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THE AXIAL VELOCITY FIELD OF AN OPTIMUM INFINITELY BLADED PROPELLER

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A. J. Tachmindji

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RESEARCH AND DEVELOPMENT REPORT
HYDROMECHANICS LABORATORY

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NOTATION

E (K)	Complete elliptic integral of the second kind
g_a	Distance factor
K(k)	Complete elliptic integral of the first kind
k	Modulus of elliptic function, Equation (9)
$\vec{P}$	Radius vector
$\vec{i},\vec{j},\vec{k}$	Unit vectors
R	Propeller radius
r'	Radius of ring vortex
r,θ,z	Cylindrical co-ordinates
$\vec{s}$	Vector tangential to the vortex ring
$\vec{v}$	Induced velocity
v_a	Axial induced velocity
x,y,z	Rectangular coordinates
Γ	Circulation
λ_i	Advance ratio

ABSTRACT

This report gives the axial induced velocity ahead of an infinitely bladed propeller. The propeller is simulated by a close succession of ring vortices whose strength vary along the propeller radius and which extend from the propeller plane to infinity. The results are compared with those obtained for a propeller represented by a uniform sink disc.

INTRODUCTION

The flow field in the vicinity of a propeller can be calculated by replacing the propeller by means of flow singularities. The usual representation has been to simulate an infinitely bladed propeller in a steady inviscid incompressible fluid by means of a distributed sink distribution over the entire propeller disc. The problem was originally formulated by Dickman[1] for a propeller having a uniform distribution of loading along the radius and the induced velocities resulting for such a distribution has been given in the literature[2]. Furthermore, a propeller with a radially varying distribution has also been simulated by superimposing sink discs with a common center[3].

[1,2,3] References are listed on page 16

The approach of the present method has been to simulate an infinitely bladed propeller with radially varying distribution of loading, by means of a series of ring vortices extending from the propeller to infinity. Such a representation is not expected to give radically different results from the sink representation for relatively large distances from the propeller plane. In the vicinity of the propeller plane, however, the magnitude of the induced velocities will be substantially affected by the mathematical representation of the propeller.

The present report is confined to the formulation and calculation of the induced axial velocities for a propeller having an optimum distribution of loading. Similar formulations, however, can also be performed for both the tangential and radial induced velocities.

Analysis

The induced velocities in the vicinity of a propeller are a result of the system of free vortices which originate at the propeller blades and constitute the propeller slipstream. A propeller with a very large number of blades (i.e., an infinite number of blades) can be considered as shedding a

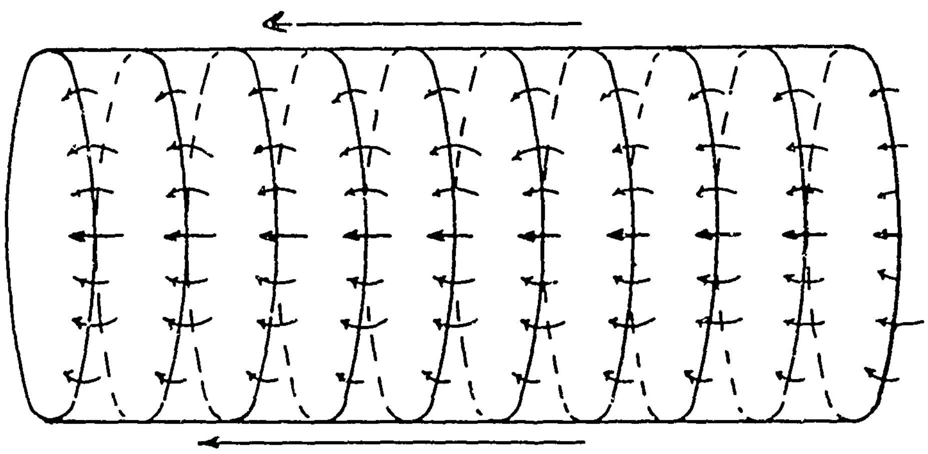

Figure 1a

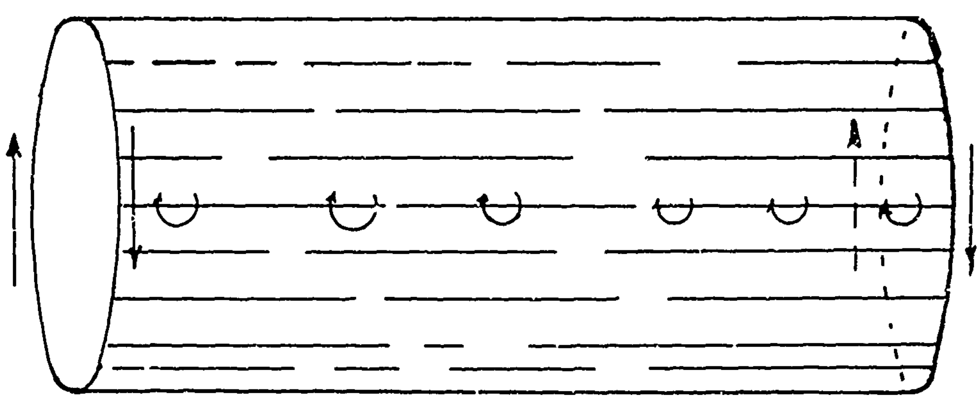

Figure 1b

system of closely spaced helical vortex surfaces which are equivalent to a compact system of helical vortex sheets. These helical surfaces can be resolved into two parts; (1) a close succession of transverse vortex rings with center at the propeller axis and (2) a system of vortex lines which are parallel to the propeller axis (Figures 1a and 1b). The axial induced velocity at any point in space is a result of only the vortex rings, as the parallel vortex lines do not induce an axial component. Similarly, the tangential induced velocities are induced only by the parallel vortex lines.

In order, therefore, to evaluate the axial velocities we will consider a vortex ring of radius r' with the coordinate system r, θ, z and x,y,z with z in the direction of the axis of the ring as shown in Figure 2.

By the law of Biot-Savart

$$d\vec{v} = \frac{\Gamma}{4\pi} \frac{\vec{\rho} \times d\vec{s}}{\rho^3} \qquad (1)$$

giving the velocity vector $d\vec{v}$ induced by an element of length ds = r'dθ' at a point (r,θ,z) or (x,y,z). With the unit vectors $\vec{i},\vec{j},\vec{k}$ in the direction of x,y,z axes we can write

$$\frac{d\vec{s}}{ds} = -\sin\phi' \, \vec{i} + \cos\phi' \, \vec{j} + 0.\vec{k}$$

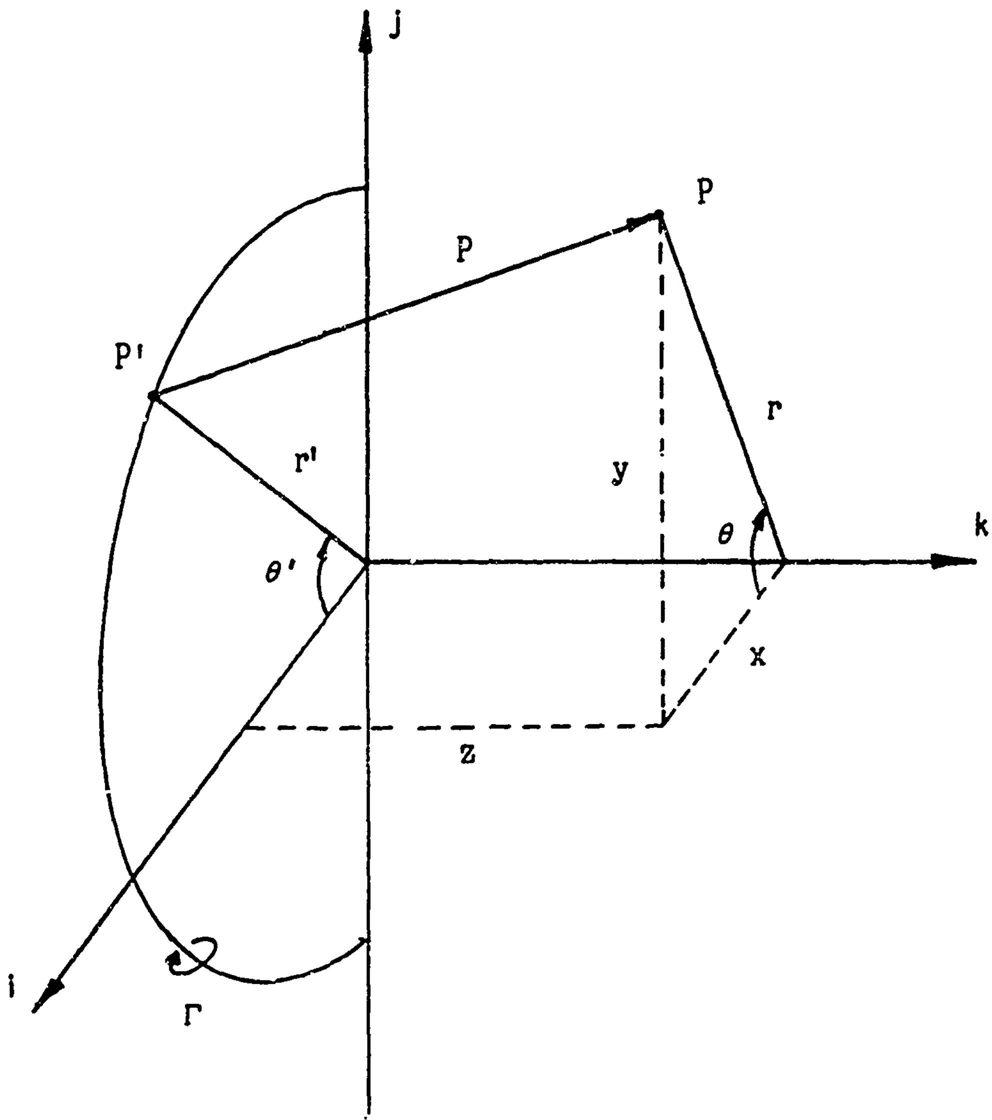

Figure 2.

and $\vec{P}$ is the radius vector from the vortex element to the point P.

$$\vec{P} = (r\cos\theta - r'\cos\theta')\vec{i} + (r\sin\phi - r'\sin\phi')\vec{j} + z\vec{k}$$

and

$$P = \sqrt{z^2 + r^2 + r'^2 - 2rr'\cos(\theta-\theta')} \tag{2}$$

Hence,

$$\vec{P} \times d\vec{s} = z\cos\theta'\,\vec{i} + z\sin\theta'\,\vec{j} - \left[r\cos(\theta-\theta') - r'\right]\vec{k}$$

The z-component of the induced velocity will be obtained by integrating the entire ring.

$$V_a(r,z) = -\frac{\Gamma}{4\pi}\int_0^{2\pi} \frac{r\cos(\theta-\theta') - r'}{\left\{\sqrt{z^2 + r^2 + r'^2 - 2rr'\cos(\theta-\theta')}\right\}^3}\,d\theta' \tag{3}$$

This integration can be performed by means of complete elliptic integrals (Ref. 4) to give

$$V_a(r,z) = \frac{\Gamma}{4\pi}\frac{1}{\sqrt{z^2 + (r+r')^2}}\left\{K(k) - E(k)\left[1 + \frac{2r'(r-r')}{z^2 + (r-r')^2}\right]\right\} \tag{4}$$

where $$k^2 = \frac{4rr'}{z^2 + (r+r')^2} \tag{5}$$

and where K(k) and E(k) are the complete elliptic integrals of the first and second kind.

For a vortex ring placed at the axial position z_0, Equation (4) becomes

$$V_a(r,z) = \frac{\Gamma}{2\pi} \frac{1}{\sqrt{(z-z_0)^2+(r+r')^2}} \left\{ K(k) - E(k)\left[1 + \frac{2r'(r-r')}{(z-z_0)^2+(r-r')^2} \right] \right\} \qquad (6)$$

where

$$k^2 = \frac{4rr'}{(z-z_0)^2+(r+r')^2} \qquad (7)$$

For a propeller having an infinite number of blades, the optimum circulation distribution along the radius is given by

$$\Gamma = C \frac{(r'/R)^2}{(r'/R)^2 + \lambda_i^2}$$

Where C is a constant, λ_i is the advance ratio and R is the propeller radius.

In order, therefore, to find the induced axial velocity for the entire system of ring vortices, Equation (6) can be written as:

$$V_a\left(\frac{r}{R},d\right) = \frac{C}{2\pi}\int_0^\infty\int_0^1 \frac{\xi^2}{\xi^2+\lambda_i^2}\,\frac{1}{[(\eta-d)^2+(r/R+\xi)^2]^{1/2}}\Big\{K(k)$$

$$-E(k)\left[1+\frac{2\xi(r/R-\xi)}{(\eta-d)^2+(r/R-\xi)^2}\right]\Big\}\,d\xi\,d\eta \qquad (8)$$

where $\xi = r'/R$

$\eta = z_o/R$

$d = z/R$

and
$$k^2 = \frac{4\xi(r/R)}{(\eta-d)^2+(r/R+\xi)^2} \qquad (9)$$

The induced velocity at any point (r/R,d) is given by Equations (8) and (9). This velocity can be normalized in terms of the axial induced velocity at the propeller disc. For points ahead of the propeller, we can write

$$g_a = 1 - \frac{V_a(r/R,d)}{V_a(r/R,0)} \qquad (10)$$

where g_a is known as a distance factor for the axial velocity. Similar expressions can easily be obtained for the tangential induced velocity by considering the system of bound vortices and line vortices.

Table I

r/R = 0.0

δ \ 1/A	1.0	2.0	3.0	4.0	5.0	6.0
0.02	0.032	0.036	0.039	0.041	0.044	0.047
0.10	0.156	0.172	0.186	0.197	0.2064	0.214
0.20	0.295	0.319	0.339	0.355	0.367	0.377
0.30	0.412	0.440	0.462	0.478	0.490	0.500
0.40	0.509	0.537	0.559	0.574	0.585	0.594
0.50	0.588	0.615	0.635	0.649	0.659	0.667
0.60	0.652	0.678	0.696	0.708	0.717	0.724
0.70	0.705	0.728	0.744	0.755	0.763	0.769
0.90	0.783	0.801	0.818	0.822	0.829	0.833
1.10	0.833	0.851	0.861	0.867	0.872	0.875
1.50	0.899	0.909	0.915	0.919	0.922	0.924
1.90	0.933	0.939	0.944	0.946	0.948	0.950

r/R = 0.2

0.02	0.036	0.040	0.043	0.045	0.047	0.047
0.10	0.168	0.184	0.196	0.203	0.209	0.212
0.20	0.309	0.331	0.349	0.360	0.366	0.371
0.30	0.425	0.452	0.469	0.480	0.487	0.492
0.40	0.521	0.546	0.564	0.574	0.581	0.586
0.50	0.598	0.622	0.638	0.648	0.654	0.658
0.60	0.660	0.683	0.697	0.706	0.712	0.715
0.70	0.711	0.731	0.745	0.752	0.757	0.761
0.90	0.787	0.803	0.813	0.819	0.823	0.826
1.10	0.838	0.851	0.859	0.864	0.867	0.869
1.50	0.899	0.908	0.913	0.917	0.919	0.920
1.90	0.933	0.939	0.942	0.944	0.946	0.947

Table I (Cont.)

r/R = 0.3

d \ 1/d_i	1.0	2.0	3.0	4.0	5.0	6.0
0.02	0.038	0.041	0.043	0.044	0.045	0.044
0.10	0.176	0.190	0.197	0.200	0.201	0.201
0.20	0.320	0.339	0.349	0.354	0.356	0.357
0.30	0.436	0.457	0.468	0.473	0.476	0.477
0.40	0.529	0.550	0.561	0.566	0.569	0.571
0.50	0.605	0.624	0.634	0.639	0.642	0.643
0.60	0.665	0.683	0.693	0.697	0.700	0.702
0.70	0.715	0.731	0.739	0.744	0.746	0.748
0.90	0.788	0.801	0.808	0.812	0.814	0.815
1.10	0.838	0.849	0.854	0.857	0.859	0.860
1.50	0.899	0.906	0.910	0.912	0.913	0.913
1.90	0.932	0.937	0.939	0.941	0.941	0.942

r/R = 0.4

0.02	0.040	0.042	0.043	0.042	0.041	0.042
0.10	0.184	0.192	0.194	0.193	0.191	0.191
0.20	0.330	0.341	0.343	0.343	0.342	0.341
0.30	0.445	0.457	0.460	0.460	0.459	0.459
0.40	0.537	0.548	0.552	0.553	0.552	0.552
0.50	0.610	0.621	0.624	0.625	0.625	0.625
0.60	0.668	0.679	0.682	0.683	0.683	0.683
0.70	0.716	0.726	0.729	0.730	0.730	0.730
0.90	0.787	0.795	0.798	0.799	0.799	0.800
1.10	0.836	0.843	0.845	0.846	0.847	0.847
1.50	0.896	0.901	0.903	0.903	0.904	0.904
1.90	0.930	0.933	0.934	0.935	0.935	0.935

Table I (Cont.)

$r/R = 0.5$

δ \ y/λ_i	1.0	2.0	3.0	4.0	5.0	6.0
0.02	0.042	0.042	0.041	0.040	0.040	0.039
0.10	0.191	0.191	0.188	0.185	0.183	0.181
0.20	0.337	0.337	0.334	0.330	0.327	0.325
0.30	0.451	0.451	0.447	0.443	0.441	0.438
0.40	0.539	0.540	0.537	0.533	0.531	0.529
0.50	0.609	0.611	0.608	0.605	0.603	0.601
0.60	0.662	0.667	0.665	0.663	0.661	0.659
0.70	0.712	0.714	0.712	0.710	0.708	0.707
0.90	0.782	0.783	0.782	0.781	0.780	0.779
1.10	0.830	0.832	0.831	0.830	0.829	0.828
1.50	0.891	0.892	0.892	0.891	0.891	0.890
1.90	0.925	0.926	0.926	0.926	0.925	0.925

$r/R = 0.6$

0.02	0.043	0.041	0.040	0.039	0.037	0.037
0.10	0.194	0.187	0.179	0.175	0.172	0.170
0.20	0.339	0.328	0.318	0.311	0.307	0.304
0.30	0.448	0.436	0.425	0.418	0.414	0.411
0.40	0.532	0.521	0.511	0.504	0.500	0.497
0.50	0.600	0.589	0.580	0.574	0.570	0.567
0.60	0.654	0.645	0.637	0.631	0.628	0.625
0.70	0.699	0.691	0.684	0.679	0.675	0.673
0.90	0.768	0.762	0.756	0.752	0.750	0.748
1.10	0.817	0.812	0.808	0.805	0.803	0.802
1.50	0.880	0.877	0.874	0.873	0.871	0.870
1.90	0.917	0.915	0.913	0.912	0.911	0.910

Table I (Cont.)

r/R = 0.7

d \ $1/\lambda_i$	1.0	2.0	3.0	4.0	5.0	6.0
0.02	0.043	0.039	0.036	0.034	0.034	0.033
0.10	0.191	0.174	0.163	0.157	0.153	0.151
0.20	0.326	0.302	0.286	0.277	0.272	0.268
0.30	0.427	0.401	0.384	0.373	0.367	0.363
0.40	0.506	0.480	0.463	0.453	0.446	0.442
0.50	0.570	0.546	0.530	0.520	0.514	0.510
0.60	0.623	0.601	0.586	0.577	0.571	0.567
0.70	0.668	0.648	0.634	0.626	0.621	0.617
0.90	0.738	0.722	0.711	0.704	0.700	0.697
1.10	0.790	0.777	0.768	0.763	0.760	0.757
1.50	0.859	0.851	0.845	0.841	0.839	0.837
1.90	0.901	0.895	0.891	0.888	0.887	0.886

r/R = 0.8

0.02	0.039	0.032	0.028	0.026	0.024	0.024
0.10	0.162	0.135	0.120	0.111	0.106	0.103
0.20	0.274	0.232	0.209	0.196	0.188	0.183
0.30	0.358	0.312	0.285	0.270	0.261	0.256
0.40	0.429	0.382	0.355	0.340	0.331	0.325
0.50	0.490	0.446	0.419	0.404	0.395	0.389
0.60	0.544	0.502	0.477	0.463	0.455	0.449
0.70	0.592	0.553	0.530	0.517	0.508	0.503
0.90	0.670	0.637	0.618	0.607	0.600	0.596
1.10	0.730	0.703	0.687	0.678	0.672	0.669
1.50	0.814	0.795	0.784	0.778	0.774	0.772
1.90	0.866	0.853	0.845	0.841	0.838	0.836

Table I (Cont.)

r/R = 0.9

d \ 1/λ	1.0	2.0	3.0	4.0	5.0	6.0
0.02	0.006	-0.016	-0.022	-0.026	-0.028	-0.029
0.10	0.011	-0.051	-0.079	-0.093	-0.101	-0.105
0.20	0.032	-0.057	-0.099	-0.121	-0.133	-0.141
0.30	0.080	-0.020	-0.069	-0.095	-0.111	-0.120
0.40	0.142	0.039	-0.012	-0.041	-0.057	-0.068
0.50	0.208	0.107	0.055	0.026	0.010	0.000
0.60	0.272	0.176	0.126	0.099	0.083	0.072
0.70	0.334	0.244	0.197	0.170	0.155	0.145
0.90	0.445	0.367	0.326	0.304	0.290	0.281
1.10	0.536	0.470	0.436	0.416	0.405	0.397
1.50	0.671	0.624	0.599	0.585	0.577	0.572
1.90	0.753	0.726	0.707	0.697	0.691	0.687

Results

The distance factor g_a, has been computed, using Equations (8), (9), and (10), on the IBM 704 Computer for values of r/R from 0 to 0.9, d from 0 to 1.9 and $1/\lambda_i$ from 1.0 to 6.0. The results are given in Table I, computed to an accuracy of 0.001.

The results can be compared with those obtained from a sink disc representation of a propeller of uniform loading. The comparison is made in Table II, and it is noticed that the difference is considerable, particularly in the vicinity of the disc. This is primarily due to relatively large change of the axial velocity component near the propeller plane, indicating that the approximation of uniform loading along the

Table II

Comparison of g_a with uniform sink disc

	r/R = 0.6			r/R = 0.8		
d	$1/\lambda_i = 1.0$	$1/\lambda_i = 6.0$	Sink disc	$1/\lambda_i = 1.0$	$1/\lambda_i = 6.0$	Sink disc
0.3	0.448	0.411	0.373	0.358	0.256	0.477
0.5	0.600	0.567	0.536	0.490	0.389	0.617
0.7	0.699	0.673	0.646	0.592	0.503	0.700

propeller radius, leads to inaccuracies of the induced velocity field. Examination of the results, shows that the distance factor g_a is larger than that obtained from a uniform disc representation for propeller radii smaller than 0.6, however, for propeller radii of 0.8 or larger the distance factor is smaller. This would indicate that although the sink distribution representation may result in the correct total effect, the radial distribution of such velocity is a sensitive function of the radial distribution of loading.

Conclusions

Comparison of the axial induced velocity in the vicinity of a propeller indicates that the loading distribution is important for points near the propeller plane. The velocities have been calculated for an optimum loading distribution which in general is not too different from the radial distribution usually used. In specific instances, however, it may be necessary to use the actual circulation distribution and this can most easily be performed by calculating the induced velocity resulting from the differences in loading between the actual and optimum distribution.

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