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NONLINEAR ESTIMATION TECHNIQUES

by

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ABSTRACT. A study was conducted in which a least squares criterion was used to solve a discrete, real-time nonlinear estimation problem. The technique centers around the derivation of a sequential algorithm which allows consideration of second-order nonlinearities in system measurements. Alternate nonlinear estimation techniques are discussed, and comparative examples of the various estimation algorithms are presented.

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FOREWORD

A major program presently being conducted by the Naval Weapons Center encompasses the design and development of experimental weapon control systems and their components. As part of this program, the various aspects of target acquisition, using direction-finding and distance-measurement estimation techniques, are being studied. This report considers a discrete, real-time nonlinear estimation problem using a least squares criterion.

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WHITE SECTION
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Nonlinear Estimation Techniques

Summary

This paper considers a discrete, real time nonlinear estimation problem using a least squares criterion. A sequential algorithm is derived which allows consideration of second order nonlinearities in system measurements. Alternate nonlinear estimation techniques are discussed and examples are given which compare the various estimation algorithms.

Introduction

A sequential solution to the optimal linear (gaussian) estimation problem is well known [3, 4, 5][1] and has been used extensively. The problem of real time (i.e. sequential) nonlinear filtering has received considerable attention with varying degrees of success [1, 2, 6, 7, 8]. The typical approach to the sequential nonlinear estimation problem consists of a local linearization of the various nonlinearities together with an application of the linear theory [1, 8]. Sridhar and Detchmendy [2] have considered a continuous nonlinear filtering problem from a least square point of view with results which allow second order nonlinearities in the observations to be considered in the computation of the filter gain matrix.

This paper first considers a discrete nonlinear estimation problem using a least squares criterion. A sequential algorithm is derived which allows consideration of second order nonlinearities in system measurements. The Sridhar-Detchmendy filter [2] is then obtained by a straightforward limiting argument.

In order to test the utility of the derived algorithm, the more popular approaches to nonlinear estimation are discussed and examples are given which compare the various estimation algorithms.

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[1] Numbers enclosed in brackets designate references.

1. Problem Formulation and Solution:

Consider a system of the form

$$x_{k+1} = f(x_k,\ u_k,\ k) + \Delta_k w_k \tag{1}$$

$$y_k = h(x_k,\ k) + v_k \tag{2}$$

where x, y, u, v, w are column vectors and h, f represent vector valued functions of their arguments. The vector u represents a deterministic system input, w represents a stochastic system input distributed into the system by the matrix Δ while v represents measurement error.

Based on the measurements $y_1, \ldots, y_n$, a sequential estimate of x_n is desired. In order to obtain a solution, this problem is imbedded in a somewhat larger problem, namely obtain those estimates

$$\{\hat{x}_{i,n}\},\ \{\hat{w}_{i,n}\}$$

which minimize

$$J_n = \|\hat{x}_{0,n} - \hat{x}_{0,0}\|^2_{P_{0,0}^{-1}} + \sum_{i=0}^{n-1}\left(\|y_{i+1} - h(\hat{x}_{i+1,n},\ i+1)\|^2_{R_{i+1}^{-1}} + \|\hat{w}_{i,n}\|^2_{Q_i^{-1}}\right) \tag{3}$$

subject to the constraints

$$\hat{x}_{i+1,n} = f(\hat{x}_{i,n},\ u_i,\ i) + \Delta_i \hat{w}_{i,n},\ i = 0, \ldots, n-1 \tag{4}$$

When the random variables w_i, v_i, x_0 are independent and gaussian with covariance matrices given respectively by Q_i, R_i, $P_{0,0}$ then (3)[2] results from a maximum likelihood criterion [1]. However, in

[2] Numbers enclosed by parentheses ( ) designate equations.

a practical situation very little is known about the statistics of the several variables. At best, estimates of the first and second moments of each component of the vector random variables are available. In such a situation, (3) may be considered as simply a "least squares" criterion. The positive definite matrices $P_{o,o}$, R_i, Q_i can be diagonal with estimates of the second moments placed at the appropriate coordinate. To simplify notation, the arguments u_k, k will normally be suppressed in the sequel.

To obtain the desired estimates, introduce multiplier vectors $\{2\lambda_{i,n}\}$ and form the functional

$$C_n = J_n + 2 \sum_{i=0}^{n-1} \lambda_{i,n}^T [\hat{x}_{i+1,n} - f(\hat{x}_{i,n}) - \Delta_i \hat{w}_{i,n}]$$

Performing the required differentiation and equating the various derivatives to zero yields the following two point boundary value problem:

$$\hat{x}_{i+1,n} = f(\hat{x}_{i,n}) + \Delta_i Q_i \Delta_i^T \lambda_{i,n}, \quad i = 0, \ldots, n-1 \tag{5}$$

$$\lambda_{i-1,n} = f_x^T(\hat{x}_{i,n})\lambda_{i,n} + h_x^T(\hat{x}_{i,n})R_i^{-1}[y_i - h(\hat{x}_{i,n})], \quad i = 1, \ldots, n \tag{6}$$

$$\hat{x}_{o,n} = \hat{x}_{o,o} + P_{o,o} f_x^T(\hat{x}_{o,n})\lambda_{o,n} \tag{7}$$

$$\lambda_{n,n} = 0 \tag{8}$$

An approximate solution to this problem can be obtained by induction on n and i [cf. Appendix A] and is given by[3]

$$\hat{x}^*_{i,n} = \hat{x}^*_{i,n-1} + P^{(n)}_{i,i} f_x^T(\hat{x}^*_{i,n-1})(\lambda_{i,n} - \lambda_{i,n-1}), \quad n \geq 1, \quad i = 0, \ldots, n-1 \tag{9}$$

---

[3] Following Cox [1], the notation A^* is used to denote an approximation of A. The superscript (n) is used to emphasize the dependence of the various variables on n.

where

$$P^{(n)}_{0,0} = P_{0,0}, \; n \geq 1 \tag{10}$$

$$P^{(n)}_{i,i} = \left| I - P^{(n)}_{i,i-1} \frac{\partial}{\partial x} [h_x^T R_i^{-1} (y_i - h)]_{\hat{x}^*_{i,n-1}} \right|^{-1} P^{(n)}_{i,i-1},$$

$$n \geq 2, \; i = 1, \; \ldots, \; n-1 \tag{11}$$

$$P^{(n)}_{1,0} = P_{1,0} = f_x(\hat{x}_{0,0}) P_{0,0} f_x^T(\hat{x}_{0,0}) + \Delta_0 Q_0 \Delta_0^T, \; n \geq 2 \tag{12}$$

$$P^{(n)}_{i,i-1} = f_x(\hat{x}^*_{i-1,n-1}) P^{(n)}_{i-1,i-1} f_x^T(\hat{x}^*_{i-1,n-1}) + \Delta_{i-1} Q_{i-1} \Delta_{i-1}^T,$$

$$n \geq 3, \; i = 2, \; \ldots, \; n-1 \tag{13}$$

The approximate solution (9-13) to the two point boundary value problem (5-8) can be used to approximate $\hat{x}_{n,n}$ [cf. Appendix A]. The results are given below for $n \geq 1$:

$$\hat{x}^*_{n,n-1} = f(\hat{x}^*_{n-1,n-1}) \tag{14}$$

$$\hat{x}^*_{n,n} = \hat{x}^*_{n,n-1} + K_n [y_n - h(\hat{x}^*_{n,n-1})] \tag{15}$$

$$K_n = P_{n,n} h_x^T(\hat{x}^*_{n,n-1}) R_n^{-1} \tag{16}$$

$$P_{n,n} = \left| I - P_{n,n-1} \frac{\partial}{\partial x} [h_x^T R_n^{-1} (y_n - h)]_{\hat{x}^*_{n,n-1}} \right|^{-1} P_{n,n-1} \tag{17}$$

$$P_{n,n-1} = f_x(\hat{x}^*_{n-1,n-1}) P_{n-1,n-1} f_x^T(\hat{x}^*_{n-1,n-1}) + \Delta_{n-1} Q_{n-1} \Delta_{n-1}^T \tag{18}$$

The computations are initiated with $\hat{x}_{0,0}$, $P_{0,0}$.

Cox [1] and others have considered the prediction of x_n based on $y_1, \ldots, y_{n-1}$ with results which furnish some justification for the definition and notation used in (14). Throughout the remainder of this paper, (14) shall be considered as one-sample prediction while (15) shall be referred to as filtering.

Assuming that certain components of (17) are invertable, alternate forms of (16, 17) can be obtained [cf. Appendix A] and are given below:

$$K_n = S_n h_x^T(\hat{x}^*_{n,n-1})[h_x(\hat{x}^*_{n,n-1})S_n h_x^T(\hat{x}^*_{n,n-1}) + R_n]^{-1} \tag{16'}$$

$$S_n = [I - P_{n,n-1} L_n]^{-1} P_{n,n-1} \tag{16''}$$

$$P_{n,n} = [I - K_n h_x(\hat{x}^*_{n,n-1})] S_n \tag{17'}$$

$$L_n = (\ell_{ij}(n)), \quad \ell_{ij}(n) = \frac{\partial^2 h(\hat{x}^*_{n,n-1})}{\partial x_j \partial x_i} R_n^{-1} [y_n - h(\hat{x}^*_{n,n-1})] \tag{18}$$

where x_i, x_j denote respectively the i^{th} and j^{th} components of the vector x.

In the last form, it is apparent that (16″) is the only additional computing required over the usual linearization of plant dynamics and observations. Essentially, (16″) represents a modification of the matrix $P_{n,n-1}$ based on an appropriate consideration of the second order nonlinearities of the observations h. It is interesting to note the close resemblance of (15-17) to the computing algorithm resulting from the minimization of $||y - h(x)||^2_{R^{-1}}$ using steepest descent.

2. A Continuous Problem

The continuous version of the estimation problem under consideration has been formulated and solved by Sridhar and Detchmendy [2]. Their results can also be established by using the classical method of finite differences and the discrete algorithm (14-18).

The dynamics and observations of the continuous problem are modeled as

$$\dot{x}(t) = g(x, u, t) + G(t)w(t), \quad 0 \le t \le T \tag{21}$$

$$y(t) = h(x, t) + v(t) \quad , \quad 0 \le t \le T \tag{22}$$

where the various variables are defined as above. The variables x, y, u, w, v, G are continuous, g is continuously differentiable with respect to x and h is twice continuously differentiable with respect to x.

Given $y(t)$, $0 \le t \le T$, estimates $\hat{x}(t;T)$, $\hat{w}(t;T)$ of $x(t)$, $w(t)$ respectively are desired which will minimize

$$J(T) = ||\hat{x}(0;T) - \hat{x}(0;0)||^2_{P_0^{-1}}$$

$$+ \int_0^T \left(||y(t) - h[\hat{x}(t;T), t]||^2_{R^{-1}(t)} + ||\hat{w}(t;T)||^2_{Q^{-1}(t)}\right)dt \quad (23)$$

subject to the constraint

$$\dot{\hat{x}}(t;T) = g[\hat{x}, u, t] + G(t)\hat{w}(t;T), \; 0 \le t \le T \quad (24)$$

A discrete approximation to the preceding problem is obtained by partitioning the interval $[0, T]$ into n equal subintervals $[t_i, t_{i+1}]$ each of length Δt. After suppressing the arguments u, t, the results are

$$x_{i+1} = f(x_i) + \Delta_i w_i \quad i = 0, \ldots, n-1 \quad (25)$$

$$y_i = h(x_i) + v_i \quad i = 1, \ldots, n \quad (26)$$

$$J_n = ||\hat{x}_{0,n} - \hat{x}_{0,0}||^2_{P_0^{-1}} + \sum_{i=0}^{n-1} ||y_{i+1} - h(\hat{x}_{i+1,n})||^2_{\Delta t R_{i+1}^{-1}}$$

$$+ \sum_{i=0}^{n-1} ||\hat{w}_{i,n}||^2_{\Delta t Q_i^{-1}} \quad (27)$$

$$\hat{x}_{i+1,n} = f(\hat{x}_{i,n}) + \Delta_i \hat{w}_{i,n}, \quad i = 0, \ldots, n-1 \quad (28)$$

where

$$f(x_i) = x_i + \Delta t\, g(x_i) + o(\Delta t) \quad (29)$$

$$\Delta_i = \Delta t\, G(t_i) + o(\Delta t) \quad (30)$$

Applying the results (14-19) to the discrete problem (25-30) obtains

$$\frac{\hat{x}^*_{n+1,n+1} - \hat{x}^*_{n,n}}{\Delta t} = g(\hat{x}^*_{n,n}) + P_{n+1,n+1} h_x^T [f(\hat{x}^*_{n,n})] R(T)^{-1} \{y_{n+1} - h[f(\hat{x}^*_{n,n})]\} + \frac{o(\Delta t)}{\Delta t}$$

Holding $T = n\,\Delta t$ constant, letting $\hat{x}^* \equiv \hat{x}^*(T) \equiv \hat{x}^*_{n,n}$, $P(T) \equiv P_{n,n}$ and taking the limit as $\Delta t \to 0$ yields

$$\dot{\hat{x}}^* = g(\hat{x}^*) + P(T) h_x^T(\hat{x}^*) R(T)^{-1} [y(T) - h(\hat{x}^*)] \tag{31}$$

To obtain P(T), use (16-19, A15) and (18, 29, 30) to obtain respectively

$$P_{n+1,n+1} = (I - K_{n+1} H_{n+1})[I + P_{n+1,n} L_{n+1}(I - P_{n+1,n} L_{n+1})^{-1}] P_{n+1,n} \tag{32}$$

$$P_{n+1,n} = P_{n,n} + \Delta t [g_x P_{n,n} + P_{n,n} g_x^T + GQG^T] + o(\Delta t) \tag{33}$$

where $H_{n+1} = h_x [f(\hat{x}^*_{n,n})]$.

Substituting (33) into (32)

$$\frac{P_{n+1,n+1} - P_{n,n}}{\Delta t} = g_x P_{n,n} + P_{n,n} g_x^T + GQG^T$$

$$+ \left| P_{n+1,n} \left(\frac{L_{n+1}}{\Delta t}\right) (I - P_{n+1,n} L_{n+1})^{-1} - \left(\frac{K_{n+1}}{\Delta t}\right) H_{n+1} \right| P_{n,n} + \frac{o(\Delta t)}{\Delta t}$$

which implies that

$$\dot{P} = g_x P + P g_x^T + GQG^T - P h_x^T R^{-1} h_x P + PLP \tag{34}$$

where

$$L = (\ell_{ij}), \quad \ell_{ij} = \frac{\partial^2 h(\hat{x}^*)}{\partial x_j \, \partial x_i} R^{-1}(T) [y(T) - h(\hat{x}^*)] .$$

Combining terms

$$\dot{P} = g_x P + P g_x^T + P \frac{\partial}{\partial x} [h_x^T R^{-1} (y - h)]_{\hat{x}^*} P + GQG^T \tag{35}$$

Equation (31) together with (34 or 35) represents an approximate solution to the continuous filtering problem. Equations (31, 35) represent the solution as derived by Sridhar and Detchmendy [2].

3. Other Nonlinear Estimation Techniques

In order to establish a basis for testing the utility of the results of Section 1, two other nonlinear estimation techniques are now discussed.

First, following [8], consider a system of the form (1, 2) where the m components of h are linearly independent functions of the m components Hx of x. Linearizing h about $H\hat{x}$ yields the approximation

$$y = h(Hx) + v \doteq h(H\hat{x}) + J(Hx - H\hat{x}) + v$$

where the Jacobian J of h at $H\hat{x}$ is nonsingular. Inverting J yields the (transformed) linear observations

$$y^{*} = J^{-1}[y - h(H\hat{x})] + H\hat{x} = Hx + J^{-1}v \quad . \tag{36}$$

Using the linear theory,

$$\hat{x}_{s,s} = \hat{x}_{s,s-1} + G_s[y_s^{*} - H\hat{x}_{s,s-1}] \tag{37}$$

where the gain sequence $\{G_s\}$ is computed using the observability matrix H and the measurement error covariance matrix $J^{-1}R(J^{-1})^T$. Substituting (36) into (37) yields

$$\hat{x}_{s,s} = \hat{x}_{s,s-1} + G_s J_s^{-1}[y_s - h(\hat{x}_{s,s-1})] \tag{38}$$

One interesting aspect of (38) is that the gain matrix

$$K_s = G_s J_s^{-1}$$

is factored into a product of a smoothing component G_s and a linearizing component J_s^{-1}. Moreover nominal values can often be chosen for $J^{-1}R(J^{-1})^T$ and f_k thus allowing the gain sequence $\{G_k\}$ to be precomputed and stored in computer memory for real time data processing.

Another approach which is often successful consists of using a one-to-one nonlinear transformation $\mathcal{F}$ to map the original nonlinear problem (1, 2) into a space in which the transformed problem is linear, i.e.

$$\mathcal{F}(x_{k+1}) = \Phi_{k+1,k}\mathcal{F}(x_k) + \Delta_k w_k \tag{39}$$

$$\mathcal{F}(y_k) = M_k\mathcal{F}(x_k) + v_k \tag{40}$$

The statistics of the measurement error $\{v_k\}$ in (40) are usually quite complicated and very nongaussian. However, a direct application of the linear theory (rationalized by the least square interpretation of Section 1) often provides excellent estimates $\hat{\mathcal{F}}(x)$ of $\mathcal{F}(x)$. Moreover, the estimate

$$\hat{x} = \mathcal{F}^{-1}[\hat{\mathcal{F}}(x)] \tag{41}$$

may also be quite good. Needless to say, the resulting estimates are not necessarily optimum.

In the sequel, nonlinear estimation using the standard linearization of system dynamics and observations, i.e. using (14-18) with L = 0, will be referred to as Type I estimation, using (14-18) as derived will be called Type II, the approach of [8] as described above is Type III while the last approach just described is $\mathcal{F}$-estimation.

4. Examples

Several controlled experiments have been conducted in order to test the utility of the various discrete nonlinear estimation techniques. Two of the experiments will be described below. (In the following two examples, lower case alphabetic characters will denote scalars.)

First consider a discrete form of example 2 of [2]. The system equations are

$$x_1 = 2,\ \dot{x}_1 = 0,\ a_1 = 2,\ T = 0.1$$

and for $n \geq 1$,

$$x_{n+1} = x_n + T\dot{x}_n \tag{42}$$

$$\dot{x}_{n+1} = (1 - 3T)\dot{x}_n - 2Tx_n - Ta_n x_n^3 + 5T \sin(nT) \tag{43}$$

$$a_{n+1} = e^{-0.1T} a_n \tag{44}$$

In order to provide a significant comparison of the estimation algorithms under consideration, output observations with significant nonlinearities should be considered. (This is a departure from the examples considered in [2].) Consequently, the output observations for this example are modeled as

$$y_n = 2 \sin\left(\frac{\pi x_n}{2}\right) + v_n \tag{45}$$

which simulates severe saturation near the maximum position amplitude. Since the dynamic range of the position ranges from about -1.5 to +1.5 after initiation, the observations are not one-to-one. Hence it was not obvious in advance that the system (42-45) was observable.

Neither the Type III algorithm nor the $\mathcal{F}$-transformation technique seem suited for this example. The Type I, II estimation equations are

$$\hat{x}_{1,0} = 0,\ \hat{\dot{x}}_{1,0} = 0,\ a_1 = 2 \ .$$

and for $n \geq 1$,

$$\hat{x}_{n,n} = \hat{x}_{n,n-1} + k_{11}(n) \left| y_n - 2 \sin\left(\frac{\pi \hat{x}_{n,n-1}}{2}\right) \right|$$

$$\hat{\dot{x}}_{n,n} = \hat{\dot{x}}_{n,n-1} + k_{21}(n) \left| y_n - 2 \sin\left(\frac{\pi \hat{x}_{n,n-1}}{2}\right) \right|$$

$$\hat{x}_{n+1,n} = \hat{x}_{n,n} + T\hat{\dot{x}}_{n,n}$$

$$\hat{x}_{n+1,n} = (1 - 3T)\hat{x}_{n,n} - 2T\hat{x}_{n,n} - Ta_n\hat{x}_{n,n}^3 + 5T \sin(nT)$$

$$a_{n+1} = e^{-0.1T}\, a_n$$

The gain matrix $\begin{pmatrix} k_{11} \\ k_{21} \end{pmatrix}$ is obtained by numerically solving (16′, 16″, 17′, 18, 19) with

$$f_x = \begin{bmatrix} 1 & T \\ -T(2+3\,a_n\hat{x}_{n,n}^2) & (1-3T) \end{bmatrix} \qquad P_{1,0} = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$

$$h_x = \begin{bmatrix} \pi \cos\left(\frac{\pi\hat{x}_{n,n-1}}{2}\right), & 0 \end{bmatrix}, \quad Q = 0, \quad R = \sigma^2 = E(v_n^2)$$

For Type I estimation, $L = 0$ while for Type II,

$$\ell_{11} = -\frac{\pi^2}{2} \sin\left(\frac{\pi\hat{x}_{n,n-1}}{2}\right) \frac{\left[y_n - \sin\left(\frac{\pi\hat{x}_{n,n-1}}{2}\right)\right]}{\sigma^2}$$

is the only nonzero component of L.

With $\sigma = 0.1$, typical results for position estimation are illustrated in Figure 1. The results for velocity estimation are similar. It seems that Type I estimation has better initial estimates while Type II is better after about 10 samples have been processed. However, there is little difference after processing 30 samples. These results were substantiated by a 50 run Monte Carlo experiment. The rms position estimation errors vs. time are given in Figure 2.

Since the examples given in [2] only used measurement error of approximately the same magnitude as above, it seemed worthwhile to consider lower signal to noise ratios. As σ was increased a rather startling result occurred for σ near 1 (S/N 0 db). For this lower signal to noise ratio, the Type II algorithm usually performed best whenever it would work, however, it would occasionally become unstable during the first few samples of tracking. Instability was never observed in the Type I algorithm. Moreover, the Type I algorithm furnished good estimates of both position and velocity for all values of σ. In fact the rms errors for the Type I algorithm with $\sigma = 1$ differed very little from the case $\sigma = 0.1$. The estimation errors seem almost entirely due to a dynamic bias resulting from poor initiation.

As a second example, consider a two dimension tracking problem. Target range and azimuth are measured every second with an accuracy of 200 yards and two milliradians respectively. (Notice that no statistical information concerning the errors is available.)

The target dynamics are modeled in a Cartesian reference frame as

$$x_{n+1} = x_n + T_n \dot{x}_n + \frac{T_n^2}{2} w_{1n}$$

$$\dot{x}_{n+1} = \dot{x}_n + T_n w_{1n}$$

with similar equations for the y-axis. The sequence $\{w_{1n}\}$ represents random acceleration components along the x-axis. The output observations are

$$r = (x^2 + y^2)^{1/2} + v_1 \qquad \theta = \text{Arctan}\left(\frac{y}{x}\right) + v_2 \ .$$

All four approaches discussed above can be used for this example. An $\mathcal{F}$-transformation for this example consists of performing a polar to Cartesian coordinate transformation on the measurements R, θ and tracking in the x-y plane ignoring the introduction of correlated errors on the synthetic measurements. The resulting x-axis estimation equations are

$$\hat{x}_{1,0} = \hat{\dot{x}}_{1,0} = 0$$

$$\hat{x}_{n,n} = \hat{x}_{n,n-1} + \alpha_n \epsilon_{xn}$$

$$\hat{\dot{x}}_{n,n} = \hat{\dot{x}}_{n,n-1} + \frac{\beta_n}{T_n} \epsilon_{xn}$$

$$\hat{x}_{n+1,n} = \hat{x}_{n,n} + T_n \lambda_{n,n}$$

$$\hat{\dot{x}}_{n+1,n} = \hat{\dot{x}}_{n,n}$$

$$\epsilon_{xn} = r_n \cos \theta_n - \hat{x}_{n,n-1}$$

with analogous equations for the y-axis. (This is a nonstationary form of the familiar α-β tracking filter.) This filter tracks very well using an initial segment of the gain sequence computed with

$$f_x = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad h_x = (1, 0) \qquad Q = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$R = 1 \qquad P_{1,0} = \begin{pmatrix} 10^6 & 0 \\ 0 & 10^6 \end{pmatrix}$$

which allows the gains to be precomputed. Estimates of range and azimuth are furnished by

$$\hat{r} = (\hat{x}^2 + \hat{y}^2)^{1/2}$$

$$\hat{\theta} = \text{Arctan}\,(\hat{y}/\hat{x}) \quad .$$

Type III estimation is performed exactly as above except that

$$\hat{x}_{1,1} = r_1 \cos\theta_1 \quad , \quad \hat{\dot{x}}_{1,1} = 0 \quad ,$$

$$\epsilon_{xn} = \frac{\hat{x}_{n,n-1}}{\hat{r}_n}(r_n - \hat{r}_n) - \hat{y}_{n,n-1}(\theta_n - \hat{\theta}_n)$$

$$\hat{r}_n = (\hat{x}_{n,n-1}^{\,2} + \hat{y}_{n,n-1}^{\,2})^{1/2} \quad , \quad \hat{\theta}_n = \text{Arctan}\left(\frac{\hat{y}_{n,n-1}}{\hat{x}_{n,n-1}}\right)$$

For Type I, II estimation

$$\hat{x}_{n,n} = \hat{x}_{n,n-1} + k_{11}(n)(r_n - \hat{r}_n) + k_{12}(n)(\theta_n - \hat{\theta}_n)$$

$$\hat{\dot{x}}_{n,n} = \hat{\dot{x}}_{n,n-1} + k_{21}(n)(r_n - \hat{r}_n) + k_{22}(n)(\theta_n - \hat{\theta}_n)$$

Attempting to avoid numerical problems in the computation of the gain matrix

$$K_n = (k_{ij}(n))$$

the position units are taken as [hundred yd] while [milliradian] is used for the angle unit. It follows that

$$R = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} \qquad h_z = \begin{pmatrix} \dfrac{\hat{x}_{n,n-1}}{\hat{r}_n} & 0 & \dfrac{\hat{y}_{n,n-1}}{\hat{r}_n} & 0 \\ -\dfrac{10^3\hat{y}_{n,n-1}}{\hat{r}_n^2} & 0 & \dfrac{10^3\hat{x}_{n,n-1}}{\hat{r}_n^2} & 0 \end{pmatrix}$$

To assure nonzero steady state gains,

$$Q = 10^{-4}\left(\begin{array}{c|c} D & 0 \\ \hline 0 & D \end{array}\right) \quad , \quad D = \begin{pmatrix} \dfrac{T^4}{4} & \dfrac{T^3}{2} \\ \dfrac{T^3}{2} & T^2 \end{pmatrix}$$

and for good transient response,

$$P_{1,0} = 10^4 I \ .$$

For Type II estimation, the nonzero components of L are

$$\ell_{11}(n) = \frac{\hat{y}_{n,n-1}{}^2}{\hat{r}_n^3}\,\frac{(r_n - \hat{r}_n)}{4} + \frac{2(10^3)\hat{x}_{n,n-1}\hat{y}_{n,n-1}}{\hat{r}_n^4}\,\frac{(\theta_n - \hat{\theta}_n)}{4}$$

$$\ell_{13}(n) = \ell_{31}(n) = -\frac{\hat{x}_{n,n-1}\hat{y}_{n,n-1}}{\hat{r}_n^3}\,\frac{(r_n - \hat{r}_n)}{4} - \frac{(10^3)\hat{x}_{n,n-1}{}^2}{\hat{r}_n^4}\,\frac{(\theta_n - \hat{\theta}_n)}{4}$$

$$\ell_{33}(n) = \frac{\hat{x}_{n,n-1}{}^2}{\hat{r}_n^3}\,\frac{(r_n - \hat{r}_n)}{4} - \frac{2(10^3)\hat{x}_{n,n-1}\hat{y}_{n,n-1}}{\hat{r}_n^4}\,\frac{(\theta_n - \hat{\theta}_n)}{4}$$

For this example, the Type I, II algorithms respond very poorly unless initiated with good initial estimates of target position and velocity. Both Type III estimation and the $\mathcal{J}$ transform filter always performed very well for both position and velocity estimation. Essentially no difference between the two was observed for the noise levels described. Typical range estimation errors for the four filters are given in Figure 3.

5. Conclusions

At this point, it is standard procedure to proclaim the usefulness of derived results after a consideration of contrived examples. While such conclusions often "look good on paper," it is very frustrating to find that the examples are unique in illustrating the utility of a particular result.

The two examples given above were designed to point out the fact that none of the popular approaches to nonlinear estimation represent a universal solution to nonlinear estimation problems. Rather, it appears that ingenuity as well as discretion is presently required in obtaining practical solutions to meaningful nonlinear estimation problems.

A few empirical results which appear to hold in general are:

a. If the signal to noise ratio is low the Type II algorithm may be unstable. Of all the approaches considered, the Type II algorithm appears the most sensitive to low signal to noise ratios.

b. If the initial estimates $\hat{x}_{1,0}$ based on apriori knowledge are poor, both Type I and Type II algorithms may exhibit poor transient response. These algorithms seem especially sensitive to initial errors in the estimate of the direction of the initial state vector x_1.

c. If either Type I or Type II estimation is used, the filter gain matrix normally must be computed on line. The main reason being that the filter gain matrix is extremely sensitive to the value of the observability matrix h_x. Instability will usually occur when attempts are made to use precomputed gains. However, it is quite possible to formulate the Type III algorithm and an $\mathcal{F}$ transformation in such a fashion as to allow the use of precomputed gains. That is, the last two techniques are considerably more amenable to real time computation.

d. When applicable, Type III estimation or appropriate $\mathcal{F}$ transformations seem to enjoy the greatest utility of the four techniques which have been considered. The resulting filters exhibit little sensitivity to initiation errors and low signal to noise ratios.

In view of the results of this study it seems that the results of [2] (or of Section 2) should be exposed to the problems of large initiation errors and low signal to noise ratios. Also, the performance of the

Sridhar-Detchmendy filter [2] should be compared with algorithms which can be considerably less complex to implement (i.e. the continuous version of the Type I and Type III algorithms).

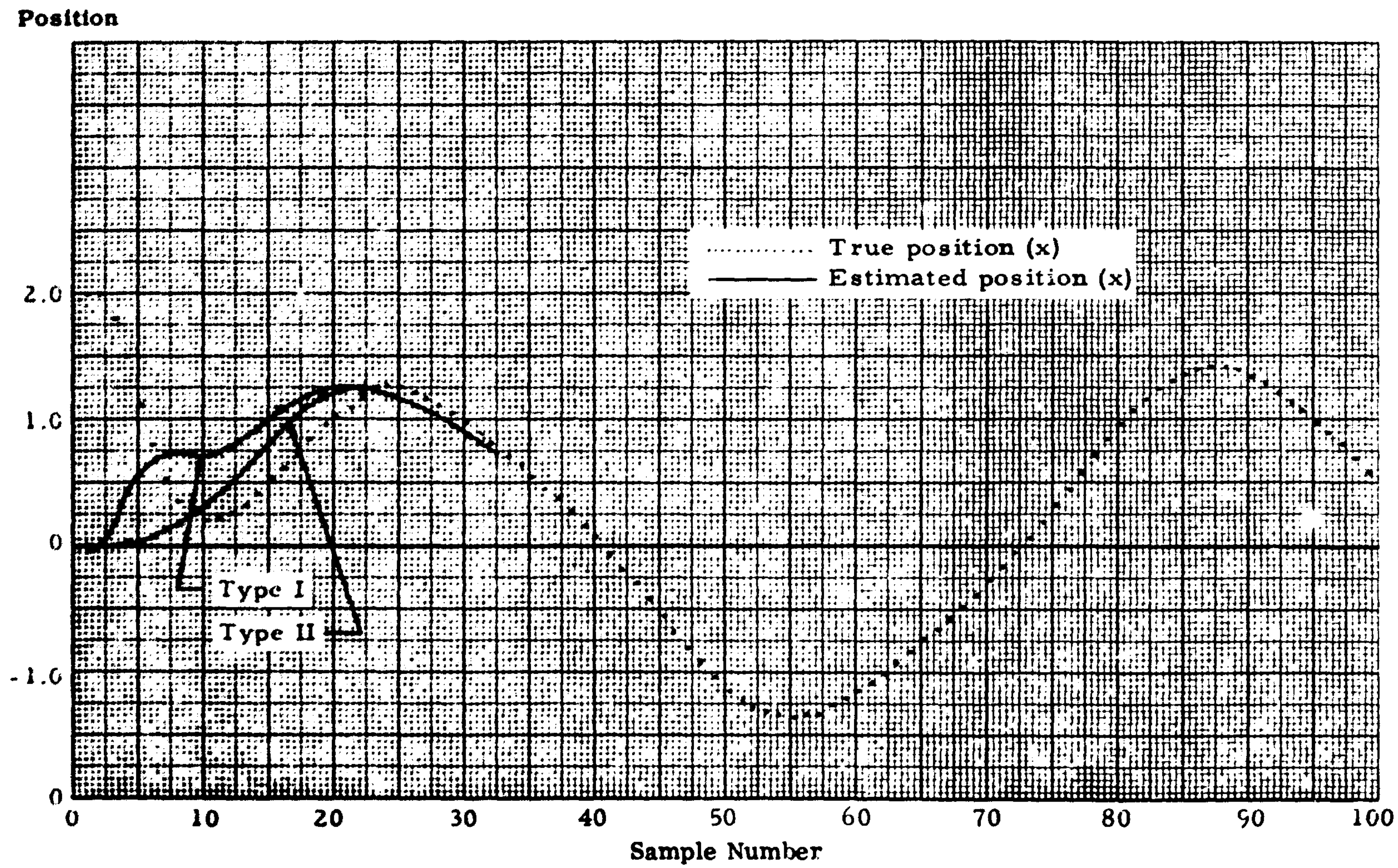

FIG. 1. True and Estimated Position vs. Sample Number.

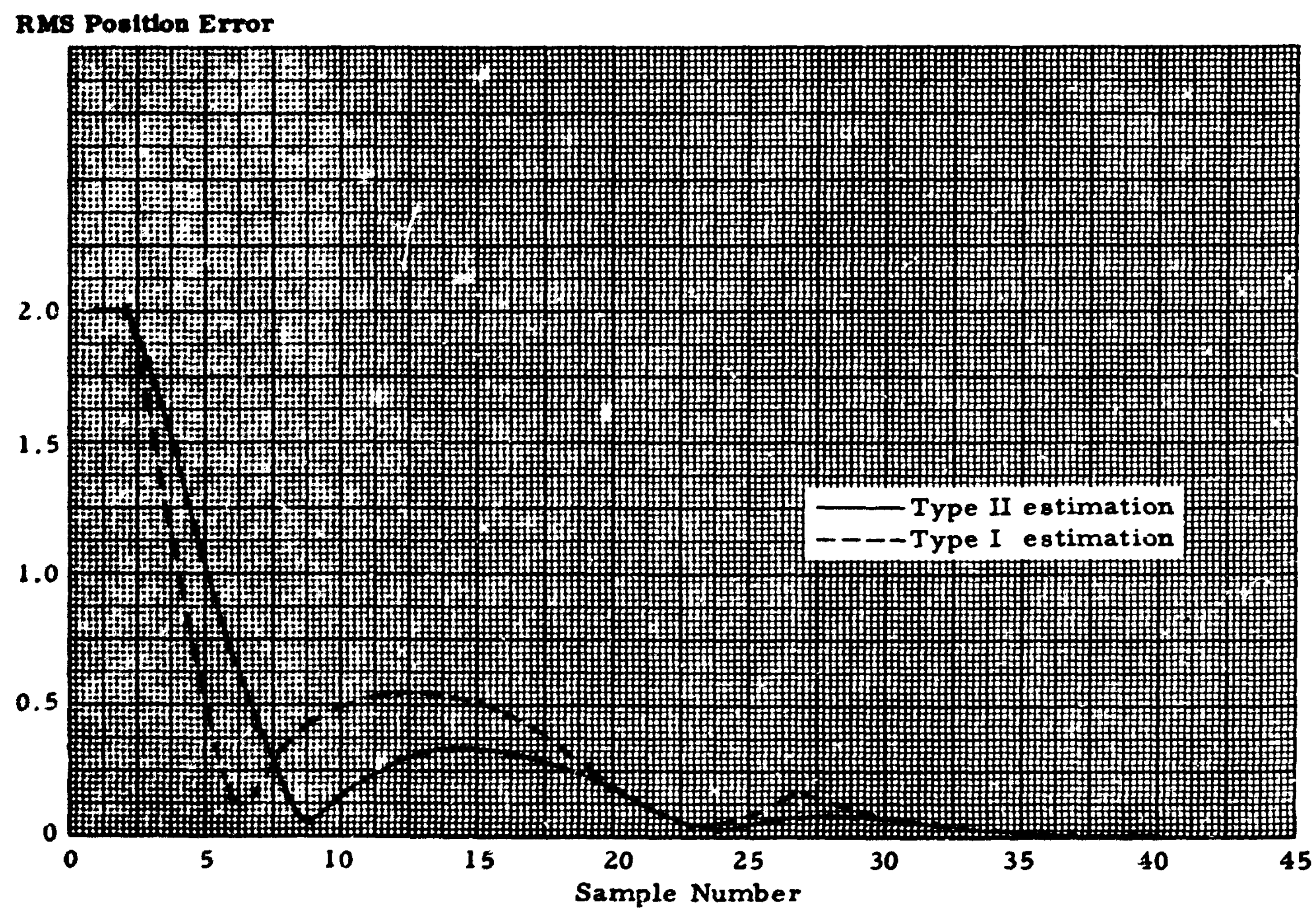

FIG. 2. RMS Position Estimation Errors vs. Sample Number.

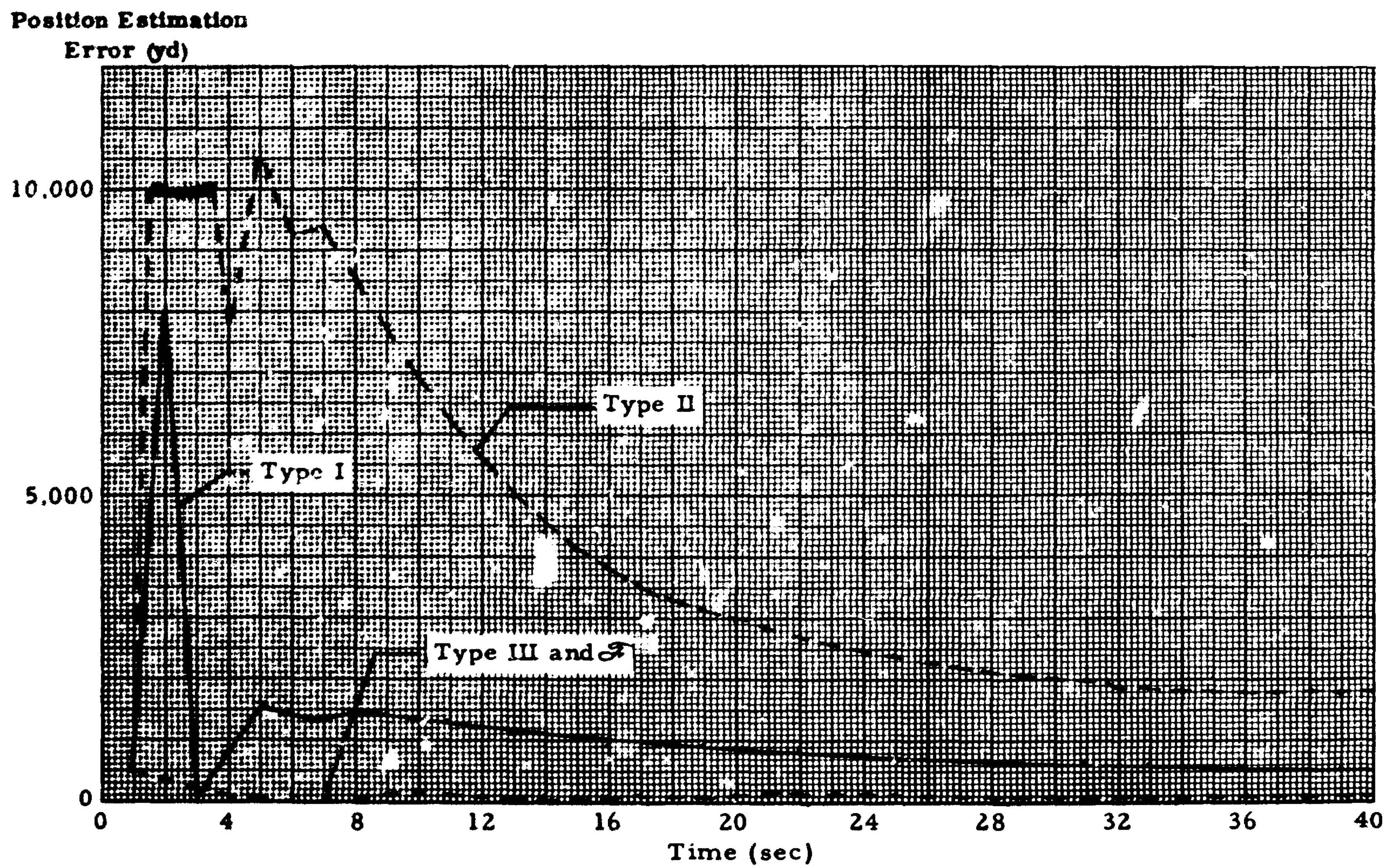

FIG. 3. Range Estimation Errors vs. Time.

Appendix A

APPROXIMATE SOLUTION TO THE TWO POINT BOUNDARY VALUE PROBLEM

Consider the two point boundary value problem derived in Section 1.

$$\hat{x}_{i+1,n} = f(\hat{x}_{i,n}) + \Delta_i Q_i \Delta_i^T \lambda_{i,n}, \quad i = 0, \ldots, n-1 \tag{A1}$$

$$\lambda_{i-1,n} = f_x^T(\hat{x}_{i,n})\lambda_{i,n} + h_x^T(\hat{x}_{i,n})R_i^{-1}[y_i - h(\hat{x}_{i,n})], \quad i = 1, \ldots, n \tag{A2}$$

$$\lambda_{n,n} = 0 \tag{A3}$$

$$\hat{x}_{o,n} = \hat{x}_{o,o} + P_{o,o} f_x^T(\hat{x}_{o,n})\lambda_{o,n} \tag{A4}$$

Setting n = 1 and linearizing $f(\hat{x}_{o,1})$ about $\hat{x}_{o,o}$ yields

$$\hat{x}^*_{1,1} = f(\hat{x}_{o,o}) + f_x(\hat{x}_{o,o})(\hat{x}_{o,1} - \hat{x}_{o,o}) + \Delta_o Q_o \Delta_o^T \lambda_{o,1}$$

Assuming

$$f_x(\hat{x}_{o,n}) \doteq f_x(\hat{x}_{o,o})$$

and using (A2-A4) yields the approximation

$$\hat{x}^*_{1,1} = f(\hat{x}_{o,o}) + P_{1,o} h_x^T(\hat{x}_{1,1})R_1^{-1}[y_1 - h(\hat{x}_{1,1})]$$

where

$$P_{1,o} = f_x(\hat{x}_{o,o})P_{o,o} f_x(\hat{x}_{o,o}) + \Delta_o Q_o \Delta_o^T$$

Linearizing $h_x^T(\hat{x}_{1,1})R_1^{-1}[y_1 - h(\hat{x}_{1,1})]$ about $\hat{x}^*_{1,0} \equiv f(\hat{x}_{0,0})$ results in a second approximation

$$\hat{x}^*_{1,1} = \hat{x}^*_{1,0} + P_{1,0} h_x^T(\hat{x}^*_{1,0}) R_1^{-1}[y_1 - h(\hat{x}^*_{1,0})]$$

$$+ P_{1,0} \frac{\partial}{\partial x}[h_x^T R_1^{-1}(y_1 - h)]_{\hat{x}^*_{1,0}} (\hat{x}^*_{1,1} - \hat{x}^*_{1,0})$$

which implies

$$\hat{x}^*_{1,1} = \hat{x}^*_{1,0} + P_{1,1} h_x^T(\hat{x}^*_{1,0}) R_1^{-1}[y_1 - h(\hat{x}^*_{1,0})]$$

where[1]

$$P_{1,1} = \left| I - P_{1,0} \frac{\partial}{\partial x}[h_x^T R_1^{-1}(y_1 - h)]_{\hat{x}^*_{1,0}} \right|^{-1} P_{1,0}$$

From (A2, A4) it follows that

$$\hat{x}^*_{0,1} = \hat{x}_{0,0} + P_{0,0} f_x^T(\hat{x}_{0,0}) h_x^T(\hat{x}^*_{1,1}) R_1^{-1}[y_1 - h(\hat{x}^*_{1,1})]$$

Now, assuming that $\{\hat{x}^*_{i,n-1};\ i = 0, \ldots, n-1\}$ have been determined, consider the following proposition:

For $n \geq 2$,

$$\hat{x}^*_{i,n} = \hat{x}^*_{i,n-1} + P^{(n)}_{i,i} f_x^T(\hat{x}^*_{i,n-1})(\lambda_{i,n} - \lambda_{i,n-1}), \quad i = 0, \ldots, n-1 \quad \text{(A5)}$$

where

$$P^{(n)}_{0,0} = P_{0,0} \quad \text{(A6)}$$

$$P^{(n)}_{i,i} = \left| I - P^{(n)}_{i,i-1} \frac{\partial}{\partial x}[h_x^T R_i^{-1}(y_i - h)]_{\hat{x}^*_{i,n-1}} \right|^{-1} P^{(n)}_{i,i-1}$$

$$i = 1, \ldots, n-1 \quad \text{(A7)}$$

---

[1]It is tacitly assumed that the appropriate inverse exists.

$$P_{1,0}^{(s)} = P_{1,0} = f_x(\hat{x}_{0,0})P_{0,0}f_x^T(\hat{x}_{0,0}) + \Delta_0 Q_0 \Delta_0^T \tag{A8}$$

$$P_{i,i-1}^{(s)} = f_x(\hat{x}_{i-1,s-1}^*)P_{i-1,i-1}^{(s)}f_x^T(\hat{x}_{i-1,s-1}^*) + \Delta_{i-1}Q_{i-1}\Delta_{i-1}^T,$$

$$n \geq 3,\ i = 2,\ \ldots,\ n-1 \tag{A9}$$

The statement A5 for $i = 0$ follows immediately from A4. Next, the proposition is established for $i = 1$, $n \geq 2$.

Setting $i = 1$ in A1 and linearizing $f(\hat{x}_{0,s})$ about $\hat{x}_{0,0}$ yields

$$\hat{x}_{1,s}^* = f(\hat{x}_{0,0}) + P_{1,0}\lambda_{0,s}$$

which implies

$$\hat{x}_{1,s}^* = \hat{x}_{1,s-1}^* + P_{1,0}(\lambda_{0,s} - \lambda_{0,s-1})$$

where

$$\lambda_{0,s} = f_x^T(\hat{x}_{1,s})\lambda_{1,s} + h_x^T(\hat{x}_{1,s})[y_1 - h(\hat{x}_{1,s})]$$

$$\lambda_{0,s-1} = f_x^T(\hat{x}_{1,s-1})\lambda_{1,s-1} + h_x^T(\hat{x}_{1,s-1})[y_1 - h(\hat{x}_{1,s-1})]$$

Linearizing $h_x^T(\hat{x}_{1,s})R_1^{-1}[y_1 - h(\hat{x}_{1,s})]$ about $\hat{x}_{1,s-1}^*$ and assuming $f_x(\hat{x}_{1,s}) \doteq f_x(\hat{x}_{1,s-1}^*)$ yields

$$\hat{x}_{1,s}^* = \hat{x}_{1,s-1}^* + P_{1,0}f_x^T(\hat{x}_{1,s-1}^*)(\lambda_{1,s} - \lambda_{1,s-1})$$

$$+ \frac{\partial}{\partial x}[h_x^T R_1^{-1}(y_1 - h)]_{\hat{x}_{1,s-1}^*}(\hat{x}_{1,s}^* - \hat{x}_{1,s-1}^*)$$

which establishes A5 for $i = 1$.

Now, from A1,

$$\hat{x}_{i,s} = f(\hat{x}_{i-1,s}) + \Delta_{i-1}Q_{i-1}\Delta_{i-1}^T\lambda_{i-1,s}, \quad n \geq 1 \quad i = 1,\ \ldots,\ n$$

Linearizing $f(\hat{x}_{i-1,n})$ about $\hat{x}_{i-1,n-1}$ obtains

$$\hat{x}^*_{i,n} = f(\hat{x}^*_{i-1,n-1}) + f_x(\hat{x}^*_{i-1,n-1})(\hat{x}^*_{i-1,n} - \hat{x}^*_{i-1,n-1}) + \Delta_{i-1}Q_{i-1}\Delta^T_{i-1}\lambda_{i-1,n}$$

By induction on i, it follows that

$$\hat{x}^*_{i,n} = f(\hat{x}^*_{i-1,n-1}) + P^{(n)}_{i,i-1}(\lambda_{i-1,n} - \lambda_{i-1,n-1}) + \Delta_{i-1}Q_{i-1}\Delta^T_{i-1}\lambda_{i-1,n-1}$$

where

$$P^{(n)}_{i,i-1} = f_x(\hat{x}^*_{i-1,n-1})P^{(n)}_{i-1,i-1}f^T_x(\hat{x}^*_{i-1,n-1}) + \Delta_{i-1}Q_{i-1}\Delta^T_{i-1}$$

Noting A1, it follows that

$$\hat{x}^*_{i,n} = \hat{x}^*_{i,n-1} + P^{(n)}_{i,i-1}(\lambda_{i-1,n} - \lambda_{i-1,n-1})$$

The remaining arguments are analogous to those used in the preceding case.

The approximate solution (A5-A9) to the two point boundary value problem (A1-A4) can be used to establish a computing algorithm for $\hat{x}^*_{n,n}$. Setting $i = n - 1$ in A5 yields

$$\hat{x}^*_{n-1,n} = \hat{x}^*_{n-1,n-1} + P_{n-1,n-1}f^T_x(\hat{x}^*_{n-1,n-1})\lambda_{n-1,n}$$

Linearizing the dynamics $f(\hat{x}_{n-1,n})$ in A1 about $\hat{x}^*_{n-1,n-1}$ and using the preceding equation implies

$$\hat{x}^*_{n,n} = f(\hat{x}^*_{n-1,n-1}) + P_{n,n-1}h^T_x(\hat{x}^*_{n,n})R_n^{-1}[y_n - h(\hat{x}^*_{n,n})]$$

where

$$P_{n,n-1} = f_x(\hat{x}^*_{n-1,n-1})P_{n-1,n-1}f^T_x(\hat{x}^*_{n-1,} \qquad \Delta_{n-1}Q_{n-1}\Delta^T_{n-1} \qquad \text{(A10)}$$

Linearizing $h^T_x(\hat{x}^*_{n,n})R_n^{-1}[y_n - h(\hat{x}^*_{n,n})]$ about $\hat{x}^*_{n,n-1} = f(\hat{x}^*_{n-1,n-1})$ obtains

$$\hat{x}^*_{n,n} = \hat{x}^*_{n,n-1} + P_{n,n}\, h_x^T(\hat{x}^*_{n,n-1}) R_n^{-1}[y_n - h(\hat{x}^*_{n,n-1})] \tag{A11}$$

with

$$P_{n,n} = \left| I - P_{n,n-1}\frac{\partial}{\partial x}[h_x^T R_n^{-1}(y_n - h)]_{\hat{x}^*_{n,n-1}} \right|^{-1} P_{n,n-1} \tag{A12}$$

Equations (A10-A12) constitute a computing algorithm for $\hat{x}^*_{n,n}$.

Alternate forms of (A12) can be obtained with moderate assumptions. Assuming that $P_{n,n-1}$ is nonsingular allows (A12) to be expressed as

$$P_{n,n} = \{P_{n,n-1}^{-1} - L_n + h_x^T(\hat{x}^*_{n,n-1}) R_n^{-1} h_x(\hat{x}^*_{n,n-1})\}^{-1},$$

$$L_n = (\ell_{ij}(n)), \quad \ell_{ij}(n) = \frac{\partial^2 h(\hat{x}_{n,n-1})}{\partial x_j \partial x_i} R_n^{-1}[y_n - h(\hat{x}^*_{n,n-1})]$$

where x_i, x_j denote respectively the i^{th} and j^{th} components of the vector x.

Using the matrix identity

$$[B^{-1} + CR^{-1}D]^{-1} = B - BC[DBC + R]^{-1}DB \tag{A13}$$

and assuming that

$$S_n^{-1} = P_{n,n-1}^{-1} - L_n$$

is invertable yields

$$P_{n,n} = S_n - S_n h_x^T[h_x S_n h_x^T + R_n]^{-1} h_x S_n \tag{A14}$$

$$S_n = P_{n,n-1} + P_{n,n-1} L_n [I - P_{n,n-1} L_n]^{-1} P_{n,n-1} \tag{A15}$$

$$= [I - P_{n,n-1} L_n]^{-1} P_{n,n-1} \tag{A16}$$

It now follows that

$$K_s \equiv P_{s,s} h_x^T R_s^{-1} = S_s h_x^T [h_x S_s h_x^T + R_s]^{-1} \tag{A17}$$

which implies that

$$P_{s,s} = [I - K_s h_x] S_s \tag{A18}$$

REFERENCES

1. Cox, H. "On the Estimation of State Variables and Parameters for Noisy Dynamic Systems," IEEE Transactions on Automatic Control, Vol. AC-9, 1964, pp. 5-12.

2. Detchmendy, D. M., and R. Sridhar. "Sequential Estimation of States and Parameters in Noisy Nonlinear Dynamical Systems," Transactions of the ASME, June 1966, pp. 362-368.

3. Carlton, A. G. "Linear Estimation in Stochastic Processes," Applied Physics Lab, Johns Hopkins University, Report No. 311, March 1962.

4. Kalman, R. E. "A New Approach to Linear Filtering and Prediction Problems," Transactions of the ASME, Journal of Basic Engineering, March 1960, pp. 35-45.

5. Kalman, R. E., and R. S. Bucy. "New Results in Linear Filtering and Prediction Theory," Transactions of the ASME, Journal of Basic Engineering, March 1961, pp. 95-108.

6. Bryson, A. E., and M. Frazier. "Smoothing for Linear and Nonlinear Dynamic Systems," Proceedings of the Optimum Systems Synthesis Conference, Wright-Patterson Air Force Base, Ohio, September 1962, AST-TDR-63-119.

7. Jazwinski, A. H. "Filtering for Nonlinear Dynamical Systems," IEEE Transactions on Automatic Control (Correspondence), Vol. AC-11, October 1966, pp. 765-6.

8. Larson, R. E., R. M. Dressler, and R. S. Ratner. "Application of the Extended Kalman Filter to Ballistic Trajectory Estimation," Stanford Research Institute, Project 5188-103, January 1967.

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13 ABSTRACT

ABSTRACT. A study was conducted in which a least squares criterion was used to solve a discrete, real-time nonlinear estimation problem. The technique centers around the derivation of a sequential algorithm which allows consideration of second-order nonlinearities in system measurements. Alternate nonlinear estimation techniques are discussed, and comparative examples of the various estimation algorithms are presented.

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