Report 1111

NAVY DEPARTMENT

DAVID TAYLOR MODEL BASIN

HYDROMECHANICS

o

AERODYNAMICS

o

STRUCTURAL MECHANICS

o

APPLIED MATHEMATICS

CALCULATION OF GOLDSTEIN FACTORS

(AML Problem 42.1-54)

by

J.W. Wrench, Jr., Ph.D.

APPLIED MATHEMATICS LABORATORY

RESEARCH AND DEVELOPMENT REPORT

September 1955 Report 1111

PRNC-TMB-648 (11-56)

28

CALCULATION OF GOLDSTEIN FACTORS

(AML Problem 42.1-54)

by

J.W. Wrench, Jr., Ph. D.

Reprint of
Applied Mathematics Laboratory
Technical Report 14

September 1955

Report 1111

CALCULATION OF GOLDSTEIN FACTORS

In his formulation of the vortex theory of screw propellers, S. Goldstein[1]* derives the following expression for the distribution of circulation about each blade section:

$$\frac{p\Gamma\omega}{2\pi w v} = \frac{8}{\pi^2}\sum_{m=0}^{\infty} \frac{T_{1,p(m+\frac{1}{2})}[(m+\frac{1}{2})p\mu]}{(2m+1)^2} + \frac{2}{\pi}\sum_{m=0}^{\infty} a_m \frac{I_{p(m+\frac{1}{2})}[(m+\frac{1}{2})p\mu]}{I_{p(m+\frac{1}{2})}[(m+\frac{1}{2})p\mu_0]} \tag{1}$$

where the coefficients a_m satisfy the following infinite system of linear equations:

$$\frac{\pi}{4}\sum_{m=0}^{\infty}\left\{\frac{(2m+1)I'_{p(m+\frac{1}{2})}[p(m+\frac{1}{2})\mu_0]/I_{p(m+\frac{1}{2})}[p(m+\frac{1}{2})\mu_0] - 2n\,K'_{pn}(pn\mu_0)/K_{pn}(pn\mu_0)}{4n^2-(2m+1)^2}\right\}a_m$$

$$= 2n\,\frac{K'_{pn}(pn\mu_0)}{K_{pn}(pn\mu_0)}\sum_{m=0}^{\infty}\frac{T_{1,p(m+\frac{1}{2})}[(m+\frac{1}{2})p\mu_0]}{(2m+1)^2[4n^2-(2m+1)^2]} - \sum_{m=0}^{\infty}\frac{T'_{1,p(m+\frac{1}{2})}[(m+\frac{1}{2})p\mu_0]}{(2m+1)[4n^2-(2m+1)^2]}, \quad n = 1, 2, 3, \ldots \tag{2}$$

The Goldstein factor K is defined in terms of the circulation Γ by the relation

$$K = \frac{1+\mu^2}{\mu^2}\,\frac{p\Gamma\omega}{2\pi w v} \tag{3}$$

---

* References are listed on page 17.

In the preceding equations, p denotes the number of blades, Γ is the circulation, ω is the angular velocity and v is the velocity of advance of the propeller, w is the velocity of the screw surface in the direction of its axis, μ represents vr/ω where r is the distance from the axis of rotation, and μ_0 is the value of μ when $r = R$, the radius of the propeller. The symbol $T_{1,p(m+\frac{1}{2})}[(m+\frac{1}{2})p\mu]$ represents Goldstein's function, which is defined by the relation

$$T_{1,n}(x) = S_{1,n}(ix) - n e^{\frac{n\pi i}{2}} K_n(x) \tag{4}$$

where $S_{1,n}(ix)$ is a Lommel function(2), and $K_n(x)$ is the modified Bessel function of the second kind, of order n. Similarly, $I_n(x)$ represents the modified Bessel function of the first kind, of order n. Primes are affixed to these functions to denote differentiation with respect to the argument.

For large values of μ_0 Goldstein showed that the individual terms of the equations comprising the infinite system (2) are closely approximated by the corresponding terms of the infinite system

$$\sum_{m=0}^{\infty} \frac{a_m^*}{2n-2m-1} = -\frac{\mu_0^2}{1+\mu_0^2}\frac{\pi}{4n}, \quad n = 1, 2, 3, \ldots, \tag{5}$$

and he succeeded in solving this system exactly, obtaining

$$a_m^* = -\frac{\mu_0^2}{1+\mu_0^2}\frac{(2m)!}{2^{2m}(m!)^2(2m+1)} \tag{6}$$

In the solution of AML Problem 42 - 54, equation (1) as modified by the substitution of a_m^* for a_m was evaluated on UNIVAC for values of μ_0 ranging from 0.25 to 6; p = 3, 4, 5, 6; and μ/μ_0 extending from 0.2 to 0.975. Subsequent desk-machine calculations have revealed that the use of a_m^* in place of a_m leads to errors in the calculated circulation that are unacceptably large when μ_0 is less than 2.

Accordingly, further research was requested, requiring an investigation of procedures for solving the infinite system (2) for a sufficient number of the successive unknowns a_m ($m = 0, 1, 2, \ldots$) to an adequate degree of accuracy to yield reliable values of the circulation.

The purpose of this report is to set forth in detail an effective procedure for solving the infinite system (2) for the leading a_m's when μ_0 exceeds unity. This limitation of the size of μ_0 is necessary to insure a satisfactory rate of convergence of the successive approximations to the unknowns a_m. Fortunately, the practically important range of μ_0 coincides with the one that we are considering.

For the sake of clarity, we shall illustrate the proposed technique by applying it to the calculation of the leading unknowns $a_0, a_1, \ldots,$ corresponding to the particular values $\mu_0 \doteq 1.5$ and $p = 3$.

The coefficients occuring in the equations of system (2) are evaluated by a variety of methods. When the arguments and orders of the Bessel functions I_n and K_n and their derivatives I_n' and K_n' are less than 20 the functional values can be obtained from existing tables[3,4]. Outside this range of arguments and orders, the following formulas[5] are effective.

$$I_n'(nx)/I_n(nx) \sim \frac{(1+x^2)^{1/2}}{x} - \frac{x}{2n(1+x^2)} + \frac{x(4-x^2)}{8n^2(1+x^2)^{5/2}} - \frac{x(4-10x^2+x^4)}{8n^3(1+x^2)^4} + \frac{x(64-560x^2+456x^4-25x^6)}{128n^4(1+x^2)^{11/2}} - \cdots \tag{7}$$

$$K_n'(nx)/K_n(nx) \sim -\frac{(1+x^2)^{1/2}}{x} - \frac{x}{2n(1+x^2)} - \frac{x(1-x^2)}{8n^2(1+x^2)^{5/2}}$$

$$- \frac{x(4-10x^2+x^4)}{8n^3(1+x^2)^4} - \frac{x(64-560x^2+456x^4-25x^6)}{128n^4(1+x^2)^{11/2}} - \cdots \quad (8)$$

The Goldstein function $T_{1,n}(x)$ is calculable by the series

$$T_{1,n}(x) = \frac{\pi n}{2\sin\frac{n\pi}{2}} I_n(x) - \frac{x^2}{2^2-n^2} - \frac{x^4}{(2^2-n^2)(4^2-n^2)} - \cdots \quad (9)$$

provided n is not an even integer.

When n is an even integer,

$$T_{1,n}(x) = 1 - \frac{n^2}{x^2} + \frac{n^2(n^2-2^2)}{x^4} - \frac{n^2(n^2-2^2)(n^2-4^2)}{x^6} + \cdots$$

$$- (-1)^{n/2} n K_n(x) \quad (10)$$

An alternative series for $T_{1,n}(n\mu_0)$ is

$$T_{1,n}(n\mu_0) \sim \tau_0(\mu_0) + \frac{\tau_2(\mu_0)}{n^2} + \frac{\tau_4(\mu_0)}{n^4} + \cdots \quad (11)$$

where
$$\tau_0(\mu_0) = \frac{\mu_0^2}{1+\mu_0^2}$$

$$\tau_2(\mu_0) = \frac{4\mu_0^2(1-\mu_0^2)}{(1+\mu_0^2)^4}$$

$$\tau_{r+2}(\mu_0) = \frac{1}{1+\mu_0^2}\left(\mu_0 \frac{d}{d\mu_0}\right)^2 \tau_r(\mu_0)$$

The values of $T'_{1,n}(x)$ can be calculated by means of series obtained by term-by-term differentiation of the preceding series for $T_{1,n}(x)$.

With the aid of such formulas, the leading coefficients of the system (2) were computed when $\mu_0 = 1.5$ and $p = 3$. The resulting system is shown on the following page.

The coefficient of a_m in the nth equation of this infinite system is closely approximated by

$$\frac{\pi}{4} \cdot \frac{(1+\mu_0^2)^{1/2}}{\mu_0(2n-2m-1)} \tag{13}$$

and the constant term is approximated by

$$-\frac{\pi^2}{16n} \frac{\mu_0}{(1+\mu_0^2)^{1/2}} \tag{14}$$

Indeed, these approximations to the coefficients and constant terms correspond to the simplified system (5).

Our procedure for solving the system (12) is as follows. We initially restrict our attention to the first equation of (12) and assume that

$$a_m = a_m^* = -\frac{\mu_0^2}{1+\mu_0^2} \cdot \frac{(2m)!}{2^{2m}(m!)^2(2m+1)}$$

for all $m \geq 1$. From equation (5) we infer that

$$\frac{\pi}{4} \frac{(1+\mu_0^2)^{1/2}}{\mu_0} \sum_{m=0}^{\infty} \frac{a_m^*}{2n-2m-1} = -\frac{\pi^2}{16n} \cdot \frac{\mu_0}{(1+\mu_0^2)^{1/2}} \tag{15}$$

and consequently the first equation of system (12) can be replaced by the following equation, with small error.

$$
\begin{aligned}
0.948605a_0 - 0.944856a_1 - 0.314805a_2 - 0.188851a_3 - 0.134882a_4 - 0.104902a_5 - 0.085827a_6 - \cdot\cdot\cdot\cdot &= -0.583667 \\
0.315529a_0 + 0.944485a_1 - 0.944225a_2 - 0.314710a_3 - 0.188816a_4 - 0.134864a_5 - 0.104892a_6 - \cdot\cdot\cdot\cdot &= -0.272021 \\
0.189156a_0 + 0.314775a_1 + 0.944140a_2 - 0.944073a_3 - 0.314679a_4 - 0.188802a_5 - 0.134857a_6 - \cdot\cdot\cdot\cdot &= -0.176764 \\
0.135050a_0 + 0.188847a_1 + 0.314698a_2 + 0.944041a_3 - 0.944014a_4 - 0.314664a_5 - 0.188797a_6 - \cdot\cdot\cdot\cdot &= -0.130837 \\
0.105009a_0 + 0.134883a_1 + 0.188813a_2 + 0.314674a_3 + 0.944001a_4 - 0.943983a_5 - 0.314659a_6 - \cdot\cdot\cdot\cdot &= -0.103832 \\
0.085900a_0 + 0.104905a_1 + 0.134864a_2 + 0.188802a_3 + 0.314664a_4 + 0.943977a_5 - 0.943972a_6 - \cdot\cdot\cdot\cdot &= -0.086064 \\
0.072674a_0 + 0.085828a_1 + 0.104892a_2 + 0.134857a_3 + 0.188796a_4 + 0.314656a_5 + 0.943967a_6 - \cdot\cdot\cdot\cdot &= -0.073481
\end{aligned}
\tag{12}
$$

$$0.948605a_0 + \left\{\frac{\pi}{4}\frac{\mu_0}{(1+\mu_0^2)^{1/2}} - \frac{\pi^2}{16}\frac{\mu_0}{(1+\mu_0^2)^{1/2}}\right\} = -0.583667 ,$$

$$\text{or} \quad 0.948605a_0 + \frac{\pi}{4}\frac{\mu_0}{(1+\mu_0^2)^{1/2}}\left(1-\frac{\pi}{4}\right) = -0.583667 ,$$

$$\text{or} \quad 0.948605a_0 = -0.723907, \tag{16}$$

whence a first approximation to a_0 is

$$a_0^{(1)} = -0.76313$$

The superscript within parentheses is used to designate the ordinal number of the approximation.

The systematic solution of (12) continues with a similar modification of the first two equations of that system. Thus, we replace the first two equations by the pair

$$0.948605a_0 - 0.944856a_1 + \frac{\pi}{4}\frac{\mu_0}{(1+\mu_0^2)^{1/2}}\left\{1 - \frac{1}{2\cdot 3} - \frac{\pi}{4}\right\} = -0.583667,$$

$$0.315529a_0 + 0.944485a_1 + \frac{\pi}{4}\frac{\mu_0}{(1+\mu_0^2)^{1/2}}\left\{\frac{1}{3} + \frac{1}{2\cdot 3} - \frac{\pi}{8}\right\} = -0.272021, \tag{17}$$

$$\text{or} \quad 0.948605a_0 - 0.944856a_1 = -0.614992,$$

$$0.315529a_0 + 0.944485a_1 = -0.342141,$$

which yield the approximations

$$a_0^{(2)} = -0.75718$$

$$a_1^{(1)} = -0.10930$$

Application of this technique to the first three equations of (12) yields the system

$$0.948605a_0 - 0.944856a_1 - 0.314805a_2 + \frac{\pi}{4}\frac{\mu_0}{(1+\mu_0^2)^{1/2}}\left\{1-\frac{1}{2\cdot3}-\frac{1}{5}\frac{1\cdot3}{2\cdot4\cdot5}-\frac{\pi}{4}\right\} = -0.583667$$

$$0.315529a_0 + 0.944485a_1 - 0.944225a_2 + \frac{\pi}{4}\frac{\mu_0}{(1+\mu_0^2)^{1/2}}\left\{\frac{1}{3}+\frac{1}{2\cdot3}-\frac{1\cdot3}{2\cdot4\cdot5}-\frac{\pi}{8}\right\} = -0.272021$$

$$0.189156a_0 + 0.314775a_1 + 0.944140a_2 + \frac{\pi}{4}\frac{\mu_0}{(1+\mu_0^2)^{1/2}}\left\{\frac{1}{5}+\frac{1}{3}\frac{1}{2\cdot3}+\frac{1\cdot3}{2\cdot4\cdot5}-\frac{\pi}{12}\right\} = -0.176764 \tag{18}$$

or,

$$0.948605a_0 - 0.944856a_1 - 0.314805a_2 = -0.598655$$

$$0.315529a_0 + 0.944485a_1 - 0.944225a_2 = -0.293129$$

$$0.189156a_0 + 0.314775a_1 + 0.944140a_2 = -0.221696$$

The roots of this system are

$$a_0^{(3)} = -0.75352$$

$$a_1^{(2)} = -0.10684$$

$$a_2^{(1)} = -0.04823$$

The next step in the iterative process involves the solution of the system:

$$\begin{aligned}
0.948605a_0 - 0.944856a_1 - 0.314805a_2 - 0.188851a_3 &= -0.592820 \\
0.315529a_0 + 0.944485a_1 - 0.944225a_2 - 0.314710a_3 &= -0.283405 \\
0.189156a_0 + 0.314775a_1 + 0.944140a_2 - 0.944073a_3 &= -0.192522 \\
0.135050a_0 + 0.188847a_1 + 0.314698a_2 + 0.944041a_3 &= -0.163174
\end{aligned} \tag{19}$$

Solution of this system yields the approximations

$$a_0^{(4)} = -0.75127$$

$$a_1^{(3)} = -0.10549$$

$$a_2^{(2)} = -0.04687$$

$$a_3^{(1)} = -0.02865$$

The next step in the refinement of the approximations to the a_i's entails the solution of the following system, where an ellipsis in each equation denotes the ommission, for typographical reasons, of the coefficients of a_1 and a_2 [unchanged from system (19)].

$$\begin{aligned}
0.948605a_0 - \cdots - 0.188851a_3 - 0.134882a_4 &= -0.589984 \\
0.315529a_0 + \cdots - 0.314710a_3 - 0.188816a_4 &= -0.279434 \\
0.189156a_0 + \cdots - 0.944073a_3 - 0.314679a_4 &= -0.185904 \\
0.135050a_0 + \cdots + 0.944041a_3 - 0.944014a_4 &= -0.143320 \\
0.105009a_0 + \cdots + 0.314674a_3 + 0.944001a_4 &= -0.128733
\end{aligned} \qquad (20)$$

From this set of equations we deduce the approximations:

$$a_0^{(5)} = -0.74978$$

$$a_1^{(4)} = -0.10463$$

$$a_2^{(3)} = -0.04610$$

$$a_3^{(2)} = -0.02779$$

$$a_4^{(1)} = -0.01953$$

Further extension of the procedure leads to the following system of six simultaneous equations, which has been abbreviated in a manner similar to system (20):

$$
\begin{aligned}
0.948605a_0 - \ldots - 0.134882a_4 - 0.104902a_5 &= -\,0.588359 \\
0.315529a_0 + \ldots - 0.188816a_4 - 0.134864a_5 &= -\,0.277345 \\
0.189156a_0 + \ldots - 0.314679a_4 - 0.188802a_5 &= -\,0.182980 \\
0.135050a_0 + \ldots - 0.944014a_4 - 0.314664a_5 &= -\,0.138447 \\
0.105009a_0 + \ldots + 0.944001a_4 - 0.943983a_5 &= -\,0.114113 \\
0.085900a_0 + \ldots + 0.314664a_4 + 0.943977a_5 &= -\,0.106107
\end{aligned}
\qquad (21)
$$

The corresponding values of the a_i's are:

$$
\begin{aligned}
a_0^{(6)} &= -\,0.74871 \\
a_1^{(5)} &= -\,0.10404 \\
a_2^{(4)} &= -\,0.045590 \\
a_3^{(3)} &= -\,0.027276 \\
a_4^{(2)} &= -\,0.018947 \\
a_5^{(1)} &= -\,0.014426
\end{aligned}
$$

Desk-machine calculations were terminated with the solution of the following system of seven simultaneous equations:

$$
\begin{aligned}
0.948605a_0 - \ldots &- 0.104902a_5 - 0.085827a_6 = -0.587328 \\
0.315529a_0 + \ldots &- 0.134864a_5 - 0.104892a_6 = -0.276085 \\
0.189156a_0 + \ldots &- 0.188802a_5 - 0.134857a_6 = -0.181360 \\
0.135050a_0 + \ldots &- 0.314664a_5 - 0.188797a_6 = -0.136179 \\
0.105009a_0 + \ldots &- 0.943983a_5 - 0.314659a_6 = -0.110333 \\
0.085900a_0 + \ldots &+ 0.943977a_5 - 0.943972a_6 = -0.094767 \\
0.072674a_0 + \ldots &+ 0.314656a_5 + 0.943967a_6 = -0.090127
\end{aligned}
\tag{22}
$$

The corresponding roots are:

$$a_0^{(7)} = -0.74793$$

$$a_1^{(6)} = -0.10361$$

$$a_2^{(5)} = -0.045235$$

$$a_3^{(4)} = -0.026940$$

$$a_4^{(3)} = -0.018580$$

$$a_5^{(2)} = -0.013992$$

$$a_6^{(1)} = -0.011220$$

Further calculations of this type, involving increasingly larger systems of equations, can be expeditiously performed by high-speed electronic computers. However, the preliminary calculation of the numerical coefficients of the equations is more formidable, and must necessarily be programmed and carried out as a separate operation.

The relatively slow rate of convergence may be deduced from the following condensed table of successive approximations to the system (12).

TABLE 1

n	$a_0(n)$	$a_1(n)$	$a_2(n)$	$a_3(n)$	$a_4(n)$	$a_5(n)$	$a_6(n)$
1	- 0.7631	- 0.1093	- 0.0482	- 0.0287	- 0.0195	- 0.0144	- 0.0112
2	- 0.7572	- 0.1068	- 0.0469	- 0.0278	- 0.0189	- 0.0140	
3	- 0.7535	- 0.1055	- 0.0461	- 0.0273	- 0.0186		
4	- 0.7513	- 0.1046	- 0.0456	- 0.0269			
5	- 0.7498	- 0.1040	- 0.0452				
6	- 0.7487	- 0.1036					
7	- 0.7479						

The convergence of each sequence $\{a_m(n)\}$ is accelerated by applying the Aitken δ^2 process (6), that is, the associated sequence $\{b_m(n)\}$ is formed according to the formula

$$b_m(n) = \frac{a_m(n-1)a_m(n+1) - a_m(n)^2}{a_m(n-1) - 2a_m(n) + a_m(n+1)} \qquad (23)$$

The sequences $\{b_m(n)\}$, $m = 0, 1, 2, \ldots$, yield the following accurate approximations to the roots of the infinite system (12):

$$a_0 = -0.7458$$
$$a_1 = -0.1024$$
$$a_2 = -0.0444$$
$$a_3 = -0.0262$$
$$a_4 = -0.0179$$
$$a_5 = -0.0133$$
$$a_6 = -0.0105$$

We define ϵ_m by the relation

$$\epsilon_m = a_m - a_m^* \tag{24}$$

In the particular numerical case under consideration, ϵ_m assumes the following values, deduced from equation (6) in conjunction with the preceding set of final approximations to the a_m's.

$$\epsilon_0 = -0.0535$$
$$\epsilon_1 = +0.0130$$
$$\epsilon_2 = 0.0075$$
$$\epsilon_3 = 0.0047$$
$$\epsilon_4 = 0.0031$$
$$\epsilon_5 = 0.0022$$
$$\epsilon_6 = 0.0015$$

The ϵ's thus derived are then used to evaluate the Kapteyn series appearing in the right member of equation (1). To this end we write

$$\frac{2}{\pi}\frac{1+\mu^2}{\mu^2}\sum_{m=0}^{\infty} a_m \frac{I_{p(m+\frac{1}{2})}\left[(m+\frac{1}{2})p\mu\right]}{I_{p(m+\frac{1}{2})}\left[(m+\frac{1}{2})p\mu_0\right]} = \frac{2}{\pi}\frac{1+\mu^2}{\mu^2}\sum_{m=0}^{\infty} a_m^* \frac{I_{p(m+\frac{1}{2})}\left[(m+\frac{1}{2})p\mu\right]}{I_{p(m+\frac{1}{2})}\left[(m+\frac{1}{2})p\mu_0\right]}$$

$$+\frac{2}{\pi}\frac{1+\mu^2}{\mu^2}\sum_{m=0}^{\infty} \epsilon_m \frac{I_{p(m+\frac{1}{2})}\left[(m+\frac{1}{2})p\mu\right]}{I_{p(m+\frac{1}{2})}\left[(m+\frac{1}{2})p\mu_0\right]} \tag{25}$$

The motivation for this representation of the Kapteyn series lies in the fact that the series comprising the first term of the right side of equation (25) has been evaluated by the UNIVAC for p = 3(1)6, μ_0 = 0.25(0.25)2.0(0.5)6.0, and μ/μ_0 = 0.2(0.1)0.8(0.05)0.95, 0.975. The multiplicative factor $\frac{1+\mu^2}{\mu^2}$ appears in equation (25) because the ultimate goal of the calculation is the evaluation of the Goldstein factor K , where, as already stated in equation (3),

$$K = \frac{1+\mu^2}{\mu^2}\frac{p\Gamma\omega}{2\pi w v}$$

The effect on K of introducing the correct coefficients a_m in place of the approximate coefficients a_m* may be inferred from an examination of the following table.

TABLE 2

Goldstein Factor K for p = 3 and μ_0 = 1.5

μ/μ_0	K (approx. value based on a_m*)	K (accurate value based on a_m)
0.2	1.4268	1.4034
0.3	1.0972	1.0756
0.4	0.9163	0.8945
0.5	0.7964	0.7731
0.6	0.7016	0.6760
0.7	0.6123	0.5837
0.8	0.5124	0.4808
0.85	0.4516	0.4191
0.9	0.3766	0.3442
0.95	0.2723	0.2428
0.975	0.1932	0.1678

The relative error in the above values of K based on the approximate coefficients a_m* increases from about 2 percent to more than 15 percent as μ/μ_0 approaches unity. On the other hand, the absolute error in the approximate values of K remains fairly constant, varying from 2.16×10^{-2} to 3.25×10^{-2}.

Appended to this report are tables of

$$\frac{1+\mu^2}{\mu^2}\,\frac{8}{\pi^2}\sum_{m=0}^{\infty}\frac{T_{1,p(m+\frac{1}{2})}[(m+\frac{1}{2})p\,\mu]}{(2m+1)^2}$$

and

$$-\frac{1+\mu^2}{\mu^2}\,\frac{2}{\pi}\sum_{m=0}^{\infty}a_m^*\,\frac{I_{p(m+\frac{1}{2})}[(m+\frac{1}{2})p\,\mu]}{I_{p(m+\frac{1}{2})}[(m+\frac{1}{2})p\,\mu_0]}$$

derived from similar results computed by UNIVAC in the solution of Applied Mathematics Laboratory Problem 42 - 54.

These data can be used as the basis for further computation of accurate values of K in the manner illustrated in this report.

An investigation was also made of the applicability of published theorems of von Koch(7) and March (8) to the problem of determining the existence of a solution to the infinite system (2). The sufficient conditions contained in these theorems are too restrictive to be satisfied by the system under consideration in this report, and consequently, the theorems do not apply. A rigorous proof of the existence of a solution of a system such as (2) apparently is not yet available.

I should like to conclude with an acknowledgment of the material assistance rendered by Mrs. Anita B. Milam by her extensive calculations of the modified Bessel functions of half-odd-integer orders, which entered into the evaluation of the coefficients of the numerical system of equations treated in this report.

REFERENCES

(1) S. Goldstein, On the Vortex Theory of Screw Propellers. Royal Society of London, Proceedings vol. 123A, 1929, p. 440-465.

(2) G. N. Watson, A Treatise on the Theory of Bessel Functions, Second Edition, Cambridge, Cambridge University Press, 1952, p. 345-352.

(3) British Association for the Advancement of Science, Mathematical Tables, vol. X, Bessel Functions, Part II, Cambridge, Cambridge University Press, 1952.

(4) C. W. Jones, A Short Table for the Bessel Functions $I_{n+\frac{1}{2}}(x)$, $\frac{2}{\pi}K_{n+\frac{1}{2}}(x)$, New York, Cambridge University Press, 1952.

(5) J. W. Wrench, Jr., The Calculation of Propeller Induction Factors, Technical Report 13, Applied Mathematics Laboratory, 11 May 1955.

(6) A. S. Householder, Principles of Numerical Analysis, New York, McGraw-Hill Book Company, 1953, p. 117-118.

(7) H. von Koch, Jahresbericht der Deutschen Mathematiker-Vereinigung, vol. 22, 1913, p. 285-291.

(8) H. W. March, American Mathematical Society, Transactions, vol. 27, 1925, p. 307.

TABLE OF $\frac{1+\mu^2}{\mu^2}\frac{8}{\pi^2}\sum_{m=0}^{\infty}\frac{T_{1,p(m+\frac{1}{2})}[(m+\frac{1}{2})\pi\mu]}{(2m+1)^2}$

p = 3

$\mu_0 \backslash \mu/\mu_0$	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.85	0.9	0.95	0.975
0.25	5.08808	4.01570	3.38187	2.95390	2.64190	2.40284	2.21316	2.13219	2.05874	1.99180	1.96050
0.50	3.38187	2.64190	2.21316	1.93054	1.73013	1.58128	1.46713	1.41975	1.37759	1.33992	1.32258
0.75	2.64190	2.05874	1.73013	1.52061	1.37759	1.27578	1.20129	1.17154	1.14576	1.12336	1.11326
1.00	2.21316	1.73013	1.46713	1.30615	1.20129	1.13048	1.08163	1.06306	1.04752	1.03450	1.02880
1.25	1.93054	1.52061	1.30615	1.18099	1.10884	1.05494	1.02359	1.01243	1.00352	0.99645	0.99349
1.50	1.73013	1.37759	1.20129	1.10384	1.04752	1.01447	0.99523	0.98899	0.98438	0.98105	0.97977
1.75	1.58128	1.27578	1.13048	1.05494	1.01447	0.99294	0.98203	0.97905	0.97718	0.97615	0.97589
2.00	1.46713	1.20129	1.08163	1.02359	0.99523	0.98203	0.97681	0.97591	0.97574	0.97609	0.97640
2.50	1.30615	1.10384	1.02359	0.99087	0.97872	0.97575	0.97679	0.97798	0.97938	0.98087	0.98163
3.00	1.20129	1.04752	0.99523	0.97872	0.97574	0.97772	0.98118	0.98298	0.98472	0.98635	0.98712
3.50	1.13048	1.01447	0.98203	0.97575	0.97772	0.98178	0.98582	0.98761	0.98922	0.99064	0.99128
4.00	1.08163	0.99523	0.97681	0.97679	0.98118	0.98582	0.98964	0.99119	0.99251	0.99364	0.99414
4.50	1.04752	0.98438	0.97574	0.97938	0.98472	0.98922	0.99251	0.99377	0.99481	0.99567	0.99604
5.00	1.02359	0.97872	0.97679	0.98239	0.98785	0.99188	0.99460	0.99558	0.99637	0.99701	0.99729
5.50	1.00684	0.97625	0.97880	0.98528	0.99044	0.99390	0.99608	0.99684	0.99744	0.99791	0.99811
6.00	0.99523	0.97574	0.98118	0.98785	0.99251	0.99540	0.99713	0.99771	0.99816	0.99851	0.99865

TABLE OF $\frac{1+\mu^2}{\mu^2}\frac{8}{\pi^2}\sum_{m=0}^{\infty}\frac{T_{1,p(m+\frac{1}{2})}[(m+\frac{1}{2})p\mu]}{(2m+1)^2}$ (Cont.)

p = 4

$\mu_0 \backslash \frac{\mu}{\mu_0}$	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.85	0.9	0.95	0.975
0.25	2.77914	2.45906	2.23636	2.06751	1.93303	1.82245	1.72950	1.68829	1.65007	1.61452	1.59765
0.50	2.23636	1.93303	1.72950	1.58136	1.46853	1.38008	1.30941	1.27932	1.25217	1.22761	1.21621
0.75	1.93303	1.65007	1.46853	1.34284	1.25217	1.18515	1.13485	1.11450	1.09675	1.08125	1.07426
1.00	1.72950	1.46853	1.30941	1.20535	1.13485	1.08619	1.05230	1.03941	1.02866	1.01969	1.01579
1.25	1.58136	1.34284	1.20535	1.12097	1.06772	1.03379	1.01224	1.00469	0.99877	0.99415	0.99225
1.50	1.46853	1.25217	1.13485	1.06772	1.02866	1.00606	0.99336	0.98943	0.98664	0.98475	0.98407
1.75	1.38008	1.18515	1.08619	1.03379	1.00606	0.99190	0.98529	0.98371	0.98288	0.98256	0.98263
2.00	1.30941	1.13485	1.05230	1.01224	0.99336	0.98529	0.98274	0.98263	0.98292	0.98354	0.98392
2.50	1.20535	1.06772	1.01224	0.99059	0.98355	0.98274	0.98434	0.98549	0.98671	0.98792	0.98853
3.00	1.13485	1.02866	0.99336	0.98355	0.98292	0.98525	0.98815	0.98959	0.99079	0.99194	0.99246
3.50	1.08619	1.00606	0.98529	0.98274	0.98525	0.98863	0.99157	0.99279	0.99385	0.99475	0.99515
4.00	1.05230	0.99336	0.98274	0.98434	0.98815	0.99157	0.99412	0.99510	0.99591	0.99657	0.99686
4.50	1.02866	0.98664	0.98292	0.98671	0.99079	0.99385	0.99591	0.99665	0.99724	0.99773	0.99793
5.00	1.01224	0.98355	0.98434	0.98911	0.99295	0.99552	0.99712	0.99768	0.99811	0.99846	0.99860
5.50	1.00096	0.98262	0.98622	0.99119	0.99463	0.99672	0.99795	0.99836	0.99868	0.99893	0.99904
6.00	0.99336	0.98292	0.98815	0.99295	0.99591	0.99758	0.99852	0.99882	0.99906	0.99925	0.99933

TABLE OF $\frac{1+\mu^2}{\mu^2}\frac{8}{\pi^2}\sum_{m=0}^{\infty}\frac{T_{1,p(m+\frac{1}{2})}[(m+\frac{1}{2})p\mu]}{(2m+1)^2}$ (Cont.)

p = 5

$\mu_0 \backslash \frac{\mu}{\mu_0}$	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.85	0.9	0.95	0.975
0.25	1.92687	1.81605	1.72633	1.65069	1.58540	1.52821	1.47760	1.45441	1.43247	1.41170	1.40172
0.50	1.72633	1.58540	1.47760	1.39200	1.32262	1.26568	1.21855	1.19806	1.17935	1.16225	1.15426
0.75	1.58540	1.43247	1.32262	1.24102	1.17935	1.13231	1.09627	1.08152	1.06860	1.05729	1.05218
1.00	1.47760	1.32262	1.21855	1.14661	1.09627	1.06090	1.03610	1.02667	1.01882	1.01231	1.00949
1.25	1.39200	1.24102	1.14661	1.08622	1.04739	1.02256	1.00692	1.00153	0.99735	0.99415	0.9926[illegible]
1.50	1.32262	1.17935	1.09627	1.04739	1.01882	1.00250	0.99361	0.99096	0.98917	0.98803	0.98765
1.75	1.26568	1.13231	1.06090	1.02256	1.00250	0.99261	0.98835	0.98746	0.98711	0.98712	0.98725
2.00	1.21855	1.09627	1.03610	1.00692	0.99361	0.98835	0.98707	0.98723	0.98766	0.98828	0.98963
2.50	1.14661	1.04739	1.00692	0.99174	0.98737	0.98741	0.98900	0.98997	0.99095	0.99188	0.99233
3.00	1.09627	1.01882	0.99361	0.98737	0.98766	0.98978	0.99206	0.99308	0.99400	0.99481	0.99517
3.50	1.06090	1.00250	0.98835	0.98741	0.98978	0.99241	0.99455	0.99540	0.99611	0.99671	0.99697
4.00	1.03610	0.99361	0.98707	0.98900	0.99206	0.99455	0.99630	0.99694	0.99747	0.99789	0.99807
4.50	1.01882	0.98917	0.98766	0.99095	0.99400	0.99611	0.99747	0.99793	0.99830	0.99861	0.99873
5.00	1.00692	0.98737	0.98900	0.99275	0.99551	0.99721	0.99822	0.99857	0.99885	0.99906	0.99915
5.50	0.99890	0.98710	0.99056	0.99428	0.99663	0.99799	0.99875	0.99900	0.99920	0.99935	0.99941
6.00	0.99361	0.98766	0.99206	0.99551	0.99747	0.99851	0.99910	0.99928	0.99942	0.99953	0.99958

TABLE OF $\frac{1+\mu^2}{\mu^2}\frac{8}{\pi^2}\sum \frac{T_{1,\frac{1}{2}(m+\frac{1}{2})}[(m+\frac{1}{2})p\mu]}{(2m+1)^2}$ (Concluded)

p = 6

$\mu_0 \backslash \frac{\mu}{\mu_0}$	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.85	0.9	0.95	0.975
0.25	1.55416	1.50985	1.46884	1.43088	1.39576	1.36326	1.33321	1.31904	1.30541	1.29231	1.28595
0.50	1.46884	1.39576	1.33321	1.27971	1.23401	1.19502	1.16179	1.14708	1.13352	1.12102	1.11515
0.75	1.39576	1.30541	1.23401	1.17774	1.13352	1.09891	1.07193	1.06080	1.05101	1.04241	1.03852
1.00	1.33321	1.23401	1.16179	1.10951	1.07193	1.04515	1.02628	1.01910	1.01315	1.00823	1.00610
1.25	1.27971	1.17774	1.10951	1.06435	1.03488	1.01599	1.00418	1.00016	0.99708	0.99476	0.99383
1.50	1.23401	1.13352	1.07193	1.03488	1.01315	1.00088	0.99437	0.99250	0.99128	0.99055	0.99032
1.75	1.19502	1.09891	1.04515	1.01599	1.00088	0.99366	0.99075	0.99021	0.99010	0.99024	0.99038
2.00	1.16179	1.07193	1.02628	1.00418	0.99437	0.99075	0.99011	0.99036	0.99080	0.99135	0.99166
2.50	1.10951	1.03488	1.00418	0.99304	0.99019	0.99056	0.99197	0.99276	0.99352	0.99424	0.99458
3.00	1.07193	1.01315	0.99437	0.99019	0.99080	0.99260	0.99438	0.99515	0.99583	0.99641	0.99667
3.50	1.04515	1.00088	0.99075	0.99056	0.99260	0.99465	0.99623	0.99684	0.99734	0.99775	0.99793
4.00	1.02628	0.99437	0.99011	0.99197	0.99438	0.99623	0.99745	0.99791	0.99827	0.99857	0.99870
4.50	1.01315	0.99128	0.99080	0.99352	0.99583	0.99734	0.99827	0.99860	0.99886	0.99906	0.99915
5.00	1.00418	0.99019	0.99197	0.99490	0.99691	0.99810	0.99881	0.99904	0.99922	0.99937	0.99943
5.50	0.99821	0.99020	0.99322	0.99603	0.99769	0.99863	0.99916	0.99933	0.99946	0.99956	0.99960
6.00	0.99437	0.99080	0.99438	0.99691	0.99827	0.99900	0.99939	0.99952	0.99961	0.99968	0.99971

TABLE CF $\quad - \frac{1+\mu^2}{\mu^2} \cdot \frac{2}{\pi} \sum_{m=0}^{\infty} a_m^* \frac{I_{p(m+\frac{1}{2})}\left[(m+\frac{1}{2})\, p\, \mu\right]}{I_{p(m+\frac{1}{2})}\left[(m+\frac{1}{2})\, p\, \mu_0\right]}$

p = 3

$\mu_0 \backslash \frac{\mu}{\mu_0}$	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.85	0.9	0.95	0.975
0.25	1.32686	1.09088	0.95565	0.86978	0.81429	0.78160	0.77078	0.77564	0.79116	0.82704	0.86377
0.50	1.09146	0.90724	0.80676	0.74814	0.71614	0.70512	0.71556	0.73136	0.75842	0.80716	0.85146
0.75	0.82757	0.70030	0.63778	0.60883	0.60245	0.61526	0.64943	0.67765	0.71805	0.78198	0.83551
1.00	0.60092	0.52097	0.48957	0.48477	0.49931	0.53191	0.58635	0.62552	0.67798	0.75618	0.81870
1.25	0.42819	0.38245	0.37300	0.38501	0.41421	0.46106	0.53085	0.57870	0.64110	0.73162	0.80227
1.50	0.30329	0.28041	0.28501	0.30749	0.34590	0.40214	0.48287	0.53736	0.60776	0.70873	0.78662
1.75	0.21486	0.20641	0.21922	0.24747	0.29099	0.35291	0.44117	0.50068	0.57751	0.68741	0.77179
2.00	0.15265	0.15282	0.16984	0.20061	0.24633	0.31124	0.40447	0.46778	0.54984	0.66748	0.75771
2.50	0.07796	0.08523	0.10384	0.13400	0.17886	0.24450	0.34241	0.41067	0.50056	0.63101	0.73155
3.00	0.04044	0.04846	0.06461	0.09075	0.13122	0.19356	0.29159	0.36237	0.45757	0.59821	0.70762
3.50	0.02126	0.02793	0.04063	0.06192	0.09677	0.15385	0.24918	0.32080	0.41950	0.56835	0.68553
4.00	0.01129	0.01625	0.02571	0.04241	0.07155	0.12256	0.21343	0.28466	0.38543	0.54093	0.66498
4.50	0.00605	0.00951	0.01633	0.02911	0.05298	0.09777	0.18310	0.25302	0.35474	0.51557	0.64575
5.00	0.00326	0.00559	0.01039	0.02001	0.03926	0.07805	0.15724	0.22518	0.32695	0.49200	0.62766
5.50	0.00177	0.00330	0.00663	0.01376	0.02910	0.06233	0.13514	0.20060	0.30166	0.46999	0.61057
6.00	0.00096	0.00195	0.00423	0.00947	0.02158	0.04980	0.11621	0.17883	0.27858	0.44936	0.59435

TABLE OF $-\dfrac{1+\mu^2}{\mu^2}\dfrac{2}{\pi}\sum_{m=0}^{\infty} a_m^* \dfrac{I_{p(m+\frac{1}{2})}'[(m+\frac{1}{2})p\mu]}{I_{p(m+\frac{1}{2})}'[(m+\frac{1}{2})p\mu_0]}$ (Cont.)

p = 4

$\mu_0 \backslash \frac{\mu}{\mu_0}$	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.85	0.9	0.95	0.975
0.25	0.58896	0.59202	0.59714	0.60527	0.61791	0.63765	0.66977	0.69415	0.72899	0.78585	0.83554
0.50	0.47535	0.48367	0.49602	0.51331	0.53703	0.56991	0.61763	0.65103	0.69608	0.76511	0.82239
0.75	0.34950	0.36273	0.38201	0.40831	0.44326	0.48992	0.55465	0.59820	0.65503	0.73855	0.80514
1.00	0.24350	0.25961	0.28314	0.31534	0.35820	0.41531	0.49394	0.54628	0.61373	0.71094	0.78674
1.25	0.16497	0.18179	0.20671	0.24134	0.28820	0.35162	0.43998	0.49908	0.57522	0.68434	0.76856
1.50	0.11024	0.12624	0.15039	0.18474	0.23244	0.29864	0.39305	0.45705	0.54005	0.65929	0.75109
1.75	0.07321	0.08749	0.10958	0.14190	0.18822	0.25462	0.35219	0.41961	0.50796	0.63584	0.73444
2.00	0.04850	0.06071	0.08009	0.10942	0.15298	0.21778	0.31637	0.38604	0.47855	0.61330	0.71859
2.50	0.02128	0.02943	0.04321	0.06567	0.10187	0.16034	0.25662	0.32825	0.42633	0.57350	0.68909
3.00	0.00938	0.01440	0.02354	0.03973	0.06824	0.11862	0.20907	0.28026	0.38124	0.53737	0.66213
3.50	0.00415	0.00710	0.01290	0.02413	0.04584	0.08796	0.17075	0.23994	0.34184	0.50464	0.63728
4.00	0.00185	0.00352	0.00710	0.01469	0.03082	0.06529	0.13966	0.20579	0.30713	0.47478	0.61423
4.50	0.00083	0.00175	0.00391	0.00895	0.02074	0.04848	0.11433	0.17672	0.27637	0.44735	0.59272
5.00	0.00037	0.00088	0.00216	0.00545	0.01395	0.03601	0.09365	0.15189	0.24899	0.42203	0.57254
5.50	0.00017	0.00044	0.00119	0.00332	0.00939	0.02674	0.07674	0.13063	0.22452	0.39857	0.55354
6.00	0.00008	0.00022	0.00066	0.00203	0.00632	0.01986	0.06288	0.11239	0.20261	0.37674	0.53557

TABLE OF $-\dfrac{1+\mu^2}{\mu^2}\,\dfrac{2}{\pi}\displaystyle\sum_{m=0}^{\infty} a_{2m}^{*}\,\dfrac{I_{p(m+\frac{1}{2})}\left[(m+\frac{1}{2})p\mu\right]}{I_{p(m+\frac{1}{2})}\left[(m+\frac{1}{2})p\mu_0\right]}$ (Cont.)

p = 5

$\mu_0 \backslash \frac{\mu}{\mu_0}$	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.85	0.9	0.95	0.975
0.25	0.26157	0.32191	0.37451	0.42353	0.47216	0.52424	0.58638	0.62556	0.67576	0.75004	0.81084
0.50	0.20696	0.25811	0.30533	0.35385	0.40527	0.46400	0.53704	0.58355	0.64276	0.72857	0.79694
0.75	0.14738	0.18784	0.22916	0.27479	0.32790	0.39276	0.47712	0.53173	0.60125	0.70083	0.77856
1.00	0.09840	0.12915	0.16377	0.20557	0.25808	0.32625	0.41901	0.48038	0.55910	0.67172	0.75878
1.25	0.06330	0.08616	0.11442	0.15142	0.20119	0.26963	0.36717	0.43342	0.51948	0.64342	0.73909
1.50	0.03987	0.05661	0.07918	0.11098	0.15657	0.22288	0.32209	0.39149	0.48313	0.61664	0.72008
1.75	0.02481	0.03692	0.05461	0.08129	0.12196	0.18453	0.28303	0.35419	0.44992	0.59147	0.70191
2.00	0.01532	0.02400	0.03765	0.05960	0.09513	0.15301	0.24910	0.32091	0.41952	0.56784	0.68456
2.50	0.00578	0.01011	0.01792	0.03213	0.05806	0.10553	0.19357	0.26431	0.36589	0.52464	0.65233
3.00	0.00216	0.00426	0.00855	0.01737	0.03551	0.07293	0.15081	0.21835	0.32015	0.48607	0.62287
3.50	0.00081	0.00180	0.00409	0.00940	0.02173	0.05042	0.11765	0.18072	0.28077	0.45134	0.59580
4.00	0.00031	0.00076	0.00196	0.00509	0.01329	0.03486	0.09184	0.14975	0.24666	0.41986	0.57075
4.50	0.00012	0.00033	0.00094	0.00275	0.00813	0.02409	0.07170	0.12417	0.21696	0.39115	0.54744
5.00	0.00005	0.00014	0.00045	0.00149	0.00497	0.01664	0.05598	0.10301	0.19101	0.36486	0.52565
5.50	0.00002	0.00006	0.00022	0.00081	0.00303	0.01149	0.04370	0.08548	0.16829	0.34068	0.50519
6.00	0.00001	0.00003	0.00011	0.00044	0.00185	0.00793	0.03411	0.07093	0.14834	0.31837	0.48592

TABLE OF $-\frac{1+\mu^2}{\mu^2}\frac{2}{\pi}\sum_{m=0}^{\infty} a_m^* \frac{I_{p(m+\frac{1}{2})}[(m+\frac{1}{2})p\mu]}{I_{p(m+\frac{1}{2})}[(m+\frac{1}{2})p\mu_0]}$ (Concluded)

p = 6

μ_0 \ μ/μ_0	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.85	0.9	0.95	0.975
0.25	0.11617	0.17512	0.23520	0.29711	0.36215	0.43301	0.51591	0.56640	0.62902	0.71810	0.78865
0.50	0.09006	0.13774	0.18871	0.24441	0.30685	0.37943	0.46922	0.52551	0.59601	0.69600	0.77407
0.75	0.06208	0.09720	0.13748	0.18517	0.24320	0.31610	0.41233	0.47483	0.55424	0.66724	0.75470
1.00	0.03969	0.06416	0.09467	0.13408	0.18631	0.25718	0.35704	0.42439	0.51155	0.63685	0.73370
1.25	0.02423	0.04076	0.06325	0.09499	0.14064	0.20737	0.30773	0.37814	0.47124	0.60716	0.71270
1.50	0.01438	0.02532	0.04162	0.06663	0.10556	0.16677	0.26504	0.33689	0.43420	0.57895	0.692[illegible]4
1.75	0.00838	0.01554	0.02716	0.04653	0.07907	0.13404	0.22837	0.30035	0.40039	0.55242	0.67286
2.00	0.00483	0.00946	0.01766	0.03243	0.05917	0.10773	0.19689	0.26799	0.36955	0.52750	0.65427
2.50	0.00156	0.00347	0.00742	0.01571	0.03310	0.06957	0.14652	0.21378	0.31558	0.48208	0.61967
3.00	0.00050	0.00126	0.00311	0.00759	0.01849	0.04489	0.10913	0.17082	0.27017	0.44173	0.58813
3.50	0.00016	0.00046	0.00130	0.00366	0.01030	0.02893	0.08128	0.13663	0.23172	0.40563	0.55921
4.00	0.00005	0.00017	0.00054	0.00176	0.00573	0.01863	0.06052	0.10933	0.19900	0.37313	0.53254
4.50	0.00002	0.00006	0.00023	0.00085	0.00319	0.01198	0.04504	0.08750	0.17105	0.34373	0.50779
5.00	0.00001	0.00003	0.00010	0.00041	0.00177	0.00769	0.03351	0.07003	0.14712	0.31702	0.48474
5.50	0.00001	0.00001	0.00004	0.00020	0.00098	0.00494	0.02492	0.05605	0.12660	0.29266	0.46317
6.00	0.00000	0.00001	0.00002	0.00010	0.00055	0.00317	0.01852	0.04485	0.10897	0.27039	0.44293