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George B. Dantzig

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ON POSITIVE PRINCIPAL MINORS

by

George B. Dantzig[1)]

When a matrix is symmetric, the property of having all its principal minors positive, is equivalent to being positive definite. When a matrix is non-symmetric this is no longer true. For example

$$(1) \qquad M = \begin{bmatrix} 1 & -7 \\ 0 & 1 \end{bmatrix}$$

has all positive principal minors, but

$$xMx = x_1^2 - 7x_1x_2 + x_2^2 < 0 \text{ for } x = (x_1, x_2) = (1,1)$$

The converse, however, as shown in Gale-Nikaido [1], Cottle [2] is true:

<u>Theorem 1</u>: The class with positive principal minors properly includes those which are positive definite.

<u>Proof</u>: Assume $\bar{M}$ is positive definite, $\bar{M}$ is non singular for if not, then there would exist $x = x^o \neq 0$ such that $\bar{M}x^o = 0$; yielding $x^o\bar{M}x^o = 0$, a contradiction. It follows that every principal submatrix of $\bar{M}$ is also nonsingular.

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1) The author acknowledges leads suggested by David Gale.

Partition $\bar{M}$ so that

$$\bar{M} = \begin{bmatrix} M_1 & c \\ r & a_{mm} \end{bmatrix}$$

choose $\quad x^* = [Y^*, -1]$ where $Y^* M_1 = r$. Then

$$x^* \bar{M} x^* = [Y^*, -1] \begin{bmatrix} M_1 & c \\ r & a_{mm} \end{bmatrix} \begin{bmatrix} Y^* \\ -1 \end{bmatrix} = (a_{mm} - Y^* c) > 0$$

But $\quad \det \begin{bmatrix} M_1 & c \\ r & a_{mm} \end{bmatrix} = \det \begin{bmatrix} M_1 & c \\ 0 & a_{mm} - Y^* c \end{bmatrix} = (\det M_1)(a_{mm} - Y^* c)$

Thus det $\bar{M}$ and its largest order principal minor have the same sign. Inductively, since first-order principal minors are positive, so must the second order ones, etc., up to the highest order. Finally, example (1) shows that the inclusion is proper.

Although positive definite matrices $\bar{M}$ do not comprise the entire class of positive principal minors, they can be used to generate a larger class by multiplying $\bar{M}$ by diagonal matrices on the right and left to form $D\bar{M}E$. For example,

$$\begin{bmatrix} 7 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1/7 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -7 \\ 0 & 1 \end{bmatrix}$$

positive definite — positive principal minors but not positive definite

It is not difficult to show that for 2 x 2 matrices the entire positive principle minor class can be so obtained from the positive definite class. The following is easily seen.

<u>Lemma:</u> Let $D = [d_{ij}]$, $E = [e_{ij}]$ be diagonal matrices with the property that $d_{ii}e_{ii} > 0$, then for any M the sign (+, -, or 0) of any principal minor of DME is the same as that of M. If $\bar{M}$ is positive definite, then $D\bar{M}E$ has positive principal minors.

<u>Theorem 2:</u> If M has positive principal minors and there exist diagonal D and E such that $\bar{M} = D^{-1}ME^{-1}$ is positive definite, then there exists a diagonal matrix F such that $F^{-1}MF$ is positive definite.

<u>Proof:</u> If $\bar{M}$ is positive definite so is $A\bar{M}A^T$ for any nonsingular A (since $x(A\bar{M}A^T)x^T = y\bar{M}y^T > 0$ where $x \neq 0$ and $y = xA \neq 0$). We consider $A\bar{M}A^T = AD^{-1}ME^{-1}A^T$ and choose A so that $AD^{-1} = (E^{-1}A^T)^{-1}$ or $A^TA = ED$. Thus $A = [a_{ij}]$ could be chosen as a diagonal matrix such that $a_{ii} = +\sqrt{d_{ii}e_{ii}}$. Then $F = [f_{ij}]$ is the diagonal matrix $F = E^{-1}A^T$ where $f_{ii} = +\sqrt{d_{ii}/e_{ii}}$.

<u>Theorem 3:</u> The characteristic roots of (any) M are the same as $F^{-1}MF$ for (any) nonsingular F.[2)]

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2) A well known result of matrix theory.

Proof: By definition, λ is a characteristic root of M if there exists an $x \neq 0$ such that $Mx = \lambda x$. Setting $Fy = x$ then $y \neq 0$ if $x \neq 0$ and we have $MFy = \lambda Fy$ or $(F^{-1}MF)\ y = \lambda y$ so that λ is a characteristic root of $(F^{-1}MF)$ also.

Theorem 4: If $\bar{M}$ is positive definite, the real part of every characteristic root is positive.[3)]

Proof: Let $\bar{M}x = \lambda x,\ x \neq 0$ and let $x = u + iv,\ \lambda = R + iS$, then by substitution and equating the real and imaginary parts

$$\bar{M}u = Ru - Sv \qquad (u,v) \neq 0$$
$$\bar{M}v = Rv + Su$$

Now $R > 0$ follows from

$$0 < u^T\bar{M}u + v^T\bar{M}v = R\ (u^2 + v^2)$$

The following is known, see Gale - Nikaido [1].

Theorem 5: If M has positive principal minors, then every real characteristic root is positive.

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3) A well known result in matrix theory that is reviewed here for non-symmetric $\bar{M}$.

Proof: The characteristic equation is obtained by setting

$$\det\ [M - \lambda I] = 0$$

this yields

$$(-\lambda)^m + C_1(-\lambda)^{m-1} + C_2\ (-\lambda)^{m-2} + \ldots + C_m = 0$$

where C_j is the sum of the j-th order principal minors. For matrices with positive principal minors, $C_j > 0$. It is not possible that $\lambda \leq 0$ because then all terms above would be non-negative and the last term positive so that the left hand side would be strictly positive, a contradiction.

Theorem 6 (Kalman*): There exist matrices M with positive principal minors that have characteristic roots not all of which have positive real parts; for such matrices no transformation $\bar{M} = DME$ exists with diagonal matrices D and E such that $\bar{M}$ is positive definite.

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*In a letter to D. Gale dated 9 July 1962, Rudolf E. Kalman showed that

$$\begin{bmatrix} 10/3 & -4 & -\sqrt{11.1} \\ 8/3 & 1/3 & -\sqrt{1.1} \\ -\sqrt{899.1} & -\sqrt{89.1} & 30 \end{bmatrix}$$

had positive minors but had complex characteristic roots with negative real parts.

Proof: If such D and E did exist, then by Theorem 2, an F would exist such that FMF^{-1} is positive definite. By Theorem 4 FMF^{-1} would have to have characteristic roots with positive real parts. By Theorem 3 the same would be true for M. However M in the example below does not have this property.

$$M = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 17 \\ 4 & 0 & 1 \end{bmatrix}$$

The first order principal minors are 1, 1, 1 whose sum $C_1 = 3$, its second order ones are 2, 1, 1 whose sum is $C_2 = 4$. Its third order one is $C_3 = 70$. Thus M has positive principal minors. Its characteristic equation is

$$\lambda^3 - C_1\lambda^2 + C_2\lambda - C_3 = 0$$

or

$$\lambda^3 - 3\lambda^2 + 4\lambda - 70 = 0$$

which factors into

$$(\lambda - 5)(\lambda^2 + 2\lambda + 14) = 0$$

The characteristic roots are

$$\lambda_1 = +5, \ \lambda_2 = -1 + i\sqrt{13}, \ \lambda_3 = -1 - i\sqrt{13}$$

Since the real parts of λ_2 and λ_3 are negative, this establishes that the class of positive principal minor matrices form a larger class than those generated from the positive definite ones by simple rescaling of the rows and columns.

In mathematical programming one seeks p-vectors $x, y \geq 0$ that satisfy $y = Mx + q$ such that the products $x_i y_i = 0$ for $i = 1,2,..,p$. The latter conditions may be replaced by the minimum value of $x^T y = x^T Mx + x^T q$, a quadratic function which is convex if and only if M is positive definite. Certain solution procedures based on convexity [2], [3], [4] turn out to be also valid even when M has positive principal minors. This leads to the speculation that by a simple change of units of x,y one could obtain a new system $\bar{y} = \bar{M}\bar{x} + \bar{q}$, $(\bar{y},\bar{x}) \geq 0$, $\bar{y}_i \bar{x}_i = 0$ for $i = 1,..,p$ in which the quadratic function $\bar{x}^T \bar{M}\bar{x} + \bar{x}^T q$ is convex. However we have shown that this is not always possible to do. The class of positive principal minor matrices does not appear to be a trivial extension of positive definite matrices. Solution techniques also valid for the latter have somehow gotten around the difficulties of local optimality usually associated with nonconvex programming problems.

An open question posed by Gale and Kalman and closely related to considerations found in a paper by Arrow and McManus [5] is the following:

Suppose for all diagonal matrices D such that $d_{ii} > 0$, that M has the property that the real part of every characteristic root of MD is positive, does this imply there exists a $D = D^\circ$ such that $D^\circ M$ is positive definite?

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The relation between matrices with all principal minors positive and positive definite matrices is explored in the non-symmetric case. It is shown that simple rescaling of rows and columns is insufficient to transform the former into the latter. As a consequence it appears that the class of problems that can be solved by complementary pivot theory of mathematical programming has been non-trivially extended beyond the convex case represented by linear and quadratic programming and takes its place along with another important extension of Lemke and Howson for the matrix associated with bi-matrix games.

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