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COMPUTERIZED EXPANSIONS IN ELLIPTIC MOTION

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SUMMARY

The functions of the Keplerian elliptic motion are expanded with respect to mean anomaly and the eccentricity by applying Poincaré's method of continuation in a direct manner to the equations of motion. Two algorithms are proposed; both lead to programs by which the classical expansions are constructed symbolically and automatically on a computer in double precision arithmetic. Even to as high a degree as 30 in the eccentricity, the procedures are remarkably swift. Special care is taken here in describing significant error controls at each step of the recurrences.

Astronomers engaged in Celestial Mechanics devote a substantial portion of their time to the routine manipulation of literal expansions. It has long been recognized that digital computers are capable in principle of easing this burden [3], [4].

As a possible start toward the practical implementation of that capability, we present here a set of algorithms which have been written into programs for an IBM 7094. The package deals with the developments for the elliptic motion in the classical Problem of Two Bodies.

The mathematical entities to be processed are (truncated) Fourier series of the mean anomaly ℓ, their coefficients being (truncated) power series of the eccentricity e:

$$(1) \quad F(\ell;e) = \sum_{j \geq 0} \left[\left(\sum_{k \geq 0} C_{k,j} e^k \right) \cos j\ell + \left(\sum_{k \geq 0} S_{k,j} e^k \right) \sin j\ell \right].$$

The coefficients $C_{k,j}$ and $S_{k,j}$ are rational numbers, and it would be in keeping with the ideal spirit of symbolic computing to store and manipulate them as such in the machine. Recently [6], this formidable task has been completed with success. Nonetheless the newly extended Cayley's tables are not likely to abate the discussion whether or not, in the present state of computing machinery, we should compromise and represent the coefficients $C_{k,j}$ and $S_{k,j}$ as floating point variables.

The solutions of the Problem of Two Bodies are known in a closed form; thus their analytical expansions in d'Alembert series of the mean anomaly and the eccentricity serve little purpose within that problem itself. But these developments are the indispensable ingredients of elaborate Perturbation Theories. Now such theories operate spontaneously in the field of real numbers and, although it is often of the utmost importance to decide how close the multiple argument of a trigonometric term is to a commensurability ratio, we cannot think of an instance where it was essential to identify the coefficient itself of such a trigonometric term as being a rational number. Thus we find some ground for submitting a decisive question: should we hold for more than a mathematical curiosity the fact that the coefficients in the Fourier expansions of a Keplerian motion are genuinely rational numbers?

Should we decide at the beginning of a Perturbation Theory to process the special functions of the Keplerian motion in their genuine arithmetic, which is that of rational numbers, then we impose on ourselves the formidable task of operating throughout with integer variables of multiple length, to track down at every step the nonrational quantities which are bound to occur and to replace them by literal symbols. These will have to be treated by the program as symbolic variables; in this way the result will involve no round off error, and the dependence on these symbols will be explicitly displayed. But thereby the number of auxiliary variables as locumtenens for irrational numbers will have proliferated at a fast pace. Keeping this growth under control should require introducing the basic concepts of Galois Theory of Fields, especially the fundamental distinction between algebraic and transcendental extensions of a field. Every time an irrational symbol is introduced, it ought to be decided whether it is transcendental or algebraic over the field of rational numbers, and in the latter case, whether or not it belongs to the algebraic extension generated by the algebraic numbers previously introduced.

Away from these artificial complexities, our aim is to generate in a straightforward manner the coefficients in double precision arithmetic for the expansions of the elliptic motion.

At the suggestion of Moulton [7], [8], development for the cosine of the eccentric anomaly E is derived immediately from the differential equation for the radial motion. The construction consists mainly in applying Poincaré's method of continuation: a transformation of coordinates exchanges the study near the circular solution for a study near an equilibrium position, so that the family of elliptic motions having a preassigned apsidal line and a given mean motion turns out to be the family of periodic orbits issued from the equilibrium point representing the circular trajectory with the same mean motion. As we adopt the eccentricity as the analytical coordinate along the family emanating from the equilibrium, the method of analytical continuation generates the expansion of $\cos E$ as a Fourier series of the mean anomaly whose coefficients are power series of the eccentricity. The outstanding merit of the algorithm is that it generates at the same time the powers $\cos^p E$ as series of the same type. Consequently only elementary manipulations of simple Poisson series are necessary to generate therefrom the classical expansions in the Problem of Two Bodies.

As an alternative, we examined another algorithm by which we expand in a direct manner the Cartesian coordinates $x = r \cos f$ and $y = r \sin f$

in terms of the mean anomaly and the eccentricity. A time dependent transformation of coordinates represents the circular motion as an equilibrium position in a conservative dynamical system with two degrees of freedom. Although two of the characteristic exponents at the equilibrium are zero, Poincaré's method of continuation establishes that the equilibrium generates a one-parameter family of periodic solutions which actually represents the elliptic motions having a preassigned line of apsides and the same mean motion as the generating circular orbit. Together with the coordinates x and y, the analytical continuation produces all monomials $x^p y^q$ up to a given degree $n = p + q$.

Both direct algorithms proposed here call upon a package of subroutines [2] to manipulate series of the type (1) Each series is denoted by a FORTRAN variable. It points to a header word in core which contains the address of the FORTRAN variable and the number of terms in the series. The miscellaneous series are stored dynamically in sequential order in a preassigned area of the core or on a disk. As it was first proposed by Herget and Musen [5], a series is considered as a mapping F of the product set $\mathbf{N} \times \mathbf{N} \times \mathbf{Z}/2$ into the set of real numbers ($\mathbf{N}$ denotes the set of natural integers, $\mathbf{Z}$ that of rational integers, and $\mathbf{Z}/2$ the set of congruences modulo 2 in $\mathbf{Z}$, thus a set reduced to the two elements 0 and 1. Each element (j,k,I) of $\mathbf{N} \times \mathbf{N} \times \mathbf{Z}/2$ are packed in one word; the fact that $F(j,k,I)$ is the image of the element (j,k,I) by the mapping F is represented physically in the core of the machine by placing the double word containing the real number $F(j,k,I)$ next to the label word containing the indices (j,k,I). The coefficients $C_{j,k}$ and $S_{j,k}$ of a series (1) are stored below its header word dynamically on the principle: first come, first served. $I = 0$ refers to a cosine term, $I = 1$ to a sine term; the multiple j of the argument ℓ is restricted to be positive for a cosine term and strictly positive for a sine term (by changing the sign of the coefficient $S_{k,j}$, when necessary).

To automatize completely Poincaré's method of continuation, we found it very convenient to split up series of type (1) into the sums

$$F(\ell;e) = \sum_{k\geq 0} F_k(\ell)e^k$$

so that the terms $F_k(\ell)$ appear as genuinely finite (and not truncated) Fourier sums of the mean anomaly ℓ. Poincaré's continuation is then accomplished not by successive iterations but by recurrence: once the coefficients of all powers of e have been determined up to order N, that of e^{N+1} is computed *once and for all*. At this stage, the key of the method is the symbolic inverse of a differential operator to be applied to a trigonometric sum. But, with respect to the natural laws of addition, multiplication and scalar multiplication, the series of the type (1) are endowed with a structure of commutative graded algebra with a unity over the field of real numbers; moreover, the differential operators which come up at various stages of the algorithms are endomorphisms if not of the whole algebra, at least of some of its most remarkable subalgebras. This is precisely the reason why Poincaré's method of continuation is so easily implemented automatically on a computer.

1. THE DIRECT EXPANSION OF cos E

In the Lagrangian function

$$(2)\qquad \mathcal{L} = \tfrac{1}{2}(\dot{x}^2 + \dot{y}^2) + \frac{\mu}{r}, \qquad r = |x^2 + y^2|^{\frac{1}{2}},$$

which describes the Problem of Two Bodies in reference to a fixed heliocentric coordinate system, we introduce new independent and dependent variables by means of the definitions:

(3_1) $x = aX,$ (3_2) $y = aY,$

(4) $n^2a^3 = \mu,$ (5) $\ell = nt;$

we also define a normalized radial distance R such that

(6_1) $r = aR,$ (6_2) $R = |X^2 + Y^2|^{\frac{1}{2}},$

and we define by D the differentiation with respect to ℓ. Thus on dividing by a^2/n^2, the transformed Lagrangian function is

$$(7)\qquad \mathcal{L} = \tfrac{1}{2}[(DX)^2 + (DY)^2] + \frac{1}{R}.$$

We change from the Cartesian coordinates (ξ,η) to the polar coordinates (R,θ) such that

$$(8)\qquad X = R\cos\theta, \qquad Y = R\sin\theta.$$

Since θ is ignorable in the Lagrangian, we substitute to $\mathcal{L}$ the Routhian function [10]:

$$\mathcal{R} = \tfrac{1}{2}(DR)^2 - \frac{1}{2}\frac{C^2}{R^2} + \frac{\mu}{R}, \tag{9}$$

where C is the constant of integration introduced by the integral of angular momentum

$$R^2 \cdot D\theta = C. \tag{10}$$

Assume that the conditions for an elliptical orbit are satisfied. Then a is the major semi-axis of the orbit, n the mean motion; the constant C is related to the eccentricity e by the relation

$$C = \sqrt{1-e^2}, \tag{11}$$

and, if we put the origin of time at an instant of passage at perigee, ℓ is the mean anomaly. Denoting by E the eccentric anomaly, we put

$$\rho = \cos E, \tag{12}$$

so that we can express the radius vector as the function

$$R = 1 - \rho e. \tag{13}$$

We substitute ρ for R in the Routhian function (9), and we expand $1/(1-\rho e)$ and $1/(1-\rho e)^2$ as power series of e. On dividing by e^2, we obtain $\mathcal{R}$ in the form of the power series

(14) $$\mathcal{R} = \tfrac{1}{2}(D\rho)^2 - \tfrac{1}{2}\rho^2 + \frac{1}{2}\sum_{n\geq 1} (n+1)(1-\rho^2)\rho^n e^n,$$

where we have omitted the constant term $(1+e^2)/2$. Thus the Lagrangian equation becomes

(15) $$(D^2+1)\rho = \frac{1}{2}\sum_{n\geq 1} (n+1)[n-(n+2)\rho^2]\rho^{n-1}e^n.$$

Obviously, when $e = 0$, the equilibrium solution $\rho \equiv 0$ represents the Keplerian circular motion with radius a and mean motion n. Moreover the variational equation around that equilibrium

(16) $$(D^2+1)\rho = 0$$

admits as a solution

(17) $$\rho_0 = \cos \ell$$

for the initial conditions

(18) $$\rho_0(0) = 1, \qquad D\rho_0(0) = 0.$$

Therefore, according to Poincaré's method of continuation, such a solution can be extended analytically into the power series

(19) $$\rho(\ell;e) = \sum_{n\geq 0} \rho_n(\ell)e^n$$

whose coefficients $\rho_n(\ell)$ for $n \geq 1$ have the following properties:

(a) $\rho_n(\ell + 2\pi) = \rho_n(\ell)$ ("condition of periodicity");

(b) $\rho_n(0) = 0$, $D\rho_n(0) = 0$ ("initial conditions").

On substituting (19) in the original differential equation, we obtain from equating like powers of e in both members a recurrent system of differential equations which determine the coefficients of (19). Before we write down this system, let us introduce a convenient notation. We put

(20) $$\rho^n(\ell;e) = \sum_{p\geq 0} \rho_p^{(n)}(\ell)e^p \qquad (n \geq 0);$$

thus $\rho_p^{(n)}$ represents the Fourier series which is the coefficient of e^p in the expansion of $(\cos E)^n$. With this convention, we can write the coefficient of e^n in the right-hand member of (15) as the function

(21) $$U_n(\ell) = \frac{1}{2} \sum_{1\leq k\leq n} (k + 1)\left[k\rho_{n-k}^{(k-1)} - (k + 2)\rho_{n-k}^{(k+1)}\right] \qquad (n \geq 1),$$

and we find that $\rho_n \equiv \rho_n^{(1)}$ is a solution of the differential equation

(22) $$(D^2 + 1)\rho_n = U_n.$$

The following facts can be established by recurrence on the index n from examining the right-hand members U_n and taking into account the initial conditions:

(a) For any $n \geq 0$, ρ_n is a *finite* trigonometric sum; more precisely, it does not contain multiples $j\ell$ of the mean anomaly beyond $j = n + 1$.

(b) For any $n \geq 0$, ρ_n is an *even* function of ℓ; thus it contains only cosine terms.

(c) For any *even* $n \geq 0$, $\rho_n(\ell + \pi) = -\rho_n(\ell)$, whereas for any *odd* $n \geq 1$, $\rho_n(\ell + \pi) = \rho_n(\ell)$.

Property (a) guarantees that the coefficients ρ_n can be computed from the recurrence in a *closed* form. In other words, they do not result from successive approximations, but once their respective differential equation (22) has been solved, they are known as accurately as it can be, save for errors resulting from cumulative rounding off in the arithmetic manipulations of the coefficients.

Property (c) implies that, at an even order $N = 2m$, the right-hand member U_{2m} contains only cosine terms of odd multiples of the mean anomaly. Thus it looks as if it would contain among others a term in $\cos \ell$. As this would prevent any solution of (22) from being periodic, we have to interpret Poincaré's theorem as meaning that, at every even order, the various contributions to U_{2m} arrange themselves so as to cancel *exactly* the term in $\cos \ell$. In the course of the recurrence, round off errors will prevent an exact cancellation. Thus the coefficient of $\cos \ell$ in U_{2m} will provide a control on the errors at every even order of the recurrence.

We list in Table I the size of the coefficient of $\cos \ell$ as it appeared in the right-hand members of U_{2n} before these spurious terms were rejected by the program.

Table I

Error control at even orders in the expansion of cos E

Order	Error	Order	Error
2	0	16	3×10^{-15}
4	0	18	-8×10^{-14}
6	0	20	1×10^{-13}
8	0	22	-4×10^{-13}
10	0	24	-2×10^{-12}
12	0	26	-4×10^{-12}
14	-2×10^{-14}	28	-1×10^{-11}

At the odd orders, the recurrence brings two controls on the error.

1°) Let us recall that the average value of r/a over the mean anomaly is equal to $1 + e^2/2$. Consequently, the coefficients ρ_{2n+1} $(n \geq 1)$ should be exempt from constant terms, which implies that the right-hand members U_{2n+1} themselves should be exempt from such terms. In point of fact, in the course of building U_{2n+1}, they cancel out exactly when the various ρ_n and their powers are produced in integer arithmetic. But, in our algorithm, as we proceed in double precision arithmetic, they might still appear, but as very small numbers. Hence they serve as tests on the adverse effects of round off errors.

2°) In order to integrate the differential equations (22), we proceed in two steps. First we generate the particular solution

(23) $$\rho_n^* = \frac{1}{D^2+1} U_n \qquad (n \geq 1)$$

by application of the inverse differential operator $1/D^2+1$ according to the rules

(24_1) $$\frac{1}{D^2+1} \cos p\ell = \frac{1}{-p^2+1} \cos p\ell, \qquad (p \neq 1),$$

(24_2) $$\frac{1}{D^2+1} \sin p\ell = \frac{1}{-p^2+1} \sin p\ell, \qquad (p \neq 1).$$

As we just observed, this operation is always possible: neither at odd nor at even orders, does U_n contain critical terms in $\cos \ell$.

Then to the particular solution ρ_n^*, we should add the general solution of the homogeneous differential equation, so as to obtain

$$\rho_n = C_n \cos \ell + S_n \sin \ell + \rho_n^*,$$

where the arbitrary coefficients C_n and S_n ought to be determined by the initial conditions.

Of course, since ρ_n is a sum of cosine terms only, we find at any order that $S_n = 0$.

At the even orders, we find that

$$C_{2n} = -\rho_{2n}^*(0).$$

But at the odd orders, we know that ρ_{2n+1} and U_{2n+1}, hence ρ^*_{2n+1}, contain only cosines of even multiples of ℓ. Therefore, not only should we make $C_{2n+1} = 0$, but we should also check that the particular solution C^*_{2n+1} meets the initial condition

$$\rho^*_{2n+1}(0) = 0.$$

In Table II, we enter, for each odd order, the size of the spurious independent term which came into U_{2n+1} and the initial values $\rho^*_{2n+1}(0)$.

Table II

Error control at odd orders in the expansion of cos E

Order	$\langle U_{2n+1} \rangle$	$\rho^*_{2n+1}(0)$	Order	$\langle U_{2n+1} \rangle$	$\rho^*_{2n+1}(0)$
3	0	0	17	2×10^{-14}	-8×10^{-14}
5	0	0	19	-8×10^{-14}	2×10^{-13}
7	-4×10^{-16}	0	21	5×10^{-14}	-2×10^{-13}
9	7×10^{-16}	-2×10^{-15}	23	-4×10^{-13}	3×10^{-13}
11	-3×10^{-15}	7×10^{-15}	25	2×10^{-13}	-4×10^{-12}
13	4×10^{-15}	-2×10^{-14}	27	-6×10^{-12}	8×10^{-12}
15	-2×10^{-14}	3×10^{-14}	29	-4×10^{-14}	-4×10^{-11}

2. DERIVED EXPANSIONS

Other special functions of the Keplerian elliptic motion may be expanded by elementary algebraic operations from the developments of $\cos^p E$ $(p \geq 1)$ that we have just obtained.

Thus, from the series

$$\cos^p E \equiv \rho^p = \sum_{n\geq 0} \rho_n^{(p)} e^n \qquad (p \geq 1),$$

we find immediately the expansion of

$$\text{(25)} \qquad R = \frac{r}{a} \equiv 1 - e \cos E = 1 - \sum_{n\geq 1} \rho_{n-1}^{(1)} e^n,$$

$$\text{(26)} \qquad \frac{1}{R} = \frac{a}{r} \equiv \frac{1}{1-e \cos E} = \sum_{n\geq 0} \rho^n e^n = \sum_{n\geq 0} \left[\sum_{p\geq 0} \rho_{n-p}^{(p)} \right] e^n,$$

$$\text{(27)} \qquad X = \frac{x}{a} \equiv e - \cos E = -1 + (1 - \rho_1^{(1)})e - \sum_{n\geq 2} \rho_n^{(1)} e^n,$$

$$\text{(28)} \qquad E \equiv \int_0^{\ell} \frac{a}{r(s)}\, ds.$$

We observe that the average of a/r over the mean anomaly is equal to 1, so that $E - \ell$ turns out to be a d'Alembert series in ℓ and e whose average over the mean anomaly is zero. Therefore, by elementary manipulations of series of the type (1), we are able to expand in the mean anomaly the function

$$\text{(29)} \qquad \sin E = \frac{1}{e} (E - \ell),$$

and this development in turn yields the expansion of

(30) $$Y = \frac{y}{a} = \sqrt{1-e^2}\,\sin E.$$

The "equation of the center" is given by the quadrature

$$f = \sqrt{1-e^2}\int_0^{\ell}\frac{1}{R^2(s)}\,ds.$$

To experiment more extensively with the simple programming techniques proposed here, we undertook to expand all powers $(r/a)^p$, $2 \le p \le 15$, in series of the mean anomaly and of the eccentricity. The computation is easy, since it reduces to repeated additions and multiplications of homogeneous components. But as the exponent p increases, the round off errors get bigger and bigger. We have two ways of watching the increasingly destructive effects of the round off errors.

First we know that, for any integer $p \ge 0$, the term of either $(r/a)^{2p-1}$ or $(r/a)^{2p}$ which is independent of ℓ is a polynomial in e^2 whose degree is 2p. Now the program ignores this property, and consequently this polynomial appears in the output as a (truncated) series in e^2. We can estimate the contaminating influence of the round off errors by inspecting these spurious coefficients. They are entered in Tables III and IV.

Another check on the calculations of the successive powers $(r/a)^p$ is suggested by Brown and Shook [1]. For any rational integer p (> 0 or < 0), we find that

Table III. Error control on the independent terms of $(\frac{r}{a})^p$

Order in e	$(\frac{r}{a})^2$	$(\frac{r}{a})^3$	$(\frac{r}{a})^4$	$(\frac{r}{a})^5$	$(\frac{r}{a})^6$	$(\frac{r}{a})^7$	$(\frac{r}{a})^8$
0	--------	--------	--------	--------	--------	--------	--------
2	--------	--------	--------	--------	--------	--------	--------
4	0	--------	--------	--------	--------	--------	--------
6	0.6×10^{-16}	0.2×10^{-15}	0.3×10^{-15}	--------	--------	--------	--------
8	-0.2×10^{-15}	-0.4×10^{-15}	-0.6×10^{-15}	-0.7×10^{-15}	-0.5×10^{-15}	--------	--------
10	0.6×10^{-16}	-0.3×10^{-15}	-0.2×10^{-14}	-0.5×10^{-14}	-0.9×10^{-14}	-0.2×10^{-13}	-0.3×10^{-13}
12	-0.6×10^{-15}	-0.9×10^{-15}	0.1×10^{-15}	0.2×10^{-14}	0.3×10^{-14}	-0.4×10^{-15}	-0.2×10^{-13}
14	0.2×10^{-15}	-0.3×10^{-14}	-0.1×10^{-13}	-0.3×10^{-13}	-0.6×10^{-13}	-0.8×10^{-13}	-0.1×10^{-12}
16	-0.6×10^{-15}	0.4×10^{-14}	0.2×10^{-13}	0.4×10^{-13}	0.5×10^{-13}	0.3×10^{-13}	-0.6×10^{-13}
18	-0.2×10^{-14}	-0.2×10^{-13}	-0.8×10^{-13}	-0.2×10^{-12}	-0.3×10^{-12}	-0.4×10^{-12}	-0.4×10^{-12}
20	-0.7×10^{-14}	0.4×10^{-14}	0.4×10^{-13}	-0.8×10^{-13}	0.4×10^{-13}	-0.2×10^{-12}	-0.9×10^{-12}
22	-0.3×10^{-13}	-0.1×10^{-12}	-0.4×10^{-12}	-0.9×10^{-12}	-0.2×10^{-11}	-0.2×10^{-11}	-0.3×10^{-11}
24	-0.3×10^{-13}	-0.3×10^{-13}	-0.2×10^{-14}	-0.1×10^{-12}	-0.8×10^{-12}	-0.3×10^{-11}	-0.7×10^{-11}
26	0.9×10^{-13}	-0.2×10^{-12}	-0.1×10^{-11}	-0.3×10^{-11}	-0.7×10^{-11}	-0.1×10^{-10}	-0.2×10^{-10}
28	-0.8×10^{-12}	-0.2×10^{-11}	-0.2×10^{-11}	-0.3×10^{-11}	-0.6×10^{-11}	-0.2×10^{-10}	-0.4×10^{-10}
30	-0.9×10^{-12}	-0.8×10^{-11}	-0.3×10^{-10}	-0.6×10^{-10}	-0.1×10^{-9}	-0.2×10^{-9}	-0.3×10^{-9}

<u>Table IV.</u> <u>Error control on the independent terms of</u> $(\frac{r}{a})^p$ (continued)

Order in e	$(\frac{r}{a})^{9}$	$(\frac{r}{a})^{10}$	$(\frac{r}{a})^{11}$	$(\frac{r}{a})^{12}$	$(\frac{r}{a})^{13}$	$(\frac{r}{a})^{14}$	$(\frac{r}{a})^{15}$
0	--------	--------	--------	--------	--------	--------	--------
2	--------	--------	--------	--------	--------	--------	--------
4	--------	--------	--------	--------	--------	--------	--------
6	--------	--------	--------	--------	--------	--------	--------
8	--------	--------	--------	--------	--------	--------	--------
10	--------	--------	--------	--------	--------	--------	--------
12	-0.6×10^{-13}	-0.1×10^{-12}	--------	--------	--------	--------	--------
14	-0.1×10^{-12}	-0.2×10^{-12}	-0.3×10^{-12}	-0.6×10^{-12}	--------	--------	--------
16	-0.3×10^{-12}	-0.6×10^{-12}	-0.1×10^{-11}	-0.2×10^{-11}	-0.3×10^{-11}	-0.6×10^{-11}	--------
18	-0.5×10^{-12}	-0.6×10^{-12}	-0.1×10^{-11}	-0.2×10^{-11}	-0.5×10^{-11}	-0.1×10^{-10}	--------
20	-0.2×10^{-11}	-0.4×10^{-11}	-0.8×10^{-11}	-0.1×10^{-10}	-0.2×10^{-10}	-0.3×10^{-10}	--------
22	-0.4×10^{-11}	-0.7×10^{-11}	-0.1×10^{-10}	-0.2×10^{-10}	-0.5×10^{-10}	-0.8×10^{-10}	--------
24	-0.1×10^{-10}	-0.3×10^{-10}	-0.5×10^{-10}	-0.8×10^{-10}	-0.1×10^{-9}	-0.2×10^{-9}	--------
26	-0.3×10^{-10}	-0.6×10^{-10}	-0.1×10^{-9}	-0.2×10^{-9}	-0.3×10^{-9}	-0.6×10^{-9}	--------
28	-0.8×10^{-10}	-0.2×10^{-9}	-0.3×10^{-9}	-0.5×10^{-9}	-0.9×10^{-9}	-0.1×10^{-8}	--------
30	-0.5×10^{-9}	-0.7×10^{-9}	-0.1×10^{-8}	-0.2×10^{-8}	-0.3×10^{-8}	-0.5×10^{-8}	--------

$$DR^p = pR^{p-1} \cdot DR,$$

$$D^2R^p = p(p-1)R^{p-2} \cdot (DR)^2 + pR^{p-1} \cdot D^2R,$$

so that, in view of the energy integral

$$(DR)^2 + \frac{1-e^2}{R^2} - \frac{2}{R} = -1 \tag{31}$$

belonging to the Routhian function (9) and of the Lagrangian equation

$$D^2R = \frac{1-e^2}{R^3} - \frac{1}{R^2} \tag{32}$$

derived from the same function (9), we obtain that

$$\frac{1}{p} D^2R^p + (p-1)R^{p-2} - (2p-3)R^{p-3} + (p-2)(1-e^2)R^{p-4} = 0. \tag{33}$$

Thus, for instance, by making p equal to 2, 3, 4 successively,

$$\frac{1}{2} D^2R^2 = -1 + \frac{1}{R},$$

$$\frac{1}{3} D^2R^3 = -2R + 3 - (1-e^2)\frac{1}{R},$$

$$\frac{1}{4} D^2R^4 = -3R^2 + 5R - 2(1-e^2).$$

The left-hand members of (33) are not as good an error control as the residuals in the power series of e which is the independent term of $(r/a)^p$. Indeed they imply that we take the second derivative of a Fourier series, and that is not, as we know, the most favorable operation we should perform on a trigonometric sum. We do not reproduce here the series which we have computed from (33) for $p = 5$ to 20; their coefficients are somewhat similar to the residuals listed in Tables III and IV.

On the whole, a detailed examination of the round off errors produced in any coefficient of the powers $(r/a)^p$ and $(a/r)^p$ $(1 \leq p \leq 20)$ has convinced us that the elementary techniques advocated here guarantee more than the accuracy which is required in Perturbation Theories from the expansions of the elliptic motion.

Other techniques, of course, are possible. But, as we shall show it on another algorithm in the next section, it is of the utmost importance that the residuals should be examined carefully before a proposed technique is adopted, and its results entered as data into a major Perturbation Theory.

3. THE DIRECT EXPANSION OF $(r/a) \cos f$ AND $(r/a) \sin f$

Let us come back to the Lagrangian function (7) expressed in normalized heliocentric Cartesian coordinates (X,Y). To these variables we substitute the variables (ξ,η) defined by the non-conservative relations

$$(34)\qquad \begin{aligned} X &= \cos \ell + \xi \cos \ell - \eta \sin \ell, \\ Y &= \sin \ell + \xi \sin \ell + \eta \cos \ell, \end{aligned}$$

so that the Lagrangian function becomes

(35) $$\mathcal{L} = \tfrac{1}{2}[(D\xi)^2 + (D\eta)^2] + (\xi \cdot D\eta - \eta \cdot D\xi) + \tfrac{1}{2}R^2 + \frac{1}{R}$$

with

(36) $$R = |1 + 2\xi + \xi^2 + \eta^2|^{\frac{1}{2}}.$$

It is obvious that $\xi = \eta = 0$ is an equilibrium solution for the system; in fact, it corresponds to the circular motion. In order to study the Keplerian motions around this equilibrium, we expand R in Taylor series in the powers of ξ and η.

Such an expansion is carried on automatically on the computer by means of programs which manipulate symbolically double power series.

We look at a series

$$A(\xi,\eta) = \sum_{j\geq 0} \sum_{k\geq 0} a_{jk}\xi^j\eta^k$$

as a sum of homogeneous components

$$A = \sum_{n\geq 0} A_n(\xi,\eta),$$

where, for any $n \geq 0$,

$$A_n = \sum_{0\leq j\leq n} a_{n-j,j}\xi^{n-j}\eta^j$$

is a homogeneous polynomial of degree n in the variables ξ and η. If we take R^2 for the series A, we would have

$$A_0 = 1,$$
$$A_1 = 2\xi,$$
$$A_2 = \xi^2 + \eta^2.$$

Since $A_0 > 0$, the series A has a formal square root, which we shall denote by B. In our case, B is the Taylor expansion of the normalized distance R in the powers of ξ and η. The homogeneous components B_n of B are determined by recurrence from the fundamental relation

$$B^2 = A$$

according to the formulae:

$$B_0 = \sqrt{A_0},$$
$$B_1 = A_1/2B_0,$$
$$B_n = \left[A_n - \sum_{1 \le j \le n-1} B_j B_{n-j}\right]/2B_0 \qquad (n \ge 2).$$

Since $B_0 = 1$, the series B has a formal inverse, which we shall denote by C. In our case, C is the Taylor expansion of $1/R$ in the powers of ξ and η. The homogeneous components C_n of C are computed recursively from the definition

$$BC = 1$$

according to the formulae

$$C_0 = 1/B_0,$$
$$C_1 = -B_1 C_0/B_0,$$
$$C_n = -\left[\sum_{0 \le j \le n-1} B_{n-j} C_j\right]/B_0 \qquad (n \ge 2).$$

Through these manipulations of polynomials, we come to the development

$$\frac{1}{R} = 1 - \xi + \xi^2 - \tfrac{1}{2}\eta^2 + \sum_{n\geq 3}\left(\sum_{0\leq j\leq n} \omega_{n-j,j}\,\xi^{n-j}\eta^j\right).$$

Accordingly, around the equilibrium, the Lagrangian function $\mathcal{L}$ is decomposed into the series

$$\mathcal{L} = \sum_{n\geq 2} \mathcal{L}_n,$$

$$\mathcal{L}_2 = \tfrac{1}{2}[(D\xi)^2 + (D\eta)^2] + (\xi \cdot D\eta - \eta \cdot D\xi) + \frac{3}{2}\,\xi^2,$$

$$\mathcal{L}_n = \sum_{0\leq j\leq n} \omega_{n-j,j}\,\xi^{n-j}\eta^j \qquad (n \geq 3).$$

We address ourselves first to the Lagrangian equations derived from $\mathcal{L}_2$, namely

$$(D^2 - 3)\xi - 2D\eta = 0,$$
$$D(D\eta + 2\xi) = 0. \tag{37}$$

They constitute a homogeneous linear system with constant coefficients. Using Putzer's algorithm [9], we find its resolvent $R(\ell)$. It is a 4×4 matrix

$$R(\ell) = (r_{i,j}(\ell))_{1\leq i,j\leq 4}$$

whose coefficients are as follows

$$r_{1,1} = 4 - 3\cos\ell, \qquad r_{2,1} = -6(\ell - \sin\ell),$$

$$r_{1,2} = 0, \qquad r_{2,2} = 1,$$

$$r_{1,3} = \sin\ell, \qquad r_{2,3} = -2(1 - \cos\ell),$$

$$r_{1,4} = 2(1 - \cos\ell), \qquad r_{2,4} = -3\ell + 4\sin\ell,$$

$$r_{3,1} = 3\sin\ell, \qquad r_{4,1} = -6(1 - \cos\ell),$$

$$r_{3,2} = 0, \qquad r_{4,2} = 0$$

$$r_{3,3} = \cos\ell, \qquad r_{4,3} = -2\sin\ell,$$

$$r_{3,4} = 2\sin\ell, \qquad r_{4,4} = -3 + 4\cos\ell.$$

The general solution of (37)

$$\xi = r_{1,1}\xi_0 + r_{1,2}\eta_0 + r_{1,3}\dot{\xi}_0 + r_{1,4}\dot{\eta}_0,$$

$$\eta = r_{2,1}\xi_0 + r_{2,2}\eta_0 \quad r_{2,3}\dot{\xi}_0 + r_{2,4}\dot{\eta}_0,$$

can thus be written in the form

$$\xi = 2A - B\cos\ell + C\sin\ell,$$

(38)

$$\eta = D - 3A\ell + 2C\cos\ell + 2B\sin\ell,$$

where the arbitrary constants A, B, C, D will have to be determined by the initial conditions.

In view of (38), the first order variational equations around the equilibrium admit a periodic solution if and only if $A = 0$. Moreover, this solution $(\xi_1(\ell), \eta_1(\ell))$ satisfy the symmetry conditions

$$\xi_1(-\ell) = \xi_1(\ell), \qquad \eta_1(-\ell) = -\eta_1(\ell)$$

if and only if $C = D = 0$. Hence B remains as a parameter, and from what we know of the Keplerian motions, we can identify B to the eccentricity e.

To summarize, a first order analysis of the motion around the equilibrium yields a one parameter family of symmetric periodic orbits

$$\xi_1 = -e \cos \ell,$$

$$\eta_1 = 2e \sin \ell.$$

The problem is to continue analytically this family into a solution of the complete differential equations, which is of the form

$$(39) \qquad \xi(\ell;e) = \sum_{k \geq 1} \xi_k(\ell) e^k, \qquad \eta(\ell;e) = \sum_{k \geq 1} \eta_k(\ell) e^k$$

where the coefficients ξ_k and η_k are periodic in ℓ with period 2π.

In view of the symmetry $R(\xi,-\eta) = R(\xi,\eta)$, the coefficients $\omega_{j,k}$ in the homogeneous components $\mathcal{L}_n (n \geq 3)$ vanish for all *odd* indices k. Hence we can impose on the coefficients $\xi_k(\ell)$ and $\eta_k(\ell)$ the symmetry conditions

(40) $\xi_k(-\ell) = \xi_k(\ell), \qquad \eta_k(-\ell) = -\eta_k(\ell) \qquad (k \geq 1).$

It implies that, for any $k \geq 1$, ξ_k is a sum of cosine terms only, and η_k a sum of sine terms only.

We shall also impose the perigee condition

(41) $\xi_k(0) = 0 \qquad (k \geq 2).$

Using Poincaré's method of the small parameter, it can be established that there exists one, and only one, family of periodic solutions in the form (39) which satisfies the symmetry conditions (40) and the initial conditions (41).

In order to outline the recursive steps that we have to take in determining the coefficients ξ_j and η_j, we shall introduce the following notations:

1° For any $j \geq 1$, and $k \geq 1$, we write the monomial $\xi^j\eta^k$ as the series

$$\xi^j\eta^k = \sum_{n \geq j+k} X_n^{j,k}(\ell)e^n.$$

2° We put

$$U = \sum_{n \geq 3} \frac{\partial \mathcal{L}_n}{\partial \xi}, \qquad V = \sum_{n \geq 3} \frac{\partial \mathcal{L}_n}{\partial \eta};$$

these partial derivatives, when expressed in terms of e and ℓ, are series of the form

$$U = \sum_{n\geq 2} U_n(\ell)e^n, \qquad V = \sum_{n\geq 2} V_n(\ell)e^n,$$

wherein

$$U_n = \sum_{2\leq m\leq n} \sum_{0\leq k\leq m} (k+1)\omega_{k+1,m-k} X_n^{k,m-k}(\ell),$$

$$V_n = \sum_{2\leq m\leq n} \sum_{0\leq k\leq m} (k+1)\omega_{m-k,k+1} X_n^{m-k,k}(\ell).$$

With these notations, it is obvious that, once all coefficients $X_{n-1}^{j,k}(\ell)$ are known up to order $n-1$ for $j \geq 1$, $k \geq 1$ and $j + k \leq n - 1$, then we can generate by multiplication of finite trigonometric sums the coefficients $X_n^{j,k}(\ell)$ of order n for $j \geq 1$, $k \geq 1$ and $j + k = n$, and consequently the expressions U_n and V_n. In this way we have built the differential equations

$$(42)\qquad \begin{aligned} (D^2 - 3)\xi_n - 2D\eta_n &= U_n, \\ D(D\eta_n + 2\xi_n) &= V_n \end{aligned}$$

which define the terms of order n in the series (39).

Before we integrate this system, let us state two properties which are satisfied by the coefficients ξ_n and η_n.

(a) For any $n \geq 0$,

$$(43)\qquad \xi_{2n+1}(\ell + \pi) = -\xi_{2n+1}(\ell), \quad \eta_{2n+1}(\ell + \pi) = -\eta_{2n+1}(\ell).$$

In other words, the coefficients at odd orders contain only odd multiples of the argument ℓ.

(b) For any $n \geq 1$,

$$(44) \qquad \xi_{2n}(\ell + \pi) = \xi_{2n}(\ell), \qquad \eta_{2n}(\ell + \pi) = \eta_{2n}(\ell).$$

At even orders, the coefficients contain only even multiples of the argument ℓ.

These properties can be established by recurrence. They are essential to the success of the recurrence scheme by which the coefficients ξ_n and η_n are computed and, as we shall point out, they provide valuable checks on the accuracy of the calculation.

We determine a particular solution of the system (42) by first performing the quadrature which is the second equation and by substituting the result into the first equation. Thus, if

$$W_n = \frac{1}{D} V_n, \qquad \phi_n = U_n - 2W_n \qquad (n \geq 2),$$

we find that the system is reduced to the equation

$$(43) \qquad (D^2 + 1)\xi_n = \Phi_n.$$

At this point, we ought to remark that, at even orders, Φ_n is a sum of cosines of even multiples of ℓ, but, at odd orders, it is a sum of cosines of odd multiples of ℓ. That, in this latter case, it does not contain terms in $\cos \ell$ is a consequence of Poincaré's theorem. In point of fact, should we carry the calculation of ϕ_{2n+1} in integer arithmetic,-- and we did it by hand up to order 5 --, we would find that this critical term vanishes by cancellation. But, as we compute in double precision arithmetic, it will appear with a small coefficient about the size of the cumulative errors caused by rounding off. As it constitutes a valuable control on the errors, we have reproduced it in Table V.

We now apply the rules (24_1) and (24_2) to construct the particular solution

(44) $$\xi_n^* = \frac{1}{D^2+1}\,\phi_n.$$

Observe that ξ_n^* contains a term independent of ℓ if and only if n is even. Thus, in the quadrature

(45) $$\eta_n^* = \frac{1}{D}(W_n - 2\xi_n^*)$$

resulting from the second of the equations (42), there occurs a secular term, say $K_n \ell$, if and only if n is even.

Because the equations (37) constitute the homogeneous system belonging to (42), the general solution of (42) is of the form

$$\xi_n = 2A_n - B_n \cos \ell + C_n \sin \ell + \xi_n^*,$$

$$\eta_n = D_n - 3A_n \ell + 2C_n \cos \ell + 2B_n \sin \ell + \eta_n^*.$$

In view of the symmetry conditions (40), we have that

$$C_n = D_n = 0.$$

The other two coefficients A_n and B_n are to be determined in different ways depending on whether the order n is even or odd.

(a) If n is even, the parity rule (44) implies that

$$B_n = 0;$$

and, for η_n to be periodic in ℓ, we must have that

$$A_n = -K_n/3.$$

In which case, the perigee condition (41) provides a check on the accuracy, namely the relation

$$2A_n - \xi_n^*(0) = 0. \tag{46}$$

The residuals on this relation are entered in Table V.

(b) If n is odd, because η_n^* is exempt from secular terms, we must have

$$A_n = 0$$

for η_n to be periodic. Then the perigee condition (41) serves to compute that

$$B_n = \xi_n^*(0).$$

Table V

Error controls in the expansions of ξ and η

Order	Error	Order	Error
2	0	3	0
4	0.6×10^{-15}	5	0
6	0.4×10^{-14}	7	0.1×10^{-13}
8	0.4×10^{-13}	9	0.7×10^{-13}
10	-0.1×10^{-12}	11	0.4×10^{-12}
12	-0.1×10^{-11}	13	0.1×10^{-10}
14	0.1×10^{-10}	15	0.1×10^{-9}
16	-0.1×10^{-9}	17	-0.2×10^{-8}

Compared with the corresponding ones in Tables I and II, the entries of Table V indicate that the algorithm we just described is less favorable than the first one in accuracy. The reason is that the series U_n and V_n are obtained by combining terms of the same size--a defect already mentioned by Brown and Shook [1].

Before we conclude the analysis of this algorithm, let us consider one more check. Once the expansions of ξ and η are known, those for $X = x/a$ and $Y = y/a$ can be calculated from the transformation formulae (34). Now we know that the average value of X over the mean anomaly is equal to $-3/2\ e$, which means that the independent term in the expansion of X reduces to that term. Actually, by composing the developments of ξ and η according to the first of the transformation formulae (34), we find for the independent term a truncated series in e. In the second column of Table VI, we enter those coefficients as they have been computed by the program; they reflect the increasingly damaging propagation of the round off errors throughout the recurrence. Indeed, that the average value of X does not contain the eccentricity beyond the first power implies that, for each $n \geq 1$, the coefficient $X_{2n+1,1}$ of $e^{2n+1}\cos \ell$ in the expansion of ξ should be equal to the coefficient $Y_{2n+1,1}$ of $e^{2n+1}\sin \ell$ in the expansion of η. For the sake of comparison, we enter the difference $|X_{2n+1,1} - Y_{2n+1,1}|$ in the second column of Table VI.

Table VI

Error controls in the expansion of x/a

Order	x/a	$\lvert X_{2n+1,1}-Y_{2n+1,1}\rvert$
3	0	0
5	-0.4×10^{-15}	0.75×10^{-15}
7	-0.1×10^{-13}	0.20×10^{-13}
9	-0.4×10^{-13}	0.74×10^{-13}
11	0.8×10^{-12}	0.16×10^{-11}
13	-0.3×10^{-11}	0.56×10^{-11}
15	-0.3×10^{-10}	0.55×10^{-10}
17	0.8×10^{-9}	0.16×10^{-8}

4. CONCLUSIONS

The classical expansions of the elliptic motion in terms of the mean anomaly can be obtained in a straightforward manner by applying Poincaré's method of continuation to finding periodic solutions to the differential motions in the neighborhood of the circular motions.

The programs that we propose here are fast. The first algorithm builds the expansions of all powers $(\cos E)^p$, $1 \leq p \leq 30$, up to the degree 30 in the eccentricity in less than 6 minutes on an IBM 7094. Because it needs a disk file to store the series representing the monomials $\xi^p \eta^q$ $(0 \leq p \leq 17,\ 0 \leq q \leq 17)$, the second algorithm takes 18 minutes of an IBM 7094 to complete the recurrence at order 17.

As long as the order of the expansions is not pushed unrealistically too high up, these elementary programs can easily become parts of major undertakings as required by the classical Perturbation Theories.

A special effort was made to provide enough error controls, so that the reader may judge the quality of accuracy he may expect from these algorithms at various orders. Thus we concede that, although it is more extensive in the intermediary results that it brings out, the second algorithm proposed here yields sensibly less accuracy at somewhat higher orders than the first one.

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Mrs. A. Deprit-Bartholomé contributed to the programs set up to implement Poincaré's method of continuation by computing long hand the solution in each case up to order 5. Stimulating discussions have been held with Jacques Henrard at the time of his stay at the Boeing Scientific Research Laboratories.

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REFERENCES

[1] BROWN, E. W. and SHOOK, C. A.

[2] DANBY, J. M. A., DEPRIT, A. and ROM. A. R. M. 1965, Boeing Scientific Research Laboratories Document D1-82-0481, Seattle, Washington.

[3] DAVIS, M. R. 1958, *Astron. Journ. 63*, 462-464.

[4] DAVIS, M. R. 1963, in *Applications of Digital Computers*, ed. by W. F. Freiberger and W. Prager, Ginn and Company, Boston, pp. 85-96.

[5] HERGET, P. and MUSEN, P. 1959, *Astron. Journ. 64*, 11-20.

[6] JARNAGIN, M. P. 1965, *Astron. Pap. Amer. Ephem. and Naut. Almanac*, Vol. *18*, U. S. Government Printing Office, Washington, D. C.

[7] MOULTON, F. R. 1907, *Astron. Journ. 25*,

[8] MOULTON, F. R. 1920, *Periodic Orbits*, Carnegie Institution of Washington, Washington, D. C., pp. 55-60.

[9] PUTZER, E. J. 1966, *Amer. Math. Monthly 73*, 2-7.

[10] WHITTAKER, E. T. 1904, *A Treatise on the Analytical Dynamics of Particles and Rigid Bodies*, Cambridge University Press, Cambridge, Great Britain, pp. 54-57.

ANNEXES

To the reader who would like to set up subroutines needed for manipulating Poisson series of the type (1,1) and to assemble therefrom a program to apply Poincaré's method of continuation to the Problem of Two Bodies, it may be of some use to check the results generated by the computer against the correct answers at the few first orders. Thus we reproduce here the calculations made by hand against which we checked our procedures. The notations in Annexes I and II adhere strictly to those introduced respectively in Sections 1 and 3.

ANNEX I

Expansion of cos E

Order 0

$$\rho_0 = \cos \ell .$$

Order 1

$$U_1 = -\frac{1}{2} - \frac{3}{2} \cos 2\ell ,$$

$$\rho_1^* = -\frac{1}{2} + \frac{1}{2} \cos 2\ell ,$$

$$C_1 = 0,$$

$$\rho_1 = -\frac{1}{2} + \frac{1}{2} \cos 2\ell .$$

Order 2

$$U_2 = -3 \cos 3\ell ,$$

$$\rho_2^* = \frac{3}{8} \cos 3\ell ,$$

$$C_2 = -\frac{3}{8} ,$$

$$\rho_2 = -\frac{3}{8} \cos \ell + \frac{3}{8} \cos 3\ell .$$

Order 3

$$U_3 = \cos 2\ell - 5 \cos 4\ell ,$$

$$\rho_3^* = -\frac{1}{3} \cos 2\ell + \frac{1}{3} \cos 4\ell ,$$

$$C_3 = 0,$$

$$\rho_3 = -\frac{1}{3} \cos 2\ell + \frac{1}{3} \cos 4\ell .$$

Order 4

$$U_4 = \frac{45}{16}\cos 3\ell - \frac{125}{16}\cos 5\ell,$$

$$\rho_4^* = -\frac{45}{128}\cos 3\ell + \frac{125}{384}\cos 5\ell,$$

$$C_4 = \frac{5}{192},$$

$$\rho_4 = \frac{5}{192}\cos \ell - \frac{45}{128}\cos 3\ell + \frac{125}{384}\cos 5\ell.$$

Order 5

$$U_5 = -\frac{3}{16}\cos 2\ell + 6\cos 4\ell - \frac{189}{16}\cos 6\ell,$$

$$\rho_5^* = \frac{1}{16}\cos 2\ell - \frac{2}{5}\cos 4\ell + \frac{189}{560}\cos 6\ell,$$

$$C_5 = 0,$$

$$\rho_5 = \frac{1}{16}\cos 2\ell - \frac{2}{5}\cos 4\ell + \frac{189}{560}\cos 6\ell.$$

These coefficients were checked by substituting the solution in the energy integral belonging to the Routhian function $\mathcal{R}$ given by formula (9) in the text.

ANNEX II

Expansion of ξ and η

Order 1

$\xi_1 = -e \cos \ell,$

$\eta_1 = 2e \sin \ell.$

Order 2

Coefficient of e^2

in ξ^2: $\frac{1}{2}e^2 + \frac{1}{2}e^2 \cos 2\ell,$

in $\xi\eta$: $-e^2 \sin 2\ell,$

in η^2: $2e^2 - 2e^2 \cos 2\ell;$

$U_2 = \frac{3}{2} \cos 2\ell,$

$V_2 = \frac{3}{2} \cos 2\ell,$

$\xi_2^* = \frac{3}{2} + \frac{1}{2} \cos 2\ell,$

$\eta_2^* = -3\ell + \frac{1}{4} \sin 2\ell,$

$A_2 = -1, \quad B_2 = 0,$

$\xi_2 = -\frac{1}{2} + \frac{1}{2} \cos 2\ell,$

$\eta_2 = \frac{1}{4} \sin 2\ell.$

Order 3

Coefficient of e^3

in ξ^2: $\frac{1}{2} \cos \ell - \frac{1}{2} \cos 3\ell,$

in $\xi\eta$: $-\frac{13}{8} \sin \ell + \frac{3}{8} \sin 3\ell,$

in η^2: $\frac{1}{2} \cos \ell - \frac{1}{2} \cos 3\ell,$

in ξ^3: $-\frac{3}{4}\cos \ell - \frac{1}{4}\cos 3\ell$,

in $\xi^2\eta$: $\frac{1}{2}\sin \ell + \frac{1}{2}\sin 3\ell$,

in $\xi\eta^2$: $-\cos \ell + \cos 3\ell$,

in η^3: $6\sin \ell - 2\sin 3\ell$;

$$U_3 = \frac{9}{4}\cos \ell - \frac{25}{4}\cos 3\ell,$$

$$V_3 = \frac{9}{8}\sin \ell - \frac{39}{8}\sin 3\ell,$$

$$\xi_3^* = \frac{3}{8}\cos 3\ell,$$

$$\eta_3^* = -\frac{9}{8}\sin \ell + \frac{7}{24}\sin 3\ell,$$

$$A_3 = 0, \quad B_3 = \frac{3}{8},$$

$$\xi_3 = -\frac{3}{8}\cos \ell + \frac{3}{8}\cos 3\ell,$$

$$\eta_3 = -\frac{3}{8}\sin \ell + \frac{7}{24}\sin 3\ell.$$

Order 4

Coefficient of e^4

in ξ^2: $\frac{3}{4} - \frac{1}{2}\cos 2\ell - \frac{1}{4}\cos 4\ell$,

in $\xi\eta$: $-\frac{5}{6}\sin 2\ell + \frac{7}{24}\sin 4\ell$,

in η^2: $-\frac{23}{32} + \frac{4}{3}\cos 2\ell - \frac{59}{96}\cos 4\ell$,

in ξ^3: $-\frac{3}{8} + \frac{3}{8}\cos 4\ell$,

in $\xi^2\eta$: $\frac{9}{8}\sin 2\ell - \frac{7}{16}\sin 4\ell$,

in $\xi\eta^2$: $-\frac{7}{4} + 2\cos 2\ell - \frac{1}{4}\cos 4\ell$,

in η^3: $\frac{3}{2}\sin 2\ell - \frac{3}{4}\sin 4\ell$,

in ξ^4: $\frac{3}{8} + \frac{1}{2}\cos 2\ell + \frac{1}{8}\cos 4\ell$, in $\xi^3\eta$: $-\frac{1}{2}\sin 2\ell - \frac{1}{4}\sin 4\ell$,

in $\xi^2\eta^2$: $\frac{1}{2} - \frac{1}{2}\cos 4\ell$,

in $\xi\eta^3$: $-2\sin 2\ell + \sin 4\ell$,

in η^4: $6 - 8\cos 2\ell + 2\cos 4\ell$;

$$U_4 = \frac{3}{64} + 4\cos 2\ell - \frac{579}{64}\cos 4\ell,$$

$$V_4 = 3\sin 2\ell - \frac{61}{8}\sin 4\ell,$$

$$\xi_4^* = \frac{3}{64} - \frac{1}{3}\cos 2\ell + \frac{67}{192}\cos 4\ell,$$

$$\eta_4^* = -\frac{3}{32}\ell - \frac{5}{12}\sin 2\ell + \frac{29}{96}\sin 4\ell,$$

$$A_4 = -\frac{1}{32}, \quad B_4 = 0,$$

$$\xi_4 = -\frac{1}{64} - \frac{1}{3}\cos 2\ell + \frac{67}{192}\cos 4\ell,$$

$$\eta_4 = -\frac{5}{12}\sin 2\ell + \frac{29}{96}\sin 4\ell.$$

Order 5

Coefficient of e^5

in ξ^2: $\frac{71}{96}\cos\ell - \frac{37}{64}\cos 3\ell - \frac{31}{192}\cos 5\ell$,

in $\xi\eta$: $\frac{37}{48}\sin\ell - \frac{175}{192}\sin 3\ell + \frac{61}{192}\sin 5\ell$,

in η^2: $-\frac{41}{48}\cos\ell + \frac{49}{32}\cos 3\ell - \frac{65}{96}\cos 5\ell$,

in ξ^3: $-\frac{15}{16}\cos\ell + \frac{27}{32}\cos 3\ell + \frac{3}{32}\cos 5\ell$,

in $\xi^2\eta$: $\frac{101}{48}\sin\ell - \frac{13}{96}\sin 3\ell - \frac{23}{96}\sin 5\ell$,

in $\xi\eta^2$: $-\frac{91}{96}\cos\ell + \frac{73}{64}\cos 3\ell - \frac{37}{192}\cos 5\ell$,

in η^3: $-\frac{65}{16}\sin\ell + \frac{95}{32}\sin 3\ell - \frac{31}{32}\sin 5\ell$,

in ξ^4: $\frac{1}{2}\cos\ell - \frac{1}{4}\cos 3\ell - \frac{1}{4}\cos 5\ell$,

$$\text{in } \xi^3\eta: \quad -\frac{13}{16}\sin \ell - \frac{15}{32}\sin 3\ell + \frac{11}{32}\sin 5\ell,$$

$$\text{in } \xi^2\eta^2: \quad \frac{5}{4}\cos \ell - \frac{13}{8}\cos 3\ell + \frac{3}{8}\cos 5\ell,$$

$$\text{in } \xi\eta^3: \quad -\frac{23}{4}\sin \ell + \frac{17}{8}\sin 3\ell - \frac{20}{8}\sin 5\ell,$$

$$\text{in } \eta^4: \quad 2\cos \ell - 3\cos 3\ell + \cos 5\ell,$$

$$\text{in } \xi^5: \quad -\frac{5}{8}\cos \ell - \frac{5}{16}\cos 3\ell - \frac{1}{16}\cos 5\ell,$$

$$\text{in } \xi^4\eta: \quad \frac{1}{4}\sin \ell + \frac{3}{8}\sin 3\ell + \frac{1}{8}\sin 5\ell,$$

$$\text{in } \xi^3\eta^2: \quad -\tfrac{1}{2}\cos \ell + \tfrac{1}{4}\cos 3\ell + \tfrac{1}{4}\cos 5\ell,$$

$$\text{in } \xi^2\eta^3: \quad \sin \ell + \tfrac{1}{2}\sin 3\ell - \tfrac{1}{2}\sin 5\ell,$$

$$\text{in } \xi\eta^4: \quad -2\cos \ell + 3\cos 3\ell - \cos 5\ell,$$

$$\text{in } \eta^5: \quad -20\sin \ell - 10\sin 3\ell + 2\sin 5\ell;$$

$$U_5 = -\frac{5}{16}\cos \ell + \frac{119}{16}\cos 3\ell - \frac{105}{8}\cos 5\ell,$$

$$V_5 = -\frac{5}{32}\sin \ell + \frac{201}{32}\sin 3\ell - \frac{185}{16}\sin 5\ell,$$

$$\xi_5^* = -\frac{13}{32}\cos 3\ell + \frac{17}{48}\cos 5\ell,$$

$$\eta_5^* = \frac{5}{32}\sin \ell - \frac{41}{96}\sin 3\ell + \frac{77}{240}\sin 5\ell,$$

$$A_5 = 0, \quad B_5 = -\frac{5}{96},$$

$$\xi_5 = \frac{5}{96}\cos \ell - \frac{13}{32}\cos 3\ell + \frac{17}{48}\cos 5\ell,$$

$$\eta_5 = \frac{5}{96}\sin \ell - \frac{41}{96}\sin 3\ell + \frac{77}{240}\sin 5\ell.$$