latter relation provides a check on the .001 curve plotted on Fig. 5 since this curve has only even components appreciable at the higher values of n . Thus, for the odd symmetry components (r/np) less than .001, $np > 12.7/p^{.24}$ or $n > 12.7/p^{1.24}$. Also, for comparison, it may be repeated that the remainder method is within .001 for all values of non-trivial difficulty (i.e. excluding $c=0$ and $c=1$), for $np > 2$ or $n > 2/p$ as long as $.1 \leq p \leq .5$. One can refer to example 9 in Appendix B for the use of this GCAr method.

POISSON PROBABILITIES

The expected number of successes in n trials is $a=np$ when p represents the probability of success in a single trial. This relation is used in the Poisson distribution.

The individual Poisson term approximating the corresponding Binomial term of eq. 8 is

$$P_i \equiv P(x, a) \equiv \frac{a^x e^{-a}}{x!} \quad (25)$$

with maxima as in Table II values for Poisson Molina table [II].

The cumulative Poisson probability which approximates the Binomial of eq. 11 is

$$P(c, a) \equiv \sum_{x=c}^{\infty} \frac{a^x e^{-a}}{x!} \quad (26)$$

* Using the symmetry nomenclature familiar in Fourier series analysis, "even" symmetry has the righthand and lefthand sides like mirror images, while "odd" symmetry requires also that the opposite sides have opposite signs, i.e., have the images inverted in addition to being reversed. For ϕ , the even and odd derivatives have even and odd symmetry when graphed against the appropriate deviate.

TABLE II

The Maximum Individual Poisson Probability $P(x,a)$
for the Tabulated Values of $a = np$.

x	a	$P(x,a)$	x	a	$P(x,a)$
0 ↓	.001	.999001	0	.7	.496585
	.002	.998002	0	.8	.449329
	.003	.997005	0	.9	.406570
	.004	.996008	↓ $X=a$ and $X=a-1$	11	.367879
	.005	.995013		2	.270671
	.006	.994018		3	.224042
	.007	.993024		4	.195367
	.008	.992032		5	.175467
	.009	.991040		6	.160623
	.01	.990050		7	.149003
	.02	.980199		8	.139587
	.03	.970446		9	.131756
	.04	.960789		10	.125110
	.05	.951229		15	.102436
	.06	.941765		20	.088835
	.07	.932394		25	.079523
	.08	.923116		30	.072635
	.09	.913932		35	.067273
	.10	.904837		40	.062947
	.15	.860708		45	.059361
	.20	.818731		50	.056325
	.25	.778801		60	.051432
	.30	.740818		70	.047626
	.40	.670320		80	.044557
	.50	.606531		90	.042013
	.60	.548812		100	.039861

From the Molina Tables [11].

For the Poisson distribution terms of eqs. 25 and 26, 6-decimal tables [11] are available for $a=np$ from .001 through 100 and for x and c , respectively, from 0 through 150. In general, values of x and c giving significant values of B_i occur in the neighborhood of $a=np$ which is graphed in a convenient form on Fig. 3.

Alternatively, identical cumulative Poisson values can be obtained less conveniently from 7-decimal tables [12] of the Incomplete Γ (Gamma) Function, for integer values of c and n for values $u_T = \frac{a}{\sqrt{c}}$ from 0 through 13.8 and for $P_T = c-1$ from 0 through 50.0, where u_T and P_T are used in entering the tables, the subscript T being used to identify table-entry terms.

Poisson individual term errors.

The maximum correction for $N_i - P_i$ is mapped on the n, p graph of Fig. 10. The curves are somewhat smoothed, especially near the line $np = 1$ for $.2 < p < .5$, the smoothing being such that the corrections are generally within the limits shown.

The correction curve for $n = 1$ is continuous, since the maximum correction occurs throughout for $x = 1$, and is nearly linear on log-log paper between corrections .001 and .196735 respectively for p 's .032 and .5. For $n = x = 1$,

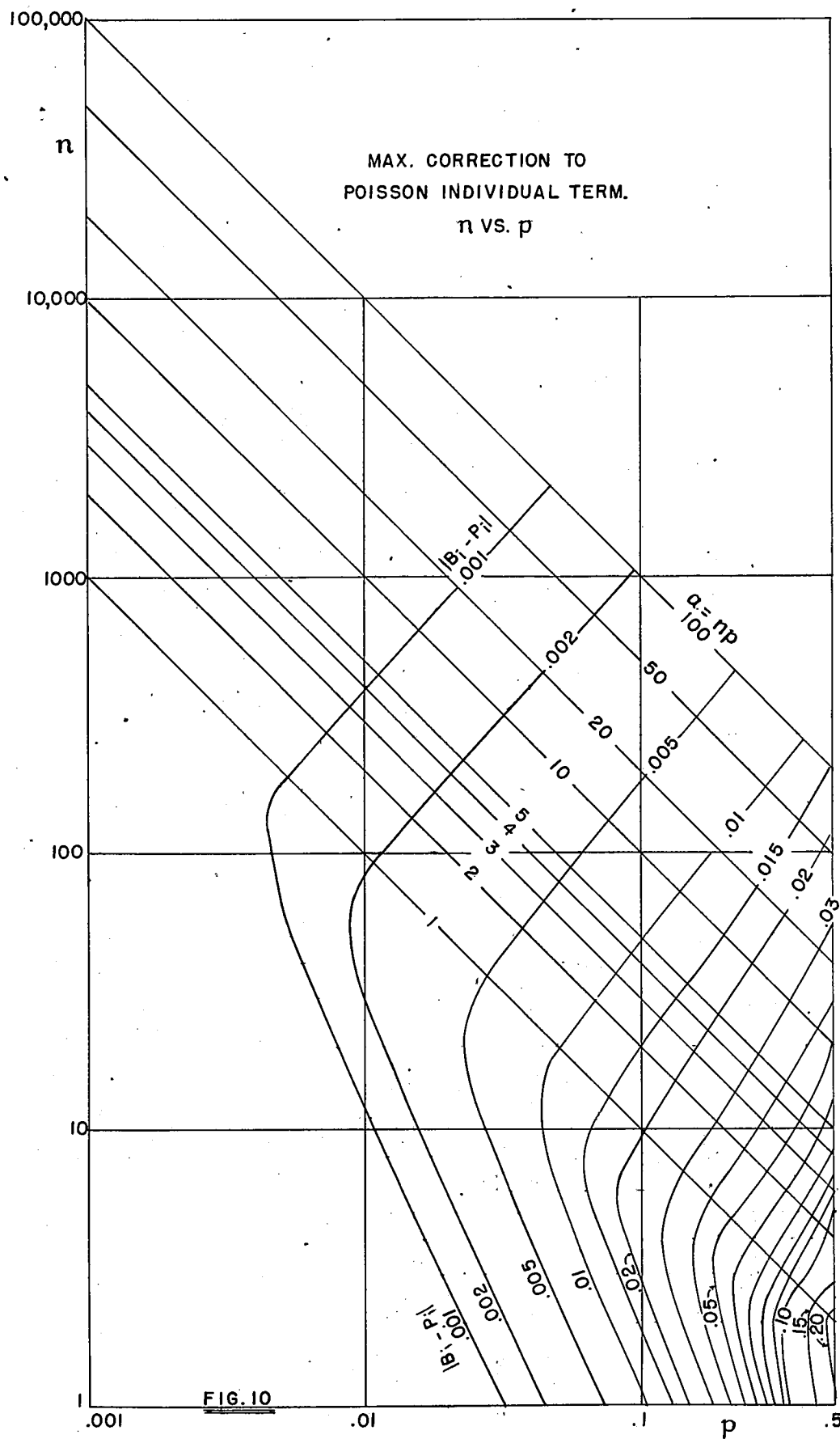
$$B_i - P_i = p(1 - e^{-p}). \quad (27)$$

The correction curve for $n = 2$ is also continuous with the maximum correction likewise at $x = 1$, the curve's higher- p end being strongly curved. But the curve for $n = 3$ is not continuous everywhere since the maximum correction occurs at $x = 1$ for low values of p , at $x = 0$ for $p = .4$, and at $x = 2$ for $p = .5$. A smoothed curve has been sketched through relatively few points for $n = 3$ since a more thorough exploration would take more time than is justified for these individual Poisson probability corrections in view of the fact that the exact values of the individual Binomial probabilities are readily found from eq. 8.

The departures of the actual corrections from the smoothed curves become less as n increases. Thus for $n \geq 20$, e.g., the maximum correction occurs for x at the expected number $a = np$ when this is an integer, and at the next higher integer when a is halfway between integers. To illustrate the use of Fig. 10, this shows that, at $n = 10$ and $p = .1$, $B_i - P_i = .02$, which closely checks the computed value of .01954.

Poisson cumulative term errors.

The Poisson cumulative values are within the 3-decimal limit throughout region "P" of Fig. 5. Fig. 11 shows values of the maximum error on an n, p map.



In an earlier work ^[13] by Ferris, the correction (B-P) was taken as independent of n for the region: $n > 50$, $p < .25$ and $np < 100$. The maximum values $(B-P)_{\max}$ of this correction are also shown on Fig. 11. A heavy dash line on Fig. 11 graphs, against p as abscissa, $(B-P)_{\max}$ for the scale along the righthand edge of the grid. Appendix A of the present work includes a discussion of the Ferris method and other alternatives to the methods recommended herein.

GRAM-CHARLIER SERIES, TYPE B

The Type B Series (first three terms) is

$$P_B(c,a) \approx P(c,a) - \frac{np^2}{2} [P(c,a) - 2P(c-1,a) + P(c-2,a)] \\ - \frac{np^3}{6} [P(c,a) - 3P(c-1,a) + 3P(c-2,a) - P(c-3,a)] - \dots \quad (28)$$

where we put $P(0,a) = P(-1,a) = P(-2,a) = P(-3,a) = \dots = 1$.

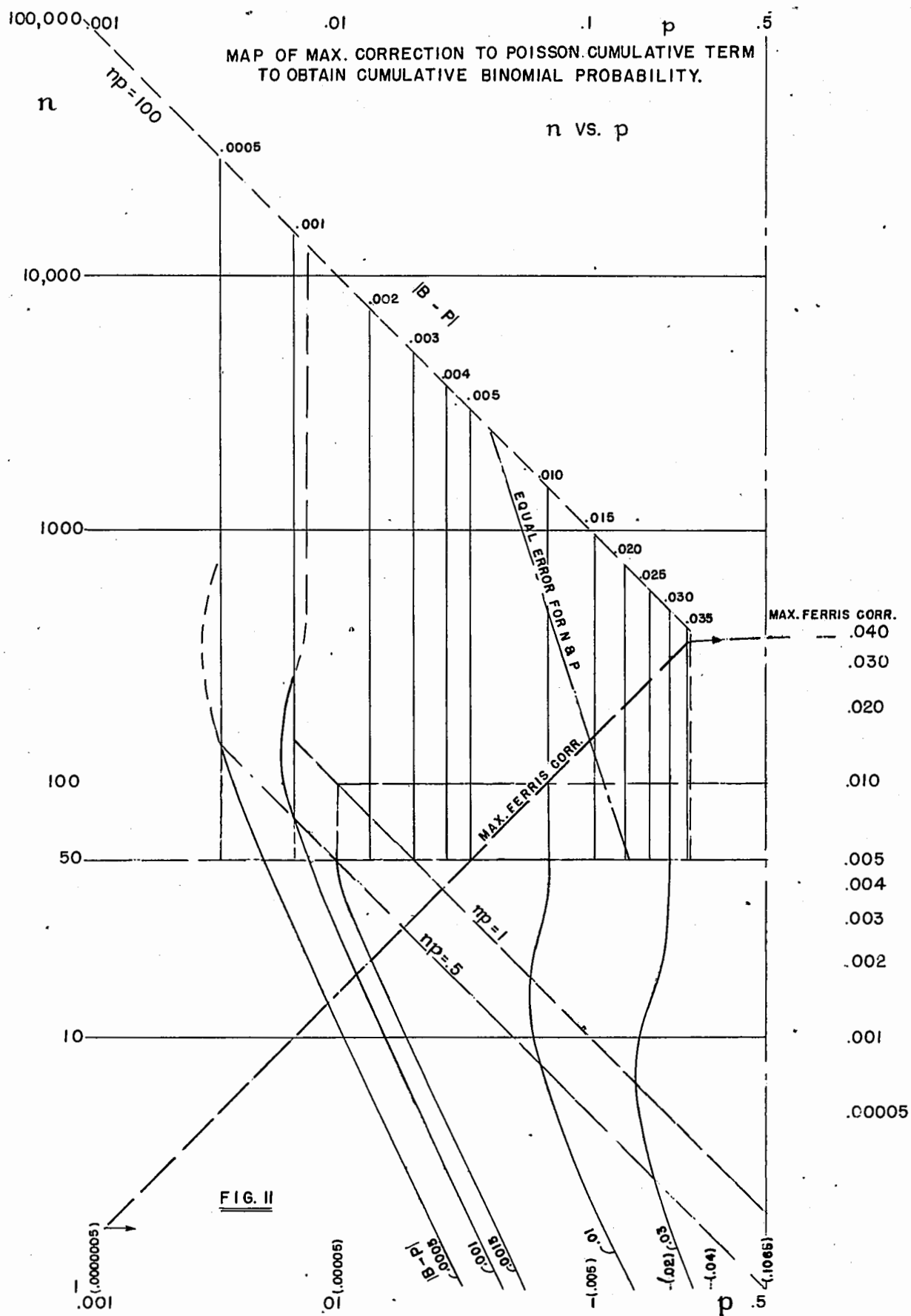
The second and third terms of this series are seen to be the second and third backward differences, respectively. The two leading terms are readily calculated and provide 3-decimal accuracy throughout the region "GCB" in Fig. 1, i.e., to the left of the $p = .1$ line for $n \geq 10$. As a practical matter, one is limited for an available table ^[11] to values of $np \leq 100$ for the Poisson cumulative terms and hence also for those of the Type B Series which depends on the Poisson.

MAP OF PROCEDURES FOR OBTAINING CUMULATIVE BINOMIAL PROBABILITIES

Fig. 12 is an n,p map for this purpose, accompanied by a cursory identification of several recommended approximations and procedures. Earlier-mentioned tables ^[1,2] are available giving the values of B for $n \leq 150$. Appendix B to this report contains a table of B for $1 \leq n \leq 20$.

The Normal approximation is seen to be within .001 for $n=28$ at $p=.5$, and from $p=.5$ to the left to the .001 bound having a straight portion for which $np \approx 4000$ for high n. The Poisson approximation is likewise seen to have this accuracy from $p=0$ up to approximately .01 for n larger than 10. A dot-dash line shows where the errors of these two approximations are equal, with a maximum error of .08 occurring at the bottom of this line, i.e., at $n = 1$ and $p = .43$. The position of the top of this line at the intersection of the .001 bounds of the Normal and Poisson was obtained by extrapolation.

The Gram-Charlier Series, Types A and B, (two terms) are respectively based on the Normal and Poisson distributions and tend to have minimum errors on the respective sides of the dot-dash line. In other words, the error of either GC series is roughly proportional to the error of its leading term. The GCA series (two terms) is seen to be within .001 for $np^{1.24} \geq 12.7$ while the 2-term GCB series is similarly accurate for p less than .1 for $n \geq 10$.



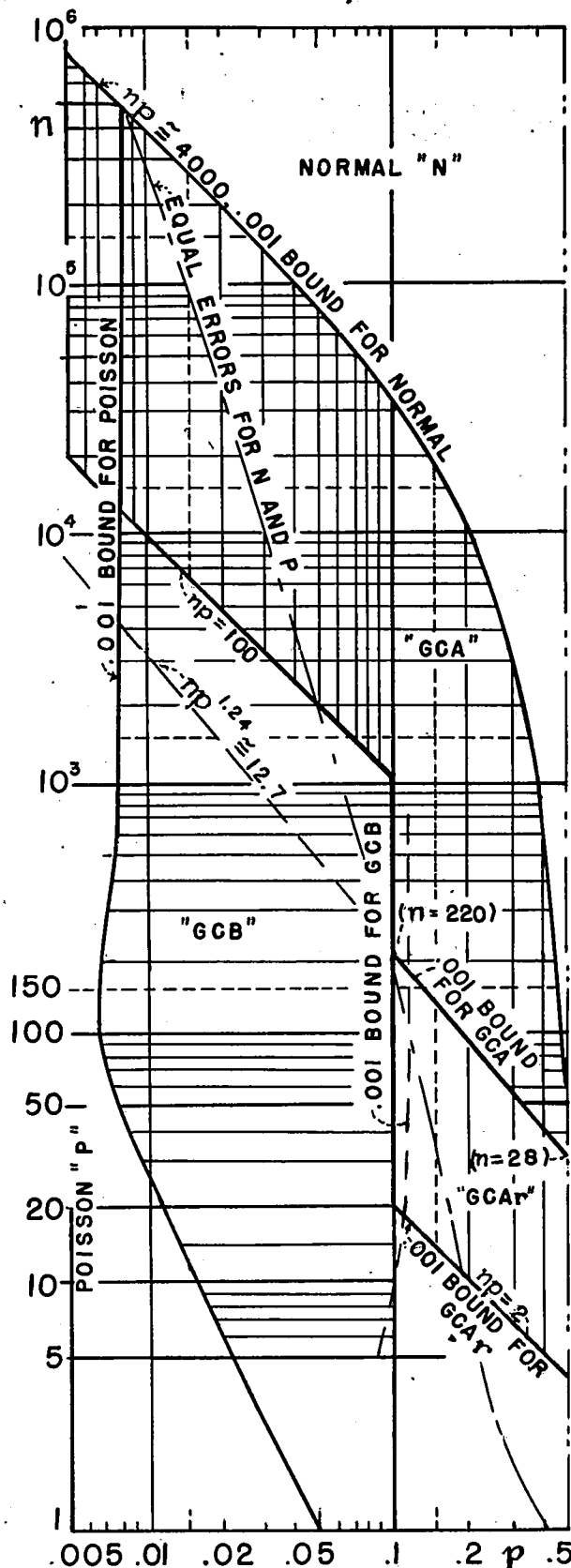


FIG. 12

TO OBTAIN VALUES WITHIN .001 OF THE CUMULATIVE BINOMIAL PROBABILITY, B .

The maximum correction required for Normal or Poisson distribution is given by Fig. 7 or 11, respectively.

For 3-decimal accuracy:

1. Proceed only if case is non-trivial, i.e. if $.001 \leq B \leq .999$, for given value of c . See Figs. 12 & 13.

2. Use available tables for B in region $1 \leq n \leq 20$, $.01 \leq p \leq .50$, see appended table C5. In region $n > 20$ & $1 \leq n-c \leq 50$, $.01 \leq p \leq .50$, one can less conveniently use Incomplete Beta Function table [11], Ex. 4.

3. In region "N", enter the Normal table C6 with $t_c = (c - a - .5) / \sigma$ (19) where $a = np$, $\sigma = \sqrt{npq}$ and $q = 1 - p$, to

obtain $\int_0^{t_c} \phi(t) dt$. Then

$$N(t_c) = .5 - \int_0^{t_c} \phi(t) dt \quad (16), \text{ Ex. 17.}$$

4. In region "GCA", likewise obtain value of $\phi^{(2)}(t_c)$ from table C6. Then use 2-term Gram-Charlier Series, Type A:

$$N_A(t_c) = N(t_c) - \frac{p-q}{6\sigma} \phi^{(2)}(t_c) \quad (21), \text{ Ex. 8.}$$

5. In region "GCAr", use equation 23 for $c > 1$:

$$N_{Ar}(t_c) = N(t_c) + \alpha \phi^{(2)}(t_c) + \frac{r(t_c)}{np}$$

with α from Fig. 8 for np , $.5 - p$ and $r(t_c)$ from Fig. 9. Use $B(0, n, p) = 1$ and $B(1, n, p) = 1 - q^n$ for $2 < a < 2.5$.

6. In region "P", use $P(c, a)$ from table C7 or the Poisson-Molina table II [11], Ex. 9. Less conveniently, one can use Incomplete Gamma Function table [12], Ex. 10.

7. In region "GCB", use 2-term Gram-Charlier Series, Type B, equation 28:

$$P_B(c, a) = P(c, a) - \frac{np^2}{2} [P(c, a) - 2P(c-1, a) + P(c-2, a)] \quad \text{where}$$

$$P(0, a) = P(-1, a) = P(-2, a) = 1, \quad \text{Ex. 11.}$$

MAP OF n, p REGIONS IN WHICH THE STATED PROCEDURES GIVE 3-DECIMAL ACCURACY.

The recommended regions of use of the GCA and GCB Series differ slightly as follows from the limits just stated. Within the upper limit, $np \leq 100$, of the Poisson-Molina tables*, [11] the Poisson approximation and the GCB Series are handier to use than the Normal approximation and the GCA Series. The first two terms of the GCA series are used for $np \geq 100$ and also for $p \geq .1$ and $np \geq 22$.

A three-term modification designated herein as "GCAR Series" gives 3-decimal accuracy for plural c ** $.1 \leq p \leq .5$ and $2 \leq np \leq 22$ by including the remainder term of the two-term GCA series. At $p \geq .1$, this GCAR modification overlaps the appended table C5 of B, with the result that 3-decimal accuracy is obtainable everywhere by the use of this report alone.

GENERAL

Limits of significant values.

There is obviously no advantage in comparing values of B smaller than the error of the approximation involved in the latter's computation. In the present work, the maximum error of the approximations was set at .001. Hence Figs. 13 and 14 are included to show respectively, least values of c for the .999 bound of B and largest values of c for the .001 bound of B. These values of Figs. 13 and 14 are respectively related with .001 and .999 percentage points of $c' = c - 1$ as follows:

$$B(c, n, p) = \sum_{x=c}^n \binom{n}{x} p^x q^{n-x} = 1 - \sum_{x=0}^{c-1} \binom{n}{x} p^x q^{n-x} = 1 - \alpha'.$$

These values of c were obtained from a table [2] of cumulative Binomial probabilities by the use of (Normal) "probability paper" for making nearly linear interpolation possible. No attempt was made to obtain fractional values of c with high accuracy since integers only are generally used in actual work.

Percentage point tables and graphs.

Percentage point tables [14] and graphs [15] can be used for checking values computed by the different methods, although percentage points*** are ordinarily used for other purposes. Since the use of the graphs is more direct than that of the percentage point tables, the graphs are useful for present checking purposes mainly in providing approximate values of c, n and p for use in the 5 significant figure percentage point tables which require interpolation. The set of tables [14] comprises separate

* A drastically condensed table of cumulative Poisson probabilities for $np \leq 100$ is included as Appendix C7 for field use.

** Use $B(0, n, p) = 1$ everywhere and $B(1, n, p) = 1 - q^n$ for $2 < a < 2.5$.

*** A percentage point is commonly given by the value of p having the stated (α') percentage chance of obtaining not more than c' successes in n trials.

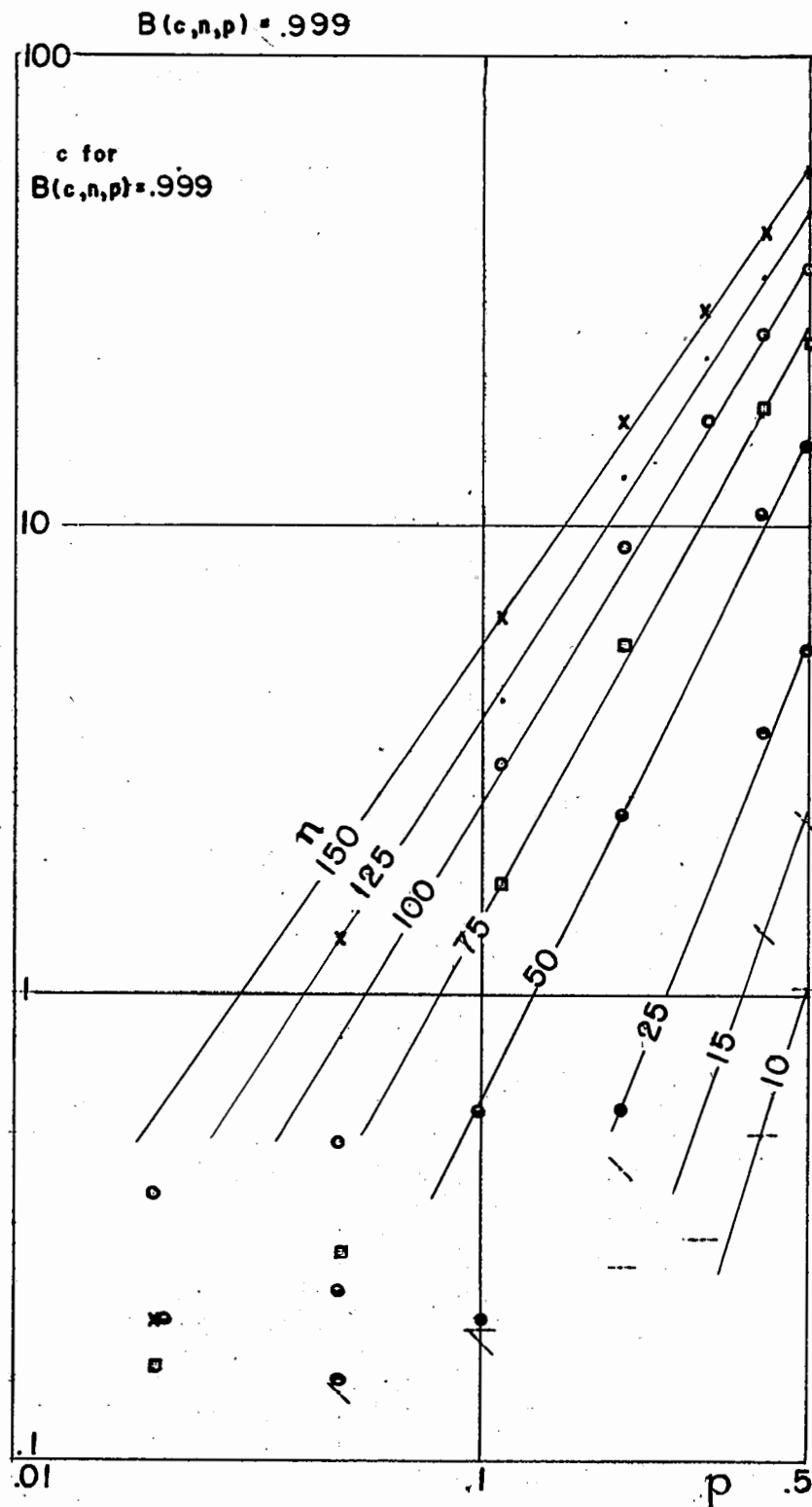


FIG. 13
VALUES OF c , n AND p FOR THE HIGHEST
VALUE (.999) OF $B(c, n, p)$ TO BE
CALCULATED.

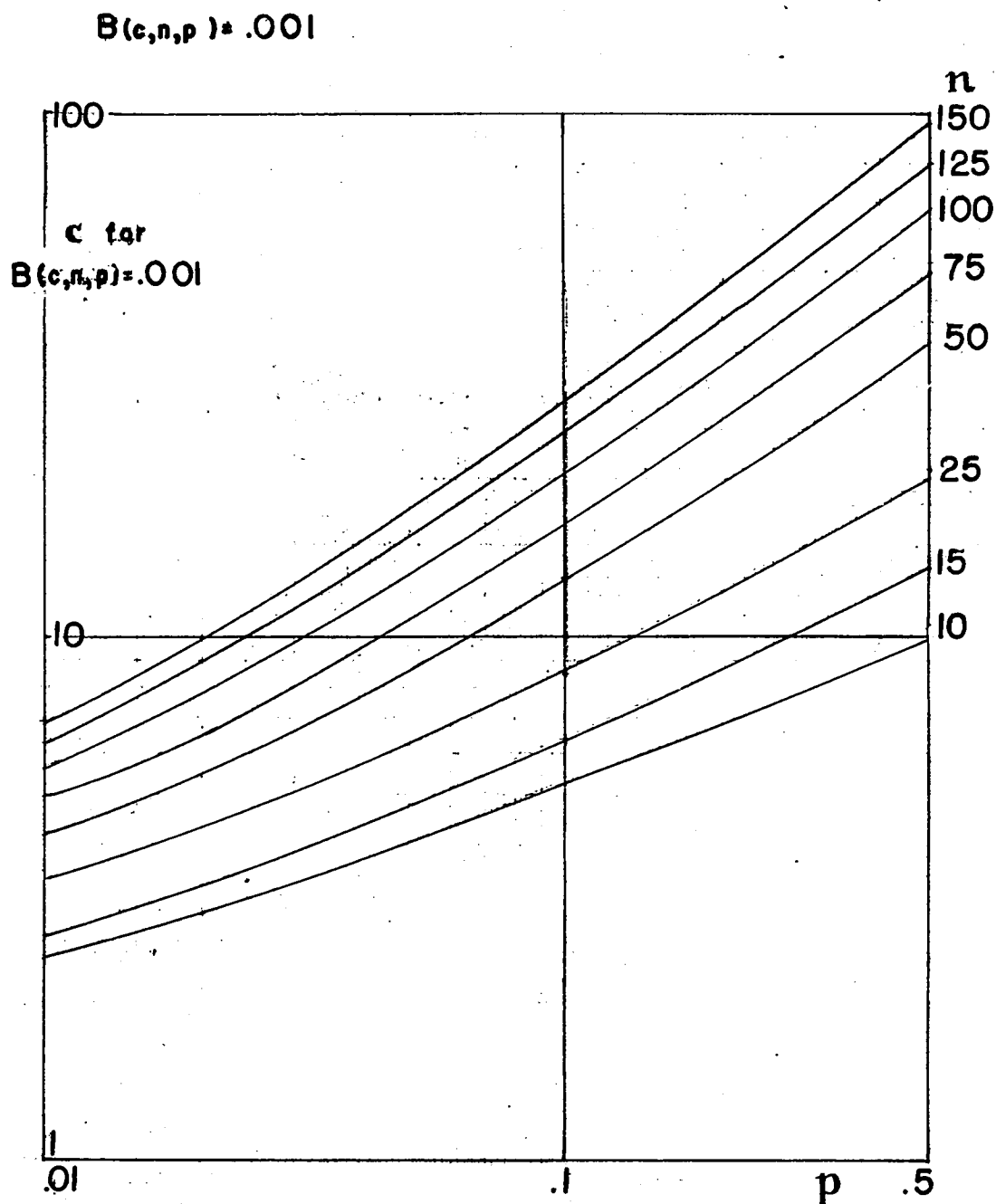


FIG. 14.
 VALUES OF c , n AND p FOR THE LOWEST
 VALUE (.001) OF $B(c, n, p)$ TO BE CALCULATED.

tables for the 50, 25, 10, 5, 2.5, 1 and 0.5 percent points of the cumulative Binomial distribution. (It may be again noted that Figs. 13 and 14 herein provide 0.1 percent points, i.e., for .001 and .999, an extension of the values of percentage points for the tables [14].) The body of each table contains the values of the single trial probabilities which correspond with the given value, e.g., .01, of the cumulative Binomial probability, B or $B(c,n,p)$, for column and row headings respectively of $\nu_1 = 2(n-c+1)$ and $\nu_2 = 2c$. The tables are entered with values of ν_1 and ν_2 . The tables also apply to values of $P(c,n,p) > .5$, or $> 50\%$ since

$$P(c,n,p) = P(\nu_1, \nu_2) = 1 - P(\nu_2, \nu_1) \quad (29)$$

p^n

The earlier mentioned relations $B(c = n, n, p) = p^n$ (5) and $B(c = 1, n, p) = 1 - q^n$ (7) can be used for checking cumulative Binomial probabilities for any values of n and p in the cases of $c=n$ and $c=1$.

The main point here is that both the percentage point tables and p^n can be used for checking approximations to the cumulative Binomial probabilities with at least 3-decimal accuracy in an n, p region for which other, more convenient tables are not available, and in which the maximum correction exceeds .001. Figs. 1 and 12 show that this n, p region is roughly a parallelogram within the sides: $np = 100$, $p = .007$, $np \approx 4000$ and $p = .37$ (this side curved). Within this region, the maximum correction is only .0065 to either the Normal or Poisson when the more accurate approximation of the two is used.

The .001 bound (having a portion for which $np \approx 4000$) for the Normal correction on Figs. 1 and 12 was determined from the .001 value of the second term of the Gram-Charlier Series, Type A, eq. 21. It was checked by means of the .001 value of the second term of the "remainder" equation 22.

CONCLUSION

1. Map. Mainly for use by engineers and mathematicians who need to obtain cumulative Binomial probabilities only occasionally, there is presented a comprehensive map (Figs. 1 and 12) which shows the regions of application of different computational procedures or tools, and the accuracies of the approximations. However, this map should also prove convenient for reference by statisticians.

2. Accuracy of the Normal and Poisson approximations. The maximum error of these approximations is about .08 if one uses the smaller of the uncorrected Normal or Poisson values, and this is for the readily computed case of $c = 1$. The maximum error, thus taken, is only .030 for $n=10$ and about .027 for $n=20$. Since the Appendices contain Table C5 of values of the cumulative Binomial probability for $1 \leq n \leq 20$, the maximum error of the Normal and Poisson approximations for higher values of n is only about .027. At $n=50$, where a portion of the Incomplete Beta Function Table [1] stops, the maximum error is only .020. At the $n=150$ limit of the cumulative Binomial table [2], the maximum error is only about .015, or one and one-half percent.

3. Two-decimal accuracy is had with the uncorrected Poisson and Normal cumulative probabilities (see Figs. 7 and 11) respectively for $p \leq .07$ and $np > 37$ (and also close to $p = .5$ for n down to 3), where Binomial probability tables are available for n through 50, the only untabulated region in which the maximum correction exceeds .01 is the small, roughly triangular region (shaded and marked "1%" on Fig. 7) having (n, p) apexes (50, .07), (500, .07) and (50, .27).

4. At least 3-decimal accuracy is obtainable everywhere by the use of tables*, formulas and graphs which are available herein for conveniently obtaining values of the cumulative Binomial probability $B(c, n, p)$ either directly from tables or from algebraically additive (two) terms of the Gram-Charlier series and three terms of the remainder modification (eq. 22) of the Gram-Charlier, type A, series. Alternative procedures, some of which are noted in the appendices, may be preferable for use in particular regions where many values are to be computed.

5. For checking values of $B(c, n, p)$, percentage point tables [14] and graphs [15], and values of p^n can be used. Normal probability paper can be conveniently used for interpolation between tabulated percentage points: 50, 25, 10, 5, 2.5, 1 and .5 per cent, where an accuracy of only two significant figures is required.

6. Appended are notes on alternative methods, examples--including some on interpolation, tables, and a list of references.

ACKNOWLEDGMENTS AND BACKGROUND

From his knowledge of the broad field of statistics, Dr. Frank E. Grubbs, of the Ballistic Research Laboratories, acquainted the author with what had already been accomplished by others and generously made many suggestions for increasing the value of the work. It is a pleasure to acknowledge the great value of his assistance and insight. Also gratefully acknowledged is the interest of General Leslie E. Simon in this work, which led to the preparation of this report.

In view of the earlier, piecemeal release of portions of the material herein, a brief history of this work is included. In the summer of 1948, the author entered a field involving many computations of the cumulative Binomial probability. While, as an engineer, he was already acquainted with the Normal and Poisson approximations to the Binomial, he was without knowledge of the accuracy of these approximations in different n, p regions. Consequently he set about "tooling up" by preparing a short "handbook" treatment for his own working notes, so that 3-decimal accuracy could readily be obtained for any desired values of c , n and p .

Sets of the author's working notes, which were circulated among his associates in late 1948, provided 3-decimal accuracy universally. This was partly through the use of different empirical relations he found applicable in different n, p regions in which $n > 50$. These 1948

*Also one can use other tables [1,2,8] if available.

notes also included the Gram-Charlier series as alternative procedures for certain regions. The author had modified the type B series from the customary form, which includes individual Poisson terms, to that of eq. 28 which involves only cumulative Poisson terms. In January 1949, there was a limited distribution of a brief memorandum excerpting the minimum material from the 1948 notes to cover all regions with 3-decimal accuracy. To reduce the number of procedures mentioned in the memorandum, it relied upon the Gram-Charlier series of two terms over as large regions as possible.

The instant report additionally includes (1) maps of accuracy of Normal and Poisson approximations to the individual Binomial probability, and (2) a remainder modification (GCAR) of the Gram-Charlier series, type A, which enables the entire n, p domain to be filled with 3-decimal accuracy by: an accompanying table of the cumulative Binomial probability for $1 \leq n \leq 20$, the Normal and Poisson approximations, the two-term Gram-Charlier series of both types and the stated GCAR modification.

This GCAR modification makes it possible for this report to be compact and self-contained.

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APPENDIX A

Alternative Methods

This appendix mentions a few of the many possible alternative methods to those recommended in the body of this work, and some reasons why the alternatives are not as generally useful for present purposes. Some of the alternatives are doubtless better for particular regions, but their inclusion in the body of this report would have complicated the mapping by adding to the number of methods already there.

Theoretical formulas of various sorts were investigated and, except for the Gram-Charlier series, found to be of little or no value for readily obtaining 3-decimal accuracy. The difficulty usually is that the rejected method is too complicated for infrequent use.

The Ferris method [13] was useful at the start of this work in that it filled a region, for $p \leq .25$, of n higher than the upper limit of an available published table [1] of the Incomplete Beta Function. In the Ferris method, the correction (B-P) of the Poisson approximation was graphed directly against the appropriate deviate

$$t_b = \frac{c-a-1}{\sigma} \quad (A1)$$

in which unity is the fitting constant. Ferris used four graphs to cover the four swings of B-P, i.e., two positive and two negative portions, although a single graph could have been used if desired, as in Fig. A-1 herein. The Ferris method failed to be useful for n down to 20, since the graphed remainder is not nearly enough independent of n . For this reason, a like method for the Normal failed to be useful for n down to 20 whether B-N was graphed against the uncorrected deviate $t = (c-a)/\sigma$ or $t_c = (c-a-.5)/\sigma$.

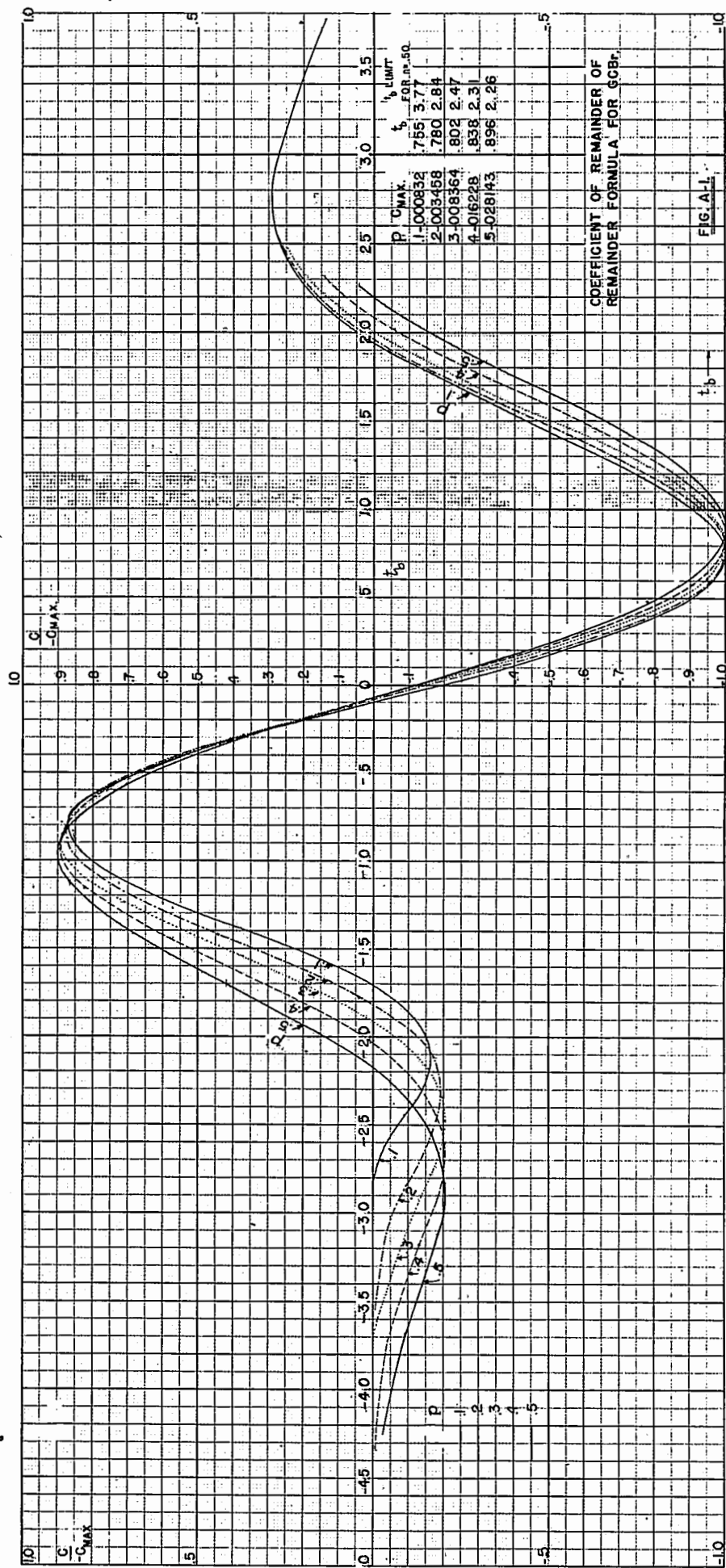
GCB remainder. A type B series modification was made in which one graph, Fig. A-1, was used with values of $C/(-C_{\max})$ of eq. A-2 plotted against the Poisson deviate t_b , for different values of p from .1 through .5. The negative value in the denominator was included so that the proper sense, or algebraic sign, of the remainder would be retained, in spite of the apparent clumsiness of this expedient. Fig. A-1, based on $n=50$, is used in connection with the formula

$$P_{Br} = P_B(c, a) + C(t_b) \quad (A2)$$

where

$$C(t_b) = B - P_B = -\left(\frac{C}{-C_{\max}}\right) (c_{\max}) \text{ is the remainder for the}$$

GCB Series having two terms. This makes the relation $C(t_b)$ nearly enough independent of n for use with 3-decimal accuracy down through



$n = 20$ for $p < .4$. Higher difference interpolation, by use of Binomial coefficients, smoothed out the relation of $C_{\max}$ to p and (n, p) so that, for $20 < n < 150$, substantially 3-decimal accuracy is obtainable with the relation

$$C_{\max} = C_{\max}(p_a) + C(n, p) \quad (A3)$$

$$\text{where } p_a = p + \frac{.001156}{.534 - p}$$

$$C_{\max}(p_a) = -.10862 p_a^{2.1406} \quad \text{within } \pm .0002 \text{ for } n = 50$$

$$\text{or } \log C_{\max}(p_a) = 2.1406 \log p_a - .96407 \text{ for } n = 50, \text{ and}$$

$$C(n, p) = .00115 \frac{.07}{.57 - p} (\log n - 1.7)$$

$$= .0000805 \frac{\log n - 1.7}{.57 - p} \quad (A4)$$

This GCB remainder method is not recommended in the body of this work since the GCA remainder method extended with 3-decimal accuracy to such low values of p , for $n \geq 20$, that no gap was left. Another reason is that a question arose as to the propriety of using negative values of P_B which occur near the upper limit of t_b for this GCB method.

The use of the Gram-Charlier Series with more than two terms proved disappointing, as compared with the remainder methods, for reasons given in the section in the body on 'The GCA "remainder" method'. The writer found that, for these reasons, the use of different derivatives of the Normal distribution with fitted coefficients failed to be generally useful, although excellent fits were had in particular limited regions.

The deviate coefficients of .5 for the Normal and unity for the Poisson were also slightly adjusted, again with excellent local fits but without general usefulness. There remains of course the problem of eliminating the "fourth dimension", i.e., any of B, c, n, p , with this expedient as with others, by finding suitable correlations. This expedient may well be promising if this be done.

A recently proposed approximation, as understood, involves entry of the Normal tables with the deviate

$$t_{Mt} = 2 \left[\sqrt{p(n-c-1)} - \sqrt{qc} \right] \quad (A5)$$

in present notation, and is in error by .0688 at $c=5, n=50, p=.1$, e.g., as compared with an error at this point of less than .001 for any of the

methods recommended herein. This was unexpected since, according to that reference [16], "The general conclusion is that the approximation is extraordinarily good near the 1% to 5% points, and remarkably good in general."

The use of other than the Normal and Poisson approximations was considered briefly, e.g., student's distribution and the use of $\frac{\sin x}{x}$ which is tabulated [17, 18] but rejected as involving less commonly available tables and as bringing in needless complications in curve fitting. Also rejected, as not sufficiently accurate, was $N(t) \approx A + B \cos Ct$, where A, B, and C are fitting constants.

Everything considered, the methods recommended in the body of this work appear to be a reasonable compromise between simplicity and accuracy, where substantially 3-decimal accuracy is the goal. It is possible to go over the same ground in more detail, using less cursory techniques than those used in the present reconnaissance. If this were done, slightly higher precision would be obtained in the location of the boundaries of the limits of 3-decimal accuracy for the several methods, but probably without much increase in the accuracy of the values of the cumulative Binomial probability obtained by the use of the herein recommended methods.

A more promising direction for future work on any short, self-contained treatment of the cumulative Binomial probability seems to lie in finding better correlations and also handier and better methods of interpolation since the abbreviated tables and graphs all require interpolation. Such a treatment of interpolation must necessarily go considerably further than the cursory treatment in Appendix B.

Specific suggestions for further work are: (1) to plot the 3-decimal accuracy limit for the Normal distribution and both types of the Gram-Charlier series for plural values of c and using $B(0, n, p) \approx 1$ and $B(1, n, p) \approx 1 - q^n$ for the others, (2) to plot the contours of equal maximum error for the regions in which the Gram-Charlier series are used, and (3) to compute a few values of $B(c, n, p)$ with less than .0001 error as check points for very high n and low p in the region in which the Gram-Charlier series, type A, is used. It may be noted that a check at $n \approx 1000$ and $p \approx .01$ showed that the errors of the several recommended approximations were well within the expected values.

APPENDIX B

Examples of Probability Computations

p^n . (Example 1): For $n = 200$, $p = .996$, find the probability of $c = 200$ successes in 200 tries. The probability of desired successes is $B(c=n, n, p) = p^n$ (5) when the desired number of successes equals the sample size,

$$B = .996^{200} = .4486$$

And when $n = 200$, $p = .997$ and $c = 200$

$$B = .997^{200} = .5483$$

Thus a difference of .001 in the single-trial probability causes the desired probability to vary by as much as 22% when the sample size is as large as 200.

Cumulative Binomial Probability, $B(c, n, p)$ (Example 2) For $n = 3$, $p = .67$, find the probability of $c = 1$ or more successes in 3 tries. From eq. (11),

$$B(c = 1, 3, .67) = \sum_{x=1}^3 \frac{3!}{x!(3-x)!} (.67)^x (.33)^{3-x} = .2189 + .4444 + .3008 = .9641$$

(Example 3): In certain cases the relationship $q = 1 - p$ simplifies the procedures for finding the desired probability. Using the same constants as in Ex. 2 we again look for the probability of one or more successes. The probability of 1 or more successes equals unity minus the probability of zero successes. From eq. (8),

$$B(x = 0, 3, .67) = \frac{3!}{0!3!} (.67)^0 (.33)^3 = q^n = (.33)^3 = .0359.$$

$B(1, 3, .67) = 1 - B(x = 0, 3, .67) = .9641$ which is identical with the result of example 2.

Incomplete Beta-function tables (Example 4): For $n = 50$ and $p = .01$, find the probability of 2 or more successes, or $B(c = 2, 50, .01)$. While the use of the Beta-function tables is less simple than of the cumulative Binomial tables [2, 8], the use of either involves only a very small part of the labor involved in computing and summing the individual Binomial terms. Alternative methods of using the Beta-function tables apply respectively to the cases of $n + 1 > 2c$ and $n + 1 < 2c$. These tables are usable for values of $n - c$ from 1 through 50 and for all values of p and q from .01 through .99 by steps of .01.

We use the subscript "t" for tabular quantities. In the present example, $n + 1 > 2c$, $q_t = c$, $p_t = n + 1 - q_t = 50 + 1 - 2 = 49$ and $x_t = q = .99$. Referring to the Incomplete Beta-function tables, page 57, for $q_t = 2$, $p_t = 49$ and $x_t = .99$,

$I_x(p_t, q_t) = I_{.99}(49, 2) = .9105647$. The probability of 2 or more successes $= B(c = 2, 50, .01) = 1 - I_x(p_t, q_t) = 1 - .9105647 = .0894353$. These tables are usable for values of n from 1 to at least 50 and for all values of p and q from zero to unity.

(Example 5): For the case of $n + 1 < 2c$: $n = 50$, $p = .40$, $c = 30$, and $n + 1 < 2c$. In this case $q_t' = p_t = n + 1 - c = 21$, $p_t' = q_t = c = 30$, $x_t' = 1 - x_t = 1 - q = p = .4$ and

$B(c=30, 50, .40) = I_{x_t}'(p_t', q_t')$. Referring to p. 350 of the Incomplete Beta-function tables, $I_{x_t}(p_t', q_t') = 1 - I_{x_t}(p_t, q_t)$. The probability of c or more successes is $B = B(30, 50, .4) = 1 - I_{x_t}(p_t, q_t) = I_{x_t}(p_t', q_t') = I_{.4}(30, 21) = .0033604$.

This figure is checked exactly by one cumulative Binomial probability table^[2] and closely by another^[8] which gives $B = 1 - .996637 = .003363$.

Cumulative Binomial Probability Tables. One set of tables^[2], for $1 \leq n \leq 150$, consists of a table for each .01 value of p or q from .01 to .99. Each table is for a particular single trial probability p' , is entered with c and n , and its body directly gives values of $B = B(c, n, p')$. No example is needed for this set of tables for $p \leq .5$. For $p > .5$, one can use the relation $B(c, n, p) = 1 - B(n - c + 1, n, q)$ and enter the tables with q instead of p .

Another set of tables^[8], for $50 \leq n \leq 100$, likewise consists of a separate table for each .01 value of p or q over the same range. However, each table is entered with $c - 1$ and n and its body contains values of $P_T = 1 - B(c, n, p)$ since each table sums individual Binomial terms for $x = 0$ to c . Since the probability of c' or more successes is 1 minus the probability of from 0 to $c' - 1$ successes, B or $B(c, n, p) = 1 - P_T$.

(Example 6): Use the latter set of tables, for the conditions and problem of example 4. To find the probability of 2 or more successes, we find 1 minus the probability of from 0 to $2 - 1 = 1$ success. Referring to the first page of the tables, for $n = 50$, $p = .01$ and $x_T = c - 1 = 1$, the tabulated probability is $P_T = .910565$. Hence $B(c, n, p) = 1 - .910565 = .089435$, which agrees with the value found in example 4 by using the Incomplete Beta-function tables.

Uncorrected Normal Cumulative Probabilities (Example 7): For $n = 100$, $p = .3$ and $c = 25$, to find the probability of 25 or more successes in 100 trials. Equation 13 can be used to obtain this probability. Values of this integral are tabulated in Normal tables. A table in which this integral is summed from the mean, is entered with $t_0 = \frac{c - a - 5}{\sigma} = \frac{25 - 30 - .5}{4.583} = -1.200$, where $a = np$ and $\sigma = \sqrt{npq}$. From such a table, e.g., Glover's, page 398, $B(25, 100, .3) \cong N = .5 - \int_0^t \phi(t) dt = .5 + .38493 = .88493$.

With a table in which the integral is summed from c to ∞ , e.g., Burington's tables^[18], page 258, the answer is found directly as .8849. This numeric is checked by a set of cumulative Binomial tables^[2] giving $B = .8864298$, or $B - N = .0015$ which is less than the maximum correction for $n = 100$, $p = .3$ on Fig. 7.

Gram-Charlier Series, Type A, (Example 8): For $n = 1300$, $p = .05$ and $c = 66$, the Gram-Charlier Series, Type A, (eq. 21) can be used to find the probability of 66 or more successes.

To use this, we have

$$t_c = \frac{c - .5 - a}{\sigma} = \frac{66 - .5 - 65}{7.8581} = \frac{.5}{7.8581} = .0636$$

Using equation (21) and Glover's Normal tables, p.394, $B(66,100,.03) \cong$

$$N_A(66,100,.03) = .50000 - .02535 - \left[\frac{.05 - .95}{(6)(7.8581)} \right] [-.39652]$$

$$= .47465 - .00757 = .46708.$$

Remainder for Gram-Charlier Series, Type A, (Example 9): To find B for $c = 2$, $n = 21$ and $p = .29$; $np = 21 \times .29 = 6.09$,

$$t_c = \frac{2 - 6.09 - .5}{(6.09 \times .71)^{.5}} = \frac{-4.59}{2.0794} = -2.2074,$$

$$\alpha \cong \frac{.351 (.5-p)^{.87}}{(np)^{.53}} = \frac{.351 \times .21^{.87}}{6.09^{.53}} = .03466 \quad \text{or can be obtained from Fig. 8.*}$$

$$\text{From } t_c = -2.2074, N = .48636 + .5 = .98636$$

$$\text{and } \phi^{(2)}(t_c) = .13515, \text{ so that } \alpha \phi^{(2)}(t_c) = .00468$$

From Fig. 9, $r = .01140$,

$$\text{so that } \frac{r(t_c)}{np} = \frac{.01140}{6.09} = .00187$$

$$N_{Ar} = .99291$$

$$B\text{-tabular} = .99279$$

$$\text{Error} = .00012, \text{ which is well within the .001 limit.}$$

This value of c was selected as providing large values of both the correction and the slope of r with respect to t_c . Also, the corresponding value of $a = np$ is nearer to an integer than to the (integer + .5) values of np used in plotting Fig. 9. In other words, this is not a particularly favorable case for this approximation.

Poisson Cumulative Probabilities (Molina tables [11]).

(Example 9): For $n=100$, $p=.004$, $c=1$, to find the probability of 1 or more successes in 100 trials. Eq. (26) can be used in obtaining this.

* To find α from Fig. 8: For $np=6.09$ on the top scale, the dash line gives $\alpha = .040$ at $.5-p=.25$. A line carried to the left from that point parallel with adjacent lines to $.5-p=.21$ for the given value of $p=.29$ gives $\alpha = .0347$ there.

probability. The results of eq. (26) are tabulated in the Poisson-Molina tables. From the P-M Table II, for $a = .4$ and $c = 1$, $P(c, a) = P(1, .4) = .3296800$. The same answer in this case can be gotten from values, obtained by using eq. (25), tabulated in the P-M Table I for the "individual term Poisson probability" and the fact that the probability of one or more successes equals $1 - (\text{probability of zero successes})$. For $x = 0$ and $a = .4$ from Table I, $P(x, a) = .6703200$. From this, the cumulative Poisson term $P(c=1, .4) = 1 - P(x=c-1, a) = 1 - P(0, .4) = 1 - .6703200 = .3296800$.

Incomplete Gamma (I)-Function Tables. (Example 10): For $n=100$, $p=.1$ and $c=4$, to find the probability of 4 or more successes.

If the Poisson-Molina tables [11] are available, they are preferably used for this purpose. As a poor alternative, incomplete Gamma-function tables can be used, the subscript "t" being used for tabular quantities.

$$u_t = \frac{np}{\sqrt{c}} = \frac{(100)(.1)}{4} = 2.5, p_t = c-1 = 4-1 = 3.$$

Referring to page 15 of Pearson's tables of the incomplete Gamma-function [12], for $u_t = 5$ and $p_t = 3$, $I(u, p) = I(5, 3) = \text{probability of 4 or more successes} = .9896639$. This value is checked by the P-M Table II which gives a value $P(4, 10) = .989664$.

This Pearson's table [12] (I) has .1 steps of u_t and p_t . Hence extensive interpolation is involved in most cases. This table includes second and fourth differences to facilitate accurate interpolation, along with instructions for the use of such differences, pp. x - xiv. Alternatively, one can use Everett's formula for interpolation [19].

Gram-Charlier Series, Type B, (Example 11): For this Type B series (eq. 28), consider the problem of example 8 for which $c=66$ and $a=(1300)(.05)=65$. Hence

$$P_B(c, a) = .467076 - \frac{(1300)(.05)^2}{2} [.467076 - (2)(.516496) + .565915] \\ .467076 - 0 = .467076$$

which agrees with the result of example 8 for the Type A series. While the simple Poisson turns out to be sufficiently accurate for this case of low p , this would not be true for a much higher p .

Remainder for Gram-Charlier Series Type B, (Example 12):
To find P_{Br} by this alternative method for $c = 19$, $n = 100$, $p = .2$.

$$a \approx 100 \times .2 = 20 \text{ and } t_b = \frac{19-20-1}{5} = -\frac{2}{5} = -.4 \\ (20 \times .8)^{.5}$$

The first term of eq. A2 is, from eq. 28,

$$P_B(19, 20) = .61858 - .01686 = .63546.$$

and the second term is $C(-.5) = \frac{C}{-C_{\max}} (C_{\max})$

From $t_b = -.5$ and $p = .2$ on Fig. A1, $\frac{C}{-C_{\max}} = .685$ or $C = -.685 C_{\max}$.

From eq. A3, $C_{\max} = C_{\max}(p_a) + C(n, p)$

$$p_a = .2 + \frac{.001156}{.534-.2} = .20346$$

$$C_{\max}(p_a) = -.10862 \times .20346^{2.1406} = -.00359 \text{ which is within } \pm .0002$$

of values for $n = 50$. The correction for the given value of n is

$$C(100, .2) = .0000805 \frac{\log 100 - 1.7}{.57 - p} = .00007.$$

Hence eq. A3 becomes $C_{\max} = -.00359 + .00007 = -.00352$

from which $C = (-.685)(-.00352) = .00241$ and $P_{Br} = .63546 + .00241 = .63787$.

Since $B = .63791$, the error of this approximation at this point is only

$$P_{Br} - B = -.00004.$$

Percentage Points. Alternative methods, of using the percentage point tables in checking values of cumulative Binomial probabilities, apply to the two cases $B(c, n, p) \leq .5$ and $B(c, n, p) \geq .5$.

In the first case (Example 13): For $n=50$, $c=20.5$ and $B(c, n, p)=.01$ to find the single-trial probability which gives .01 as cumulative probability for 20.5 or more successes*. To enter the percentage point table [14], use $\mathcal{V}_1 = 2(n-c+1) = 2(50-20.5+1) = 61$ and $\mathcal{V}_2 = 2c = 2(20.5) = 41$.

In using these tables, page 179, harmonic interpolation is used for $\mathcal{V}_1$ and linear interpolation for $\mathcal{V}_2$. The result is $p=.247$ which closely checks the Binomial probability line $p=.25$ on Fig. 3, where this line passes through the point $c=20.5$ and $p=.01$.

In the second case, (Example 14): For $B(c, n, p) \geq .5$: for $n=9$, $c=5$ and $B(c, n, p)=.9$ to find the initial probability p so that the final probability of 5 or more successes is $P=.9$. To enter the tables [14] for this case, $\mathcal{V}_1 = 2(5) = 10$ and $\mathcal{V}_2 = 2(9-5+1) = 10$. Since $I_{1-p}(n-c+1, c) = 1 - I_p(c, n-c+1)$, then the table of percentage points (.005, .01, .025, .05, .10, .25, .50) also can be used to give values of $p = 1 - p_t$ for various values of $\mathcal{V}_1 = 2c$ and $\mathcal{V}_2 = 2(n-c+1)$ for which $P(c, n, p) = .995, .99, .975, .90$ and $.75$ respectively. Referring to these tables, p.173,

* In general, of course only integer values of c are used.

$p_T = .30$. The desired value $p = 1 - .30 = .70$. This checks with the results derived from the incomplete Beta-function tables.

In this reference [14], these percentage point tables are followed by 5-significant-figure tables of Lagrangian coefficients for both linear and harmonic interpolation which are required for accurate use of the above-mentioned tables of said percentage points. Harmonic interpolation is "applicable to any table of percentage points (depending on a parameter n with an infinite range) in which the statistic can be adequately represented as a polynomial in $1/n$, a property of any 'studentized' statistic. Incidentally, the percentage points can be used to obtain other values of $B(c, n, p)$ within a few percent by plotting the tabulated percentage point values on probability paper where they lie on smooth, nearly linear curves".

INTERPOLATION

Extensive interpolation is required in obtaining probabilities with the required accuracy from the several tables. Hence interpolation procedures form an essential part of the "examples" portion of a work on methods of obtaining values of probabilities within .001.

Using tables, it generally saves time to plot a few tabular values from adjacent rows and columns, transforming entry parameters if necessary to a basis which gives lines that are nearly straight -- so that linear numerical interpolation can be used. Often the interpolation can be made on graphs by eye alone with sufficient accuracy, although occasionally a "cross plot" or "section" graph may be required.

Experience in mapping contours permits one to save considerable time in interpolating with the necessary accuracy. And this may be the only practical way of proceeding where a family of curves is involved, each of which is based on too few points for accurate interpolation but with enough points altogether so that reliable curves can be drawn. It may seem unscientific to use a set of freehand curves, but this may be the only reasonably rapid method. For example, it was used in drawing the "contours" on Fig. 9.

Some methods [20,21] give different slopes on opposite sides of evenly spaced ordinates. Osculatory interpolation, i.e., giving a continuous derivative, requires more points than are commonly available or convenient, especially near the ends of tables. And this is particularly true where one or more points of inflection are involved.

A knowledge of the curve type or form usually reduces the number of points below that otherwise needed. For example, if one knows that a curve is a circle, only three points are needed to determine it. If it is known only that it is one of the second degree equations, or that the curve is a conic of some sort, five points are required. And higher numbers of points are needed of course for the higher degree polynomials. The use of logarithm paper is occasionally helpful where an exponential can be put through a base point* where two others are known on the same side of the base point, proceeding in the direction consistent with the

*Using departures from the tangent to the base point.

type of curve, i.e., toward a portion of less curvature. Methods of interpolation also furnish enough knowledge of relations to facilitate both extrapolation and integration. The literature is so extensive that a question sometimes exists as to whether it is not less time-consuming to proceed from fundamental considerations than to go into the literature. So much for generalities.

Linear interpolation is commonly adequate, the adequacy being readily checked by taking second and adjacent higher differences which can be used in taking care of the non-linearities.

Higher-Difference Interpolation. A convenient and adequate formula for non-linear interpolation in a table of y as a function of x , for x tabulated with equal intervals, is

$$y = y_1 + \left(\frac{m}{1}\right) D' + \left(\frac{m}{2}\right) D'' + \left(\frac{m}{3}\right) D''' + \dots + \left(\frac{m}{x}\right) D^x$$

$$= y_1 + m D' + \frac{m(m-1)}{2!} D'' + \frac{m(m-1)(m-2)}{3!} D''' + \dots$$

where $x = x_1 + md$, values y_1 and y_2 respectively are tabulated for x_1 and x_2 , the constant tabular difference is $d = x_2 - x_1$, and D' , D'' , D''' , ... are the successive differences in the series of y 's starting with y_1 , and higher-order differences can be neglected.

The Binomial coefficients $\left(\frac{m}{x}\right)$ are tabulated [20] for proper fractional values of m and the lower orders of differences. Of course x_1 may be at either end of the series of tabulated values of x .

Central Interpolation [23] is useful where a value of y must be found near the center of a tabulated series of a relatively few values.

Harmonic Interpolation, as noted earlier herein, is useful with percentage point tables [14]. (Example 15: For the case of example 12, the percentage point tables are entered with $\nu_1 = 61$ and $\nu_2 = 41$.)

Using harmonic interpolation: For $\nu_1 = 60$ and $\nu_2 = 40$, $p = .24819$. From this:

$$\frac{p(\nu_1 = 60)}{p(\nu_1 = 61)} = \frac{\nu_1 = 61}{\nu_1 = 60}, \quad \frac{.24819}{p(\nu_1 = 61)} = \frac{61}{60}$$

and $p(\nu_1 = 61) = .24412$. For $\nu_1 = \nu_2 = 60$, $p = .35258$.

$$\frac{p(\nu_1 = 60)}{p(\nu_1 = 61)} = \frac{\nu_1 = 61}{\nu_1 = 30}, \quad \frac{.35258}{p(\nu_1 = 31)} = \frac{61}{60}, \quad p(\nu_1 = 31) = .34670$$

At $\nu_1 = 61$, these results .24412 and .34680 are respectively for $\nu_2 = 40$ and 60. Using linear interpolation $p = .24412 + \frac{(.34680 - .24412)}{41} = .2466$ or .247 for $\nu_1 = 61$ and $\nu_2 = 41$, which

.247 is close to the actual value of .250. The actual value is closely approximated by the use of either non-linear interpolation methods noted above or the Lagrangian coefficients following these percentage point tables [14].

Interpolation by use of auxiliary table, (Example 16):
For example, to find the cumulative Poisson probability $P(11, 20)$ from Table C7, using the Normal Table C6 as the auxiliary table: From Table C7, $P(10, 20) = .99501$ and $P(12, 20) = .97861$. From Table C6, the corresponding deviates are 2.5764 and 2.0259, respectively; from which $t(11) \approx 2.30115$, and by re-entering table C6,

$$P_{\text{int}}(11, 20) \approx .98931$$

$$P(11, 20) = .98919 \text{ from the Poisson-Molina tables.}$$

$$\text{Error} = .00012$$

This tabular method of interpolation directly corresponds with the graphical method illustrated by Figs. 3 and 4, which is useful where only 2-decimal accuracy is adequate.

Of course this method of interpolation can be used whenever values of a fast-moving variable, such as $P(c, a)$, can be transformed into those of a finely tabulated variable, such as t, N , in which a nearly linear relation exists between the parameters, c and t in this example. For further example, the Normal Table, or even an extensive table of cosines, can be thus used as the auxiliary table in interpolating in a cumulative Binomial table. Also appended are several references [24 on] of related interest.

APPENDIX C

Tables

- C1 $\log n$, $n = 1(.01)10$, 10-places.
- C2 $n!$ and $\log n!$, $n = 0(1)200$, 10-places.
- C3 $\binom{n}{x}$ and $\log \binom{n}{x}$, $n = 1(1)50$, 5-places.
- C4 e^{-x} , $x = 0(.001)1(1)100$.
- C5 $B(c,n,p)$, $n = 1(1)20$ and $p = .01(.01).5$.
- C6 Normal tables: integral, density $\phi(t)$ and 2nd derivative
 $\phi^{(2)}(t)$, $t = 0(.01)4$.
- C7 $P(c,a)$, $c = 1(1)22$, $a = .001(.001).01(.01).1(.1)1(1)10$
 and $c/a = .1(.1)2.2$, $a = 10(10)100$.

Table G1, log n, n=1(.01)10.

	n	0	1	2	3	4	5	6	7	8	9	n										
58	1.0	.00000	00000	.00432	13738	.00860	01718	.01283	72247	.01703	33393	.02118	92991	.02530	58653	.02938	37777	.03342	37555	.03742	64979	1.0
	1.1	.04139	26852	.04532	29788	.04921	80227	.05307	84435	.05690	48513	.06069	78404	.06445	79892	.06818	58617	.07188	20073	.07554	69614	1.1
	1.2	.07918	12460	.08278	53703	.08635	98307	.08990	51114	.09342	16852	.09691	00130	.10037	05451	.10380	37210	.10720	99696	.11058	97103	1.2
	1.3	.11394	33523	.11727	12957	.12057	39312	.12385	16410	.12710	47984	.13033	37585	.13353	89084	.13672	05672	.13987	90864	.14301	48003	1.3
	1.4	.14612	80357	.14921	91127	.15228	83444	.15533	60375	.15836	24921	.16136	80022	.16435	28558	.16731	73347	.17026	17154	.17318	62684	1.4
	1.5	.17609	12591	.17897	69473	.18184	35879	.18469	14308	.18752	07208	.19033	16982	.19312	45984	.19589	96524	.19865	70870	.20139	71243	1.5
	1.6	.20411	99827	.20682	58760	.20951	50145	.21218	76044	.21484	38480	.21748	39442	.22010	80880	.22271	64711	.22530	92817	.22788	67046	1.6
	1.7	.23044	89214	.23299	61104	.23552	84469	.23804	61031	.24054	92483	.24303	80487	.24551	26678	.24797	32664	.25042	00023	.25285	30310	1.7
	1.8	.25527	25051	.25767	85749	.26007	13880	.26245	10897	.26481	78230	.26717	17284	.26951	29442	.27184	16065	.27415	78493	.27646	18042	1.8
	1.9	.27875	36010	.28103	33672	.28330	12287	.28555	73090	.28780	17299	.29003	46114	.29225	60714	.29446	62262	.29666	51903	.29885	30764	1.9
	2.0	.30102	99957	.30319	60574	.30535	13694	.30749	60379	.30963	01674	.31175	38611	.31386	72204	.31597	03455	.31806	33350	.32014	62861	2.0
	2.1	.32221	92947	.32428	24553	.32633	58609	.32837	96034	.33041	37733	.33243	84599	.33445	37512	.33645	97338	.33845	64936	.34044	41148	2.1
	2.2	.34242	26808	.34439	22737	.34635	29745	.34830	48631	.35024	80183	.35218	25181	.35410	84391	.35602	58572	.35793	48470	.35983	54823	2.2
	2.3	.36172	78360	.36361	19799	.36548	79849	.36735	59210	.36921	58574	.37106	78623	.37291	20030	.37474	83460	.37657	69571	.37839	79009	2.3
	2.4	.38021	12417	.38201	70426	.38381	53660	.38560	62736	.38738	98263	.38916	60844	.39093	51071	.39269	69533	.39445	16808	.39619	93471	2.4
	2.5	.39794	00087	.39967	37215	.40140	05408	.40312	05212	.40483	37166	.40654	01804	.40823	99653	.40993	31233	.41161	97060	.41329	97641	2.5
	2.6	.41497	33480	.41654	05073	.41830	12913	.41995	57485	.42160	39269	.42324	58739	.42488	16366	.42651	12614	.42813	47940	.42975	22800	2.6
	2.7	.43136	37642	.43296	92909	.43456	89040	.43616	26470	.43775	05628	.43933	26938	.44090	90821	.44247	97691	.44404	47959	.44560	42033	2.7
	2.8	.44715	80313	.44870	63199	.45024	91083	.45178	64355	.45331	83400	.45484	48600	.45636	60331	.45788	18967	.45939	24878	.46089	78428	2.8
	2.9	.46239	79979	.46389	29890	.46538	28514	.46686	76204	.46834	73304	.46982	20160	.47129	17111	.47275	64493	.47421	62641	.47567	11883	2.9
3.0	.47712	12547	.47856	64956	.48000	69430	.48144	26285	.48287	35836	.48429	98393	.48572	14265	.48713	83755	.48855	07165	.48995	84794	3.0	
3.1	.49136	16938	.49276	03890	.49415	45940	.49554	43375	.49692	96481	.49831	05538	.49968	70826	.50105	92622	.50242	71200	.50379	06831	3.1	
3.2	.50514	99783	.50650	50324	.50785	58717	.50920	25223	.51054	50102	.51188	33610	.51321	76001	.51454	77527	.51587	38437	.51719	58979	3.2	
3.3	.51851	39399	.51982	79938	.52113	80837	.52244	42335	.52374	64668	.52504	48070	.52633	92774	.52762	99009	.52891	67003	.53019	96982	3.3	
3.4	.53147	89170	.53275	43790	.53402	61061	.53529	41200	.53655	84426	.53781	90951	.53907	60988	.54032	94748	.54157	92439	.54282	54270	3.4	
3.5	.54406	80444	.54530	71165	.54654	26635	.54777	47054	.54900	32620	.55022	83531	.55144	99980	.55266	82161	.55388	30266	.55509	44486	3.5	
3.6	.55630	25008	.55750	72019	.55870	85705	.55990	66250	.56110	13836	.56229	28645	.56348	10854	.56466	60643	.56584	78187	.56702	63662	3.6	
3.7	.56820	17241	.56937	39096	.57054	29399	.57170	88318	.57287	16022	.57403	12677	.57518	78449	.57634	13502	.57749	17998	.57863	92100	3.7	
3.8	.57978	35966	.58092	49757	.58206	33629	.58319	87740	.58433	12244	.58546	07295	.58658	73047	.58771	09650	.58883	17256	.58994	96013	3.8	
3.9	.59106	46070	.59217	67574	.59328	60670	.59439	25504	.59549	62218	.59659	70956	.59769	51859	.59879	05068	.59988	30721	.60097	28957	3.9	
4.0	.60205	99913	.60314	43726	.60422	60531	.60530	50461	.60638	13651	.60745	50232	.60852	60336	.60959	44092	.61066	01631	.61172	33080	4.0	
4.1	.61278	38567	.61384	18219	.61489	72160	.61595	00517	.61700	03411	.61804	80967	.61909	33306	.62013	60550	.62117	62818	.62221	40230	4.1	
4.2	.62324	92904	.62428	20958	.62531	24510	.62634	03674	.62736	58566	.62838	89301	.62940	95991	.63042	78750	.63144	37690	.63245	72922	4.2	
4.3	.63346	84556	.63447	72702	.63548	37468	.63648	78964	.63748	97295	.63848	92570	.63948	64893	.64048	14370	.64147	41105	.64246	45202	4.3	
4.4	.64345	26765	.64443	85895	.64542	22693	.64640	37262	.64738	29701	.64836	00110	.64933	48587	.65030	75231	.65127	80140	.65224	63410	4.4	
4.5	.65321	25138	.65417	65419	.65513	84348	.65609	82020	.65705	58529	.65801	13967	.65896	48427	.65991	62001	.66086	54780	.66181	26855	4.5	
4.6	.66275	78317	.66370	09254	.66464	19756	.66558	09910	.66651	79806	.66745	29529	.66838	59167	.66931	68806	.67024	58531	.67117	28427	4.6	
4.7	.67209	78579	.67302	09071	.67394	19986	.67486	11407	.67577	83417	.67669	36096	.67760	69527	.67851	83790	.67942	78966	.68033	55134	4.7	
4.8	.68124	12374	.68214	50764	.68304	70382	.68394	71308	.68484	53616	.68574	17386	.68663	62693	.68752	89612	.68841	98220	.68930	88591	4.8	
4.9	.69019	60800	.69108	14921	.69196	51028	.69284	69193	.69372	69489	.69460	51989	.69548	16765	.69635	63887	.69722	93428	.69810	05456	4.9	
5.0	.69897	00043	.69983	77259	.70070	37171	.70156	79851	.70243	05364	.70329	13781	.70415	05168	.70500	79593	.70586	37123	.70671	77823	5.0	
5.1	.70757	01761	.70842	09001	.70926	99610	.71011	73651	.71096	31190	.71180	72290	.71264	97016	.71349	05431	.71432	97597	.71516	73578	5.1	
5.2	.71600	33436	.71683	77233	.71767	05030	.71850	16889	.71933	12870	.72015	93034	.72098	57442	.72181	06152	.72263	39225	.72345	56720	5.2	
5.3	.72427	58696	.72509	45211	.72591	16323	.72672	72090	.72754	12570	.72835	37820	.72916	47897	.72997	42857	.73078	22757	.73158	87652	5.3	
5.4	.73239	37598	.73319	72651	.73399	92865	.73479	98296	.73559	88997	.73639	65023	.73719	26427	.73798	73263	.73878	05585	.73957	23445	5.4	

Table C2, n! and log n!, n=0(1)200.

n	n!	log n!	n	n!	log n!	n	n!	log n!	n	n!	log n!
0	1	0.00000	50	30414 09320	64.48307 48725	100	93326 21544	157.97000 36547	150	57133 83956	262.75689 34109
1	1	0.00000	51	15511 18753	66.19064 50486	101	94259 47760	159.97432 50285	151	86272 09774	264.93537 03582
2	2	0.30102 99957	52	80658 17517	67.90664 83922	102	96144 66715	161.98292 52003	152	13113 35886	267.11771 39432
3	6	0.77815 12504	53	42748 83284	69.63092 42618	103	99029 00716	163.99576 24250	153	20063 43905	269.30240 53770
4	24	1.38021 12417	54	23084 36973	71.36331 80216	104	10299 01675	166.01279 57643	154	30897 69614	271.48992 60976
5	120	2.07918 12460	55	12696 40335	73.10368 07111	105	10813 96758	168.03398 50633	155	47891 42901	273.58025 77960
6	720	2.85733 24964	56	71099 85878	74.85186 87381	106	11462 80564	170.05929 09286	156	74710 62926	275.87338 23943
7	5040	3.70243 05364	57	40526 91950	76.60774 35938	107	12265 20203	172.08867 47063	157	11729 56879	278.06928 20458
8	40320	4.60552 05234	58	23505 61331	78.37117 15874	108	13246 41819	174.12209 84618	158	18532 71869	280.26793 91337
9	3 62880	5.55976 30329	59	13868 31185	80.14202 35990	109	14438 59583	176.15952 49597	159	29467 02272	282.46933 62560
10	36 28800	6.55976 30329	60	83209 87113	81.92017 48494	110	15882 45542	178.20091 76449	160	47147 23636	284.67345 62407
11	399 15800	7.60115 57180	61	50758 02139	83.70550 46844	111	17629 52551	180.24624 06237	161	75907 05054	286.88028 21167
12	4790 01600	8.68033 69641	62	31469 97326	85.49789 67339	112	19745 06857	182.29545 86463	162	12296 94219	289.08979 71313
13	62270 20800	9.79428 03164	63	19826 08315	87.29723 69234	113	22311 92749	184.34853 70898	163	20044 01577	291.30196 47357
14	87178 29120	10.94040 83521	64	12688 69322	89.10341 58973	114	25435 59733	186.40544 19411	164	32872 16586	293.51682 85837
15	13076 74368	12.11649 96111	65	82476 50592	90.91633 02540	115	29250 93693	188.46613 97815	165	54239 10666	295.73431 25279
16	20922 78989	13.32061 95938	66	54434 49391	92.73587 41895	116	33931 08684	190.53059 77707	166	90036 91706	297.95442 06160
17	35568 74281	14.55106 85152	67	36471 11092	94.56194 89922	117	39699 37161	192.59878 36325	167	15036 16515	300.17713 70871
18	64023 73706	15.80634 10203	68	24800 35542	96.39445 79049	118	46845 25850	194.67066 56398	168	25260 75745	302.40244 63686
19	12164 51004	17.08509 46212	69	17112 24524	98.23330 69957	119	55745 85761	196.74621 26012	169	42690 68009	304.63033 30735
20	24329 02008	18.38612 46159	70	11978 57167	100.07840 50357	120	66895 02913	198.82539 38472	170	72574 15615	306.86078 19948
21	51090 94217	19.70834 39116	71	85047 85886	101.92966 33844	121	80942 98525	200.90817 92175	171	12410 18070	309.09377 61052
22	11240 00728	21.05076 65924	72	61234 45838	103.78699 58808	122	98750 44201	202.99453 90482	172	21345 51081	311.32930 55521
23	25852 01674	22.41249 44285	73	44701 15462	105.65031 87410	123	12146 30437	205.08444 41597	173	36927 73370	313.56735 26551
24	62044 84017	23.79270 56702	74	33078 85442	107.51955 04607	124	15061 41742	207.17786 58448	174	64254 25663	315.80790 19035
25	15511 21004	25.19064 56788	75	24809 14081	109.39461 17241	125	18826 77177	209.37477 58578	175	11244 49491	318.05093 99522
26	40329 14611	26.60561 90268	76	18854 94702	111.27542 53164	126	23721 73243	211.37514 64029	176	19790 31104	320.29645 26200
27	10888 86945	28.03698 27910	77	14518 30920	113.16191 60415	127	30126 60018	213.47895 01239	177	35028 85055	322.54442 58864
28	30488 83446	29.48414 08223	78	11324 28118	115.05401 06442	128	38562 04824	215.58616 00935	178	62351 53597	324.79484 58837
29	88417 61994	30.94553 88202	79	89461 52131	116.95163 77355	129	49745 04222	217.69674 98038	179	11160 89236	327.04769 89137
30	26525 28598	32.42366 00749	80	71569 45705	118.85472 77225	130	64668 55489	219.81069 31561	180	20089 60625	329.30297 14248
31	82228 38654	33.91502 17688	81	57971 26021	120.76321 27414	131	84715 80691	221.92796 44518	181	36362 18731	331.56064 99997
32	26313 08369	35.42017 17471	82	47536 43337	122.67702 65938	132	11182 48651	224.04853 83830	182	66179 18091	333.82072 13876
33	86833 17619	36.93868 56870	83	39455 23970	124.59610 46861	133	14872 70706	226.17239 00240	183	12110 79011	336.08317 24774
34	29523 27990	38.47016 46040	84	33142 40135	126.52036 39722	134	19929 42746	228.29949 48223	184	22283 65380	338.34799 03004
35	10333 14797	40.01423 26484	85	28171 04114	128.44980 28979	135	26904 72707	230.42982 85908	185	41225 12952	340.61516 20288
36	37199 33268	41.57053 51431	86	24227 09538	130.38430 13492	136	36590 42882	232.56336 74992	186	76678 74091	342.88467 49730
37	13763 75309	43.13873 68732	87	21077 57298	132.32382 06018	137	50128 83748	234.70008 80664	187	14358 92455	345.15651 65795
38	52302 26175	44.71852 04698	88	18548 26423	134.26830 32739	138	69177 86473	236.83996 71528	188	26957 17815	347.43067 44233
39	20397 88208	46.30958 50768	89	16507 95516	136.21769 32806	139	96157 23197	238.98298 19530	189	50949 06671	349.70713 62330
40	81591 52832	47.91164 50682	90	14857 15964	138.17193 57900	140	13462 01248	241.12910 99887	190	96803 22675	351.98588 98339
41	33452 52561	49.52442 89249	91	13520 01528	140.13097 71823	141	18981 43759	243.27832 91014	191	18489 41631	354.26592 32012
42	14050 06118	51.14767 82153	92	12438 41405	142.09476 50097	142	26953 64138	245.43061 74457	192	35499 67931	356.55022 44200
43	60415 26306	52.78114 66709	93	11567 72507	144.06324 79582	143	38543 70717	247.58595 34832	193	68514 38108	358.83578 17389
44	26582 71575	54.42459 93473	94	10873 66157	146.03637 58118	144	55502 93833	249.74431 59753	194	13291 78993	361.12358 34688
45	11962 22209	56.07781 18611	95	10329 97849	148.01409 94171	145	80479 26057	251.90568 39775	195	25918 99036	363.41361 80802
46	55026 22160	57.74056 96928	96	99167 79349	149.99637 06502	146	11749 97204	254.07773 68333	196	50801 22111	365.70587 41515
47	25862 32415	59.41266 75507	97	96192 75968	151.98314 23644	147	17272 45890	256.23735 41681	197	10007 84056	368.00034 03777
48	12413 91559	61.09390 87881	98	94268 90449	153.97436 84601	148	25563 23918	258.40761 58835	198	19815 52432	370.29700 55680
49	60828 18640	62.78410 48681	99	93326 21544	155.97000 36547	149	38089 22638	260.50080 21519	199	39432 89337	372.59585 86444
									200	78865 78674	374.89688 86400

Table C4. e^{-x} , $x=0(.001)1$ and $1(1)100$.

x	0	1	2	3	4	5	6	7	8	9	x										
.00	1.00000	00000	.99900	04998	.99800	19987	.99700	44955	.99600	79893	.99501	24792	.99401	79641	.99302	44429	.99203	19148	.99104	03788	.00
.01	.99004	98337	.98906	02788	.98807	17129	.98708	41350	.98609	75443	.98511	19396	.98412	73201	.98314	36846	.98216	10324	.98117	93622	.01
.02	.98019	86733	.97921	89646	.97824	02351	.97726	24838	.97628	57098	.97530	99120	.97433	50896	.97336	12415	.97238	83668	.97141	64645	.02
.03	.97044	55335	.96947	55731	.96850	65821	.96753	85596	.96657	15046	.96560	54163	.96464	02935	.96367	61353	.96271	29409	.96175	07091	.03
.04	.96078	94392	.95982	91299	.95886	97806	.95791	13901	.95695	39575	.95599	74818	.95504	19622	.95408	73976	.95313	37871	.95218	11297	.04
.05	.95122	94245	.95027	86705	.94932	88668	.94838	00125	.94743	21065	.94648	51480	.94553	91359	.94459	40694	.94364	99474	.94270	67692	.05
.06	.94176	45336	.94082	32398	.93988	28868	.93894	34737	.93800	49995	.93706	74634	.93613	08643	.93519	52013	.93426	04736	.93332	66801	.06
.07	.93239	38199	.93146	18921	.93053	08958	.92960	08300	.92867	16938	.92774	34863	.92681	62066	.92588	98536	.92496	44265	.92403	93244	.07
.08	.92311	63464	.92219	36914	.92127	19587	.92035	11472	.91943	12561	.91851	22844	.91759	42312	.91667	70956	.91576	08767	.91484	55736	.08
.09	.91393	11853	.91301	77109	.91201	51495	.91119	35003	.91028	27622	.90937	29345	.90846	40161	.90755	60061	.90664	89038	.90574	27080	.09
.10	.90483	74180	.90393	30329	.90302	95517	.90212	69735	.90122	52974	.90032	45226	.89942	46481	.89852	56730	.89762	75964	.89673	04175	.10
.11	.89583	41353	.89493	87489	.89404	42575	.89315	06601	.89225	79559	.89136	61439	.89047	52233	.88958	51932	.88869	60526	.88780	78008	.11
.12	.88692	04367	.88603	39596	.88514	83685	.88426	36626	.88337	98409	.88249	69026	.88161	48468	.88073	36726	.87985	33791	.87897	39655	.12
.13	.87809	54309	.87721	77744	.87634	09951	.87546	50921	.87459	00646	.87371	59117	.87284	26325	.87197	02261	.87109	86917	.87022	80285	.13
.14	.86935	82354	.86848	93117	.86762	12565	.86675	40689	.86588	77481	.86502	22931	.86415	77032	.86329	39774	.86243	11149	.86156	91149	.14
.15	.86070	79764	.85984	76987	.85898	82807	.85812	97218	.85727	20210	.85641	51775	.85555	91904	.85470	40588	.85384	97820	.85299	63590	.15
.16	.85214	37890	.85129	20711	.85044	12045	.84959	11884	.84874	20219	.84789	37041	.84704	62342	.84619	96113	.84535	38347	.84450	89034	.16
.17	.84366	48166	.84282	15735	.84197	91732	.84113	76148	.84029	68977	.83945	70208	.83861	79833	.83777	97845	.83694	24235	.83610	58994	.17
.18	.83527	02114	.83443	53587	.83360	13404	.83276	81557	.83193	58038	.83110	42839	.83027	35950	.82944	37364	.82861	47072	.82778	65067	.18
.19	.82695	91339	.82613	25882	.82530	68685	.82448	19741	.82365	79043	.82283	46581	.82201	22347	.82119	06333	.82036	98531	.81954	98933	.19
.20	.81873	07531	.81791	24316	.81709	49279	.81627	82414	.81546	23712	.81464	73164	.81383	30763	.81301	96500	.81220	70367	.81139	52356	.20
.21	.81058	42460	.80977	40669	.80896	46976	.80815	61372	.80734	83850	.80654	14402	.80573	53019	.80492	99693	.80412	54417	.80332	17182	.21
.22	.80251	87980	.80171	66803	.80091	53643	.80011	48493	.79931	51344	.79851	62188	.79771	81017	.79692	07823	.79612	42598	.79532	85335	.22
.23	.79453	36025	.79373	94660	.79294	61233	.79215	35735	.79136	18159	.79057	08496	.78978	06739	.78899	12880	.78820	26911	.78741	48824	.23
.24	.78662	78611	.78584	16264	.78505	61776	.78427	15138	.78348	76343	.78270	45382	.78192	22249	.78114	06935	.78035	99433	.77957	99734	.24
.25	.77880	07831	.77802	23716	.77724	47381	.77646	78818	.77569	18020	.77491	64980	.77414	19688	.77336	82138	.77259	52321	.77182	30230	.25
.26	.77105	15858	.77028	09196	.76951	10237	.76874	18973	.76797	35397	.76720	59500	.76643	91275	.76567	30715	.76490	77811	.76414	32556	.26
.27	.76337	94943	.76261	64964	.76185	42611	.76109	27876	.76033	20753	.75957	21232	.75881	29308	.75805	44971	.75729	68215	.75653	99032	.27
.28	.75578	37415	.75502	83355	.75427	36845	.75351	97879	.75276	66447	.75201	42543	.75126	26159	.75051	17288	.74976	15922	.74901	22054	.28
.29	.74826	35676	.74751	56780	.74676	85360	.74602	21407	.74527	64914	.74453	15875	.74378	74280	.74304	40124	.74230	13397	.74155	94094	.29
.30	.74081	82207	.74007	77727	.73933	80649	.73859	90964	.73786	08665	.73712	33744	.73638	66195	.73565	06009	.73491	53180	.73418	07700	.30
.31	.73344	69562	.73271	38759	.73198	15282	.73124	99126	.73051	90282	.72978	88743	.72905	94502	.72833	07551	.72760	27884	.72687	55493	.31
.32	.72614	90371	.72542	32510	.72469	81903	.72397	38544	.72325	02424	.72252	73536	.72180	51874	.72108	37430	.72036	30197	.71964	30167	.32
.33	.71892	37334	.71820	51690	.71748	73229	.71677	01942	.71605	37822	.71533	80864	.71462	31058	.71390	88399	.71319	52879	.71248	24491	.33
.34	.71177	03228	.71105	89082	.71034	82047	.70963	82116	.70892	89280	.70822	03535	.70751	24871	.70680	53283	.70609	88762	.70539	31303	.34
.35	.70468	80897	.70398	37539	.70328	01220	.70257	71934	.70187	49674	.70117	34432	.70047	26202	.69977	24977	.69907	30750	.69837	43514	.35
.36	.69767	63261	.69697	89985	.69628	23678	.69558	64335	.69489	11947	.69419	66509	.69350	28012	.69280	96450	.69211	71817	.69142	54105	.36
.37	.69073	43306	.69004	39416	.68935	42425	.68866	52328	.68797	69118	.68728	92788	.68660	23330	.68591	60739	.68523	05007	.68454	56127	.37
.38	.68386	14092	.68317	78896	.68249	50532	.68181	28993	.68113	14272	.68045	06362	.67977	05257	.67909	10949	.67841	23433	.67773	42700	.38
.39	.67705	68745	.67638	01560	.67570	41140	.67502	87476	.67435	40562	.67368	00392	.67300	66959	.67233	40256	.67166	20277	.67099	07014	.39
.40	.67032	00460	.66965	00610	.66898	07457	.66831	20993	.66764	41213	.66697	68109	.66631	01674	.66564	41903	.66497	88788	.66431	42323	.40
.41	.66365	02501	.66298	69316	.66232	42761	.66166	22828	.66100	09513	.66034	02807	.65968	02705	.65902	09199	.65836	22284	.65770	41953	.41
.42	.65704	68198	.65639	01014	.65573	40394	.65507	86331	.65442	38819	.65376	97851	.65311	63421	.65246	35522	.65181	14148	.65115	99292	.42
.43	.65050	90947	.64985	89108	.64920	93767	.64856	04918	.64791	22555	.64726	46671	.64661	77259	.64597	14314	.64532	57829	.64468	07796	.43
.44	.64403	64211	.64339	27066	.64274	96355	.64210	72071	.64146	54208	.64082	42760	.64018	37721	.63954	39083	.63890	46840	.63826	60987	.44
.45	.63762	81516	.63699	08422	.63635	41697	.63571	81336	.63508	27332	.63444	79679	.63381	38371	.63318	03401	.63254	74762	.63191	52449	.45
.46	.63128	36455	.63065	26774	.63002	23399	.62939	26325	.62876	35545	.62813	51052	.62750	72840	.62688	00904	.62625	35237	.62562	75832	.46
.47	.62500	22683	.62437	75784	.62375	35129	.62313	00712	.62250	72526	.62188	50565	.62126	34822	.62064	25293	.62002	21970	.61940	24847	.47
.48	.61878	33918	.61816	49177	.61754	70618	.61692	98234	.61631	32019	.61569	71968	.61508	18073	.61446	70329	.61385	28730	.61323	93270	.48
.49	.61262	63942	.61201	40740	.61140	23658	.61079	12691	.61018	07831	.60957	09073	.60896	16411	.60835	29838	.60774	49349	.60713	74937	.49

65

.50	.60653	.06597	.60592	.44322	.60531	.88106	.60471	.37944	.60410	.93829	.60350	.55754	.60290	.23715	.60229	.97705	.60169	.77718	.60109	.63747	.50
.51	.60049	.55788	.59989	.53834	.59929	.57878	.59869	.67916	.59809	.83941	.59750	.05946	.59690	.33927	.59630	.67876	.59571	.07789	.59511	.53659	.51
.52	.59452	.05480	.59392	.63246	.59333	.26951	.59273	.96590	.59214	.72156	.59155	.53644	.59096	.41047	.59037	.34360	.58978	.33576	.58919	.38690	.52
.53	.58860	.49697	.58801	.66589	.58742	.9362	.58684	.18008	.58625	.52524	.58566	.92901	.58508	.39136	.58449	.91221	.58391	.49152	.58333	.12921	.53
.54	.58274	.82524	.58216	.57954	.58158	.39206	.58100	.26274	.58042	.19151	.57984	.17833	.57926	.22314	.57868	.32587	.57810	.48647	.57752	.70488	.54
.55	.57694	.98104	.57637	.31489	.57579	.70639	.57522	.15546	.57464	.66206	.57407	.22612	.57349	.84759	.57292	.52641	.57235	.26252	.57178	.05586	.55
.56	.57120	.90638	.57063	.81403	.57006	.77874	.56949	.80045	.56892	.87912	.56836	.01468	.56779	.20707	.56722	.45624	.56665	.76214	.56609	.12470	.56
.57	.56552	.54387	.56496	.01959	.56439	.55181	.56383	.14047	.56326	.78551	.56270	.48688	.56214	.24452	.56158	.05837	.56101	.92838	.56045	.85450	.57
.58	.55989	.83666	.55933	.87481	.55877	.96889	.55822	.11885	.55766	.32463	.55710	.58618	.55654	.90344	.55599	.27636	.55543	.70487	.55488	.18893	.58
.59	.55432	.72847	.55377	.32345	.55321	.97381	.55266	.67949	.55211	.44043	.55156	.25659	.55101	.12790	.55046	.05431	.54991	.03577	.54936	.07222	.59
.60	.54881	.16361	.54826	.30988	.54771	.51097	.54716	.76684	.54662	.07742	.54607	.44266	.54552	.86252	.54498	.33692	.54443	.86582	.54389	.44917	.60
.61	.54335	.08691	.54280	.77898	.54226	.52533	.54172	.32591	.54118	.18066	.54064	.08953	.54010	.05246	.53956	.06941	.53902	.14031	.53848	.26511	.61
.62	.53794	.44376	.53740	.67620	.53686	.96239	.53633	.30226	.53579	.69577	.53526	.14285	.53472	.64346	.53419	.19755	.53365	.80505	.53312	.46592	.62
.63	.53259	.18010	.53205	.94754	.53152	.76819	.53099	.64199	.53046	.56889	.52993	.54883	.52940	.58177	.52887	.66765	.52834	.80642	.52781	.99802	.63
.64	.52729	.24240	.52676	.53952	.52623	.88931	.52571	.29172	.52518	.74671	.52466	.25421	.52413	.81418	.52361	.42656	.52309	.09131	.52256	.80836	.64
.65	.52204	.57768	.52152	.39919	.52100	.27286	.52048	.19863	.51996	.17645	.51944	.20626	.51892	.28802	.51840	.42167	.51788	.60716	.51736	.84443	.65
.66	.51685	.13345	.51633	.47415	.51581	.86648	.51530	.31040	.51478	.80585	.51427	.35277	.51375	.95112	.51324	.60085	.51273	.30190	.51222	.05423	.66
.67	.51170	.85778	.51119	.71250	.51068	.61834	.51017	.57524	.50966	.58317	.50915	.64206	.50864	.75187	.50813	.91254	.50763	.12403	.50712	.38628	.67
.68	.50661	.69924	.50611	.06286	.50560	.47709	.50509	.94189	.50459	.45719	.50409	.02296	.50358	.63913	.50308	.30566	.50258	.02250	.50207	.78960	.68
.69	.50157	.60691	.50107	.47437	.50057	.39194	.50007	.35957	.49957	.37721	.49907	.44480	.49857	.56230	.49807	.72966	.49757	.94682	.49708	.21375	.69
.70	.49658	.53038	.49608	.89667	.49559	.31257	.49509	.77803	.49460	.29300	.49410	.85743	.49361	.47127	.49312	.13447	.49262	.84698	.49213	.60876	.70
.71	.49164	.41975	.49115	.27990	.49066	.18917	.49017	.14751	.48968	.15486	.48919	.21118	.48870	.31642	.48821	.47053	.48772	.67346	.48723	.92517	.71
.72	.48675	.22560	.48626	.57470	.48577	.97243	.48529	.41874	.48480	.91358	.48432	.45690	.48384	.04865	.48335	.68878	.48287	.37725	.48239	.11401	.72
.73	.48190	.89901	.48142	.73220	.48094	.61353	.48046	.54295	.47998	.52043	.47950	.54590	.47902	.61932	.47854	.74064	.47806	.90982	.47759	.12681	.73
.74	.47711	.39155	.47663	.70401	.47616	.06433	.47568	.47186	.47520	.92717	.47473	.42999	.47425	.98029	.47378	.57802	.47331	.22312	.47283	.91556	.74
.75	.47236	.65527	.47189	.44223	.47142	.27637	.47095	.15766	.47048	.08604	.47001	.06147	.46954	.08390	.46907	.15329	.46860	.26958	.46813	.43273	.75
.76	.46766	.64270	.46719	.89943	.46673	.20289	.46626	.55301	.46579	.94976	.46533	.39310	.46486	.88296	.46440	.41932	.46394	.00211	.46347	.63130	.76
.77	.46301	.30683	.46255	.02867	.46208	.79676	.46162	.61106	.46116	.47152	.46070	.37810	.46024	.33075	.45978	.32942	.45932	.37408	.45886	.46466	.77
.78	.45840	.60113	.45794	.78344	.45749	.01155	.45703	.28540	.45657	.60496	.45611	.97018	.45566	.38101	.45520	.83740	.45475	.33932	.45429	.88671	.78
.79	.45384	.47953	.45339	.11773	.45293	.80128	.45248	.53012	.45203	.30420	.45158	.12349	.45112	.98794	.45067	.89750	.45022	.85213	.44977	.85178	.79
.80	.44932	.89641	.44887	.98597	.44843	.12042	.44798	.29972	.44753	.52361	.44708	.79266	.44664	.10621	.44619	.46443	.44574	.86727	.44530	.31468	.80
.81	.44485	.80662	.44441	.34305	.44396	.92392	.44352	.54919	.44308	.21881	.44263	.93274	.44219	.69093	.44175	.49334	.44131	.33993	.44087	.23064	.81
.82	.44043	.16545	.43999	.14430	.43955	.16715	.43911	.23395	.43867	.34466	.43823	.49925	.43779	.69765	.43735	.93984	.43692	.22576	.43648	.55537	.82
.83	.43604	.92863	.43561	.34550	.43517	.80593	.43474	.30987	.43430	.85729	.43387	.44814	.43344	.08238	.43300	.75996	.43257	.48085	.43214	.24499	.83
.84	.43171	.05234	.43127	.90287	.43084	.79652	.43041	.73326	.42998	.71304	.42955	.73582	.42912	.80156	.42869	.91020	.42827	.06172	.42784	.25607	.84
.85	.42741	.49319	.42698	.77307	.42656	.09563	.42613	.46086	.42570	.86870	.42528	.31911	.42485	.81205	.42443	.34747	.42400	.92534	.42358	.54561	.85
.86	.42316	.20823	.42273	.91317	.42231	.66039	.42189	.44984	.42147	.28148	.42105	.15526	.42063	.07115	.42021	.02911	.41979	.02908	.41937	.07103	.86
.87	.41895	.15492	.41853	.28071	.41811	.44835	.41769	.65780	.41727	.90902	.41686	.20197	.41644	.53660	.41602	.91288	.41561	.33076	.41519	.79021	.87
.88	.41478	.29117	.41436	.83361	.41395	.41749	.41354	.04276	.41312	.70939	.41271	.41733	.41230	.16654	.41188	.95698	.41147	.78861	.41106	.66139	.88
.89	.41065	.57528	.41024	.53023	.40983	.52620	.40942	.56316	.40901	.64106	.40860	.75986	.40819	.91953	.40779	.12001	.40738	.36127	.40697	.64328	.89
.90	.40656	.96597	.40616	.32933	.40575	.73330	.40535	.17785	.40494	.66293	.40454	.18851	.40413	.75454	.40373	.36099	.40333	.00781	.40292	.69496	.90
.91	.40252	.42240	.40212	.19010	.40171	.99801	.40131	.84609	.40091	.73430	.40051	.66261	.40011	.63097	.39971	.63933	.39931	.68767	.39891	.77595	.91
.92	.39851	.90411	.39812	.07212	.39772	.27995	.39732	.52755	.39692	.81488	.39653	.14191	.39613	.50859	.39573	.91488	.39534	.36074	.39494	.84614	.92
.93	.39455	.37104	.39415	.93539	.39376	.53915	.39337	.18230	.39297	.86478	.39258	.58655	.39219	.34759	.39180	.14784	.39140	.98728	.39101	.86586	.93
.94	.39062	.78354	.39023	.74028	.38984	.73604	.38945	.77079	.38906	.84449	.38867	.95709	.38829	.10856	.38790	.29886	.38751	.52795	.38712	.79579	.94
.95	.38674	.10235	.38635	.44757	.38596	.83144	.38558	.25390	.38519	.71492	.38481	.21446	.38442	.75248	.38404	.32894	.38365	.94380	.38327	.59704	.95
.96	.38289	.28860	.38251	.01845	.38212	.78655	.38174	.59286	.38136	.43735	.38098	.31997	.38060	.24070	.38022	.19948	.37984	.19629	.37946	.23107	.96
.97	.37908	.30381	.37870	.41445	.37832	.56297	.37794	.74932	.37756	.97346	.37719	.25536	.37681	.53497	.37643	.87227	.37606	.24722	.37568	.65977	.97
.98	.37531	.10989	.37493	.59753	.37456	.12268	.37418	.68528	.37381	.28529	.37343	.92269	.37306	.59744	.37269	.30949	.37232	.05881	.37194	.84536	.98
.99	.37157	.66910	.37120	.53001	.37083	.42803	.37046	.36314	.37009	.33529	.36972	.34445	.36935	.39059	.36898	.47366	.36861	.59363	.36824	.75046	.99
(1.00	.36787	.94412)																			
x	0		2		3		4		5		6		7		8		9		x		

For larger x, multiply values tabulated above by values in table for x=1(1)100

x		e^{-x}	x		e^{-x}
1	(-1)	3.67879 44117	51	(-23)	7.09547 41623
2	(-1)	1.35335 28324	52	(-23)	2.61027 90697
3	(-2)	4.97870 68368	53	(-24)	9.60268 00545
4	(-2)	1.83156 38889	54	(-24)	3.53262 85722
5	(-3)	6.73794 69991	55	(-24)	1.29958 14250
6	(-3)	2.47875 21767	56	(-25)	4.78089 28839
7	(-4)	9.11881 96555	57	(-25)	1.75879 22024
8	(-4)	3.35462 62790	58	(-26)	6.47023 49256
9	(-4)	1.23409 80409	59	(-26)	2.38026 64087
10	(-5)	4.53999 29762	60	(-27)	8.75651 07627
11	(-5)	1.67017 00790	61	(-27)	3.22134 02860
12	(-6)	6.14421 23533	62	(-27)	1.18506 48642
13	(-6)	2.26032 94070	63	(-28)	4.35961 00001
14	(-7)	8.31528 71910	64	(-28)	1.60381 08905
15	(-7)	3.05902 32050	65	(-29)	5.90009 05416
16	(-7)	1.12535 17472	66	(-29)	2.17052 20113
17	(-8)	4.13993 77188	67	(-30)	7.98490 42457
18	(-8)	1.52299 79745	68	(-30)	2.93748 21117
19	(-9)	5.60279 64375	69	(-30)	1.08063 92777
20	(-9)	2.06115 36224	70	(-31)	3.97544 97359
21	(-10)	7.58256 04279	71	(-31)	1.46248 62273
22	(-10)	2.78946 80929	72	(-32)	5.38018 61600
23	(-10)	1.02618 79632	73	(-32)	1.97925 98779
24	(-11)	3.77513 45443	74	(-33)	7.28129 01783
25	(-11)	1.38879 43865	75	(-33)	2.67863 69618
26	(-12)	5.10908 90281	76	(-34)	9.85415 46861
27	(-12)	1.87952 88165	77	(-34)	3.62514 09191
28	(-13)	6.91440 01069	78	(-34)	1.33361 48155
29	(-13)	2.54366 56474	79	(-35)	4.90609 47306
30	(-14)	9.35762 29688	80	(-35)	1.80485 13878
31	(-14)	3.44247 71085	81	(-36)	6.63967 71996
32	(-14)	1.26641 65549	82	(-36)	2.44260 07377
33	(-15)	4.65888 61451	83	(-37)	8.98582 59440
34	(-15)	1.71390 84315	84	(-37)	3.30570 06268
35	(-16)	6.30511 67601	85	(-37)	1.21609 92993
36	(-16)	2.31952 28302	86	(-38)	4.47377 93062
37	(-17)	8.53304 76257	87	(-38)	1.64581 14311
38	(-17)	3.13913 27920	88	(-39)	6.05460 18954
39	(-17)	1.15482 24173	89	(-39)	2.22736 35618
40	(-18)	4.24835 42553	90	(-40)	8.19401 26240
41	(-18)	1.56288 21893	91	(-40)	3.01440 87851
42	(-19)	5.74952 22643	92	(-40)	1.10893 90193
43	(-19)	2.11513 10376	93	(-41)	4.07995 86672
44	(-20)	7.78113 22411	94	(-41)	1.50078 57627
45	(-20)	2.86251 85805	95	(-42)	5.52108 22770
46	(-20)	1.05306 17358	96	(-42)	2.03109 26627
47	(-21)	3.87399 76287	97	(-43)	7.47197 23373
48	(-21)	1.42516 40827	98	(-43)	2.74878 50079
49	(-22)	5.24288 56634	99	(-43)	1.01122 14926
50	(-22)	1.92874 98480	100	(-44)	3.72007 59760

The numbers in parentheses indicate the power -20 of 10 by which tabulated values are to be multiplied; e.g. $e^{-20} = .0000000020611536224$.

Table C5, B(c,n,p) for n=1(1)20 and p=.01(.01).50.

n	p	01	02	03	04	05	06	07	08	09
1	0	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
1	1	0100	0200	0300	0400	0500	0600	0700	0800	0900
2	0	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
2	1	0199	0396	0591	0784	0975	1164	1351	1536	1719
2	2	0001	0004	0009	0016	0025	0036	0049	0064	0081
(Note: Unity values for c=0 omitted from here on.)										
3	1	02970	05881	08733	11526	14263	16942	19564	22131	24643
3	2	00030	00118	00265	00467	00725	01037	01401	01818	02284
3	3	00000	00001	00003	00006	00013	00022	00034	00051	00073
4	1	03940	07763	11471	15065	18549	21925	25195	28361	31425
4	2	00059	00234	00519	00910	01402	01991	02673	03443	04296
4	3	00000	00003	00011	00025	00048	00083	00130	00193	00272
4	4		00000	00000	00000	00001	00001	00002	00004	00007
5	1	04901	09608	14127	18463	22622	26610	30431	34092	37597
5	2	00098	00384	00847	01476	02259	03187	04249	05436	06738
5	3	00001	00008	00026	00060	00116	00197	00308	00453	00634
5	4	00000	00000	00000	00001	00003	00006	00011	00019	00030
5	5				00000	00000	00000	00000	00000	00001
6	1	05852	11416	16703	21724	26491	31013	35301	39364	43213
6	2	00146	00569	01246	02155	03277	04592	06082	07729	09515
6	3	00002	00015	00050	00117	00223	00376	00584	00851	01183
6	4	00000	00000	00001	00004	00009	00018	00032	00054	00085
6	5			00000	00000	00000	00000	00001	00002	00003
7	1	06793	13187	19202	24855	30166	35152	39830	44215	48324
7	2	00203	00786	01709	02938	04438	06178	08127	10259	12548
7	3	00003	00026	00086	00198	00376	00629	00969	01401	01933
7	4	00000	00001	00003	00008	00019	00039	00071	00118	00184
7	5		00000	00000	00000	00001	00001	00003	00006	00011
8	1	07726	14924	21626	27861	33658	39043	44042	48678	52975
8	2	00269	01034	02234	03815	05724	07916	10347	12976	15768
8	3	00005	00042	00135	00308	00579	00962	01470	02110	02889
8	4	00000	00001	00005	00016	00037	00075	00134	00220	00341
8	5		00000	00000	00001	00002	00004	00008	00015	00026
8	6				00000	00000	00000	00000	00001	00001
9	1	08648	16625	23977	30747	36975	42701	47959	52784	57207
9	2	00344	01311	02816	04777	07121	09784	12705	15832	19117
9	3	00003	00061	00198	00448	00836	01380	02091	02979	04048
9	4	00000	00002	00009	00027	00064	00128	00227	00372	00570
9	5		00000	00000	00001	00003	00008	00017	00031	00055
9	6				00000	00000	00000	00001	00002	00004

n	p	10	15	20	25	30	35	40	45	50
1	0	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
1	1	1000	1500	2000	2500	3000	3500	4000	4500	5000
2	0	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
2	1	1900	2775	3600	4375	5100	5775	6400	6975	7500
2	2	0100	0225	0400	0625	0900	1225	1600	2025	2500
(Note: Unity values for c=0 omitted from here on.)										
3	1	27100	38587	48800	57812	65700	72537	78400	83362	87500
3	2	02800	06075	10400	15625	21600	28175	35200	42525	50000
3	3	00100	00337	00800	01562	02700	04287	06400	09112	12500
4	1	34390	47799	59040	68359	75990	82149	87040	90849	93750
4	2	05230	10952	18080	26172	34830	43702	52480	60902	68750
4	3	00370	01198	02720	05078	08370	12648	17920	24148	31250
4	4	00010	00051	00160	00391	00810	01501	02560	04101	06250
5	1	40951	55629	67232	76270	83193	88397	92224	94967	96875
5	2	08146	16479	26272	36719	47178	57159	66304	74378	81250
5	3	00856	02661	05792	10352	16308	23517	31744	40687	50000
5	4	00046	00223	00672	01563	03078	05402	08704	13122	18750
5	5	00001	00008	00032	00098	00243	00525	01024	01845	03125
6	1	46856	62285	73786	82202	88235	92458	95334	97232	98437
6	2	11427	22352	34464	46606	57983	68092	76672	83643	89063
6	3	01585	04734	09888	16943	25569	35291	45568	55848	65625
6	4	00127	00589	01696	03760	07047	11742	17920	25526	34375
6	5	00005	00040	00160	00464	01093	02232	04096	06920	10937
6	6	00000	00001	00006	00024	00073	00184	00410	00830	01563
7	1	52170	67942	79028	86652	91765	95098	97201	98478	99219
7	2	14969	28342	42328	55505	67058	76620	84137	89758	93750
7	3	02569	07377	14803	24359	35293	46772	58010	68356	77344
7	4	00273	01210	03334	07056	12604	19985	28979	39171	50000
7	5	00018	00122	00467	01288	02880	05561	09626	15293	22656
7	6	00001	00007	00037	00134	00379	00901	01884	03571	06250
7	7	00000	00000	00001	00006	00022	00064	00164	00374	00781
8	1	56953	72751	83223	89989	94235	96814	98320	99163	99609
8	2	18690	34282	49668	63292	74470	83087	89362	93682	96484
8	3	03809	10521	20308	32146	44823	57219	68461	77987	85547
8	4	00502	02135	05628	11382	19410	29360	40591	52304	63672
8	5	00043	00285	01041	02730	05797	10609	17367	26038	36328
8	6	00002	00024	00123	00423	01129	02532	04981	08846	14453
8	7	00000	00001	00008	00038	00129	00357	00852	01812	03516
8	8		00000	00000	00002	00007	00023	00066	00168	00391
9	1	61258	76838	86578	92492	95965	97929	98992	99539	99805
9	2	22516	40052	56379	69966	80400	87891	92946	96148	98047
9	3	05297	14085	26180	39932	53717	66273	76821	85050	91016
9	4	00833	03393	08564	16573	27034	39111	51739	63862	74609
9	5	00089	00563	01958	04893	09881	17172	26657	37858	50000
9	6	00006	00063	00307	00999	02529	05359	09935	16582	25391
9	7	00000	00005	00031	00134	00429	01118	02503	04977	08984
9	8		00000	00002	00011	00043	00140	00380	00908	01953
9	9			00000	00000	00002	00008	00026	00076	00195

n	p	01	02	03	04	05	06	07	08	09
10	1	09562	18293	26258	33517	40126	46138	51602	56561	61058
	2	00427	01618	03451	05815	08614	11759	15173	18788	22545
	3	00011	00086	00276	00621	01150	01884	02834	04008	05404
	4	00000	00003	00015	00044	00103	00203	00358	00580	00883
	5		00000	00001	00002	00006	00015	00031	00059	00101
	6			00000	00000	00000	00001	00002	00004	00008
	7						00000	00000	00000	00000

11	1	10466	19927	28470	36176	43120	49370	54990	60036	64563
	2	00518	01951	04135	06923	10189	13822	17723	21810	26011
	3	00016	00117	00372	00829	01524	02476	03698	05190	06947
	4	00000	00005	00023	00067	00155	00304	00531	00854	01290
	5		00000	00001	00004	00011	00026	00054	00100	00171
	6			00000	00000	00001	00002	00004	00009	00016
	7					00000	00000	00000	00001	00001
	8							00000	00000	00000

69

12	1	11362	21528	30616	38729	45964	52408	58140	63233	67752
	2	00617	02311	04865	08094	11836	15955	20332	24868	29481
	3	00021	00154	00485	01073	01957	03157	04680	06520	08662
	4	00000	00007	00033	00098	00224	00434	00753	01201	01799
	5		00000	00002	00006	00018	00043	00088	00161	00272
	6			00000	00000	00001	00003	00008	00016	00030
	7					00000	00000	00000	00001	00003
	8							00000	00000	00000

13	1	12248	23098	32697	41180	48666	55263	61071	66175	70655
	2	00725	02695	05637	09319	13542	18142	22978	27937	32925
	3	00027	00197	00616	01354	02451	03925	05775	07987	10536
	4	00001	00010	00047	00137	00310	00598	01028	01627	02417
	5	00000	00000	00003	00010	00029	00067	00134	00244	00410
	6			00000	00001	00002	00006	00013	00027	00052
	7				00000	00000	00000	00001	00002	00005
	8							00000	00000	00000

n	p	10	15	20	25	30	35	40	45	50
10	1	65132	80313	89263	94369	97175	98654	99395	99747	99902
	2	26390	45570	62419	75597	85069	91405	95364	97674	98926
	3	07019	17980	32220	47441	61722	73839	83271	90044	94531
	4	01280	04997	12087	22412	35039	48617	61772	73396	82813
	5	00163	00987	03279	07813	15027	24850	36690	49560	62305
	6	00015	00138	00637	01973	04735	09493	16624	26156	37695
	7	00001	00013	00086	00351	01059	02602	05476	10199	17187
	8	00000	00001	00008	00042	00159	00482	01229	02739	05469
	9		00000	00000	00003	00014	00054	00168	00450	01074
	10				00000	00001	00003	00010	00034	00098

11	1	68619	83266	91410	95776	98023	99125	99637	99861	99951
	2	30264	50781	67788	80290	88701	93942	96977	98607	99414
	3	08956	22119	38260	54480	68726	79987	88108	93478	96729
	4	01853	06944	16114	28670	43044	57445	70372	80888	88672
	5	00275	01589	05041	11463	21030	33169	46723	60286	72559
	6	00030	00266	01165	03433	07822	14868	24650	36688	50000
	7	00002	00032	00197	00756	02162	05014	09935	17380	27441
	8	00000	00003	00024	00119	00429	01224	02928	06096	11328
	9		00000	00002	00013	00058	00204	00592	01480	03271
	10			00000	00001	00005	00021	00073	00221	00586

11				00000	00000	00001	00004	00015	00049	
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12	1	71757	85776	93128	96832	98616	99431	99782	99923	99976
	2	34100	55654	72512	84162	91497	95756	98041	99171	99683
	3	11087	26418	44165	60932	74718	84871	91656	95786	98071
	4	02564	09221	20543	35122	50748	65335	77466	86553	92700
	5	00433	02392	07256	15764	27634	41665	56182	69557	80615
	6	00054	00464	01941	05440	11785	21274	33479	47307	61279
	7	00005	00067	00390	01425	03860	08463	15821	26069	38721
	8	00000	00007	00058	00278	00949	02551	05731	11174	19385
	9		00001	00006	00039	00169	00561	01527	03557	07300
	10			00000	00004	00021	00085	00281	00788	01929

11				00000	00000	00002	00008	00032	00108	00317
12					00000	00000	00000	00002	00007	00024

13	1	74581	87909	94502	97624	99031	99630	99869	99958	99988
	2	37866	60172	76635	87329	93633	97042	98737	99510	99829
	3	13388	30804	49835	66740	79752	88681	94210	97309	98877
	4	03416	11800	25268	41575	57939	72173	83142	90708	95386
	5	00646	03416	09913	20604	34569	49950	64696	77205	86658
	6	00092	00753	03004	08021	16540	28411	42560	57319	70947
	7	00010	00127	00700	02429	06238	12947	22884	35626	50000
	8	00001	00016	00125	00565	01822	04620	09767	17877	29053
	9	00000	00002	00017	00099	00403	01257	03208	06985	13342
	10		00000	00002	00013	00065	00251	00779	02034	04614

11			00000	00001	00007	00035	00132	00414	01123	
12				00000	00000	00003	00014	00052	00171	
13						00000	00001	00003	00012	

n	p	01	02	03	04	05	06	07	08	09
14	c									
1	1	13125	24636	34716	43533	51233	57948	63796	68881	73296
2	2	00840	03103	06449	10593	15299	20369	25645	30996	36321
3	3	00034	00247	00767	01672	03005	04778	06980	09583	12551
4	4	00001	00014	00064	00185	00417	00797	01360	02136	03148
5	5	00000	00001	00004	00015	00043	00098	00197	00354	00590
6			00000	00000	00001	00003	00009	00022	00045	00084
7					00000	00000	00001	00002	00004	00009
8							00000	00000	00000	00001
9										00000
15	1	13994	26143	36675	45791	53671	60471	66330	71370	75699
2	2	00963	03534	07297	11911	17095	22624	28315	34027	39649
3	3	00042	00304	00937	02029	03620	05713	08286	11297	14690
4	4	00001	00018	00085	00245	00547	01036	01753	02731	03994
5	5	00000	00001	00006	00022	00061	00140	00278	00497	00820
6			00000	00000	00001	00005	00015	00034	00070	00130
7					00000	00000	00001	00003	00008	00016
8							00000	00000	00001	00002
9									00000	00000
16	1	14854	27620	38575	47960	55987	62843	68687	73661	77886
2	2	01093	03986	08179	13266	18924	24895	30976	37015	42893
3	3	00051	00369	01128	02424	04294	06728	09688	13115	16937
4	4	00002	00024	00110	00316	00700	01317	02211	03417	04957
5	5	00000	00001	00008	00031	00086	00194	00381	00676	01106
6			00000	00000	00002	00008	00022	00051	00104	00192
7					00000	00001	00002	00005	00013	00026
8						00000	00000	00000	00001	00003
9									00000	00000

n	p	10	15	20	25	30	35	40	45	50
14	c									
1	1	77123	89723	95602	98218	99322	99760	99922	99977	99994
2	2	41537	64333	80209	89903	95252	97948	99190	99711	99908
3	3	15836	35209	55195	71887	83916	91607	96021	98299	99353
4	4	04413	14651	30181	47866	64483	77950	87569	93678	97131
5	5	00923	04674	12984	25847	41580	57728	72074	83281	91022
6		00147	01153	04385	11167	21948	35949	51415	66268	78802
7		00018	00221	01161	03827	09328	18359	30755	45388	60474
8		00002	00033	00240	01031	03147	07534	15014	25864	39526
9		00000	00004	00038	00215	00829	02434	05832	11886	21198
10			00000	00005	00034	00167	00604	01751	04262	08978
11				00000	00004	00025	00111	00391	01143	02869
12					00001	00003	00014	00061	00215	00647
13						00000	00001	00006	00025	00092
14							00000	00000	00001	00006
15	1	79411	91265	96482	98664	99525	99844	99953	99987	99997
2	2	45096	68141	83287	91982	96473	98582	99483	99831	99951
3	3	18406	39577	60198	76391	87317	93827	97289	98935	99631
4	4	05556	17734	35184	53871	70313	82730	90950	95758	98242
5	5	01272	06171	16423	31351	48451	64806	78272	87960	94077
6		00225	01681	06105	14837	27838	43572	59678	73924	84912
7		00031	00361	01806	05662	13114	24516	39019	54784	69638
8		00003	00061	00424	01730	05001	11323	21310	34650	50000
9		00000	00008	00078	00419	01524	04219	09505	18176	30362
10			00001	00011	00079	00365	01244	03383	07693	15088
11			00000	00001	00012	00067	00283	00935	02547	05923
12				00000	00001	00009	00048	00193	00633	01758
13					00000	00001	00006	00028	00111	00369
14						00000	00000	00003	00012	00049
15							00000	00000	00001	00003
16	1	81470	92575	97185	98998	99668	99898	99972	99993	99998
2	2	48527	71610	85926	93652	97389	99024	99671	99901	99974
3	3	21075	43862	64816	80289	90064	95491	98166	99338	99791
4	4	06841	21011	40187	59501	75414	86614	93485	97187	98936
5	5	01700	07905	20175	36981	55010	71079	83343	91469	96159
6		00330	02354	08169	18965	34022	51004	67116	80240	89494
7		00050	00559	02666	07956	17531	31185	47283	63397	77275
8		00006	00106	00700	02713	07435	15941	28394	43710	59819
9		00001	00016	00148	00747	02567	06706	14227	25589	40181
10		00000	00002	00025	00164	00286	02286	05832	12410	22725
11			00000	00003	00029	00157	00620	01914	04862	10506
12				00000	00004	00027	00130	00490	01494	03841
13					00000	00003	00020	00094	00346	01064
14						00000	00002	00013	00057	00209
15							00000	00001	00006	00026
16								00000	00000	00002

n	p										
		01	02	03	04	05	06	07	08	09	
17	1	15706	29068	40417	50041	58188	65072	70879	75768	79876	
	2	01231	04459	09090	14654	20777	27171	33616	39946	46042	
	3	00061	00441	01339	02858	05025	07818	11178	15027	19273	
	4	00002	00031	00141	00401	00880	01641	02734	04192	06035	
	5	00000	00002	00011	00042	00116	00261	00509	00895	01453	
	6		00000	00001	00003	00012	00032	00074	00149	00274	
	7			00000	00000	00001	00003	00009	00020	00041	
	8					00000	00000	00001	00002	00005	
	9							00000	00000	00000	
	10										
18	1	16549	30486	42205	52040	60279	67168	72917	77706	81688	
	2	01376	04951	10030	16069	22648	29445	36224	42812	49088	
	3	00073	00521	01572	03330	05813	08979	12749	17020	21682	
	4	00003	00039	00177	00499	01087	02012	03325	05059	07226	
	5	00000	00002	00015	00057	00155	00344	00665	01159	01865	
	6		00000	00001	00005	00017	00046	00105	00209	00380	
	7			00000	00000	00002	00005	00013	00030	00062	
	8					00000	00000	00001	00004	00008	
	9							00000	00000	00001	
	10									00000	

n	p	10	15	20	25	30	35	40	45	50	
17	1	83323	93689	97748	99248	99767	99934	99983	99996	99999	
	2	51821	74755	88178	94989	98072	99330	99791	99943	99986	
	3	23820	48024	69038	83630	92261	96727	98768	99591	99883	
	4	08264	24439	45112	64698	79809	89721	95358	98155	99364	
	5	02214	09871	24178	42611	61131	76516	87400	94042	97548	
	6	00467	03187	10570	23469	40318	58030	73607	85293	92827	
	7	00078	00828	03766	10708	22478	38122	55216	70976	83385	
	8	00011	00174	01093	04024	10464	21276	35949	52569	68547	
	9	00001	00030	00258	01238	04028	09938	19894	33744	50000	
	10	00000	00004	00049	00310	01269	03833	09190	18341	31453	
18	11		00000	00008	00063	00324	01203	03481	08259	16615	
	12			00001	00010	00066	00301	01059	03010	07173	
	13			00000	00001	00059	00252	00862	02452	05173	
	14				00000	00001	00009	00045	00187	00636	
	15					00000	00001	00006	00029	00117	
	16						00000	00000	00003	00014	
	17							00000	00000	00001	
	18										
	19										
	20										
18	1	84991	94635	98199	99436	99837	99957	99990	99998	1.000	
	2	54972	77595	90092	96054	98581	99541	99868	99967	99993	
	3	26620	52034	72866	86469	94005	97638	99177	99749	99934	
	4	09820	27976	49897	69431	83545	92173	96722	98800	99623	
	5	02819	12056	28365	48133	66735	81138	90583	95893	98456	
	6	00642	04190	13292	28255	46562	64500	79124	89230	95187	
	7	00117	01182	05127	13898	27830	45090	62572	77419	88106	
	8	00017	00272	01628	05695	14068	27172	43656	60852	75966	
	9	00002	00051	00425	01935	05959	13906	26316	42215	59274	
	10	00000	00008	00091	00542	02097	05969	13471	25272	40726	
18	11		00001	00016	00124	00607	02123	05765	12796	24034	
	12		00000	00002	00023	00143	00617	02028	05372	11894	
	13			00000	00003	00027	00144	00575	01829	04813	
	14				00004	00026	00128	00491	01544	04544	
	15					00000	00004	00021	00100	00377	
	16						00000	00003	00014	00066	
	17							00000	00001	00007	
	18								00000	00000	
	19										
	20										

[illegible][illegible]

Table C6:
Normal Tables, integral, density $\phi(t)$, 2nd derivative $\phi^{(2)}(t)$; $t=0(.01)4$.

t	$\int_0^t \phi(t)dt$	$\phi(t)$	$\phi^{(2)}(t)$	t	$\int_0^t \phi(t)dt$	$\phi(t)$	$\phi^{(2)}(t)$	t	$\int_0^t \phi(t)dt$	$\phi(t)$	$\phi^{(2)}(t)$	t	$\int_0^t \phi(t)dt$	$\phi(t)$	$\phi^{(2)}(t)$
.00	.00000	.39894	-.39894	.50	.19146	.35207	-.26405	1.00	.34134	.24197	.00000	1.50	.43319	.12952	.16190
.01	.00399	.39892	-.39888	.51	.19497	.35029	-.25918	1.01	.34375	.23955	.00482	1.51	.43448	.12758	.16332
.02	.00798	.39886	-.39870	.52	.19847	.34849	-.25426	1.02	.34614	.23713	.00950	1.52	.43574	.12566	.16467
.03	.01197	.39876	-.39840	.53	.20194	.34667	-.24929	1.03	.34850	.23471	.01429	1.53	.43699	.12376	.16595
.04	.01595	.39862	-.39799	.54	.20540	.34482	-.24427	1.04	.35083	.23230	.01896	1.54	.43822	.12188	.16717
.05	.01994	.39844	-.39745	.55	.20884	.34294	-.23920	1.05	.35314	.22988	.02356	1.55	.43943	.12001	.16831
.06	.02392	.39822	-.39679	.56	.21226	.34105	-.23409	1.06	.35543	.22747	.02812	1.56	.44062	.11816	.16939
.07	.02790	.39797	-.39602	.57	.21566	.33912	-.22894	1.07	.35769	.22506	.03261	1.57	.44179	.11632	.17040
.08	.03188	.39767	-.39512	.58	.21904	.33718	-.22375	1.08	.35993	.22265	.03705	1.58	.44295	.11450	.17135
.09	.03586	.39733	-.39411	.59	.22240	.33521	-.21853	1.09	.36214	.22025	.04143	1.59	.44408	.11270	.17222
.10	.03983	.39695	-.39298	.60	.22575	.33322	-.21326	1.10	.36433	.21785	.04575	1.60	.44520	.11092	.17304
.11	.04380	.39654	-.39174	.61	.22907	.33121	-.20797	1.11	.36650	.21546	.05001	1.61	.44630	.10915	.17379
.12	.04776	.39608	-.39038	.62	.23237	.32918	-.20265	1.12	.36864	.21307	.05420	1.62	.44738	.10741	.17447
.13	.05172	.39559	-.38890	.63	.23565	.32713	-.19729	1.13	.37076	.21069	.05834	1.63	.44845	.10567	.17509
.14	.05567	.39505	-.38731	.64	.23891	.32506	-.19192	1.14	.37286	.20831	.06241	1.64	.44950	.10396	.17565
.15	.05962	.39448	-.38560	.65	.24215	.32297	-.18652	1.15	.37493	.20594	.06641	1.65	.45053	.10226	.17615
.16	.06356	.39387	-.38379	.66	.24537	.32086	-.18110	1.16	.37698	.20357	.07035	1.66	.45154	.10059	.17659
.17	.06749	.39322	-.38186	.67	.24857	.31874	-.17566	1.17	.37900	.20121	.07423	1.67	.45254	.09893	.17697
.18	.07142	.39253	-.37981	.68	.25175	.31659	-.17020	1.18	.38100	.19886	.07803	1.68	.45352	.09728	.17729
.19	.07535	.39181	-.37766	.69	.25490	.31443	-.16473	1.19	.38298	.19652	.08177	1.69	.45449	.09566	.17755
.20	.07926	.39104	-.37540	.70	.25804	.31225	-.15925	1.20	.38493	.19419	.08544	1.70	.45543	.09405	.17775
.21	.08317	.39024	-.37303	.71	.26115	.31006	-.15376	1.21	.38686	.19186	.08904	1.71	.45637	.09246	.17790
.22	.08706	.38940	-.37056	.72	.26424	.30785	-.14826	1.22	.38877	.18954	.09257	1.72	.45728	.09089	.17799
.23	.09095	.38853	-.36798	.73	.26730	.30563	-.14276	1.23	.39065	.18724	.09603	1.73	.45818	.08933	.17803
.24	.09483	.38762	-.36529	.74	.27035	.30339	-.13725	1.24	.39251	.18494	.09942	1.74	.45907	.08780	.17802
.25	.09871	.38667	-.36250	.75	.27337	.30114	-.13175	1.25	.39435	.18265	.10274	1.75	.45994	.08628	.17795
.26	.10257	.38568	-.35961	.76	.27637	.29887	-.12624	1.26	.39617	.18037	.10599	1.76	.46080	.08478	.17783
.27	.10642	.38466	-.35662	.77	.27935	.29659	-.12074	1.27	.39796	.17810	.10916	1.77	.46164	.08329	.17766
.28	.11026	.38361	-.35353	.78	.28230	.29431	-.11525	1.28	.39973	.17585	.11226	1.78	.46246	.08183	.17744
.29	.11409	.38251	-.35035	.79	.28524	.29200	-.10976	1.29	.40147	.17360	.11529	1.79	.46327	.08038	.17717
.30	.11791	.38139	-.34706	.80	.28814	.28969	-.10429	1.30	.40320	.17137	.11824	1.80	.46407	.07895	.17685
.31	.12172	.38023	-.34369	.81	.29103	.28737	-.09883	1.31	.40490	.16915	.12113	1.81	.46485	.07754	.17648
.32	.12552	.37903	-.34022	.82	.29389	.28504	-.09338	1.32	.40658	.16694	.12393	1.82	.46562	.07614	.17607
.33	.12930	.37780	-.33666	.83	.29673	.28269	-.08795	1.33	.40824	.16474	.12667	1.83	.46638	.07477	.17562
.34	.13307	.37654	-.33301	.84	.29955	.28034	-.08253	1.34	.40988	.16256	.12933	1.84	.46712	.07341	.17512
.35	.13683	.37524	-.32927	.85	.30234	.27798	-.07714	1.35	.41149	.16038	.13192	1.85	.46784	.07206	.17458
.36	.14058	.37391	-.32545	.86	.30511	.27562	-.07177	1.36	.41309	.15822	.13443	1.86	.46856	.07074	.17399
.37	.14431	.37255	-.32155	.87	.30785	.27324	-.06643	1.37	.41466	.15608	.13687	1.87	.46926	.06943	.17337
.38	.14803	.37115	-.31756	.88	.31057	.27086	-.06111	1.38	.41621	.15395	.13923	1.88	.46995	.06814	.17270
.39	.15173	.36973	-.31349	.89	.31327	.26848	-.05582	1.39	.41774	.15183	.14152	1.89	.47062	.06687	.17200
.40	.15542	.36827	-.30935	.90	.31594	.26609	-.05056	1.40	.41924	.14973	.14374	1.90	.47128	.06562	.17126
.41	.15910	.36678	-.30586	.91	.31859	.26369	-.04533	1.41	.42073	.14764	.14588	1.91	.47193	.06438	.17048
.42	.16276	.36526	-.30083	.92	.32121	.26129	-.04013	1.42	.42220	.14556	.14795	1.92	.47257	.06316	.16966
.43	.16640	.36371	-.29646	.93	.32381	.25888	-.03497	1.43	.42364	.14350	.14995	1.93	.47320	.06195	.16881
.44	.17003	.36213	-.29203	.94	.32639	.25647	-.02985	1.44	.42507	.14146	.15187	1.94	.47381	.06077	.16793
.45	.17364	.36053	-.28752	.95	.32894	.25406	-.02477	1.45	.42647	.13943	.15372	1.95	.47441	.05959	.16701
.46	.17724	.35889	-.28295	.96	.33147	.25164	-.01973	1.46	.42786	.13742	.15550	1.96	.47500	.05844	.16607
.47	.18082	.35723	-.27831	.97	.33398	.24923	-.01473	1.47	.42922	.13542	.15721	1.97	.47558	.05730	.16509
.48	.18439	.35553	-.27362	.98	.33646	.24681	-.00977	1.48	.43056	.13344	.15884	1.98	.47615	.05618	.16408
.49	.18793	.35381	-.26886	.99	.33891	.24439	-.00486	1.49	.43189	.13147	.16040	1.99	.47670	.05508	.16304

t	$\int_0^t \phi(t)dt$	$\phi(t)$	$\phi^{(2)}(t)$	t	$\int_0^t \phi(t)dt$	$\phi(t)$	$\phi^{(2)}(t)$	t	$\int_0^t \phi(t)dt$	$\phi(t)$	$\phi^{(2)}(t)$	t	$\int_0^t \phi(t)dt$	$\phi(t)$	$\phi^{(2)}(t)$
2.00	.47725	.05399	.16197	2.50	.49379	.01753	.09202	3.00	.49865	.00443	.03545	3.50	.49977	.00087	.00982
2.01	.47778	.05292	.16088	2.51	.49396	.01709	.09060	3.01	.49869	.00430	.03466	3.51	.49978	.00084	.00954
2.02	.47831	.05186	.15976	2.52	.49413	.01667	.08919	3.02	.49874	.00417	.03389	3.52	.49978	.00081	.00927
2.03	.47882	.05082	.15862	2.53	.49430	.01625	.08779	3.03	.49878	.00405	.03312	3.53	.49979	.00079	.00900
2.04	.47932	.04980	.15745	2.54	.49446	.01585	.08639	3.04	.49882	.00393	.03237	3.54	.49980	.00076	.00874
2.05	.47982	.04879	.15626	2.55	.49461	.01545	.08501	3.05	.49886	.00381	.03163	3.55	.49981	.00073	.00849
2.06	.48030	.04780	.15504	2.56	.49477	.01506	.08364	3.06	.49889	.00370	.03090	3.56	.49981	.00071	.00824
2.07	.48077	.04682	.15381	2.57	.49492	.01468	.08227	3.07	.49893	.00358	.03019	3.57	.49982	.00068	.00800
2.08	.48124	.04586	.15255	2.58	.49506	.01431	.08092	3.08	.49897	.00348	.02949	3.58	.49983	.00066	.00777
2.09	.48169	.04491	.15128	2.59	.49520	.01394	.07957	3.09	.49900	.00337	.02880	3.59	.49983	.00063	.00754
2.10	.48214	.04398	.14998	2.60	.49534	.01358	.07824	3.10	.49903	.00327	.02813	3.60	.49984	.00061	.00732
2.11	.48257	.04307	.14867	2.61	.49547	.01323	.07692	3.11	.49906	.00317	.02746	3.61	.49985	.00059	.00710
2.12	.48300	.04217	.14735	2.62	.49560	.01289	.07560	3.12	.49910	.00307	.02681	3.62	.49985	.00057	.00689
2.13	.48341	.04128	.14600	2.63	.49573	.01256	.07431	3.13	.49913	.00298	.02617	3.63	.49986	.00055	.00669
2.14	.48382	.04041	.14464	2.64	.49585	.01223	.07302	3.14	.49916	.00288	.02555	3.64	.49986	.00053	.00649
2.15	.48422	.03955	.14327	2.65	.49598	.01191	.07174	3.15	.49918	.00279	.02493	3.65	.49987	.00051	.00629
2.16	.48461	.03871	.14188	2.66	.49609	.01160	.07048	3.16	.49921	.00271	.02433	3.66	.49987	.00049	.00610
2.17	.48500	.03788	.14049	2.67	.49621	.01130	.06923	3.17	.49924	.00262	.02374	3.67	.49988	.00047	.00592
2.18	.48537	.03706	.13907	2.68	.49632	.01100	.06799	3.18	.49926	.00254	.02316	3.68	.49988	.00046	.00574
2.19	.48574	.03626	.13765	2.69	.49643	.01071	.06676	3.19	.49929	.00246	.02259	3.69	.49989	.00044	.00556
2.20	.48610	.03547	.13622	2.70	.49653	.01042	.06555	3.20	.49931	.00238	.02203	3.70	.49989	.00042	.00539
2.21	.48645	.03470	.13478	2.71	.49664	.01014	.06435	3.21	.49934	.00231	.02148	3.71	.49990	.00041	.00522
2.22	.48679	.03394	.13333	2.72	.49674	.00987	.06316	3.22	.49936	.00224	.02095	3.72	.49990	.00039	.00506
2.23	.48713	.03319	.13188	2.73	.49683	.00961	.06199	3.23	.49938	.00216	.02042	3.73	.49990	.00038	.00491
2.24	.48745	.03246	.13041	2.74	.49693	.00935	.06082	3.24	.49940	.00210	.01991	3.74	.49991	.00037	.00475
2.25	.48778	.03174	.12894	2.75	.49702	.00909	.05968	3.25	.49942	.00203	.01940	3.75	.49991	.00035	.00461
2.26	.48809	.03103	.12747	2.76	.49711	.00885	.05854	3.26	.49944	.00196	.01891	3.76	.49992	.00034	.00446
2.27	.48840	.03034	.12599	2.77	.49720	.00861	.05742	3.27	.49946	.00190	.01843	3.77	.49992	.00033	.00432
2.28	.48870	.02965	.12450	2.78	.49728	.00837	.05631	3.28	.49948	.00184	.01795	3.78	.49992	.00031	.00419
2.29	.48899	.02898	.12301	2.79	.49736	.00814	.05522	3.29	.49950	.00178	.01749	3.79	.49992	.00030	.00405
2.30	.48928	.02833	.12152	2.80	.49744	.00792	.05414	3.30	.49952	.00172	.01704	3.80	.49993	.00029	.00392
2.31	.48956	.02768	.12003	2.81	.49752	.00770	.05308	3.31	.49953	.00167	.01659	3.81	.49993	.00028	.00380
2.32	.48983	.02705	.11854	2.82	.49760	.00748	.05202	3.32	.49955	.00161	.01616	3.82	.49993	.00027	.00368
2.33	.49010	.02643	.11704	2.83	.49767	.00727	.05099	3.33	.49957	.00156	.01573	3.83	.49994	.00026	.00356
2.34	.49036	.02582	.11554	2.84	.49774	.00707	.04996	3.34	.49958	.00151	.01532	3.84	.49994	.00025	.00344
2.35	.49061	.02522	.11405	2.85	.49781	.00687	.04895	3.35	.49960	.00146	.01491	3.85	.49994	.00024	.00333
2.36	.49086	.02463	.11256	2.86	.49788	.00668	.04795	3.36	.49961	.00141	.01451	3.86	.49994	.00023	.00322
2.37	.49111	.02406	.11106	2.87	.49795	.00649	.04697	3.37	.49962	.00136	.01413	3.87	.49995	.00022	.00312
2.38	.49134	.02349	.10957	2.88	.49801	.00631	.04600	3.38	.49964	.00132	.01375	3.88	.49995	.00021	.00302
2.39	.49158	.02294	.10808	2.89	.49807	.00613	.04505	3.39	.49965	.00127	.01338	3.89	.49995	.00021	.00292
2.40	.49180	.02239	.10660	2.90	.49813	.00595	.04411	3.40	.49966	.00123	.01301	3.90	.49995	.00020	.00282
2.41	.49202	.02186	.10512	2.91	.49819	.00578	.04318	3.41	.49968	.00119	.01266	3.91	.49995	.00019	.00273
2.42	.49224	.02134	.10364	2.92	.49825	.00562	.04227	3.42	.49969	.00115	.01231	3.92	.49996	.00018	.00264
2.43	.49245	.02083	.10217	2.93	.49831	.00545	.04137	3.43	.49970	.00111	.01197	3.93	.49996	.00018	.00255
2.44	.49266	.02033	.10070	2.94	.49836	.00530	.04048	3.44	.49971	.00107	.01164	3.94	.49996	.00017	.00247
2.45	.49286	.01984	.09924	2.95	.49841	.00514	.03961	3.45	.49972	.00104	.01132	3.95	.49996	.00016	.00238
2.46	.49305	.01936	.09778	2.96	.49846	.00499	.03875	3.46	.49973	.00100	.01100	3.96	.49996	.00016	.00230
2.47	.49324	.01889	.09633	2.97	.49851	.00485	.03791	3.47	.49974	.00097	.01070	3.97	.49996	.00015	.00223
2.48	.49343	.01842	.09489	2.98	.49856	.00471	.03708	3.48	.49975	.00094	.01040	3.98	.49997	.00014	.00215
2.49	.49361	.01797	.09345	2.99	.49861	.00457	.03626	3.49	.49976	.00090	.01010	3.99	.49997	.00014	.00208
												4.00	.49997	.00013	.00201

Table C7, Cumulative Poisson Probability, $P(c,a)$.

c	a	.001	.002	.003	.004	.005	.006	.007	.008	.009
1		.00100	.00200	.00300	.00399	.00499	.00598	.00698	.00797	.00896
	a	.01	.02	.03	.04	.05	.06	.07	.08	.09
1		.00995	.01980	.02955	.03921	.04877	.05824	.06761	.07688	.08607
2		.00005	.00020	.00044	.00078	.00121	.00173	.00234	.00303	.00381
	a	.1	.2	.3	.4	.5	.6	.7	.8	.9
1		.09516	.18127	.25918	.32968	.39347	.45119	.50341	.55067	.59343
2		.00468	.01752	.03694	.06155	.09020	.12190	.15581	.19121	.22752
3		.00015	.00115	.00360	.00793	.01439	.02311	.03414	.04742	.06286
4		.00000	.00006	.00027	.00078	.00175	.00336	.00575	.00908	.01346
5			.00000	.00002	.00006	.00017	.00039	.00079	.00141	.00234
	a	1	2	3	4	5	6	7	8	9
1		.63212	.86467	.95021	.98168	.99326	.99752	.99909	.99967	.99988
2		.26424	.59399	.80085	.90842	.95957	.98265	.99271	.99698	.99877
3		.08030	.32332	.57681	.76190	.87535	.93803	.97036	.98625	.99377
4		.01899	.14288	.35277	.56653	.73497	.84880	.91824	.95762	.97877
5		.00366	.05265	.18474	.37116	.55951	.71494	.82701	.90037	.94504
6		.00059	.01656	.08392	.21487	.38404	.55432	.69929	.80876	.88431
7		.00008	.00453	.03351	.11067	.23782	.39370	.55029	.68663	.79322
8		.00001	.00110	.01191	.05113	.13337	.25602	.40129	.54704	.67610
9		.00000	.00024	.00380	.02136	.06809	.15276	.27091	.40745	.54435
10			.00005	.00110	.00813	.03183	.08392	.16950	.28338	.41259
11			.00001	.00029	.00284	.01370	.04262	.09852	.18411	.29401
12				.00007	.00092	.00545	.02009	.05335	.11192	.19699
13				.00002	.00027	.00202	.00883	.02700	.06380	.12423
14				.00000	.00008	.00070	.00363	.01281	.03418	.07385
15					.00002	.00023	.00140	.00572	.01726	.04147
16					.00001	.00007	.00051	.00241	.00823	.02204
17					.00000	.00002	.00018	.00096	.00372	.01111
18						.00001	.00006	.00036	.00159	.00532
19						.00000	.00002	.00013	.00065	.00243
20							.00001	.00004	.00025	.00106
21							.00000	.00001	.00009	.00044
22								.00001	.00003	.00018

c/a	10	20	30	40	50	60	70	80	90	100
.1	.99995									
.2	.99950	1.000								
.3	.99723	.99993	1.000	1.000						
.4	.98966	.99922	.99994	.99999	1.000	1.000				
.5	.97075	.99501	.99908	.99982	.99997	.99999	1.000	1.000	1.000	1.000
.6	.93291	.97861	.99273	.99745	.99908	.99967	.99988	.99996	.99998	.99999
.7	.86986	.93387	.96471	.98066	.98922	.99392	.99654	.99802	.99886	.99934
.8	.77978	.84349	.88535	.91448	.93543	.95082	.96230	.97095	.97752	.98255
.9	.66718	.70297	.73266	.75759	.77896	.79759	.81403	.82867	.84181	.85365
1.0	.54207	.52974	.52428	.52103	.51881	.51717	.51589	.51487	.51402	.51330
1.1	.41696	.35630	.31546	.28378	.25769	.23551	.21623	.19925	.18413	.17056
1.2	.30322	.21251	.15738	.11958	.09227	.07193	.05650	.04464	.03543	.02823
1.3	.20844	.11219	.06484	.03874	.02360	.01457	.00908	.00570	.00360	.00228
1.4	.13554	.05248	.02211	.00968	.00433	.00197	.00091	.00042	.00020	.00009
1.5	.08346	.02182	.00627	.00188	.00058	.00018				
1.6	.04874	.00809	.00149	.00029						
1.7	.02704	.00269	.00030							
1.8	.01428	.00080								
1.9	.00719									
2.0	.00345									
2.1	.00159									
2.2	.00070									

Footnote: For values of $P(c,a)$ from the above cumulative Poisson table, one can linearly interpolate, by using Normal table C6 with the fitting t , within .001 for $a < 40$ and within .002 for $a \leq 100$.

APPENDIX D

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APPENDIX E

SUMMARY OF RECOMMENDED PROCEDURES FOR OBTAINING VALUES OF THE CUMULATIVE BINOMIAL PROBABILITY B WITHIN 3-DECIMAL ACCURACY UNIVERSALLY.

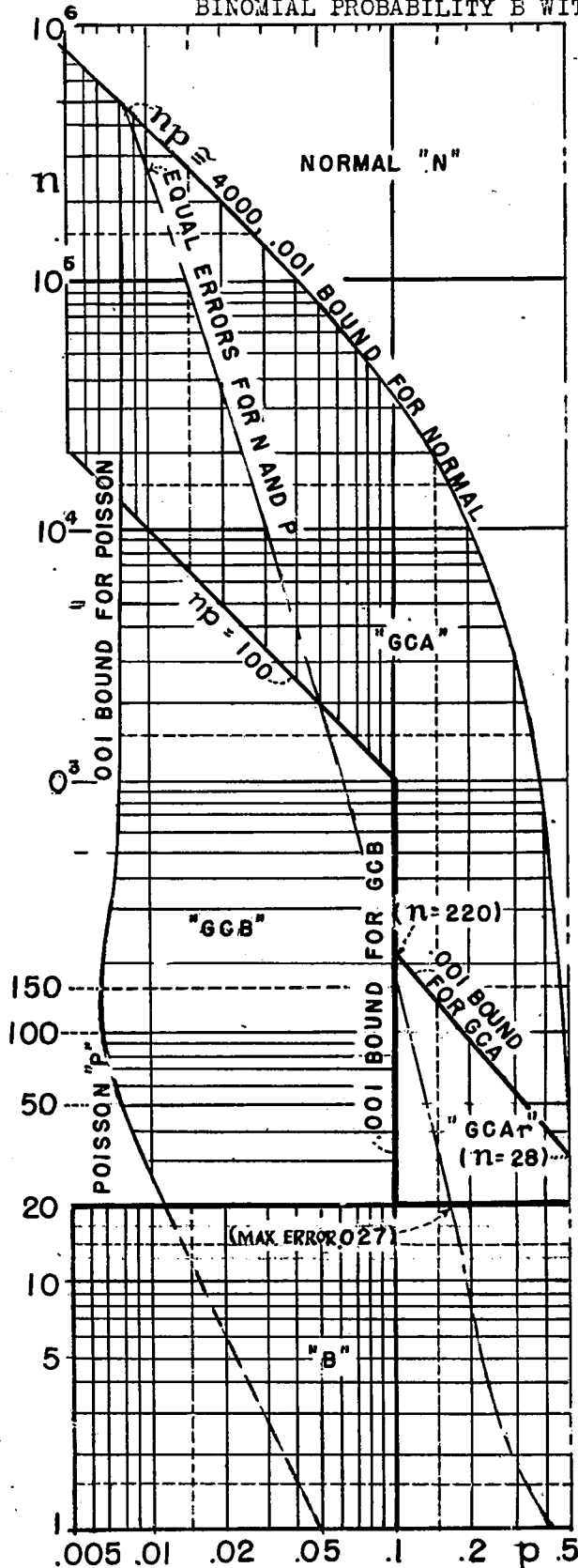


FIG. E1.

$B = \sum_{x=c}^n \frac{n!}{x!(n-x)!} p^x q^{n-x}$ is the chance of obtaining at least c successes in n trials for probability p of success in a single trial, where $q=1-p$ and B is the sum of chances of obtaining $c, c+1, \dots, n$ successes in n trials.

From percentage point values, see if the value of c is such that $.001 < B < .999$. If so, proceed as follows:

In region "B" and adjoining regions for which a table of B is available, use the table as far as it goes. For other significant regions, use approximations to B .

In region "N", use Normal probability table to obtain the Normal approximation

$$N(t_c) = .5 - \int_0^{t_c} \phi(t) dt \text{ where}$$

$$t_c = (c - a - .5) / \sigma, \quad a = np, \quad \sigma = \sqrt{npq} \text{ and}$$

$$\phi(t) = \frac{1}{\sqrt{2\pi}} e^{-t^2/2}$$

In region "GCA", use Normal tables to obtain the 2-term Gram-Charlier series, type A, approximation:

$$N_A(t_c) = N(t_c) - \frac{p-q}{6\sigma} \phi^{(2)}(t_c) \text{ where the}$$

$$\text{2nd derivative } \phi^{(2)}(t_c) = (t_c^2 - 1)\phi(t_c).$$

In region "GCAr", use the remainder modification of the preceding equation for $c > 1$:

$$N_{Ar} = N(t_c) + \alpha \phi^{(2)}(t_c) + r(t_c)/np$$

$$\text{where } \alpha \approx .351 \frac{(.5-p)^{.87}}{(np)^{.53}} \text{ and } r(t_c) \text{ can}$$

be obtained from a graph (Fig. 9). Use $B(0, n, p) = 1$ and $B(1, n, p) = 1 - q^n$ for $2 < a < 2.5$.

In region "P", use a table of cumulative Poisson probabilities

$$P \text{ or } P(c, a) = \sum_{x=c}^{\infty} \frac{a^x e^{-a}}{x!}$$

In region "GCB", use table of P with the 2-term Gram-Charlier series, type B:

$$P_B(c, a) = P(c, a)$$

$$- \frac{np^2}{2} [P(c, a) - 2P(c-1, a) + P(c-2, a)]$$

where $P(0, a) = P(-1, a) = P(-2, a) = 1$.

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