Report 2419

AD657404

NAVAL SHIP RESEARCH AND DEVELOPMENT CENTER

WASHINGTON, D.C. 20007

LINEAR VERSUS LOGARITHMIC AVERAGING

by

LCDR Henry Cox, USN

Distribution of this document is unlimited.

ACOUSTICS AND VIBRATION LABORATORY
RESEARCH AND DEVELOPMENT REPORT

June 1967

Report 2419

LINEAR VERSUS LOGARITHMIC AVERAGING

LINEAR VERSUS LOGARITHMIC AVERAGING

by

LCDR Henry Cox, USN

Reprinted from The Journal of Acoustical Society of America, Vol. 39, No. 4, 688-690, April 1966.

June 1967 Report 2419

TABLE OF CONTENTS

Linear versus Logarithmic Averaging

HENRY COX

David Taylor Model Basin, Washington, D. C. 20007

Consider n data samples $\{x_1, \cdots, x_n\}$ such that $0<L\leq x_i\leq U<\infty$. Let $K=U/L$; then it is shown that independent of n a lower bound on the ratio of the geometric mean to the arithmetic mean of the data samples is given by $[\ln K/(K-1)]K^{\{(1/\ln K)-1/(K-1)\}}$. This bound is useful in acoustic signal processing since it limits the amount of deviation that can be attributed to averaging logarithms vice taking the logarithm of the average of data samples. Both methods are currently in use at facilities specializing in the processing of acoustic data. For a K of 10 dB, for example, the geometric mean is less than 1.5 dB below the arithmetic mean.

INTRODUCTION

In dealing with acoustic data, it is customary to express quantities in logarithmic form (decibels). The output from a data-processing operation can depend upon the point at which the conversion from a linear to a logarithmic scale is made. In particular, certain data-processing facilities average before taking logarithms and so obtain the logarithm of the arithmetic mean of the data sample, while other facilities convert to a logarithmic scale before averaging, thereby obtaining the logarithm of the geometric mean. The question naturally arises, "How much difference can the type of averaging used make in the final result?" The purpose of this paper is to provide a bound on the amount of deviation that can be attributed to geometric averaging vice arithmetic averaging.

The data are assumed to consist of n samples that are confined to lie between an upper limit U and a positive lower limit L. The quantity studied is the ratio of the geometric mean to the arithmetic mean; that is, the function

$$F(x)=[\prod_{i=1}^{n} x_i]^{1/n}\Big/\frac{1}{n}\sum_{i=1}^{n} x_i, \tag{1}$$

where $0<L\leq x_i\leq U<\infty$. It is well-known that this ratio is equal to or less than unity, and equal to unity only if all the x_i's are equal. The problem is to bound this ratio as closely as possible from below.

I. MAIN RESULT

The main result of this paper can be summarized in the following Theorem. Let $F(x)$ be defined as in Eq. 1. If $0<L\leq x_i\leq U<\infty$ for $i=1, 2, \cdots, n$, and $K=U/L>1$, then $B(K)\leq F(x)\leq 1$, where

$$B(K)=[\ln K/(K-1)]K^{\{(1/\ln K)-1/(K-1)\}}.$$

The proof of this theorem is given in Appendix A. For convenience, values of the lower bound $B(K)$ are given in Table I and a plot of $B(K)$ is given in Fig. 1.

From Table I, we see, for example, that, for $K=2$ (a spread in data of 6 dB), then $F(x)>0.942$ and the geometric mean is less than 0.52 dB below the arithmetic mean.

II. DISCUSSION

The function $B(K)$ is a lower bound on the ratio of the geometric mean to arithmetic mean. As is shown in Appendix A, $F(x)$ takes on its minimum value *only* when a certain percentage of the data points lie on the

TABLE I. Values of lower bound.

K	$20\log_{10}K$	$B(K)$	$20\log_{10}B(K)$
$\sqrt{2}$	3.010 3	0.985 12	− 0.130 19
2	6.020 6	0.942 08	− 0.518 20
4	12.041	0.791 30	− 2.033 2
8	18.062	0.599 97	− 4.437 4
16	24.082	0.417 65	− 7.583 7
32	30.103	0.271 75	−11.316
64	36.124	0.167 98	−15.495
128	42.144	0.099 959	−20.004
256	48.165	0.057 840	−24.755
512	54.185	0.032 782	−29.687
1024	60.206	0.018 294	−34.754

upper limit U and the remainder lie on the lower limit L. Hence, it is extremely unlikely in an actual data-processing operation that $F(x)$ will take on its minimum value. In fact, the deviation between the geometric mean and the arithmetic mean will frequently be much less than the amount of the bound. Further, it must be emphasized that, in a data-processing operation, it is the spread of the data at the point in the process at which conversion from linear to logarithmic scale occurs rather than the spread in the raw data that determines the amount of deviation.

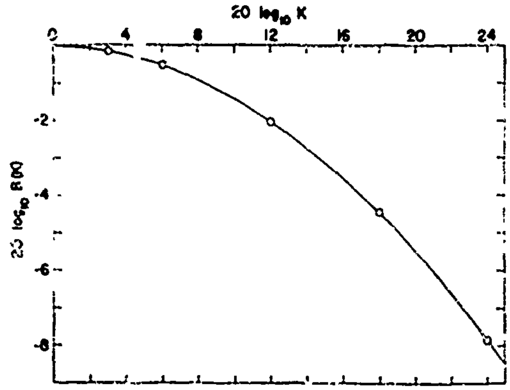

FIG 1. Plot of lower bound $B(K)$ vs K.

III. CONCLUSION

A bound on the deviation of the geometric mean from the arithmetic mean has been presented. This bound depends only on the ratio of the maximum value to the minimum value of the data sample and is independent of the number of sample points. This result has direct application in the processing of acoustic data.

Appendix A: Proof of Theorem

Let x be the vector $\{x_1, x_2, \cdots, x_n\}$, and consider the function

$$F(x)=[\prod_{i=1}^{n} x_i]^{1/n} \Big/ \frac{1}{n}\sum_{i=1}^{n} x_i, \tag{A1}$$

defined on the hypercube $H=\{x\colon 0<L\leq x_i\leq U<\infty\}$. Note that F is continuous on H, and H is compact so that a minimum of F on H exists, and that this minimum must be either in the interior of H or on the boundary of H. Differentiating Eq. A1 with respect to x_i and rearranging terms, yields

$$\partial F/\partial x_i=(n-1)[\prod_{k=1}^{n} x_k]^{1/n}\left[\frac{1}{n-1}\sum_{k\neq i} x_k-x_i\right] \Big/ x_i[\sum_{k=1}^{n} x_k]^2. \tag{A2}$$

Examining Eq. A2, we see that, for $x\epsilon H$,

$$\frac{\partial F}{\partial x_i}=0 \quad \text{for} \quad x_i=\frac{1}{n-1}\sum_{k\neq i} x_k, \tag{A3}$$

$$\frac{\partial F}{\partial x_i}<0 \quad \text{for} \quad x_i>\frac{1}{n-1}\sum_{k\neq i} x_k, \tag{A4}$$

$$\frac{\partial F}{\partial x_i}>0 \quad \text{for} \quad x_i<\frac{1}{n-1}\sum_{k\neq i} x_k \tag{A5}$$

At a stationary point, the relation $\partial F/\partial x_i=0$ must be satisfied for $i=1, 2, \cdots, n$. From Eq. A3, it is obvious that, for $x\epsilon H$, this happens if and only if all the x_i's are equal, i. e., if and only if $x_i=x_j$ for $i=1, \cdots, n$, $j=1, \cdots, n$. In this case, $F(x)=1$ and the stationary point is a maximum.

We can obtain information about the minimum of $F(x)$ on H from Eqs. A4 and A5, which imply that, ***independent of the values of the other coordinates***, $F(x)$ can always be made smaller by decreasing x_i for

$$x_i<\frac{1}{n-1}\sum_{k\neq i} x_k \tag{A6}$$

and by increasing x_i for

$$x_i>\frac{1}{n-1}\sum_{k\neq i} x_k. \tag{A7}$$

This implies that F must take on its minimum at one of the vertices of H; that is, at the minimum of F, each coordinate must equal either the upper limit U or the lower limit L.

Let us now examine the value of F at the vertices of H. Suppose that λn coordinates are equal to U and $(1-\lambda)n$ coordinates are equal to L; then, substituting these values into Eq. A1, we see that $F(x)$ is equal to

$$f(\lambda)=\frac{K^\lambda}{1+\lambda(K-1)}; \quad \lambda=0, 1/n, 2/n, \cdots, 1, \tag{A8}$$
$$K=U/L>1.$$

At $\lambda=0$ and $\lambda=1$, all coordinates are equal and $f(0)=f(1)=1$, which is the maximum value of $F(x)$.

Now, consider the problem of minimizing Eq. A8 with respect to λ or, equivalently, minimizing

$$\ln(f(\lambda))=\lambda\ln K-\ln\{1+\lambda(K-1)\}. \tag{A9}$$

For the moment, suppose that λ could take on continuous values on the interval $0<\lambda<1$. Then, setting

$$d\{\ln(f(\lambda))\}/d\lambda=\ln K-(K-1)/[1+\lambda(K-1)] \quad (A10)$$

equal to zero and solving for λ, we obtain the minimizing value

$$\lambda^{*}=(1/\ln K)-(1/(K-1)). \quad (A11)$$

This value of λ lies in the interval $0<\lambda<1$ as required. To verify that λ^{*} actually corresponds to a minimum, we note that

$$d^{2}[\ln[f(\lambda)]]/d\lambda^{2} =(K-1)^{2}/[1+\lambda(k-1)]^{2}>0 \quad \text{FOR} \quad 0<\lambda<1, \quad (A12)$$

which shows that $\ln[f(\lambda)]$ is convex on the interval $0<\lambda<1$. Substituting λ^{*} from Eqs. A11 in Eq. A8, yields the basic result

$$F(x)\geq[\ln K/(K-1)]K^{[(1/\ln K)-1/(K-1)]}, \quad (A13)$$

which was to be proven.

REMARK

Although we have treated λ as a continuous variable in deriving Eq. A13 in the original problem, λ could only take on discrete values $0, 1/n, 2/n, \cdots, 1$. If none of these values correspond to the minimizing value λ^{*} given in Eq. A11, then the inequality in Eq. A13 becomes a strict inequality. If, for example, $(m/n)<\lambda^{*}<(m+1)/n$ for some $m\epsilon\{0, 1, \cdots, n-1\}$, then, from Eq. A12, we see that the minimum of $F(x)$ is equal to either $f(m/n)$ or $f(m+1/n)$, whichever is smaller.

UNCLASSIFIED

Security Classification

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author)
Naval Ship Research and Development Center
Washington, D.C. 20007 (See Item 11)

2a. REPORT SECURITY CLASSIFICATION
UNCLASSIFIED

2b. GROUP

3. REPORT TITLE
LINEAR VERSUS LOGARITHMIC AVERAGING

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)
Final

5. AUTHOR(S) (First name, middle initial, last name)
Cox, Henry LCDR, USN

6. REPORT DATE
June 1967

7a. TOTAL NO. OF PAGES
7

7b. NO. OF REFS

8a. CONTRACT OR GRANT NO.

b. PROJECT NO.

c.

d.

9a. ORIGINATOR'S REPORT NUMBER(S)
2419

9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)

10. DISTRIBUTION STATEMENT
Distribution of this document is unlimited.

11. SUPPLEMENTARY NOTES
Successor to
David Taylor Model Basin and
Navy Marine Engineering Laboratory

12. SPONSORING MILITARY ACTIVITY
Naval Ship Research and Development Center

13. ABSTRACT

Consider n data samples $\{x_1, \ldots, x_n\}$ such that $0<L \leq x_i \leq U < \infty$. Let $K = U/L$; then it is shown that independent of n a lower bound on the ratio of the geometric mean to the arithmetic mean of the data samples is given by $[\ln K/(K-1)]K^{\{(1/\ln K)-1/(K-1)\}}$. This bound is useful in acoustic signal processing since it limits the amount of deviation that can be attributed to averaging logarithms vice taking the logarithm of the average of data samples. Both methods are currently in use at facilities specializing in the processing of acoustic data. For a K of 10 dB, for example, the geometric mean is less than 1.5 dB below the arithmetic mean.

DD FORM 1 NOV 65 1473 (PAGE 1)

S/N 0101-807-6801

UNCLASSIFIED

Security Classification