

AD 653071

California Institute of Technology  
Department of Mathematics

**REDUCED POWERS  
OF THE REAL NUMBER SYSTEM  
AND  
EQUIVALENTS OF THE  
HAHN-BANACH EXTENSION THEOREM**

by

W. A. J. Luxemburg

Technical Report No. 2

Prepared under Contract Nonr N00014-66-C0009-A04

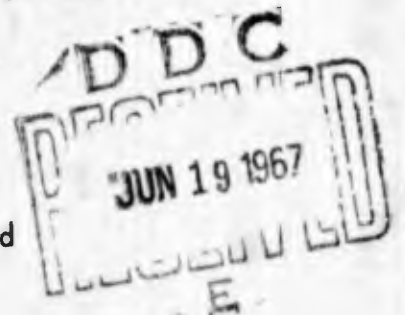
(Nr-041-339)

For

Office of Naval Research

April 1967

Distribution of this document is unlimited



**ARCHIVE COPY**

CALIFORNIA INSTITUTE OF TECHNOLOGY  
Department of Mathematics

REDUCED POWERS OF THE REAL NUMBER SYSTEM  
AND  
EQUIVALENTS OF THE HAHN-BANACH EXTENSION THEOREM

by

W. A. J. Luxemburg

Technical Report 2

Prepared under Contract Nonr N00014 - 66 - C0009 - A04

(Nr.-041-339)

For

Office of Naval Research

May, 1967

Distribution of this document is unlimited.  
Reproduction in whole or in part is permitted for any purpose of the  
United States Government.

## 1. Introduction

There are a number of axioms in set theory such as the axiom of choice (AC), the continuum hypothesis (CH) and the prime ideal theorem for Boolean algebras (PI) which have been studied extensively from various points of view. In particular, there exist detailed studies concerning the question which other axioms of set theory are effectively equivalent to these axioms. For such results the reader is referred to the book of Sierpinski on the continuum hypothesis (see [1]) and Rubin and Rubin's book on the axiom of choice (see [2]).

One of the main principles of functional analysis is the so-called Hahn-Banach extension theorem (HB). It was shown in [3] and [4] by different methods that (PI) implies (HB). The earlier proofs of (HB) were based on (AC). Thus we have  $(AC) \Rightarrow (PI) \Rightarrow (HB)$ . It is known that the former implication cannot be reversed (see [5]) whether the same holds for the latter implication seems to be an open question. In the present paper we are mainly interested in the problem which other statements in mathematics can be shown to be effectively equivalent to (HB)? A number of such results are given below which culminate in the result that (HB) is effectively equivalent to the statement that every Boolean algebra admits a non-trivial real measure. We do not include the well-known equivalent forms of (HB) which deal with results concerning separation of convex sets in locally convex spaces.

Many of the proofs given below are based on the basic

properties of reduced powers, so that Sections 2, 3 and 4 are devoted to this concept.

For the present paper it is of importance to state explicitly which axioms of set theory we shall assume to hold. These are the axiom groups A, B, C and D of Gödel (see [6]). We shall not make any additional assumption if not otherwise stated. Thus the (AC) (Axiom E in [6]) or any of its weaker forms are not assumed to hold. For the sake of simplicity we shall denote the system of axioms of groups A, B, C and D of [6] by  $\Sigma$ . If a theorem in this paper is stated without further specification it means that it holds in  $\Sigma$ . Furthermore, we shall say (see above) that two statements are effectively equivalent whenever their logical equivalence can be shown in  $\Sigma$ .

## 2. Reduced powers of mathematical structures.

For terminology and notation not explained below we refer the reader to Chapter 1 of [7].

Let  $\mathcal{M}$  be a mathematical structure and let L denote a higher order language with type structure  $\mathcal{T}$ . By  $K(K_0)$  we shall denote all sentences of L which are defined (hold) in  $\mathcal{M}$  with respect to some correspondence between the constants of L and the entities of all finite types of  $\mathcal{M}$ .

Let  $\mathcal{D}$  be a proper filter of subsets of an infinite set I. We shall now describe a new structure  $\mathcal{M}'$  which in some weak sense has the same properties as  $\mathcal{M}$ . As in Chapter 1 of [7], we shall denote by  $M_\sigma$  the set of all entities of type  $\sigma$  of  $\mathcal{M}$  and by

$M = \cup(M_\sigma : \sigma \in J)$  the set of all entities of all finite types of  $\mathcal{M}$ . Thus  $M_0$  denotes here the set of individuals of  $\mathcal{M}$ .

For the sake of convenience we shall denote by  $'M$  the set of all mappings of  $I$  into  $M$ . An element  $a \in 'M$  will be called an admissible entity of  $'\mathcal{M}$  whenever there exists a type  $\sigma \in J$  such that  $\{i : T_\sigma(a(i)) \text{ holds in } \mathcal{M}_i\} \in \mathcal{D}$ , and in that case  $a$  will be called an admissible entity of type  $\sigma$ . As usual we shall say that an admissible entity  $a$  of type  $\sigma$  of  $'\mathcal{M}$  is standard whenever there exists an entity  $b$  of type  $\sigma$  of  $\mathcal{M}$  such that

$\{i : a(i) = b\} \in \mathcal{D}$ . With this definition a 1-1 imbedding of  $\mathcal{M}_i$  into  $'\mathcal{M}$  is obtained. An entity  $a$  of  $\mathcal{M}$  of type  $\sigma \neq 0$  is usually denoted by  $'a$  in  $'\mathcal{M}$  under this imbedding. Individuals, i.e., entities of type 0, of  $\mathcal{M}$  are usually denoted in  $'\mathcal{M}$  by the same symbol. The reader should observe that if  $A$  is a set of entities of type  $\sigma$  of  $\mathcal{M}$ , then  $a \in 'A$  holds in  $'\mathcal{M}$  if and only if  $\{i : a(i) \in A \text{ in } \mathcal{M}_i\} \in \mathcal{D}$ . In a similar fashion the atomic formulae are extended. For instance, if  $\beta$  is a relation of type  $(\sigma_1, \dots, \sigma_n)$  in  $\mathcal{M}$ , then  $\mathcal{R}_{n+1}(\beta, a_1, \dots, a_n)$  holds in  $'\mathcal{M}$  if and only if  $\{i : \mathcal{R}_{n+1}(\beta, a_1(i), \dots, a_n(i)) \text{ holds in } \mathcal{M}_i\} \in \mathcal{D}$ .

It is obvious that all the individuals of  $'\mathcal{M}$  are admissible entities of type 0. A set  $A$  whose elements are admissible entities of type  $\sigma$  of  $'\mathcal{M}$  need not be itself admissible. It is admissible if and only if there exists an admissible entity  $B$  of type  $(\sigma)$  in  $'\mathcal{M}$  such that  $a \in A \Leftrightarrow \{i : a(i) \in B(i) \text{ in } \mathcal{M}_i\} \in \mathcal{D}$ .

Since  $\mathcal{D}$  is not necessarily an ultrafilter all the sentences of  $K_0$  do not necessarily hold in  $\mathcal{M}$ . It can be shown, however, that the fundamental theorem holds in the following restricted sense (see [8], Lemma 2.1). A sentence  $\phi$  of  $K$  is called a Horn sentence whenever written in its prenex normal form (see Section 4 of Chapter 2 of [9]) has a matrix which is a conjunction of (qualifier free) wff which are disjunctions of atomic formulas or negation of atomic formulas, but where at least one atomic formula occurs unnegated.

A Horn sentence of  $K$  holds in  $\mathcal{M}$  if and only if it holds in  $\mathcal{M}$ . Unfortunately, the proof of this result requires the axiom of choice. Since we will be studying equivalent of (HB) in a set theory without (AC) the above result cannot be used. We shall, in fact, only use the fact that the atomic formulas of  $K$  hold in  $\mathcal{M}$  if and only if they hold in  $\mathcal{M}$ . A statement which holds by definition.

### 3. Concurrent binary relations and reduced powers.

Let  $\mathcal{M}$  be a mathematical structure. We recall (see [7], Section 3 of Chapter 1) that a binary relation  $\beta$  of  $\mathcal{M}$  of a certain type is said to be concurrent whenever for every finite set of entities  $a_1, \dots, a_n \in \text{dom}(\beta)$  there is an entity  $b \in \text{ran}(\beta)$  such that  $R_3(\beta, a_1, b)$  and  $\dots$  and  $R_3(\beta, a_n, b)$  holds in  $\mathcal{M}$ .

We continue with the following definition.

Let  $\mathcal{M}$  be a mathematical structure and let  $\beta$  be a concurrent binary relation of say type  $(\sigma_1, \sigma_2)$  of  $\mathcal{M}$ . A higher order reduced power  $\mathcal{M} = \mathcal{M}(I, \mathcal{D})$  of  $\mathcal{M}$  is said to satisfy  $\beta$  whenever there exists an admissible entity  $b$  of type  $\sigma_2$  of  $\mathcal{M}$  such that  $\mathcal{R}_3(\beta, 'a, b)$  holds in  $\mathcal{M}$  for all standard elements  $'a$  of the domain of  $\beta$ , i.e.,  $\{i : \mathcal{R}_3(\beta, a, b(i)) \text{ holds in } \mathcal{M}\} \in \mathcal{D}$  for all standard  $a$  in the domain of  $\beta$ .

We shall now prove the following simple theorem.

**THEOREM 3.1.** Let  $\mathcal{M}$  be a mathematical structure and let  $\beta$  be a concurrent binary relation of  $\mathcal{M}$  of type  $(\sigma_1, \sigma_2)$ . Then the family of sets  $(B_a = \{b : \mathcal{R}_3(\beta; a, b) \text{ holds in } \mathcal{M}\} : a \in \text{dom}\beta)$  has the finite intersection property. Furthermore, the higher or order reduced power  $\mathcal{M} = \mathcal{M}(I, \mathcal{D})$ , where  $I = M_{\sigma_2}$  and  $\mathcal{D}$  is the filter generated by the family of sets  $\{B_a : a \in \text{dom}\beta\}$  satisfies  $\beta$ .

**PROOF.** Let  $a_1, \dots, a_n \in \text{dom}\beta$ . Then  $\beta$  being concurrent there is an element  $b \in M_{\sigma_2}$  such that  $\mathcal{R}_3(\beta, a_i, b)$  holds in  $\mathcal{M}$  for  $i = 1, 2, \dots, n$ , and so  $b \in \bigcap_{i=1}^n B_{a_i}$ . This shows that the family  $(B_a : a \in \text{dom}\beta)$  has the finite intersection property. In order to prove the second part of the theorem consider the identity map  $e$  of  $I = M_{\sigma_2}$  onto itself. Then  $e$  is an admissible entity of type  $\sigma_2$  of  $\mathcal{M}$ . Furthermore, if  $a \in \text{dom}\beta$ , then  $\{i : \mathcal{R}_3(\beta, a, e(i)) \text{ holds in } \mathcal{M}\} = B_a \in \mathcal{D}$ . Thus  $\mathcal{M}$  satisfies  $\beta$  on the standard elements of its domain, and the proof is complete.

It is easy to derive from the above theorem that if  $\beta_1, \dots, \beta_p$  are a finite number of concurrent relations of type, then there exists a reduced power  $\mathcal{M}$  with the property that  $\mathcal{M}$  satisfies  $\beta_1, \dots, \beta_p$  simultaneously in the sense defined above. For this purpose consider the binary relation  $\beta = \beta_1 \times \dots \times \beta_p$  with domain  $\text{dom}\beta_1 \times \dots \times \text{dom}\beta_p$ . It is obvious that  $\beta$  is concurrent. Furthermore, any reduced power which satisfies  $\beta$  on its domain also satisfies  $\beta_1, \beta_2, \dots,$  and  $\beta_p$ . Without the axiom of choice it does not seem possible to show that there are reduced powers which satisfies the elements of an infinite set of concurrent binary relation. If the axiom of choice holds, then the adequate filters introduced in Section 3 of Chapter 1 of [7] can be used to satisfy all the concurrent binary relations in a reduced power. Fortunately, in the proofs which will follow below we shall only use one concurrent binary relation at a time.

REMARK. If the set of individuals of a mathematical structure  $\mathcal{M}$  is infinite, then the binary relation "not ( $x = y$ )" restricted to the set of individuals is concurrent, and conversely. Thus a reduced power which satisfies "not ( $x = y$ )" is a proper extension of the given mathematical structure. In that case, the filter  $\mathcal{F}$  is  $\mathfrak{S}$ -incomplete (see [7], Section 4 of Chapter 1).

#### 4. Reduced powers of the real number system.

Let  $R$  denote the real number system and consider a higher order reduced power of the theory of real numbers with respect

to some proper filter  $\mathfrak{D}$  of subsets of some set  $I$ . The set of individuals of this reduced power will be denoted by  $'R$ . Thus  $R \subset 'R$ . The operations of addition and multiplication of  $R$  are extended to  $'R$  as well as the ordering of  $R$ . We recall the definition. If  $a, b, c \in 'R$ , then  $a + b = c$  holds in  $'R$  if and only if  $\{i : a(i) + b(i) = c(i)\} \in \mathfrak{D}$ , and  $ab = c$  if and only if  $\{i : a(i)b(i) = c(i)\} \in \mathfrak{D}$ ; and  $a \leq b$  if and only if  $\{i : a(i) \leq b(i)\} \in \mathfrak{D}$ . The reader can easily check that under these operations  $'R$  is an ordered commutative ring with a unit element, namely the unit 1 of  $R$ , of which  $R$  is a subfield. Since  $\mathfrak{D}$  is not necessarily an ultrafilter we cannot show that  $'R$  has no divisors of zero. In general,  $'R$  will have divisors of zero. Furthermore, the ordering of  $'R$  does not in general totally order  $'R$  but it does satisfy the two basic properties of being translation invariant and  $a \geq 0, b \geq 0$  implies  $ab \geq 0$ . It is important to observe that  $'R$  is a lattice. Indeed, let  $a, b \in 'R$ , then the mapping  $i \rightarrow c(i) = \max(a(i), b(i))$  of  $I$  into  $R$  determines an element  $c \in 'R$  such that  $c = \max(a, b)$ , and similarly  $\min(a, b)$  exists. For every  $a \in 'R$  we shall write  $a^+ = \max(a, 0)$ ,  $a^- = \max(-a, 0)$ . Then  $a = a^+ - a^-$  and we denote  $a^+ + a^-$  by  $|a|$ . To the structure of  $'R$  one usually refers to as the structure of a commutative Riesz ring with unit, which in our case has the additional property that it is an extension of  $R$ .

$'R$  as a Riesz space is not Archimedean ( $a \geq 0$  and  $na \leq 1$  for all  $n = 1, 2, \dots$  implies  $a = 0$ ) if and only if  $\mathfrak{D}$  is  $\delta$ -incomplete.

This justifies the following definition. An element  $a \in 'R$  will be called an infinitesimal whenever  $|a| < r$  for all real numbers  $0 < r \in R$ . The set of all infinitesimal elements will be denoted by  $\mathcal{L}_1$ . An element  $a \in 'R$  is called finite whenever there exists a positive real number  $0 < r \in R$  such that  $|a| < r$ . The set of all finite elements of  $'R$  will be denoted by  $\mathcal{L}_0$ . It is easy to see that  $\mathcal{L}_0$  is a Riesz subring of  $'R$  and that  $\mathcal{L}_1$  is an ideal of  $\mathcal{L}_0$  with the property that  $a \in \mathcal{L}_1$  and  $|b| < |a|$  implies  $b \in \mathcal{L}_1$ . From now on we shall consider  $\mathcal{L}_0$  as a lattice ordered vector space (= Riesz space) over  $\mathbb{R}$ .

In the Riesz space  $\mathcal{L}_0$  the multiplicative unit 1 is a strong order unit  $e$ , i.e., for every  $a \in \mathcal{L}_0$  there exists a real number  $0 < r \in \mathbb{R}$  such that  $|a| < re$ . This shows that  $\mathcal{L}_0$  is a very special Riesz space. We can introduce on  $\mathcal{L}_0$  a Riesz semi-norm by means of the following definition.

(\*) For every  $a \in \mathcal{L}_0$  we define  $\rho(a) = \inf\{r : 0 < r \in \mathbb{R} \text{ and } |a| \leq r\}$

Then it is easy to verify that  $\rho$  has the following properties.

- i) For all  $a \in \mathcal{L}_0$ ,  $\rho(a) = \rho(|a|)$  and for all  $0 < r \in \mathbb{R}$ ,  $\rho(r) = r$
- ii)  $\rho$  is a seminorm and  $\rho(a) = 0$  if and only if  $a \in \mathcal{L}_1$ .
- iii)  $\rho$  is submultiplicative, i.e.,  $\rho(ab) \geq \rho(a)\rho(b)$  for all  $a, b \in \mathcal{L}_0$ .
- iv)  $\rho(\max(|a|, |b|)) = \max \rho(|a|, |b|)$  for all  $a, b \in \mathcal{L}_0$ .

The reader will recognize immediately that the Riesz space  $(\mathcal{L}_0, \rho)$  is an abstract M-space in the sense of Kakutani (see [10], p. 100), and  $\mathcal{L}_0/\mathcal{L}_1$  is the corresponding Archimedean

quotient space containing  $R$ . The space  $(\mathcal{L}_0, \rho)$  will play a fundamental role in what follows.

5. The Hahn-Banach extension theorem.

The Hahn-Banach extension theorem which we have been referring to is the following result.

THEOREM (HB). Let  $V$  be a linear subspace of a real linear space  $E$  and let  $p$  be a sublinear functional on  $E$  (i.e.,  $p(x + y) \leq p(x) + p(y)$  for all  $x, y \in E$  and  $p(rx) = rp(x)$  for all  $0 \leq r \in \mathbb{R}$  and  $x \in E$ ). If  $\varphi$  is a linear functional defined on  $V$  such that  $\varphi(x) \leq p(x)$  for all  $x \in V$ , then there exists a linear functional  $\psi$  on  $E$  such that  $\psi = \varphi$  on  $V$  and  $\psi(x) \leq p(x)$  for all  $x \in E$ .

The proof of this result is based on the following weaker form of it due to Banach, (see [11], p.27).

THEOREM (B). Under the same hypothesis as in the H.B. extension theorem, if  $x_1, \dots, x_n \in E$  and if  $V_n$  is the linear subspace generated by  $V$  and the elements  $x_1, \dots, x_n$ , then there exists a linear functional  $\psi_n$  on  $V_n$  such that  $\varphi = \psi_n$  on  $V$  and  $\psi_n(x) \leq p(x)$  for all  $x \in V_n$ .

(HB) follows from (B) upon the application of (AC) or (PI) (see [3] and [4]). Whereas the proof of theorem (B) is constructive in the sense that the existence of  $\psi_n$  can be shown in  $\Sigma$ .

If we examine Theorem B a little closer then we will

recognize that it expresses the fact that the following binary relation is concurrent. "x is an element of E and  $\psi$  is a linear functional defined on some linear subspace of E such that  $V \subset \text{dom}\psi$  and  $\varphi = \psi$  on V and  $x \in \text{dom}\psi$  and  $\psi \leq p$  on  $\text{dom}\psi$ ."

Thus from Theorem 3.1 the following kind of extension theorem holds in  $\Sigma$ .

**THEOREM 5.1.** Let V be a linear subspace of E and let p be a sublinear functional defined on E. If  $\varphi$  is a linear functional defined on V such that  $\varphi(x) \leq p(x)$  for all  $x \in V$ , then there exists a higher order reduced power structure of the theory of real numbers and a linear mapping of T of E into  $\mathcal{L}_0$  such that  $T = \varphi$  on V and  $T(x) \leq p(x)$  for  $x \in E$ .

This shows that in  $\Sigma$  we can only prove the above weaker form of (HB).

#### 6. On the existence of positive linear functionals on $\mathcal{L}_0$

Let 'R be the set of numbers of some higher order reduced power of the theory of real numbers. Since  $(\mathcal{L}_0, \rho)$  is a linear space with a semi-norm it is natural to inquire whether there there do exist linear functionals on  $\mathcal{L}_0$  which are bounded by  $\rho$ . The answer is contained in the following theorem.

**THEOREM 6.1.** The Hahn-Banach extension theorem is effectively equivalent to the statement. For every reduced power of the theory of real numbers there exists a real linear functional

$\ell$  on the Riesz space  $(\mathcal{L}_0, \rho)$  such that  $\ell(a) \geq 0$  for all  $0 \leq a \in \mathcal{L}_0$ ,  $|\ell(a)| \leq \rho(a)$  for all  $a \in \mathcal{L}_0$  and  $\ell(1) = 1$ .

PROOF. Assume first that (HB) holds. Then for every  $a \in \mathcal{L}_0$  we define  $p(a) = \rho(a^+)$ . Then it is easy to see that  $p$  is sublinear. Indeed, for all  $a, b \in \mathcal{L}_0$  and  $0 \leq r \in \mathbb{R}$  we have  $p(a + b) = \rho(a + b)^+ \leq \rho(a^+ + b^+) \leq \rho(a^+) + \rho(b^+) = p(a) + p(b)$  and  $p(ra) = \rho((ra)^+) = \rho(ra^+) = r\rho(a^+) = rp(a)$ . Furthermore,  $p(1) = \rho(1) = 1$ , and  $a \in \mathcal{L}_1$  implies  $p(a) = \rho(a^+) = 0$ . Let  $V$  be the linear subspace of  $E$  generated by  $1 = e$  and let  $\varphi$  be the linear functional on  $V$  satisfying  $\varphi(r) = r$  for all  $r \in V = \mathbb{R}$ . Then  $\varphi \leq p$  on  $V$ , and so, by (HB), there exists a real linear functional  $\ell$  on  $\mathcal{L}_0$  such that  $\ell(a) \leq p(a)$  for all  $a \in \mathcal{L}_0$  and  $\ell = \varphi$  on  $\mathbb{R}$ , and so  $\ell(1) = 1$ . In order to prove that  $\ell$  is positive observe that if  $0 \leq a \in \mathcal{L}_0$ , then  $(-a)^+ = 0$ , and so  $-\ell(a) = \ell(-a) \leq p(-a) = \rho(1-a)^+ = \rho(0) = 0$  implies  $\ell(a) \geq 0$ . This completes the proof of the first half. For the proof of the second half we use Theorem 5.1. Thus there is a linear transformation  $T$  of  $E$  into  $(\mathcal{L}_0, \rho)$  satisfying  $T = \varphi$  on  $V$  and  $T \leq p$ . Now let  $\ell$  be a positive linear functional on  $\mathcal{L}_0$  such that  $\ell(1) = 1$ . Then  $\psi = \ell \circ T$  is a real linear functional on  $E$  satisfying  $\psi = \ell \circ T = \ell \circ \varphi = \varphi$  on  $V$  and  $\psi(x) = \ell(T(x)) \leq \ell(p(x)) = p(x)$  for all  $x \in E$ . Thus (HB) holds; and the proof is finished.

The above proof also shows that the following result holds.

THEOREM 6.2. The Hahn-Banach extension theorem is effectively equivalent to the following statement. For every reduced power of the theory of  $R$  and every  $0 < a \in \mathcal{L}_0$  such that  $a \notin \mathcal{L}_1$  there exists a positive linear functional  $l$  on  $\mathcal{L}_0$  such that  $l(a) > 0$ ,  $l(1) = 1$  and  $l \leq \rho$ .

7. The main theorem.

The purpose of this section is to establish the equivalence in  $\Sigma$  of (HB) and the statement there exists a (non-trivial) measure on every Boolean algebra. To this end, we shall first present the following introductory remarks.

For the terminology concerning Boolean algebras which is not explained below we refer to [12]. We shall consider only non-degenerate Boolean algebras. A mapping  $m$  of a Boolean algebra  $\mathcal{B}$  into  $R$  is called a measure whenever  $m$  satisfies the following conditions: i)  $m(x) \geq 0$  for all  $x \in \mathcal{B}$ . ii)  $x \wedge y = 0$  implies  $m(x \vee y) = m(x) + m(y)$  for all  $x, y \in \mathcal{B}$  (additivity). iii)  $m(1) > 0$ , where  $1$  is the unit element of  $\mathcal{B}$ .

We shall first prove the following Lemma.

LEMMA 7.1. The following statements are effectively equivalent.

- i) There exists a measure on every Boolean algebra.
- ii) For every proper filter  $\mathcal{J}$  of elements of a Boolean algebra  $\mathcal{B}$  there exists a measure  $m$  on  $\mathcal{B}$  such that  $m(x) = 1$  for all  $x \in \mathcal{J}$ .
- iii) For every proper ideal  $\mathcal{I}$  of elements of a Boolean algebra  $\mathcal{B}$  there exists a measure  $m$  on  $\mathcal{B}$  such that  $m(x) = 0$  for all  $x \in \mathcal{I}$ .

iv) For every Boolean algebra  $\mathcal{B}$  and for every element  $0 \neq x \in \mathcal{B}$  there exists a measure  $m$  on  $\mathcal{B}$  such that  $m(x) \neq 0$ .

PROOF. The only implication which needs to be shown is  $i \Rightarrow ii$ ). If  $\mathcal{J}$  is a proper filter of elements of a Boolean algebra  $\mathcal{B}$ , then the set of the complements of the elements of  $\mathcal{J}$  is a proper ideal  $\mathcal{I}$ . Then  $i$ ) implies that there exists a measure  $m$  on the quotient algebra  $\mathcal{B}/\mathcal{I}$  such that  $m(1) = 1$ . Then this measure defines a measure on  $\mathcal{B}$  by saturation with respect to  $\mathcal{I}$  which is equal to 1 on the elements of  $\mathcal{J}$ , and the proof is finished.

According to a result of Tarski it can be shown in  $\Sigma$  that to every Boolean algebra  $\mathcal{B}$  there corresponds a set algebra  $\Lambda(\mathcal{B})$ , i.e., a subalgebra of the Boolean algebra of all subsets of a set, and a Boolean homomorphism  $h$  of  $\Lambda(\mathcal{B})$  onto  $\mathcal{B}$ . Indeed, we may take for  $\Lambda(\mathcal{B})$  the subalgebra of the Boolean algebra of all subsets of  $\mathcal{B}$  generated by the principal ideals of  $\mathcal{B}$  with an obvious definition of  $h$ . On the basis of this result we have the following theorem.

**THEOREM 7.2.** The following two statements are effectively equivalent.

- 1) There exists a measure on every Boolean algebra.
- ii) For every non-empty set  $X$  and for every proper filter  $\mathcal{J}$  of subsets of  $X$ , there exists a measure  $m$  on the Boolean algebra of all subsets of  $X$  such that  $m(E) = 1$  for all  $E \in \mathcal{J}$ .

iii) For every non-empty set  $X$  and for every proper ideal  $\mathcal{I}$  of subsets of  $X$ , there exists a measure  $m$  on the Boolean algebra of all subsets of  $X$  such that  $m(A) = 0$  for all  $A \in \mathcal{I}$ .

PROOF. We have only to show that ii)  $\Rightarrow$  i). To this end, let  $\mathcal{B}$  be a Boolean algebra. Then by Tarski's theorem there exists a subalgebra  $\Lambda$  of the Boolean algebra of all subsets of  $\mathcal{B}$  such that  $\mathcal{B}$  is a Boolean homomorphic image of  $\Lambda$ . Let  $\mathcal{I}_0$  be the kernel of that homomorphism  $h$ . Then  $\mathcal{I}_0$  is a proper ideal of  $\Lambda$  ( $\mathcal{B}$  is non-degenerate). Let  $\mathcal{I}$  be the ideal in the Boolean algebra of all subsets of  $\mathcal{B}$  generated by  $\mathcal{I}_0$ . Then  $\mathcal{I}$  is also a proper ideal. Hence, by iii) which is obviously equivalent to ii) there is a measure  $m$  on the Boolean algebra of all subsets of  $\mathcal{B}$  such that  $m(\mathcal{I}) = 0$ . Then the restriction  $m_0$  of this measure  $m$  to  $\Lambda$  is non-trivial since  $m(\mathcal{B}) \neq \{0\}$ ; and  $m_0(\mathcal{I}_0) = 0$ . Since  $\Lambda/\mathcal{I}_0$  is isomorphic to  $\mathcal{B}$  it follows that  $\mathcal{B}$  admits a measure in the sense defined above; and the proof is finished.

REMARK. It is not without interest to observe that the above results in this section are analogous to the following list of effectively equivalent statements (see [12], paragraph 47).

- a) Every Boolean algebra has a prime ideal.
- b) Every Boolean algebra has a two-valued measure.
- c) Every ideal is contained in a prime ideal.

d) Every proper filter of subsets of a non-empty set is contained in an ultrafilter.

We are now in a position to prove the main theorem of this paper.

**THEOREM 7.3.** The Hahn-Banach extension theorem is effectively equivalent to the statement that every Boolean algebra admits a (non-trivial) measure.

**PROOF.** Assume that the Hahn-Banach extension theorem holds. In order to prove that every Boolean algebra admits a measure it is sufficient to show according to Theorem 7.2 that for every non-empty set  $X$  and every proper ideal  $\mathcal{J}$  of subsets of  $X$  there exists a measure  $m$  on the Boolean algebra of all subsets of  $X$  such that  $m(A) = 0$  for all  $A \in \mathcal{J}$ . To this end, consider the Banach space  $B(X)$  of all bounded real functions on  $X$  with the sup norm. Let  $V = \{f: f \in B(X) \text{ and } \{x: f(x) \neq 0\} \in \mathcal{J}\}$ . Then  $V$  is a proper closed linear subspace of  $B(X)$  such that  $f \in V$  implies  $|f| \in V$ . Hence, it follows from the Hahn-Banach extension theorem applied to  $B(X)$ ,  $V$  and  $p(f) = \sup\{f^+(x) : x \in X\}$  that there exists a positive linear functional  $\varphi$  on  $B(X)$  such that  $\varphi(1) = 1$  and  $\varphi(V) = 0$ . Define  $m(E) = \varphi(c_E)$ , where  $c_E$  is the characteristic function of  $E \subset X$ . Then  $m$  is the required measure.

The converse is a little harder to show. According to Theorem 6.1 we have to show that for every  $(I, \mathcal{D})$ -reduced

power of the theory of  $R$  there exists a positive linear functional  $\ell$  on  $(\mathcal{L}_0, \rho)$  such that  $\ell(\mathcal{L}_1) = 0$  and  $\ell(1) = 1$ . From the hypothesis it follows that there exists a measure  $m$  on the Boolean algebra of all subsets of  $I$  such that  $m(D) = 0$  for all  $D \in \mathcal{D}$ . For every finitely valued real function  $s = \sum_{i=1}^n s_i c_{E_i}$ ,  $E_i \subset I$   $i = 1, 2, \dots, n$  we define  $\varphi(s) = \sum_{i=1}^n s_i m(E_i)$ . Then  $\varphi$  defines a positive linear functional on the Riesz space  $S(I)$  of all the real step functions. Since every bounded real function can be uniformly approximated by elements of  $S$  it follows immediately that  $\varphi$  can be extended uniquely by limits to  $B(I)$ . Then for every  $a \in \mathcal{L}_0$  we define  $\ell(a) = \varphi(a)$ . In order to justify this definition we have to show that  $a = b \in \mathcal{L}_0$  implies  $\ell(a) = \ell(b)$ . But  $a = b$  if and only if  $\{i : a(i) = b(i)\} \in \mathcal{D}$ , and so  $\varphi(a) = \varphi(b)$ . Furthermore, it is easy to see that  $\ell$  is a positive linear functional on  $(\mathcal{L}_0, \rho)$  such that  $\ell(1) = 1$ . Then also  $\ell(a) = 0$  whenever  $a \in \mathcal{L}_1$  follows readily. This completes the proof of the theorem.

8. On Tarski's extension theorem for measures on Boolean algebras.

In [13], Tarski showed that if  $m_0$  is a measure on a subalgebra  $\mathcal{B}_0$  of a Boolean algebra  $\mathcal{B}$  then there exists a measure  $m$  on  $\mathcal{B}$  such that  $m = m_0$  on  $\mathcal{B}_0$  and the range  $m(\mathcal{B})$  of  $m$  is contained in the closure of the range of  $m_0$ . In fact, he showed that this extension theorem for measures is

effectively equivalent to the prime ideal theorem for Boolean algebras. For a proof of Tarski's result using ultrapowers the reader is referred to [14], Section 4 of Chapter 6.

Tarski's result is based on the following theorem which is related to Tarski's theorem as Theorem (B) is related to (HB).

Let  $\mathcal{B}_0$  be a subalgebra of a Boolean algebra  $\mathcal{B}$  and let  $m_0$  be a measure on  $\mathcal{B}_0$ . Then for every finite set of elements  $x_1, \dots, x_k$  of  $\mathcal{B}$  there exists a measure  $m_k$  on the Boolean algebra  $\mathcal{B}_k$  generated by  $\mathcal{B}_0$  and the set  $\{x_1, \dots, x_k\}$  such that  $m_k = m_0$  on  $\mathcal{B}_0$ , and the range of  $m_k$  is contained in the closure of the range of  $m_0$ .

As in the case of Theorem B of Section 5 we recognize that the above theorem is equivalent to the statement that the following binary relation is concurrent.

" $x$  is an element of  $\mathcal{B}$  and  $m$  is a measure defined on some subalgebra  $\mathcal{B}'$  of  $\mathcal{B}$  such that  $\mathcal{B}_0 \subset \mathcal{B}'$ ,  $m = m_0$  on  $\mathcal{B}_0$  and  $x \in \mathcal{B}'$  and  $m(\mathcal{B}') \subset \overline{m_0(\mathcal{B}_0)}$ ".

Thus using Theorem 3.1 we obtain the following weaker revision of Tarski's theorem in  $\Sigma$ .

**THEOREM 8.1.** Let  $\mathcal{B}_0$  be a subalgebra of a Boolean algebra  $\mathcal{B}$  and let  $m_0$  be a measure on  $\mathcal{B}_0$ . Then there exists a reduced power of the theory of  $R$  and a measure  $m$  on  $\mathcal{B}$  with values in  $(\mathcal{L}_0, \rho)$  such that  $m = m_0$  on  $\mathcal{B}_0$  and  $m(\mathcal{B}) \subset \overline{(m_0(\mathcal{B}_0))}$ , where the latter denotes the closure of the range of  $m_0$  in  $R$ .

If (HB) holds, then according to Theorem 6.1 there exists a positive linear functional  $l$  on the Riesz space of the finite elements of the  $(I, \mathfrak{A})$ -reduced powers of  $R$  such that  $l(1) = 1$  and  $|l(a)| \leq \rho(a)$  for all  $a \in \mathcal{L}_0$ . The positive linear functional

$\ell$  induces on  $B(I)$  the space of bounded real functions a positive linear functional  $\ell'$  by saturation with respect to  $\mathcal{D}$ . Using then the simple fact that  $\ell'(a)$  is contained in the closed convex hull of the range of  $a$  we obtain immediately the following lemma.

LEMMA 8.2. For every  $A \subset \mathbb{R}$  the set  $\{\ell(a) : a \in 'A \cap \mathcal{L}_0\}$  is equal to the closed convex hull of  $A$ .

From Theorem 8.1, Theorem 7.3, Theorem 6.1 and Lemma 8.1 the following result follows.

THEOREM 8.3. The following two statements are effectively equivalent.

- 1) The Hahn-Banach extension theorem.
- ii) Let  $\beta_0$  be a subalgebra of a Boolean algebra  $\beta$  and let  $m_0$  be a real measure on  $\beta_0$ . Then there exists a real measure  $m$  on  $\beta$  such that  $m = m_0$  on  $\beta_0$  and the range of  $m$  is contained in the closed convex hull of the range of  $m_0$ .

It is interesting to compare Theorem 8.3 with Tarski's result quoted above. The only difference is that on the basis of (HB) we can no longer show that there is an extension whose range is in the closure of the range of the given measure. This somehow illustrates the difference between the existence of measures on Boolean algebras and the existence of two-valued measures (= prime ideals) on Boolean algebras.

#### 9. On the existence of strictly positive measures.

A measure  $m$  on a Boolean algebra is called strictly positive

whenever  $m(a) = 0$  implies  $a = 0$ .

It is well-known that there do exist Boolean algebras which do not admit strictly positive real-valued measures (see Section 5 of Chapter 6 of [14]). Nikodym showed, however, that for every Boolean algebra  $\mathcal{B}$  there exists a totally ordered field  $F$  with the property that  $\mathcal{B}$  admits a strictly positive  $F$ -valued measure (see [15] and [14]). It was pointed out by Professor A. Tarski at the meeting on Boolean algebras in Oberwolfach in the summer of 1962 that Nikodym's result is an immediate consequence of the compactness principle of model theory. This bases Nikodym's result on the prime ideal theorem for Boolean algebras rather than on the axiom of choice as in the case in Nikodym's proof. Tarski's proof in the form of ultrapowers was presented in [4] and [14].

If we now assume that the Hahn-Banach extension theorem holds, then we can still prove the following weaker form of Nikodym's theorem.

**THEOREM 9.1.** If the Hahn-Banach extension theorem holds, then for every Boolean algebra  $\mathcal{B}$  there exists a  $(I, \rho)$  - reduced power of  $R$  such that  $\mathcal{B}$  admits a strictly positive measure with values in  $(\mathcal{L}_0, \rho)$ .

**PROOF.** From Lemma 7.1 and Theorem 7.3 it follows that the following binary relation is concurrent.

" $x$  is an element of  $\mathcal{B}$  and  $m$  is a real-valued measure on  $\mathcal{B}$  such that  $m(x) > 0$ ".

Thus the result follows immediately from Theorem 3.1.

10. A lemma of D. Scott

Let  $X$  be a non-empty set. Then by  $\Omega$  we shall denote the Riesz space of all real functions on  $X$  which vanish off a finite set. For every  $f \in R^X$  and every  $\omega \in \Omega$  we define  $\omega(f) = \sum_{x \in X} \omega(x)f(x)$ . Then by means of this definition  $\omega$  defines an order bounded linear functional on the Riesz space  $R^X$ .

Some time ago, Professor Scott kindly pointed out to me that the Hahn-Banach extension theorem is a special case of the following lemma which itself is an immediate consequence of the Tychonov product theorem for compact Hausdorff space (= the prime ideal theorem for Boolean algebras (see [16])).

LEMMA 10.1 (D. Scott). Let  $\Omega_0$  be an arbitrary subset of  $\Omega$  and let  $g_0, f_0 \in R^X$  be such that  $g_0(x) \leq f_0(x)$  for all  $x \in X$ . If for every finite subset  $\{\omega_1, \dots, \omega_n\}$  of  $\Omega_0$  there exists an element  $h \in H = \{h : g_0 \leq h \leq f_0\}$  such that  $\omega_i(h) \geq 0$  for  $i = 1, 2, \dots, n$ , then there exists an element  $h_0 \in H$  such that  $\omega(h_0) \geq 0$  for all  $\omega \in \Omega_0$ .

From the Tychonov product theorem for compact Hausdorff spaces it follows immediately that  $H$  is a compact subset of  $R^X$  in its product topology. Since for every  $\omega \in \Omega$  the set  $H_\omega = \{f : f \in H \text{ and } \omega(f) \geq 0\}$  is a closed convex subset of the compact subset  $H$  it follows from the hypothesis, namely the family  $\{H_\omega : \omega \in \Omega_0\}$  has the finite intersection property, that  $\bigcap_{\omega \in \Omega_0} H_\omega \neq \emptyset$ , and so the conclusion of the lemma follows.

We shall now prove, however, that D. Scott's lemma is effectively equivalent to the Hahn-Banach extension theorem.

THEOREM 10.2. The Hahn-Banach extension theorem is effectively equivalent to D. Scott's lemma.

PROOF. We shall first present Scott's proof of the fact that the Hahn-Banach extension theorem is a consequence of Scott's lemma. To this end, assume that  $\varphi_0$  is a linear functional defined on a linear subspace  $V$  of a real linear space  $E$  such that  $\varphi_0 \leq p$  on  $V$ , where  $p$  is a sublinear functional on  $E$ . Then in order to apply the lemma we take for  $X$  the set  $E$ ;  $g_0$  and  $f_0$  are defined as follows:  $g_0(x) = -p(-x)$ ,  $x \notin V$ ,  $g_0(x) = \varphi_0(x)$ ,  $x \in V$ ,  $f_0(x) = p(x)$ ,  $x \in V$ , and  $f_0(x) = \varphi_0(x)$  for  $x \in V$ ; the system  $\Omega_0$  consists of all linear expressions of the type  $\sum_{i=1}^k r_i f(x_i) - f(\sum_{i=1}^k r_i x_i) = 0$ , where  $x_1, \dots, x_k \in E$  and  $r_1, \dots, r_k \in R$ . Then Theorem B of section 5 shows that the hypothesis of Scott's lemma are satisfied, and so there exists a mapping  $\varphi$  of  $E$  into  $R$  such that  $g_0(x) \leq \varphi(x) \leq f(x)$  for all  $x \in E$  and  $\sum_{i=1}^k r_i \varphi(x_i) = \varphi(\sum_{i=1}^k r_i x_i)$  for all  $x_1, \dots, x_k \in E$  and  $r_1, \dots, r_k \in R$ . Thus  $\varphi$  is linear,  $x \in V$  implies  $\varphi(x) = \varphi_0(x)$  and  $\varphi(x) \leq p(x)$  for all  $x \in E$ . Hence (HB) holds.

In order to prove the converse we observe that Scott's lemma is equivalent to the statement that the following binary relation is concurrent.

" $\omega$  is an element of  $\Omega_0$  and  $f$  is an element of  $H$  such that  $\omega(f) \geq 0$ ."

Hence, it follows from Theorem 3.1 that there exists a reduced power  $'R$  of  $R$  and a mapping  $h$  of  $X$  into  $'R$  such that  $g_0(x) \leq h(x) \leq f(x)$  for all  $x \in X$  and  $\omega(h) \geq 0$  for all  $\omega \in \Omega_0$ . The former condition implies that  $h(x) \in \mathcal{L}_0$  for all  $x \in X$ . Since (HB) holds, there exists, by Theorem 6.1, a positive linear functional  $l$  on  $\mathcal{L}_0$  such that  $l(1) = 1$ . Hence,  $h_0 = l \circ h$  is a mapping of  $X$  into  $R$  such that  $g_0(x) \leq h_0(x) \leq f_0(x)$  for all  $x \in X$ , i.e.,  $h_0 \in H$  and  $\omega(h_0) = \omega(l \circ h) = l(\omega(h)) \geq 0$  for all  $\omega \in \Omega_0$ . This shows that Scott's lemma holds and the proof is finished.

#### 11. Convex compactness.

Let  $E$  be a normed linear space over  $R$  and let  $E'$  be the Banach dual of  $E$ , i.e.,  $E'$  is the Banach space of all bounded linear functionals on  $E$ . Then a well-known and important theorem of Bourbaki and Alaoglu (see [17] and [18]) states that the unit ball  $E'_1 = \{x' : x' \in E' \text{ and } \|x'\| \leq 1\}$  is compact in the weak dual topology  $\sigma(E', E)$ . It is interesting, however, that the theorem of Bourbaki and Alaoglu is effectively equivalent to the prime ideal theorem for Boolean algebras as was shown in [16]. This suggests the question whether it is possible to formulate a property of  $E'_1$  which will turn out to be effectively equivalent to the Hahn-Banach extension theorem? The purpose of this section is to give an answer to this question. The details are as follows.

A subset  $A$  of a topological linear space  $E$  will be called convex compact whenever for any family of closed convex sets  $C_\tau$ ,  $\tau \in T$ , of  $E$  such that the family  $\{A \cap C_\tau : \tau \in T\}$  has the finite intersection property  $\bigcap (A \cap C_\tau : \tau \in T) \neq \emptyset$ .

The reader who is familiar with Smulian's notion of convex compactness which deals with decreasing sequences of closed convex sets should observe that the above notion of convex compactness is stronger.

We shall now formulate an answer to the question raised above.

**THEOREM 11.1** The Hahn-Banach extension theorem is effectively equivalent to the statement that for every normed space  $E$  the unit ball of  $E'$  is convex compact in the  $\sigma(E', E)$ -topology.

**PROOF.** We shall first assume that (HB) holds. Let  $\{C_\tau : \tau \in T\}$  be a family of  $\sigma(E', E)$ -closed subsets of  $E'_1$ , the unit ball of  $E'$ . We shall first show that for every  $\tau \in T$  there exists a subset  $\Omega_\tau$  of  $E$  such that

$$C_\tau = \bigcap \{ x' : x' \in E' \text{ and } (x, x') \leq 1 \text{ for all } x \in \Omega_\tau \}$$

To this end, we have to show that for every  $x'_0 \notin C_\tau$  there exists an element  $x_0 \in E$  such that  $1 < (x_0, x'_0)$  and  $(x_0, x') \leq 1$  for all  $x' \in C_\tau$ . Since  $C_\tau$  is  $\sigma(E', E)$ -closed and  $x'_0 \notin C_\tau$  there exist elements  $x_1, \dots, x_n \in E$  such that

$\max(|(x_1, x' - x'_0)| : 1 = 1, 2, \dots, n) \leq 1$  implies  $x' \notin C_\tau$ .

It is obvious that there is no loss in generality to assume that the elements  $x_1, \dots, x_n$  are linearly independent.

Consider then the mapping  $x' \rightarrow ((x_1, x'), \dots, (x_n, x'))$  of  $E'$  into  $R^n$ . This mapping maps  $C_\tau$  into a convex subset  $K_\tau$  of  $R^n$  and  $x'_0$  is mapped into an element  $y_0$  of  $R^n$  such that  $y_0 \notin K_\tau$  and there exists a sphere with center  $y_0$  which does not intersect  $K_\tau$ . Then the Minkowski separation theorem for finite

dimensional spaces implies that there exists an element  $z_0 \in R^n$

such that  $(y_0, z_0) > 1$  and  $(y, z_0) \leq 1$  for all  $y \in K_\tau$ . Thus

there exist real constants  $c_1, \dots, c_n$  in  $R$  such that

$(y, z_0) = (\sum_{i=1}^n c_i x_i, x')$  for all  $x' \in E'$ , where  $y$  is the image of  $x'$  under the mapping defined above, and so  $\sum_{i=1}^n c_i x_i$  is the

required element. Now, let  $\Omega_0 = \cap (\Omega_\tau : \tau \in T)$ . Then since

the family  $\{C_\tau : \tau \in T\}$  has the finite intersection property

it follows that  $\Omega_0$  has the following property. For every finite

subset  $\{x_1, \dots, x_k\}$  of  $\Omega_0$  there exists an element  $x' \in E'_1$

such that  $(x_1, x') \leq 1, \dots, (x_k, x') \leq 1$ . Then it follows

from Lemma 10.1 and Theorem 10.2 that there exists an element

$x' \in E'_1$  such that  $(x, x') \leq 1$  for all  $x \in \Omega_0$ . Hence,

$x' \in \cap (C_\tau : \tau \in T)$  which shows that  $E'_1$  is convex compact.

In order to prove the converse we have to show by Theorem 7.3

and Theorem 7.2 that for every proper filter of subsets of a

non-empty set  $X$  there exists a measure  $m$  on the Boolean algebra

of all subsets of  $X$  such that  $m(F) = 1$  for all  $F \in \mathcal{J}$ . To this

end, let  $B(X)$  be the Banach space of all bounded real functions

on  $X$ , and let  $B'_1$  denote the unit ball of the dual space of  $B(X)$ . It is easy to see that the mapping  $x \rightarrow f(x)$ ,  $f \in B$  maps  $X$  into  $B'_1$  in a one-to-one manner. Then consider the family of those subsets of  $B'_1$  which are the  $\sigma(B', B)$  - closed convex hulls of the elements of  $\mathcal{J}$ . Since this family has the finite intersection property ( $\mathcal{J}$  is proper) and  $B'_1$  is  $\sigma(B', B)$  - convex compact there exists an element  $\varphi \in B'_1$  which is an element of every set of the family. Then we define  $m(E) = \varphi(c_E)$ , where  $c_E$  is the characteristic function of  $E \subset X$ . Since  $\varphi$  is linear and positive, the latter follows from the fact that  $\varphi$  is a  $\sigma(E', E)$  - limit point of a set of positive linear point functionals,  $m$  is a measure on the Boolean algebra of all subsets of  $X$ . In order to complete the proof we have to show that  $m(F) = 1$  for all  $F \in \mathcal{J}$ . To this end, we shall denote by  $F_c$  the  $\sigma(B', B)$  - closed convex hull of  $F$  as a subset of  $B'_1$ . Since  $\varphi \in F_c$  it follows that for every  $\varepsilon > 0$  and every  $f \in B$  there exist elements  $x_1, \dots, x_n$  in  $F$  and positive constants  $c_1, \dots, c_n$  such that  $\sum_{i=1}^n c_i = 1$  and  $|\sum c_i f(x_i) - \varphi(f)| < \varepsilon$ . For  $f = c_F$ , we obtain  $|\sum c_i c_F(x_i) - \varphi(c_F)| = |1 - \varphi(c_F)| = |1 - m(F)| < \varepsilon$  for all  $\varepsilon > 0$ , and so  $m(F) = 1$ ; and the proof is finished.

REMARKS. 1. The same method can be used to show that the prime ideal theorem for Boolean algebras is effectively equivalent to the theorem of Bourbaki and Alaoglu.

2. In the theory of locally convex spaces the theorem of Bourbaki and Alaoglu takes the form. If  $U$  is a neighbor-

hood of 0 of a locally convex space  $E$ , then its polar set  $U^0$  is compact in the  $\sigma(E', E)$  - topology.

With the technique of this section it can readily be shown that this result is effectively equivalent to (PI). Furthermore, if we replace compact by convex compact in the above statement, then it becomes a statement which is effectively equivalent to (HB).

References

- [1] Sierpinski, W., *Hypothèse du Continu*, Chelsea, New York second edition (1956).
- [2] Rubin H. and J. Rubin, *Equivalents of the Axiom of Choice*, Studies in Logic, Amsterdam (1963).
- [3] Łos, J. and C. Ryll-Nardzewski, On the applications of Tychonov's theorem in mathematical proofs, *Fund. Math.* 38, 233-237 (1951).
- [4] Luxemburg, W. A. J., Two applications of the method of construction by ultrapowers to analysis, *Bull. Amer. Math. Soc.* 68, 416-419 (1962).
- [5] Halpern, J. D., The independence of the axiom of choice from the Boolean prime ideal theorem, *Fund. Math.* 55, 57-66 (1964).
- [6] Gödel, K., *The Consistency of the Continuum Hypothesis*, Annals of Mathematics Studies, Princeton (1940).
- [7] Luxemburg, W. A. J., A new approach to the theory of monads, *These Proceedings*.
- [8] Frayne, T., C. Morel and D. S. Scott, Reduced direct products, *Fund. Math.* 51, 195-228 (1962).
- [9] Robinson, A., *Non-standard Analysis*, Studies in Logic, Amsterdam (1966).
- [10] Day, M., *Normed Linear Spaces*, *Ergebnisse der Mathematik*, Berlin (19 ).
- [11] Banach, S., *Théorie des Opérations Linéaires*. Warszawa (1932).
- [12] Sikorski, R., *Boolean Algebras*, *Ergebnisse der Mathematik* Bund 25, Berlin (1964) second edition.
- [13] Tarski, A., Une contribution à la théorie de la mesure, *Fund. Math.* 15, 42-50 (1930).
- [14] Luxemburg, W. A. J., *Non-standard Analysis*, California Institute of Technology, Pasadena (1962) and revised edition (1964).

- [15] Nikodym, O., Sur le mesure non-archimédienne effective sur une tribu de Boole arbitraire, C. R. Acad. Sci. Paris 251, 2113-2115 (1960).
- [16] Rubin, J. E. and D. Scott, Some topological theorems equivalent to the Boolean prime ideal theorem, Bull. Am. Math. Soc. 60, 389 (1954).
- [17] Bourbaki, N., Sur les espaces de Banach, C. R. Acad. Sci. Paris 206, 1701-1704 (1938).
- [18] Alaoglu, L., Weak topologies of normed linear spaces, Ann. of Math. 41, 252 - 267 (1940).

Unclassified

Security Classification

DOCUMENT CONTROL DATA - R & D		
<i>Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified.</i>		
1. ORIGINATING ACTIVITY (Corporate author)  California Institute of Technology	2a. REPORT SECURITY CLASSIFICATION <b>Unclassified</b>	
2b. GROUP		
3. REPORT TITLE  Reduced powers of the real number system and equivalents of the Hahn-Banach extension theorem.		
4. DESCRIPTIVE NOTES (Type of report and inclusive dates) Technical Report, April 1967.		
5. AUTHOR(S) (First name, middle initial, last name)  Luxemburg, W. A. J.		
6. REPORT DATE April 1967	7a. TOTAL NO. OF PAGES 28	7b. NO. OF REFS 18
8a. CONTRACT OR GRANT NO. Nonr N00014-66-C0009-A04	9a. ORIGINATOR'S REPORT NUMBER(S) Nr 2	
b. PROJECT NO. Task Nr. NR-041-339	9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)	
c.		
d.		
10. DISTRIBUTION STATEMENT  Releasable without limitations on dissemination.		
11. SUPPLEMENTARY NOTES  None	12. SPONSORING MILITARY ACTIVITY Mathematics Branch Office of Naval Research Washington, D. C. 20360	
13. ABSTRACT  One of the main principles of functional analysis is the so-called Hahn-Banach extension theorem. In the present paper we are mainly interested in the question which other statements in mathematics can be shown to be effectively equivalent to the Hahn-Banach extension theorem. The main result can be phrased as follows:  The Hahn-Banach extension theorem is effectively equivalent to the statement every non-degenerate Boolean algebra admits a non-trivial measure. The method being used is to consider the linear ring of finite elements in a reduced power of the real number system. It is shown that this linear ring is an abstract M-space in the sense of Kakutani. Some of the consequences of this fact are given in the present paper.		

DD FORM 1473 (PAGE 1)

1 NOV 65  
S/N 0101-807-6801

Security Classification

Unclassified

Security Classification

14 KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Functional Analysis Model theory Boolean algebras						