NORTHWESTERN UNIVERSITY
The Technological Institute
Department of Civil Engineering
Structural Mechanics Laboratory
Evanston, Illinois

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by J. D. Achenbach and G. Herrmann

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J. D. Achenbach*

and

G. Herrmann**

Northwestern University, Evanston, Illinois

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* Associate Professor, Department of Civil Engineering, Mem. AIAA

** Professor, Department of Civil Engineering, Mem. AIAA

Abstract

A set of displacement equations of motion is proposed that is suitable for the dynamic analysis of fiber-reinforced composites. In deriving the equations, representative elastic moduli are used for the binder, and the elastic and geometric properties of the fibers are combined into effective stiffnesses. With the aid of certain assumptions regarding the deformation of the fibers, and by employing a smoothing operation, approximate kinetic and strain energy densities for the fiber-reinforced composite are obtained. Application of Hamilton's principle yields the displacement equations of motion. The proposed set of equations is employed to study the propagation of plane harmonic waves propagating in the direction of the fibers and normal to the fiber direction. Plane transverse waves propagating in the direction of the fibers are dispersive, and dispersion curves are shown.

Introduction

In a previous report[1] the authors presented a method of deriving displacement equations of motion for fiber-reinforced composites. The method yields equations of motion according to what was termed the "effective stiffness theory," which differs from the well-known effective modulus theory, primarily because bending, shear, torsional and extensional stiffnesses of the reinforcing elements, together with appropriate functions describing the deformation, enter the strain energy density. For a laminate, the equations of motion were used to study the dispersion of plane time-harmonic transverse waves propagating in the direction of the layering.

In the present report the equations of motion according to the effective stiffness theory are worked out in detail for a fiber-reinforced composite. The method as presented earlier[1] is modified in order that wave propagation in the direction normal to the direction of the fibers can be included.

We consider a cylindrical element of length unity and cross-sectional area S of a fiber-reinforced composite. The fibers are unidirectional, randomly spaced, see Fig. 1, and the axis of the cylindrical element is parallel to the fibers. Suppose the element contains n fibers, each of circular cross-sectional area A. The fiber density η is then defined as

$$\eta = nA/S \tag{1}$$

Effective Modulus Theory

According to the effective modulus theory, the gross-scale elastic behavior of the fiber-reinforced composite can be studied by considering a homogeneous but anisotropic material. The effective Hooke's law for the composite is defined as

$$\sigma_{ij} = C_{ijkl}^{*} \epsilon_{kl} \tag{2}$$

The stresses and strains in Eq. (2) are averages over the volume considered, and the constants C_{ijkl}^{*} are effective elastic moduli whose number is determined by symmetry considerations. For random fiber arrangement, transverse isotropy may be assumed,[2] and the stress-strain relations may be written in terms of five elastic moduli,

$$\sigma_{11} = C_{11}^{*} \epsilon_{11} + C_{12}^{*} \epsilon_{22} + C_{12}^{*} \epsilon_{33} \tag{3}$$

$$\sigma_{22} = C_{12}^{*} \epsilon_{11} + C_{22}^{*} \epsilon_{22} + C_{23}^{*} \epsilon_{33} \tag{4}$$

$$\sigma_{33} = C_{12}^{*} \epsilon_{11} + C_{23}^{*} \epsilon_{22} + C_{22}^{*} \epsilon_{33} \tag{5}$$

$$\sigma_{12} = 2\, C_{44}^{*} \epsilon_{12} \tag{6}$$

$$\sigma_{13} = 2\, C_{44}^{*} \epsilon_{13} \tag{7}$$

$$\sigma_{23} = (C_{22}^{*} - C_{23}^{*})\epsilon_{23} \tag{8}$$

The propagation of time-harmonic waves in a transversely isotropic medium was discussed by Postma.[3] By substituting the stress-strain relations (3-8) in the stress equations of motion, we obtain the displacement equations of motion in the form

$$L_i^*[u_1,u_2,u_3] = \rho^* \ddot{u}_i \tag{9}$$

where i = 1, 2 and 3, and

$$L_1^*[\;] = C_{11}^* u_{1,11} + C_{44}^*(u_{1,22} + u_{1,33}) + C_{12}^* u_{2,21} + C_{44}^* u_{2,12} + (C_{12}^* + C_{44}^*)u_{3,13} \tag{10}$$

$$L_2^*[\;] = (C_{44}^* + C_{12}^*)u_{1,21} + C_{44}^* u_{2,11} + C_{22}^* u_{2,22} + \frac{1}{2}(C_{22}^* - C_{23}^*)u_{2,33} + \frac{1}{2}(C_{22}^* + C_{23}^*)u_{3,23} \tag{11}$$

The operator $L_3^*[\;]$ is obtained from Eq. (11) by replacing 3 by 2 and 2 by 3 in the subscripts of the displacement u. The effective density ρ^* is defined as

$$\rho^* = \eta\, \rho_f + (1 - \eta)\rho_b \tag{12}$$

where ρ_f and ρ_b are the mass densities of the fiber and binder materials, respectively, and η is the fiber density as defined by Eq. (1).

In studying the propagation of time-harmonic waves according to the effective modulus theory we consider waves propagating in the x_1-direction and in either the x_2- or the x_3-direction.

(a) Longitudinal waves propagating in the x_1-direction,

$$u_1 = U_1 \exp[ik(x_1 - c_{L1} t)] \tag{13}$$

where U_1 is a constant amplitude, and k and c_{L1} are the wave number and the phase velocity, respectively. By substitution of (13) into (9), we easily obtain

$$c_{L1} = (C_{11}^*/\rho^*)^{\frac{1}{2}} \tag{14}$$

In the following we list only the phase velocities of waves that are considered, namely,

(b) Transverse waves propagating in the x_1-direction,

$$c_{T1} = (C_{44}^*/\rho^*)^{\frac{1}{2}} \tag{15}$$

It can be checked that (15) is also the phase velocity for transverse waves in the x_1x_2 or x_1x_3 planes, propagating in the x_2- or x_3-directions, respectively.

(c) Longitudinal waves propagating in either the x_2- or the x_3-direction,

$$c_{L2} = (C_{22}^*/\rho^*)^{\frac{1}{2}} \tag{16}$$

(d) Transverse waves in the x_2x_3 plane propagating in either the x_2- or the x_3-direction,

$$c_{T2} = \left[\frac{1}{2}(C_{22}^* - C_{23}^*)/\rho^*\right]^{\frac{1}{2}} \tag{17}$$

For random fiber arrangement, approximate expressions for the constants C_{11}^*, C_{22}^* and C_{44}^*, and bounds for $C_{22}^* - C_{23}^*$ were presented by Hashin and Rosen.[2]

For waves that are infinitely long, the averaging on which the effective modulus theory is based is as justifiable for dynamic as for static problems. The expressions (14-17) may therefore be considered as the limit cases at infinite wave length of the actual phase velocities.

Effective Stiffness Theory

In the effective stiffness theory the fibers are considered as long and slender structural elements of circular cross section, which are endowed with stiffnesses relative to flexure, torsion and extension. In the actual composite the fibers are embedded in an isotropic elastic binder, and interaction of the fiber and binder deformations takes place through continuity of displacements and tractions at the fiber-binder interfaces. In the present approximate theory, however, we allow interaction only through the displacement of the centerline of the fiber and the displacement of the binder. Because of this restricted interaction the fiber does not affect deformations of the composite in planes normal to the fiber direction. Rather than accept this as an inadequacy of the present theory we assume that the contribution of the fibers to the stiffnesses of the composite in planes normal to the fiber direction can be included by redefining the binder as a transversely isotropic material. In this manner this fictitious binder, henceforth referred to as the effective matrix, includes part of the effect of the fibers, namely, the part which is not coupled to deformation of the fibers in flexure, torsion and extension. In the sequel it is shown that four of the five elastic constants of the effective matrix are equal to the corresponding effective moduli of the effective modulus theory. The constant related to longitudinal deformation in the fiber direction differs from the corresponding constant of the effective modulus theory.

We consider the kth fiber, whose centerline is defined by $x_2 = x_2^k$, $x_3 = x_3^k$, and we define a local coordinate system x_1, x_2', x_3' with axes parallel to x_1, and with origin at $x_1 = 0$, $x_2 = x_2^k$ and $x_3 = x_3^k$, see Figs. 1,2. The displacements at the centerline of the kth fiber are denoted by $u_{0i}^{fk}\left(x_1, x_2^k, x_3^k, t\right)$. The fibers are of circular cross-sectional area, with radius a. It is assumed that the cross section of a fiber can transmit bending moments M_2 and M_3, a torsional moment M_1, shear forces Q_2 and Q_3 and a normal force Q_1. The moments and forces are shown with positive sense in Fig. 2.

In relating the bending moments and the shear forces to the deformation of the fiber we include the effect of transverse shear through the well-known Timoshenko theory for the flexure of beams. For the kth fiber we write

$$M_2^k = E_f I_2 \, \partial\psi_2^k/\partial x_1 \tag{18}$$

$$Q_2^k = \kappa \, \mu_f A \left(\partial u_{02}^{fk}/\partial x_1 - \psi_3^k\right) \tag{19}$$

$$M_3^k = E_f I_3 \, \partial\psi_3^k/\partial x_1 \tag{20}$$

$$Q_3^k = \kappa \, \mu_f A \left(\partial u_{03}^{fk}/\partial x_1 + \psi_2^k\right) \tag{21}$$

$$M_1 = \mu_f I_p \, \partial\psi_1^k/\partial x_1 \tag{22}$$

$$Q_1 = E_f A \, \partial u_{01}^{fk}/\partial x_1 \tag{23}$$

where

$$I_2 = I_3 = \frac{1}{4}\pi a^4, \qquad I_p = \frac{1}{2}\pi a^4 \tag{24a,b}$$

In Eqs. (18-23), E_f, μ_f and A are Young's modulus, the shear modulus and the cross-sectional area, respectively. The rotations ψ_1^k, ψ_2^k and ψ_3^k are shown with positive sense in Fig. 2. The shear forces Q_2^k and Q_3^k are related to the corresponding shear deformations through the shear stiffness $\varkappa\mu_f A$, where $\varkappa$ is the Timoshenko shear coefficient, which for a fiber of circular cross section[4] assumes the value $\varkappa = 0.847$. The torsional moment M_1^k and axial force Q_1^k are related to the cross-sectional rotation ψ_1^k and the elementary longitudinal strain through the torsional stiffness $\mu_f I_p$ and the uniaxial stiffness $E_f A$, respectively. These assumptions correspond to essentially a state of one-dimensional stress in the fiber. This assumption is related to the assumption, to be elaborated upon in the sequel, that interaction with the surrounding medium takes place only through the displacement of the centerline of the fiber, $u_{01}^{fk}\left(x_1, x_2^k, x_3^k, t\right)$.

For the shear-deformation theory the fiber displacements are of the form

$$u_1^{fk} = u_{01}^{fk} - x_2' \psi_3^k + x_3' \psi_2^k \tag{25}$$

$$u_2^{fk} = u_{02}^{fk} - x_3' \psi_1^k \tag{26}$$

$$u_3^{fk} = u_{03}^{fk} + x_2' \psi_1^k \tag{27}$$

The moment- and force-deformation relations (18-23) and the displacement distributions (25-27) can be used to compute approximate expressions for the kinetic energy T_f^k and the strain energy V_f^k per unit length of the kth fiber. The energy densities are defined at n discrete points $\left(x_2^k, x_3^k\right)$, and the total kinetic and strain energies stored in the n fibers of the cylindrical element S are obtained by addition. It is now assumed that the summation over the n discrete points $\left(x_2^k, x_3^k\right)$ may be replaced by an integration,

$$\sum^{n} T_f^{k}\left(x_1, x_2^{k}, x_3^{k}\right) \simeq \frac{\eta}{A}\int_S T_f(x_1, x_2, x_3)dx_2dx_3 \tag{28}$$

$$\sum^{n} V_f^{k}\left(x_1, x_2^{k}, x_3^{k}\right) \simeq \frac{\eta}{A}\int_S V_f(x_1, x_2, x_3)dx_2dx_3 \tag{29}$$

where η is the fiber density as defined by Eq. (1), and $A = \pi a^2$ is the cross-sectional area of a fiber. By means of the smoothing operations (28) and (29) the field variables u_{0i}^{fk}, ψ_1^{k}, etc., which were previously defined only at discrete points $\left(x_2^{k}, x_3^{k}\right)$, now have become continuously varying functions of x_2 and x_3 as well as of x_1. To indicate this change we omit, henceforth, the superscript k, i.e., we write u_{0i}^{f}, ψ_1, etc. By means of Eqs. (28) and (29) we also have introduced energy densities T_f and V_f, whose functional dependence on u_{0i}^{f}, ψ_i, etc., is the same as that of T_f^{k} and V_f^{k} on u_{0i}^{fk}, ψ_i^{k}, etc.

For the effective matrix, the kinetic energy density is of the form

$$T_m = \frac{1}{2}\rho_b \dot{u}_i^{m}\dot{u}_i^{m} \tag{30}$$

Similarly, the strain energy density of the effective matrix is written

$$V_m = \frac{1}{2}\sigma_{ij}^{m}\epsilon_{ij}^{m} \tag{31}$$

where

$$\epsilon_{ij}^{m} = \frac{1}{2}\left(u_{i,j}^{m} + u_{j,i}^{m}\right) \tag{32}$$

Since the effective matrix is transversely isotropic, the stresses σ_{ij}^{m} are related to the strains ϵ_{ij}^{m} through five constants C_{11}, C_{12}, C_{22}, C_{23} and C_{44} by relations of the form (3-8).

The inertias of the effective matrix and the fiber being purely additive, we assume that the kinetic energy of the effective matrix stored in the cylindrical element S, Fig. 1, may be written

$$(1 - \eta) \int_B T_m(x_1, x_2, x_3) dx_1 dx_2 dx_3 \tag{33}$$

where T_m is defined by Eq. (30). The strain energy of the effective matrix includes, however, part of the effect of the fibers, and the strain energy stored in the element is thus

$$\int_B V_m(x_1, x_2, x_3) dx_1 dx_2 dx_3 \tag{34}$$

From Eqs. (28), (29), (33) and (34) it follows that the approximate kinetic and strain energy densities of the fiber-reinforced composite may be written as

$$T = \frac{\eta}{A} T_f + (1 - \eta) T_m \tag{35}$$

$$V = \frac{\eta}{A} V_f + V_m \tag{36}$$

The energy densities (35) and (36) involve both $u_{0i}{}^{f}$ and $u_i{}^{m}$. We now define, however, these two field variables as being one and the same, henceforth referred to as u_i. In this manner we secure interaction of the fibers and the matrix, essentially through continuity of the displacements at the centerlines of the fibers.

To obtain the field equations for $u_i(\underset{\sim}{x}, t)$, $\psi_i(\underset{\sim}{x}, t)$ we invoke Hamilton's principle

$$\delta \int_{t_0}^{t} \int_B (T - V) dx_1 dx_2 dx_3 dt = 0 \tag{37}$$

where T and V are defined by Eqs. (35) and (36). It is well established in the calculus of variations that the condition (37) is satisfied if the dependent variables are solutions of the following set of partial differential equations

$$\sum_{r=1}^{4} \frac{\partial}{\partial q_r} \left[\frac{\partial(T - V)}{\partial(\partial f_s/\partial q_r)}\right] - \frac{\partial(T - V)}{\partial f_s} = 0 \qquad (38)$$

where f_s are the dependent variables, and q_r are the independent variables x_i and t. There are thus six displacement equations of motion. Certain stiffnesses of the fibers and the elastic constants of the effective matrix appear as coefficients in the set of equations. The equations that are obtained are similar in form to equations for elasticity with microstructure that were obtained by Mindlin.[5] For fiber-reinforced composites an analogous system of equations for static problems, and including elementary bending, torsion and extension only, was proposed earlier by V. V. Bolotin.[6]

Displacement Equations of Motion

The kinetic energy in an element of unit length of the kth fiber is

$$T_f^{\,k} = \int_A \frac{1}{2} \rho_f \, \dot{u}_i^{\,fk} \, \dot{u}_i^{\,fk} \, dA \qquad (39)$$

By substituting Eqs. (25-27) into Eq. (39), and by invoking Eq. (28), we obtain for a circular cylindrical fiber of radius a,

$$T_f/A = \frac{1}{2} \rho_f \left\{(\dot{u}_1)^2 + (\dot{u}_2)^2 + (\dot{u}_3)^2 + r^2 \left[2(\dot{\psi}_1)^2 + (\dot{\psi}_2)^2 + (\dot{\psi}_3)^2\right]\right\} \qquad (40)$$

In Eq. (40), r is the radius of gyration,

$$r^2 = I_2/A = I_3/A = a^2/4 \qquad (41)$$

The components of strain in the kth fiber are computed from the approximate displacement distributions (25-27) by equations of the type (32). By integrating the general expression for the strain energy density of the type Eq. (31) over the cross section of the fiber, we obtain

$$2V_f^{\,k} = M_1^{\,k}\,\frac{\partial\psi_1^{\,k}}{\partial x_1} + M_2^{\,k}\,\frac{\partial\psi_2^{\,k}}{\partial x_1} + M_3^{\,k}\,\frac{\partial\psi_3^{\,k}}{\partial x_1} + Q_1^{\,k}\,\frac{\partial u_{01}^{\,fk}}{\partial x_1} + Q_2^{\,k}\left(\frac{\partial u_{02}^{\,fk}}{\partial x_1} - \psi_3^{\,k}\right) + Q_3^{\,k}\left(\frac{\partial u_{03}^{\,fk}}{\partial x_1} + \psi_2^{\,k}\right) \qquad (42)$$

By invoking Eq. (29) and the discussion following that equation, we obtain, after substitution of Eqs. (18-23) in Eq. (42),

$$V_f/A = \frac{1}{2}\,E_f\left\{(u_{1,1})^2 + r^2\left[(\psi_{2,1})^2 + (\psi_{3,1})^2\right]\right\} + \frac{1}{2}\,\kappa\mu_f\left[(u_{2,1} - \psi_3)^2 + (u_{3,1} + \psi_2)^2\right] + \mu_f r^2(\psi_{1,1})^2 \qquad (43)$$

The energy densities according to the effective stiffness theory are obtained by substitution of Eqs. (40) and (43) into Eqs. (35) and (36). The system of six field equations for u_i and ψ_i obtained from the Euler equations, (38), is

$$L_1[u_1, u_2, u_3] + \eta\,E_f\,u_{1,11} = \rho^*\,\ddot{u}_1 \qquad (44)$$

$$L_2[u_1, u_2, u_3] + \eta\kappa\mu_f(u_{2,11} - \psi_{3,1}) = \rho^*\,\ddot{u}_2 \qquad (45)$$

$$L_3[u_1, u_2, u_3] + \eta\kappa\mu_f(u_{3,11} + \psi_{2,1}) = \rho^*\,\ddot{u}_3 \qquad (46)$$

$$\mu_f\,\psi_{1,11} = \rho_f\,\ddot{\psi}_1 \qquad (47)$$

$$E_f r^2\,\psi_{2,11} - \kappa\mu_f(u_{3,1} + \psi_2) = \rho_f r^2\,\ddot{\psi}_2 \qquad (48)$$

$$E_f r^2 \psi_{3,11} + \kappa\mu_f(u_{2,1} - \psi_3) = \rho_f r^2 \ddot{\psi}_3 \quad (49)$$

The operators $L_1[\ ]$, $L_2[\ ]$ and $L_3[\ ]$ are defined by Eqs. (10,11), except that the constants C_{ij}^* must be replaced by constants C_{ij}. The effective mass density ρ^* is defined by Eq. (12).

Equations (44-49) are the displacement equations of motion for the fiber-reinforced composite according to the effective stiffness theory. The merits of the effective stiffness theory, as well as its inadequacies, can be shown by considering the propagation of plane time-harmonic waves in the principal directions.

Propagation of Plane Waves

It is noted from Eqs. (44-49) that the angle of rotation ψ_1 is not coupled to the other dependent field variables, and that ψ_1 may arbitrarily depend on x_2 and x_3, but can propagate in the x_1-direction only. The phase velocity for propagation in the x_1-direction is

$$c_{Tf} = (\mu_f/\rho_f)^{\frac{1}{2}} \quad (50)$$

According to the present approximate theory, torsional waves can propagate in the fibers without interaction with the surrounding matrix. This result, which demonstrates an obvious inadequacy of the present theory, is a consequence of the assumptions principally embodied in Eqs. (25-27), implying that fiber and matrix interact only through continuity of displacement at the centerline of the fiber. The absence of interaction between fiber and matrix becomes more acceptable the higher the ratio μ_f/μ_m and the smaller the fiber density η.

We now consider solutions of Eqs. (44-46) and (48-49) representing plane waves propagating in the x_1-direction. If the field variables are independent of x_2 and x_3, Eqs. (48) and (49) are unchanged and Eqs. (44-46) reduce to

$$[C_{11} + \eta E_f]u_{1,11} = \rho^* \ddot{u}_1 \tag{51}$$

$$[C_{44} + \eta\kappa\mu_f]u_{2,11} - \eta\kappa\mu_f \psi_{3,1} = \rho^* \ddot{u}_2 \tag{52}$$

$$[C_{44} + \eta\kappa\mu_f]u_{3,11} + \eta\kappa\mu_f \psi_{2,1} = \rho^* \ddot{u}_3 \tag{53}$$

It is apparent that the system of equations for plane waves, Eqs. (51-53) and (48-49), splits up into three subsystems. Longitudinal motion is governed by Eq. (51), and transverse motion is governed by two equivalent systems (52,49) and (53,48).

According to Eq. (51), longitudinal waves propagating in the x_1-direction are not dispersive, and the phase velocity is obtained as

$$c = [(C_{11} + \eta E_f)/\rho^*]^{\frac{1}{2}} \tag{54}$$

Time-harmonic waves propagating in the x_1-direction representing transverse motion in the x_2-direction are defined by solutions of the form

$$u_2 = U_2\exp[ik(x_1 - ct)] \tag{55}$$

$$\psi_3 = \Psi_3\exp[ik(x_1 - ct)] \tag{56}$$

In Eqs. (55) and (56), k is the wave number, c is the phase velocity, and U_2 and Ψ_3 are constants. The dispersion equation is obtained by substituting Eqs. (55,56) into Eqs. (52) and (49), and by requiring that the determinant of the coefficients of U_2 and Ψ_3 vanish. We obtain

$$\left[C_{44} + \eta\varkappa\mu_f - c^2\rho^*\right]\left[E_f r^2 k^2 + \varkappa\mu_f - \rho_f r^2 k^2 c^2\right] - \eta\varkappa^2\mu_f^2 = 0 \tag{57}$$

In the limit of vanishing wave number we obtain

$$\lim_{k \to 0} c = [C_{44}/\rho^*]^{\frac{1}{2}} \tag{58}$$

The equations governing the propagation of plane waves propagating in the x_2-direction are obtained from Eqs. (44-49) as

$$C_{44}\, u_{1,22} = \rho^* \ddot{u}_1 \tag{59}$$

$$C_{22}\, u_{2,22} = \rho^* \ddot{u}_2 \tag{60}$$

$$\frac{1}{2}\,(C_{22} - C_{23})u_{3,22} = \rho^* \ddot{u}_3 \tag{61}$$

$$-\varkappa\mu_f\, \psi_2 = \rho_f r^2 \ddot{\psi}_2 \tag{62}$$

$$-\varkappa\mu_f\, \psi_3 = \rho_f r^2 \ddot{\psi}_3 \tag{63}$$

Propagation of plane waves in the x_3-direction is governed by a system of equations equivalent to Eqs. (59-63). It is noted that the propagation of plane waves in the x_2- and x_3-directions is not dispersive, and the phase velocities are obtained by inspection from Eqs. (59-61). The remaining two equations (62,63) show that in this case the rotations ψ_2 and ψ_3 are uncoupled time-periodic standing shear vibrations, apparently of arbitrary dependence on x_2 and x_3.

The elastic constants of the effective matrix are obtained from the observation that the phase velocities at vanishing wave numbers should equal the phase velocities according to the effective modulus theory. From Eqs. (14) and (54) we thus conclude

$$C_{11} = C_{11}^{*} - \eta E_f \tag{64}$$

Comparison of Eqs. (15) and (58) yields

$$C_{44} = C_{44}^{*} \tag{65}$$

The effective stiffness theory does not account for dispersion of waves propagating in directions normal to the x_1-axis, and by choosing $C_{22} = C_{22}^{*}$, $C_{23} = C_{23}^{*}$ we obtain the same constant phase velocities as predicted by the effective modulus theory. The remaining constant C_{12} cannot be obtained from a comparison of a wave velocity in an unbounded solid. This constant, which gives the stress in x_2- or x_3-direction for a state of one-dimensional strain in x_1-direction, is also chosen as $C_{12} = C_{12}^{*}$, since the fibers in the present theory are then in a state of one-dimensional stress.

Discussion of Results

Dispersion curves for transverse waves propagating in the x_1-direction, relating the dimensionless phase velocity and the dimensionless wave number, are easily computed from Eq. (57). To compute C_{44} from Eq. (65), we use an approximate expression for C_{44}^{*} derived by Hashin and Rosen,[2]

$$C_{44} = \mu_b \frac{\gamma(1 + \eta) + (1 - \eta)}{\gamma(1 - \eta) + (1 + \eta)} \tag{66}$$

where η is defined by Eq. (1), and

$$\gamma = \mu_f/\mu_b \tag{67}$$

In Eq. (67), μ_b is the shear modulus of the binder. The dispersion equation (57) may then be rewritten as

$$\left\{ \frac{\gamma(1 + \eta) + (1 - \eta)}{\gamma(1 - \eta) + (1 + \eta)} + \eta\kappa\gamma - \beta^2(\eta\theta + 1 - \eta) \right\}$$

$$\times \left\{ \frac{1}{2} (1 + \nu_f)\gamma\xi^2 + \kappa\gamma - \frac{1}{4} \theta\beta^2\xi^2 \right\} - \eta\kappa^2\gamma^2 = 0 \qquad (68)$$

where

$$\theta = \rho_f/\rho_b \qquad (69)$$

The dimensionless phase velocity is defined as

$$\beta = c/(\mu_b/\rho_b)^{\frac{1}{2}} \qquad (70)$$

The dimensionless wave number is

$$\xi = ka \qquad (71)$$

The independent parameters in Eq. (68) are γ, θ, η and ν_f. Numerical values for the parameters are employed that are representative for boron-epoxy and glass-epoxy composites.[7] For all computations we use $\nu_f = 0.20$ and $\theta = 3$. The ratio of the shear moduli, γ, was chosen as 100 for a boron-epoxy composite and 20 for a glass-epoxy composite. For various values of the fiber density, η, dispersion curves for transverse motion are shown in Fig. 3. It is noted that the dispersion curves depart at small values of the dimensionless wave number ξ from the horizontal lines representing the phase velocities according to the effective modulus theory. The dispersion is particularly notable for the boron-epoxy composite. It must be realized, however, that for boron filament small values of the dimensionless wave number ka correspond to quite short waves. The Texaco Experiment Incorporated, who manufactures boron filament under the trademark <u>borofil</u>, reports values for a of about 0.004 inch, and thus at $\xi = 1$, where for $\eta = 0.5$ the phase velocity is about 1.9 times the value according to the effective modulus theory, the actual wave length is about 0.025 inch.

In the approximate description of the motion of the kth fiber, the effect of thickness stretch motion can be included without much additional difficulty. If this is done, the approximate theory can account for dispersion of longitudinal waves propagating in the direction of the fibers. It was found, however, that such dispersion is very weak in the range of dimensionless wave numbers over which most of the dispersion of transverse waves occurs. It can also be argued that if the first mode of thickness stretch is included, the first symmetrical thickness shear mode should also be included. To be consistent, one should thus include both, which would give rise to additional complications, or neither one, and realize that the theory is valid for a limited range of wave lengths. Since the diameters of fibers are small, this range may include quite short actual wave lengths.

For a fiber-reinforced composite a comparison of the present approximate dispersion curves with exact curves obtained by employing elasticity equations for fiber and binder is virtually out of the question. A comparison of dispersion curves according to the effective stiffness theory for a laminate with curves obtained from a study of time-harmonic waves in a stratified medium[8] was, however, carried out in an earlier paper.[1] For transverse motion, good agreement was found in the range of dimensionless wave numbers in which dispersion is particularly pronounced. It was also found that the dispersion of longitudinal waves for small wave numbers is negligible as compared to dispersion of transverse waves, which for laminates justifies the omission of thickness stretch motion of the reinforcing elements.

References

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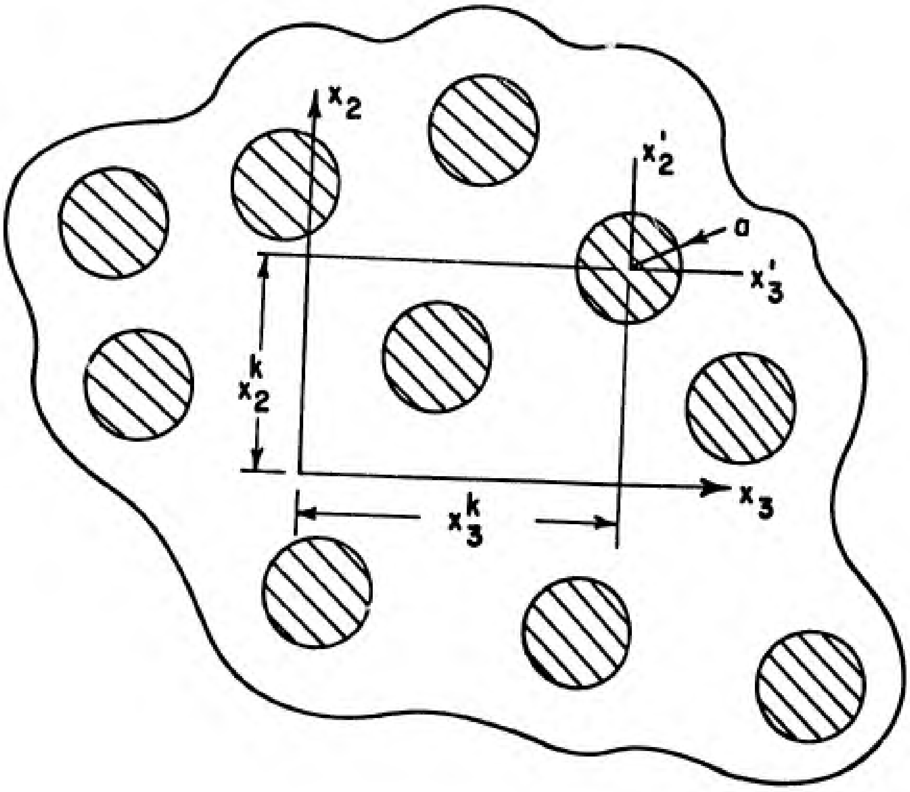

Fig. 1 Fiber-Reinforced Composite

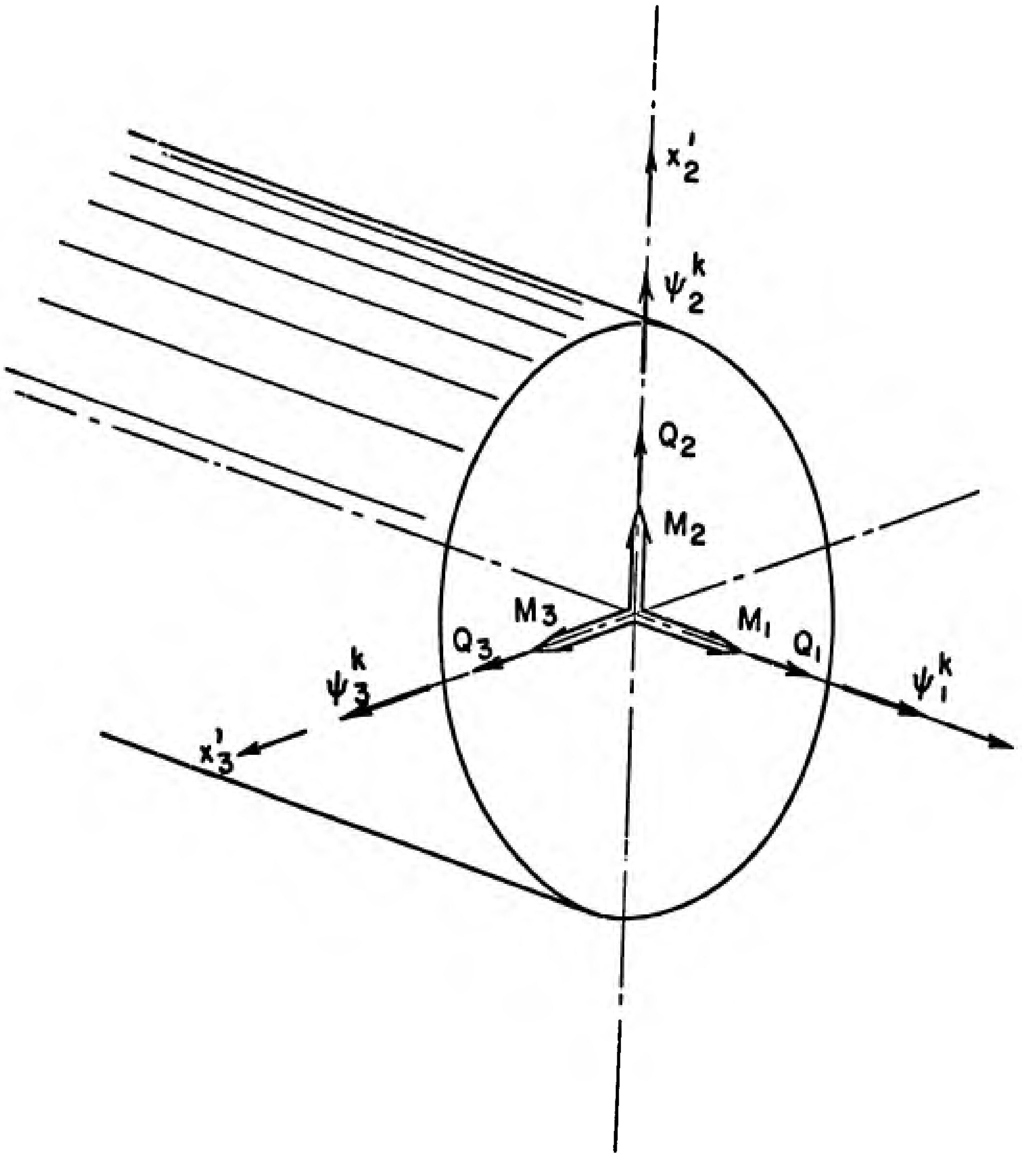

Fig. 2 Forces and Moments Acting in Cross Section of Fiber

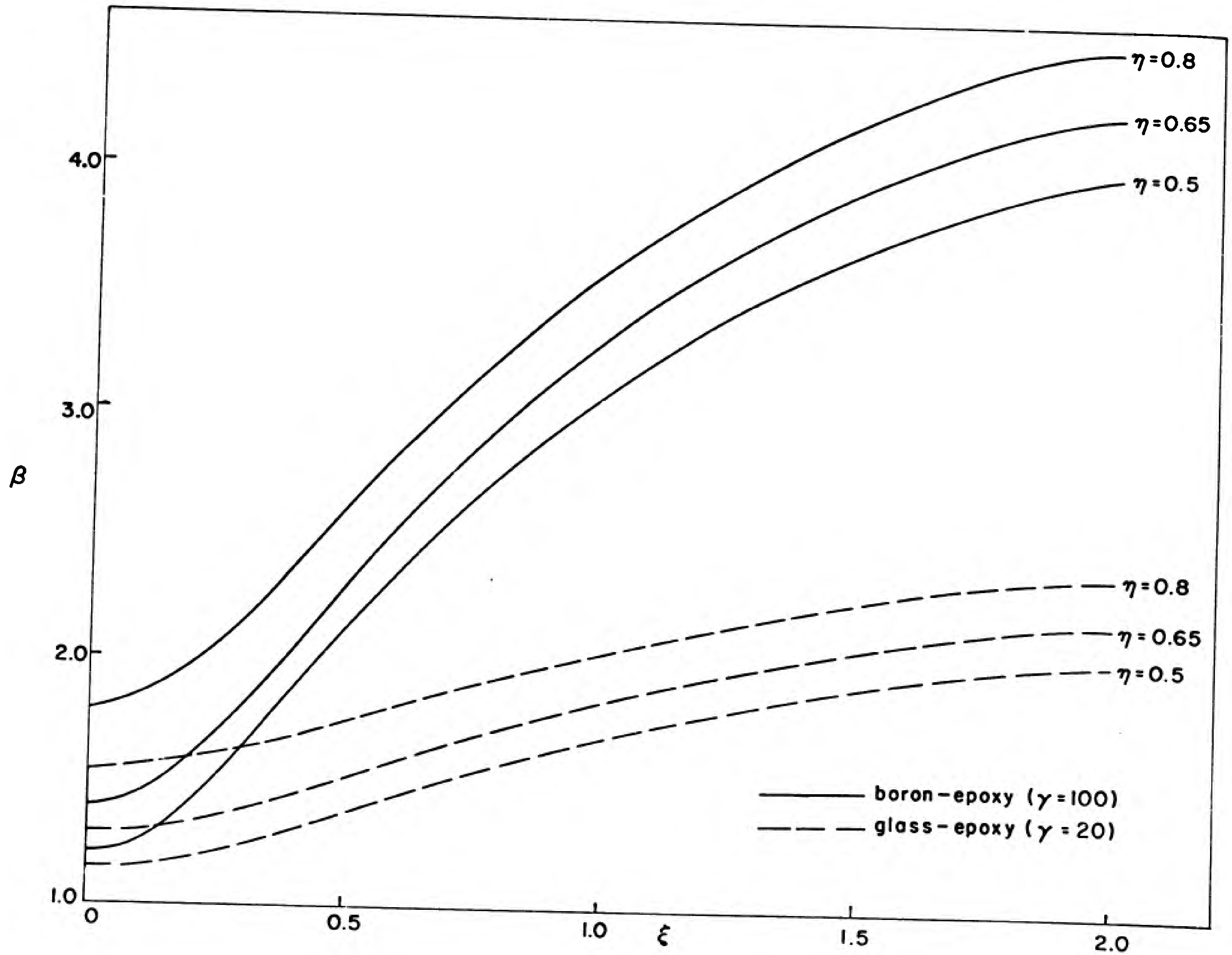

Fig. 3 Dispersion Curves for Transverse Waves