

ARL 67-0027
FEBRUARY 1967



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POINT INTERVAL ESTIMATION, FROM ONE-ORDER STATISTIC,
OF THE LOCATION PARAMETER OF AN EXTREME-VALUE
DISTRIBUTION WITH KNOWN SCALE PARAMETER AND OF
THE SCALE PARAMETER OF A WEIBULL DISTRIBUTION
WITH KNOWN SHAPE PARAMETER

(Reprint from IEEE Transactions on Reliability, Vol. R-15, No. 3,
December 1966)

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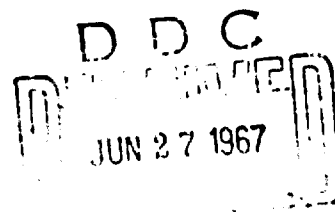
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Project No. 7071

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Point and Interval Estimation, From One-Order Statistic, of the Location Parameter of an Extreme-Value Distribution with Known Scale Parameter and of the Scale Parameter of a Weibull Distribution with Known Shape Parameter

ALBERT H. MOORE AND H. LEON HARTER

Abstract—This paper derives a one-order statistic estimator u_{mn} for the location parameter of the (first) extreme-value distribution of smallest values with cumulative distribution function $F(x; u, b) = 1 - \exp\{-\exp[(x-u)/b]\}$ using the minimum-variance unbiased one-order statistic estimator for the scale parameter of an exponential distribution, as was done in an earlier paper for the scale parameter of a Weibull distribution. It is shown that exact confidence bounds, based on one-order statistic, can be easily derived for the location parameter of the extreme-value distribution and for the scale parameter of the Weibull distribution, using exact confidence bounds for the scale parameter of the exponential distribution. The estimator for u is shown to be $b \ln c_{mn} + x_{mn}$ where x_{mn} is the m th order statistic from an ordered sample of size n from the extreme-value distribution with scale parameter b and c_{mn} is the coefficient for a one-order statistic estimator of the scale parameter of an exponential distribution. Values of the factor c_{mn} , which have previously been tabulated for $n = 1(1)20$, are given for $n = 21(1)40$. The ratios of the mean-square-errors of the maximum-likelihood estimators based on m order statistics to those of the one-order statistic estimators for the location parameter of the extreme-value distribution and the scale parameter of the Weibull distribution are investigated by Monte Carlo methods. The use of the table and related tables is discussed and illustrated by numerical examples.

I. INTRODUCTION

IN A PREVIOUS PAPER, Harter and Moore [1] have derived a maximum-likelihood and an unbiased estimator $\hat{a}/b$ and $\hat{a}/b$ of the location parameter of the extreme-value distribution with known scale parameter, based on the first m out of n ordered observations. However, in many practical applications an inefficient estimator may be chosen for its inherent simplicity. Harter [2] found the minimum-variance unbiased one-order statistic estimator for the scale parameter σ of the exponential distribution, from a sample of size n . Moore and Harter [3] tabulated the coefficient c_{mn} of the minimum-variance unbiased one-order statistic estimator for the scale parameter of the exponential distribution from a censored sample of size m from a life test of n items [$n = 1(1)20$] and showed how it

could be used to obtain a one-order statistic estimator for the scale parameter of Weibull populations with known shape parameter. By use of the coefficient of the m th order statistic, computed for estimation of the scale parameter of the exponential distribution, it is shown in Section II that a consistent one-order statistic estimator for the scale parameter of the extreme-value distribution with known scale parameter can be obtained. The values of c_{mn} are given in Table I, along with the relative efficiencies of the one-order statistic estimators of the scale parameter σ of an exponential distribution as compared with the unbiased m -order statistic estimators, for $n = 21(1)40$, $m = 1(1)n$, and $k = \min(m, r)$, where the r th order statistic is optimal for the complete sample. In Sections III and IV it is shown that exact confidence bounds, based on one-order statistic, can be derived for the location parameter of the extreme-value distribution and the scale parameter of the Weibull distribution using the coefficients of the exact confidence bounds, found by Harter [4], for the scale parameter of an exponential distribution. In Section V a Monte Carlo comparison of the relative merits of the one-order statistic estimators and the maximum-likelihood estimators is given. In Section VI, the use of Table I and related tables is discussed and illustrated by numerical examples.

II. MATHEMATICAL FORMULATION

If Y has an exponential distribution with scale parameter σ and location parameter zero, then $X = b \ln Y$ has the (first) extreme-value distribution with $u = b \ln \sigma$ as location parameter and b as scale parameter. A one-order statistic estimator for the scale parameter of the exponential distribution is given by

$$\hat{\sigma} = c_{mn} y_{mn} \quad (1)$$

where

$$c_{mn} = 1 / \left[\sum_{i=1}^m 1 / (n - i + 1) \right] \quad (2)$$

and y_{mn} is the m th order statistic from an ordered sample of size n from the exponential distribution. Therefore, an

Manuscript received March 16, 1966.
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TABLE I

COEFFICIENT OF OPTIMUM (K TH) ORDER STATISTIC IN UNBIASED ESTIMATOR, BASED ON ONE ORDER STATISTIC, OF
PARAMETER θ OF ONE-PARAMETER EXPONENTIAL POPULATION, GIVEN FIRST M -ORDER STATISTICS OF SAMPLE OF SIZE N ,
AND ITS EFFICIENCY RELATIVE TO BEST M -ORDER-STATISTIC ESTIMATOR

N	K	M	$C(K, N)$	EFF.	N	K	M	$C(K, N)$	EFF.
21	1	1	21.00000	100.00	24	10	10	1.00000	97.77
21	2	2	10.24300	99.94	24	11	11	1.07835	97.12
21	3	3	0.65555	99.83	24	12	12	1.48644	96.34
21	4	4	4.85804	99.07	24	13	13	1.32261	95.41
21	5	5	3.77887	99.44	24	14	14	1.18065	94.27
21	6	6	3.05689	99.14	24	15	15	1.05598	92.90
21	7	7	2.53938	98.75	24	16	16	0.94500	91.23
21	8	8	2.14950	98.25	24	17	17	0.84524	89.18
21	9	9	1.84451	97.62	24	18	18	0.75417	86.64
21	10	10	1.59877	96.84	24	19	19	0.66906	83.45
21	11	11	1.39589	95.88	24	20	20	0.59080	79.36
21	12	12	1.22490	94.68	24	20	21	0.59080	75.58
21	13	13	1.07817	93.19	24	20	22	0.59080	72.14
21	14	14	0.95012	91.33	24	20	23	0.59080	69.01
21	15	15	0.83657	89.99	24	20	24	0.59080	66.13
21	16	16	0.73420	86.00	25	1	1	25.00000	100.00
21	17	17	0.64019	82.10	25	2	2	12.24490	99.96
21	17	18	0.64019	77.54	25	3	3	7.99074	99.88
21	17	19	0.64019	73.46	25	4	4	5.86168	99.77
21	17	20	0.64019	69.78	25	5	5	4.58256	99.62
21	17	21	0.64019	66.46	25	6	6	3.72830	99.42
22	1	1	22.00000	100.00	25	7	7	3.11672	99.16
22	2	2	10.74419	99.95	25	8	8	2.65671	98.85
22	3	3	6.98941	99.85	25	9	9	2.29764	98.46
22	4	4	5.10073	99.70	25	10	10	2.00012	97.98
22	5	5	3.97993	99.50	25	11	11	1.77181	97.41
22	6	6	3.22493	99.23	25	12	12	1.57276	96.73
22	7	7	2.68396	98.88	25	13	13	1.40302	95.90
22	8	8	2.27660	98.44	25	14	14	1.25615	94.92
22	9	9	1.95818	97.80	25	15	15	1.12741	93.74
22	10	10	1.70183	97.21	25	16	16	1.01318	92.31
22	11	11	1.49046	96.37	25	17	17	0.91066	90.59
22	12	12	1.31260	95.34	25	18	18	0.81759	88.48
22	13	13	1.16030	94.08	25	19	19	0.73209	85.89
22	14	14	1.02780	92.53	25	20	20	0.65248	82.64
22	15	15	0.91078	90.61	25	20	21	0.65248	78.71
22	16	16	0.80502	88.19	25	20	22	0.65248	75.13
22	17	17	0.71049	85.13	25	20	23	0.65248	71.86
22	18	18	0.62209	81.16	25	20	24	0.65248	68.87
22	18	19	0.62209	76.80	25	20	25	0.65248	66.12
22	18	20	0.62209	73.04	26	1	1	26.00000	100.00
22	18	21	0.62209	69.56	26	2	2	12.74510	99.96
22	18	22	0.62209	66.40	26	3	3	8.32444	99.89
23	1	1	23.00000	100.00	26	4	4	6.11223	99.79
23	2	2	11.24444	99.95	26	5	5	4.78329	99.65
23	3	3	7.32323	99.86	26	6	6	3.89500	99.47
23	4	4	5.36044	99.73	26	7	7	3.26073	99.24
23	5	5	4.18089	99.54	26	8	8	2.78310	98.95
23	6	6	3.39283	99.30	26	9	9	2.41041	98.60
23	7	7	2.82835	98.90	26	10	10	2.11108	98.17
23	8	8	2.40349	98.50	26	11	11	1.86501	97.66
23	9	9	2.07155	98.11	26	12	12	1.65877	97.05
23	10	10	1.80454	97.51	26	13	13	1.48305	96.32
23	11	11	1.58458	96.78	26	14	14	1.33119	95.46
23	12	12	1.39975	95.89	26	15	15	1.19826	94.43
23	13	13	1.24174	94.81	26	16	16	1.08055	93.20
23	14	14	1.10458	93.49	26	17	17	0.97518	91.73
23	15	15	0.98383	91.88	26	18	18	0.87985	89.95
23	16	16	0.87609	89.80	26	19	19	0.79267	87.80
23	17	17	0.77864	87.41	26	20	20	0.71204	85.15
23	18	18	0.68920	84.28	26	21	21	0.63650	81.86
23	19	19	0.60571	80.24	26	21	22	0.63650	78.14
23	19	20	0.60571	76.23	26	21	23	0.63650	74.74
23	19	21	0.60571	72.60	26	21	24	0.63650	71.62
23	19	22	0.60571	69.30	26	21	25	0.63650	68.76
23	19	23	0.60571	66.29	26	21	26	0.63650	66.11
24	1	1	24.00000	100.00	27	1	1	27.00000	100.00
24	2	2	11.74468	99.95	27	2	2	13.24528	99.96
24	3	3	7.65700	99.87	27	3	3	8.65812	99.90
24	4	4	5.61109	99.75	27	4	4	6.36273	99.81
24	5	5	4.38177	99.58	27	5	5	4.98396	99.68
24	6	6	3.56062	99.36	27	6	6	4.06342	99.51
24	7	7	2.97260	99.08	27	7	7	3.40464	99.30
24	8	8	2.53018	98.73	27	8	8	2.90937	99.04
24	9	9	2.18470	98.30	27	9	9	2.52303	98.72

TABLE I (Cont.)

N	K	M	C(K, N)	EFF.	N	K	M	C(K, N)	EFF.
27	10	10	2 21286	98.33	30	3	3	9 65000	99.92
27	11	11	1 95799	97.87	30	4	4	7 11402	99.85
27	12	12	1 74451	97.33	30	5	5	5 58509	99.74
27	13	13	1 56276	96.68	30	6	6	4 56561	99.61
27	14	14	1 40583	95.92	30	7	7	3 83580	99.45
27	15	15	1 26864	95.01	30	8	8	3 28759	99.25
27	16	16	1 14734	93.94	30	9	9	2 86018	99.00
27	17	17	1 03897	92.66	30	10	10	2 51732	98.71
27	18	18	0 94119	91.15	30	11	11	2 23590	98.36
27	19	19	0 85208	89.33	30	12	12	2 00048	97.96
27	20	20	0 77006	87.13	30	13	13	1 80039	97.49
27	21	21	0 69374	84.43	30	14	14	1 62708	96.93
27	22	22	0 62184	81.00	30	15	15	1 47763	96.29
27	22	23	0 62184	77.56	30	16	16	1 34513	95.54
27	22	24	0 62184	74.33	30	17	17	1 22721	94.67
27	22	25	0 62184	71.36	30	18	18	1 12136	93.66
27	22	26	0 62184	68.61	30	19	19	1 02553	92.47
27	22	27	0 62184	66.07	30	20	20	0 93607	91.08
28	1	1	28 00000	100.00	30	21	21	0 85762	89.45
28	2	2	13 74545	99.97	30	22	22	0 78301	87.50
28	3	3	8 99177	99.91	30	23	23	0 71320	85.18
28	4	4	6 61319	99.82	30	24	24	0 64725	82.36
28	5	5	5 18458	99.70	30	24	25	0 64725	79.06
28	6	6	4 23087	99.55	30	24	26	0 64725	76.02
28	7	7	3 54846	99.35	30	24	27	0 64725	73.21
28	8	8	3 03553	99.12	30	24	28	0 64725	70.59
28	9	9	2 63552	98.82	30	24	29	0 64725	68.16
28	10	10	2 31448	98.48	30	24	30	0 64725	65.89
28	11	11	2 05078	98.06	31	1	1	31 00000	100.00
28	12	12	1 83002	97.57	31	2	2	15 24580	99.97
28	13	13	1 64219	96.99	31	3	3	9 99259	99.93
28	14	14	1 48015	96.31	31	4	4	7 36440	99.86
28	15	15	1 33862	95.50	31	5	5	5 78618	99.76
28	16	16	1 21365	94.56	31	6	6	4 73290	99.64
28	17	17	1 10218	93.45	31	7	7	3 97951	99.49
28	18	18	1 00180	92.13	31	8	8	3 41351	99.30
28	19	19	0 91058	90.57	31	9	9	2 97237	99.07
28	20	20	0 82602	88.71	31	10	10	2 61858	98.81
28	21	21	0 74945	86.47	31	11	11	2 32826	98.49
28	22	22	0 67697	83.73	31	12	12	2 08548	98.12
28	23	23	0 60833	80.34	31	13	13	1 87921	97.69
28	23	24	0 60833	76.99	31	14	14	1 70157	97.19
28	23	25	0 60833	73.91	31	15	15	1 54675	96.61
28	23	26	0 60833	71.07	31	16	16	1 41041	95.93
28	23	27	0 60833	68.44	31	17	17	1 28919	95.16
28	23	28	0 60833	65.99	31	18	18	1 18048	94.25
29	1	1	29 00000	100.00	31	19	19	1 08221	93.21
29	2	2	14 24561	99.97	31	20	20	0 99269	91.99
29	3	3	9 32539	99.91	31	21	21	0 91052	90.56
29	4	4	6 86362	99.83	31	22	22	0 83453	88.89
29	5	5	5 38516	99.72	31	23	23	0 76372	86.91
29	6	6	4 39827	99.58	31	24	24	0 69716	84.55
29	7	7	3 69221	99.40	31	25	25	0 63402	81.70
29	8	8	3 16160	99.18	31	25	26	0 63402	78.55
29	9	9	2 74790	98.92	31	25	27	0 63402	75.65
29	10	10	2 41596	98.60	31	25	28	0 63402	72.94
29	11	11	2 14341	98.22	31	25	29	0 63402	70.43
29	12	12	1 91534	97.78	31	25	30	0 63402	68.08
29	13	13	1 72139	97.25	31	25	31	0 63402	65.88
29	14	14	1 55418	96.64	32	1	1	32 00000	100.00
29	15	15	1 40827	95.93	32	2	2	15 74603	99.97
29	16	16	1 27956	95.09	32	3	3	10 32616	99.93
29	17	17	1 16490	94.11	32	4	4	7 61475	99.87
29	18	18	1 06182	92.96	32	5	5	5 98665	99.78
29	19	19	0 96835	91.61	32	6	6	4 90015	99.66
29	20	20	0 88266	90.01	32	7	7	4 12308	99.52
29	21	21	0 80399	88.10	32	8	8	3 53936	99.35
29	22	22	0 73057	85.81	32	9	9	3 08448	99.14
29	23	23	0 66153	83.04	32	10	10	2 71974	98.89
29	24	24	0 59583	79.61	32	11	11	2 42051	98.60
29	24	25	0 59583	76.42	32	12	12	2 17035	98.26
29	24	26	0 59583	73.48	32	13	13	1 95788	97.87
29	24	27	0 59583	70.76	32	14	14	1 77498	97.41
29	24	28	0 59583	68.24	32	15	15	1 61566	96.88
29	24	29	0 59583	65.88	32	16	16	1 47544	96.28
30	1	1	30 00000	100.00	32	17	17	1 35087	95.58
30	2	2	14 74576	99.97	32	18	18	1 23926	94.77

TABLE I (Cont.)

V	K	M	C(K, N)	EFF.	N	K	M	C(K, N)	EFF.
32	19	19	1.13848	93.84	34	28	41	0.59044	72.07
32	20	20	1.04681	92.76	34	28	32	0.59044	69.82
32	21	21	0.98282	91.51	34	28	33	0.59044	67.70
32	22	22	0.88533	90.15	34	28	34	0.59044	65.71
32	23	23	0.81332	88.34	35	1	1	35.00000	100.00
32	24	24	0.74501	86.33	35	2	2	17.24638	99.98
32	25	25	0.68230	83.95	35	3	3	11.32080	99.94
32	26	26	0.62170	81.05	35	4	4	8.30560	99.89
32	26	27	0.62170	78.05	35	5	5	6.58786	99.82
32	26	28	0.62170	75.26	35	6	6	5.40168	99.72
32	26	29	0.62170	72.66	35	7	7	4.55352	99.61
32	26	30	0.62170	70.24	35	8	8	3.91658	99.47
32	26	31	0.62170	67.98	35	9	9	3.42042	99.30
32	26	32	0.62170	65.85	35	10	10	3.02276	99.10
33	1	1	33.00000	100.00	35	11	11	2.69670	98.87
33	2	2	16.24615	99.98	35	12	12	2.42430	98.60
33	3	3	10.65972	99.93	35	13	13	2.19314	98.30
33	4	4	7.86597	99.87	35	14	14	1.99433	97.94
33	5	5	6.18708	99.79	35	15	15	1.82136	97.54
33	6	6	5.06736	99.69	35	16	16	1.66933	97.08
33	7	7	4.26660	99.55	35	17	17	1.53451	96.55
33	8	8	3.66515	99.39	35	18	18	1.41397	95.95
33	9	9	3.19652	99.20	35	19	19	1.30580	95.27
33	10	10	2.82082	98.97	35	20	20	1.20692	94.49
33	11	11	2.51266	98.70	35	21	21	1.11705	93.60
33	12	12	2.25510	98.39	35	22	22	1.03450	92.59
33	13	13	2.03642	98.03	35	23	23	0.95825	91.43
33	14	14	1.84823	97.61	35	24	24	0.88739	90.09
33	15	15	1.68438	97.13	35	25	25	0.82114	88.54
33	16	16	1.54025	96.58	35	26	26	0.75883	86.75
33	17	17	1.41229	95.94	35	27	27	0.69983	84.64
33	18	18	1.29774	95.22	35	28	28	0.64353	82.16
33	19	19	1.19441	94.38	35	28	29	0.61353	79.32
33	20	20	1.10052	93.42	35	28	30	0.61353	76.68
33	21	21	1.01462	92.31	35	28	31	0.61353	74.21
33	22	22	0.93552	91.03	35	28	32	0.61353	71.89
33	23	23	0.86220	89.54	35	28	33	0.61353	69.71
33	24	24	0.79376	87.80	35	28	34	0.61353	67.66
33	25	25	0.72943	85.76	35	28	35	0.61353	65.72
33	26	26	0.66848	83.33	36	1	1	36.00000	100.00
33	27	27	0.61020	80.41	36	2	2	17.74648	99.98
33	27	28	0.61020	77.54	36	3	3	11.66032	99.95
33	27	29	0.61020	74.87	36	4	4	8.61594	99.89
33	27	30	0.61020	72.37	36	5	5	6.78822	99.83
33	27	31	0.61020	70.04	36	6	6	5.56879	99.74
33	27	32	0.61020	67.85	36	7	7	4.69692	99.63
33	27	33	0.61020	65.79	36	8	8	4.04223	99.50
34	1	1	34.00000	100.00	36	9	9	3.53229	99.34
34	2	2	16.74627	99.98	36	10	10	3.12364	99.16
34	3	3	10.99326	99.94	36	11	11	2.78861	98.94
34	4	4	8.11538	99.88	36	12	12	2.50877	98.70
34	5	5	6.38548	99.80	36	13	13	2.27135	98.41
34	6	6	5.23453	99.71	36	14	14	2.06720	98.08
34	7	7	4.41008	99.58	36	15	15	1.88964	97.71
34	8	8	3.79089	99.43	36	16	16	1.73964	97.29
34	9	9	3.30850	99.25	36	17	17	1.59536	96.80
34	10	10	2.92183	99.04	36	18	18	1.47178	96.25
34	11	11	2.60472	98.79	36	19	19	1.36053	95.63
34	12	12	2.33975	98.50	36	20	20	1.25972	94.93
34	13	13	2.11483	98.17	36	21	21	1.16777	94.13
34	14	14	1.92134	97.79	36	22	22	1.09343	93.21
34	15	15	1.75294	97.35	36	23	23	1.03561	92.18
34	16	16	1.60487	96.84	36	24	24	0.98340	90.99
34	17	17	1.47350	96.27	36	25	25	0.93694	89.62
34	18	18	1.35597	95.61	36	26	26	0.89283	88.05
34	19	19	1.25003	94.85	36	27	27	0.74317	86.23
34	20	20	1.15387	93.99	36	28	28	0.68648	84.10
34	21	21	1.06601	93.00	36	29	29	0.63223	81.58
34	22	22	0.98522	91.87	36	29	30	0.63223	78.86
34	23	23	0.91047	90.56	36	29	31	0.63223	76.32
34	24	24	0.84087	89.04	36	29	32	0.63223	73.94
34	25	25	0.77565	87.27	36	29	33	0.63223	71.70
34	26	26	0.71411	85.20	36	29	34	0.63223	69.59
34	27	27	0.65559	82.74	36	29	35	0.63223	67.60
34	28	28	0.59944	79.79	36	29	36	0.63223	65.72
34	28	29	0.59944	77.04	37	1	1	37.00000	100.00
34	28	30	0.59944	74.47	37	2	2	18.24658	99.98

TABLE I (Cont.)

N	K	M	(K, N)	EFF.	N	K	M	(K, N)	EFF.
37	3	3	11.00383	99.95	39	4	4	9.36866	99.91
37	4	4	8.80619	99.90	39	5	5	7.38918	99.85
37	5	5	6.98056	99.84	39	6	6	6.06949	99.78
37	6	6	5.73588	99.75	39	7	7	5.12695	99.69
37	7	7	4.84029	99.65	39	8	8	4.41896	99.58
37	8	8	4.16784	99.53	39	9	9	3.86764	99.45
37	10	10	3.64411	99.38	39	10	10	3.42596	99.30
37	11	11	3.22446	99.21	39	11	11	3.06399	99.12
37	12	12	2.88146	99.01	39	12	12	2.76177	98.92
37	13	13	2.59317	98.78	39	13	13	2.50549	98.69
37	14	14	2.34967	98.51	39	14	14	2.28527	98.43
37	15	15	2.13998	98.21	39	15	15	2.09387	98.13
37	16	16	1.95782	97.86	39	16	16	1.92585	97.79
37	17	17	1.79783	97.47	39	17	17	1.77705	97.41
37	18	18	1.65605	97.03	39	18	18	1.64424	96.99
37	19	19	1.52941	96.53	39	19	19	1.52485	96.59
37	20	20	1.41547	95.96	39	20	20	1.41683	95.96
37	21	21	1.31228	95.31	39	21	21	1.31856	95.36
37	22	22	1.21824	94.59	39	22	22	1.22852	94.67
37	23	23	1.13204	93.76	39	23	23	1.14572	93.90
37	24	24	1.05261	92.83	39	24	24	1.06916	93.04
37	25	25	0.97900	91.76	39	25	25	0.99402	92.08
37	26	26	0.91044	90.55	39	26	26	0.93161	90.95
37	27	27	0.84623	89.16	39	27	27	0.86931	89.69
37	28	28	0.78578	87.57	39	28	28	0.81059	88.26
37	29	29	0.72853	85.72	39	29	29	0.75496	86.61
37	30	30	0.67398	83.56	39	30	30	0.70196	84.72
37	31	31	0.62161	81.02	39	31	31	0.65117	82.51
37	32	32	0.62161	78.41	39	32	32	0.65117	79.94
37	33	33	0.62161	75.96	39	33	33	0.65117	77.51
37	34	34	0.62161	73.66	39	34	34	0.65117	75.23
37	35	35	0.62161	71.49	39	35	35	0.65117	73.06
37	36	36	0.62161	69.45	39	36	36	0.65117	71.05
37	37	37	0.62161	67.52	39	37	37	0.65117	69.13
38	1	1	38.00000	100.00	39	38	38	0.65117	67.31
38	2	2	18.74667	99.98	39	39	39	0.65117	65.59
38	3	3	12.32733	99.95	40	1	1	40.00000	100.00
38	4	4	9.11643	99.91	40	2	2	19.74684	99.98
38	5	5	7.18888	99.85	40	3	3	12.99430	99.96
38	6	6	5.90295	99.77	40	4	4	9.61688	99.92
38	7	7	4.98363	99.67	40	5	5	7.58946	99.86
38	8	8	4.29341	99.56	40	6	6	6.23702	99.79
38	9	9	3.75589	99.42	40	7	7	5.27024	99.71
38	10	10	3.32523	99.26	40	8	8	4.54447	99.60
38	11	11	2.97225	99.07	40	9	9	3.97934	99.48
38	12	12	2.67750	98.85	40	10	10	3.52664	99.34
38	13	13	2.42752	98.61	40	11	11	3.15568	99.17
38	14	14	2.21266	98.32	40	12	12	2.84599	98.99
38	15	15	2.02589	98.00	40	13	13	2.58340	98.77
38	16	16	1.86189	97.64	40	14	14	2.35780	98.52
38	17	17	1.71661	97.23	40	15	15	2.16177	98.24
38	18	18	1.58689	96.77	40	16	16	1.98971	97.93
38	19	19	1.47024	96.25	40	17	17	1.83739	97.58
38	20	20	1.36464	95.68	40	18	18	1.70146	97.18
38	21	21	1.26847	94.99	40	19	19	1.57932	96.74
38	22	22	1.18040	94.24	40	20	20	1.46885	96.24
38	23	23	1.09930	93.40	40	21	21	1.36436	95.68
38	24	24	1.02423	92.44	40	22	22	1.27643	95.05
38	25	25	0.95441	91.36	40	23	23	1.19191	94.35
38	26	26	0.88913	90.12	40	24	24	1.11382	93.56
38	27	27	0.82780	88.71	40	25	25	1.04133	92.68
38	28	28	0.76986	87.09	40	26	26	0.97373	91.68
38	29	29	0.71483	85.21	40	27	27	0.91041	90.55
38	30	30	0.66223	83.03	40	28	28	0.85082	89.27
38	31	31	0.61160	80.47	40	29	29	0.79449	87.81
38	32	32	0.61160	77.96	40	30	30	0.74097	86.14
38	33	33	0.61160	75.60	40	31	31	0.69086	84.23
38	34	34	0.61160	73.37	40	32	32	0.64074	82.00
38	35	35	0.61160	71.28	40	33	33	0.64074	79.52
38	36	36	0.61160	69.30	40	34	34	0.64074	77.18
38	37	37	0.61160	67.42	40	35	35	0.64074	74.97
38	38	38	0.61160	65.65	40	36	36	0.64074	72.89
39	1	1	39.00000	100.00	40	37	37	0.64074	70.92
39	2	2	19.24675	99.98	40	38	38	0.64074	68.06
39	3	3	12.66082	99.95	40	39	39	0.64074	67.28
					40	40	40	0.64074	65.60

estimator for u is given by

$$\begin{aligned}\hat{u} &= b \ln \hat{\sigma} \\ &= b \ln (c_{mn}/y_{mn}) \\ &= b \ln c_{mn} + b \ln y_{mn}\end{aligned}$$

and since $x = b \ln y$ we obtain

$$\hat{u} = b \ln c_{mn} + x_{mn} \quad (3)$$

where c_{mn} is given by (2) and x_{mn} is the m th order statistic from an ordered sample of size n from the extreme-value distribution. Now u is a consistent estimator of the location parameter u of the extreme-value distribution, since $u = b \ln \sigma$ and $\hat{\sigma}$ is a consistent estimator for the scale parameter σ of the exponential population.

Similarly, as was shown in an earlier paper [3], if Y has an exponential distribution with scale parameter σ then $T = Y^{1/K}$ has a Weibull distribution with shape parameter K , scale parameter $\theta = \sigma^{1/K}$, and

$$\hat{\theta} = c_{mn}^{1/K} t_{mn} \quad (4)$$

is a consistent estimator for θ and t_{mn} is the m th order statistic from a Weibull distribution with shape parameter K .

The coefficient c_{mn} has been tabulated for $n = 2(1)20$ by Moore and Harter [3]. The values of the coefficient are given for $n = 21(1)40$ in Table I, in which it is called c_{kn} , where $k = \min(m, r)$, the k th order statistic being optimal for estimation from the first m order statistics of a sample of size n and the r th order statistic being optimal for estimation from the complete sample.

III. EXACT CONFIDENCE BOUNDS FOR THE LOCATION PARAMETER OF THE EXTREME-VALUE DISTRIBUTION

Harter [4] has obtained exact upper and lower bounds and central confidence intervals for the scale parameter of the one-parameter exponential distribution for a wide range of confidence levels based on the m th order statistic y_{mn} of a sample of size n . The coefficients B_{mn}/y_{mn} have been tabulated for $n = 1(1)20(2)40$ for all m optimal. Let us introduce the notation

$$D_{lm} = B_{lm}/y_{mn} \text{ and } D_{un} = B_{un}/y_{mn}.$$

Now the exact confidence interval based on one-order statistic is given by

$$D_{lm}y_{mn} < \sigma < D_{un}y_{mn}. \quad (5)$$

Let $X = b \ln Y$ and we obtain

$$D_{lm}e^{x_{mn}} < \sigma < D_{un}e^{x_{mn}}$$

where x_{mn} is the m th order statistic in a sample of size n from the extreme-value distribution. Now take the natural logarithm of the terms of the inequality, multiply by b and we obtain

$$b \ln D_{lm} + x_{mn} < b \ln \sigma < b \ln D_{un} + x_{mn}.$$

But $u = b \ln \sigma$; therefore, by substitution, we obtain the following

$$b \ln D_{lm} + x_{mn} < u < b \ln D_{un} + x_{mn} \quad (6)$$

which gives an exact central confidence interval with the same level of confidence given by the tabulated values of $B_{lm}/y_{mn} = D_{lm}$ and $B_{un}/y_{mn} = D_{un}$. Therefore we have a simple method of computing exact central confidence intervals or upper and lower confidence bounds for the location parameter of the extreme-value distribution, with scale parameter b , based on one-order statistic.

IV. EXACT CONFIDENCE BOUNDS FOR THE SCALE PARAMETER OF THE WEIBULL DISTRIBUTION WITH KNOWN SHAPE PARAMETER

If the random variable T has a Weibull distribution with shape parameter K then it is easily shown that $Y = T^K$ has an exponential distribution with $\theta = \sigma^{1/K}$. In inequality (5) replace y_{mn} by t_{mn}^K , the K th power of the m th order statistic from a Weibull distribution with shape parameter K , and we obtain

$$D_{lm}t_{mn}^K < \sigma < D_{un}t_{mn}^K. \quad (7)$$

Take the K th root of each member of (7) and obtain

$$D_{lm}^{1/K}t_{mn} < \sigma^{1/K} < D_{un}^{1/K}t_{mn}. \quad (8)$$

But $\theta = \sigma^{1/K}$ and therefore

$$D_{lm}^{1/K}t_{mn} < \theta < D_{un}^{1/K}t_{mn} \quad (9)$$

gives an exact central confidence interval for the scale parameter of Weibull distributions with known shape parameter K .

V. MONTE CARLO STUDY OF RATIOS OF MEAN-SQUARE-ERRORS

It seemed reasonable to the authors to conjecture that the ratios of the mean-square-errors of the m -order-statistic estimator and of the one-order statistic estimator for both the scale parameter of a two-parameter Weibull distribution with known shape parameter and the location parameter of an extreme-value distribution with known scale parameter are closely approximated by the relative efficiency of the one-order statistic estimator of the scale parameter of an one-parameter exponential distribution as compared with the m order statistic estimator, which has been tabulated by Moore and Harter [3] for $n = 1(1)20$ and in Table I of the present paper for $n = 21(1)40$. (It should be noted that one may speak of relative efficiency in the case of the exponential distribution, since the estimators are unbiased, but only of ratios of mean-square-errors in the cases of the Weibull and extreme-value distributions, for which the estimators are biased.) In order to check the validity of the conjecture, a Monte Carlo study of the ratios of mean-square-errors was performed. One thousand random samples each of size n [$n = 1(1)40$] from a one-parameter exponential distribution with scale

parameter 1 were generated in the IBM 7094 computer. These were transformed into samples from a 2-parameter Weibull distribution with shape parameter 2 and from an extreme-value distribution with scale parameter 1. From each sample, the one-order statistic estimate and the m order statistic estimate, based on the first m order statistics [$m = 1(1)n$], of the scale parameters of the exponential and Weibull distributions and of the location parameter of the extreme-value distribution were computed. For each distribution and for each combination of m and n , the ratio of the mean-square-error of the m order statistic estimates to that of the one-order statistic estimates was calculated. Except for fluctuations due to random sampling, the ratio of mean-square-errors in the case of the exponential distribution should agree with the tabulated relative efficiencies, and it was found that the agreement was quite good. Moreover, it was found that the ratios of mean-square-errors in the cases of the Weibull and extreme-value distributions agreed with the tabulated relative efficiencies almost as well as did those for the exponential distribution, thus confirming the conjecture.

VI. USE OF TABLE I AND RELATED TABLES, WITH NUMERICAL EXAMPLES

Table I gives the coefficient of the optimum single-order statistic (the k th) in an unbiased estimator of the scale parameter of a one-parameter exponential distribution from the first m order statistics of a sample of size n [$n = 21(1)40$], and the relative efficiency of the one-order statistic estimator as compared with the m order statistic estimator. It is a condensed extension of the similar table for $n = 1(1)20$ given by Moore and Harter [3], which also includes columns giving the variances of the two estimators. These columns have been omitted from Table I to save space, which can be done without loss of information, since the variance of the m order statistic estimator is simply $1/m$ and that of one-order statistic estimator can be found by dividing $1/m$ by the relative efficiency. These two tables can also be used to obtain consistent one-order statistic estimators of the scale parameter of a two-parameter Weibull distribution with known shape parameter and of the location parameter of an extreme-value distribution with known scale parameter, together with their approximate "efficiencies" (ratios of mean-square-errors) relative to the m order statistic estimators. Harter [4] has tabulated coefficients of optimum order statistics in exact upper and lower confidence bounds, based on one-order statistic, for the scale parameter of a one-parameter exponential distribution. These may also be used to obtain exact confidence bounds, based on one-order statistic, for the scale parameter of a two-parameter Weibull distribution with known shape parameter and for the location parameter of an extreme-value distribution with known scale parameter.

As an example of the previously mentioned uses of Table I and related tables, consider the following tabulation of data (observed failure times in hours) resulting from a simulated life test on forty components:

5	33	55	65	82	102	114	142
10	34	58	65	85	103	116	143
17	36	58	66	90	106	117	151
32	54	61	67	92	107	124	158
32	55	64	68	92	114	139	195

Suppose that the experimenter knows that these data have come from a two-parameter Weibull distribution with shape parameter $K = 2.0$, and that he wishes to find a point estimate and 80 percent lower and upper confidence bounds on the scale parameter θ . Harter and Moore [5] have previously done this for estimates based on the first m order statistics [$m = 8(8)40$]. From Table I of the present paper and from Table I of Harter [4], one finds that, for a one-parameter exponential distribution, the optimum-order statistic for obtaining a point estimator and the 80 percent lower and upper confidence bounds on the scale parameter is 32, with coefficients 0.64074, 0.553447, and 0.768717, respectively. Substituting these values in (4) and (9), one finds that the point estimate of the scale parameter of the Weibull distribution from which the above sample came is $\sqrt{0.64074(116)} = 92.9$, the 80 percent lower confidence bound is $\sqrt{0.553447(116)} = 86.3$ and the 80 percent upper confidence bound is $\sqrt{0.768717(116)} = 101.7$, as compared with results 93.7, 87.6, and 101.7 obtained from the first 32 order statistics, 93.3, 87.8, and 100.3 obtained from all 40 observations, and the true population parameter of 100.

Now consider the same data transformed to data from an extreme-value distribution with scale parameter $b = 0.5$ by taking natural logarithms.

1.609	3.497	4.637	4.174	4.407	4.625	4.736	4.956
2.303	3.526	4.060	4.174	4.443	4.635	4.754	4.963
2.833	3.584	4.060	4.190	4.500	4.663	4.762	5.017
3.466	3.989	4.111	4.205	4.522	4.673	4.820	5.063
3.466	4.007	4.159	4.220	4.522	4.736	4.934	5.273

Using the same tabular values, one finds by substituting in (3) and (6) that the point estimate of the location parameter of the extreme-value distribution is $0.5 \ln 0.64074 + 4.754 = 4.531$, the 80 percent lower confidence bound is $0.5 \ln 0.553447 + 4.754 = 4.458$, and the 80 percent upper confidence bound is $0.5 \ln 0.768717 + 4.754 = 4.623$, as compared with results [1] 4.541, 4.474, and 4.624 based on the first 32 order statistics, 4.537, 4.476, and 4.610 based on all 40 observations, and the true population parameter of 4.605 ($= \ln 100$).

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Vol. R-15, No. 3, December 1966
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PRINTED IN THE U.S.A.
pp. 120-126

UNCLASSIFIED

Security Classification

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author) Aerospace Research Laboratories Applied Mathematics Research Laboratory Wright-Patterson AFB, Ohio 45433		2a. REPORT SECURITY CLASSIFICATION UNCLASSIFIED	
		2b. GROUP	
3. REPORT TITLE POINT INTERVAL ESTIMATION, FROM ONE-ORDER STATISTIC, OF THE LOCATION PARAMETER OF AN EXTREME-VALUE DISTRIBUTION WITH KNOWN SCALE PARAMETER AND OF THE SCALE PARAMETER OF A WEIBULL DISTRIBUTION WITH KNOWN SHAPE PARAMETER			
4. DESCRIPTIVE NOTES (Type of report and inclusive dates) Scientific. Interim.			
5. AUTHOR(S) (First name, middle initial, last name) Albert H. Moore H. Leon Harter			
6. REPORT DATE February 1967		7a. TOTAL NO. OF PAGES 7	7b. NO. OF REFS 5
8a. CONTRACT NO. 33(616)585-68481-100 In-House-Research		8b. ORIGINATOR'S REPORT NUMBER(S)	
9. PROJECT NO. 7071-00-11			
c. DoD Element 61445014		9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)	
d. DoD Subelement 681304		ARL 67-0027	
10. DISTRIBUTION STATEMENT 1. Distribution of this document is unlimited			
11. SUPPLEMENTARY NOTES IEEE Transactions on Reliability, Vol. R-15, No. 3, December 1966		12. SPONSORING MILITARY ACTIVITY Aerospace Research Laboratories (ARL) Wright-Patterson AFB Ohio 45433	
13. ABSTRACT This paper derives a one-order statistic estimator U_{nb} for the location of the (first) extreme-value distribution of smallest values with cumulative distribution function $F(x;u,b) = 1 - \exp\{-\exp[(x-u)/b]\}$ using the minimum-variance unbiased one-order statistic estimator for the scale parameter of an exponential distribution, as was done in an earlier paper for the scale parameter of a Weibull distribution. It is shown that exact confidence bounds, based on one-order statistics, can be easily derived for the location parameter of the extreme-value distribution and for the scale parameter of the Weibull distribution, using exact confidence bounds for the scale parameter of the exponential distribution. The estimator for u is shown to be b in $c_{mn} + x_{mn}$ is the m th order statistic from an ordered sample of size n from the extreme-value distribution with scale parameter b and c_{mn} is the coefficient for a one-order statistic estimator of the scale parameter of an exponential distribution. Values of the factor c_{mn} , which have previously been tabulated for $n = 1(1)20$, are given for $n = 21(1)40$. The ratios of the mean-square errors of the maximum-likelihood estimators based on m order statistics to those of the one-order statistic estimators for the location parameter of the extreme-value distribution and the scale parameter of the Weibull distribution are investigated by Monte Carlo methods. The use of the table and related tables is discussed and illustrated by numerical examples.			

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