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HIGH SPEED DIGITAL FILTERING BY CONVOLUTION APPROXIMATION

E. B. WEIS, JR., MD

FEBRUARY 1967

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Foreword

This work was conducted in support of Project 7233, "Biological Information Handling Systems and Their Functional Analogs," Task 723301, "Biological Mechanisms for Signal Analysis," and Task 723305, "Theory of Information Handling."

The period of conduct of this work was January 1964 to June 1966.

This technical report has been reviewed and is approved.

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Abstract

This report discusses a technique for digital filtering by convolution approximation which is an acceptable compromise between accuracy and speed. This technique is applicable where high accuracy is not necessary and where a digital computer with elaborate processing capability is not available.

Most requirements for digital filtering can be developed in terms of the approximations suggested in this report. By casting the required filter in a form that is most amenable to numerical computation, the accuracy of approximation is maximized. If power-of-two accuracy is insufficient there is a continuous trade-off between accuracy and speed which involves range division and more bits in the approximation.

SECTION I.

Introduction

It is frequently desirable to be able to accomplish high-speed digital filtering. The need arises for preprocessing of data for online calculations, studies of nonstationary signals where the nonstationarity is primarily due to a certain frequency spectrum and as an alternative to detailed harmonic analysis (to specify three examples) (ref 1, ref 3).

A great deal of effort has been expended to make digital filtering practical in computation time, cost, and accuracy of the technique of digital filtering. There are two general techniques known to the author. These are the convolution or Ormsby filters (ref 1) and the weighted average filters which include the recursive filters.

This discussion is not intended to review the entire problem of digital filtering, but rather to explore a technique for implementing the convolution technique in a manner that is relatively accurate and very fast in computation time for certain digital computers.

SECTION II.

The Theory of Convolution Filters

The output of a linear filter (not necessarily realizable) can be given as follows:

$$F^*(w) = H(w) F(w) \quad (1)$$

where $F^*(w)$ is the Fourier Transform of the filter output

$F(w)$ is the Fourier Transform of the filter input

$H(w)$ is the Fourier Transform of the filter impulse response

Equation 1 has a time domain representation as follows:

$$f^*(\gamma) = \int_{-\infty}^{\infty} h(t) f(t-\gamma) dt \quad (2)$$

where $f^*(\gamma)$ is the filter output

$f(t)$ is the filter input

$h(t)$ is the filter impulse response

A numerical approximation of equation (2), using the trapezoidal rule for integration, can be written as follows:

$$f^*(\gamma_j) = \sum_i [h(t_i) f(t_i - \gamma_j) + h(t_{i+1}) f(t_{i+1} - \gamma_j)] \left[\frac{t_{i+1} - t_i}{2} \right] \quad (3)$$

If the t_i are equally spaced:

$$t_{i+1} - t_i = \Delta t \quad (4)$$

and therefore:

$$f^*(\gamma_j) = \frac{\Delta t}{2} \sum_i [h(t_i) f(t_i - \gamma_j) + h(t_{i+1}) f(t_{i+1} - \gamma_j)] \quad (5)$$

In general $h(t_i)$ is nonzero almost everywhere. However, let us restrict the considerations to the following case:

$$h(t_i) \approx 0 \quad I_1 \Delta t = T_{min} < t_i < T_{max} = I_2 \Delta t \quad (6)$$

Therefore,

$$f^*(\gamma_j) = \frac{\Delta t}{2} \sum_{I_1}^{I_2} [h(t_i) f(t_i - \gamma_j) + h(t_{i+1}) f(t_{i+1} - \gamma_j)] \quad (7)$$

This development is accurate for an arbitrary $f(t)$ but in fact, practical considerations limit the application to cases where:

$$\int_{-\infty}^{\infty} f(t) dt = 0 \quad (8)$$

If the $f(t)$ does not fulfill this condition then write $f(t)$ as follows:

$$f(t) = f_1(t) + A \quad (9)$$

where $\int_{-\infty}^{\infty} f_1(t) dt = 0$

and $\int_{-\infty}^{\infty} f(t) dt = A$

Then equation 7 can be rewritten as follows:

$$f^*(\gamma_j) = \frac{\Delta t}{2} \frac{I_2}{I_1} \sum [h(t_i) (f_1(t_i - \gamma_j) + A) + h(t_{i+1}) (f_1(t_{i+1} - \gamma_j) + A)] \quad (10)$$

$$\begin{aligned} f^*(\gamma_j) = & A \frac{\Delta t}{2} \frac{I_2}{I_1} \sum [h(t_i) + h(t_{i+1})] \\ & + \frac{\Delta t}{2} \frac{I_2}{I_1} \sum [h(t_i) f_1(t_i - \gamma_j) + h(t_{i+1}) f_1(t_{i+1} - \gamma_j)] \end{aligned} \quad (11)$$

or

$$f^*(\gamma_j) = f_1^*(\gamma_j) + A \int_{I_1}^{I_2} h(t) dt \quad (12)$$

This shows that one can correct for a nonzero mean process by subtracting the mean in the beginning and adding the mean times the integral of the impulse response of the filter at the end.

The previous development will suffice for low-pass operations and the following for high-pass operations.

$$f_h^*(t) = f(t) - f_l^*(t) \quad (13)$$

where $f_h^*(t)$ is a desired high-pass operation

$f_l^*(t)$ is the complementary low-pass operation

$f(t)$ is the unfiltered data.

Therefore, a high-pass operation can be accomplished by subtracting from the data the result of passing the data through a low-pass filter whose transfer characteristic is the complement (in terms of the band-pass) of the desired high-pass filter.

Implementation of the convolution filter in this form on a digital computer involves a multiplication and a sum for each t_i for each γ_j . The speed of such a process in certain computers is often a severe limitation of the acceptability of this filtering technique.

SECTION III.

Approximations to the Filter

The primary disadvantage of the previous form lies in the time required for multiplication. However, it is obvious that if the values of the filter impulse response were integral powers of two, slow multiply steps could be replaced with fast shift steps in the digital computer. Therefore, the question arises; how well does the ideal filter characteristic compare with the filter that has values which are the nearest (in some sense) integral power of two?

It is clearly possible to expand $h(t)$ in a binary series to any accuracy desired.

$$h(t_m) = u(t_m) \sum_j a_j 2^j \quad j = 0, 1, 2 \dots \quad (14)$$

where $u(t_m) = \begin{cases} 1 & h(t_m) \geq 0 \\ -1 & h(t_m) < 0 \end{cases}$

$a_j = \begin{cases} 0 & \text{otherwise} \\ 1 & \text{for each } j \text{ corresponding to the} \\ & \text{presence of the } j^{\text{th}} \text{ power of two} \end{cases}$

This is, of course, the way a number is represented in a digital computer and so implementation of a multiplication by expansion in a binary series is equivalent to binary multiplication with N bits of accuracy. Our discussion concerns implementation with only a few bits of accuracy.

Another point is that it is possible to logically divide the range of $h(t)$ in implementing the convolution so that one might expand the filter weights as follows:

$$\begin{aligned} h(t_m) &= u(t_m) \sum_j a_j 2^j & h(t_m) \leq S_1 \\ &= u(t_m) (S_1 + \sum_j a_j 2^j) & S_1 < h(t_m) \leq S_2 \\ &\vdots \\ &= u(t_m) (S_n + \sum_j a_j 2^j) & S_n < h(t_m) \leq S_{n+1} \end{aligned} \quad (15)$$

Where the $\{S_i\}$ are arbitrary level divisions.

For the sake of simplicity consider nearest in the following sense. Choose n_i to be the integer value such that:

$$| h(t_i) - u(t_i) a_i 2^{n_i} | \text{ is a minimum}$$

An integral square error criterion has not been considered. It may be that the approximation could be improved thereby.

Using the power-of-two approximation one has:

$$f_1^*(\gamma_j) = \frac{\Delta t}{2} \sum_{I_1}^{I_2} [u(t_i) a_i 2^{n_i} f_1(t_i - \gamma_j) + u(t_{i+1}) a_{i+1} 2^{n_{i+1}} f_1(t_{i+1} - \gamma_j)] \quad (17)$$

Using the logical level division and the power-of-two approximation one has:

$$f_1^*(\gamma_j) = \frac{\Delta t}{2} \sum_{I_1}^{I_2} [u(t_i) (S_n + a_i 2^{n_i}) f_1(t_i - \gamma_j) + u(t_{i+1}) (S_n + a_{i+1} 2^{n_{i+1}}) f_1(t_{i+1} - \gamma_j)] \quad (18)$$

or

$$\begin{aligned} f_1^*(\gamma_j) = & \frac{\Delta t}{2} S_n \sum_{I_1}^{I_2} [u(t_i) f_1(t_i - \gamma_j) + u(t_{i+1}) f_1(t_{i+1} - \gamma_j)] \\ & + \frac{\Delta t}{2} \sum_{I_1}^{I_2} [u(t_i) a_i 2^{n_i} f_1(t_i - \gamma_j) + u(t_{i+1}) a_{i+1} 2^{n_{i+1}} f_1(t_{i+1} - \gamma_j)] \end{aligned} \quad (19)$$

where

$$S_n < h(t_i) \leq S_{n+1}$$

A computer program has been written for the IBM 7044-7094 direct coupled system to generate the approximation for a filter of the following type:

$$h(t) = \frac{w_c}{\pi} \frac{\sin(w_c t)}{(w_c t)} \quad (20)$$

where $w_c = 2\pi f_c$

f_c is the cutoff frequency

The Fourier Transform for this function is:

$$H(w) = \begin{cases} 1 & -w_c \leq w \leq w_c \\ 0 & \text{otherwise} \end{cases}$$

The time domain graphs for the function and its approximation are illustrated in figure 1.

Using an existing computer program, SYSTRAN (ref 2), the Fourier transforms of the function shown in equation 20 and its power-of-two approximation were calculated. The results are illustrated in figures 2A and 2B.

The filter accuracy for the division of the range of the filter values has not been evaluated.

There is a further consideration in decreasing the time involved in convolving filters of the $(\sin x)/x$ type. This function is even, therefore, the number of shift operations can be reduced by half. Further, one can reduce the number of shift operations again since the function crosses some bit levels more than once. The number of shift operations can also be reduced by using subtract instructions (instead of add) when the filter weight is negative.

$$f_1^*(\gamma_j) = \frac{\Delta t}{2} \sum_{n_i} a_i 2^{n_i} \sum_{2^{n_i}} u(t_i) f_1(t_i - \gamma_j) \quad (21)$$

The internal sum in equation 21 is for all values of $f_1(t_i - \gamma_j)$ which are to be multiplied by a particular 2^{n_i} . The external sum is over all possible 2^{n_i} .

SECTION IV.

The Accuracy of the Approximate Filter

The power-of-two approximation has been found to show an average error (relative to the exact filter) of about 5% in the passband and rolloff range (over a limited frequency range). In the stop-band range one finds that the power-of-two filter admits somewhat more signal energy than the exact filter. However, the approximate filter is still as much as 60 dB down.

The rate of rolloff is primarily determined by the length of the convolution (i.e., I_1 and I_2). Using $2\frac{1}{4}$ cycles of $(\sin x)/x$, the rolloff approaches 140 dB/octave.

One can generate an exact expression for the error of the approximate filter, but it is always in the form of a series. Therefore, short of evaluating the series for an actual case, the error cannot be given a detailed numerical bound.

The appendix shows the results of decreasing the length of the convolution and increasing the interval between samples in the convolution. This material is presented in the form of graphs on which the pertinent parameters are listed (see figures 1 through 6B).

Appendix

Examples of the Spectra of a Filter and its Power-of-Two Approximation

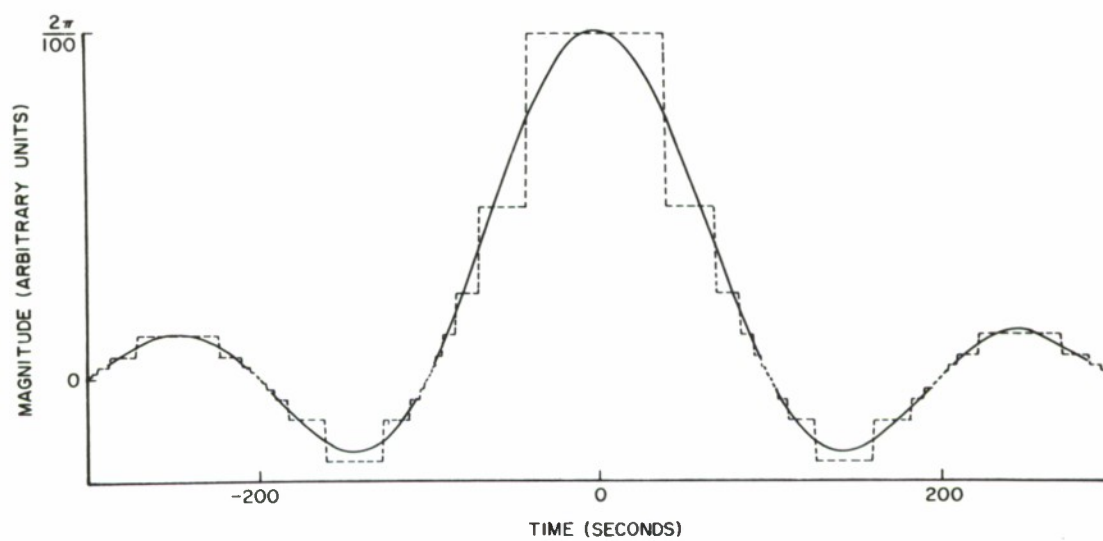


FIGURE 1. Filter Function and Approximation

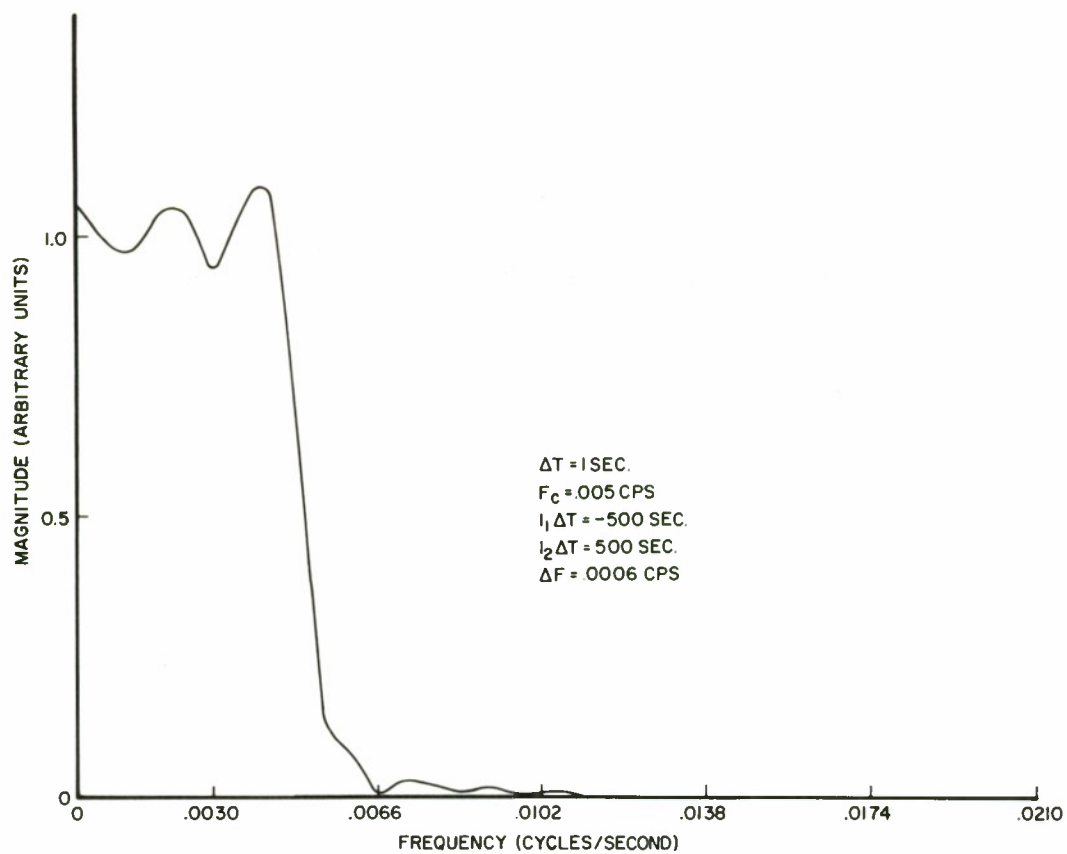


FIGURE 2A. Fourier Spectrum of the Filter

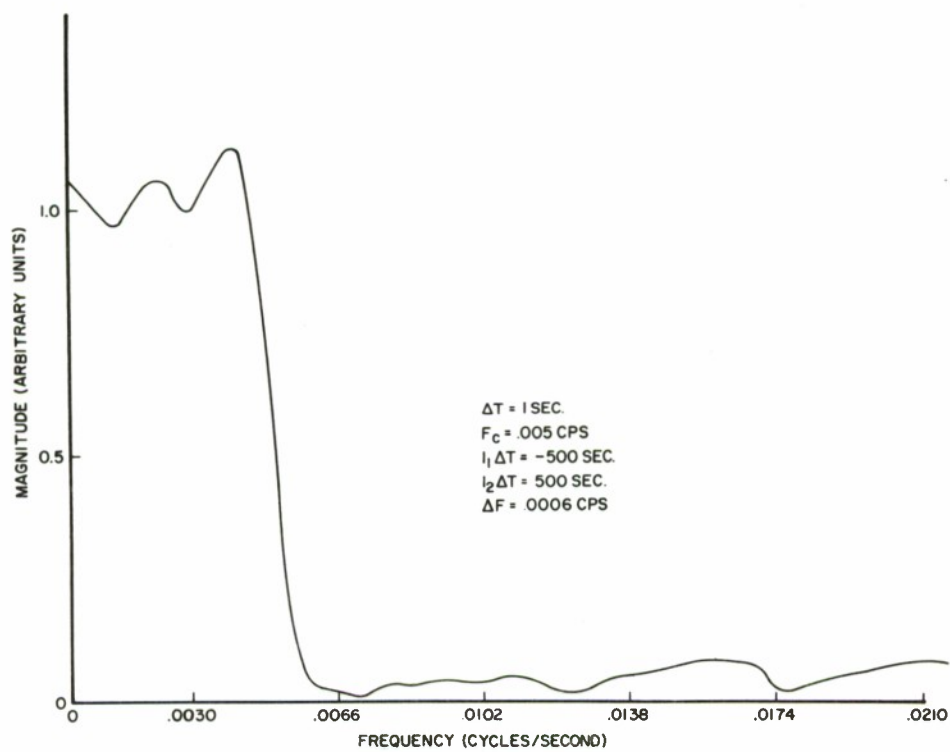


FIGURE 2B. Fourier Spectrum of the Filter Approximation

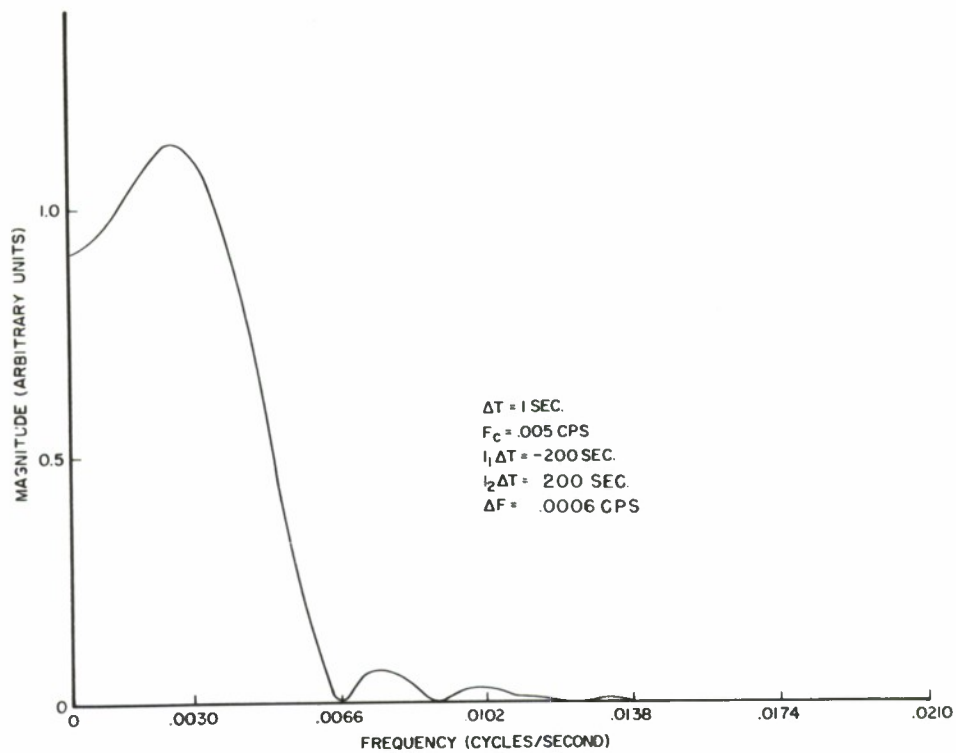


FIGURE 3A. Fourier Spectrum of the Filter

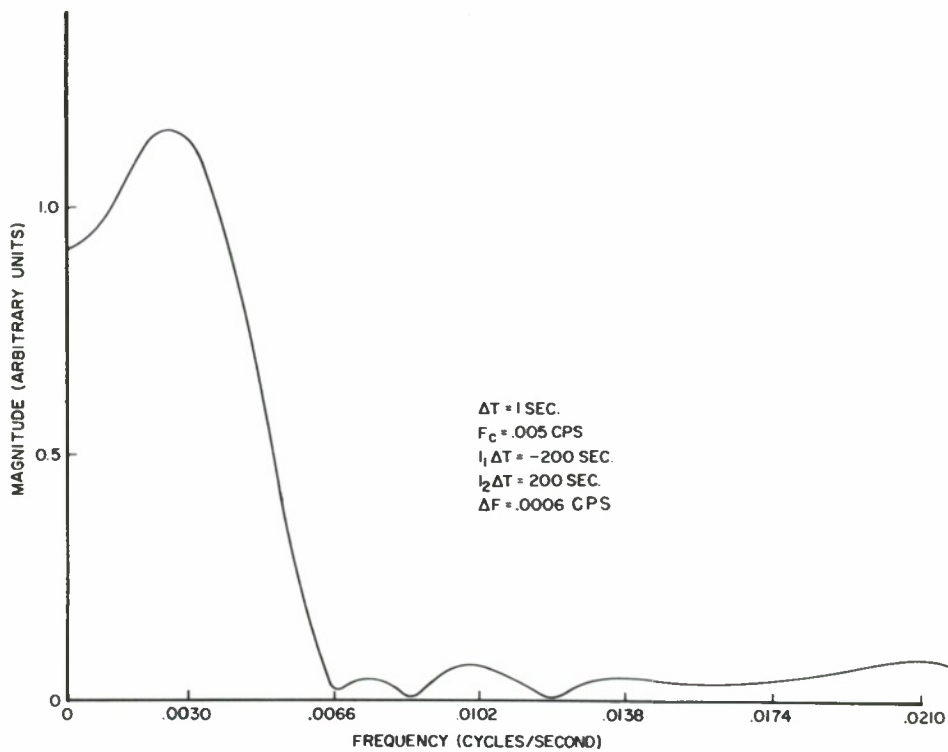


FIGURE 3B. Fourier Spectrum of the Filter Approximation

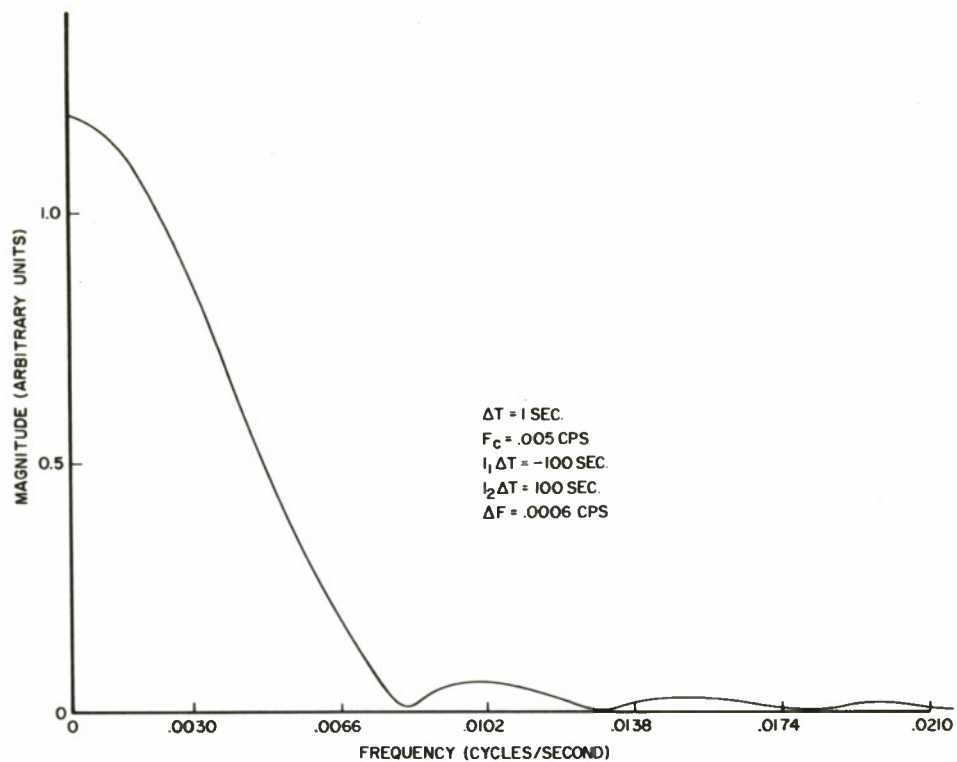


FIGURE 4A. Fourier Spectrum of the Filter

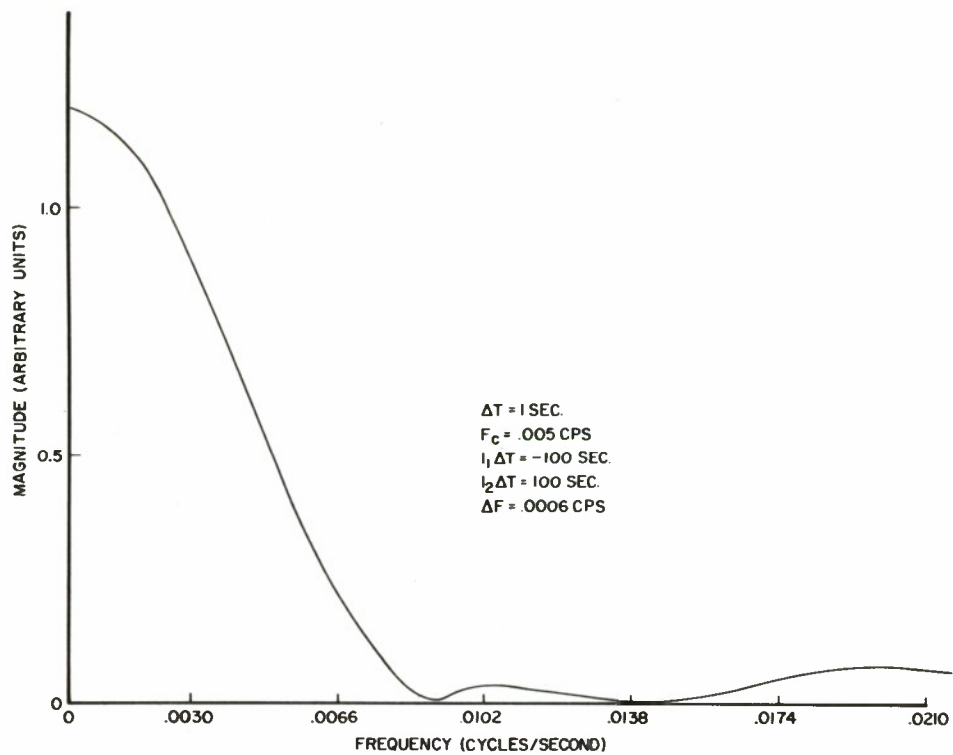


FIGURE 4B. Fourier Spectrum of the Filter Approximation

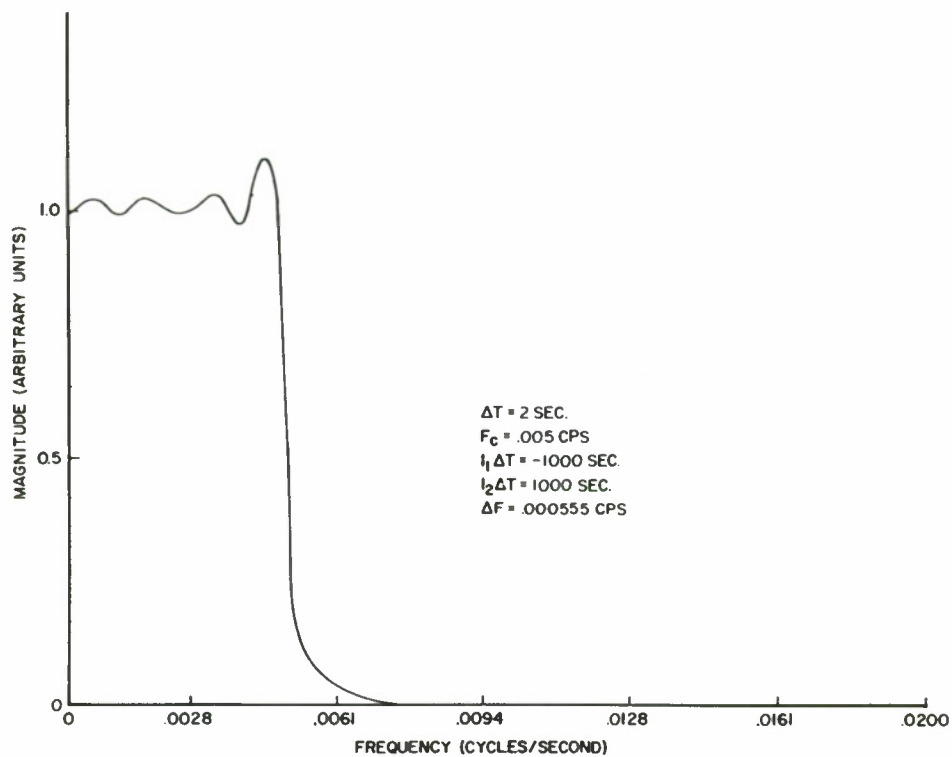


FIGURE 5A. Fourier Spectrum of the Filter

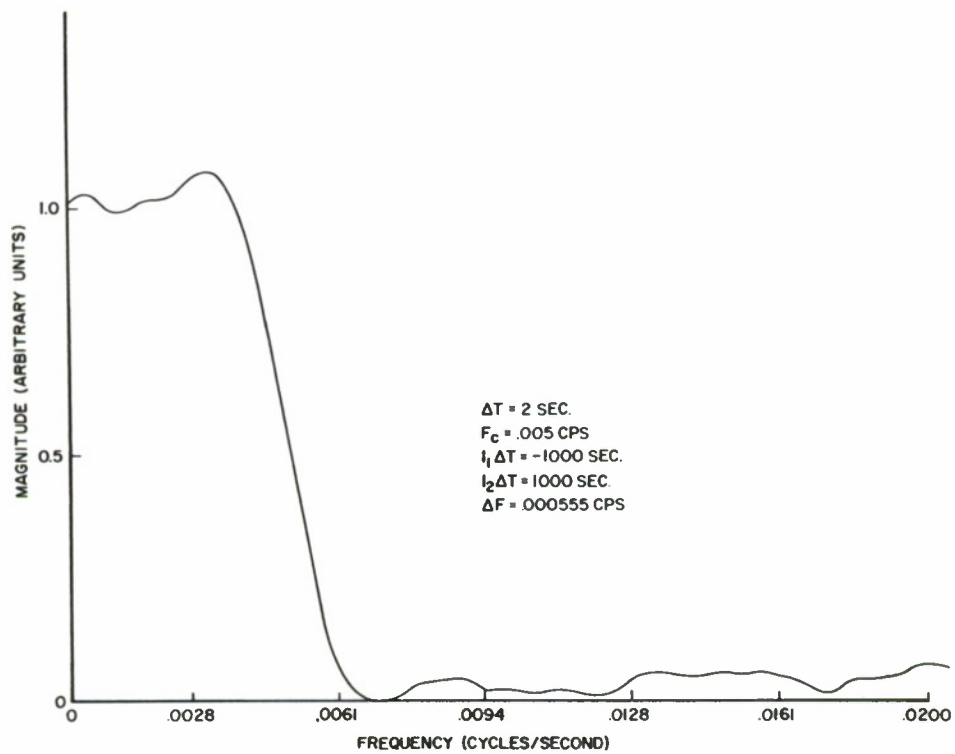


FIGURE 5B. Fourier Spectrum of the Filter Approximation

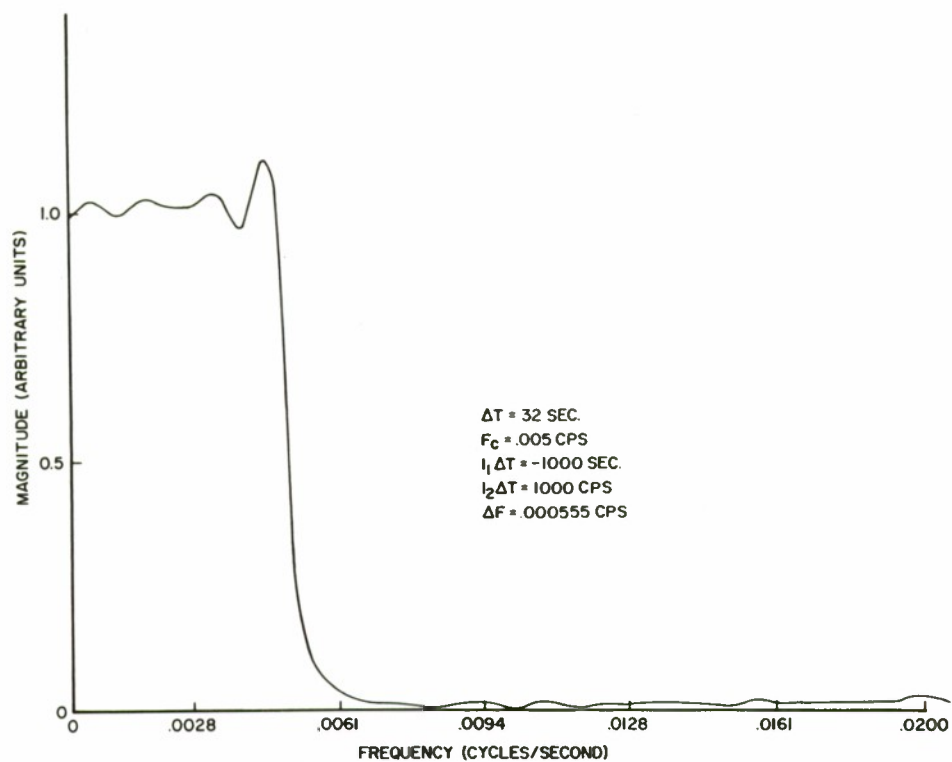


FIGURE 6A. Fourier Spectrum of the Filter

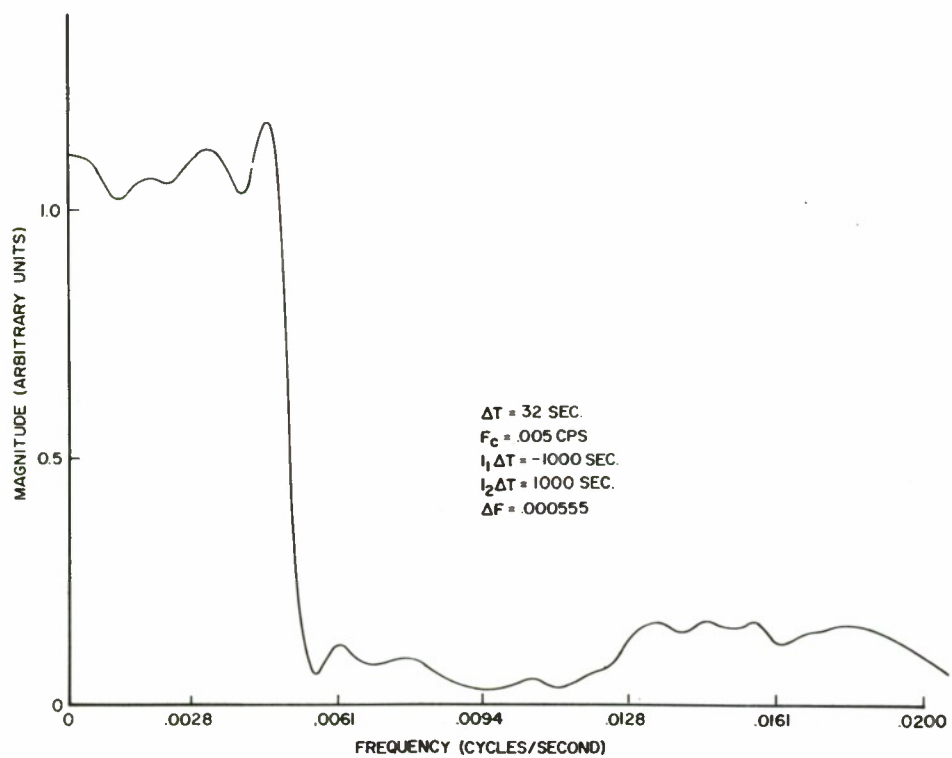


FIGURE 6B. Fourier Spectrum of the Filter Approximation

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