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THE PROPAGATION OF RADIO WAVES ALONG THE SURFACE OF THE EARTH

by

Ye. L. Feynberg

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By: Ye. L. Feynberg

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ABSTRACT: This monograph covers all the basic subdivisions of the theory of surface radiowave propagation, that is, of radiowaves, for which the role of the ionosphere is immaterial. A theory is expounded for the plane and the spherical radiowaves, alongside with the theory of events induced by the electrical inhomogeneity and the irregularity of the surface constituting the divide between the ground and the atmosphere, and that of the influence of the stratified inhomogeneity of the troposphere; described also are the methods of accounting for the effects connected with its turbulence. The book is designed for scientific workers, engineers, aspirants and graduate students in specializations covering radiophysics and radioengineering.

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PREFACE

The study of the propagation process of radiowaves emitted by ground sources has been linked from the very beginning and for already more than half a century with complex theoretical and, in particular, mathematical problems. Even after the detection of the ionosphere influence and the successful separation of a range of phenomena not connected with the ionosphere, the restricted area of the problem which we specifically define as the theory of radiowave propagation along the terrestrail surface still remains vast. Solutions of major subdivisions of the entire problem have been arrived at; however, each time new problems emerged. Rather than speaking in terms of refinements or of "finish," there was always a question about new events, leading to a serious change or to a substantial enrichment of the earlier-acquired knowledge.

The cause of the unexpected appearance of new major questions was to a significant extent connected with the fact that correct representations of the process of radiowave propagation near the terrestrial surface were absent here. A practical and descriptive approximate theory of diffraction was created at the outset by Fresnel-Kirchhoff in and one electrodynamic problem (diffraction on an absolutely conducting semiplane) could be resolved only many years later. The rigorous solution found was utilized mainly for the estimate of the limits of applicability of the Fresnel-Kirchhoff theory. Contrary to that, in the problem of radiowave propagation along the ground surface, differing from the standard optical problems "only" in that the source and the point of observations are near the earth, descriptive representations were either absent or erroneous. Here the theory was developed <u>at the outset</u> in a rigorous mathematical formulation for sharply idealized schemes (homogenous plane or spherical earth). Despite the participation of front-line mathematicians, the development was painful and was attended for years by persistent errors of principle. It could only succeed in the forties by working out an approximate method and descriptive representations that allowed to obtain easily the earlier correct results of complex theories and, above all, to resolve a series of new problems (such as inhomogenous soil and so forth).

However, even after this, the transition in practice to shorter waves has led to the revelation of entirely new factors, namely refraction and superrefraction and scattering on chaotic inhomogeneities in the troposphere. This, on the one hand, made the theoretical problems more complex, while strengthening the role of the statistical treatment on the other. It should be stressed that even now, the relative role of various factors is far from being always clear, and the theory continues to develop.

Therefore, even neglecting the influence of the ionosphere, that is, considering the <u>tropospheric waves</u> (term proposed by M.P. Dolukhanov), we are confronted with complex theoretical problems requiring a more profound study. The practical questions that have to be answered by the theory are very diversified. They are specific for radio broadcasting and radio communications (choice of the route, level fluctuations); for radar (distortion on account of diffraction around the earth, refraction, scattering on the irregularities of the earth's surface and air inhomogeneities); for television (diffraction, refraction, scattering in the troposphere and enhancement because of obstacles); for radiogeology and radiometeorology.

About ten years ago, Ya.L. Al'pert, V.L. Ginzburg and the author of the present monograph wrote the book "Radiowave Propagation" (Gostekhizdat, 1953). Reviewed in it were also the questions of radiowave propagation in the ionosphere and the questions directly connected with experiment. It is clear at present that it is nearly impossible to encompass this entire range of problems in a single book. As a result of revision and expansion of the second part of the book referred to, there appeared V.L. Ginzburg's monograph "Propagation of Electromagnetic Waves in the Plasma" (Fizmatgiz, 1960), and from the third and the fourth parts there emerged the book by Ya.L. Al'pert "Propagation of Radiowaves and the Ionosphere" (Izd-vo AN SSSR, 1960), the first part having served as a basis for the present monograph. An attempt is made here to expound all the fundamental questions within the framework of the above-indicated theme (propagation processes of radiowaves not subject to the influence of the ionosphere.

The only exception is probably the section on the influence of troposphere's turbulent inhomogeneity, to which broad literature is devoted (Chapter 10). Here we were compelled to limit ourselves to the consideration of methods of theory. However, as is easy to notice, the equilibrium is not always observed, even in systematic exposition of the remaining sections. In particular, relatively considerable space is alloted to questions with which the author's scientific interests are to a greater degree linked (the role of the nonuniformity of the terrestrial surface).

As to the character of the exposition a certain duality may be noted here. On the one hand, the full theory, which may be accessible to and required only by the specialist, that is, the scientific worker, is the object of detailed treatment only in certain paragraphs. On the other hand, there are included paragraphs and entire chapters (Chapter 2, §§33-37 and so forth), in which the qualitative traits of the event are analyzed in detail. These spots are nearly superfluous for the specialist. However, they may contribute to the development of descriptive physical representations allowing the orientation even in cases when there is a lack of complete theory (or when it is too complex), and it must facilitate the understanding of the sense of the strict theory's formal results. Such a duality is, generally speaking, risky. But it was admitted, and in the above-referred to book by three authors, a significant part of which was utilized here, as well as according to testimonials obtained by me, it was justified in general by the practice of this book's use. At the same time, one is tempted to believe that the unique style of the exposition was not then disrupted.

When working on this book, I made use of remarks by F.G. Bass, D Ye. Vakman, M.A. Isakovich, V.G. Nosov, V.I. Tatarskiy and Ye.V. Chayevskiy, who read a series of sections of the manuscript, and also of the counsel and the wishes of L.A. Vaynshteyn, M.P. Dolukhanov, Yu.K. Kalinin and M.M. Kobrin. To all of them I am very grateful.

Ye.L. Feynberg

Chapter 1

BASIC EQUATIONS

A. DIFFERENTIAL RELATIONSHIPS

§1. Introduction. The Complex Dielectric Constant

Taken literally, the problem of the theory of radiowave propagation ought to be reduced to the determination of the field of radiowaves in a certain region when given, for example, at any part of its boundary. It is indeed often required to find the field on one side of an imaginary surface, on which, as may be asserted according to specific causes, it coincides with great precision with the field of a plane wave arriving from the other side. However, in many cases such a statement of the problem is impossible, the field over the surface is unknown, and the question evolves about the necessity to find a radiowave field emitted by either source for a specific disposition and properties of the surrounding media.

The current in the antenna depends not only on the supplied electromotive force, but also on the geometry and physical properties of the antenna itself, as well as of the surrounding bodies. This is why the problem of radiowave propagation is undesirably intertwined with questions of the theory of antennas. In order to avoid going far into the latter, we shall assume that the current in the emitting antenna (or, to be more precise, its distribution) is known (it may, for example, be measured).

There lay at the basis of the theory of radiowave propagation the Maxwellian equations linking the strengths $\vec{E}$ and $\vec{H}$ of the electric and magnetic fields and their inductions $\vec{D}$ and $\vec{B}$ with the densitites of the current and of the charge:

$$\mathrm{rot}\, H = \frac{1}{c}\frac{\partial D}{\partial t} + \frac{4\pi}{c} j + \frac{4\pi}{c} j^{stor}, \qquad (1.I)$$

$$\mathrm{rot}\, E = -\frac{1}{c}\frac{\partial B}{\partial t},$$

$$\mathrm{div}\, D = 4\pi(\rho + \rho^{stor}),$$

$$\mathrm{div}\, B = 0.$$

Clearly separated here from the densities of the current and of the charge are their parts $\vec{j}^{stor}$ and ρ^{stor} (different from zero in the region of the emitting antenna), which differ from the remaining parts $\vec{j}$ and ρ in that they are determined by alien forces. In our case these also are electromagnetic fields (they are, however, not included in $\vec{E}$ and $\vec{H}$), sustaining the "outside" current and charge, which, owing to that, do not depend on conditions of radiowave propagation in the surrounding space.

From (1.I) and (1.III) follows the continuity equation (law of charge preservation)

$$\mathrm{div}\, j = -\frac{\partial \rho}{\partial t}, \qquad (1.V)$$

evidently also separately valid for $\vec{j}^{stor}$ and ρ^{stor}.

For the system of equations to be specific, three links must be known: between $\vec{E}$ and $\vec{D}$, $\vec{H}$ and $\vec{B}$, $\vec{j}$ and $\vec{E}$. For media encountered in the theory of radiowave propagation along the Earth, it may be universally estimated that

$$B = H. \qquad (1.VI)$$

At the same time, here Eq. (1.VI) is in essence superfluous: it stems from Eq. (1.II), provided we apply to it the operator div. As to the remaining links, it should be taken into account that the assignment of two functions — the conductivity σ and the dielectric constant ε is sufficient only for infinitely slow processes in linear and isotropic medium:

$$D = \varepsilon(x, y, z)E, \quad j = \sigma(x, y, z)E. \qquad (1.VII)$$

As to the region of fast-varying fields (for radiowaves in the ground in the ultra-short and shorter waves), it is, generally speaking, impossible, inasmuch as the dielectric polarization, for example, may lag behind the field. Consequently, the character of the link between $\vec{D}$ and $\vec{E}$, $\vec{J}$ and $\vec{E}$ (and also between $\vec{B}$ and $\vec{H}$, provided $\vec{B} \neq \vec{H}$) depends on the temporal regime. Generally speaking, it is impossible to make use of the field equations in the form (1.I)-(1.VII). This is why a temporal regime of specific type, the harmonic, is chosen, its use being limited to its study, and then, taking advantage of field equations linearity, any other process is represented as a superimposition of harmonic processes. Obviously, from this standpoint an assortment of any other regimes, mathematically described by a complete orthogonal system of functions, would be admissible. However, the harmonic regime has the advantage that, in the first place, many other real generator regimes are very close to it; secondly, the simple correlations (1.VII) are valid for it (in a sufficiently uniform medium); in these, ε and σ are functions of the numerical parameter characterizing the given regime of frequency ω:

$$\varepsilon = \varepsilon(\omega), \quad \sigma = \sigma(\omega).$$

This is closely linked with the fact that, under the influence of the harmonic force, the charges of the medium too effect harmonic oscillations. This is why in the following we shall consider only the propagation of a field harmonically dependent on time, and characterized in accord with this, by a specific frequency ω.

We shall utilize complex quantities, describing the dependence on time by the multiplier $e^{-i\omega t}$. The real parts of the quantities utilized will have a real sense. It is well known that, precisely because of that, we may utilize with equal right $e^{+i\omega t}$ also. Both methods of description are as frequently encountered in literature. However, when considering the propagation of radiowaves along the Earth (just as in general electrodynamic equations), the dependence $e^{-i\omega t}$, chosen by us, is utilized more often, whereas during the consideration of radiowave propagation in the ionosphere, $e^{+i\omega t}$ is used nearly always. Aside from the tradition, there is no reasonable basis for either choice. For the sake of uniformity, a specific form of dependence was chosen in [1], namely, exp $(+i\omega t)$, for in the latter work it was a joint consideration of both questions. In order to pass from formulas obtained with the time dependence exp $(-i\omega t)$ to those obtained with the selection of exp $(+i\omega t)$, one should simply pass to complex conjugate quantities, substituting i by $-i$. This is why the formulas of the present book will differ from the corresponding formulas of the first part of the book [1] by the substitution of i by $-i$.

Now the field equations will take the form

$$\operatorname{rot} H = \left(\frac{4\pi}{c}\sigma(\omega) - \frac{i\omega}{c}\varepsilon(\omega)\right)E + \frac{4\pi}{c}j^{\text{стор}},$$

$$\operatorname{rot} E = \frac{i\omega}{c}H,$$

$$\operatorname{div}(\varepsilon E) = 4\pi(\rho + \rho^{\text{стор}}),$$

$$\operatorname{div} B = 0,$$

with the continuity equation

$$j(\omega) = \sigma(\omega)E, \quad D = \varepsilon(\omega)E. \tag{1.1e}$$

Here it is taken into account that for the given frequency

$$\operatorname{div}(\sigma E) = i\omega\rho. \tag{1.1f}$$

We shall not, for the present, take into account the alien forces.

If σ and ε are real, we may consider from the phenomenological viewpoint the right-hand part of Eq. (1.1a) as composed of vortex sources d of two types: those being in phase with rot H [term (4π)/

$/(c)\sigma E]$ and those leading rot H in phase by $\pi/2$ [term $-(i\omega)/(c)\varepsilon E =$ $= (\omega)/(c)Ee^{-i(\pi)/(2)}$]. In the general case of inertia conductors and dielectrics, the conduction current and the displacement current may not be able to keep up with the field, and this is why the phase shift of these two vortex sources relative to rot H itself, may differ from the values 0 and $\pi/2$. Thus, for example, if the real fields $\vec{E}$ and $\vec{H}$ depend on time, as cos ωt, both the induction and the current may lag behind them:

$$D = |\varepsilon| E_0 \cos(\omega t - \varphi_\varepsilon), \quad J = |\sigma| E_0 \cos(\omega t - \varphi_\sigma),$$

where φ_ε, φ_σ are certain phase shifts. As may be easily verified, this may also be expressed in the complex notation by simply estimating ε and σ in Eqs. (1.1a, c and e) as complex quantities. In this case, two sources of vortex H in (1.1a) differ from one another by their phase, but no longer by $\pi/2$. But then there is no longer any sense in making a distinction between ε and σ. In reality, having written $\sigma =$ $= \sigma' + i\sigma''$, $\varepsilon = \tau' + i\varepsilon$, we may rewrite Eq. (1.1a) also in the form:

$$\mathrm{rot}\, H = \left(\frac{4\pi}{c}\sigma' + \frac{\omega}{c}\varepsilon''\right)E - \left(\frac{\omega}{c}\varepsilon' - \frac{4\pi}{c}\sigma''\right)iE, \qquad (1.2)$$

where the right-hand part is again expanded in the sources being in phase with rot $\vec{H}$, and sources leading rot $\vec{H}$ by $\pi/2$ (thus, here the most general case is taken into account). This is why the presence of the dielectric constant of the imaginary part $i\varepsilon''$ is equivalent to the presence of the effective ohmic conduction:

$$\frac{\omega}{c}\varepsilon'' = \frac{4\pi}{c}\sigma_{\text{эфф}}, \quad \sigma_{\text{эфф}} = \frac{\omega}{4\pi}\varepsilon'', \qquad (1.3)$$

and, on the other hand, the presence of inertia in ohmic conduction is equivalent to a certain real dielectric constant:

$$-\frac{4\pi}{c}\sigma'' = \frac{\omega}{c}\varepsilon_{\text{эфф}}, \quad \varepsilon_{\text{эфф}} = -\frac{4\pi}{\omega}\sigma''. \qquad (1.4)$$

This is why only the total current $\vec{J}_p = \vec{J} - (i\omega)/(4\pi)\vec{D} = (\sigma - i\omega\varepsilon/4\pi)\vec{E}$

has any physical physical sense, and not either of the displacement or conduction currents taken alone. It may be described by either introduction in the form $\vec{J}_p = \sigma\vec{E}$, the total complex conductance σ in place of the sum $\sigma - i\omega g/4\pi$, or, which is customary, by introducing the total complex permeability

$$\varepsilon + \frac{4\pi i}{\omega}\sigma \to \varepsilon \qquad (1.5)$$

in the form

$$j_n = -\frac{i\omega}{4\pi}\varepsilon E. \qquad (1.6)$$

One might think that the equivalence of the imaginary part ε and the real part σ is disrupted by Eq. (1.1) clearly including precisely ε and not σ. However, in reality this is not so. Whenever desired, we may relate to the number of basic equations either of the two Eqs. (1.1c) or (1.1e). If, for example, instead of ε and σ we substitute certain ε and σ, postulating at the same time $\tilde{\sigma} = 0$, whereas $\tilde{\varepsilon}$ differs from ε, $\tilde{\varepsilon} = \varepsilon + \Delta\varepsilon$, where $\Delta\varepsilon = (4\pi i)/(\omega)\sigma$, we shall respectively obtain from Eqs. (1.1c) and (1.1e)

$$\operatorname{div}(\tilde{\varepsilon}E) = 4\pi(\rho + \rho^{\text{стор}}), \qquad (1.7)$$

that is,

$$\rho = 0,$$

$$\operatorname{div}(\varepsilon E) = 4\pi\rho^{\text{стор}} - \operatorname{div}\left(\frac{4\pi\sigma}{\omega} iE\right).$$

Therefore, there will be no "real current," whereas there will appear a complementary polarization

$$P_{\text{доп}} = -\frac{1}{4\pi}\Delta\varepsilon E = i\frac{\sigma}{\omega}E$$

with the corresponding polarization current

$$j_{\text{поляр. доп}} = \frac{\partial P_{\text{доп}}}{\partial t} = -i\omega P_{\text{доп}} = \sigma E,$$

including the same magnetic effects as the conduction current does. But, if, to the contrary, we postulated $\tilde{\varepsilon} = 1$, $\tilde{\sigma} = \sigma + \Delta\sigma$, $\Delta\sigma = -i\omega/$

$/4\pi(\varepsilon - 1)$, it would follw from the continuity equation tnat ρ increases by a certain quantity $\Delta\rho$,

$$\Delta\rho = \frac{1}{i\omega}\operatorname{div}(\Delta\sigma \cdot E), \tag{1.8}$$

so that for $\varepsilon \to \tilde{\varepsilon} = 1$ and $\rho \to \rho + \Delta\rho$, we would obtain

$$\operatorname{div} E = 4\pi(\rho + \rho^{\text{cтop}}) + \operatorname{div}\left(\frac{4\pi}{i\omega}\,\Delta s \cdot E\right), \tag{1.7a}$$

which, as is easy to be convinced, will again be equivalent to the initial Eq. (1.1c) at $\varepsilon \neq 1$. Appearing here instead of the polarization $\varepsilon - 1/4\pi\,\vec{E}$ is the "real" charge $\Delta\rho$.

As is well known, according to field equations, the real conduction of the medium and the currents (term $(4\pi)/(c)\vec{J} = (4\pi)/(c)\sigma\vec{E}$ linked with it in (1.1), lead to the fact that, as an average, the work $\overline{dW}/at = 1/2\ (\vec{J}\vec{E}^*) = (1)/(2)\overline{\sigma\vec{E}\vec{E}^*}$ is performed per unit of volume in the unit of time. The Joule-Lenz law indicates that this work is liberated in the form of heat:

$$\frac{\overline{dW}}{dt} = \frac{1}{2}\sigma_{\text{эфф}} EE^* = \frac{\omega\varepsilon''}{8\pi}EE^*. \tag{1.9}$$

The presence of the inertial part $i\varepsilon''$ in the polarization naturally leads also to the fact that the field in the dielectric performs the work

$$\frac{\varepsilon''}{\varepsilon'} = \operatorname{tg}\delta, \tag{1.10}$$

Introducing the notation

$$\frac{\overline{dW}}{dt} = \frac{\omega}{4\pi}\varepsilon'\operatorname{tg}\delta\cdot\frac{1}{2}EE^*. \tag{1.10a}$$

where δ is the angle on the phase diagram showing the lag of vector $\vec{D}$ with respect to vector $\vec{E}$, is written sometimes also as follows:

$$\frac{\overline{dQ}}{dt} = \frac{\overline{dW}}{dt} = \frac{1}{2}\sigma EE^*. \tag{1.10b}$$

For small δ, we have $\varepsilon' \approx |\varepsilon|$. Hence it may be seen why δ is designated as the loss angle. Morever, except for the rare case taking place only in quite rarefied gases, where the polarization inertia is conditioned only by the so-called radiation reaction (radial friction), this is also converted to heat and leads to heating of the dielectric. In case of action by only radial friction, this energy is again radiated and leads to the scattering of the incident wave. Thus, for example, in real gases the energy transferred to atoms from the electromagnetic wave transforms to heat as a result of excited atoms' collisions. In case of oriented polarization, the transfer to heat is conditioned by the friction of oscillating dipoles, etc.

Thus, if we admit the possibility of dispersion events, that is, if we admit the complexity of conduction and of the dielectric constant, the distinction between these two physical quantities becomes strictly conventional: the imaginary component ε does not physically differ in any way from the real component σ, and vice-versa. Thus, for example, (see §15), it has been experimentally revealed that for quite high frequencies ($f \sim 10^9$-10^{10} Hz) the dry sand behaves as if its conduction exceeded sharply that found during measurements in a permanent field, and attains 10^{10} CGSE. Obviously, this simply means that at such frequencies the polarization inertia is manifest substantially and that, according to (1.3), it becomes $\varepsilon'' \gg 4\pi\sigma/\omega$. However, this event is not distinguishable from the variation of conduction and is described as the increase of σ. It is obvious that the subdivision of the total current into the displacement and conduction current, admitted in the theory of radiowave propagation, would be appropriately understood only as the subdivision of the total (complex) dielectric constant into the real (ε') and imaginary ($\varepsilon'' = 4\pi\sigma/\omega$) parts in which σ is the real parameter. It makes sense to introduce separately ε' and $\sigma = (\omega)/(4\pi)\varepsilon''$ only because the dependence of these quantities on the frequency in the radioband appears only at high frequencies, when the wavelength in the air becomes substantially less than 1 m, and only in certain media (see Table 1 in §15). However, inasmuch as such a dependence still exists and the superhigh frequency band plays a still increasing role, we shall use everywhere the notion of <u>total complex dielectric constant</u> ε, understanding by ε' <u>its real part</u>, and by $4\pi\sigma/\omega = \varepsilon''$ its imaginary part.*

Therefore, Eqs. (1.1a)-(1.1d) will be written in the following form:

$$\operatorname{rot} H = -\frac{i\omega}{c}\varepsilon(\omega)E + \frac{4\pi}{c}j^{\text{стор}},$$

$$\operatorname{rot} E = \frac{i\omega}{c}H,$$

$$\operatorname{div}(\varepsilon' E) = 4\pi(\rho + \rho^{\text{стор}}),$$

$$\operatorname{div} B = 0,$$

$$\varepsilon = \varepsilon' + 4\pi i\frac{\sigma}{\omega};\quad \varepsilon' = \operatorname{Re}\varepsilon.$$

Let us take into account the continuity equation

$$\operatorname{div} j = \operatorname{div}(\sigma E) = i\omega\rho. \tag{1.12}$$

Then, Eq. (1.11c) will be transformed into the equation

$$\operatorname{div}(\varepsilon E) = 4\pi\rho^{\text{стор}}. \tag{1.12a}$$

and will be a corollary of Eq. (1.11a) (it could be obtained from it by applying the operator rot, provided it is taken into account that for $\vec{j}^{\,stor}$ and ρ^{stor} the continuity equation is evidently valid also).** This is why Eqs. (1.11c) and (1.11d) may already no longer be considered among the basic ones and, denoting

$$k = \frac{\omega}{c}, \tag{1.12b}$$

we shall obtain the system of field equations in the form

$$\operatorname{rot} H = -ik\varepsilon E + \frac{4\pi}{c}j^{\text{стор}}, \tag{1.13a}$$

$$\operatorname{rot} E = ikH. \tag{1.13b}$$

The continuity equation (1.12), by which they should be complemented so as to obtain Eq. (1.11c), may be simply considered as the determination of the quantity ρ which no longer appears anywhere.

The solution of these equations depends upon the material characteristics of the medium only through a unique parameter ε. Even with σ and ε' being independent from frequency, ε clearly depends on ω. This is why we might have expected that for an ideal radiotransmission, when the field constitutes a superimposition of fields of various frequencies (for example, during artistic radiobroadcast $\Delta\omega \sim 2\pi \cdot 10^4$), a difference in the conditions of propagation for various frequencies will lead to the distortion of radiotransmission. In principle, this is obviously what actually takes place. However, usually the quantitative correlations are such that, as we shall see in §27, the distortion remains entirely immaterial, and only in special cases (propagation of pulses and so forth), the dispersion affects the shape of the wave.

In the absence of alien forces the volume charges cannot be present in a medium with conduction. Even if they were introduced at a certain moment of time with the density $\rho_0(x, y, z)$, then, as follows from the equations

$$\operatorname{div} E = \frac{4\pi}{\varepsilon'}\rho,$$

$$\frac{\partial\rho}{\partial t} = -\operatorname{div} j = -\operatorname{div}(\sigma E) = -\frac{4\pi\sigma}{\varepsilon'}\rho, \qquad (1.14)$$

their density would decrease exponentially with time at every point

$$\rho(x, y, z, t) = \rho(x, y, z)\, e^{-\frac{4\pi\sigma}{\varepsilon'}t}, \qquad (1.15)$$

independently from how the part of the field $\vec{E}$, devoid of sources, would vary at every point.

However, this conclusion is invalid for a nonuniform medium if ε' and σ become functions of the point.

In this case the volume charges are also subject to the continuity equation

$$\frac{\partial\rho}{\partial t} = -\operatorname{div} j = -\operatorname{div}(\sigma E) = -\operatorname{div}\left(\frac{\sigma}{\varepsilon'} D\right) = -\frac{\sigma}{\varepsilon'}\operatorname{div} D - \left(D,\ \operatorname{grad}\frac{\sigma}{\varepsilon'}\right).$$

and, according to (1.III),

$$\frac{\partial\rho}{\partial t} = -\frac{4\pi\sigma}{\varepsilon'}\rho - \left(D,\ \operatorname{grad}\frac{\sigma}{\varepsilon'}\right). \tag{1.14a}$$

From the solution of this equation we may separate the part decreasing with time

$$\rho = \rho_0 e^{-\frac{4\pi\sigma}{\varepsilon'}t} + \rho_1, \tag{1.15a}$$

at the same time, ρ_1 itself satisfies Eq. (1.14a).

Inasmuch as we consider fields harmonically dependent on time, we may postulate $\partial\rho_1/\partial t = -i\omega\rho_1$, so that Eq. (1.14a) takes the form

$$\rho_1 = \frac{1}{i\omega}\left(D,\ \operatorname{grad}\frac{\sigma}{\varepsilon'}\right)\frac{\varepsilon'}{\sigma} = -\frac{1}{4\pi}\left(D,\ \operatorname{grad}\ln\left(\frac{\sigma}{\varepsilon'}\right)\right). \tag{1.15b}$$

Consequently, the volume density contains a fluctuating part ρ_1, disappearing only in a medium in which the σ to ε' ratio is constant (or varies only in a direction perpendicular to $\vec{D}$). Thus, for example, at the boundary between a conducting body and a dielectric σ/ε' the ratio varies from a finite value to zero in a jump-like fashion. This is why there must arise at the jump point (that is, on the surface) an infinitely great volume density. The integral from it over the normal is finite and provides the superficial density of the charge varying with time. Therefore, at boundaries of the divide between uniform media, the charges may exist in their superficial form. As to the charges in uniform media, they may exist only in places where they are sustained by alien forces.

§2. System of Differential Equations

We shall consider an antenna in a medium which may be considered as uniform at points closely adjacent to it.

Assume that the antenna, being a region of action by alien fields, has a conduction σ_1 and a real dielectric constant ε_1', whereas the alien fields are absent in the surrounding medium and the characteristics of the medium are σ and ε'.

The field equations (1.13a), (1.13b) in the region occupied by the antenna, differ from the equations in the surrounding medium by the presence of the term $\vec{j}^{\,stor}$ and by the substitution of ε by

$$\varepsilon_1 = \varepsilon_1' + \frac{4\pi i\sigma_1}{\omega}.$$

For the antenna we shall represent the right-hand part of Eq. (1.13a) in the form

$$-ik\varepsilon_1 E + \frac{4\pi}{\varepsilon} j^{\text{стор}} = -ik\varepsilon E + \frac{4\pi}{\varepsilon} j_0, \qquad (2.1)$$

where

$$j_0 = j^{\text{стор}} - (\varepsilon_1 - \varepsilon)\frac{i\omega E}{4\pi}. \qquad (2.2)$$

The antenna setup is always such that the currents in the antenna are significantly greater than in the regions directly adjacent to it. Thus, for example, in the case of a conducting antenna, its conduction is knowingly great by comparison with the conduction of the surrounding medium. Even for a copper antenna immersed in sea water, $\sigma_1 \sim 10^{17}$, $\sigma \sim 10^{11}$ CGSE, the σ_1 to σ ratio has the order of 10^6. This condition must be always observed, for in the opposite case, alien forces would not be exciting currents as much in the antenna as in the surrounding medium (the same is obviously valid for dielectric antennas). This is why we have approximately

$$\dot{j}_0 \approx \dot{j}^{\text{стор}} - ik\varepsilon_1 E. \qquad (2.3)$$

In the region occupied by the antenna the system of equations takes the form

$$\operatorname{rot} H = -ik\varepsilon E + \frac{4\pi}{\varepsilon} \dot{j}_0, \qquad (2.4a)$$

$$\operatorname{rot} E = ikH. \qquad (2.4b)$$

In this way we separated the main part of the current $\vec{J}_0$ in the antenna into a separate term, which evidently is much greater than the expression $-ik\varepsilon\vec{E}$ (which by order of magnitude is equal to the total current density in the surrounding medium).

Now it is obvious that the system of equations for the field inside the antenna (2.4) differs from the system of equations for the field in the surrounding medium only by the presence in the right-hand part of Eq. (2.4a) of the term $4\pi/c\vec{J}_0$. Consequently, we may consider the entire space, including the volume of the antenna, as a uniform space, characterized by the electric constants σ and ε' of the surrounding medium; at the same time we must take into account the presence in the region occupied by the antenna of the current $\vec{J}_0$ partially sustained by the field, partially by alien forces (and also the presence of densities of both, the volume ρ_0 and the surface δ_0 charges, unilaterally determined by it (see below (4.3a)).

The determination of the current, flowing in the antenna at given lateral forces, of the date on geometry and properties of the media, constitutes in itself a subject of antenna theory, and we shall estimate this part of the problem as resolved, that is, we shall consider the distribution of currents $\vec{J}_0$ (2.2) in the antenna as known.

The object of the theory of radiowave propagation is the determination of the fields on the condition that the current, flowing in the antenna, is given. This is why the problem is reduced to the solution of the system of Maxwellian equations (2.4) in a uniform medium with properties of the surrounding medium and sources given in the form of a known function $\vec{J}_0$.

Let us note that it is sufficient to find only one of the two fields, for example, $\vec{E}$. The other is then obtained by simple differentiation (from (2.4b)).

§3. Potentials

1. As is always the case in electrodynamics, the search for the solution of Maxwellian equations for a problem formulated in the case of a uniform medium may be substantially alleviated if we pass from these equations to certain potentials. They may be introduced by a method, which is standard for the theory of the electromagnetic field, and namely, by postulating, for example,

$$\boldsymbol{H} = \operatorname{rot} \boldsymbol{A}, \tag{3.1}$$

$$\boldsymbol{E} = -\operatorname{grad} \varphi - \frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t} = -\operatorname{grad} \varphi + ik\boldsymbol{A}. \tag{3.2}$$

Equations (2.4) will be satisfied if the vector potential $\vec{A}$ and the scaler potential φ satisfy the equations:

$$\nabla^2 \boldsymbol{A} + \varepsilon k^2 \boldsymbol{A} = -\frac{4\pi}{\varepsilon} \boldsymbol{j}_0, \tag{3.3}$$

$$\varphi = \frac{1}{i\varepsilon k} \operatorname{div} \boldsymbol{A}. \tag{3.4}$$

(If here and in the following the expressions of the type $\nabla^2 \vec{A}$ are recognized as the result of action of the operator ∇^2 on every component $\vec{A}$, and not as the abbreviated denotation for grad div $\vec{A}$ – rot rot $\vec{A}$, the expression (3.3) will be valid only in Descartes coordinates).

Hence it is easy to obtain for φ the equation

$$\nabla^2 \varphi + \varepsilon k^2 \varphi = -\frac{4\pi}{\varepsilon} \rho_0, \tag{3.4a}$$

where

$$\rho_0 = \frac{1}{i\omega} \operatorname{div} \boldsymbol{j}_0. \tag{3.4b}$$

Inasmuch as according to Eq. (3.4) φ is, however, unambiguously determined through $\vec{A}$, so that

$$\boldsymbol{E} = \frac{1}{-i\varepsilon k} \operatorname{grad} \operatorname{div} \boldsymbol{A} + ik\boldsymbol{A}, \tag{3.5}$$

there is no necessity of introducing a special equation for φ.

At the same time a further simplification is possible in the free space, inasmuch as, taking advantage of the certain remaining liberty in the choice of potentials ([16], §1.9) ("gauge invariance" of field equations), they may be so chosen that

$$\operatorname{div} A = \varphi = 0. \qquad (3.6)$$

However, the tradition has been established in the theory of radio-wave propagation to introduce somewhat different potentials. In essence, there is question only of different terminology, as we now shall be convinced. It is linked with the fact that formerly we had to be concerned in most of the problems with a special form of currents' $\vec{J}_0$ distribution, either in the form of a short, by comparison with the wavelength, rectilinear antenna, equivalent to an oscillating dipole, or in the form of a small frame (equivalent to a magnetic dipole). At the present time, with the passage to shorter waves and to emitters of complex configurations, even that justification stands no longer. Nevertheless, potentials slightly different from A and φ are broadly accepted as formerly. Inasmuch as the distinction amounts, in essence, to change of denotations, this does not bring about additional complications.

Depending upon the antenna configuration these potentials are introduced by one of the following two methods.

If the emitter is rectilinear, it is appropriate to consider the current $\vec{J}_0$ as a current of certain polarization $\vec{P}_0$, distributed about the emitter's volume. Formally this means that we introduce an auxiliary vector $\vec{P}_0$, linked with the current $\vec{J}_0$ by a formula usually determining the polarization current:

$$j_0 = \frac{\partial P_0}{\partial t} = -i\omega P_0. \qquad (3.7)$$

Then, it follows from the continuity equation (3.4b) that

$$\rho_0 = -\operatorname{div} P_0. \qquad (3.8)$$

This polarization $\vec{P}_0$ generates an identical electromagnetic field to the one that would be generated by the true charge ρ_0 and current $\vec{J}_0$.

Let us now introduce the auxiliary vector $\vec{\Pi}$, the so-called electric Herz vector, according to formulas

$$H = -ik\varepsilon \operatorname{rot} \Pi, \tag{3.9}$$

$$E = \operatorname{grad} \operatorname{div} \Pi + \varepsilon k^2 \Pi = \operatorname{rot} \operatorname{rot} \Pi + \nabla^2 \Pi + \varepsilon k^2 \Pi. \tag{3.10}$$

Comparison of (3.1) and (3.2) shows that we simply have

$$A = -ik\varepsilon \Pi, \quad \varphi = -\operatorname{div} \Pi. \tag{3.11}$$

It is therefore natural that at substitution of (3.9), (3.10) into the field equations, they are satisfied, provided

$$\nabla^2 \Pi + \varepsilon k^2 \Pi = -\frac{4\pi}{\varepsilon} P_0 = -\frac{4\pi i}{\omega\varepsilon} j_0 \tag{3.12}$$

(cf. (3.3)).

Therefore, the utilization of the Hertz vector instead of the vector potential is in essence a terminological factor indeed (the constant factor $-ik\varepsilon$ in (3.11), is obviously immaterial).

Thus, considering the current in the antenna as given, we must resolve three differential equations for three components of $\vec{\Pi}$ (at boundary conditions with which we shall be confronted subsequently), and then determine with their aid $\vec{E}$ and $\vec{H}$ according to Formulas (3.9) and (3.10).

The convenience of the conducted transformation is manifest in the case of rectilinear antenna, when $\vec{J}_0$ contains only one component, and this is why for infinitely extended media, and also in numerous cases not disrupting the symmetry, in a nonuniform medium (vertical antenna on a plane or spherical ground) $\vec{\Pi}$ is also reduced to a single component.

Before passing to other cases, let us note that, acting upon Eq. (3.12) by the operator $ik\varepsilon$ rot, we shall obtain by the strength of

Eq. (3.9)

$$(\nabla^2 + k^2\varepsilon)H = 4\pi ik \operatorname{rot} P_0 = -\frac{4\pi}{\varepsilon} \operatorname{rot} j_0. \qquad (3.9a)$$

But acting upon Eq. (3.12) by the operator grad div + εk^2, we shall have according to Eq. (3.10)

$$(\nabla^2 + k^2\varepsilon)E = -\frac{4\pi}{\varepsilon}(\operatorname{grad}\operatorname{div} P_0 + \varepsilon k^2 P_0) =$$

$$= -\frac{4\pi i}{\omega\varepsilon}(\operatorname{grad}\operatorname{div} j_0 + \varepsilon k^2 j_0). \qquad (3.9b)$$

This gives us two separate differential equations for the fields $\vec{E}$ and $\vec{H}$. Having resolved one of them, the second field vector may be found by either (2.4a) or (2.db).

2. In case of antennas of more complex configuration Eq. (3.12) will obviously be also valid; however, the correlations of the components may become more complex. This is why it is appropriate to provide, at least for a form of antennas often encountered in practice, a more convenient transformation. We have in mind loop antennas (of which the dimensions are small by comparison with the wavelength).

Here it is appropriate to substitute each current contour by a magnetic shell, that is, to estimate that it is tightened by a thin layer, within the bounds of which there exists a certain magnetization $\vec{M}_0$ (varying with the same frequency ω).

Inasmuch as the total current I is constant along the contour's length, the integral from $\vec{M}_0$ over the volume of the shell, giving the total magnetic moment of the contour, will be simply equal to (1)/ /(c)$I\vec{S}$ where S is the surface of the contour.

This "magnetization" is estimated to be given; at the given point it is not proportional to $\vec{H}$. This is why one should write

$$B = H + 4\pi M_0. \qquad (3.13)$$

Therefore, we shall be compelled to return to Eqs. (1.I)-(1.IV), containing $\vec{B}$. Outside the loop we shall have, as previously, Eqs. (2.4) with $\vec{J}_0 = 0$; but within the bounds of the imaginary magnetization layer, instead of (2.4) the field equations will give

$$\operatorname{rot} H = -ik\varepsilon E,$$
$$\operatorname{rot} E = ik(H + 4\pi M_0),$$
$$\operatorname{div} E = 0.$$
$$\operatorname{div} H = -4\pi \operatorname{div} M_0,$$

where evidently, the last two equations will be corollaries of the first ones.

Inasmuch as now $\vec{E}$ is solenoidal, and not $\vec{H}$ (as in the case of rectilinear antenna), we shall introduce the Hertz magnetic vector $\vec{\Pi}_m$ by the formulas

$$E = ik \operatorname{rot} \Pi_m, \qquad (3.15)$$
$$H = \operatorname{grad} \operatorname{div} \Pi_m + \varepsilon k^2 \Pi_m = \operatorname{rot} \operatorname{rot} \Pi_m + \nabla^2 \Pi_m + \varepsilon k^2 \Pi_m, \qquad (3.16)$$

(the last equality assuming again the decomposition of $\vec{\Pi}_m$ into rectangular components).

At substitution of these expressions Eq. (3.14a) is fulfilled identically, whereas (3.14b) is satisfied, if $\vec{\Pi}_m$ is subject to the equation

$$\nabla^2 \Pi_m + \varepsilon k^2 \Pi_m = -4\pi M_0. \qquad (3.17)$$

It is obvious that this equation is well adpated to the case of small flat loop, when $\vec{M}_0$ is reduced to a single component; $\vec{\Pi}_m$ will also be reduced to a single component.

Analogously to that we could also introduce the magnetic, vectorial and scalar potentials $\vec{A}_m$ and φ_m, that are well adapted to the case. It is inoed easy to be convinced that by postulating

$$E = \operatorname{rot} A_m, \qquad (3.18)$$
$$H = \varepsilon \operatorname{grad} \varphi_m - ik\varepsilon A_m, \qquad (3.19)$$

we shall satisfy Eqs. (3.14) if we subject $\vec{A}_m$ and φ_m to equations

$$\nabla^2 A_m + \varepsilon k^2 A_m = -\frac{4\pi}{c} i\omega M_0 = -\frac{4\pi}{c} j_{0m}, \tag{3.20}$$

$$\varphi_m = \frac{1}{ik\varepsilon} \operatorname{div} A_m. \tag{3.21}$$

Hence follows for φ_m the equation, equivalent to Eq. (3.14d):

$$\nabla^2 \varphi_m + \varepsilon k^2 \varphi_m = -\frac{4\pi}{c} \operatorname{div} M_0 = -\frac{4\pi}{c} \rho_{0m}, \tag{3.22}$$

where the fictitious magnetic current $\vec{j}_{0m}$ and charge ρ_{0m} are introduced:

$$j_{0m} = -\frac{\partial M_0}{\partial t}, \quad \rho_{0m} = \operatorname{div} M_0. \tag{3.22a}$$

However, Eq. (3.22) is superfluous, inasmuch as there exists a simple relation (3.21). The correspondence between the correlations (3.1)-(3.4a), on the one hand, and between (3.18)-(3.22) on the other, is obvious. The equations for $\vec{A}$ and φ formally coincide with equations for $\vec{A}_m$ and φ_m.

This correspondly establishes the relation of $\vec{A}_m$ and φ_m with the Hertz magnetic vector $\vec{\Pi}_m$:

$$A_m = ik\Pi_m, \tag{3.23}$$

$$\varphi_m = \frac{1}{\varepsilon} \operatorname{div} \Pi_m, \tag{3.24}$$

which out to be compared with Formulas (3.11).

The explicit expressions for the fields through the derivatives of the Hertz vector are as follows:

<u>for the electric Hertz vector</u>:

$$\begin{aligned} E_x &= -\nabla^2 \Pi_x + \frac{\partial^2 \Pi_x}{\partial x^2} + \frac{\partial}{\partial x}\left(\frac{\partial \Pi_y}{\partial y} + \frac{\partial \Pi_z}{\partial z}\right), & H_x &= -ik\varepsilon\left(\frac{\partial \Pi_z}{\partial y} - \frac{\partial \Pi_y}{\partial z}\right), \\ E_y &= -\nabla^2 \Pi_y + \frac{\partial^2 \Pi_y}{\partial y^2} + \frac{\partial}{\partial y}\left(\frac{\partial \Pi_x}{\partial x} + \frac{\partial \Pi_z}{\partial z}\right), & H_y &= -ik\varepsilon\left(\frac{\partial \Pi_x}{\partial z} - \frac{\partial \Pi_z}{\partial x}\right), \\ E_z &= -\nabla^2 \Pi_z + \frac{\partial^2 \Pi_z}{\partial z^2} + \frac{\partial}{\partial z}\left(\frac{\partial \Pi_x}{\partial x} + \frac{\partial \Pi_y}{\partial y}\right), & H_z &= -ik\varepsilon\left(\frac{\partial \Pi_y}{\partial x} - \frac{\partial \Pi_x}{\partial y}\right); \end{aligned} \tag{3.25}$$

for the magnetic Hertz vector:

$$E_x = ik\left(\frac{\partial\Pi_{mz}}{\partial y} - \frac{\partial\Pi_{my}}{\partial z}\right),\quad H_x = -\nabla^2\Pi_{mx} + \frac{\partial^2\Pi_{mx}}{\partial x^2} + \frac{\partial}{\partial x}\left(\frac{\partial\Pi_{my}}{\partial y} + \frac{\partial\Pi_{mz}}{\partial z}\right),$$

$$E_y = ik\left(\frac{\partial\Pi_{mx}}{\partial z} - \frac{\partial\Pi_{mz}}{\partial x}\right),\quad H_y = -\nabla^2\Pi_{my} + \frac{\partial^2\Pi_{my}}{\partial y^2} + \frac{\partial}{\partial y}\left(\frac{\partial\Pi_{mx}}{\partial x} + \frac{\partial\Pi_{mz}}{\partial z}\right),$$

$$E_z = ik\left(\frac{\partial\Pi_{my}}{\partial x} - \frac{\partial\Pi_{mx}}{\partial y}\right),\quad H_z = -\nabla^2\Pi_{mz} + \frac{\partial^2\Pi_{mz}}{\partial z^2} + \frac{\partial}{\partial z}\left(\frac{\partial\Pi_{mx}}{\partial x} + \frac{\partial\Pi_{my}}{\partial y}\right). \quad (3.25a)$$

3. In case of spherical system of coordinates, which must be chosen if we consider, for example, the emission of a vertical dipole (either electric or magnetic), placed above a spherical ground, $\nabla^2\vec{\Pi}$ in Eq. (3.10) and in other analogous formulas must be recognized as an abbreviated denotation for grad div Π – rot rot $\vec{\Pi}$, and so forth. The equations for $\vec{\Pi}$ become considerably complicated, and it is convenient to choose for the potentials special Hertz vectors, the so-called Debye potentials.

Let us consider, for example, Eq. (3.12). For a source, having the form of a vertical electric current, we have in the spherical system of coordinates r, ϑ, φ, $P = P_r$, and introducing new scalar functions u and ψ, we may assume

$$\Pi = ru + \operatorname{grad}\psi. \quad (3.26)$$

Ho rot grad $\psi = 0$. Consequently, the last addend has no effect on the values of the field $\vec{H} = -ik\varepsilon$ rot $\vec{\Pi}$ and $\vec{E}$ = rot rot $\vec{\Pi} - (4\pi)/(\varepsilon)\vec{P}_0$ (see (3.9) and (3.10), taking into account (3.12)). If we substitute (3.26) into (3.12), we may, utilizing the formulas of vectorial analysis (in particular, formulas rot [grad u, $\vec{r}$] = $(r\nabla)$ grad u – (grad u∇) $\vec{r}$ + grad u·div $\vec{r}$ – $\vec{r}$ div grad u, and so forth), arrive in the last resort at the following result.

If we superimpose on ψ the condition

$$\nabla^2\psi + \varepsilon k^2\psi = -2u, \quad (3.27)$$

u also must satisfy the equation

$$r(\nabla^2 u + \varepsilon k^2 u) = -\frac{4\pi}{\varepsilon} P_0. \qquad (3.28)$$

Consequently, first of all, the substitution of (3.26) is possible only in the case when $\vec{P}_0$ is directed along $\vec{r}$, that is, for the vertical current $\vec{J}_0$. Secondly, inasmuch as all the reasonings may be also transferred to $\vec{P}_m$, a similar substitution in the Eq. (3.17) is possible in the case of a vertical magnetic moment. For example, in the electric case

$$\nabla^2 u + \varepsilon k^2 u = \frac{4\pi P_0}{\varepsilon r}. \qquad (3.28a)$$

Thirdly, if u has been found, ψ is determined from the condition (3.27). However, inasmuch as no matter what value is obtained for ψ, it does not affect the values of the fields and may therefore be disregarded.

This is why it is possible to state that in case of vertical linear antenna we have the right to estimate simply that

$$\Pi = r u(r), \qquad (3.29)$$

$u(\vec{r})$ satisfying at the same time Eq. (3.28a). As to the case of horizontal loop, we shall obtain analogously

$$\Pi_m = r v(r), \qquad (3.30)$$

$v(\vec{r})$ satisfying at the same time the equation

$$\nabla^2 v + \varepsilon k^2 v = -4\pi \frac{M_0}{r}. \qquad (3.31)$$

The functions u and v are designated as Debye functions (for details see, for example, [4]).

In these two cases, the fields, that is, the "electric" and "magnetic" types, are endowed with different symmetry properties. It is appropriate to write out their explicit expressions in free space (at points, where P_0 and M_0 are zero). From Formulas (3.29), (3.9) and (3.10) (for example, for an emitting antenna with a vertical electric current), it follows that

$$E_r = \frac{\partial^2}{\partial r^2}(ru) + \varepsilon k^2 ur = \frac{-\frac{\partial}{\partial \vartheta}\left(\sin\vartheta \frac{\partial u}{\partial \vartheta}\right)}{r \sin\vartheta}, \quad H_r = 0,$$

$$E_\vartheta = \frac{1}{r}\frac{\partial^2 (ru)}{\partial r \partial \vartheta}, \qquad H_\vartheta = -\frac{ik\varepsilon}{r\sin\vartheta}\frac{\partial (ru)}{\partial \varphi},$$

$$E_\varphi = \frac{1}{r\sin\vartheta}\frac{\partial^2 (ru)}{\partial r \partial \varphi}, \qquad H_\varphi = \frac{ik\varepsilon}{r}\frac{\partial (ru)}{\partial \vartheta}. \quad (3.32).$$

On the other hand, from Formulas (3.30), (3.15) and (3.16) (for example, for the horizontal loop) it follows that

$$E_r = 0, \qquad H_r = \frac{\partial^2 (rv)}{\partial r^2} + \varepsilon k^2 vr = -\frac{\frac{\partial}{\partial \vartheta}\left(\sin\vartheta \frac{\partial v}{\partial \vartheta}\right)}{r\sin\vartheta},$$

$$E_\vartheta = \frac{ik}{r\sin\vartheta}\frac{\partial (rv)}{\partial \varphi}, \quad H_\vartheta = \frac{1}{r}\frac{\partial^2 (rv)}{\partial r \partial \vartheta},$$

$$E_\varphi = -\frac{ik}{r}\frac{\partial (rv)}{\partial \vartheta}, \quad H_\varphi = \frac{1}{r\sin\vartheta}\frac{\partial^2 (rv)}{\partial r \partial \varphi}. \qquad (3.33)$$

4. Up until now we considered the potentials for a field in uniform space. However, as will be seen in the following, the inhomogeneity of the electric properties of the terrestrial atmosphere (and also of the soil) must be taken into account in numerous cases. Generally speaking, the transition from Maxwellian to wave equations becomes here impossible. There exists, nonetheless, an important particular case of inhomogeneity which admits such a transition. It takes place when the inhomogeneities are characterized by a laminar structure, that is, when the parameter ε, characterizing the electrical properties of the medium, depends only on a single coordinate, the altitude above the terrestrial surface or the depth under that surface. If its is sufficient to consider the ground as plane, this implies that ε depends only on a single Descartes coordinate, say

$$\varepsilon = \varepsilon(z), \qquad (3.34)$$

whereas when the ground's sphericity is significant, it depends on the radial coordinate

$$\varepsilon = \varepsilon(r). \qquad (3.35)$$

Let us consider the Maxwellian equations in free space ($\vec{J}_0 = 0$), using at the outset the Descartes coordinates. The field, having the character of the "electric" field, that is, induced, for example, by a current in a vertical antenna, may again be described by the Hertz vector $\vec{\Pi}$, as was done for a uniform medium, with a single component $\Pi = \Pi_z$, different from zero. To that effect it is sufficient to postulate

$$\bar{E} = \frac{k}{k_1^2} \operatorname{rot} \operatorname{rot} (\Pi k_1) = \frac{k}{k_1^2} (\operatorname{grad} \operatorname{div} (k_1 \Pi) - \nabla^2 (k_1 \Pi)), \tag{3.36a}$$

$$H = i \operatorname{rot} (k_1 \Pi) \tag{3.36b}$$

$$k_1^2 = k^2 \varepsilon, \tag{3.36c}$$

and to substitute these expressions into Eq. (1.13b) (Eq. (1.13a) being identically satisfied under such a substitution). One can easily be persuaded that by substitution of (3.36) the problem is resolved if $\vec{\Pi}$ is the solution of the wave number, depending on $\underline{z}$:

$$\nabla^2 \Pi + k_1^{*2}(z) \Pi = 0, \tag{3.37a}$$

$$k_1^{*2} = k_1^2 - \frac{2}{k_1^2} \left(\frac{dk_1}{dz} \right)^2 + \frac{1}{k_1} \frac{d^2 k_1}{dz^2} = k_1^2 - k_1 \frac{d^2}{dz^2} \frac{1}{k_1}, \quad k_1^2 = k^2 \varepsilon (z). \tag{3.37b}$$

If, however, the field is generated by a horizontal loop (vertical magnetic dipole), the position is found to be still simpler. As is easy to be persuaded, formulas (3.15), (3.16) and (3.17) preserve their strength; at the same time, in Eq. (3.17) one must simply take into account the dependence of ε on $\underline{z}$:

$$\nabla^2 \Pi_m + k_1^2 (z) \Pi_m = 0. \tag{3.37c}$$

Let us now pass to the case of spherical symmetry, when $k_1^2(r) = k^2 \varepsilon(r)$.

For the electric vertical current we may postulate

$$E = \frac{k}{k_1^2} \operatorname{rot} \operatorname{rot} (k r u). \tag{3.38a}$$

$$H = i k_1 \operatorname{rot} (r u). \tag{3.38b}$$

with $\underline{u}$ being, at the seme time, subject to equation

$$\nabla^2 u + k_1^{*2}(r) u = 0, \tag{3.39a}$$

$$k_1^{*2}(r) = k_1^2 - k_1 \frac{d^2}{dr^2} \frac{1}{k_1}, \tag{3.39b}$$

which suggests in an obvious fashion the case of plane lamination (3.36), (3.37).

For a vertical magnetic dipole the situation is again simplified. As is easy to be persuaded, at spherical lamination Formulas (3.29), (3.30), (3.31) conserve their strength, and one must simply estimate there $\varepsilon = \varepsilon(r)$, that is,

$$\nabla^2 v + k_1^2(r) v = 0. \tag{3.39c}$$

Therefore, at laminar inhomogeneities, the field may be found from the solution of wave-type equations for a plane, as well as for a spherical structure, but with a wave number dependent on a single coordinate. For a "magnetic type" field the square of the wave number is simply $k^2\varepsilon$, as to a field of "electric type," k_1^{*2}.

§4. Boundary Conditions

The differential equations must be complemented by boundary conditions on the surface of the divide between different media, for example, between air and ground. As is well known, the following conditions stem from the field equations for the fields.

a) The components of electric field's strength, tangential to the surface of the divide, are identical on both sides of that surface:

$$E_{1t} = E_{2t}. \tag{4.1}$$

Inasmuch as in the tangential plane the field may be decomposed into two independent components, two conditions are, in essence, included here. For example, if the surface of the divide is normal to the axis $\underline{z}$, we have $E_1 = E_{2x}$, $E_{1y} = E_{2y}$.

This condition is obtained at the ultimate transition from equation rot $E = ik\vec{H}$, not containing electric parameters of the medium, and thus valid, for example, at as great a conduction of one of the media as desired.

b) The normal vector $\vec{D}$ component satisfies the condition for a "superficial divergence" of the shift:

$$\mathrm{Div}\, D = D_{n_1} + D_{n_2} = 4\pi\delta, \tag{4.2}$$

where the indices n_1 and n_2 indicate that the projection of the vector on the normal to the surface of the divide is taken, respectively directed either into the first medium (n_1), or into the second one (n_2) (Fig. 4.1); δ is the density of the surface charge.

c) The surface charge δ may, generally speaking, be present independently from that of the current in either medium. However, we shall assume everywhere that such a statistical charge is absent simply because the statistical field, induced by it (for example, the always existing statistical field of the ground) alongside with the fields induced by it on various objects of statistical charges, is superimposed on the high-frequency radiowave field, but is not sensed by radiore-

Fig. 4.1. Boundary conditions for the electric induction.

ceivers. That is why the surface density of the charge δ may be determined from the continuity equation, which for the surface density has the form

$$\frac{\partial\delta}{\partial t} = -\mathrm{Div}\, j, \qquad (4.3)$$

that is,

$$\delta = -\frac{i}{\omega}(j_{1n_1} + j_{2n_2}) = -\frac{i}{\omega}(\sigma_1 E_{1n_1} + \sigma_2 E_{2n_2}). \qquad (4.3a)$$

Here again j_{1n_1} and j_{2n_1} are the current projections in the first and second media on the normals to the surface, respectively directed to the first and second medium. If only one of these media has a noticeable conduction, for example $\sigma_1 = 0$, the surface charge

$$\delta = -\frac{i\sigma_2}{\omega} E_{2n_2} \qquad (4.3b)$$

is determined by the field in the conducting medium.

The conditions (4.2) and (4.3a) may appropriately be united, excluding from them δ. We shall then obtain an important correlation, which, on the other hand, stems directly from Eq. (1.12a) for the case $\rho^{stor} = 0$ after the usual ultimate transition from the volume to the surface divergence:

$$\varepsilon_1 E_{1n_1} + \varepsilon_2 E_{2n_2} = 0. \qquad (4.4)$$

d) In correspondence with Eq. (1.1d) for $\vec{B} = \vec{H}$, the surface divergence of the magnetic field intensity vanishes:

$$H_{1n_1} + H_{2n_2} = 0. \qquad (4.5)$$

When projecting on one and the same normal, for example, the first one, we have

$$H_{1n_1} = H_{2n_1}, \qquad (4.5a)$$

that is, the normal component of the magnetic field is continuous.

e) In case of finite conduction of the media, and, consequently, of finite volume density of the current, from Eq. (2.4a) follows the continuity of any of the tangential components of the magnetic field:

$$H_{1t} = H_{2t}. \qquad (4.6)$$

This condition is disrupted for an infinitely great conduction of any of the media (1, 2) or, to be more precise, for such great a conduction, that the skin-layer becomes extremely thin and any measurement near the surface is knowingly still conducted beyond the limits of this skin-layer (Fig. 4.2). It is natural that in such a case a jump of the tangential component of the magnetic field is possible. As is well

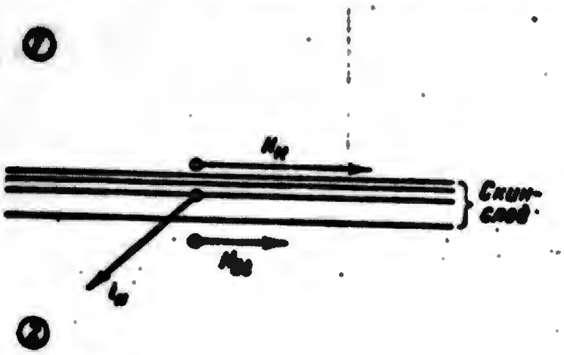

Fig. 4.2. Boundary conditions for the magnetic field. A) Skin-layer.

known, according to (2.4a), it is equal to (the denotations being those of Fig. 4.2):

$$H_{1t} - H_{2t} = \frac{4\pi}{c} i_N, \qquad (4.6a)$$

where i_N is the total current flowing in the direction perpendicular to the direction $\underline{t}$ (which in the drawing is directed at the reader), computed for a unit of length laid in the direction $\underline{t}$. However, the representation on the surface surrent at medium's boundary with infintely great conduction is practical only under ideal conditions used in certain special cases. But in the case of radiowave propagation along the ground the skin-layer is usually great, and the variations of the field within its bounds offer an independent interest. This is why it is preferable to account everywhere for the continuous variation of tangential components of $\vec{H}$, considering the current as distributed about a skin-layer of finite thickness and thus always having a finite volume density.

It would be natural to transfer the formulated boundary conditions for the fields $\vec{E}$ and $\vec{H}$ on the Hertz vectors $\vec{\Pi}$ and $\vec{\Pi}_m$. However, in the general case these conditions are obtained overly complex. They are simplified only in special cases owing to problem's symmetry. Thus, for example, if the vector $\vec{\Pi}$ has only one component, $\Pi = \Pi_z$, which takes place for a vertical antenna and a plane boundary of the divide, $z = 0$, of a uniform ground and atmosphere (ε_1 = const, ε_2 = const). Then it follows from Formula (3.25)

$$H_x = -ik\varepsilon \frac{\partial \Pi_z}{\partial y}.$$

Hence, as a consequence of continuity of the tangential components of $\vec{H}$ (4.6), we obtain

$$\varepsilon_1 \frac{\partial \Pi_z^{(1)}}{\partial y} = \varepsilon_2 \frac{\partial \Pi_z^{(2)}}{\partial y}.$$

Integrating this correlation over y to $\pm\infty$ and taking into account that far off the source $\Pi_z^{(1)} = \Pi_z^{(2)} = 0$, we obtain

$$\varepsilon_1 \Pi_z^{(1)} = \varepsilon_2 \Pi_z^{(2)} \quad \text{at } z = 0. \tag{4.7}$$

On the other hand, the continuity of the tangential components of $\vec{E}$, means in particular, according to (3.25)

$$\frac{\partial^2 \Pi_z^{(1)}}{\partial x \, \partial z} = \frac{\partial^2 \Pi_z^{(2)}}{\partial x \, \partial z} \quad \text{at } z = 0.$$

Integrating over x, we obtain the continuity condition for the normal derivatives of Π_z

$$\frac{\partial \Pi_z^{(1)}}{\partial z} = \frac{\partial \Pi_z^{(2)}}{\partial z} \quad \text{at } z = 0 \tag{4.8}$$

(the utilization of the equality of E_y components contributes nothing new). The conditions (4.7) and (4.8) are used essentially in the following.

Finally, let us bring forth still another form of boundary conditions, which we shall utilize in the furthest.

From the continuity of field's tangential components, for example, of $\vec{E}$, it follows, that their derivatives in the tangential direction are continuous too, and this is why

$$\frac{\partial E_x^{(1)}}{\partial x}+\frac{\partial E_y^{(1)}}{\partial y}=\frac{\partial E_x^{(2)}}{\partial x}+\frac{\partial E_y^{(2)}}{\partial y} \text{ at } z = 0.$$

At the same time, Eqs. div $E^{(1)} = 0$, div $E^{(2)} = 0$ take place. Subtracting them from one another, we obtain

$$\frac{\partial E_z^{(1)}}{\partial z}=\frac{\partial E_z^{(2)}}{\partial z} \text{ for } z = 0. \qquad (4.9)$$

On the other hand, from the continuity of the tangential components of $\vec{H}$, and, consequently, of their derivatives in the tangential direction, stems

$$\mathrm{rot}_z H^{(1)} = \mathrm{rot}_z H^{(2)} \text{ at } z = 0,$$

inasmuch as here enter only these derivatives. But then the Maxwellian equations (1.I), taken for each of the media, give

$$\varepsilon_1 E_z^{(1)} = \varepsilon_2 E_z^{(2)}. \qquad (4.10)$$

For a vertical magnetic dipole above flat ground, vector $\vec{\Pi}_m$ is reduced to a single vertical component. Requiring the continuity of tangential components of $\vec{E}$ and $\vec{H}$ at z = 0, we obtain exactly in the same way

$$\Pi_{mz}^{(1)} = \Pi_{mz}^{(2)}, \text{ at } z = 0; \qquad (4.11)$$

$$\frac{\partial \Pi_{mz}^{(1)}}{\partial z}=\frac{\partial \Pi_{mz}^{(2)}}{\partial z}, \text{ at } z = 0. \qquad (4.12)$$

For a dipole above a spherical surface r = a = const, requiring the continuity of E_ϑ, E_φ, H_ϑ and H_φ, from Formulas (3.32) and (3.33) we obtain by the same method (here the integration is effected over ϑ) for the vertical electric dipole

$$\frac{1}{r}\frac{\partial (r u^{(1)})}{\partial r}=\frac{1}{r}\frac{\partial (r u^{(2)})}{\partial r}, \text{ at } r = a; \qquad (4.13)$$

$$\varepsilon^{(1)}u^{(1)} = \varepsilon^{(2)}u^{(2)}, \text{ at } r = a; \qquad (4.14)$$

for the vertical magnetic dipole

$$v^{(1)} = v^{(2)}, \text{ at } r = a; \qquad (4.15)$$

$$\frac{1}{r}\frac{\partial(rv^{(1)})}{\partial r} = \frac{1}{r}\frac{\partial(rv^{(2)})}{\partial r}, \text{ at } r = a. \qquad (4.16)$$

The boundary conditions for the Hertz vertical magnetic vector do not therefore contain medium's constants.

The important correlation (4.10) shows that the electric field component, normal to the surface of the divide, drops sharply at transition to the medium with great $|\varepsilon|$. This is why, for example, the reception over a vertical antenna will sharply deteriorate during the immersion of a submarine carrying such an antenna ($\varepsilon_1 = 0$), air; $|\varepsilon_2| \gg\gg 1$, sea water; see Table 1 in §15). However, the sinking of a vertical antenna into an extremely dry soil may not induce such bad consequences.

On the other hand, the horizontal components $\vec{E}$ are uninterrupted at transition through the boundary of the divide. This is why during reception on a horizontal loop, we shall not feel the transition through the plane $z = 0$. Hence, it is understood that it does not follow at all as yet (though in reality it is correct, see §30) that the reception under water on a horizontal loop will give a better result than the reception on a vertical antenna. The choice depends also on the kind of correlation E_t and E_n are subject to above the surface of the divide; this is a question requiring special consideration (with which we shall deal below in §21).

B. INTEGRAL CORRELATIONS

§5. Integral Form of the Wave Equation

Therefore, in case of uniform space our problem is reduced to the solution of three independent equations (for example, (3.3) or (3.12), etc.) of the form

$$\nabla^2 u + k^2 u = U, \tag{5.1}$$

where u, for example, is one of the components of the Hertz vector; $k^2 = \varepsilon(\omega^2)/(c^2)$; U is the given function, describing the sources (in this section u and v are not necessarily denoting the Debye functions at all).

Assume that we are required to find u at the point $\vec{R}$, $u = u(\vec{R})$.

Let us call attention to the fact that the function of the coordinates of two points is

$$v(R, R') = \frac{e^{ikr}}{r}, \quad r = |R - R'|. \tag{5.2}$$

if U = 0 satisfies the equation (5.1) for all $\vec{R}$ and $\vec{R}'$ (in relation to the differentiation with respect to the components of $\vec{R}$, as well as that with respect to the components of $\vec{R}'$), aside from the case $\vec{R} \rightarrow \vec{R}'$, $r \rightarrow 0$, when the differentiation is impossible.

Understanding by $\nabla^2_{\vec{R}'}$ the Laplace operator containing the differen tiation with respect to the components of vector $\vec{R}'$, we may write

$$(\nabla^2_{R'} + k^2)\, u(R') = U(R'), \tag{5.3}$$

$$(\nabla^2_{R'} + k^2)\, v(r) = 0, \text{ aside from the point } \vec{R} = \vec{R}'. \tag{5.4}$$

According to the Green theorem, any two functions (and this means also u and v), that are continuous alongside with their first derivatives in a certain volume V, satisfy the correlation

$$\int_V \{v(r)\, \nabla^2_{R'} u(R')\, \nabla^2_{R'} v(r)\}\, dV = \int_S \left\{v(r) \frac{\partial u(R')}{\partial n} - u(R') \frac{\partial v(r)}{\partial n}\right\} dS', \tag{5.5}$$

where the integration from the right is spread over the surface encompassing the volume V, and $\partial/\partial n$ denotes the differentiation with respect to the external normal. We shall choose this volume in such a way that the point $\vec{R}$, at which we search for u, lay inside it. We shall further require that this volume be entirely situated in a uniform region of space, in which Eqs. (5.3) and (5.4) would be valid for a constant k.

Then it will be possible to substitute $\nabla^2_{\vec{R}'} u$ and $\nabla^2_{\vec{R}'} v$ by their expressions from (5.3) and (5.4). However, the point $\vec{R}'$, coinciding with the observation point $\vec{R}$, at which we wish to determine $u(\vec{R})$, must be excluded from the integration region beforehand, for here $\underline{v}$ is not continuous and Eq. (5.4) is not valid. To that effect we shall surround the

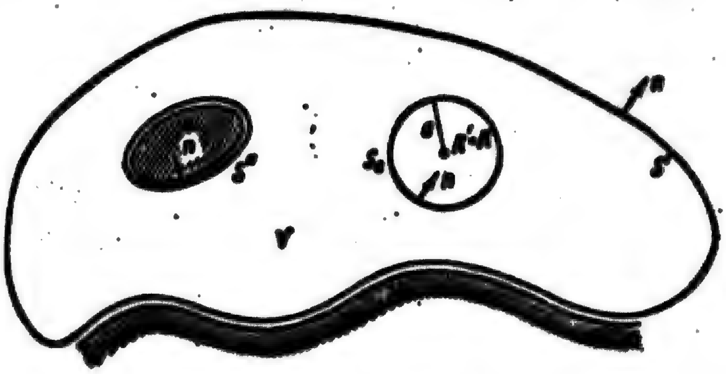

Fig. 5.1. Integration volume.

point $\vec{R}' = \vec{R}$ by a small sphere of radius $\underline{a}$ and surface S_0, so that we may recognize a uniform (by electrical properties) part of space, included between the surfaces S and S', where S' is the external surface. Now the integral over S is decomposed into two parts, respectively related to S_0 and S', with the normal $\underline{n}$ being each time directed outward toward the center of the sphere. Evidently, the surface S' may also consist in its turn of separate parts. If, for example, we wish to take into account that certain parts of space have different electric properties, so that Eqs. (5.3) and (5.4) are invalid in them, we may surround these parts by new surfaces S", S"', ... Then, the volume V will be included between the surfaces S_0, S', S", ..., as in Fig. 5.1.

Let us consider first of all the integral over S_0 and, at the same time, within limits when the radius of the sphere approaches zero. Here the differentiation over $\underline{n}$ is reduced to $-\partial/\partial a$, that is, to $-\partial/\partial r$ (inasmuch as $\underline{a}$ coincides with $r = |\vec{R} - \vec{R}'|$). The functions $u(\vec{R}')$ and

$\partial u(\vec{R}')/\partial r$ are finite within the bounds of the whole sphere, for according to the assumption, Eq. (5.3) is valid everywhere. This means that both these functions may be replaced by their mean values on the sphere and taken out of the integral. To the contrary, v(r) and

$$\frac{\partial v}{\partial n} = -\frac{\partial v}{\partial r} = e^{ika}\left(-\frac{ik}{a} + \frac{1}{a^2}\right) \qquad (5.6)$$

increase unlimitedly with the decrease of a. However, $dS_0 = a^2 d\Omega$ (where $d\Omega$ is the solid angle element) decreases at the same time at least as rapidly, so that there will remain, at limit, a result different from zero only from the term with $1/a^2$. We finally find

$$\lim_{a\to 0}\oint\left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}\right)dS = \lim_{a\to 0}\left\{\left(\frac{\partial u}{\partial n}\right)_{R'\to R}\oint\frac{e^{ika}}{a}a^2 d\Omega -\right.$$

$$\left. - u(R)\oint e^{ika}\left(\frac{-ik}{a} + \frac{1}{a^2}\right)a^2 d\Omega\right\} = -4\pi u(R). \qquad (5.7)$$

This is why, by substituting (5.3), (5.4) into the correlation (5.5), we obtain

$$u(R) = -\frac{1}{4\pi}\int U(R')v(R, R')dV' +$$

$$+\frac{1}{4\pi}\oint\left\{v(R, R')\frac{\partial u(R')}{\partial n} - u(R')\frac{\partial v(R, R')}{\partial n}\right\}dS', \qquad (5.8)$$

where by S we shall understand all the surfaces encompassing the volume V in which the point of observation $\vec{R}$ is situated (in Fig. 5.1 these are the surfaces S', S", S"', ...).

If, in particular, the uniform medium would spread boundlessly, S would be reduced to a single external surface and it should be possible to draw it apart to infinity, considering it, for instance, as a sphere, described by radius r around the observation point. However, it is not seen outright that the influence of such a surface at the observation point would disappear as this takes place in analogous cases of the theory of the potential. In reality, although v would then decrease as $1/r$, the differentiation of $\underline{v}$ with respect to the normal, which would amount here to $\partial/\partial r$ gives for great $\underline{r}$

$$\frac{\partial v}{\partial r} = \frac{e^{ikr}}{r}\left(ik - \frac{1}{r}\right) \approx ik\frac{e^{ikr}}{r}.$$

On the other hand, dS rises as r^2, and this is why the requirement

$$\lim_{r\to\infty} (ur) \to 0. \qquad (5.9)$$

should be superimposed to the rate of $\underline{u}$ decrease.

For real $\underline{k}$, that is, in a nonconducting medium, this requirement cannot be fulfilled, because, as is well known (and will be obtained below from computations), the radiation field decreases only as $1/r$, the field has the character e^{-ikr}/r for the outgoing wave, and e^{ikr}/r for the incoming wave.

The solution, found long ago in optics, consists in the superimposition of a new requirement, designated as the principle or condition of radiation. It is analogous to the rejection of leading potentials. It is precisely required that at infinity, the field $\underline{u}$ pass to the outgoing wave, that is, it behaves as e^{ikr}/r.

However, in reality it is entirely sufficient to limit oneself to the requirement (5.9), as the purely material $\underline{k}$ is an idealization. We may always assume the existence of a certain as low a conduction (complexity of ε) as is desired, owing to which the condition (5.9) will be fulfilled for the outgoing wave. As to the incoming wave, the field at infinity will increase boundlessly, the condition (5.9) will be disrupted and it will have to be rejected.

In the furthest we shall consider this condition as fulfilled everywhere, and therefore we shall reject all the integrals over an infinitely remote surface. In particular, for a boundless uniform medium there remains

$$u(R) = \frac{1}{4\pi}\int_V Uv\,dV'. \qquad (5.10)$$

In this way, the physical sense of the first addend in the correlation (5.8) consists in that it expresses a field that the given sources would have induced at the point $\vec{R}$ of interest to us, had the medium been uniform and boundless. Thus, utilizing the Hertz electric vector, we shall have, according to Eq. (3.12), for any antenna in a boundless medium, the field

$$\Pi(R) = \frac{1}{-i\omega\varepsilon}\int_V \frac{e^{ikr}}{r} j_0(R')\,dV' = \frac{1}{\varepsilon}\int_V \frac{e^{ikr}}{r} P(R')\,dV'$$

$$\left(r = |R - R'|,\ k^2 = \varepsilon\frac{\omega^2}{c^2}\right). \qquad (5.11)$$

Utilizing the Hertz magnetic vector, and in accord with Eq. (3.17), we have for any combination of loop antennas, under the condition that each one of them may only be described by a constant magnetization M_0,

$$\Pi_m(R) = \int_V \frac{e^{ikr}}{r} M_0(R')\,dV'. \qquad (5.12)$$

The application of the very same formula (5.10) to Eqs. (3.9a) and (3.9b) gives (for a uniform medium) an explicit expression for the fields through the charge and the current in the antenna:

$$H = \frac{1}{c}\int \frac{e^{ikr}}{r}\,\mathrm{rot}\,j_0\,dV' = \frac{1}{c}\int \left[j_0,\ \mathrm{grad}\frac{e^{ikr}}{r}\right]dV', \qquad (5.13)$$

$$E = \int\left\{\frac{i\omega}{c^2} j_0 + \frac{1}{-i\omega\varepsilon}\,\mathrm{grad\,div}\,j_0\right\}\frac{e^{ikr}}{r}\,dV' = \int\left\{\frac{i\omega}{c^2} j_0\frac{e^{ikr}}{r} + \frac{\rho_0}{\varepsilon}\,\mathrm{grad}\frac{e^{ikr}}{r}\right\}dV'. \qquad (5.13a)$$

Thus, in case of boundlessly extending uniform medium we reached the complete solution of the problem stated at the end of §2: to find the field by preassigned sources. However, in reality this is insufficient in many cases. Expanding the boundaries of the integration region we inescapably come across, on the one hand, the terrestrial surface, and the other hand, the ionosphere. Here it is not in any way evident, in which case the integral may be neglected. For a finite medium the conditions (5.9) cannot be superimposed, for reflected waves will knowingly be arriving from the boundary line of the media to the observation point. This is why, generally speaking, it will be necessary for us to utilize formula (5.8) in its complete form.

Assembling in vectorial form three formulas for the components of $\vec{\Pi}$, we may write

$$\Pi(R) = \frac{1}{\varepsilon}\int_V \frac{P(R')}{r} e^{ikr}\, dV' + \frac{1}{4\pi}\int_S \left\{ \frac{e^{ikr}}{r} \frac{\partial \Pi(R')}{\partial n} - \frac{\partial \frac{e^{ikr}}{r}}{\partial n} \Pi(R') \right\} dS'. \quad (5.14)$$

In such a formula there is no longer any solution of the problem stated by us, since the function $\vec{\Pi}$, of interest to us, enters also in the right-part under the integral. In essence, the whole characteristic of either concrete problem of radiowave propagation for the given sources is included in this integral. Depending upon the properties and the disposition of boundaries with other properties than the region in which the observation point $\vec{R}$, is situated, that is, those characterized by another value of k, and on field distributions over these surfaces, we shall be obtaining different results.

Note that if in the medium in which we search for the field there are no sources, that is, they are taken out into another medium, the first term to the right is absent and the field is determined exclusively by its values (and the values of its normal derivative) on the surface S, encompassing the point of observation and entirely disposed in the same medium (in particular, in the one passing along the surface bounding this medium).

The vectorial formula (5.14) may be also written for the field strength. Thus, for example, in the absence of charges and currents, each component of $\vec{E}$ satisfies the wave equation, and this is why, assembling three formulas for the components, we have

$$E(R)=\frac{1}{4\pi}\oint_S\left\{v\frac{\partial E(R')}{\partial n}+\frac{\partial v}{\partial n}E(R')\right\}dS'. \qquad (5.14a)$$

However, this formula is not always practical, inasmuch as it contains the derivative of the vector $\vec{E}$ along the direction of the normal, generally varying from point to point. There exists an equivalent formula, which may be obtained by direct integration of vectorial Maxwellian equations, if we take advantage of a method, in principle quite analogous to that utilized when deriving formulas (5.8). Without expounding all the operations (see, for example [16], §8.14), we shall indicate the final result (in the absence of sources!)

$$E(R)=\frac{1}{4\pi}\oint_S\{i\omega\mu\,[H(R')\,n]+[[E(R')\,n]\,\mathrm{grad}\,v]-(nE(R'))\,\mathrm{grad}\,v\}\,dS'; \qquad (5.14b)$$

$$H(R)=\frac{1}{4\pi}\oint_S\{-i\omega\varepsilon\,[E(R')\,n]+[[H(R')\,n]\,\mathrm{grad}\,v]-(nH(R'))\,\mathrm{grad}\,v\}\,dS'. \qquad (5.14c)$$

The integral equation (5.8) may be somewhat generalized for the case of nonuniform medium. Assume that we are confronted with the wave equation (5.3) with k^2 depending on the point

$$\nabla^2_{R'}u+k^2(R')u=U, \quad k^2=\frac{\omega^2}{c^2}\varepsilon(R'). \qquad (5.15)$$

as this takes place in a laminar medium (see §3). We shall utilize a Green function $v(|\vec{R}-\vec{R}'|)$ of the form (5.2)

$$v\sim\frac{e^{iKr}}{r}, \qquad (5.16)$$

where K is a certain constant number, having if only a small positive imaginary part, that is, v satisfies the radiation condition and, besides, also the wave equation

$$\nabla^2 v+K^2 v=0. \qquad (5.17)$$

Thus, here, by repeating the reasoning for uniform space, we obtain

$$u(R)=\frac{1}{4\pi}\int_V U(R')\,v(R,R')\,dV'+\frac{1}{4\pi}\oint_S\left\{v(R,R')\frac{\partial u(R')}{\partial n}-u(R')\times\right.$$
$$\left.\times\frac{\partial v(R,R')}{\partial n}\right\}dS+\frac{1}{4\pi}\int_V u(R')\,(\nabla^2_{R'}v(R,R')+k^2(R')\,v(R,R'))\,dV'. \qquad (5.18)$$

According to Eq. (5.17), the last integral in the Eq. (5.18) is

$$\frac{1}{4\pi}\int_V (k^2(R') - K^2)u(R')v(R, R')dV'. \qquad (5.18a)$$

This integral constitutes the measure of influence of space nonuniformity. If k^2 = const, we may choose $\underline{v}$ with $K = k$, and the integral will vanish. But in the general case, the question evolves about the integral equation for $u(\vec{R})$. It may be utilized for various aims, if only for estimating the error arising when the medium's nonuniformity is neglected, or when a nonuniform medium is substituted by a certain uniform medium with modified parameters, etc.

§6. Point Source

In the furthese we shall essentially base ourselves upon Formula (5.14). But, for the moment, we shall somewhat simplify it suitably to one important case. Assume that the sources of the field are concentrated in a very small volume, so small that the multiplier e^{ikr}/r in the volume integral may be taken out of the integral sign. To that effect it is obviously necessary, on the one hand, that for the given $\vec{R}$, the exponent $ikr = ik|\vec{R} - \vec{R}'|$ vary little at the change of $\vec{R}$ near

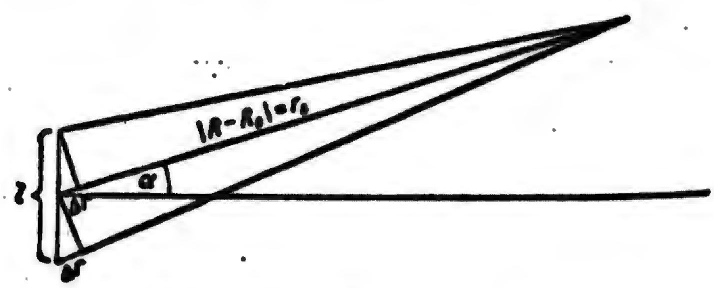

Fig. 6.1. The dipole.

its certain mean value $\vec{R}_0$, i.e., that the phases of the waves arriving at the observation point from various parts of the source, differ little among themselves. As to the factor $1/r$, it will be, on the other hand, sufficiently constant and close to $1/|\vec{R} - \vec{R}_0|$, if the possible values of $|\vec{R}' - \vec{R}_C|$, that is, if the dimensions of the source are small by comparison with r.

The second requirement means that we shall be considering only the observations points $\vec{R}$ far off the source, by a distance which is great by comparison with its dimensions. The first requirement will be satisfied if the dimensions of the source are small by comparison with the magnitude of the order of the wavelength in the given medium. In reality, if, for example, the distance between the source and the linear antenna is l, we may see that (Fig. 6.1) by selecting for the median point $\vec{R}_0$ its center, r varies on both sides and near $r_0 = |\vec{R} - \vec{R}_0|$ by the quantity $\Delta r \approx l/2 \sin \alpha$.

Consequently, ikr varies by a quantity lesser than $ikl(\sin\alpha)/(2) = i\pi(l)/(\lambda)\sin\alpha$. If the latter is quite small, we shall have $e^{ikr} \approx e^{ikr_0}$. This is why it is necessary that we have

$$l \ll \frac{\lambda}{\pi \sin\alpha}. \qquad (6.1)$$

This means that it is sufficient to require for all angles $l \ll \lambda/\pi$, that is, the smallness of linear dimensions by comparison with the wavelength. But for the observation points located near the plane perpendicular to the antenna and passing through its center ($\alpha = 0$), this requirement is sufficiently easing off, and even a large antenna may be considered as a point antenna. It goes without saying that the correlation (6.1) is here no longer valid, for the difference $r - r_0$ is equal to $l/2 \sin\alpha$ only with a precision to the terms of second order in l. For points near $\alpha = 0$ we shall obtain $r^2 \sim r_0^2 + l^2$, so that, inasmuch as $r_0 \gg l$,

$$k(r - r_0) \sim k(\sqrt{r_0^2 + l^2} - r_0) \approx \frac{kl^2}{2r_0}.$$

Consequently, in order that the various points of the antenna emit waves nearly in a single phase, it must be required that

$$r \ll \frac{r_0 \lambda}{\pi}, \quad l < \sqrt{\frac{r_0 \lambda}{\pi}}. \tag{6.2}$$

This characteristic correlation will be encountered more than once. Obviously, for sufficiently great r_0 even very large antennas may be treated as small ones for the observation points near the plane $\alpha = 0$. Let us stress that this is correct only for a rectilinear antenna. Generally speaking, however, we must have

$$l \ll \frac{\lambda}{\pi}. \tag{6.3}$$

If these conditions are observed, we may take out e^{ikr}/r from the integral sign, and having denoted

$$\int P(R') dV = p, \tag{6.4}$$

where $\vec{p}$ is the total dipole moment of the system, we shall obtain for a boundless medium at the distance r_0 from the source with a dipole moment $\vec{p}$

$$\Pi = \frac{p}{\varepsilon} \frac{e^{ikr_0}}{r_0}. \tag{6.5}$$

The source, which is fully characterized by a single vector $\vec{p}$, placed at an accordingly selected point, is called a point electrid dipole.

If the vertical linear antenna is such a source, we may perform the integration over the cross section S of the conducting wire by substituting the polarization $\vec{P}$ by the current according to Formula (3.7); this will give instead of the density of the current $\vec{J}_0$ the total current $\vec{I}(h)$ flowing through the given cross section, where h is counted along the axis of the wire. Postulating

$$dV' = dS\, dh,$$

we have

$$p = \int \frac{1}{-i\omega} J_0 \, dV' = \frac{1}{-i\omega} \int I(h)\, dh. \tag{6.4a}$$

If the length of the dipole is small by comparison with the length of the wave $2\pi/k$, the quantity I is constant along the wire, as this is well known from the theory of quasistationary currents, and it can be taken out of the integral sign:

$$p = \frac{Ih}{\omega} i_0 \tag{6.4b}$$

where h is the length of the antenna. In the opposite case we may introduce the maximum value of the current I_{maks}, and denote the remaining integral by

$$h_{эфф} = \int \frac{I(h)}{I_{макс}} dh. \tag{6.4c}$$

This quantity is designated as the reduced or acting height. It is always smaller than h. Usually, the subscripts "maks" and "eff" are rejected, implying under I and h the indicated quantities.

In the more general case the acting height of the antenna is determined by the formula (see Fig. 6.1)

$$h_{эфф} = \frac{1}{I_{макс}} \int I(h) e^{ik|R - R_0|} dh \approx \frac{1}{I_{макс}} \int I(h) e^{ikh \sin\alpha} dh, \tag{6.4d}$$

or by the real part of this expression.

For medium and long waves and in numerous cases for short waves too, the real antennas may be considered as point ones, particularly if the observation point is located near the plane $\alpha = 0$ and the antenna is rectilinear, so that the criterion for judging it is the formula (6.2) and not the formula (6.3). This last requirement must be fulfilled if the antenna consists of several rectilinear portions, arbitrarily disposed in space. In particular, it is always fulfilled if we consider the field of the frame and apply the general correlation (5.8) to the Hertz magnetic vector. Indeed, the very substitution of the contour by the magnetic moment is possible only when the dimensions of the frame l are small by comparison with λ (see §3, page 2). This is why the various portions of the surface of the magnetic shell will be given at the observation point by nearly inphase oscillations, and taking e^{ikr}/r out of the integral sign, we may designate

$$\int M_0 dV' = m, \tag{6.6}$$

where $\vec{m}$ is the total magnetic moment of the frame, so that

$$\Pi_m = m\frac{e^{ikr_0}}{r_0}. \tag{6.7}$$

Such a source is designated as a point magnetic dipole.

We shall consider in the following only point sources. More complex cases may be always obtained by superimposition of the elementary ones.

Thus, for example, if we have to electric dipoles of equal power and opposite sign, shifted relative to one another by a small vector $\vec{l}$ (Fig. 6.2), the aggregate Hertz vector is

$$\Pi = \frac{p}{\varepsilon}\left(\frac{e^{ik\left|r_0-\frac{l}{2}\right|}}{\left|r_0-\frac{l}{2}\right|} - \frac{e^{ik\left|r_0+\frac{l}{2}\right|}}{\left|r_0+\frac{l}{2}\right|}\right) \approx -\frac{p}{\varepsilon}l\frac{\partial}{\partial r_0}\frac{e^{ikr_0}}{r_0}\cos(lr_0), \tag{6.8}$$

where $\vec{r}_0$ is the distance from the median point of the vector $\vec{l}$ to the point of observation.

Such a system is designated as the electric quadrupole. If vector $\vec{l}$ is directed along the axis $\vec{p}$, for the axis $\underline{z}$, we may see that the Hertz vector has a component

$$\Pi_z = -\frac{pl}{\varepsilon}\frac{\partial}{\partial z_Q}\frac{e^{ikr_0}}{r_0}, \tag{6.8a}$$

different from zero, where the index "Q" at $\underline{z}$ shows that the differentiation takes place along the "point of outflow," that is, along the coordinate of the origin $\vec{r}_0$.

The quantity

$$q = 2pl \tag{6.8b}$$

is designated as the quadrupole moment. If l is directed, for example, along the axis $\underline{x}$ (and $\vec{p}$ along the axis $\underline{z}$, as previously), we have

$$\Pi_z = -\frac{pl}{\varepsilon}\frac{\partial}{\partial x_Q}\frac{e^{ikr_0}}{r_0}. \tag{6.9}$$

Fig. 6.2. Quadrupole.

§7. The Huygens Principle

The basic integral equation (5.14) allows a graphic interpretation.

The first (volume) integral has an obvious sense: each volume element dV' with total dipole moment $d\vec{p} = \vec{P}dV'$ sends to the observation point the same spherical wave as if we had a point dipole in infinite uniform space.

Superimposed on that field are fields emitted from the surface limiting the volume. At the same time, each element of the surface dS' has a dipole moment $\partial\vec{\Pi}/\partial n\, dS'$ and a quadrupole moment formed by the combination of dipole moments $\vec{\Pi}(dS')/(\ell)$, equal in magnitude and of opposite signs, shifted relative to one another along the normal to the segment ℓ, equalling zero at limit.

If, for example, we consider a field above ground, induced by a source situated in the atmosphere, it is possible to trace the surface S in such a way that it encompasses both the source O and the observation point A (Fig. 7.1), consisting at the same time of a plane adjacent to the ground surface (which, for the moment, we shall consider as plane), and of an infinite hemisphere. At limit, the integral over the surface of the hemisphere vanishes (see §5), whereas the integral over the ground surface remains. It precisely describes the influence of the ground on the field in space.

According to the above interpretation we may consider that sources in the form of dipoles and quadrupoles are induced on the terrestrial surface. These secondary sources induce a field which is superimposed to the field of primary source. Since in the atmosphere $\varepsilon = 1$, $k = k_0 = \omega/c$, we have

$$\Pi(A) = \frac{1}{4\pi}\int_V P\frac{e^{ikr}}{r}dV + \frac{1}{4\pi}\oint_S\left\{\frac{\partial\Pi}{\partial n}\frac{e^{ikr}}{r} - \Pi\frac{\partial}{\partial n}\frac{e^{ikr}}{r}\right\}dS. \quad (7.1)$$

But if we are interested in the field underground, we may construct a surface S' consisting of a plane ground surface and an infinite hemisphere, closing the volume from the other side.* In the given

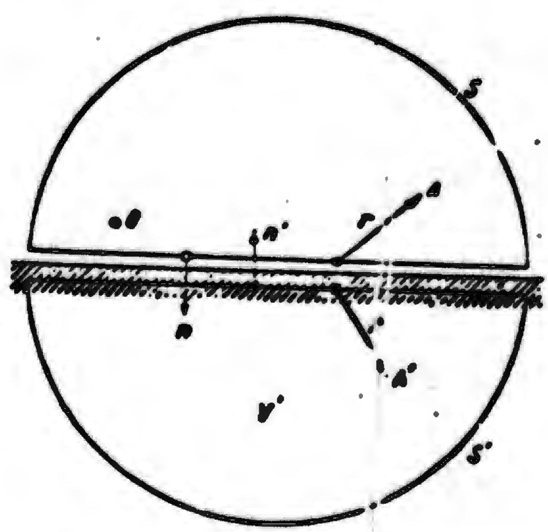

Fig. 7.1. To the formulation of the Huygens principle in the presence of ground.

case, there are no sources in the volume encompassed by the surface S', the volume integral is absent and the field is

$$\Pi(A') = \frac{1}{4\pi}\oint_{S'}\left\{\frac{\partial\Pi}{\partial n'}\frac{e^{ik'r}}{r} - \Pi\frac{\partial}{\partial n'}\frac{e^{ik'r}}{r}\right\}dS', \quad (7.2)$$

where n' is the external normal relative to the surface S', that is, on the ground surface it is directed upward; k' is the wave number in the soil: $k' = \sqrt{\varepsilon}k_0$; $\underline{r}$ is the distance from the current point of the surface to the observation point A'.

Here the field is induced exclusively by secondary dipoles and quadrupoles, themselves induced on the ground surface.

It is obvious that in reality the currents are excited not only on the ground, but also along the entire thickness of the skin-layer, so that in reality, the radiation is emitted to the point A' by all points

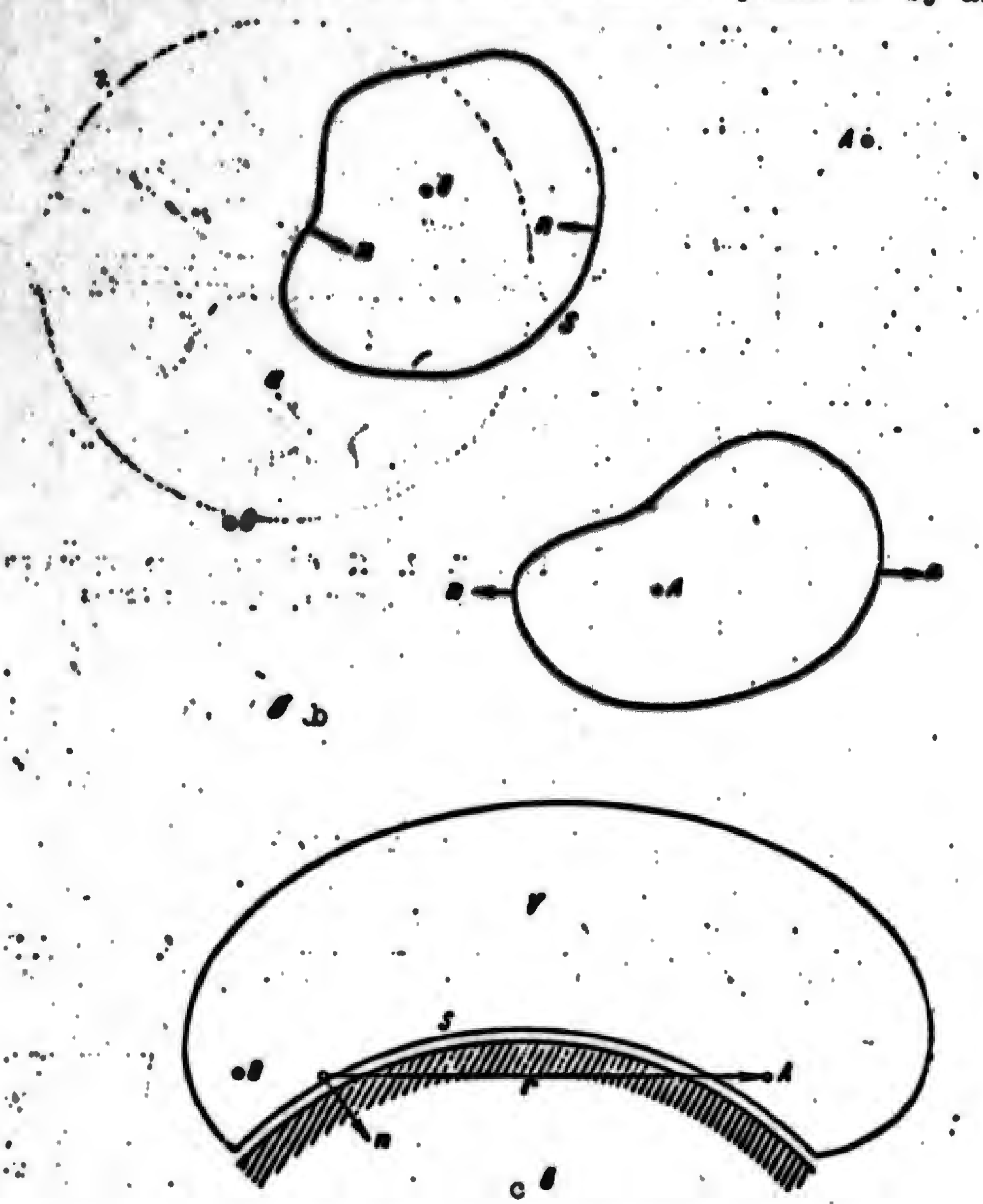

Fig. 7.2. Surface of virtual sources: a) surrounds the emitter; b) surrounds the observation point; c) in the case of spherical earth.

of the volume V'. The interference of these waves with the emission by secondary surface dipoles and quadrupoles alters the velocity of radiowaves at their propagation in the ground. This circumstance is already manifest in that k' differs from k_0. Thus we may indeed estimate the field in A' as being created by the secondary sources distributed over the terrestrial surface.

However, formula (5.14) permits a still broader interpretation. Let us figure out the surface S as lying entirely in the atmosphere devoid of conduction and, at the same time, in such a fashion that the source O and the point of observation A be on different sides of it (Fig. 7.2a and 7.2b). In that case the field in A is

$$\Pi(A)=\frac{1}{4\pi}\int_S\left\{\frac{\partial\Pi}{\partial n}\frac{e^{ik_0r}}{r}-\Pi\frac{\partial}{\partial n}\frac{e^{ik_0r}}{r}\right\}dS. \qquad (7.3)$$

Since, generally speaking, the field $\Pi(A)$ is not zero at all, this means that such a surface should be visualized as filled by secondary emitters in vacuum too.

The surface S may be deformed arbitrarily, so that it pass through any point of the field. This is why every point of the field is in itself a source of secondary spherical waves, as this should be the case according to the Huygens principle.

It should be stressed that the integration must be performed over the entire surface S, for example, in the case represented in Fig. 7.2a, and on its side "invisible" from the point A.

Note also that between the current point in the integral over the surface and the observation point, another medium may wedge in, through which electromagnetic waves do not really pass, for example, in the case of spherical earth, as this is shown in Fig. 7.2c. Nevertheless, here too r is counted along the line linking the points of the surface S with the observation point A (at the same time the obviousness of the physical interpretation is lost). Subsequently, however, we shall see (Chapter 2) that not all the points of the surface S are by any means essential for field formation at the point A. If the dimensions and the conduction of the "wedged in" body are sufficiently great, so that only a feebly diffracted field may reach the point of observation, the integration over the part of the surface invisible from A will provide an insignificant contribution.

§8. The Green Function

Equation (5.8) allows to compute the field at any point of the volume V, provided the sources located in it are known, and also the values of the function u sought for and of its derivative $\partial u/\partial n$ on the surface (or on surfaces) S encompassing the volume V. The deduction of this formula was found to be possible because, when applying the Green theorem, the auxiliary function v was chosen in a special fashion, namely as being such a function of the distance r between two points x, y, z and x', y', z', which a) satisfies the wave equation $\nabla^2 v + k^2 v = 0$; b) satisfies the radiation condition and c) when r becomes zero, it approaches the infinity in an entirely special fashion, that is, as $1/r$. Had v approached the infinity more rapidly than $1/r$ and, this means its derivative, more rapidly than $1/r^2$, the integration over the small sphere S_0, surrounding the point of observation, would have given the infinity; if v approached the infinity slower than $1/r$, the integration would have given zero. Only because $\lim\limits_{r\to 0} (r^2 (\partial v)/(\partial r))$ is a finite number, different from zero, could we obtain at limit the term $4\pi u(\overline{R})$ by taking out of the integral sign the mean value of $u(\overline{R'})$ on the sphere S_0.

Obviously, the enumerated conditions are not only satisfied by the function $v = v_0 = (1)/(r)e^{ihr}$, but also, for example, any other function satisfying the wave equation and differing from v_0 by a multiplier remaining finite at $r \to 0$. True, if this multiplier should increase to infinity, we would not be able to draw apart boundlessly the surface S' that closes the volume. At the indicated limitations there still remains an ample choice allowing the simplification of the solution of numerous problems by a successful choice of <u>v</u>, and we shall take advantage of it in the following (see also, for example, Chapter 7).

Consequently, we may take for the function <u>v</u> a function of the form

$$v = v_0 + \varphi(x, y, z, x', y', z') \tag{8.1}$$

under the condition that φ satisfy the wave equation $(\nabla^2 + k^2)\varphi = 0$) and, furthermore, remain finite everywhere inside the volume V.* When integrating over the sphere S_0, the addend φ will give zero at limit, when $a \to 0$. Instead of (5.8), we shall obtain

$$u(R) = -\frac{1}{4\pi}\int_V U(R')(v_0 + \varphi)\,dV' + \frac{1}{4\pi}\int_S \left\{(v_0 + \varphi)\frac{\partial u}{\partial n} - u\frac{\partial(v_0 + \varphi)}{\partial n}\right\} dS'. \tag{8.2}$$

If we succeeded in so choosing a complementary function φ that S convert to zero everywhere on the surface S, for example, the sum $v_0 +$ $+ \varphi$, the first addend in the surface integral would disappear and for the determination of u(R) at any point we would have to know only the sources U in the volume and the field <u>u</u> on the surface, but the knowledge of $\partial u/\partial n$ would not be required. But if we assorted φ in such a way that $\partial/\partial n(v_0 + \varphi)$ vanish on the surface, the second term would drop off and only $\partial u/\partial n$ would have to be known on the surface. In the first case $v = v_0 + \varphi$ is sometimes designated as the Green function, in the second case as the second Green function or the characteristic Neumann function.

It is essential that such an assortment of the function φ should be performed for the given form of a surface S and for the given medium once. Knowing φ, it is possible to find $\underline{u}$ for any dispositions of the source and for any field $u(\vec{R})$ assignments (or, correspondingly, of $\partial u/\partial n$ on the surface). This is why it is customary to speak of the Green function for the space inside the sphere, of the Green function for the half-space, etc. Thus, the Green function may serve as the characteristic of the surface S. This shows, on the other hand, that in the general case too the field is determined unambiguously if, besides sources and limiting surfaces, we either assign on the surface $\underline{u}$ or $\partial u/\partial n$, but not both these quantities simultaneously, which could appear at the outset as compulsory. If we proceed by way of integral correlation (5.8), only our incapability to select a Green function for any configuration complets us to assign, besides $\underline{u}$, also $\partial u/\partial u$ or the relationship between them. It finally follows therefrom that two functions $\underline{u}$ and $\partial u/\partial n$ cannot be assigned on the surface independently from one another. We shall find the Green function for S of the following form. Assume that S consists of a plane $z = 0$ and of a hemisphere of infinitely great radius, closing the half-space $z > 0$.

Selecting φ, we may be guided by the following considerations. Since the function φ must satisfy the wave equation, it can be interpreted as the field of some fictive sources in a medium with properties of the one filling our volume V (half-space $z > 0$). On the other hand, the field of φ must not have sources within the limits of V, inasmuch as the equation $(\nabla^2 + k^2)\,\varphi = 0$ must be valid at all points of the volume.

Consequently, φ must be, for example, the field of imaginary sources placed by us outside the volume V. Finally, these imaginary sources should be so assorted that they either quench the field $v_0 = e^{ikr}/r$ on the surface S or vanish on it alongside with v_0.

For the case of half-space such sources are not difficult to select. If we search for the field at a point $A(\overline{R})$, it is sufficient to have a representation of a point source at a point $A_1(\overline{R}_1)$ being the mirror image of A relative to the plane z = 0, and at the same time, such a source that its field in the plane z = 0 be exactly equal to $-e^{ikr}/r$ where $\underline{r}$ is the distance to the given point of the plane z = 0 from the point of observation. Thus we assume (Fig. 8.1):

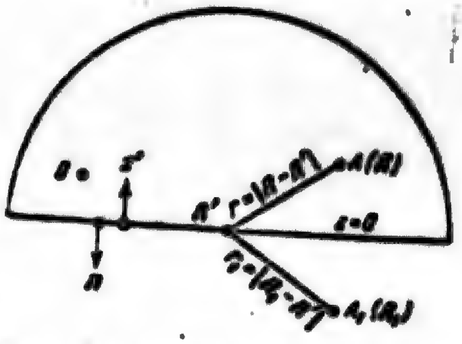

Fig. 8.1. Deriving the Green function for the plane.

$$\varphi = -\frac{e^{ik|R_1-R'|}}{|R_1-R'|}, \quad R = R(x, y, z), \quad R_1 = R_1(x, y, -z),$$

$$v_0 + \varphi = v_- = \frac{e^{ik|R-R'|}}{|R-R'|} - \frac{e^{ik|R_1-R'|}}{|R_1-R'|} = \frac{e^{ikr}}{r} - \frac{e^{ikr_1}}{r_1},$$

$$r = \sqrt{(x-x')^2+(y-y')^2+(z-z')^2},$$

$$r_1 = \sqrt{(x-x')^2+(y-y')^2+(z+z')^2}. \qquad (8.3)$$

Indeed, when $\overline{R}'$ denotes one of the points of the surface z = 0, v_- becomes zero. As to the surface of the hemisphere, both addends satisfy on it the radiation condition and thus convert the integral to zero.

Finally, as may already be seen from the considerations of symmetry, with R' denoting any point of the surface z = 0, we have

$$\frac{\partial}{\partial n}\frac{e^{ikr_1}}{r_1} = -\frac{\partial}{\partial n}\frac{e^{ikr}}{r}. \qquad (8.4)$$

In reality, if by $\underline{n}$ we understand $-z'$, the correlation

$$\left(\frac{\partial r}{\partial z'}\right)_{z'=0} = -\left(\frac{\partial r_1}{\partial z'}\right)_{z'=0}$$

is valid, inasmuch as z' enters in $\underline{r}$ and r' with different signs. On the other hand $r|_{z'=0} = r_1|_{z'=0}$. Hence follows also Formula (8.4).

Therefore, the field of the arbitrary sources located in the upper half-space is*

$$u(\vec{R}) = -\frac{1}{4\pi}\int_V U(R')v_-(R,R')dV' - \frac{1}{2\pi}\int_{(z=0)} u(R')\frac{\partial}{\partial n}\frac{e^{ik|R-R'|}}{|R-R'|}dS. \qquad (8.5)$$

It is now sufficient to know (besides sources) the field in the plane z = 0, in order to determine the field universally.

On the other hand we may postulate

$$v_0 + \varphi = v_+ = \frac{e^{ik|R-R'|}}{|R-R'|} + \frac{e^{ik|R_1-R'|}}{|R_1-R'|} = \frac{e^{ikr}}{r} + \frac{e^{ikr_1}}{r_1}. \qquad (8.6)$$

In the given case, when $\vec{R}'$ is the radius-vector of one of the points of the surface z = 0, $\partial v_+/\partial n$ vanishes on account of Formula (8.4). This is why the integral containing the unknown function $u(\vec{R}')$ on the surface S

$$u(\vec{R}) = -\frac{1}{4\pi}\int U(R')v_+(R,R')dV' + \frac{1}{2\pi}\int_{(z=0)} \frac{\partial u(R')}{\partial n}\frac{e^{ik|R-R'|}}{|R-R'|}dS. \qquad (8.7)$$

will disappear also.

If in particular the point of observation is situated in the plane z = 0 itself (for example, if the question is about a plane separating the earth from the atmosphere, on the terrestrial surface), we have

$$r = r_1, \quad v_+ = 2\frac{e^{ikr}}{r} = 2v_0,$$

and this is why

$$u(\vec{R})\big|_{z=0} = -\frac{1}{2\pi}\int U(R')v_0(R,R')dV + \frac{1}{2\pi}\int \frac{\partial u}{\partial n}\frac{e^{ik|R-R'|}}{|R-R'|}dS.$$

Let us compare this expression with Formula (5.8). We recall that the first integral in Formula (5.8) expresses a field, which the given sources U would have induced at the given point $\vec{R}$, had the entire space been uniform and characterized by the propagation constant $\underline{k}$ (that is, if the source S could be moved to infinity). Denoting this "unperturbed" field by $u_0(\vec{R})$, we shall obtain in our case

$$u(R)_{z=0} = 2u_0(R) + \frac{1}{2\pi}\int\limits_{(z=0)} \frac{\partial u(R')}{\partial n}\frac{e^{ikr}}{r}\,dS, \qquad (8.7a)$$

where the integral is spread over the plane z = 0. In the following this equation will be broadly utilized by us.

The functions v_+ and v_- are called Green function for the half-space.

However, the possibility of selection of special Green functions, facilitating the solution of particular problems, is by no means exhausted by these exmples. As is clear from the foregoing, any solution of the wave equation satisfying either boundary conditions on the surface encompassing a region of space of interest to us, and having a peculiarity of the 1/r type characteristic for really materialized point source fields, may serve as a Green function. Thus, any function describing a field, really possible in the presence of a point source in a volume surrounded by a surface with either physical properties, may serve as a Green function. For example, v_0 constitutes a point source field in boundless vacuum. As will be seen in §20, v_+ describes a vertically polarized electric field of a point source in half-space above an ideally conducting plane, and v_- is a horizontally polarized field in tre same space.

It will be shown in §41 that a wide range of problems for a field above an electrically inhomogeneous surface may be investigated provided we choose for the Green function a function describing a field in a plane (and also spherical) surface of finite conduction (these functions are derived in Chapters 5 and 6).

§9. Reciprocity Theorem

We shall complete the consideration of general correlations by the exposition of the reciprocity theorem. Its quite broad applicability makes this theorem quite useful in a series of practical cases. Examples of its utilization will be encountered more than once in the following.

Assume that we have an arbitrary volume filled with any combination of media, of which we shall require only that the Maxwellian equations, with values of ε' and σ independent from the field, be valid at any of their points. We shall consider the field $\mathbf{E}^{(1)}$, $\mathbf{H}^{(1)}$, induced by some system of alien forces distributed in the antenna and harmonic in time. In a particular case the electromotive force may be concentrated in a point dipole of moment $p^{(1)}$, placed at a certain point A_1.

Assume, on the other hand, that by switching off this electromotive force we may apply a certain other alien field of same frequency, for example, concentrated in the form of a dipole of moment $p^{(2)}$, at a certain point A_2 (it is essential that the switching on and off on these electromotive forces not be attended by any rearrangement of bodies, for example, antennas). We shall obtain a certain new distribution of the fields $\mathbf{E}^{(2)}$ (x, y, z) and $\mathbf{H}^{(2)}$(x, y, z). It is found that the fields $\mathbf{E}^{(1)}$ and $\mathbf{E}^{(2)}$ may be linked with outside electromotive forces (in a particular case, with the moments $p^{(1)}$ and $p^{(2)}$) by a certain useful correlation derived below. This correlation precisely includes the reciprocity theorem. Since we wish the final result to be valid for variables in space ε' and σ, we cannot, generally speaking, start from wave equations, but we must revert to field equations, for example, in the form (1.I), (1.II) (assuming at the same time $\frac{1}{c}\frac{\partial}{\partial t} = -ik_0$):

$$
\begin{aligned}
&\mathrm{rot}\,H^{(1)} = -i\varepsilon k_0 E^{(1)} + \frac{4\pi\sigma}{c} E^{(1)\,\text{стор}}, && +E^{(2)}\\
&\mathrm{rot}\,E^{(1)} = ik_0 H^{(1)}, && +H^{(2)}\\
&\mathrm{rot}\,H^{(2)} = -i\varepsilon k_0 E^{(2)} + \frac{4\pi\sigma}{c} E^{(2)\,\text{стор}}, && -E^{(1)}\\
&\mathrm{rot}\,E^{(2)} = ik_0 H^{(2)}. && -H^{(1)}
\end{aligned}
\qquad (9.1)
$$

Here we assumed $\vec{J}^{(1)stor}$, where $\vec{E}^{(1)stor}$ is the intensity of the field of outside forces, of which the integral provides the outside electromotive force, which is valid so long as the antenna conduction (that is, the conduction of the region where the outside forces are applied) is great by comparison with that of the surrounding medium.

Multiplying scalarly these four equations by the vectors written out in the right-hand column and adding, we shall obtain by way of formula $\vec{A}$ rot $\vec{B}$ – $\vec{B}$ rot $\vec{A}$ = div$[\vec{B}\vec{A}]$

$$
\mathrm{div}\,[H^{(1)}E^{(2)}] - \mathrm{div}\,[H^{(2)}E^{(1)}] = \frac{4\pi\sigma}{c}(E^{(1)\,\text{стор}}E^{(2)} - E^{(2)\,\text{стор}}E^{(1)}). \qquad (9.2)
$$

Let the outside forces be distributed at the outset about the volume in a continuous fashion, so that the fields $\vec{E}$ and $\vec{H}$ nowhere convert to infinity. The obtained equality will then be valid at all points, including also the antenna volume, where $\vec{E}^{(1)stor}$ and $\vec{E}^{(2)stor}$ are not zero.

Let us integrate Eq. (9.2) over the entire space. We shall show that the integral from the left-hand part vanishes. To that effect we shall transform to the integral from $[\vec{H}^{(1)}\vec{E}^{(2)}]_n$ and $[\vec{H}^{(2)}\vec{E}^{(1)}]_n$ over the surface consisting of: a) the surfaces Σ', Σ" outlining the disruption surface of electric constants S', S", ... (boundaries of medium divide), and b) the infinitely remote outer sphere S_0 (Fig. 9.1). The integral over Σ', Σ", ... vanishes, for though at transition through S', S" the field vectors do undergo a break, the normal components of $[\vec{E}^{(i)}\vec{H}^{(k)}]_n$ of vectorial products are continuous, inasmuch as they contain only the tangential components of field vectors remaining continuous, inasmuch as they contain only the tangential components of field vectors remaining continuous. As to the integral over the infinitely remote sphere S_0, it vanishes, as usually, when all the sources are located at a finite mutual distance. Therefore, integrating the correlation (9.2) over the volume, we shall obtain

$$\int E^{(1)\,\text{стор}} E^{(2)} \sigma dV = \int E^{(2)\,\text{стор}} E^{(1)} \sigma dV. \qquad (9.3)$$

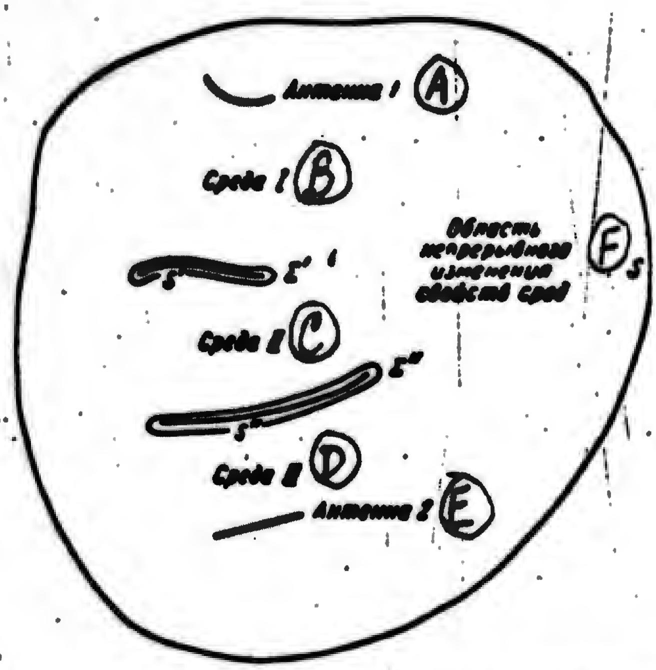

Fig. 9.1. Proof of the reciprocity theorem. A) Antenna 1; B) medium 1; C) medium II; D) medium III; E) antenna 2; F) region of continuous variation of medium properties.

Here the integrals are taken over the entire space. In fact, however, they are distributed only in the region of action of alien electromotive forces: the left-hand integral over the "first" source, the right-hand one, over the "second." This is why it is convenient to rewrite the equality (9.3) as follows:

$$\int_{(1)} \sigma(1) E^{(1)\,\text{стор}}(1) E^{(2)\,\text{стор}}(1) dV_1 = \int_{(2)} \sigma(2) E^{(2)\,\text{стор}}(2) E^{(1)\,\text{стор}}(2) dV_2.$$

Here $\sigma(1)$ and $\sigma(2)$ are the conductions in the region of action of respectively the first and the second outside electromotive forces; $\vec{E}^{(2)}(1)$ is the field in the volume of the "first" source induced at switching on the outside forces of $E^{(2)\,\text{стор}}$ in the "second" source; $\vec{E}^{(1)}(2)$ is the field in the "second" source generated by the electromotive force switched on to the "first" source.

This equality constitutes precisely the reciprocity theorem's content in the most general form. In practical cases it may be substantially simplified and concretized.*

The distribution of the current, just as the distribution of $\vec{E}^{stor}$ by the antenna cross section, may obviously be nonuniform, since, for example, because of skin-effect, the external field of the first source does not always penetrate into the second conductor.

If, however, it were possible to estimate that $\vec{E}^{stor}$ is uniformly distributed along the cross section in both antennas, we might, after having written the volume element dV_1 of the conductor in the form

$$dV_1 = dh_1 dS_1, \tag{9.4}$$

where dh_1 is an element of conductor's length, dS_1 is the cross section element, we may, for any distribution $j^{(2)}(1) = \sigma(1)E^{(2)}(1)$, integrate over the cross section:

$$\int \sigma E^{(2)} E^{(1)\,\text{стор}} dV_1 = \int E^{(1)\,\text{стор}} dh_1 \int j^{(2)} dS_1 = \int E^{(1)\,\text{стор}}(h_1) I^{(2)}(h_1) dh_1, \tag{9.5}$$

where

$$I^{(2)}(h_1) = \int j^{(2)}(1) dS_1 \tag{9.6}$$

is the total current force in the first conductor, appearing in its cross section at the height h_1, when the electromotive force is switched on in the second conductor. Thus, we obtain for the reciprocity theorem the following form:

$$\int_{(1)} I^{(2)}(h_1) E^{(1)\,\text{стор}}(h_1) dh_1 = \int_{(2)} I^{(1)}(h_2) E^{(2)\,\text{стор}}(h_2) dh_2, \tag{9.7}$$

where it is implied that the vectors are projected over the direction of the conductor's axis.

If finally the electromotive force is applied over a very small area of the conductor, $I^{(2)}(h_1)$ and $I^{(1)}(h_2)$ may be taken out of the integral sign, while the quantity

$$\int_{(i)} E^{(i)\,\text{стор}}\, dh_i = \mathcal{E}^{(i)} \qquad (9.8)$$

may be denoted as an outside electromotive force. Then

$$I^{(2)}(h_2^0)\,\mathcal{E}^{(1)} = I^{(1)}(h_1^0)\,\mathcal{E}^{(2)}, \qquad (9.9)$$

where h^0_1 and h^0_2 are the points of application of outside electromotive forces in the respective conductors.

Therefore, the formulations (9.7, (9.9) are obtained only at specific assumptions on the electromotive forces fed and the distribution of currents. However, in the reality these assumptions may no longer correspond to reality if only on account of skin-effects. Nevertheless, we may interpret the formula (903a), derived for the general case, if only we limit ourselves to the consideration of electromotive forces of, simplest of all, point electric dipoles. Then, on the other hand, we shall benefit by the generality of applications, for, in the final result, we shall have to deal with fields induced by antenna emission in space, rather than with fields, induced by one antenna in the material of another one.

For deriving the reciprocity theorem in this case, we might, as is usually done, start as previously from Eq. (9.2) but, integrating over the volume of the left-hand part, it would be necessary to outline the points A_1 and A_2 by small spheres, and then have the radii of the sphere approach zero. At the same time, we ought to substitute in the integral over small spheres the expresion for the point dipole field, surrounded by a medium with certain ε_1 or ε_2, characteristic of the media surrounding the points A_1 and A_2 (see (6.5)). But we may also act differently, namely, by effecting the ultimate transition from the case of distributed sources.

We know that if the conduction of antennas is sufficiently great by comparison with that of the surrounding medium, Eqs. (2.4) take place; they are different from Eqs. (9.1) only in that it must be understood everywhere by ε the complex dielectric constant of the medium at the given point, and that the total current $\vec{j}_0$ in the given antenna must be substituted instead of $\sigma\vec{E}^{stor}$.

In such a case Formula (9.3) takes the form

$$\int\limits_{(1)} j_0^{(1)}E^{(2)}(1)\,dV_1 = \int\limits_{(2)} j_0^{(2)}E^{(1)}(2)\,dV_2. \qquad (9.10)$$

Here $\vec{E}^{(i)}(k)$ is the field strength at the location of the k<u>th</u> antenna, when the outside electromotive force is switched on only at the place of location of the i<u>th</u> antenna, however computed in the assumption that the properties of the material in the volume of both antennas coincide with the properties of the surrounding media (see §2). It may therefore be said that, for example, $\vec{E}^{(2)}(1)$ is the field at the place of location of the first antenna which would have been established, had the second antenna and the current $\vec{j}_0^{(2)}$ been incuced at the place of disposition of the second antenna.*

For the indicated sense of the quantity $\vec{E}^{(i)}(k)$ it is clear that having to do with a point source, it is possible to take it out of the integral sign. On the other hand, the remaining integral is expressed, according to Formula (6.4a), by a dipole moment of the source. Abbreviating by i/ω both sides of the equations, we finally obtain

$$p^{(1)}E^{(2)}(A_1) = p^{(2)}E^{(1)}(A_2). \qquad (9.11)$$

For two dipoles with identical effective height this correlation may be rewritten as follows:

$$I^{(1)}E^{(2)}(A_1) = I^{(2)}E^{(1)}(A_2), \qquad (9.12)$$

with $I^{(i)}$ being the current amplitudes in the dipoles.

These two formulae represent the most widespread form of the reciprocity theorem. Let us expound its content once more.*

Let us place at points A_1 and A_2 of a space arbitrarily nonuniform identical point dipoles directed along certain axes. Assume that having induced in one of them an alternating moment with an amplitude value of current $\dot{I}^{(1)}$, we obtain at the point of disposition of the second dipole A_2 a field component $E^{(1)}(A_2)$ along its axis. Then, having excited in the second dipole a moment of <u>same frequency</u> with an amplitude current value $\dot{I}^{(2)}$ (when the electromotive forces of the first dipole are switched off), we obtain at the point of disposition of the first dipole A_1 a field $E^{(2)}(A_1)$ in the direction of its axis. These fields and the amplitude values of the currents are precisely linked by the correlation (9.12). If, in particular, these amplitude values of currents are equal, so are the fields. In other words, the dipole placed in A_1 and directed along h_1 induces at the point A_2 along the direction h_2 an electric field equal to the one which would be created at the point A_1 in the direction h_1 by that dipole placed in A_2 and directed along h_2. This theorem is valid for any conductions of the media, for which our initial field Eqs. (2.4) remain valid. As is shown in §2, they are correct if the conduction of the antenna is much greater than that of the surrounding media.** They are in principle incorrect, for example, in the ionosphere, where the properties of the medium depend on the amplitude of the field and, therefore, the equations may be nonlinear.

It should be noted that at times a somewhat different context is ascribed to the concept of "reciprocity theorem." Two fields $\dot{E}_1$, $\dot{H}_1$ and $\dot{E}_2$, $\dot{H}_2$ are considered, created, shall we say, by different sources, or for various body dispositions. Thus, for example, one of them may be the radar field in the absence of aim, and the other may constitute its field in the presence of a reflecting target. Considering the region of space in which the body disposition remains invariable and in which outside forces are absent, we may again take advantage of the equation (9.2). Integrating it over the volume and transforming the volume integral into surface, we shall again obtain zero from integrals over discontinuity surfaces (Σ', Σ'' etc.), so that there remains

$$\int ([H^{(1)}E^{(2)}]_n - [H^{(2)}E^{(1)}]_n)\, dS = 0, \qquad (9.15)$$

where the integral extends over the surface encompassing the volume (and in case of necessity, over surfaces outlining inside the volume those regions in which alien currents are present, or in which the conditions vary, for example, when there appears a new medium reflecting the target etc. at transition from the field (1) to the field (2). If the field $\vec{E}^{(1)}$, $\vec{H}^{(1)}$ is known, we extract therefrom, the results for the field $\vec{E}^{(2)}$, $\vec{H}^{(2)}$, provided we act with dexterity (see, for example, [13], Appendix A, p. 686).

The reciprocity theorem may also be formulated for the fields induced by magnetic dipoles. In this case the Maxwellian equations have the form (3.14) and, moreover, there exist the correlations (6.6), (6.7). The above reasonings, giving the reciprocity theorem (9.11), may be fully repeated also in this case. They obviously will give

$$m^{(1)}H^{(2)}(A_1) = m^{(2)}H^{(1)}(A_2), \qquad (9.16)$$

that is, two magnetic dipoles $\vec{m}^{(1)}$ and $\vec{m}^{(2)}$, placed at various points A_1 and A_2, each giving such a magnetic field $\vec{H}^{(1)}(A_2)$ or $\vec{H}^{(2)}(A_1)$ at the point of location of another dipole that the scalar product of this field by the moment of another dipole here located is one and the same in both cases.

Manuscript Page No.	[Footnotes]
14*	Let us stress that <u>these denotations are contrary to the prevalent ones</u>. Thus, in [1] and, generally in numerous books on radiophysics, ε' is recognized as the <u>complex</u> dielectric constant, which we shall denote here by ε (without the prime), and by ε its real part, which we shall denote here by ε'.
14**	It is sometimes assumed that the third Maxwellian equation should be $\operatorname{div}(\varepsilon' E) = 0$, as it follows from (1.11) for $\rho = 0$ and not Eq. (1.12a), which in the absence of outside forces has the form $\operatorname{div}(\varepsilon E) = 0$. However, according to (1.12), $\rho = 0$ occurs only at $\operatorname{div} \vec{j} = 0$, that is, either in the absence of conduction or generally of the imaginary part of ε, or in a uniform medium (see below (1.156)). Then $\operatorname{div}(\varepsilon' E) = \operatorname{div}(\varepsilon E) = 0$, and both formulations coincide. In the general case, however, one should utilize Formula (1.12a).
49	Let us recall that in Formula (5.14) the medium must be described by the same complex dielectric constant in the entire volume encompassed by the surface S.
53	Or, to be more precise, if it does indeed go to infinity at some point B, it must do so slower than $1/r$, where <u>r</u> is the distance to the point B. In such a case, having outlined this point by the sphere S'_0, we shall at any rate obtain zero at limit, when drawing S'_0 to the point B.
56	We write $v_-(R, R')$ rather than $v_-(R. R'. R_1)$, as R_1 is unambiguously determined by the assignment of R.
61	It evidently does not follow from Formula (9.3a) that the integrands are equal (as is sometimes assumed). This may be seen in particular from the fact that antennas 1 and 2 may have different shapes and in that case no sense of any kind can be ascribed to equality of the integrands.
63	Strictly speaking, we neglect the influence of the idle i<u>th</u> antenna upon the operation of the transmitting k<u>th</u> antenna. If both antennas are sufficiently small and far apart, this influence is negligibly small.
64*	Let us call attention to the fact that the constants of the media at points of antenna location are not clearly part of Relation (9.12). Of the correctness of that result one may find easily evidence as follows. Assume that the antennas are in reality surrounded by arbitrary media. If then we should remove these media from the very thin layers surrounding each of the antennas, it is evident that this cannot influence the field of each antenna far from it. On the other hand, after that the antennas will be found to be in vacuum and the equality (9.12) must be valid. This means that it generally may not contain medium's constants.

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64** Formula (9.12) differs from the one sometimes brought up in literature as the expression of the reciprocity theorem for point dipoles in a medium with conduction, when instead of (9.12) one generally writes

$$\frac{\varepsilon_1'}{\varepsilon_1} f^{(1)} E^{(2)}(A_1) = \frac{\varepsilon_2'}{\varepsilon_2} f^{(2)} E^{(1)}(A_2). \qquad (9.13)$$

Here ε_1, ε'_1 and ε_2, ε'_2 are respectively the complex dielectric constant and its real part at the location points of the first and second dipoles. Thus, the difference in the formulation arises only in the case when the media at points A_1 and A_2 have conduction properties. The cause of this distinction consists in that, in deriving (9.13), the electric field in the immediate vicinity of the dipole with the instantaneous value of the moment p was assumed to be equal to the electric field in the corresponding dielectric, i.e., for example, at the distance $\vec{R}_1$ from the point $\vec{A}_1$ with a dipole moment $\vec{p}$, the field (at $|kR_1\sqrt{\varepsilon_1}| \ll 1$) was assumed to be equal to

$$E(R_1) = \frac{1}{\varepsilon_1'}\left\{\frac{3(pR_1)R_1}{R_1^5} - \frac{p}{R_1^3}\right\}. \qquad (9.14)$$

Meanwhile, as may be seen from Formula (6.5), $1/\varepsilon_1$ should in reality stand in the formula (9.14) instead of $1/\varepsilon'_1$. It is obvious that Formula (9.12) could be obtained also be outlining the dipoles by a small sphere and having their radii approach zero, provided only we utilize for $\vec{E}$ the expression (9.14) with ε'_1, substituted by ε_1. The application of the incorrect formula may lead to significant errors (cf. §31).

[Transliterated Symbols]

7 стор = stor = storonnyy = extraneous

10 эфф = eff = effektivnyy = effective

11 п = p = polnyy = total

11 поляр.. = polyar. = polyarizatsiya = polarization

46 макс = maks = maksimal'nyy = maximum

Chapter 2

REGION OF SPACE ESSENTIAL FOR THE PROCESS OF RADIOWAVE PROPAGATION

The conditions in wh ich the propagation of radiowaves takes place near the ground are diversified and complex even if we limit ourselves to the cases when the influence of the ionosphere may be disregarded. Various factors (such as the presence of the ground surface, its relief, the inhomogeneity of the electrical properties of the soil, the air density varying with the altitude, the turbulent inh omogeneities of the atmosphere, etc.) cannot, and usually need not be accounted for at once. They are manifest in various ways, depending upon the wavelength of the radiation and the disposition of the transmitter and receiver.

In order to estimate the relative role of the various factors, it is necessary to know, first of all, which is the region of space substantially encompassed by the propagation process under given conditions. To that effect we shall consider certain questions linking the theory of radiowave propagation in a uniform medium with wave optics.

The propagation of radiowaves near the ground is undoubtedly similar in a series of cases to those phenomena with which wave optics is concerned. Thus, the propagation of centimeter waves between points raised to the altitude of several hundred meters (towers of television and relaying stations) must to a considerable extent suggest the propagation of a ray in free space. However, in this example we are already beset with peculiarities that become still more substantial when the

the question evolves toward longer waves. The principal peculiarities are the proximity of the source and (or) of the point of observation to the interface between media with different electrical properties. We shall see that usually the mere smallness of the wavelength in one of the media by comparison with the distance to the interface cannot serve as a sufficient criterion: the corresponding points may be remote from the ground surface by many wavelengths, but if the distance betwen them is very great, the conditions differ essentially from those usually encountered in optics. True, while investigating the problems of radiotelegraphy, L.I. Mandel'shtam pointed as early as in 1914 to an optical phenomenon in which analogous peculiarities appear owing to the proximity of the source to the interface between the two media [1]. This however, was an especially contrived exception.

Despite these peculiarities, the requirement of involving the optical theory of diffraction is nearly obvious. Attempts were made a long time ago to utilize the theory of diffraction from the edge of a plane screen in order to describe the effects of hills upon the propagation of ultrashort waves. However, the idealization applied was too far reaching. This path may be successful only in a limited number of cases (see §53).

In the present chapter we shall extract from the theory of diffraction not the complete solutions of separate problems, but only some semiquantitative criteria that may be subsequently required for estimating the influence of variations in the conditions of radiowave propagation on the wave field.

§10. RECTILINEAR PROPAGATION OF LIGHT AND THE STATIONARY PHASE METHOD

1. We shall deal with the integral equation (7.1) for the Hertz vector:

$$\Pi(R) = \int_V P v\, dV + \frac{1}{4\pi}\oint_S \left\{\frac{\partial \Pi}{\partial n} v - \Pi \frac{\partial v}{\partial n}\right\} dS. \qquad (10.1)$$

Here the volume integral extends over an arbitrary volume including the observation point $A(\vec{R})$. Of that volume it is required, generally speaking, that it be filled by a uniform medium characterized by the propagation constant k; $\vec{P}$ is the polarization density linked with the current density in the sources by the formula $j_e = -i\omega P$; n is the normal to the surface S enclosing the volume V, and is external to the volume; v is the Green function, which may always be postulated equal to $\frac{1}{r}e^{ikr}$; it may, however, be taken in a different form in separate cases. The same equation is valid for any field vector subject to the wave equation; in particular, one may substitute $\vec{E}$ and $\vec{H}$ instead of $\vec{\Pi}$ if $\vec{P}$ is substituted by the sources of the fields $\vec{E}$ and $\vec{H}$ in accord with Eqs. (3.9a, b).

As already mentioned, this formula contains the Huygens principle; a certain limiting case there stems from it, in particular, rectilinear propagation of light. Disruptions and limitations of propagation rectilinearity, linked with diffraction phenomena, also follow from it.

The Huygens principle may be obtained, for example, in the formulation following below.

If it is known that there is located at a point O of uniform space a source of oscillations, giving on a certain surface S, closed around O, the field of the vector $\vec{\Pi} = \vec{p}\cdot e^{ik\rho}/\rho$, where ρ is the distance between that point S of the surface, and O, the field of vector $\vec{\Pi}$ may be considered as induced at any point A, situated outside the surface S at the distance D from O, by virtual dipoles and quadrupoles distributed in a peculiar manner over the surface S, and not by the real source. They also give $\vec{\Pi} = \vec{p}e^{ikD}/\vec{D}$. Therefore, at this external point relative to S, the field may arbitrarily be considered either as the result of superimposition of spherical waves emitted by various elements of the surface S, or to have arrived here directly along the line OA.

We shall obtain this result from Eq. (10.1) and take advantage of the conclusion itself in order to illustrate the application of one mathematical procedure having a simple physical sense and being extremely important for the following.

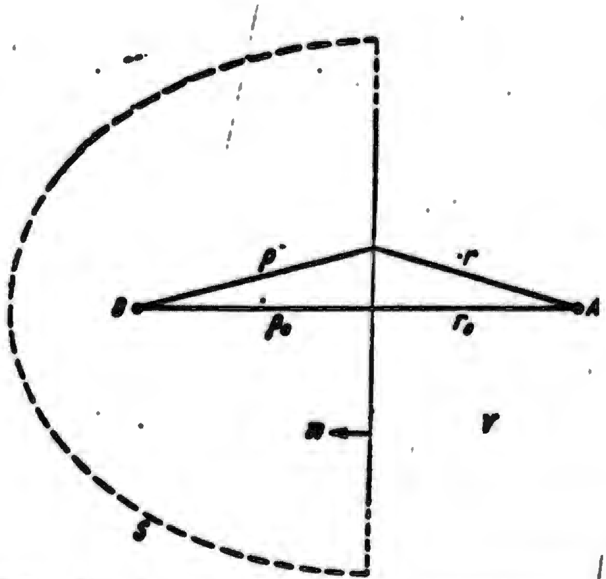

Fig. 10.1. Rectilinear propagation of radiowaves.

For definiteness we shabl consider at the outset <u>k</u> as real. Assume that the surface S consists of an infinite plane perpendicular to the segment OA, and of an infinite hemisphere, resting upon that plane and closing the volume in which the source is located (Fig. 10.1).

In the volume V, where the observation point is located, there are no sources, $\vec{p} = 0$. This is why, having selected the Green function by either of the two methods ($v = v_-$ or $v = v_+$), as is shown in §8, we may obtain two equally correct expressions for $\vec{\Pi}$, see (8.5) and (8.7a):

$$\Pi(R) = -\frac{1}{2\pi}\int \Pi(R')\frac{\partial}{\partial n}\frac{e^{ik|R-R'|}}{|R-R'|}dS', \qquad (10.2)$$

$$\Pi(R) = \frac{1}{2\pi}\int \frac{\partial \Pi(R')}{\partial n}\frac{e^{ik|R-R'|}}{|R-R'|}dS', \qquad (10.3)$$

where the integration is performed over the entire surface, with, at the same time, $|R-R'|=r, \Pi(R')=p\frac{e^{ik\rho}}{\rho}$ (in this way, owing to the special shape of the surface, we may succeed in selecting a Green function such that the virtual sources are either only quadrupoles or only dipoles).

Let us consider Expressions (10.3). Here (see Fig. 10.1)

$$\frac{\partial\rho}{\partial n}=-\frac{\rho_0}{\rho},\quad \frac{\partial}{\partial n}\frac{e^{ik\rho}}{\rho}=-\left(ik-\frac{1}{\rho}\right)\frac{e^{ik\rho}}{\rho}\cdot\frac{\rho_0}{\rho},$$

where ρ_0 is the distance between the point O and the surface. Since the observation point in our case is always located in the wave zone, it may be estimated that, though ρ_0 (and perhaps also r_0) is great by comparison with the wavelength, $\rho_0/\lambda >> 1$. This means, a fortiori, that $\rho >> \lambda$, and since $k = 2\pi/\lambda$, we have the more so

$$k\rho \gg 1. \qquad (10.4)$$

Consequently, $1/\rho$ is small by comparison with ik and may be rejected. This is why

$$\Pi(R)=-\frac{ikp}{2\pi}\int\frac{\rho_0}{\rho}\frac{e^{ik(r+\rho)}}{r\rho}dS'. \qquad (10.5)$$

This integral has a peculiar character. If at integration $\vec{r}$ and ρ should vary, then owing to the condition (10.4) even a relatively small variation of ρ would induce a rapid fluctuation of the multiplier $\exp\{ik(r+\rho)\}$, that is, a frequent variation of the sign of both the real as well as the imaginary parts of the integrand. On the other hand, the remaining multipliers of the integrand too vary relatively little at relatively small variation of ρ and r.

As will be seen in the following, these peculiar properties of the integrable function are quite general for all those cases with which one has to deal in the theory of radiowave propagation. Nearly always if only one of the distances, either from the source or from the observation point to the surface over which one has to integrate applying the relation (10.1), is great by comparison with the wavelength. This is still further stressed by the fact that the distance must be great by comparison with the wavelength distributed by 2π, and not with the wavelength as is, for the great value must be the product $k\rho_0 = 2\pi\rho_0/\lambda$.

We shall often denote the wavelength divided by 2π by the letter $\lambda\!\!\!^{-}$:

$$k\rho_0 = \frac{\rho_0}{\lambda\!\!\!^{-}} \gg 1, \quad \lambda\!\!\!^{-} = \frac{\lambda}{2\pi}. \qquad (10.4a)$$

In the remote limit case, when the ratios of all distances to $\lambda\!\!\!^{-}$ are so great that all the combinations of all possible distances are also extremely great, provided $\lambda\!\!\!^{-}$ stands in them in the denominator in some positive order, we shall in the very end eventually arrive at geometric optics. In the theory of radiowave propagation along the ground the situation is generally intermediate: the relations of the type (10.4) are valid, but their left-hand parts are not great to the extent that the wave phenomena (diffraction) can be neglected. To the contrary, the accounting for these phenomena precisely constitutes the subject of the theory. This is why the consideration of integrals with similar integrands has fundamental significance.

2. It was found that these integrals may be computed with sufficient (and at the same time controllable) precision with the help of the stationary phase method. This method shows, in particular, that the most substantial integration region is that where the exponent of the oscillating function has an extreme, that is, where the phase is stationary relative to small shifts. Let us pause at this method at further length.

At the outset we shall consider it for a single variable, namely by examining the integral

$$G=\int_{-\infty}^{+\infty} e^{ip\varphi(\xi)} f(\xi)\, d\xi, \tag{10.6}$$

in which p is a large number: $p \gg 1$; $\varphi(\xi)$ and $f(\xi)$ are dimensionless functions of dimensionless variable ξ. It is convenient to assume that they have an absolute value of the order of the unity and derivatives of same order. However, the authentic conditions, which they must satisfy, are described below. Assume that $\varphi(\xi)$ has a unique extreme at a certain point ξ_0, and let this be, for concreteness, the minimum:

$$\varphi'(\xi_0)=0, \quad \varphi''(\xi_0)>0. \tag{10.7}$$

Let us introduce the new variable

$$u^2=p\{\varphi(\xi)-\varphi(\xi_0)\}, \tag{10.8}$$

$$2u\,du=p\varphi'(\xi)\,d\xi, \quad \frac{du}{d\xi}=\frac{1}{2u}p\varphi'(\xi), \tag{10.8a}$$

so that the integration limits will become

$$u_{1,2}=\sqrt{p\{\varphi(\mp\infty)-\varphi(\xi_0)\}}. \tag{10.8b}$$

Inasmuch as the function φ increases monotonically from the point ξ_0 on both sides, u_1 and u_2 are positive numbers and they are at the same time great, since p is great. The integral over u may be broken in two parts. Note that $\varphi'(\xi)$ has different signs on both sides from the minimum point ξ_0. This is why in the interval $0 < u < u_1$, obtained from the interval $-\infty < \xi < \xi_0$, we have $\varphi'(\xi(u))=-|\varphi'(\xi)|<0$, and in the second integral $0 < u < u_2$ will be $\varphi'(\xi(u))>0$. Consequently, utilizing the equality (10.8), we may write

$$G=e^{ip\varphi(\xi_0)}\left\{\int_0^{u_1} e^{iu^2} f(\xi(u))\frac{du}{\frac{1}{2}\frac{p|\varphi'(\xi(u))|}{u}}+\right.$$
$$\left.+\int_0^{u_2} e^{iu^2} f(\xi(u))\frac{du}{\frac{1}{2}\frac{p\varphi'(\xi(u))}{u}}\right\}=G_1+G_2. \tag{10.6a}$$

Let us consider, for example, the first of these integrals, G_1:

$$G_1 = -e^{ip\varphi(\xi_0)} \int_0^{u_1} e^{iu^2} \frac{u\,du}{\frac{p\varphi'(\xi(u))}{2f(\xi(u))}}. \qquad (10.9)$$

We shall expand the denominator in series in u near the point u = 0. It will be necessary to know the quantity

$$\left(\frac{d\xi}{du}\right)_{\xi=\xi_0}.$$

Evaluating the indeterminate form, we obtain from (10.8a)

$$\left(\frac{d\xi}{du}\right)_{\xi=\xi_0} = \lim_{\xi\to\xi_0} \frac{2u}{p\varphi'(\xi)} = \lim_{\xi\to\xi_0} \frac{2\frac{du}{d\xi}}{p\varphi'(\xi)} = \frac{}{p\varphi''(\xi_0)} \left(\frac{du}{d\xi}\right)_{\xi=\xi_0},$$

so that

$$\left(\frac{d\xi}{du}\right)_{\xi=\xi_0} = \sqrt{\frac{2}{p\varphi''(\xi_0)}}. \qquad (10.10)$$

According to (10.10), we find (the prime meaning everywhere the differentiation with respect to ξ)

$$\frac{\varphi'}{f} = \sqrt{\frac{2\varphi''(\xi_0)}{p}} \frac{u}{f(\xi_0)} + \left[\frac{\varphi'''(\xi_0)}{pf(\xi_0)\varphi''(\xi_0)} - \frac{2f'(\xi_0)}{pf^2(\xi_0)}\right] u^2 + \ldots. \qquad (10.10a)$$

Consequently,

$$-G_1 = \frac{e^{ip\varphi(\xi_0)} f(\xi_0)}{\sqrt{\frac{p\varphi''(\xi_0)}{2}}} \int_0^{u_1} \frac{e^{iu^2}\,du}{1 + \sqrt{\frac{1}{2p\varphi''(\xi_0)}} \left(\frac{\varphi'''(\xi_0)}{\varphi''(\xi_0)} - \frac{2f'(\xi_0)}{f(\xi_0)}\right) u + \ldots}. \qquad (10.11)$$

Thus, the denominator is found to be expanded in powers of the quantity

$$\sqrt{\frac{2}{p\varphi''(\xi_0)}}\, u.$$

Regarding p as a very large number, we obtain for zero approximation and having left in the denominator the unity only

$$G_1 \approx G_1^0 = -\sqrt{\frac{2}{p\varphi''(\xi_0)}} f(\xi_0) e^{ip\varphi(\xi_0)} \int_0^{u_1} e^{iu^2} du. \qquad (10.12)$$

The integr.l appearing here is reduced to the well known Fresnel integrals. It is important to us that as u_1 increases, it rapidly approaches a certain limit:

$$\int_0^x e^{it^2}dt = \frac{1}{2}\sqrt{\pi i} + \frac{e^{ix^2}}{2ix}\left[1+\frac{1}{2ix^2}+\frac{1\cdot 3}{(2ix^2)^2}+\dots\right], \quad (10.13)$$

where

$$\sqrt{i} = e^{i\frac{\pi}{4}}.$$

We obtain this expansion by separating the integral from 0 to ∞ and performing multiple integration by parts, estimating each time that $e^{it^2}dt = \frac{de^{it^2}}{2it}$. Thus, if $x \to \infty$, this integral is equal to $\frac{1}{2}\sqrt{\pi i}+O\left(\frac{1}{x}\right)$.

If we postulate $t^2 = \frac{\pi}{2}\xi^2$, we shall obtain another form:

$$\sqrt{\frac{\pi}{2}}\int_0^x e^{i\frac{\pi}{2}\xi^2}d\xi = \frac{\sqrt{i}}{\sqrt{2}} + \sqrt{\frac{2}{\pi}}\,\frac{e^{ix^2}}{2ix}\left[1+\frac{1}{2ix^2}+\frac{1\cdot 3}{(2ix^2)^2}+\dots\right], \quad (10.13a)$$

or, at $\sqrt{\frac{2}{\pi}}x = v$,

$$F(v) = \int_0^v e^{i\frac{\pi}{2}\xi^2}d\xi = \frac{\sqrt{i}}{\sqrt{2}} + \frac{e^{i\frac{\pi}{2}v^2}}{\pi i v}\left[1+\frac{1}{\pi i v^2}+\frac{1\cdot 3}{(i\pi v^2)^2}+\dots\right]. \quad (10.13b)$$

This expression is called the Fresnel integral. Designated as Fresnel integrals (or, respectively, Fresnel cosine-integral and Fresnel sing-integral) are also its real and imaginary parts taken separately:

$$C(v) = \int_0^v \cos\frac{\pi\xi^2}{2}d\xi, \quad S(v) = \int_0^v \sin\frac{\pi\xi^2}{2}d\xi,$$
$$F(v) = C(v) + iS(v). \quad (10.13c)$$

Obviously [see (10.13b)]

$$C(\infty) = S(\infty) = \frac{1}{2},$$

and generally

$$\int_{-\infty}^{+\infty} e^{ia\xi^2}\, d\xi = \sqrt{\frac{\pi i}{a}} = \sqrt{\frac{\pi}{a}}\, e^{\frac{\pi i}{4}}. \qquad (10.13d)$$

This formula is correct even for complex a, provided $\operatorname{Im} a > 0$. We may indeed compute the square of this integral, passing at computation to polar coordinates, $d\xi\, d\xi' = r\, dr\, d\varphi$, $\xi^2 + \xi'^2 = r^2$, and then, assuming $r^2 = t$:

$$I^2 = \int_{-\infty}^{+\infty} d\xi \int_{-\infty}^{+\infty} d\xi'\, e^{ia(\xi^2+\xi'^2)} = \int_0^{\infty} r\, dr \int_0^{2\pi} d\varphi \cdot e^{iar^2} = \pi \int_0^{\infty} e^{iat}\, dt = \frac{\pi i}{a}.$$

Hence follows anew the relation (10.13d).

Let us return to our integral G (10.6), (10.6a), (10.9). If it is admissible to limit ourselves to the unity in the denominator of the integral (10.11), and provided u_1 and u_2 are sufficiently great, we have, according to Formula (10.13d)

$$G \approx \sqrt{\frac{2\pi i}{p\varphi''(\xi_0)}}\, f(\xi_0)\, e^{ip\varphi(\xi_0)}. \qquad (10.14)$$

Desiring to obtain correction of the order $1/u_1$ or $1/u_2$, we must take into account not only the correction terms in the expansion (10.13), but also the inequality of the denominator in (10.11) to unity.

We may write

$$\int_0^{u_2} = \int_0^{u_2} e^{iu^2} du \left\{ 1 - \frac{1}{\sqrt{2p\varphi''(\xi_0)}} \left(\frac{\varphi'''(\xi_0)}{\varphi''(\xi_0)} - \frac{2f'(\xi_0)}{f(\xi_0)} \right) u + \ldots \right\}. \qquad (10.14a)$$

Here the correction term gives an elementary integral. The condition for its smallness formulates the requirements set forth to functions φ and f in the considered method. The correction ensuing therefrom vanishes when $u_1 = u_2$, as is easy to find evidence. But, moreover this term is generally absent whenever φ and f are even functions of u (see 10.10a)). This is precisely the case which materializes in the problem of interest to us, as will be evident and borne in mind in the following, and which is related to the type- (10.5) integrals. (This is conditioned by the fact that r and ρ depend on the square of the integration variable x or y; see below.) The subsequent terms of the expansion (10.10a) have already the order $1/p\varphi''(\xi_0)$ and, if φ is limited, they are immaterial by comparison with the terms of the order

$\frac{1}{u_{1,2}} \sim \frac{1}{\sqrt{p|\varphi(\xi_{1,2})-\varphi(\xi_0)|}}$, occurring in the expansion (10.13), which have to be taken into account only in this case:

$$G \approx \sqrt{\frac{2\pi}{p\varphi''(\xi_0)}} f(\xi_0) e^{ip\varphi(\xi_0)} \cdot e^{\frac{\pi}{4}i}\left[1+\left(\frac{e^{iu_1^2}}{\sqrt{\pi}u_1}+\frac{e^{iu_2^2}}{\sqrt{\pi}u_2}\right)e^{-\frac{3\pi}{4}i}\right]. \quad (10.15)$$

The neglecting of the square bracket is equivalent to the case $u_1 = u_2 = \infty$. This means that we may shift in the integrals (10.6a) or (10.12) the limits to infinity with a precision to correction terms.

We considered the particular and simplest case of type- (10.6) integrals. Had the point ξ_0 been maximum and not minimum for $\varphi(\xi)$, $\varphi''(\xi_0) < 0$, we might have introduced instead of u^2 (10.8) the variable

$$v^2 = p(\varphi(\xi_0) - \varphi(\xi)), \quad (10.16)$$

$$\frac{dv}{d\xi} = -\frac{1}{2}\frac{p\varphi'(\xi)}{v}, \quad (10.16a)$$

and, repeating all the preceding reasonings, we would have arrived at the conclusion that

$$G \approx \sqrt{\frac{2\pi}{p|\varphi''(\xi_0)|}} f(\xi_0) e^{ip\varphi(\xi_0)} \cdot e^{-i\frac{\pi}{4}}\left[1+\left(\frac{e^{-iv_1^2}}{\sqrt{\pi}v_1}+\frac{e^{-iv_2^2}}{\sqrt{\pi}v_2}\right)e^{\frac{3\pi}{4}i}\right]. \quad (10.17)$$

By its complexity the following case is that when the integral in Formula (10.6) is taken in finite limits

$$G = G(a, b) = \int_a^b f(\xi) e^{ip\varphi(\xi)} d\xi. \quad (10.18)$$

Acting as before, that is, introducing a new variable u according to Formula (10.8) or v according to Formula (10.16), we shall be concerned with the very same integrals (10.6a) or (10.12), with their integration limits, however, now being equal to

$$u_1 = \sqrt{p(\varphi(a) - \varphi(\xi_0))}, \quad u_2 = \sqrt{p(\varphi(b) - \varphi(\xi_0))} \qquad (10.19)$$

(and analogously in Formulas (10.16), (10.17). We assume that u_1 and u_2 are great, inasmuch as p is great. For a more general case see the works [2, 3].

The inequality to the unity of the square bracket in the relation (10.15) will now reflect the influence of the finiteness of the integration interval.

If the extreme lies inside the integration interval, but is not unique, the stationary phase method may then be sometimes applied. To that effect the interval should be divided into smaller intervals in such a way that within the limits of each interval only one extreme remains; then to each of them either Formula (10.15) or (10.17) should be applied, which is admissible at limitations obvious from the above said. First of all it is necessary that u_1 and v_1, corresponding to the boundaries of each small interval, be sufficiently great.

The integral (10.13) differs from its value at infinite limits by less than 10% already at x = 8. Therefore, if this precision is sufficient, the region of values of u near zero, in which $u \leq 10$ is essential at integration over u. This is why, determining the limits of integration over u it may be said that the essential region is the one in which $\varphi(\xi)$ differs from the value $\varphi(\xi_0)$ in the extreme by a quantity of the order 100/p. If p is great, this points in the scale of ξ to quite narrow limits for the essential region.

Consequently, according to Formula (10.14), *for integrals of the type (10.18), containing the product of a rapidly fluctuating multiplier $f(\xi)$, the latter may be taken at the point where the exponent has an extreme, and taken out of the integral sign; besides, in view of the immaterialness of the remote integration regions, the limits may be expanded to infinity.*

Note that the application of the stationary phase method requires in essence that $p\varphi''(\xi_0)$ and not p be great. This is why, even if the phasesdoes not have the form $p\varphi(\xi)$, but has an extreme at a certain point ξ_0, and varies rapidly over length $\Delta\xi \ll 1$, this method is also valid. This, for example, if instead of $p\varphi(\xi)$ there stands in the exponent $\varphi(\sqrt{p}\xi)$, where $\varphi(x)$ is recognized as a function having as also its derivatives a value, generally speaking, of the order of the unity, and at the same time such that at variation of x by a quantity of the order of the unity, $\varphi(x)$ also varies by a quantity of the order of the unity, but for a great p at variation of ξ by a small quantity the function $\varphi(\sqrt{p}\xi)$ may be subject to numerous oscillations. This is why, repeating the previous reasonings, we shall obtain instead of Formula (10.9) the expression

$$G = \int_{-\infty}^{+\infty} e^{i\varphi(\sqrt{p}\xi)} f(\xi)\, d\xi = \int e^{i\varphi(\sqrt{p}\xi_0)+iu^2} \frac{u\,du}{\frac{\sqrt{p}\varphi'(\sqrt{p}\xi)}{2f}}, \qquad (10.20)$$

where $u^2 = \varphi(\sqrt{p}\xi) - \varphi(\sqrt{p}\xi_0)$, and the prime denotes the differentiation with respect to the argument. Taking into account that now, instead of (10.10), the following relation will be valid:

$$\left(\frac{d\xi}{du}\right)_{\xi=\xi_0} = \frac{2}{p\varphi''(\sqrt{p}\xi_0)} \qquad (10.21)$$

and effecting the same transformations, we shall obtain again

$$G \approx G_0 = \sqrt{\frac{2\pi i}{p\varphi''(\sqrt{p}\xi_0)}}\, e^{i\varphi(\sqrt{p}\xi_0)} f(\xi_0). \qquad (10.22)$$

Such an approach to an integral containing a rapidly oscillating function is also possible when the stationary point lies beyond the (finite) integration interval $a < \xi < b$ (for details see [2, 3]; for the case of double integrals see [I, 12; I, 11], [5-7]).

In practice the stationary phase method is applied as follows. If the integral (10.18) is preassigned, $\varphi(\xi)$ is expanded in series in the neighborhood of the point ξ_0 with limitation to quadratic terms ($\varphi'(\xi_0) = 0$):

$$\varphi(\xi) = \varphi(\xi_0) + \frac{1}{2}\varphi''(\xi_0)(\xi - \xi_0)^2. \qquad (10.23)$$

Then

$$G = e^{ip\varphi(\xi_0)} \int_a^b f(\xi)\, e^{\frac{i}{2}p\varphi''(\xi_0)(\xi - \xi_0)^2}\, d\xi. \qquad (10.24)$$

Taking the value of $f(\xi)$ at the point $\xi = \xi_0$ out of the integral sign and applying Formula (10.13) for the Fresnel integrals, we obtain the result (10.17). It is evident that taking $f(\xi)$ at the extreme point out of the integral sign and at the same preserving the second addend in parenthesis in (10.17) (or applying (10.15) is possible only in the case when the expansions of $\varphi(\xi)$ as well as of $f(\xi)$ do not contain terms that would make a contribution of same order as the correction in (10.17) (cf. the remark made after Formula (10.14a)).

If $f(\xi)$ varies not too weakly within the limits of integration, the result may be improved by setting $f(\xi) = \exp(\ln f(\xi))$ and

$$ip\varphi(\xi) + \ln f(\xi) = i\Phi(\xi) \approx i\left\{\Phi(\xi_1) + \frac{\Phi''(\xi_1)}{2}(\xi - \xi_1)^2\right\}, \qquad (10.25)$$

where ξ_1 is a stationary point of the total phase, $\Phi'(\xi_1) = 0$. It is then possible to apply the previous method. However, fictitiousness of the phase Φ may then spring up and we must ealize this refining within the framework of a more general method (corresponding to the case of complex p or $\varphi(\xi)$), that is, *the saddle point* [or steepest descent] method.

It may be found that $\varphi''(\xi_0)$ is small or even becomes zero. In such a case the expansion (10.23) must be extended, for example,

$$\varphi(\xi) = \varphi(\xi_0) + \frac{1}{2}\varphi''(\xi_0)(\xi - \xi_0)^2 + \frac{1}{6}\varphi'''(\xi_0)(\xi - \xi_0)^3. \qquad (10.26)$$

By substitution

$$\xi - \xi_0 = \sqrt{\frac{2}{p\varphi'''(\xi_0)}}\, x - \frac{\varphi''(\xi_0)}{\varphi'''(\xi_0)} \qquad (10.27)$$

the expansion of $\varphi(\xi)$ is brought to a form generally not containing quadratic terms. After that, taking out of the integral sign the value of $f(\xi)$ at $\xi = \xi_0$ (the admissibility of this would require, generally speaking, an additional basis, similarly to what was done above), we obtain [3b; 2]

$$G = e^{ip\left(\varphi(\xi_0) + \frac{1}{3}\frac{[\varphi''(\xi_0)]^3}{[\varphi'''(\xi_0)]^2}\right)} f(\xi_0) \sqrt[3]{\frac{2}{p\varphi'''(\xi_0)}} \int e^{i\left(tx + \frac{1}{3}x^3\right)}\, dx,$$

$$t = -\frac{1}{2}\frac{p[\varphi''(\xi_0)]^2}{\varphi'''(\xi_0)} \sqrt[3]{\frac{2}{p\varphi'''(\xi_0)}}. \qquad (10.28)$$

The integral figuring here is the Airy integral. Detailed tables are available for it. At $\varphi''(\xi_0) = 0$ (the stationary point is the inflexion point), taking into account that (see [VI, 4a]

$$\int e^{\frac{i}{3}x^3} dx = 2\int_0^\infty \cos\frac{x^3}{3} dx = 2\sqrt{\pi}\, v(0) = \frac{2\pi}{3^{\frac{2}{3}}\Gamma\left(\frac{2}{3}\right)}, \qquad (10.28a)$$

where $v(0) = \operatorname{Im} w(0)$, $w(t)$ is an Airy function, $v(0) = 0.62927\ldots$, we obtain

$$G = f(\xi_0)\, e^{ip\varphi(\xi_0)} \sqrt{\frac{2}{p\varphi'''(\xi_0)}} \cdot \frac{2\pi}{3^{\frac{2}{3}}\Gamma\left(\frac{2}{3}\right)}. \qquad (10.28b)$$

However, another approach is also possible [4]. Instead of reducing this particular case to the tabulated integral, we may substitute φ by the sum of the trigonometrical and linear functions, inasmuch as, at any rate, the expansion (10.26) is valid only with a precision to cubic terms so that the integral will take the form of integral representation of a Bessel function. As a result, G will be expressed by clindrical functions of great order from a great argument. Convenient asymptotic representations for them are well known.

Thus, for example, we may postulate

$$\varphi(\xi) = \varphi(\xi_0) + b(\xi - \xi_0) + c(\sin\alpha\xi - \sin\alpha\xi_0). \qquad (10.28c)$$

where ξ_0 is determined by the condition $\varphi'(\xi_0) = 0$, while the three constants b, c and α are determined from the requirement that this approximation convey $\varphi(\xi)$ with a precision to the terms of third order relative to $\xi - \xi_0$

$$\varphi'(\xi_0) = b + c\alpha\cos\alpha\xi_0 = 0,$$
$$\varphi''(\xi_0) = -c\alpha^2\sin\alpha\xi_0,$$
$$\varphi'''(\xi_0) = -c\alpha^3\cos\alpha\xi_0.$$

Assuming after that

$$\alpha\xi = \theta, \quad \alpha\xi_0 = \beta, \quad -\frac{b}{\alpha} = c\cos\alpha\xi_0 = v,$$

we bring the integral to the form

$$\int e^{ip\varphi(\xi)}\,d\xi = \frac{\xi_0}{\beta}\, e^{ip\{\varphi(\xi_0) - v(\operatorname{tg}\beta - \beta)\}} \int e^{ipv(\sec\beta\sin\theta - \theta)}\,d\theta. \qquad (10.28d)$$

With the corresponding choice of limits the remaining integral be a cylindrical function $Z_{pv}(pv\sec\beta)$. [11]; at the same time

$$\beta\operatorname{ctg}\beta = \xi_0\frac{\varphi'''(\xi_0)}{\varphi''(\xi_0)}, \quad v = -\left(\frac{\xi_0}{\beta}\right)^2\varphi''(\xi_0). \qquad (10.28e)$$

3. Up until now we spoke of integrals G (10.6) with purely real p. If p should be a purely imaginary number with positive sign, the above assertions would be valid to a still greater measure. Indeed, if in the integral

$$G(i|p|) = \int^{\infty} e^{-i|p|\varphi(\xi)} f(\xi)\,d\xi. \qquad (10.29)$$

the function $f(\xi)$ does not influence substantially the convergence of the integral, the function $\varphi(\xi)$ must be positive. Here $f(\xi)$ may (contrary to the case of real p) even increase so long as this increase is not too rapid. For such an integral it is evident that the essential region is the one adjacent to the extreme and at the same time necessarily to the minimum, that is, the function $\varphi(\xi)$. Introducing again the variable $u^2 = |p|\,|\varphi(\xi) - \varphi(\xi_0)|$, this circumstance is easy to relate to the rapid convergence of the Gaussian integral

$$\int_0^x e^{-t^2}\,dt = \frac{\sqrt{\pi}}{2} - \frac{e^{-x^2}}{2x}\left[1 - \frac{1}{2x^2} + \frac{1\cdot 3}{(2x^2)^2} - \dots\right], \qquad (10.29a)$$

obtained from (10.13) upon the substitution $-it^2 = t'^2$.

The difference between the cases of real and imaginary p thus consists in that at imaginary p the minimum region is important, whereas at real p both the minimum and maximum regions are equally important.

Essential also is the fact that the influence of remote regions in the case of real p decreases only as $1/p\xi$, whereas in the case of imaginary p it does so much more rapidly; exponentially. This is why if the integration interval is not infinite, the position of the integration limit for a real p is manifest in the terms of the order $1/p\xi_1$, where ξ_1 is the distance to the limit, whereas for an imaginary p it is manifest in the terms of the order $\exp(-p\xi^2{}_1)$, that is, much more feebly.

In reality, both these cases are particular cases of a more general consideration, valid for integrals of the form (10.6) with p being neither real nor imaginary. Moreover, there is no longer any sense here in limiting ourselves to the integration over real values of ξ. Let us therefore consider the integral

$$G(p) = \int_C f(\xi)\,e^{\varphi(\xi)}\,d\xi, \quad \varphi = \varphi_1 + i\varphi_2, \qquad (10.30)$$

taken over a certain contour C in the complex plane ξ. The function φ into which we now have included the great parameter p, will be postulated as rapidly varying with the variation of ξ.

We are interested in the case when there exists in the plane ξ at a finite distance from the origin, a point ξ_0 such that the derivative φ in it becomes zero

$$\left(\frac{d\varphi}{d\xi}\right)_{\xi=\xi_0} = 0. \qquad (10.31)$$

It is well known from the theory of complex variable functions that, generally speaking, grad φ_1 (that is, the vector pointing at the direction of fastest variation of φ_1 at each of the given points, is directed along the line over which φ_2 is constant, and vice versa. The point $\xi = \xi_0$ differs in that in it (provided $\varphi'' \neq 0$) two lines intersect at a right angle (and more than two if $\varphi'' = 0$), along which φ_1 is constant and, simultaneously, two (or more) lines alone which φ_2 is constant. Consequently (see Fig. 10.2), the neighborhood of the point ξ_0 is firstly divided in four sectors separated by the lines $\varphi_1 = \varphi_1(\xi_0) =$ = const, in which $\varphi_1(\xi)$ is alternatively greater (a and c) or smaller (b and d) than at the point ξ_0. If we represent $\varphi_1(\xi)$ in the form of relief above the plane ξ, the surface $\varphi_1(\xi)$ will descend from the point ξ_0 onto two opposite sectors (b and d) and ascend into two others disposed crosswise relative to the first ones (at the point ξ_0 itself both functions φ_1 and φ_2 have an extreme). Consequently, $\varphi_1(\xi)$ forms a saddle-like surface that suggests at the same time the relief of a mountain pass; φ_2 also forms a saddle-like surface; however, at the point ξ_0 the lines φ_2 = const pass relative to lines φ_1 = const at angles equal to 45°.

Assume that the contour C passes through the point ξ_0 from one sector with $\varphi_1 < \varphi_1(\xi_0)$ (say, from d) into another identical sector (b). The absolute value of the function $\exp(\varphi(\xi))$, will pass through a maximum, and, evidently, the principal contribution to the integral will be made by the neighborhood of the point ξ_0.

It was mentioned above that the gradient direction, that is the direction of the most rapid variation of φ_1 generally described as *the direction of the steepest descent*, is also that of the invariable $_2$. The contour C may pass exactly along that direction C = C_1. But even if this is not so, it may be shifted in most cases till it coincides with that direction (as is well known, one should bear in mind only the poles crossed at such a shift). Then we shall have at any point of the contour C_1 $\varphi_2(\xi)=\varphi_2(\xi_0)$ and this is why $\varphi(\xi)=\varphi(\xi_0)+\varphi_1(\xi)-\varphi_1(\xi_0)$. Consequently, we may write

$$G = e^{\varphi(\xi_0)}\int_{\xi_0} e^{\varphi_1(\xi)-\varphi_1(\xi_0)} f(\xi)\,d\xi. \qquad (10.32)$$

There stands under the integral an exponential function, a rapidly varying real exponent which has a zero maximum at $\xi = \xi_0$. If $f(\xi)$ varies comparatively slowly, only the direct neighborhood of the point ξ_0 will be essential, and the value of $f(\xi)$ at the point $\xi = \xi_0$ may be taken out of the integral.

Moreover, the integration may be performed along an infinite line, tangent to C_1 at the point ξ_0 at which we may denote

$$\xi-\xi_0 = xe^{i\psi_1},\quad x=|\xi-\xi_0|,\quad \psi_1=\arg(\xi-\xi_0)=\text{const}. \qquad (10.33)$$

Expanding $_1$ in series on this line and limiting ourselves to quadratic terms (according to the condition $\left(\frac{d\varphi_1}{d\xi}\right)_{\xi=\xi_0} = \left(\frac{d\varphi_1}{e^{i\psi_1}dx}\right)_{x=x_0}=0$), we obtain

$$\varphi_1(\xi)-\varphi_1(\xi_0) = \frac{1}{2}\left(\frac{d^2\varphi_1}{d\xi^2}\right)_{\substack{\text{along } C_1\\ \xi=\xi_0}}(\xi-\xi_0)^2 =$$

$$=\frac{1}{2}\left(\frac{d^2\varphi_1}{e^{2i\psi_1}dx^2}\right)_{x=x_0} e^{2i\psi_1}x^2 = \frac{1}{2}\ddot{\varphi}_1(\xi_0)x^2, \qquad (10.33a)$$

$$d\xi = e^{i\psi_1}dx,$$

where $\ddot{\varphi}_1(\xi_0) = \frac{d^2\varphi_1(\xi_0)}{dx^2}<0$. Therefore, the integral (10.32) is reduced to the Gaussian integral

$$G = e^{\varphi(\xi_0)} f(\xi_0) e^{i\psi_1}\int_{-\infty}^{+\infty} e^{\frac{1}{2}\ddot{\varphi}_1(\xi_0)x^2}dx = \sqrt{\frac{2\pi}{|\ddot{\varphi}_1(\xi_0)|}}\, e^{\varphi(\xi_0)} f(\xi_0) e^{i\psi_1}. \qquad (10.34)$$

On the other hand, the contour of integration might have been given along the line C_2, $\varphi_1=\text{const}$, or it might have been made to coincide with the line C_2 (taking the same precautions relative to poles). Then, reasoning quite analogously, we obtain

$$G = e^{\varphi(\xi_0)} \int\limits_{\xi_0} e^{i(\varphi_2(\xi)-\varphi_2(\xi_0))} f(\xi)\, d\xi. \tag{10.35}$$

The oscillating function again separates as the most material region the neighborhood of the point ξ_0 if $\ _2(\xi)$ varies rapidly (it is true, however, that here there is no longer any exponential decrease), and the integral may be taken along an infinite line, tangent to C_2.

Assuming on C_2

$$\xi - \xi_0 = x e^{i\psi_2}, \quad \psi_2 = \text{const}, \tag{10.36}$$

at the same time, since the angle between C_1 and C_2 at the point ξ_0 is $\pi/4$, we have

$$\psi_2 = \psi_1 - \frac{\pi}{4}, \tag{10.36a}$$

expanding $\varphi_2(\xi)$ in series to quadratic terms and taking $f(\xi)$ at the point ξ_0 out of the integral, we arrive at the Fresnel integral (10.13)

$$\begin{aligned} G &= e^{\varphi(\xi_0)} f(\xi_0) e^{i\psi_2} \int\limits_{-\infty}^{+\infty} e^{\frac{i}{2}\ddot{\varphi}_2(\xi_0)x^2} dx = \\ &= \sqrt{\frac{2\pi}{\ddot{\varphi}_2(\xi_0)}}\, e^{\varphi(\xi_0)} f(\xi_0) e^{i\psi_1}, \end{aligned} \tag{10.37}$$

where ψ_1 emerged because there stands in Formula (10.13) a general factor $\sqrt{i}$, while the equality (10.36a) is taken into account.

Thus, the stationary phase method is one of the extreme cases for an integral of a rapidly varying function. Another extreme case is constituted by the integral of the exponentially decreasing function (10.29). If the contour C should occupy an intermediate position, it is possible, by shifting it, to take advantage of any of these extreme cases; at the same time, as may be seen from Formulas (10.34) and (10.37), the results will certainly coincide. Recognizing by φ'' a derivative with respect to ξ, and not x, and taking into account that $\left(\frac{d^2\varphi}{dx^2}\right) = e^{2i\psi_2} \frac{d^2\varphi}{d\xi^2}$, the result may be rewritten in the form

$$G = \int_{C_1} e^{\varphi(\xi)} f(\xi)\, d\xi = \sqrt{\frac{2\pi}{-\varphi''(\xi_0)}}\, e^{\varphi(\xi_0)} f(\xi_0). \qquad (10.34a)$$

A more detailed consideration of the saddle point [or steepest descent] method can be found in the books [7].

In practice, when applying this method one must only search for the point ξ_0 and utilize Formula (10.34a). At times, and in order to make the question of sign more precise when extracting the root from the complex quantity, one must still find the direction of the steepest descent C_1, which will determine ψ_1 in Formula (10.34). A particular case may then be met with if the pole of the integrand is located near the contour (this precisely takes place in the case of radiowave propagation along the ground). This question is considered in detail in the work [8].

4. Let us return to the question of rectilinear propagation of light and the Huygens principle. It was reduced to the consideration of the integral (10.5), that precisely incited us to turn to the stationary phase method. This integral is two-dimensional, as a consequence of which the problem becomes somewhat more complex. Now we have to seek on the integration surface a point, at which the phase would be stationary at shift in any direction, and to apply subsequently the method in each of the two variables over which integration must be performed. Let us illustrate this procedure using the example of the integral (10.5).

We shall introduce on the plane S the rectangular coordinates x, y, z with origin at the point where the plane S is intersected by the ray OA. The exponent $k(r + \rho)$ may be written, for example, in either of the two forms

$$k(r+\rho) = k\left(\sqrt{x^2+y^2+r_0^2} + \sqrt{x^2+y^2+\rho_0^2}\right) = kr_0\left\{\sqrt{1+\frac{x^2+y^2}{r_0^2}} + \right.$$

$$+\sqrt{\frac{\rho_0^2}{r_0^2}+\frac{x^2+y^2}{r_0^2}}\Bigg\}-k\rho_0\Bigg\{\sqrt{\frac{r_0^2}{\rho_0^2}+\frac{x^2+y^2}{\rho_0^2}}+\sqrt{1+\frac{x^2+y^2}{\rho_0^2}}\Bigg\}.$$

where r_0 and ρ_0 are the distances from the points O and A to the origin of the coordinates.

Considering kr_0 or $k\rho_0$ as a great parameter p, entering in the stationary phase method, it is possible to transfer here all the results obtained for this method. Thus, in the integrand taken, for example, as a function of x, essential will be the region of values of x, near the point at which

$$\frac{d}{dx}(r+\rho)=\frac{x}{r}+\frac{x}{\rho}=0,$$

that is near the point $x = 0$, where the phase has a minimum. Considering, on the other hand, the exponent as a function of y, we obtain in exactly the same way that at integration over y the essential region will be that near the point $y = 0$. This is why all the factors, with the exception of the oscillating one, may be taken at the point $x = y = 0$, and taken out of the integral. Considering that $kr_0 \gg 1$ and k $_0 \gg 1$ (the fundamental case; concerning the case $kr_0 \leq 1$ see below), we shall expand the exponent in series

$$\varphi=k(r+\rho)=k\left\{r_0+\frac{x^2+y^2}{2r_0}+\ldots+\rho_0+\frac{x^2+y^2}{2\rho_0}+\ldots\right\}; \quad (10.38)$$

and limiting ourselves to terms, quadrative relative to x and y, we have

$$\Pi(A)\approx\frac{p}{2\pi}\frac{-ik}{r_0\rho_0}e^{ik(r_0+\rho_0)}\int\limits_{-\infty}^{+\infty}\int dx\,dy\,e^{\frac{ik}{2}\left(\frac{1}{r_0}+\frac{1}{\rho_0}\right)x^2}e^{\frac{ik}{2}\left(\frac{1}{r_0}+\frac{1}{\rho_0}\right)y^2}. \quad (10.39)$$

For each of the integrals we may utilize Formula (10.13d). This is why

$$\Pi(A)\approx\frac{-ik}{2\pi r_0\rho_0}pe^{ik(r_0+\rho_0)}\frac{i\pi}{\frac{k}{2}\left(\frac{1}{r_0}+\frac{1}{\rho_0}\right)}=p\frac{e^{ikD}}{D}. \quad (10.40)$$

Therefore, if the source O had indeed induced in the plane S the field $p\frac{1}{\rho}e^{ik\rho}$, the virtual dipoles at the point A, distrubuted over S, would induce the field $\Pi(A)=p\frac{e^{ikD}}{D}$, the same as at direct reaching of the wave. This is precisely where the content of the Huygens principle can be seen. The above demonstration is valid, inasmuch as we neglect in either ormula (10.11) or (10.14a) the terms of higher order relative to $1/\sqrt{p\varphi''(\xi_0)}$ and, in particular, the terms of the order

$$\frac{1}{\sqrt{2p\varphi''(\xi_0)}}\left(\frac{\varphi'''}{\varphi''}-\frac{2f'}{f}\right).$$

taken at the extreme point. But here, according to Formula (10.38)

$$\varphi'''=f'=0,\quad p\varphi''\sim k\left(\frac{1}{r_0}+\frac{1}{\rho_0}\right).$$

This is a corollary of the parity of φ as functions of x and y, to which it was referred above. Consequently, the correction of the order $1/\sqrt{p\varphi''}$ vanishes and all the consideration is rightful with a precision to terms of the order $1/p\varphi''(\xi_0)$. It is obvious that, by the strength of the general theorem (10.3), the result is exactly correct in reality.

As is well known from elementary optics, the region near the point x = y = 0 is isolated in its way of being essential, because the action of more remote regions is mutually annihilated.

In reality, in the integral (10.5), extended over S,

$$\Pi(A)=p\frac{-ik}{2\pi}\int\frac{\rho_0}{\rho}\frac{e^{ik(r+\rho)}}{r\rho}dS, \tag{10.5a}$$

the plane is broken up into concentric rings, such that at transition from one ring to the other either the real or the imaginary part of the integral changes sign. The boundaries of these rings are determined by the condition

$$k\{(r+\rho)-(r_0+\rho_0)\}=m\frac{\pi}{2},\quad m=1,2,\ldots. \tag{10.41}$$

The virtual dipoles, located within the bounds of one ring, emit to the observation point radiations, of which the phases have an identical sign, whereas the dipoles of the two neighboring rings emit radiations mutually compensating one another. An elementary, but not too rigorous a consideration shows that a nonvanishing result remains only from the first, central ring. The stationary phase method provides a rigorous demonstration of this. The rings themselves are called Fresnel zones.* Let us compute their width.

Assuming $x^2 + y^2 = a^2$, we have [see (10.38)]

$$\frac{k}{2}a^2\left(\frac{1}{r_0}+\frac{1}{\rho_0}\right)=\frac{\pi}{2}m, \tag{10.41a}$$

that is, for the first zone, m = 1

$$a_0=\sqrt{x^2+y^2}=\sqrt{\frac{\pi}{k}\frac{1}{\frac{1}{r_0}+\frac{1}{\rho_0}}}\sim\sqrt{\frac{\pi R}{k}}\sim\sqrt{\lambda R}, \tag{10.41b}$$

where we denoted by R the smallest of the two instances, r_0 and ρ_0.

Therefore, the width of the first zone is significantly greater than the wavelength if $kr_0 \gg 1$ and $k\rho_0 \gg 1$: $\quad a\sim\sqrt{\lambda R}\gg\lambda;$ (10.42)

For more remote zones, for which the width Δa is much smaller than the distance to the center of zones a, we shall obtain from Formula (10.41a)

$$ka\Delta a\left(\frac{1}{r_0}+\frac{1}{\rho_0}\right)=\frac{\pi}{2}, \tag{10.42a}$$

i.e.,

$$\Delta a\sim\frac{\pi}{2k}\frac{R}{a}\sim\frac{a_0^2}{a}\ll a_0. \tag{10.42b}$$

Here R is everywhere the smallest of the two distances between the respective points and the plane S. Thus, the width of these rings decreases with increase of their number. As to the area, it remains constant:

$$2\pi a\Delta a\sim\frac{\pi}{2}R\lambda. \tag{10.43}$$

Let us pause at one mathematical question, essential for some of the computations in the following.

We might compute the integral (10.5) over S, taking into account Formula (10.38) in polar coordinates, having postulated

$$x = a\cos\psi, \quad y = a\sin\psi.$$

Then we would have

$$\Pi(A) \approx \frac{-ik}{2\pi}\frac{p}{r_0\rho_0}e^{ikD}\int_0^\infty a\,da\int_0^{2\pi}\frac{d\psi\, e^{\frac{ika^2}{2}\left(\frac{1}{r_0}+\frac{1}{\rho_0}\right)}}{\sqrt{1+\frac{a^2}{r_0^2}}\sqrt{1+\frac{a^2}{\rho_0^2}}}. \qquad (10.44)$$

Effecting the substitution $a^2 = t$, we obtain upon integration over ψ

$$\Pi(A) = -\frac{ik}{2}p\frac{e^{ikD}}{r_0\rho_0}\int_0^\infty\frac{e^{\frac{ikt}{2}\left(\frac{1}{r_0}+\frac{1}{\rho_0}\right)}}{\left(1+\frac{t}{\rho_0^2}\right)^{1/2}\left(1+\frac{t}{r_0^2}\right)^{1/2}}dt. \qquad (10.44a)$$

This integral from an oscillating function would, contrary to Integral (10.13) be improper at real k, if in the denominator no factors were found to be increasing, though slowly, but nevertheless increasing with the rise of t (that is, if we simply substituted in the denominator r and ρ by r_0 and ρ_0, as was done earlier). Owing to them, the integral (10.44a) converges, but its value still does not depend on the exact value of these factors if they vary little, when the phase succeeds in changing by a quantity of the order of the unity. If, for example, $r_0 < \rho_0$, the independence of the value of the integral from the exact form of slow factors takes place when

$$\frac{k}{2}\frac{1}{r_0} \gg \frac{1}{r_0^2},$$

that is, when $kr_0 \gg 1$, which is what we assumed from the very beginning.

In this case, neglecting t/r_0^2 and t/ρ_0^2 by comparison with the unity, we may compute the integral by introducing, for insuring convergence, a factor of another form: $e^{-\alpha t}$ with an infinitely small value of the parameter α, that is, postulating*

$$\int_0^\infty e^{ik\frac{t}{2}\left(\frac{1}{r_0}+\frac{1}{\rho_0}\right)}dt = \lim_{\alpha\to 0}\int_0^\infty e^{ik\frac{t}{2}\left(\frac{1}{r_0}+\frac{1}{\rho_0}\right)-\alpha t}dt = \frac{1}{-\frac{ik}{2}\left(\frac{1}{r_0}+\frac{1}{\rho_0}\right)},$$

which brings us exactly to the previous result (10.40).

In the following we shall more than once encounter integrals containing a slowly varying function, which is the only one to assure the convergence of the integral. Replacing this function by its value in the essential region, we will obtain sometimes, at real k and contrary to the case of the Fresnel integral, the improper integral

$$I = \int_a^\infty e^{ikr}dr.$$

We shall always recognize it precisely according to the following terms:

$$I = \int_a^\infty e^{ikr}dr = \lim_{\alpha\to 0}\int_a^\infty e^{ikr-\alpha r}dr = \frac{e^{ika}}{-ik}. \qquad (10.45)$$

Up until now we assumed that not only ρ_0, but also r_0 is great by comparison with the wavelength. If such is not the case, it is easy to show that the conclusion derived is also correct (see [I.1], p.   ).

Thus, we have traced how this particular case of the Huygens principle (10.1) stems from its general formulation. If, namely, we consider that virtual dipoles (virtual sources of spherical waves), distributed over the surface S, send waves to each given observation point, their aggregate field provides precisely the field of a directly arriving wave. It is obvious that these calculations maintain their strength even in the case when the surface S is not a plane. It is true, however, that for more complex configurations the computations may not always be performed as simply, but the validity of the Huygens principle is guaranteed by the validity of the general relation (10.1).

In order to study in detail the role of the essential region, we shall pass to the case of a physically bounded plane, namely to the passage of electromagnetic oscillations through an aperture in an opaque screen.

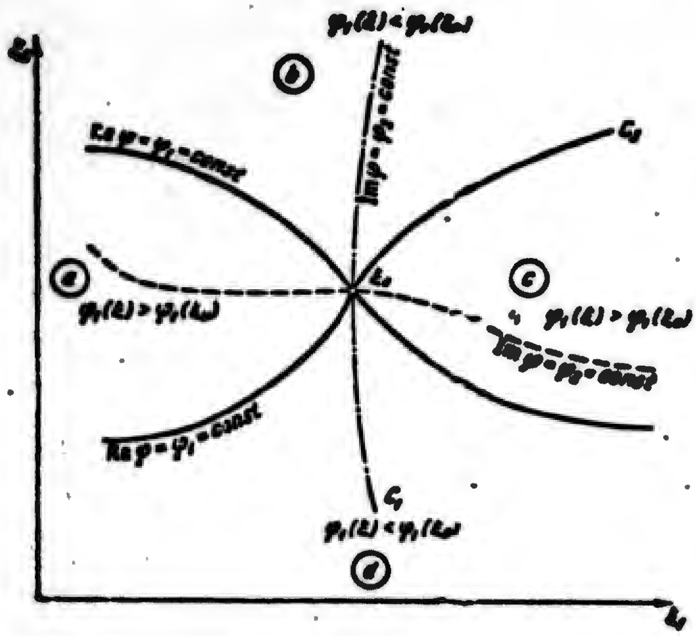

Fig. 10.2. Saddle point and stationary phase methods.

§11. DIFFRACTION FROM AN APERTURE IN A SCREEN (ESSENTIAL REGION FOR THE PASSAGE OF RADIATION IN A UNIFORM MEDIUM)

Let the plane S represent a real, entirely opaque screen,* in which is cut an aperture assumed rectangular at the outset and extending along the axis x from 0 to +a, along the axis y from −b to +b (Fig. 11.1). Let us trace the axis z perpendicularly to the screen and postulate for simplicity that the source 0 is also located somewhere on the axis z at the point $z = -z_0$. We shall place the point of observation at a certain point $A(x_A, y_A, z_A)$, generally speaking not being on the axis z. In this way, the plane of the screen here is not perpendicular to the line OA. We shall assume the distances from the source to the aperture ($\rho_0 \equiv z_0$) and from A to the screen (r_0) quite great by comparison with the dimensions of the aperture.

Applying the general formula (10.1), it is possible, by selecting

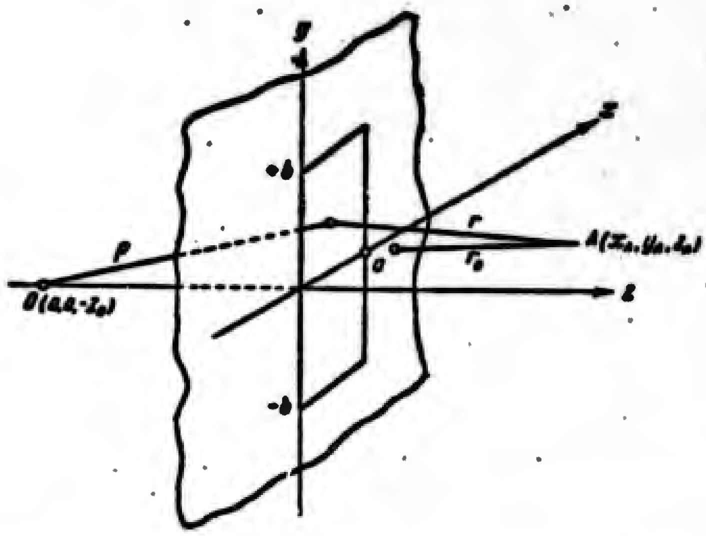

Fig. 11.1. Diffraction on a rectangular aperture.

a special form of the Green function, to lay at the foundation Eq. (10.2). Further, we have

$$\frac{\partial}{\partial n}\frac{e^{ikr}}{r} \approx ik\frac{r}{r_0}\frac{e^{ikr}}{r} \approx ik\frac{e^{ikr}}{r},$$

so that

$$\Pi(A) = -\frac{ik}{2\pi}\int\frac{e^{ikr}}{r}\Pi(S)dS, \qquad (11.1)$$

where $\vec{\Pi}(S)$ is the field in the plane of the screen S, which is unknown to us. The problem standing before us is extremely complex in rigorous formulation. The total solution for it was found by Sommerfeld [9] only for the case of a semi-infinite screen (for example, $a, b \to \infty$).

However, according to the Kirchhoff proposal, all classical problems of the diffraction theory are resolved with the help of the following approximate method (we shall discuss the limits of its applicability subsequently, in §13).

Assume that at each point of the aperture's plane the field coincides with the unperturbed field, which would take place here if the screen did not exist in general, and that the field on the dark side of the screen's surface is zero. Obviously, these assumptions may not entirely correspond to the reality, since, for example, because of diffraction, the shaded side of the screen must be partly illuminated, whereas in the plane of the aperture, if only near its edges, the field must undergo a perturbation. However, it may be expected that in some cases (we shall subsequently discuss in which ones) a good approximation can be obtained along this path. Comparison with the above-referred-to exact solution shows the validity of this expectation.

Having admitted the Kirchhoff approximation, we must assume in the integral of (11.1)

$$\Pi(S)=\begin{cases} 0 & \text{outside the aperture region;} \\ p\dfrac{e^{ik\rho}}{\rho} & \text{in the aperture region.} \end{cases} \qquad (11.2)$$

For our rectangular aperture we obtain

$$\Pi(A)=\frac{-ik}{2\pi}p\int_{0}^{a}dx\int_{-b}^{+b}dy\,\frac{e^{ik(r+\rho)}}{r\rho}. \qquad (11.3)$$

Estimating the dimensions of the aperture (a, b) small by comparison with r_0 and ρ_0, we shall obtain

$$r\approx r_0+\frac{(x-x_A)^2+(y-y_A)^2}{2r_0},\quad \rho\approx\rho_0+\frac{x^2+y^2}{2\rho_0}. \qquad (11.4)$$

Therefore we again arrive at the Fresnel integrals. Taking the slowly varying factors out of the integral, we obtain

$$\Pi(A)=\frac{-ik}{2\pi}p\,\frac{e^{ik(r_0+\rho_0)}}{r_0\rho_0}\int_{0}^{a}e^{\frac{ik}{2}\left(\frac{(x-x_A)^2}{r_0}+\frac{x^2}{\rho_0}\right)}dx\int_{-b}^{+b}e^{\frac{ik}{2}\left(\frac{(y-y_A)^2}{r_0}+\frac{y^2}{\rho_0}\right)}dy. \qquad (11.4a)$$

Let us postulate

$$\frac{k}{2}\left(x-\frac{x_A}{1+\frac{r_0}{\rho_0}}\right)^2\left(\frac{1}{r_0}+\frac{1}{\rho_0}\right)=\frac{\pi}{2}u^2;\ \frac{k}{2}\left(y-\frac{y_A}{1+\frac{r_0}{\rho_0}}\right)\left(\frac{1}{r_0}+\frac{1}{\rho_0}\right)=\frac{\pi}{2}v^2 \qquad (11.5)$$

and correspondingly

$$\left.\begin{aligned}
&\sqrt{\frac{k}{2}\left(\frac{1}{r_0}+\frac{1}{\rho_0}\right)}\left(-\frac{x_A}{1+\frac{r_0}{\rho_0}}\right)=-\sqrt{\frac{\pi}{2}}u_1;\\
&\sqrt{\frac{k}{2}\left(\frac{1}{r_0}+\frac{1}{\rho_0}\right)}\left(-b-\frac{y_A}{1+\frac{r_0}{\rho_0}}\right)=-\sqrt{\frac{\pi}{2}}v_1;\\
&\sqrt{\frac{k}{2}\left(\frac{1}{r_0}+\frac{1}{\rho_0}\right)}\left(a-\frac{x_A}{1+\frac{r_0}{\rho_0}}\right)=\sqrt{\frac{\pi}{2}}u_2;\\
&\sqrt{\frac{k}{2}\left(\frac{1}{r_0}+\frac{1}{\rho_0}\right)}\left(b-\frac{y_A}{1+\frac{r_0}{\rho_0}}\right)=\sqrt{\frac{\pi}{2}}v_2.
\end{aligned}\right| \qquad (11.5a)$$

Thus, u_1, u_2, v_1, v_2 measure in a specific scale the distance from the edges of the aperture of the plane's S point which is the projection on S of the point of observation. All of them may have any sign, but they are positive when the projection lies inside the aperture.

The sense of the admitted scale is easy to establish. For example, if $\rho_0 \gg r_0$, that is, if the source is quite remote (in other words, if it may be considered that a plane wave is incident upon the screen), we have, according to Formula (10.41b),

$$u_1=\frac{x_A}{a_0},\quad v_1=\frac{b+y_A}{a_0},\quad u_2=\frac{a-x_A}{a_0},\quad v_2=\frac{b-y_A}{a_0}, \qquad (11.5b)$$

i.e., in this case the scale is the dimension of the first Fresnel zone.

Besides, taking into account that

$$r_0+\rho_0+\frac{x_A^2+y_A^2}{2(r_0+\rho_0)}\approx\sqrt{(r_0+\rho_0)^2+x_A^2+y_A^2}=D,$$

where D is the distance from the point O to the point A along the straight line, we may rewrite (effecting at the same time the substitution in the denominator $r_0+\rho_0=D$)

$$\Pi(A) = \frac{-ik}{2\pi} p \frac{\pi}{k\left(\frac{1}{r_0}+\frac{1}{\rho_0}\right)} \frac{e^{ikD}}{r_0\rho_0} \int_{-u_1}^{+u_2} e^{i\frac{\pi}{2}u^2} du \int_{-v_1}^{+v_2} e^{i\frac{\pi}{2}v^2} dv = \tag{11.6}$$

$$= \frac{-i}{2} p \frac{e^{ikD}}{D} (F(u_2) - F(-u_1)) (F(v_2) - F(-v_1)),$$

where

$$F(u) = \int_0^u e^{i\frac{\pi}{2}u^2} du = -F(-u)$$

is the Fresnel integral. It is obvious that

$$p \frac{e^{ikD}}{D} = \Pi_0(A) \tag{11.7}$$

is the field at the point A that would take place if the screen were removed and the entire space were thus uniform.

The presence of the screen induces a deviation from that ideal field. However, so long as u_1, u_2, v_1 and v_2, are positive and great by comparison with the unity, the perturbation is not great, for in this case expansion of the type (10.13b), in which the series containing the oscillating factor, constitute a small correction, are valid. Indeed, in that case

$$(F(u_2) - F(-u_1)) (F(v_2) - F(-v_1)) \approx 2\frac{\sqrt{i}}{\sqrt{2}} \cdot 2\frac{\sqrt{i}}{\sqrt{2}} = 2i,$$

so that $\Pi = \Pi_0$. But if, for example $u_2 > 0$, with $u_1 < 0$ and they are

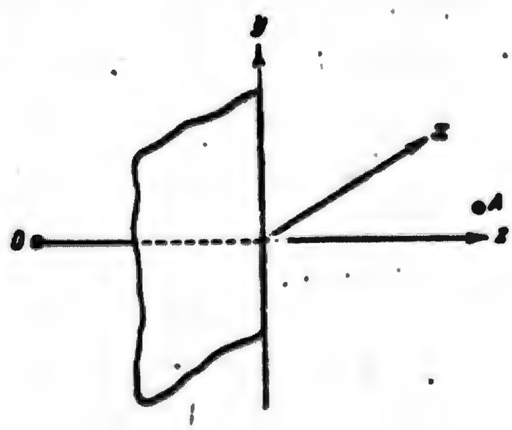

Fig. 11.2. Diffraction on a semiplane.

again great by module, we shall obtain in the expression $F(u_2) - F(-u_1)$ according to Formula (10.13b)

$$F(|u_2|) - F(|u_1|) \approx \frac{e^{i\frac{\pi}{2}u_2^2}}{i\pi|u_2|} - \frac{e^{i\frac{\pi}{2}u_1^2}}{i\pi|u_1|}, \qquad (11.7a)$$

which by module is quite small.

According to Formula (11.5a), the negative value of u_1 is obtained when $x_A < 0$. It is easy to see from Fig. 11.1 that at such a position of the point A it is situated in the geometric shadow. At the same time, the greater $|u_1|$, the deeper the shadow obtained. Therefore, the boundary between light and shadow is washed off.

We shall write the field in the form

$$\left.\begin{aligned} w_x(x_A) &= \frac{e^{-i\pi/4}}{\sqrt{2}}(F(u_2) - F(-u_1)), \\ w_y(y_A) &= \frac{e^{-i\pi/4}}{\sqrt{2}}(F(v_2) - F(-v_1)), \end{aligned}\right\} \qquad \begin{matrix}(11.8)\\ \\ (11.8a)\end{matrix}$$

where $w_x(x_A)$, $w_y(y_A)$ are the field attenuation factors, conditioned by the presence of the screen. Here, for example, $|w_A(x_A)|^2$ indicates the course of intensity distortion at shift along the axis x.

Let us consider the particular case of Fig. 11.2. Assume that the half-space $x < 0$ is shielded by the screen and the region $x > 0$ is free. In this case we have $a = \infty$, $b = \infty$, so that $F(u_2) = \frac{\sqrt{i}}{\sqrt{2}} = \frac{e^{i\pi/4}}{\sqrt{2}}$ and, moreover, $w_y(y_A) = 1$,

$$w_x(x_A)\,w_y(y_A) = \frac{e^{-i\frac{\pi}{4}}}{\sqrt{2}}\left\{\frac{e^{i\frac{\pi}{4}}}{\sqrt{2}} + F(u_1)\right\} = \frac{1}{2} + \frac{e^{-i\frac{\pi}{4}}}{\sqrt{2}}F(u_1). \qquad (11.8b)$$

At $u_1 \to 0$, $F(u_1)$ becomes zero. This corresponds to the geometrical boundary of the shadow. As may be seen, here $w_x = 1/2$, and, consequently, the intensity is equal to one-quarter of the intensity of the unperturbed wave. The course of the function $w_x(x_A)|^2$ is represented in

Fig. 11.3.

Other coordinates are often used for the diffraction on the rectilinear edge of a plane screen. Instead of considering the plane of the screen Σ_0 to be perpendicular to the line OT, drawn from the source to

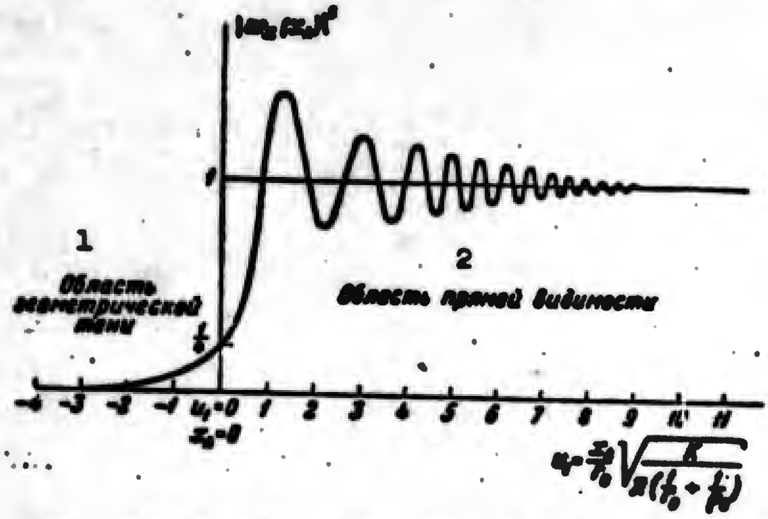

Fig. 11.3. Distribution of intensity in case of diffraction on a semi-plane. 1) Region of geometric shadow; 2) region of direct visibility.

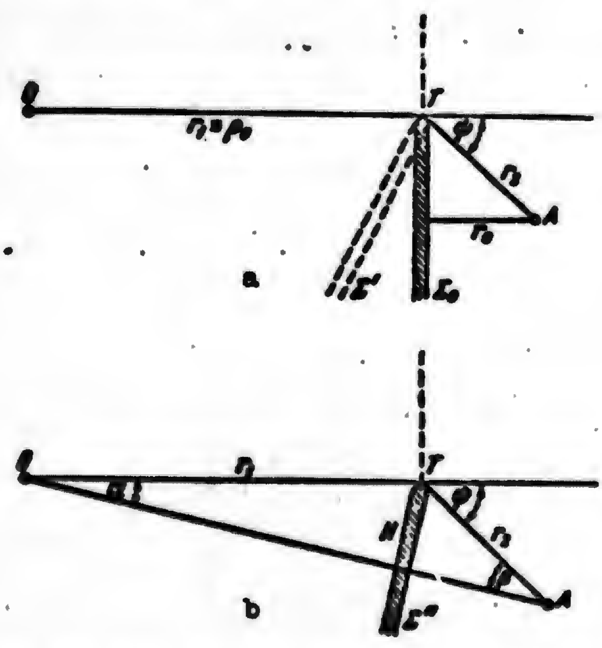

Fig. 11.4. Diffraction on the rectilinear edge of a plane screen. a) When the plane of the screen is perpendicular to the line drawn from the source O to the edge of the screen T; b) when the plane of the screen is perpendicular to the line linking the source O with the observation point A.

its edge (Fig. 11.4a), as was done above (Fig. 11.2), we may, as previously, integrating over the semi-plane S, arbitrarily lean the screen, for example, shift it into the position Σ' (dashed lines in Fig. 11.4a) provided only the position of the edge T remains unchanged. It is then obvious that nothing will be modified in the above-expounded deduction of the attenuation formula (11.8b). However, for the coordinates chosen above we would have been compelled to draw anew the perpendiculars ρ_0 and r_0 from O and A to the screen for every position of the latter. This is why, rather than express u_1 by x_A and r_0, it is preferable to do it, say, by $\rho_0 \equiv r_1$ and r_2 by the distance from the source O and the observation point A from the edge of the screen T, and by the diffraction angle ψ:

$$r_2 = r_0 \cos\psi \approx r_0; \; x_A = -r_2 \sin\psi.$$

Consequently,

$$u_1 = \frac{x_A}{r_0}\sqrt{\frac{k}{\pi}\frac{\rho_0 r_0}{\rho_0 + r_0}} \approx -\sin\psi \sqrt{\frac{k}{\pi}\frac{r_1 r_2}{r_1 + r_2}}. \qquad (11.8c)$$

Here we substituted $\cos\psi \approx 1$, inasmuch as the whole theory is correct only for small ψ (cf. below §13). At times another expression is applied for u_1. If we take for the base the line linking O and A, and if we dispose the screen perpendicularly to it (see Fig. 11.4b), assuming as previously $r_1 \sin\alpha = r_2 \sin\beta$, and introducing the distance H from the edge of the screen T to the line OA, we shall find from the relations $\psi = \alpha + \beta, \cos\alpha \approx \cos\psi \approx 1, \sin\psi \approx \frac{r_1 + r_2}{r_2}\sin\alpha = \frac{r_1 + r_2}{r_1 r_2}H$, and this is why

$$u_1 = -H\sqrt{\frac{k}{\pi}\frac{r_1 + r_2}{r_1 r_2}}. \qquad (11.8d)$$

In the cases when the integral (10.13) is designated as the Fresnel integral, and not (10.13b), the argument u_1 is substituted by $\sqrt{\frac{\pi}{2}}u_1$.

Let us now turn back to the general case of a rectangular aperture, and consider a field on the axis $\underline{z}$, assuming $x_A = y_A = 0$.

At the same time we shall consider, however, that the aperture extends along the axis x from $-a_1$ to $+a_2$, along the axis y from $-b_1$ to $+b_2$, that is, the aperture may possibly be asymmetrically disposed relative to the line OA, the screen's plane remaining, however, perpendicular to the line OA. Formulas (11.8), (11.8a) will then remain valid, but u_1, u_2, v_1, v_2 must then be recognized as somewhat different from (11.5), namely:

$$u_i = \pm a_i \sqrt{\frac{k}{\pi}\left(\frac{1}{r_0}+\frac{1}{\rho_0}\right)},\quad v_i = \pm b_i \sqrt{\frac{k}{\pi}\left(\frac{1}{r_0}+\frac{1}{\rho_0}\right)},\quad i = 1, 2. \tag{11.9}$$

The proximity of attenuation functions to the unity is determined by arguments' u_1 and v_1 excess over unity. As is easy to see, these numbers simply show how many zones can be fitted from the line OA in the direction of each of the axes to the corresponding boundary of the aperture.

The number m_1 of such zones is indeed determined by a formula analogous to Formula (10.41):

$$k[(r_i+\rho_i)-(r_0+\rho_0)] = m_i\frac{\pi}{2}, \tag{11.10}$$

where r_1, ρ_1 are respectively the distances from the points O and A to the corresponding edge of the aperture. Expanding r_1 and ρ_1 in series, we shall exactly obtain therefrom ($r_i = \sqrt{r_0^2+x_i^2+y_i^2}$, $\rho_i = \sqrt{\rho_0^2+x_i^2+y_i^2}$):

$$k\frac{x_i^2+y_i^2}{2}\left(\frac{1}{r_0}+\frac{1}{\rho_0}\right) = m_i\frac{\pi}{2},$$

and, consequently, respectively

$$u_i = \pm\sqrt{m_i} \text{ or } v_i = \pm\sqrt{m_i} \tag{11.11}$$

for the points $x_1 = a_1$, $y_1 = 0$ or $x_1 = 0$, $y_1 = b_1$. *The argument of the attenuation functions is simply the square root of the number of Fresnel zones that fit in the aperture plane in the direction of the given axis between the direct ray OA and the boundary of the aperture* (this number may be different in the directions x and y).

Thus, the attenuation factor depends on the distances to the screen and on the dimensions of the aperture, taken in a specific combination.

Let us figure that, having taken the points O and A in free space for two foci, we will construct a system of confocal ellipsoids

$$k(r+\rho)-kD=m\frac{\pi}{2} \tag{11.12}$$

(here D is the distance between the points O and A; r, ρ are the distances of the current point to each of them), imparting to m in succession the values 1, 2, ... (Fig. 11.5). If kD is very great, we shall obtain at the outset very prolate ellipsoids.*

Let us place perpendicularly to the line OA an opaque screen with a transparent aperture in it in such a way that the axis OA pass through the screen. The diaphragm will condition a certain distortion at the observation point, determined by the number of zones encompassed by the aperture, as has just been derived by us. Consequently, if we displace the screen along the line OA so that the dimensions of the aperture change simultaneously, allowing the transmission of the same zones at any position of the screen, the field observed will not vary.

Assume that the aperture is at the side of the axis. Then the observed field (this is the field in the geometrical shadow) will be found to be very weak. If, however, by shifting the screen the aperture is deformed in such a way that its edges slide along invariable ellipsoids, the field will remain invariable in this case too. This may be considered as a certain similarity law.

Note still one more peculiarity of Formulas (11.6). They have a form as if each of the aperture edges induced a certain perturbing effect, the effects from opposite edges then adding up in a specific fashion. Meanwhile we started from the consideration of the action of the entire aperture area. This peculiarity is a particular case of the general theorem, according to which the aggregate effect of action of

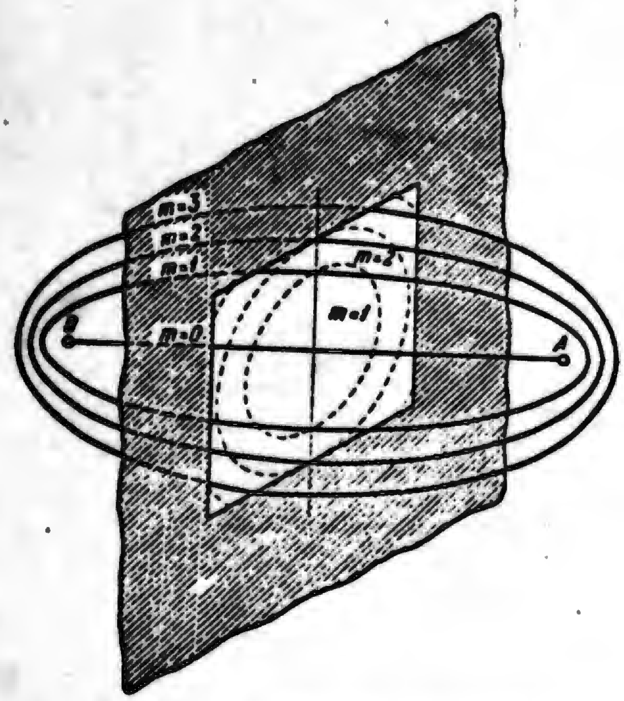

Fig. 11.5. Essential region in case of diffraction on a rectangular aperture.

virtual dipoles, disposed over a portion of the surface, may be always represented as a result of integration of a certain function over its periphery. Both representations are equivalent. Because of this additivity of actions of edges in case of eccentric position of the aperture, the predominating role will be played, generally speaking, by aperture regions adjacent to ellipsoids with the least m. This is why, upon transferring the aperture from one point to another, we shall, generally speaking, obtain a field amplification, if as a result of transfer, ellipsoids with m smaller than prior to transfer hit the aperture, and, vice versa, attenuation of the field will set in if the least m for ellipsoids encompassed by the aperture, should rise.

§12. REGION ESSENTIAL FOR REFLECTION

The cases of passage through apertures are of little interest to us in the theory of radiowave propagation. However, the notion of zones conserves even here its significance when determining the region, essential for the formation of a perturbation.

Let us consider, for instance, the field at the point $A(x_A, y_A, z_A)$ above the surface of a uniform and plane ground, when the source is located at the point $O(x_0, y_0, z_0)$, also above the interface. We may, without any limitation of the generality, draw the axis of coordinates in such a way that we have $x_0 = 0$, $y_A = y_0 = 0$.

As previously, we shall apply Formula (10.1), understanding by integration volume the entire space above the interface between the two media. The surface S consists of (plane) ground surface and of an infinitely remote hemisphere resting upon the interface and including in itself the entire open half-space.

In this case, contrary to the former ones, the integral over field sources does not drop off. According to Formula (10.1) the field $\Pi_0(A)$, which would have taken place for the given distribution of sources in a uniform atmosphere (in the absence of ground), and of a field, emitted

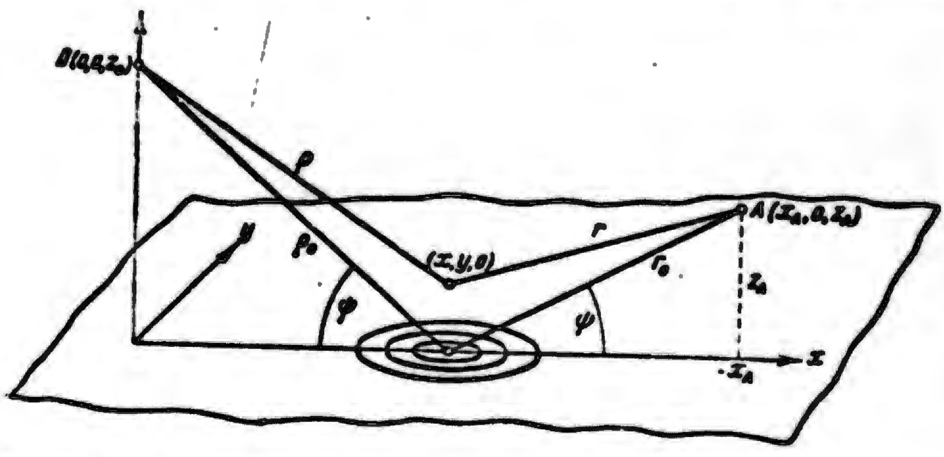

Fig. 12.1. Essential region at reflection from a plane, when both corresponding points are raised.

by virtual dipoles (and quadrupoles) distributed over the surface S. The integral over the hemisphere vanishes; as to that over the ground surface it provides the "reflected" field, which is superimposed on the field directly arriving from the source. This integral will always contain a product of the type $(e^{ikr}/\vec{r})\vec{\Pi}$, no matter how the Green function is selected by us.

The field $\vec{\Pi}$ on the ground may always be considered to be the product of an unperturbed field at the given point $\Pi_0 = p\frac{e^{ik\rho}}{\rho}$ and a slowly varyinf function – the attenuation factor $\underline{w}$. In this case we again are concerned with an integral from rapidly oscillating function of a well known type. This is why we may again attempt to isolate the most substantial region of the surface.

Let us consider the exponent of the oscillating function $i\varphi = ik(r+\rho)$ on the integration surface (x, y) (Fig. 12.1), and find the point x_0, y_0 at which it attains the extreme.

Since (see Fig. 12.1)

$$r = \sqrt{(x_A - x)^2 + y^2 + z_A^2}\,,\ \rho = \sqrt{x^2 + y^2 + z_0^2}$$

the extreme's conditions have the form

$$\left.\begin{aligned}\left(\frac{\partial\varphi}{\partial x}\right)_{\substack{x=x_0\\ y=y_0}} &= k\left(\frac{x}{\rho_0} - \frac{x_A - x}{r_0}\right)_{\substack{x=x_0\\ y=y_0}} = 0.\\ \left(\frac{\partial\varphi}{\partial y}\right)_{\substack{x=x_0\\ y=y_0}} &= k\left(\frac{y}{\rho_0} + \frac{y}{r_0}\right)_{\substack{x=x_0\\ y=y_0}} = 0.\end{aligned}\right\} \qquad (12.1)$$

Consequently, first of all, the extreme point is situated on the axis $\underline{x}$ ($y_0 = 0$); secondly, at this point

$$\frac{x_0}{\rho_0} = \frac{x_A - x_0}{r_0} = \cos\psi, \qquad (12.2)$$

"the angle of incidence is equal to the angle of reflection" (ψ is the gliding angle). Therefore, the essential region lies near the correct mirror reflection (x_0, y_0, 0). Let us investigate the behavior of φ

near it. To that effect we postulate

$$\xi = x - x_0,\ \eta = y - y_0 \tag{12.3}$$

and, considering $\xi, \eta \ll r, \rho$, let us expand r and ρ in series, limiting ourselves to the terms of second order in ξ and η:

$$\begin{aligned} r &= \sqrt{(x_A - x_0 - \xi)^2 + \eta^2 + z_A^2} = \\ &= \sqrt{[(x_A - x_0)^2 + z_A^2]\left[1 + \frac{-2(x_A - x_0)\xi + \xi^2 + \eta^2}{(x_A - x_0)^2 + z_A^2}\right]} \approx \\ &\approx r_0 - \xi\cos\psi + \frac{\xi^2\sin^2\psi + \eta^2}{2r_0}, \end{aligned} \tag{12.4a}$$

$$\rho \approx \rho_0 + \xi\cos\psi + \frac{\xi^2\sin^2\psi + \eta^2}{2\rho_0}. \tag{12.5}$$

Thus, the phase has the form

$$\varphi = k(r + \rho) \approx k\left\{(r_0 + \rho_0) + \frac{\xi^2\sin^2\psi + \eta^2}{2}\left(\frac{1}{r_0} + \frac{1}{\rho_0}\right)\right\}, \tag{12.5}$$

and the extreme point is the point of phase minimum. The lines of equal phase $\varphi(\xi, \eta) = \text{const}$ have the form of ellipses described around the point of correct reflection. The oscillating factor changes the sign of either the real or the imaginary part when transiting through the ellipse

$$\frac{k}{2}\left(\frac{1}{r_0} + \frac{1}{\rho_0}\right)(\xi^2\sin^2\psi + \eta^2) = m\frac{\pi}{2}, \quad m = 1, 2, \dots \tag{12.6}$$

These equations may be rewritten as follows

$$\frac{\xi^2}{\dfrac{\pi m}{k\sin^2\psi\left(\dfrac{1}{r_0} + \dfrac{1}{\rho_0}\right)}} + \frac{\eta^2}{\dfrac{\pi m}{k\left(\dfrac{1}{r_0} + \dfrac{1}{\rho_0}\right)}} = 1, \tag{12.6a}$$

or

$$\frac{\xi^2}{m\dfrac{a_0^2}{\sin^2\psi}} + \frac{\eta^2}{m a_0^2} = 1, \tag{12.6b}$$

where a_0 is the dimension of the first Fresnel zone at passage of light through the aperture (10.41b).

The semiaxes of these ellipses are:

along the axis x

$$a_m = \frac{1}{\sin\psi}\sqrt{\frac{\pi}{k}\frac{m}{\frac{1}{r_0}+\frac{1}{\rho_0}}} = \sqrt{m}\,\frac{a_0}{\sin\psi};$$

along the axis y

$$b_m = \sqrt{\frac{\pi}{k}\frac{m}{\frac{1}{r_0}+\frac{1}{\rho_0}}} = \sqrt{m}\,a_0;$$

$$m = 1, 2, \ldots \quad (12.7)$$

These ellipses serve as boundaries of the zones for reflection. At gliding incidence, when ψ is small, $a_m \gg b_m$, that is, the ellipses are strongly prolate along the axis x, i.e., along the propagation direction. But if a and b are small by comparison with the intervals over which the remaining ("slowly varying") comultipliers entering into the integral undergo notable variations, the rapidly oscillating function $e^{i\varphi}$ determines the integration region that is most essential. The integration over the plane (x, y) may be performed in variables (ξ, η), and then the factor

$$e^{i\varphi} = e^{ik(r_0+\rho_0)}e^{\frac{ik}{2}\left(\frac{1}{r_0}+\frac{1}{\rho_0}\right)(\xi^2\sin^2\psi+\eta^2)} \qquad (12.8)$$

determines for such an essential region the area of the plane near the point of correct reflection, encompassing the first zones. Taking out of the integral the slowly varying factors taken at the point $\xi = \eta = 0$, we shall again obtain the Fresnel integrals.

If, for example, we consider as essential the first eight zones, $m = 8$ (indeed, when the argument F(v) is equal to $v = \sqrt{m} = \sqrt{8}$, the value of the integral differs, according to Formula (10.13b), from its value at infinite limits by the quantity

$$\left|\frac{\frac{i}{\pi v}e^{i\frac{\pi}{2}v^2}}{\sqrt{\frac{i}{2}}}\right| = \frac{1}{\pi}\sqrt{\frac{2}{m}} = \frac{1}{2\pi} \approx 0{,}16,$$

that is, only by 16%), the dimensions of the essential region constitute along the axis x

$$2a_0 = \frac{4}{\sin\psi}\sqrt{\frac{r_0\rho_0}{r_0+\rho_0}\lambda} \sim \frac{4}{\sin\psi}\sqrt{R\lambda},$$

and along the axis y

$$2b_0 \sim 4\sqrt{R\lambda},$$

where R is the smallest of the two distances, r_0 and ρ_0. These reasonings loose strength in two cases: firstly, if either r_0 or ρ_0 becomes of the order $\sqrt{R\lambda}$ (or, which is the same, of the order λ), for in this case the value of ξ/R is not small and the expansion of the phase (12.4) is invalid; secondly, if the attenuation factor for the ground is such that it varies notably within the limits of one zone. The oscillating factor does not appear to be more determining in either case.

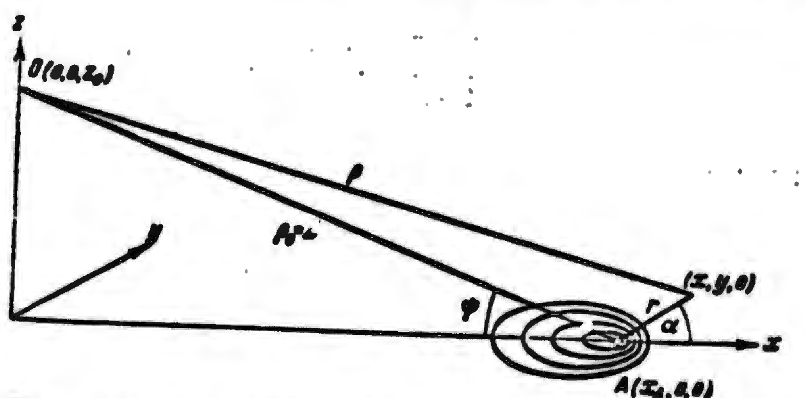

Fig. 12.2. Essential region at reflection, when one of the corresponding points has been lowered to the plane.

The second limitation is related to the properties of the soil. We shall deal with it subsequently (Chapter 7) and ascertain that it is immaterial in numerous practically encountered cases. As to the first limitation, it bears a purely geometric character and may be already considered now.

The zone may acquire a dimension of the order R, provided one of the corresponding points, O or A draws nearer to ground. Let us consider the extreme case: assume A to be situated quite near the ground, so that we may estimate $z_A = 0$. In this case $r + \rho$ attains the least value at the point A, that is, when $x = x_A$, $y = y_A$. Let us postulate $y_A = 0$ and introduce for the points near A polar coordinates with the center at A (Fig. 12.2)

$$x = x_A + r\cos\alpha, \quad y = r\sin\alpha. \qquad (12.9)$$

Expanding ρ and the whole phase in series by r near its minimum value, we obtain, limiting ourselves to terms linear relative to r,

$$\rho = \sqrt{x^2 + y^2 + z_0^2} = \sqrt{(x_A + r\cos\alpha)^2 + y^2 + z_0^2} \approx \rho_0 + r\cos\alpha\cos\psi,$$

$$i\varphi = ik(r + \rho) \approx ik\rho_0 + ikr(1 + \cos\alpha\cos\psi).$$

At integration over the plane in polar coordinates r, α, the factor $e^{i\varphi}$ emerges again as a fast oscillating multiplier, determining the most essential region. The boundaries of the zones are determined by the condition

$$kr(1 + \cos\alpha\cos\psi) = m\frac{\pi}{2}, \quad m = 1, 2, \ldots;$$

$$r = \frac{m\pi/2k}{1 + \cos\alpha\cos\psi}. \qquad (12.10)$$

This is the ellipse equation with focus at the point $r = 0$ (at the point of observation A) and with the eccentricity $\varepsilon = \cos\psi$. It is determined by the sliding angle of radiowaves. The major semiaxis of the ellipse is directed along the axis x and equal to

$$a_m = \frac{\frac{m\pi}{2k}}{1 - \varepsilon^2} = \frac{m\pi}{2k\sin^2\psi}. \qquad (12.11)$$

The minor semiaxis is directed along the axis y and is equal to

$$b_m = \sqrt{1 - \varepsilon^2}\, a_m = \frac{\pi m}{2k\sin\psi}. \qquad (12.11a)$$

These ellipses are strongly prolate in the direction toward the field source (see Fig. 12.2). Counting along the axis x in the direction at the source, the distance from to consecutive ellipses is

$$(r)_{\alpha=\pi} = \frac{m\pi}{2k(1 - \cos\psi)}.$$

In the direction from the source it is

$$(r)_{\alpha=0}=\frac{m\pi}{2k(1+\cos\psi)}.$$

In the transverse direction the dimension of the ellipse near the observation point is also fairly small: at $\alpha = \pi/2$ we obtain $r = m\pi/2k = m\lambda/4$.

Thus, at gliding fall (the low-placed transmitter, ψ is small), the first zones may already encompass a great part of the course on the segment between 0 and A. Outside this segment ("beyond the limits of the course") the ellipses converge quite closely, the distance between their boundaries having the order $(\cos\psi\approx 1)$

$$\frac{\pi}{4k}=\frac{\lambda}{8}=\frac{\pi}{4}\lambda, \qquad (12.12)$$

that is, the order of one-eighth of the wavelength.

Thus, estimating the essential region by the number $m = 8$, we shall obtain for $\psi = 15°$, its dimensions in the direction toward the transmitter equal to $\frac{8\pi}{2k}\frac{1}{\psi^2/2}\approx 64\lambda$, and from the transmitter $\sim\lambda$; at $\psi = 30°$, we shall respectively obtain 16λ and $\sim\lambda$, etc.

It should, however, be noted that if the details of the reflected ray offer no interest to us, the minimum required area for reflection may already be estimated from the values of the dimensions of one-two first ellipses if, for example, we postulate $m = 2$.

These considerations have a significance in case, for example, of selection of the area for radar installation (Leontovich, 1942). The radiowaves received after reflection from the target may undergo substantial distortions if the ground surrounding the receiver is insufficiently smooth and uniform. Obviously, similar requirements must be set forth mainly to a few first zones (this is naturally valid only in the case when no sharp irregularities, such as high houses, masts, etc., exist over more substantial distances, for reflections from them might combine with the useful signal).

We may see that the region behind the receiver is not as essential as that ahead of it. As the object draws farther away, when ψ decreases, it becomes more and more difficult to satisfy the conditions of good work. Hence may follow a limitation of the installation's range of action.

It is evident that the surface acts as a plane only in the case, whereby it does not diverge from the plane within the bounds of the essential region (to be more precise, when it diverges in such a way that the path length of the rays varies by less than λ on account of the divergence). In particular, a correct mirror reflection over a well reflecting surface may occur only in the case when its plane area near the point of reflection encompasses if only a few Fresnel zones. Similar considerations help to estimate, for example, in which cases a hilly surface or the surface of separate areas of sea waves may be replaced by a correspondingly inclined plane surface.

The formation of bright shines on snow may serve here as an example known to all. In order to be able to see from a height of the order of man's size (~200 cm) the correct reflection of solar rays, ($\lambda \approx 5 \cdot 10^{-5}$ cm) while the Sun is not too high, from a group of conglomerate snow crystals, it is necessary that the dimension of the plane of such a group be of the order of several zones, that is, of the order

$$\sqrt{5 \cdot 10^{-5} \cdot 2 \cdot 10^{2}} \approx 0{,}1 \text{ см}.$$

It is obvious that such a case materializes rather often.

Finally, let us pass to the consideration of the configuration of zones forming when both, the transmitting and the receiving points are disposed at quite low heights, at limit – on the plane $z = 0$. In this case the boundaries of equiphase zones on the plane are determined by the same requirement:

$$k(r+\rho)=\text{const}, \tag{12.13}$$

where ρ, r are the rays traced from two points O and A of the same plane. But this is the equation of an ellipse with foci at O and A. Since the least value of the constant is attained at $r+\rho=x_A$, the boundaries of the zones are defined by the condition

$$k(r+\rho)=kx_A+m\frac{\pi}{2},\quad m=1,2,\ \ldots . \tag{12.13a}$$

The major semiaxis of the mth ellipse is equal to

$$a_m=\frac{\rho+r}{2}=\frac{x_A}{2}+m\frac{\pi}{4k}=\frac{x_A}{2}+m\frac{\pi}{4}\lambda. \tag{12.14}$$

Behind the points O and A these ellipses draw quite closely to one another and they are quite elongated. In reality, the minor semiaxis (Fig. 12.3; see the footnote to page 103; the scale has not been observed in this drawing either) is equal to

$$b_m=\frac{r+\rho}{2}\sin\alpha=\frac{r+\rho}{2}\sqrt{1-\left(\frac{x_A}{r+\rho}\right)^2}=\sqrt{\frac{m\pi}{4k}\left(x_A+\frac{m\pi}{4k}\right)}. \tag{12.15}$$

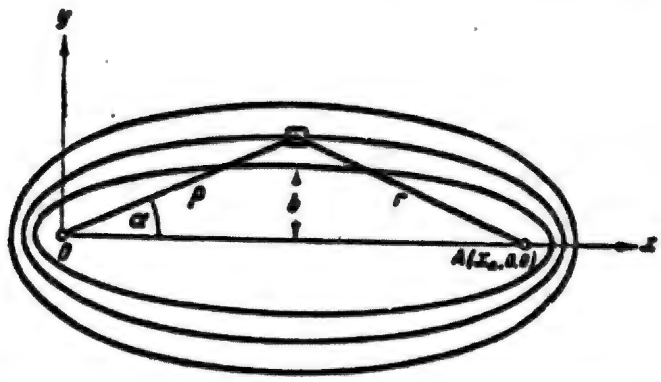

Fig. 12.3. Essential region at reflection, when both corresponding points have been lowered on the plane.

For small m

$$b_m\approx\sqrt{\frac{m\pi}{4k}x_A}=\sqrt{\frac{m}{8}\lambda x_A}=\sqrt{\frac{m\pi}{4}\lambda x_A}. \tag{12.15a}$$

In this way the transverse dimensions of the elliptic zone are determined by relations well known to us. As always, if $x_A \gg \lambda$, which evidently is assumed, $b_m \ll x_A$, that is, the zone is narrow.

It is obvious that the first several ellipses, encompassing the points of disposition of the receiver and transmitter by a narrow loop, determine again the essential region. The distances between the consecutive ellipses behind the receiver and the transmitter are quite small. They are determined by the accretion of the quantity $\frac{1}{2}(\rho + r - x_A) = \frac{m\pi}{4k} = m\frac{\lambda}{8}$, that is, they constitute one-eighth of the wavelength. The distance between ellipses near 0 and A in the transverse direction is also small. Thus, the distance q_m along the axis y from 0 to the mth ellipse is determined by the relation $(\rho = q_m)$

$$k\left\{\sqrt{x_A^2 + q_m^2} + q_m\right\} = kx_A + m\frac{\pi}{2}.$$

Since $q_m \ll x_A$, we hence obtain, neglecting $q^2m/2x_A$ by comparison with q_m,

$$q_m = \frac{m\pi}{2k} = m\frac{\lambda}{4}. \qquad (12.15b)$$

These considerations allow us to introduce an important, though somewhat conditional *idea of radiowave course. The region encompassed by the first* (or one of the first) *ellipse* may be regarded as being such a region.

Thus we have found the essential zones for three cases: when both corresponding points are disposed sufficiently high, when one of them has been lowered to ground and when they both are on the ground. Unifying all these cases we may proceed in the following fashion.

At arbitrary position of the points 0 and A in space above ground we must figure a system of confocal ellipsoids of revolution in space, having their common foci in 0 and A and being determined by the equation

$$k(r+\rho)=kD+M\frac{\pi}{2}, \quad M=1,2,\ \ldots, \qquad (12.16)$$

where D is the distance between the points O and A, and r and ρ are the distances from A and O to the current points in space. Starting from a certain $M = M_1$, these ellipsoids will be intersected by the ground surface. The point where the first contact takes place will be the one in which takes place the correct reflection

$$k(r_0+\rho_0)=kD+M_1\frac{\pi}{2}$$

(according to the properties of the ellipse, the rays, traced from the foci to any of its points, form identical angles with the normal), and the subsequent values $M = M_1 + m$, $m = 1, 2, \ldots$, provide in the cross section with the ground surface the contours of Fresnel zones (we disregard here the fact that M_1 is not a whole number).

If the heights of the points O and A are equal, concentric ellipses will be obtained in the cross section (for small m) (Fig. 12.4a). But if either one or both points are situated on the ground, the cross sections will have the shape of ellipses with respectively either one (Fig. 12.4b) or two (Fig. 12.4c) common foci.

As already stressed, it is obvious that the remark concerning the preferential value of the first zone region should be understood conditionally: they are more substantial under other equal conditions. But if, for example, particularly well conducting portions of the surface are encountered in the more remote zones (this means, as will be seen further, that they are particularly well reflecting), they may exert in known cases a notable influence. The same refers to particularly large towers, masts, mountains, etc., situated in the more remote zones (see above).

In conclusion we shall enumerate once more the formulas in which are shown the transverse and the longitudinal dimensions of the essential zones (ellipses): for both corresponding points raised – (12.7); for one point raised and the other lowered to ground – (12.11) and (12.11a); for both points on the ground – (12.14), (12.15) and (12.15a).

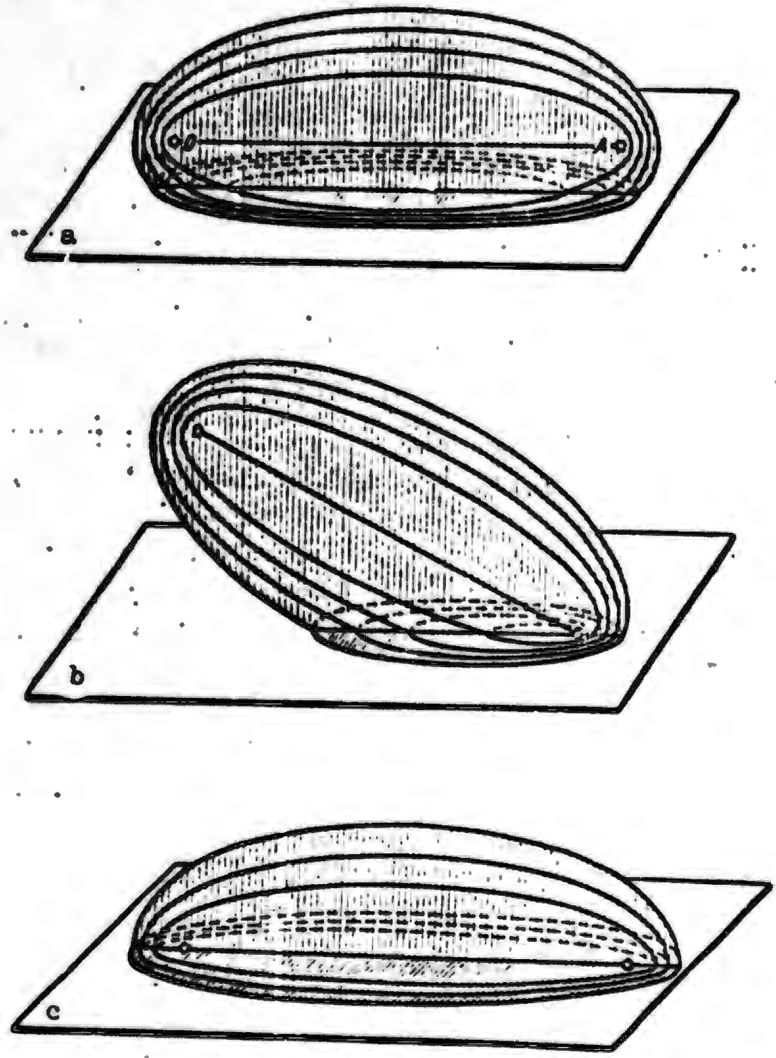

Fig. 12.4. Ellipsoids of the Fresnel zones at reflection. a) When both corresponding points are raised over the plane; b) when one of the points is lowered to the plane; c) when both corresponding points are located on the plane.

§13. LIMITS OF APPLICABILITY OF THE KIRCHHOFF APPROXIMATION IN THE CASE OF DIFFRACTION ON A SCREEN

We shall consider as an example of utilization of the conception of the most essential region the question as to when the Kirchhoff method, applied by us when reviewing the diffraction from an aperture, is valid.

To that effect we shall consider the simplest case, that is, the diffraction from a rectilinear edge of a semi-infinite screen (Fig. 13.1), when the source is located at the point O, and the observer somewhere near the boundary of the geometric shadow, or deeper in the shadow, at the point A_1.

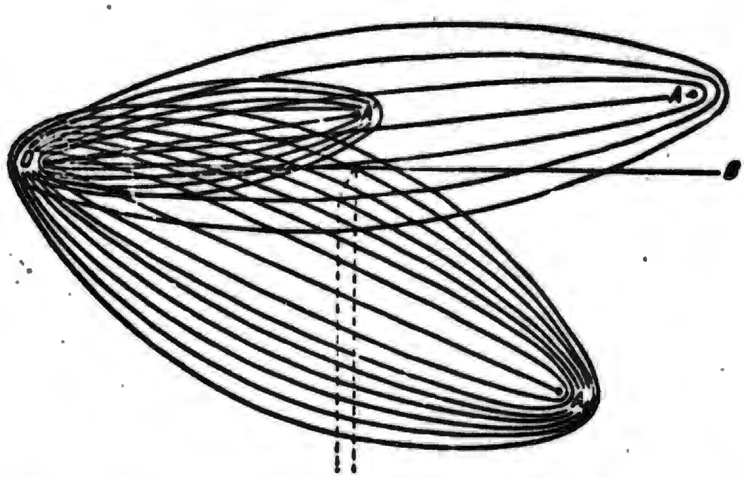

Fig. 13.1. Role of various Fresnel zones for various positions of the point of observation.

When computing by the Kirchhoff method, it is assumed that the field is zero on its dark side, whereas in the plane of the aperture the field is equal to the unperturbed field

$$\Pi = \Pi_0 = p\frac{e^{ik\rho}}{\rho}, \qquad (13.1)$$

that would take place here had the screen been absent. In reality, near the screen's edge the field in the aperture plane is always distorted. Let us examine this distortion.

First of all we shall ascertain to what extent the assumption that the field is unperturbed in aperture plane, is valid.

In accord with the Huygens principle, the field forms at a certain point of the aperture A', distant from the edge of the screen by the quantity d, by waves reaching here from various points. Most essential are the points of space lying within the limits of the first ellipsoids, constructed with the points O and A' as foci (see Fig. 13.1).

The distance to the mth ellipse in the aperture plane is determined by Formula (12.15b). From it we may see that near A' the ellipses converge quite closely to one another, a distance of the order λ. This is why we may assert that if the point A' is removed from the edge of the screen by a distance, substantially exceeding the wavelength λ, the field in it is practically unperturbed by the presence of the screen. Th is, in particular, takes place in the case described in Fig. 13.1.

Indeed, direct calculation, carried out in the assumption that this perturbation is small, shows ([I.1], pp. 90-91, Formula (13.8)) that at the distance d from the screen's edge the field in the aperture plane is

$$\Pi \approx p \frac{e^{ik\rho_0}}{\rho_0} \left\{ 1 + \frac{e^{i\left(kd - \frac{3\pi}{4}\right)}}{\sqrt{2\pi kd}} \right\}. \qquad (13.2)$$

Therefore, the correction for an unperturbed field has already the order $d \sim \lambda$ at a distance $(2\pi)^{-1} \sim 15\%$.

It remains to be clarified, to what extent is manifest at the observation point, that is, at the point A or A_1 behind the screen, the fact that the field in the aperture, near its edge, still remains perturbed. To that effect let us construct the first substantial ellipsoids with points O and A, and also with points O and A_1 for foci. These ellipsoids define the region of space which participates in the formation of the field respectively at the points A and A_1.

In the aperture plane the dimensions of the first zones for the point A closely situated to the geometrical boundary of the shadow, are rather great; they have an order $\sqrt{R\lambda} \gg \lambda$, where R is the least of the distances from the points O and A to the screen. If, for example, as is shown in Fig. 13.1, the first such zone touches the screen by its edge only, the field is in truth unperturbed over nearly the entirety of that zone. It cannot be given by Formula (13.1) only in its marginal part, at a distance of the order λ. But this region constitutes only a fraction of the order $\lambda/\sqrt{R\lambda} = \sqrt{\lambda/R} \ll 1$ of the entire area of the zone. Inasmuch as all the points of one zone emit nearly cophasal fields, the error hence emerging is correspondingly small.

In this way one may consider that for points of observation near the boundary of the shadow (and, <u>a fortiori</u> in a well "illuminated" region, considerably above the line OB in Fig. 13.1) the application of the Kirchhoff method is justified at distances $R \gg \lambda$.

If, however, the point of observation is located deep in the shadow (point A_1), the distance between the essential ellipsoids (dimensions of the essential zones in the aperture plane) may become small. It corresponds to large numbers <u>m</u> and, according to Formulas (10.42b), (10.41a), it is

$$\Delta a \sim \frac{\pi}{2k} \frac{R}{\sqrt{\frac{\pi}{2} m R}} \sim \sqrt{\frac{R\lambda}{m}}.$$

It is thus large by comparison with λ only in the case when $R \gg m\lambda$. Consequently, deep in the shadow the Kirchhoff method may prove to be inaccurate. This also is seen from the simpler construction brought out in Fig. 13.2. The rays, traced here from the point of observation to that of the aperture differ in their wavelength. Consequently, the segments cut by them on the axis <u>x</u> define the Fresnel zones. These zones have themselves dimensions of the order λ, provided the angle between the rays and the plane of the screen β is not near $\pi/2$. Consequently, the field in the entire first of the essential zones is distorted by the presence of the screen, and this is why the Kirchhoff method provides the field in A incorrectly. This will happen when

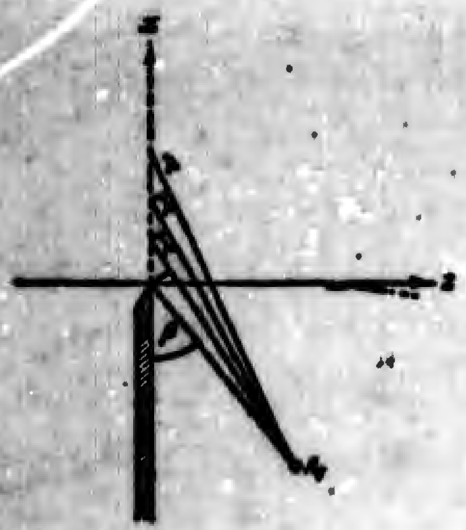

Fig. 13.2. Diffraction over great angles.

$$\frac{\pi}{2}-\beta\sim 1.$$

Note that the electrodynamic problem of diffraction on a semi-infinite screen is fully resolved in [9] and the results may be compared with the approximate solution by the Kirchhoff method. The comparison confirms the validity of the criterion derived by us:* for a semi-infinite screen the Kirchhoff method is inapplicable only at $(\pi/2) - \beta \sim 1$.

We have not discussed the second condition of the Kirchhoff method, namely that of field vanishing on the shadow side of the opaque part of the screen. It may be shown by analogous computations that its consideration adds nothing new.

§14. ESSENTIAL REGION AT REFRACTION

Let us now consider briefly the question which boundary region of the interface between two media is essential at passage of radiation from one medium to the other. We shall analyze this question only for the case when the wave number k in one medium (say, in that where the observation point is located) is much greater by module than in the other.

We shall consider the field in the medium 2, characterized by the constant k, for example, in the soil, when the source is disposed in the medium 1, characterized by the wave number $k_0 \ll |k|$, for example, in the air. In this case the radiation arrives on the surface, spreading mostly in the medium 1, whereas the waves, arriving through the soil, damp rapidly (this will be shown in §31).

If the source is placed at the point 0 $(0,0,z_0)$, the field at a certain point A $(x_A, 0, z_A = -d)$, may be expressed by Formula (10.2), more than once applied by us, having assumed for the surface S the plane of the interface $z = 0$, complemented by the infinite hemisphere that closes the lower half-space (Fig. 14.1):

$$\Pi^{(2)}(x_A, 0, -d) = -\frac{1}{2\pi}\int \Pi^{(2)}(x, y, -0)\frac{\partial}{\partial z}\frac{e^{ikr}}{r}dS, \tag{14.1}$$

$$r = \sqrt{(x_A - x)^2 + y^2 + d^2}, \quad dS = dx\,dy, \quad k = k_1 + ik_2.$$

By the strength of boundary conditions, $\vec{\Pi}^{(2)}$ are determined on the plane $z = -0$ by the values of $\vec{\Pi}^{(1)}$ at the corresponding points on the other side of the surface. This is why $\vec{\Pi}^{(2)}$ varies substantially at shift along the surface $z = 0$ on the segment of the order of the wavelength in the air $\lambda_0 = 1/k_0$, while e^{ikr} may oscillate (or exponentially decrease) at shift along the surface on an interval of the order of the width of the corresponding Fresnel zone, that is of the order $\sqrt{|\lambda| d}$. It may be considerably less than the wavelength in the air if we limit ourselves to not too great depths d, $d \ll \left|\frac{k}{k_0}\right|\lambda_0$ (which are, however, still great by comparison with the wavelength in the soil). Let us break down the exponent into real and imaginary parts:

$$ikr = ik_0 r\sqrt[4]{\varepsilon'^2 + \left(\frac{4\pi\sigma}{\omega}\right)^2}\cos\frac{\chi}{2} - k_0 r\sqrt[4]{\varepsilon'^2 + \left(\frac{4\pi\sigma}{\omega}\right)^2}\sin\frac{\chi}{2}, \tag{14.2}$$

$$\chi = \operatorname{arctg}\frac{4\pi\sigma}{\varepsilon'\omega}.$$

Two cases are possible:

a) The soil may be "almost nonconducting," $\frac{4\pi\sigma}{\varepsilon'\omega} \ll 1$. This takes place for ultrashort waves. In such a case the exponential decrease of the factor e^{ikr} may be small, and the principal role will be played by oscillations.

b) If the soil is sufficiently high conducting or the frequencies sufficiently low, $\frac{4\pi\sigma}{\varepsilon\omega} \gg 1$, then the exponential and the oscillatory dependences will be manifest over segments of single order.

In any case, there separates in the integral (14.1) as the essential region of the surface z = 0 a Fresnel zone near the point, where the exponent is minimum. This region lies near the point for which $r = d$.

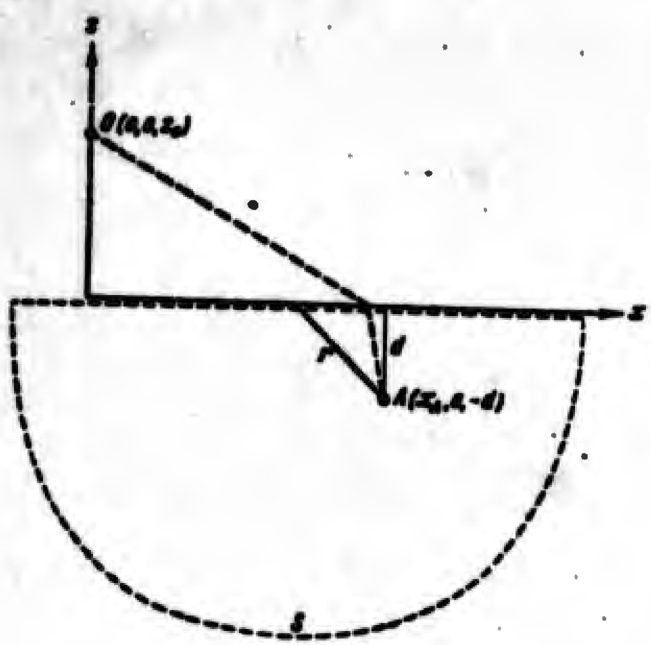

Fig. 14.1. Essential region at refraction.

Thus, at passage into a highly conducting (or "strongly dielectric"): $\varepsilon' \gg 1$) medium, the essential role at observation at points sunk to the depth d, not exceeding $\left|\frac{k}{k_0}\right|\lambda_0$, is played by the region of the surface situated above the observation point and having dimensions much smaller than the wavelength in the air. This is why it dispatches in its entirety nearly cophasal oscillations to the observation point.

Let us now obtain more rigorously the above-described results using Formula (14.1). We shall consider that the emitter is a vertical dipole and this is why $\vec{\Pi}$ is reduced to a single vertical component, and also that $x_A \gg z_0$ (comparatively low-disposed emitter), but still having $z_0|\sqrt{\varepsilon}| \gg \lambda_0$.

Applying the boundary condition (4.7), we shall express $\vec{\Pi}^{(2)}$ in (14.1) by $\vec{\Pi}^{(1)}$ on the surface of the soil in the air. It is not equal to the field in free space, for superimposed on it are the waves reflected from the interface and created by the currents, generated in the soil by the primary radiation incident from the air. We shall subsequently see §19) that owing to that the total field on the surface of the divide differs from the incident field only by the factor 1 + f, where f is a constant number, the reflection factor dependent on frequency, on the angle of incidence, on the polarization of the wave and on the properties of the soil and air. This is why we shall assume

$$\Pi^{(1)}(x, y, 0) = p(1+f)\frac{e^{ik_0\rho}}{\rho}, \quad \rho = \sqrt{x^2+y^2+z_0^2}. \qquad (14.3)$$

Further, since we may consider $|k|d \gg 1$, and for that reason, the more so $|k|r \gg 1$, we shall obtain from (14.1) and (4.7):

$$\Pi^{(2)}(x_A, 0, -d) = -\frac{ik}{2\pi}p(1+f)\int\frac{e^{i(k_0\rho+kr)}}{\rho r}dx\,dy. \qquad (14.4)$$

Here, we assume in correspondence with the above-expounded qualitative consideration that $\frac{\partial r}{\partial n} \approx 1$. For the same reasons we may take out of the integral ρ and r, taken at a certain effective point of the surface near the point of observation (it is determined below with more precision). Therefore, we again arrive at the integral

$$I = \frac{1}{r\rho}\int e^{i(k_0\rho+kr)}dx\,dy. \qquad (14.5)$$

which, however, differs from the earlier encountered in that $k \neq k_0$ and, moreover, $\underline{k}$ is complex.

The integration over $\underline{y}$ is easily performed by the former method. The phase extreme over $\underline{y}$ is disposed at $y = 0$, since the condition

$$\frac{\partial(k_0\rho + kr)}{\partial y} = k_0\frac{y}{\rho} + k\frac{y}{r} = 0 \qquad (14.6)$$

is satisfied only at $y = 0$. This is why we may expand the phase by $\underline{y}$, limit ourselves to the expansion's quadratic term and obtain with the aid of (10.13d)

$$I = \frac{1}{r\rho}\int e^{i\left(k_0\sqrt{x^2+z_0^2}+k\sqrt{(x_A-x)^2+d^2}\right)} \times \frac{\sqrt{2i\pi}}{\sqrt{\frac{k_0}{\sqrt{x^2+z_0^2}}+\frac{k}{\sqrt{(x_A-x)^2+z_A^2}}}}\,dx. \qquad (14.7)$$

The integration over $\underline{x}$ is not as simple, inasmuch as the phase extreme is disposed at complex $\underline{x}$. We shall somewhat simplify the further operations (they might be conducted by exactly the same scheme without this simplification), if we took into account, first of all, that at $x_A \gg z_0$ we may postulate $\sqrt{x^2+z_0^2} = x + \frac{z_0^2}{2x}$, secondly, that because of the factor $\underline{k}$ in the second term of the exponent the effective values $|x_A - x| \ll d$, and this is why we may consider $\sqrt{(x_A-x)^2+d^2} \approx d + \frac{(x_A-x)^2}{2d}$. The extreme value $x = x_S$ is obtained, if the first phase derivative with respect to $\underline{x}$ is equated to zero in the form:

$$x_S = x_A - \frac{k_0 d}{\sqrt{k^2-k_0^2}} = x_A - \frac{d}{\sqrt{\varepsilon-1}}. \qquad (14.8)$$

If, as we consider, $|\varepsilon| = |k|^2/k_0^2 \gg 1$, we have $x_S \approx x_A$. Therefore, within the limit for very great $|\varepsilon|$ the quantity x_S is real and the most essential region of the interface is disposed exactly above the point of observation. However, at finite $|\varepsilon|$ the real part of x_S is somewhat displaced toward the side of the source (so much the more when $\underline{d}$ is greater and $|\varepsilon|$ is smaller), and, besides, the extreme point is somewhat shifted into the complex plane. This is why we deform the integration contour passing in the integral (14.7) along the materal axis $\underline{x}$, so that it pass through the point x_S, and we apply the saddle point method. First of all we shall search for the direction of the steepest descent. To that effect we postulate

$$x = x_S + \xi + i\eta, \tag{14.9}$$

where ξ and η are real, and we shall consider the whole function on the line passing in that direction (subject to determination). We shall namely postulate $\eta = \gamma\xi$, where γ is the tangent of the inclination angle of the line with axis ξ. Now, expanding the exponent in series by ξ, we shall obtain (neglecting at the same time the quantity k_0^2 in certain terms, which is small by comparison with k^2, and the quantity $k_0^2 d/2k$, small by comparison with the unity)

$$i\varphi = i\left(k_0\sqrt{x^2+z_0^2} + k\sqrt{(x_A-x)^2+d^2}\right) \approx i\left(k_0\sqrt{x_S^2+z_0^2} + k\sqrt{(x_A-x_S)^2+d^2}\right) + \frac{\xi^2}{4d}\{i[k_1(1-\gamma^2)-2k_2\gamma] - [k_2(1-\gamma^2)+2k_1\gamma]\}. \tag{14.10}$$

The direction of the steepest descent is determined by the fact that for it we have a decrease without oscillations. This is why we shall determine γ from the requirement

$$k_1(1-\gamma^2) - 2k_2\gamma = 0, \text{ т. е. } \gamma = -\frac{k_2}{k_1} + \sqrt{\frac{k_2^2}{k_1^2}+1}. \tag{14.11}$$

The sign before the radical was so chosen that at $k_2 = 0$ we obtain $\gamma = +1$ (at $\gamma = -1$ this would have been the direction of the most rapid rise, and not decrease for). In particular, at $k_2 \gg k_1$ we may consider $\gamma = k_1/2k_2$. Assuming further in the pre-exponential factors $x = x_S$, we finally obtain (at $\gamma \approx \frac{k_1}{2k_2}$, $|k\rho_S| \gg |kr_S| \approx |kd|$ etc.):

$$I = \frac{\sqrt{2\pi i}}{\sqrt{(k_0 r_S + k r_S)\rho_S r_S}}\, e^{i(k_0 r_S + k r_S)},$$

$$\times \int_{-\infty}^{+\infty} e^{-\frac{|k|^2}{2k_0}\gamma\xi^2}(1+i\gamma)d\xi \approx 2\pi i \frac{e^{i(k_0\rho_S + kr_S)}}{\rho_S}. \qquad (14.12)$$

Here $\rho_S = \sqrt{x_S^2 + z_0^2}$ и $r_S = \sqrt{(x_A - x_S)^2 + d^2}$ $(|r_S| \ll |\rho_S|)$ are complex "distances" from the source to the point $(x_s, 0, 0)$ and from that point to the point of observation. If in the formula (14.12) we divide the quantity $k_0\rho_S + kr_S$ into the real and imaginary parts, the former will provide the phase incidence over the path from the source to the most essential region of the interface lying above the point of observation, but displaced toward the side of the source, and then, from there into the point of observation (dashed line in Fig. 14.1). The imaginary part of that sum, proportional to d, provides the damping as it spreads in the soil from this region to the point of observation.

The width of the effective region of the surface may be estimated from the form of the integral over ξ in Formula (14.12); for $k_2 \gg k_1$ we hence obtain

$$\Delta\xi_{eff} \sim \Delta x_{eff} \sim \sqrt{\frac{k_0 d}{|k|^2\gamma}} \sim \sqrt{\frac{2d}{|k|}} \sim \sqrt{\frac{\lambda_0 d}{|\sqrt{\varepsilon}|}} = \sqrt{\lambda d}. \qquad (14.13)$$

Therefore, this width has the order of the width of the Fresnel zone relative to the "wavelength in the soil" $\lambda = 1/|k|$. Within the framework of the above reserved condition we have $d \ll \sqrt{|\varepsilon|}\lambda_0$; hence it follows that $\Delta x_{eff} \ll \lambda_0$.

The parts lying beyond this essential region will dispatch to the point of observation waves damping still more as a consequence of the long path. Their relative role will decrease exponentially with the rise of the number of their zone.

In case of purely dielectric soil the more remote zones will dispatch mutually extinguishable radiation, which, however, will not undergo exponential attenuation, and this is why their role will decrease more slowly with the rise of their number (inversely to the number, cf.

Formulas (10.13) and (10.29a).

However, in the wavelength band and soil properties of interest to us the conduction still takes place, and the exponential damping is found to be quite significant.

As ascertained by us, the width of the region dispatching cophasal oscillations has the order λ_0 and therefore, is much greater than the depth d of the point of observation. This is why we may visualize graphically that a plane wave arrives at the point of observation from the surface, damping as it propagates in the depth of the soil in the same fashion as at propagation in a uniform medium with properties of the soil.

Therefore, the field in the soil is determined by its values in the nearby-lying points of the surface, i.e., it is somehow tied up to the field in the air on the boundary of the interface and is carried over alongside with it. In particular, along the axis x the field in the ground varies notably on the segment of the order of the wavelength in the air.

Let us stress that these considerations are valid with certain reservations even in case of "almost dielectric" soil (or very high frequencies ω). It is true, that inasmuch as here the role of remote zones is reduced only by a rather weak exponential damping, it may appear that the variation of field strength in the air, for example, its amplification as it drifts toward the side of the source, would compensate the quenching action of the zones. A more detailed analysis (§31) ows, however, that it is only necessary that the wave, moving in the soil from the source, have time to damp (in other words, that the exponential damping have time to manifest itself). This condition is observed even at low conduction of the soil at sufficiently great distance from the source.

The considerations expounded lay at the basis of an extremely effective approximate method which was successfully applied to a great number of problems linked with the propagation of radiowaves above a surface with sufficiently great $|\varepsilon|$, or, in any case, at a sufficiently great distance from the source. We shall examine it in Chapter 4 and we will make use of it subsequently more than once.

Manuscript Page No. [Footnotes]

91 The regions defined by a somewhat different condition:

$$k[(r+\rho)-(r_0+\rho_0)]=m\pi,\ m=1,\ 2,\ \dots. \qquad (10.41^*)$$

are often called Fresnel zones. However, the determination (10.41) appears to us in certain respects as being more practical (see, for example, below (11.11)).

93 The accuracy of such a determination of an improper integral may be confirmed, on the one hand, by the fact that at such a determination the results is obtained coinciding with that of a knowingly correct calculation in rectangular coordinates; on the other hand, the case of real k is an idealization. We may always consider that the medium is always endowed with but quite low a conduction, as a result of which there will appear at k a small imaginary part $i\alpha$, assuring the convergence of the integral. It may be neglected after the computation.

94 This means that it is made of infinitely conducting material.

103 For the sake of clarity of construction, ellipsoids are drawn in Fig. 11.5, as well as in subsequent drawings, very wide, which corresponds to such a wavelength to distance OA ratio, that is not met with in the case of radiowave propagation. Usually the transverse dimensions of the first ellipsoids are quite small.

120 A detailed analysis of certain cases of diffraction and analysis of the admissibility of various approximations is included in the book by A.I. Potekhin [10].

[Transliterated Symbols]

126 эфф = eff = effektivnyy = effective

Chapter 3

PROPAGATION OF RADIOWAVES IN UNIFORM MEDIA AND REFRACTION ON THE PLANE BOUNDARY OF THE INTERFACE

§15. ELECTRICAL PROPERTIES OF MEDIA ENCOUNTERED IN THE THEORY OF RADIOWAVE PROPAGATION

1. The media, with which we are confronted in the theory of radiowave propagation along the ground are the soil (to which we may also refer the water of seas, lakes and rivers) and the atmosphere.

So long as practical interest was offered only by long, medium and shortwaves, it was considered during a long period of time that each of these media may be approximately regarded as uniform. Then it was shown that even in this band the presence of large-scale soil inhomogeneities (such as sea – dry land, etc.) may *qualitatively* modify the character of the process of radiowave propagation. Yet it was still felt that if we digress from the influence of the ionosphere, the air, of which the dielectric constant is under standard conditions $\varepsilon = 1.0003$, and the conduction is extremely small, may be identified with vacuum. However, as the role of shorter wave bands increased, it was still better ascertained that the inhomogeneities of the air play a fundamental role. Recognized first of all was the role of normal decrease of air density with height, which always enhances the penetration of radiowaves beyond horizon. Then the influence was revealed of various kinds of inversions in this altitude course. Under specific conditions, and namely, for ultrashort, decimeter and centimeter waves, stratified inhomogeneities in the form of waveguide channels may determine the ultraremote propagation. Finally, as of 1950 it was already found that irregular, turbulent inhomogeneities, always present in the atmosphere, influence most substantially the passage of decimeter and centimeter waves in both the limits of direct visibility, when the radiowave scattering induced by them hinders fundamentally the propagation, and beyond the horizon, where the scattered radiation penetrates, with the result that the field there is much stronger than might have been expected. It is not always easy to break down the influence of these factors (though for sufficiently long waves the situation is in any case simplified). Before studying their joint action it is appropriate to conduct a separate consideration, and analyze first of all the propagation of radiowaves in a uniform medium with properties corresponding to the mean properties of either the atmosphere or the soil.

2. The conduction of the lower layers of the atmosphere and the correspondingly imaginary part ε are extremely small. It is standard practice to consider them zero, $\varepsilon = \varepsilon'$. Because ε is not equal to the unity, the supplementary phase incidence over the path l is $\Delta\varphi = kl - k_0 l = k_0(\sqrt{\varepsilon} - 1)l$. Under standard conditions, when $\varepsilon = 1.003$, we have $\Delta\varphi = 2\pi \cdot 1{,}5 \cdot 10^{-4} \frac{l}{\lambda_0} \approx 10^{-3} \frac{l}{\lambda_0}$. Therefore, the phase shifts notably only over a path of the order $10^3\lambda_0$. However, in itself this shift cannot lead to anything substantial. It is equivalent to a certain variation of the distance l (by fractions of percent) and can be easily neglected. Only systematic or chaotic inhomogeneities ε may lead to the distortion of the propagation direction and to radiowave scattering.

The value of ε or of the refraction coefficient $n = \sqrt{\varepsilon}$ in the atmosphere is determined experimentally by special refractometers. It is dependent on the density of the air (in its turn dependent on temperature) and on water vapor content. The index of refraction is given by the following formula

$$n = 1 + \frac{79}{T}\left(p_{\kappa\delta} + \frac{4800 e_{\kappa\delta}}{T}\right)\cdot 10^{-6}. \qquad (15.1)$$

where $p_{\kappa\delta}$ and $e_{\kappa\delta}$ are the pressure of air and water vapor in millibars, T is the absolute temperature. The reduced index of refraction

$$N = 10^6(n-1). \qquad (15.2)$$

is oten utilized instead of n. Utilizing the barometric formula for the dependence of $p_{\kappa\delta}$ on the altitude h above the sea level, we may hence obtain for the sea level at T = 273°

$$\left(\frac{dn}{dh_{M}}\right)_{\substack{h=0\\T=273^\circ}} = -\left\{0{,}037 + (1{,}05 + 0{,}037 e_{\kappa\delta})\frac{dT}{dh_{M}} - 5{,}05\frac{de_{\kappa\delta}}{dh_{M}}\right\}\cdot 10^{6} \qquad (15.3)$$

(here the altitudes h_M are taken in meters).

The first term in braces describes the density decrease in the equilibrium dry atmosphere. Usually in dry (but nonequilibrium) atmosphere

$$\frac{dT}{dh_{M}} \sim -0{,}0098 \approx -0{,}01 \text{ deg/m}$$

According to international agreement the "standard atmosphere" or the "radioatmosphere" is the atmosphere in a state, in which at sea level

$$T = 273 + 15^\circ,\ \frac{dT}{dh_{M}} = -0{,}0065 \text{ deg/m},\ \frac{de_{\kappa\delta}}{dh_{M}} = -0{,}0033 \text{ mb/m},\ e = 10_{\kappa\delta}.$$

Hence is obtained the gradient of the index of refraction, constant in height

$$\frac{dn}{dh} = -4\cdot 10^{-8} M^{-1},\ \frac{d\varepsilon}{dh_{M}} = 2\frac{dn}{dh_{M}} = -8\cdot 10^{-8} M^{-1}. \qquad (15.4)$$

Similar conditions do indeed correspond to a certain averaged state of the atmosphere. However, in the altitude range of the order 10-16 km even the mean linear course of ε is disrupted (which is understandable, since here n attains the value of the unity). Besides, the entire pattern varies strongly with the season and is generally very often distorted. At certain altitudes gradient ε may differ in absolute value by several factors from that indicated by Formula (15.4), and at times it may even differ in sign. Meanwhile, the presence of levels with different signs of gradient ε induce the emergence of layers having relative to sufficiently short waves "waveguide" properties. Here a peculiar phenomenon sets in, of radiowave propagation inside such a layer, channeling them to quite large distances (see §57). Such inversions in the course of ε are often observed near sea shores, particularly in the tropics, and they are conditioned to a significant measure by water vapor shift. At the same time the presence of one-sign gradient (15.4) is already the cause of normal atmospheric refraction of radiowaves facilitating their penetration beyond horizon (see §56).

Small scale oscillations ε, which are also measured directly by the devices, are superimposed to the large-scale inhomogeneities of the atmosphere. Contrary to the latter, these oscillations fluctuate rapidly also in time. It is customary to relate them with the development of turbulence in the atmosphere. According to the statistical theory of uniform turbulence by A.N. Kolmogorov and A.M. Obukhov (see, for example, [1, 2], large-scale motions with great Reynolds number in a viscous medium must break up and pass to turbulent vortices of still smaller dimensions so long as the Reynolds number does not drop to the unity. After that the kinetic energy will no longer be expended on the creation of tinier vortices, but will pass to heat at the expense of viscosity. Thus, for the given density ρ and viscosity η the characteristic velocity v_s and the characteristic dimensions ℓ_s in the smallest possible vortices are determined by the condition

$$\mathrm{Re}=\frac{\rho v_s l_s}{\eta}=1. \qquad (15.5)$$

Here, as everywhere in such a treatment, $\underline{v}$ has both the order of a certain mean velocity in the vortex of the given scale and that of the characteristic difference of velocities over a distance of the order of the dimensions of the vortex l, $\overline{(\Delta v_l)^2} \sim \overline{v_l^2}$.

At birth of a large-scale vortex with dimensions l_0 and the characteristic velocity v_0, a kinetic energy of the order $T_0 \sim \frac{\rho v_0^2}{2}$ is introduced into the system per unit of volume. This may, for example, be the energy of the wind. Then v_0 is the wind velocity. The energy T_0, not being expended on heat, passes into smaller vortices; at the same time for vortices of scale l their total energy per unit of volume evidently has the order $T_l \sim \frac{\rho v_l^2}{2}$. Since this energy is not expended on anything else (so long as $l > l_s$) and this transition is materialized during the time of the order $\tau_l \sim \frac{l}{v_l}$, we conclude that there takes place of each l the condition of constance of the redistribution rate of energy S_l,

$$S_l = \frac{T_l}{\tau_l} = \text{const}.$$

Consequently,

$$\frac{v_l^3}{l} = \text{const}, \quad v_l^2 \sim l^{2/3}. \tag{15.6}$$

Therefore, the mean square of velocity fluctuation or of the difference of velocities is proportional to the distance l, over which this distance is measured, taken in power 2/3 ("law of two-thirds"). Taking into account (15.5) we hence obtain also the least dimension of the vortex:

$$l \sim \left(\frac{\eta}{\rho}\right)^{3/4} \frac{l_0^{1/4}}{v_0^{3/4}}, \tag{15.7}$$

where l_0 is the value of l for the initial large-scale inhomogeneity. For more details on the theory of isotropic turbulence see [1, 2; X, 2].

The velocity fluctuations in the scale l are inescapably linked with the fluctuations of pressure, temperature and density. Consequently, they lead to fluctuations of the dielectric constant. Therefore, there sets in an electrical inhomogeneity of the medium influencing the propagation of radiowaves (and also of light) and an acoustic inhomogeneity influencing the propagation of sound. The square of fluctuations of ε at the given point $\overline{(\delta\varepsilon)^2}$, and also the *correlation function* F of fluctuations at two points distant by $R = |r - r'|$ from one another may serve as a measure of electrical inhomogeneity (in case of *uniform* and *isotropic turbulence* depending only on R and to which we limit ourselves)

$$\overline{\delta\varepsilon(r)\cdot\delta\varepsilon(r')} = \overline{(\delta\varepsilon)^2}F_\varepsilon(R), \qquad (15.7a)$$

just as may do so the so-called *structural function* for the fluctuations of ε,

$$D_\varepsilon(R) = \overline{(\varepsilon(r) - \varepsilon(r'))^2}. \qquad (15.8)$$

If we consider that

$$\varepsilon(r) = \varepsilon_0 + \delta\varepsilon(r), \quad \varepsilon_0 = \overline{\varepsilon(r)}, \quad \overline{\delta\varepsilon(r)} = 0, \qquad (15.8a)$$

it is easy to see that

$$D_\varepsilon(R) = 2\overline{(\delta\varepsilon)^2}(1 - F_\varepsilon(R)). \qquad (15.8b)$$

The theory of isotropic turbulence foretells for velocities and parameters not affecting the development of turbulence (for "passive admixtures"), like ε or $n = \sqrt{\varepsilon}$, a uniform structural function expressed, for example, by the formula

$$D_\varepsilon(R) = \begin{cases} C_\varepsilon^2 l_s^{2/3}\left(\dfrac{R}{l_s}\right)^2 & \text{at } R \ll l_s, \\ C_\varepsilon^2 R^{2/3} = C_\varepsilon^2 l_s^{2/3}\left(\dfrac{R}{l_s}\right)^{2/3} & \text{at } l_s \ll R \ll l_0. \end{cases} \qquad (15.8c)$$

Here C_ε^2 is a constant. For other physical quantities of the type indicated only the constant factor C^2 will vary. At $R \gg l_0$ the structural function $D_\varepsilon(R)$ remains constant.

For the determination of the constant C it is found to be extremely essential to take into account the vertical gradients p, T and e, really taking place in the atmosphere. The mixing linked with them thus influences the absolute value of fluctuations. Namely, if we assume, as is shown in the work, [X, 2],

$$C_\varepsilon^2 = a_\varepsilon^2 l_0^{4/3} M^2 \tag{15.8d}$$

and introduce the *specific moisture* $q = 0.62\ e/p$, we have

$$M = -\frac{79\cdot10^{-6}p_{mb}}{T^2}\left(1+\frac{15500q}{T}\right)\left(\frac{dT}{dh}+\gamma_a-\frac{7800}{1+\frac{15500q}{T}}\frac{dq}{dh}\right). \tag{15.8e}$$

The quantity M has the dimensionality of the inverse length. If l_0 and R are measured in meters, h too must be measured in meters and so forth; a_ε^2 is a numerical factor of the order of unity, remaining indeterminate; γ_a is the above-mentioned temperature gradient for the dry (nonequilibrium) atmosphere; $\gamma_a = 0.0098$ grad/m.

In the *standard atmosphere* the principal role in the second parenthesis of Formula (15.8e) is played by the first two terms, as is easy to be convinced; according to the above-said, we have $(dT/dh) + \gamma_a = -0.0163$ grad/m, that is, for $T = 288°$ at sea level ($p \approx 1000$ mb)

$$M \approx \frac{0{,}79\cdot1{,}63p_{mb}}{T^2}\cdot10^{-4}\,m^{-1} \sim 2\cdot10^{-6}\,m^{-1} = 2\cdot10^{-8}\,cm^{-1}.$$

For turbulences induced by the wind the following example may be considered. Assume that $l_0 \sim 0.1$ km, $v_0 \sim 1$ m/sec. For the air $\eta \sim 2\cdot10^{-4}$ g/cm, $\rho \sim 10^{-3}$ g/cm^3 we have at the outset

$$Re \sim \frac{10^{-3}\cdot10^{3}\cdot10^{4}}{2\cdot10^{-4}} \sim 10^7.$$

The dimension of the smallest vortice is

$$l_0 \sim \left(\frac{2\cdot10^{-4}}{10^{-3}}\right)^{3/4}\frac{(10^4)^{1/4}}{10^{2\cdot3/4}} \sim 0{,}1\ cm$$

(near the reality we apparently have $l_s \sim 1$ cm). Farther,

$$C_\varepsilon^2 \sim 10^{\frac{16}{3}}\cdot 4\cdot 10^{-20} \sim 10^{-14}\,\text{cm}^{-\frac{2}{3}}.$$

For the determination of $\overline{(\delta\varepsilon)^2}$ it is necessary, according to Equality (15.6b), to take D_ε at the maximum distance, when $F \to 0$, that is, when $R \sim l_0$. Consequently,

$$\overline{(\delta\varepsilon)^2} \approx \frac{1}{2}D_\varepsilon(l_0) \sim \frac{1}{2}\cdot 10^{-14}(10^4)^{2/3} \sim 2\cdot 10^{-11}.$$

The real values of these quantities are to a strongest degree dependent on meteorological conditions at the given moment of time. Thus, direct measurements in the near the ground layer of the atmosphere give

$$\overline{(\delta n)^2} = \frac{1}{4}\overline{(\delta\varepsilon)^2} \sim 10^{-11} \div 10^{-12},$$

with the characteristic dimensions of inhomogeneities having at the same time an order 10^3-10^4 cm, and at the altitude of the order of 2-5 km [4], they are $\overline{(\delta n)^2} \sim 10^{-13}$-$2\cdot10^{-12}$. At the same time, the important parameter for theory, $\overline{(\delta n)^2}/l$, varies from 10^{-16} to 10^{-19} cm^{-1} (see Chapter 10). The close orders for the values of $\overline{(\delta n)}/l$ stem from measurements in the near the ground layer [3], in which it was found, as previously stated, $\overline{(\delta n)^2} \sim 10^{-11}$-$10^{-12}$ for $l \sim 10^3$-10^4 cm, which gives $\overline{(\delta n)^2}/l \sim 10^{-15}$-$10^{-15}$ cm^{-1}. This shows we may take for l_0 values constituting fractions of kilometer with the then measured values of $\overline{(\delta n)^2}$ agreeing not too poorly with those forecast by theory. However, the variation of this quantity by one, two or even three orders as a function of the meteorological environment and altitude above ground is quite normal.

As will be seen below, such fluctuations of ε should be considered quite significant; they may exert a very substantial influence upon the propagation of decimeter and shorter radiowaves.

The question as to what extent the representations of the theory of isotropic turbulence correspond to the true pattern of turbulent fluctuations ε cannot be considered as entirely clarified experimentally as yet. There exist direct measurements, in which an indication has been obtained that the distribution of fluctuations by dimensions follow the forecasts of theory. However, this question has not been studied to the end.

Finally, a combined action of both forms of inhomogeneities is possible, namely of the systematic course with the altitude and the turbulent fluctuations.

It may already be seen from the above that statistical methods of treatment must be applied at study of the influence of air turbulence on the propagation of radiowaves. The essential region of space (see §10) will inescapably include many such inhomogeneities, and the observed field will always transmit a certain average result of their ac-

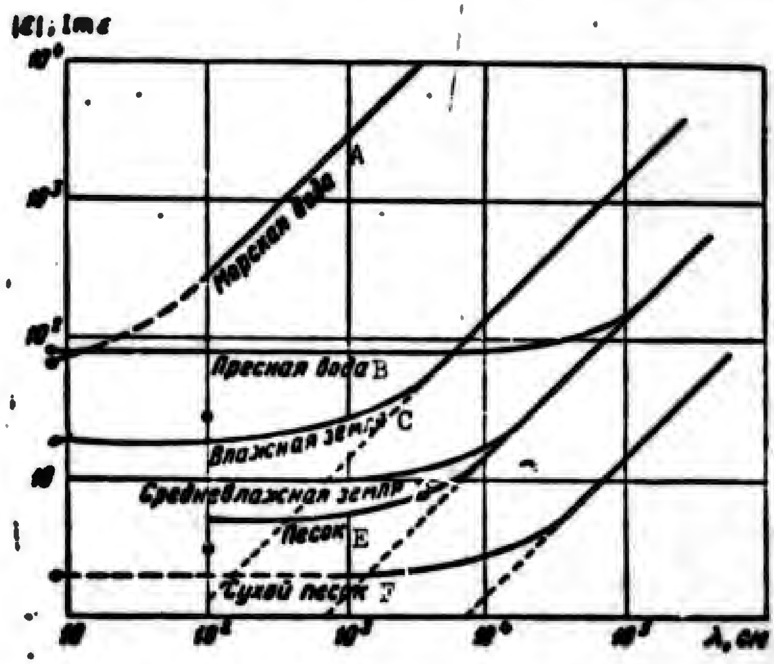

Fig. 15.1. Absolute value of the dielectric constant ε and its imaginary part Im ε for various soils and wavelengths. The inclined lines indicate Im ε. In the region of shorter waves ε differs from Im ε. A) Sea water; B) fresh water; C) moist ground; D) medium moist ground; E) sand; F) dry sand.

tion. This result will depend not only on the mean statistical characteristics of the medium, but also on others: on dispersion, on the correlation function, etc. This is why the consideration of these questions forms the chapter of statistical radiophysics.

3. As to the soil, there is observed here an extreme diversification of properties. It is mainly determined by the low conduction of dry ground ($\sigma \sim 10^{6}$ CGSE). Such a soil is usually rich in salts and this is why its conduction varies sharply as the amount of moisture increases. As will be seen, a comparatively thin superficial layer takes part in the radiowave propagation along the ground, in which the moisture content in various places is quite different. A sharp distinction in the con itions of propagation is contributed by regions covered by the sea. The conduction of salt sea water is thousands and tens of thousand times greater than that of the soils. The experimental data are compiled in Table 1 and in Fig. 15.1 (wherever the experimental points provide the scattering, the solid curves have quite an approximate character). These data must be considered as certain average, quite tentative ones. The oscillations σ two-three times in either side must be regarded as standard. Moreover, it should be borne in mind that these data are obtained on various installations, by different investigators, often characterizing the studied soil rather subjectively. This is why we point in references to characteristics brought out by the authors.

A systematic determination of "effective conductions" is effected in certain countries, as a result of which a chart is composed of the effective conductions for the entire country. Thus, for example, in USA such a chart was composed in 1954 on the basis of measurements of field attenuation of a given transmitter and comparison of this attenuation with the theoretically forecast (for a uniform surface) at different conductions σ [5]. 7000 layouts were studied. The authors note that the σ_{eff} found are not only fluctuating as a function of season, weath-

TABLE 1

Tentative Values of Electrical Parameters of Various Media

1 Среда	2 Длинные, средние и короткие волны				3 УКВ					
					λ=100 см		λ=9 см		λ=3,2 см	
	ε'	σ	$\lambda^{(1)}=\frac{c\varepsilon'}{2\sigma}$	$\lambda^{(2)}$	ε'	σ	ε'	σ	ε'	σ
4 Сухой песок	2	$2\cdot10^{8}$	150	700	4*	$9\cdot10^{8}$	2*	$3\cdot10^{9}$*		
5 Песок . . .	5	$2\cdot10^{7}$	40	70						
6 Средневлажная земля	10	$2\cdot10^{7}$	80	0						
7 Влажная земля . .	20	$2\cdot10^{9}$	15	0	30	$9\cdot10^{9}$	24	$6\cdot10^{9}$		
8 Пресная вода . . .	80	$2\cdot10^{7}$	600	0	80	$9\cdot10^{6}$—$9\cdot10^{7}$	80	$2\cdot10^{10}$**		
9 Морская вода . . .	80	$4\cdot10^{10}$	0,3	0			80	$5{,}5\cdot10^{10}$***	65	$1{,}4\cdot10^{10}$****

*"Dry soil" (sandy clayey soil).
**"Water from water conduit."
***Four percent solution. "Salt water."
****Sea water at 28°C.

$\frac{4\pi\sigma}{\omega}=\varepsilon'$ for $\lambda=\lambda^{(1)}$

$|\varepsilon|>10$ for $\lambda>\lambda^{(2)}$.

1) Medium; 2) long, medium and shortwaves; 3) ultrashort waves; 4) dry sand; 5) sand; 6) medium-moist ground; 7) moist ground; 8) fresh water; 9) sea water.

er, etc., but they differ too (by more than a factor of two), depending upon the direction of the course, the frequency the applied equipment, the method of measurements' interpretation, etc., even on the very same visibly uniform surface. As will be shown in Chapter 8 (§§48 and 51), the influence of surface irregularities is reduced in a series of cases to the variability of conduction, the substitution of the real conduction by a certain effective conduction, specifically dependent upon the wavelength. On the other hand, the presence of electrical inhomogeneities, even over a comparatively small part of the course, may notably change the character of propagation (Chapter 7). This is why it is not surprising that the values of σ_{eff} obtained by the above-described method differ that much among themselves. At the same time it is noted that for the given type of soils the values of the logarithms of the effective conductions are distributed according to the normal law with standard error of 0.266 around the mean $\bar{\sigma}_{eff}$. This means that in two-thirds of cases the value of $\bar{\sigma}_{eff}$ lie between $\bar{\sigma}_{eff}/1.85$ and $1.85\ \bar{\sigma}_{eff}$.

It may be seen from Table 1 that in the $\omega/2\pi \geq 10^8$ frequency band the dispersional properties of certain media and the dependence of the dielectric constant on frequency begin to manifest themselves. For these frequencies the term $4\pi\sigma/\omega$, computed by the value of σ for $\omega = 0$, becomes too small for the soil; this is why it is considered that there emerges the complex part of ε, connected with the polarization inertia, which, as already stated in §1, is not distinguishable from the conduction. Conducting, for example, measurements of the factor of ultra-shortwaves' reflection from the soil, this effect is interpreted as a rise of conduction. (If we take into account the dissimilarity of molecular mechanisms, we may state, as pointed out in §1, that here σ_{eff} is measured.) As may be seen from Table 1, this conduction may exceed that measured at $\omega = 0$ by 100 times.

However, there enter in the equation of theory not the ε' and σ themselves, but the complex dielectric constant $\varepsilon = \varepsilon' + 4\pi i \frac{\sigma}{\omega}$, which depends strongly on frequency even for constant and real ε' and σ. This is why the surface layer of the ground always behaves entirely differently relative to waves of various frequencies. In a certain context we may say that it always constitutes a strongly dispersing medium.

For sufficiently short waves (but prior to onset of dispersion ε') and small conductions the imaginary part of ε becomes quite small by comparison with ε'. Correspondingly, ε nearly ceases to depend on wavelength, and the conditions of propagation for waves of various wavelengths become, generally speaking, close. This remains valid even for centimeter waves, when σ rises by many factors only in very high frequencies (fresh water, sand). This case may be chosen as a criterion for the difference between long and short waves [I, 5].

Since the ratio of conduction currents' density σE to that of displacement currents $\frac{\omega\varepsilon'}{4\pi}E$ is exactly equal to $4\pi\sigma/\varepsilon'\omega$, this partition points to the one playing the greater role: the displacement or the conduction currents.

From this viewpoint "short" waves are those, for which the greater role for a given soil conduction is played by the displacement currents

$$\frac{4\pi\sigma}{\omega} \ll \varepsilon'. \tag{15.9}$$

For the "long" waves, to the contrary, the principal role is played by the conduction currents:

$$\frac{4\pi\sigma}{\omega} \gg \varepsilon'. \tag{15.10}$$

The wavelength $\lambda^{(1)}$, at which lies the boundary between "short" and "long" waves, approximately equal to $2\pi c/\omega^{(1)}$, where $\omega^{(1)} = 4\pi\sigma/\varepsilon'$, is also brought up in Table 1 for certain types of soils.

In one extreme case (usually of very short but not ultrashort waves), when the dispersion disappears completely, the medium behaves as if it were "almost dielectric." In another extreme case (of very long waves), when the real part in ε may be entirely neglected, the medium behaves as ideally conducting ($\sigma \to \infty$).

This method of subdivision of waves into "long" and "short" evidently does not in many cases coincide at all with the accepted well known radioengineering terminology.

Finally, all the media encountered may be considered as nonmagnetic. In connection with this we shall always postulate

$$\mu = 1. \tag{15.1}$$

We shall see in the following that the value of the complex dielectric constant plays a great role in the process of theoretical investigation. If $|\varepsilon|$ is great, a multitude of formulas are simplified. Generally, numerous results of theory are valid only at $|\varepsilon| \gg 1$. This is why it is appropriate to determine for each type of media the wavelength for which $|\varepsilon|$ really becomes great.

We shall conditionally choose for the criterion of the notion "much greater" the increase by 10 times. Therefore, we introduce a certain critical wavelength $\lambda^{(2)}$ such that for it $|\varepsilon| = 10$. Obviously if $\varepsilon' \geq 10$, $|\varepsilon|$ is "quite great" for all wavelengths. In the remaining cases ($\varepsilon' < 10$)

$$\lambda^{(2)} = \sqrt{100 - \varepsilon'^2}\,\frac{c}{2\sigma}. \qquad (15.12)$$

The value of $\lambda^{(2)}$ for various soils is shown in Table 1.

Therefore, in the region of short and longer waves nearly all the media encountered in the problem of radiowave propagation may be considered as having "quite great" $|\varepsilon|$. Only soils with little moisture content ($\varepsilon' < 5$, $\sigma < 10^7$ and the still drier ones) do not fit this criterion.

§16. PLANE WAVES IN A UNIFORM MEDIUM AND THEIR LINK WITH SPHERICAL WAVES

In a uniform medium, outside the region occupied by the sources, the equation for the Hertz vector

$$\nabla^2 \Pi + k^2 \Pi = 0, \qquad (16.1)$$

$$k^2 = \varepsilon' k_0^2, \quad \varepsilon = \varepsilon' + 4\pi i \frac{\sigma}{\omega}, \quad k_0 = \frac{\omega}{c}, \qquad (16.1a)$$

admits the harmonic in time solution of the form

$$\Pi = \Pi_0 e^{i(k_x x + k_y y + k_z z - \omega t)}, \qquad (16.2)$$

where k_x, k_y, k_z are certain, generally speaking, complex numbers. The substitution into the equation shows that the only requirement that must be satisfied by these numbers has the form

$$k_x^2 + k_y^2 + k_z^2 = k^2. \tag{16.3}$$

Consequently, one two of them are arbitrary. Since these are complex numbers, the partial solution (16.2) contains four arbitrary parameters (besides the amplitude of $\vec{\Pi}_0$), allowing to satisfy the boundary conditions. Let us break down all k's into real and imaginary parts, that is, assume

$$\left.\begin{aligned} k_x &= k_{1x} + ik_{2x}, \\ k_y &= k_{1y} + ik_{2y}, \\ k_z &= k_{1z} + ik_{2z}, \\ k &= k_1 + ik_2 \end{aligned}\right\} \tag{16.4}$$

and, moreover, conceive the conjunction of three numbers (k_{1x}, k_{1y}, k_{1z}) as being a certain real vector $\vec{q}$, and the conjunction of the numbers (k_{2x}, k_{2y}, k_{2z}) a certain real vector $\vec{p}$:

$$\left.\begin{aligned} k_{1x} &= q\sin\vartheta\cos\varphi, & k_{2x} &= p\sin\alpha\cos\beta, \\ k_{1y} &= q\sin\vartheta\sin\varphi, & k_{2y} &= p\sin\alpha\sin\beta, \\ k_{1z} &= q\cos\vartheta, & k_{2z} &= p\cos\alpha. \end{aligned}\right\} \tag{16.5}$$

In other words, instead of six parameters k_{1x}, ..., k_{2z}, linked with the condition (16.3), we shall introduce six parameters $q, \vartheta, \varphi, p, \alpha, \beta$, linked with the conditions stemming from the substitution of (16.5) into the relation (16.3):

$$pq\cos\xi = k_1k_2, \tag{16.6a}$$

$$q^2 - p^2 = k_1^2 - k_2^2, \tag{16.6b}$$

where $\cos\xi = \cos(pq) = \cos\vartheta\cos\alpha + \sin\vartheta\sin\alpha\cos(\varphi - \beta)$, that is, ξ is the angle between vectors $\vec{q}$ and $\vec{p}$. Then the direction of each of them remains arbitrary.

Now the solution of (16.2) has the form

$$\Pi = \Pi_0 e^{i(qR - \omega t) - pR}, \tag{16.7}$$

where $\vec{R}$ is the radius-vector of the point $[R = R(x, y, z)]$.

Selecting various directions of vectors $\vec{q}$ and $\vec{p}$ in space, we may sort all the possible solutions of the type (16.2).

The factor $e^{i(qR-\omega t)}$ indicates that each such solution determines the plane wave, of which the phase remains constant on surfaces

$$qR-\omega t=\mathrm{const}, \tag{16.7a}$$

that is, on surfaces $qR=\omega t+\mathrm{const}$, perpendicular to the vector $\vec{q}$. The position of these surfaces at two different moments of time is shown in Fig. 16.1. It is obvious that the motion velocity of this surface (in a direction normal to it), that is, the phase velocity, is

$$v=\frac{\omega}{|q|}. \tag{16.7b}$$

Thus, the factor $e^{i(qR-\omega t)}$ indicates that we have to do with a plane wave, propagating in a direction defined by the vector $\vec{q}$.

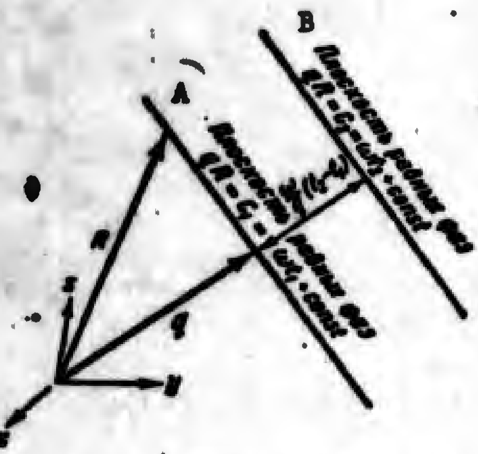

Fig. 16.1. Propagation of the wave front. A) Plane of equal phases; B) plane of equal phases.

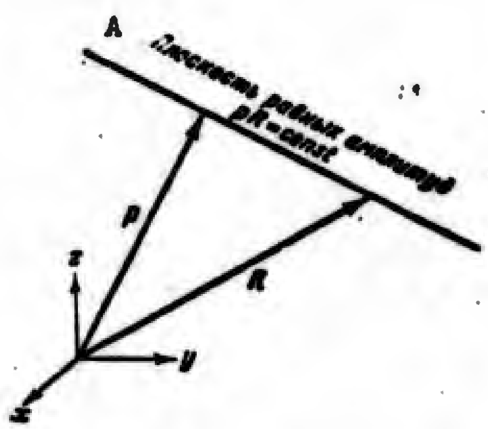

Fig. 16.2. Displacement of the plane of equal amplitudes. A) Plane of equal magnitudes.

The factor $e^{-\vec{p}\vec{R}}$ describes the exponential decrease of wave amplitude. It takes place as the product $\vec{p}\vec{R}$ rises; consequently, we have on all planes

$$pR=\mathrm{const} \tag{16.7c}$$

a constant amplitude. This equation determines the plane perpendicular to the vector $\vec{p}$ (Fig. 16.2). Therefore, the factor $e^{-\vec{p}\vec{R}}$ indicates that the exponential damping takes place in a direction defined by the vector $\vec{p}$.

We see that the wave equation admits the existence of solutions in the form of plane waves, whose amplitude decrease takes place <u>not</u> in the direction of propagation.

At times such solutions are written in another form, assuming

$$k_x x + k_y y + k_z z = |k|(x\cos u + y\cos v + z\cos w), \qquad (16.8)$$

where cos u, cos v, cos w are recognized as complex numbers (generally speaking not less than the unity in absolute value), whose values are easy to determine through k_x, k_y and k_z. It is stated in such a case that complex direction cosines of a plane wave are introduced. Their real parts determine, in conjunction, the direction and the velocity of phase propagation (direction and value of the vector $\vec{q}$), and their real parts the direction and the rate of amplitude decrease (direction and value of the vector $\vec{p}$). At the same time, according to Formula (16.3), they must satisfy the standard condition for the real direction cosines

$$\cos^2 u + \cos^2 v + \cos^2 w = 1.$$

One should not be led to believe that such solutions are possible only in a conducting medium, that is, only in case of complex <u>k</u>. According to the equality (16.6a), for a real <u>k</u> the planes of equal phases and the planes of equal amplitudes are mutually orthogonal. Thus, for example, the function

$$\Pi = \Pi_0 e^{i(k_x x - \omega t) - \varkappa y} \qquad (16.9)$$

is the solution of the wave equation for real k_x, $\varkappa$ and k^2, if only

$$k_x^2 - \varkappa^2 = k^2. \qquad (16.9a)$$

However, this function does not remain finite at $y \to -\infty$. This is why it may describe the field only in a limited region of values of <u>y</u>, for example, at $y > 0$. But if we write

$$\Pi = \begin{Bmatrix} \Pi_0 e^{i(k_x x - \omega t) - \varkappa y} \text{ for } y > 0, \\ \Pi_0 e^{i(k_x x - \omega t) + \varkappa y} \text{ for } y < 0, \end{Bmatrix} \qquad (16.9b)$$

such a function will be a possible solution; however, for $y = 0$ it undergoes a break and it thus corresponds to the presence of sources distributed by an infinitely th in layer in the plane $y = 0$, and this is why it is the solution of an inhomogenous wave equation and not of Eq. (16.1).

The case, whereby at $y > 0$ the function (16.9) is in itself a solution of an inhomogenous equation is not surprising, inasmuch as it may be represented in the form of the integral

$$\Pi = \Pi_0 \int_{-\infty}^{+\infty} \frac{dk_y}{2\pi i(k_y - i\varkappa)} e^{i(k_x x + k_y y) - i\omega t}, \qquad (16.9c)$$

composed of nondamping plane waves. (In order to be convinced of that it is sufficient to close in the integral (16.9c) the contour $-\infty < k_y < +\infty$ in the upper half-plane of the complex variable k_y by a semicircumference of large radius and take the deduction at $k_y = i\varkappa$.) However, the plane waves of normal type entering here respond to *different* wavelengths, inasmuch as for them the conditions $k_x^2 + k_y^2 = k^2$ do not exist (the integral is extended to infinite limits). It is obvious that similar fields may be represented as a superimposition of standard plane waves with different wavelengths, each of which is the solution of the wave equation for the same frequency (inasmuch as k^2 in Eq. (16.1) emerged as a denotation of ω^2/c^2, where ω is the single frequency for all wavelengths); however, the phase velocities of these waves are different for the various wavelengths (in the case (16.9c) $v = \frac{\omega}{\sqrt{k_x^2 + k_y^2}}$)

Any field of given frequency ω, satisfying not only the wave equation (16.1), but also the boundary conditions, can be formed from plane waves by way of superimposition. These boundary conditions are sharply divided into three groups:

1) the conditions at infinity, requiring that the condition of emission be observed (and in any case that the field do not increase with the rise of R). This means that for large R the aggregate field must pass to an outgoing wave;

2) the conditions at interfaces of uniform media are particularly simple, provided these interfaces are planes;

3) the conditions at points where point emitters are located. If, for example, the emitter is a point dipole placed at the point $\vec{R}_0$, near it the field increases boundlessly and it thus must pass into the field of the given emitter, stabilizing when it is situated in a boundless uniform medium. Therefore, the field must pass into the field of a spherical wave

$$\Pi \sim \frac{1}{r} e^{i(kr-\omega t)}, \quad r = |R - R_0|.$$

not only at infinity, but also when $R \to R_0$

Now we shall be convinced that the conditions 1) and 3) cannot be satisfied by an assortment of plane waves in which are present only waves with real k_x, k_y and k_z and with one specific wavelength.

In reality the three-variable function e^{ikR}/R may be expanded into a triple Fourier integral

$$\frac{e^{ikR}}{R} = \iiint\limits_{-\infty}^{+\infty} A(q') e^{iq'R} dq'. \qquad (16.10)$$

in which each of the superimposed plane waves is a solution of the homogenous equation

$$(\nabla^2 + q'^2) e^{iq'R} = 0, \quad q'^2 = q_x'^2 + q_y'^2 + q_z'^2.$$

Inasmuch as in the integral q_x', q_y', q_z' run independently through all the real values, the waves with q', not equal to a certain unique specific q', for example q' = k, might drop out only at a special form of $A(\vec{q}')$. However, in the given case they do not drop out. Indeed, $A(\vec{q}')$ will be determined uniquely, if both parts of Formula (16.10) are multiplied by $e^{-i\vec{q}\vec{R}}$ and integrated over the entire space. We shall obtain at the right-hand side

$$\iiint\limits_{-\infty}^{+\infty} A(q')\,dq' \iiint\limits_{-\infty}^{+\infty} e^{i(q'-q)R}\,dR = (2\pi)^3 A(q).$$

while on the left we may always postulate $qR = qR\cos\bar{\theta}$ by taking advantage of the materiality of q, and integrate in spherical coordinates R, θ, φ. This gives

$$\int_0^\infty e^{ikR} R\,dR \int_0^\pi e^{iqR\cos\theta} \sin\theta\, d\theta \int_0^{2\pi} d\varphi = 2\pi \int_0^\infty e^{ikR}\left(e^{iqR} - e^{-iqR}\right)\frac{dR}{iq}.$$

In reality the medium is always conducting, if only to an insignificant degree. This is why k always contains a small imaginary part, ahead of which the sign is still not determined by Formula (16.1a), but must be so chosen that the field would not increase at infinity; this means that $k = k_1 + ik_2$, where $k_2 > 0$. This is why we may formally substitute k by $k + i\alpha$, considering k to be real as previously, $\alpha > 0$. Since such a small conduction in the region of finite distances from the source cannot manifest itself in the physical result, we may postulate at the end of all calculations $\alpha = 0$.

Owing to this substitution the integral becomes convergent and the substitution of the upper limit gives zero. But substituting the lower limit we shall obtain

$$A(q) = \frac{1}{2\pi^2}\,\frac{1}{q^2 - (k + i\alpha)^2}$$

and the expansion of the spherical wave into plane ones acquires the form

$$\frac{e^{ikR}}{R} = \frac{1}{2\pi^2} \iiint\limits_{-\infty}^{+\infty} \frac{e^{i(q_x x + q_y y + q_z z)}}{q_x^2 + q_y^2 + q_z^2 - (k + i\alpha)^2}\, dq_x\, dq_y\, dq_z \qquad (16.11)$$

where α will be subsequently postulated to be zero.

Therefore, plane waves with different $q=\sqrt{q_x^2+q_y^2+q_z^2}$ enter essentially into the expansion.

There is nothing surprising in this. Had the inverse situation taken place, that is, if all the plane waves belonged to the only wavelength $\lambda = 2\pi/k$, they would constitute the solutions of the very same homogenous equation (16.1). Meanwhile, the spherical wave, having a peculiarity at $R = 0$, is the solution of the corresponding inhomogenous equation, precisely such an equation, in the right-hand part of which stands a point source.

However, we may formally represent this expansion, as was done in the case of mutually equivalent formulas (16.9) and (16.9b), as a superimposition of plane waves with *complex* wave numbers being the solutions of one and the same homogenous equation (16.1).

First of all we shall perform one of the integrations entering into the expansion (16.11), say, the integration over q_z. To that effect it is sufficient to supplement the contour $-\infty<q_z<+\infty$ by a semicircumference, the integral of which vanishes, and take the deduction at the pole. There are two poles:

$$q_z^0=\pm i\sqrt{q_x^2+q_y^2-(k+i\alpha)^2}.$$

For definiteness we shall consider that at extraction of the root, of the two of its values we shall take the one for which the real part is positive.

When we apply the result in the region $z > 0$, the integral over the circumference vanishes if we close the contour in the upper half-plane. But here is located only one of the poles. Taking into account that near the pole the denominator has the form $q_z^2-(q_z^0)^2\approx 2q_z^0(q_z-q_z^0)$, and directing α to zero if only now, we shall obtain (assuming $\sqrt{q^2-k^2}=-i\sqrt{k^2-q^2}$)

$$\frac{e^{ikR}}{R}=\frac{1}{2\pi}\iint\limits_{-\infty}^{+\infty}\frac{e^{i(q_xx+q_yy)-z\sqrt{q_x^2+q_y^2-k^2}}}{\sqrt{q_x^2+q_y^2-k^2}}dq_xdq_y= \tag{16.11a}$$

$$=\frac{e^{i\frac{\pi}{2}}}{2\pi}\iint\frac{e^{i(q_xx+q_yy+z\sqrt{k^2-q_x^2-q_y^2})}}{\sqrt{k^2-q_x^2-q_y^2}}dq_xdq_y.$$

This expression may be considered as the expansion into plane waves with complex $\vec{q}$ (in the region $q_x^2+q_y^2>k^2$, q_z becomes imaginary).

If we introduce in the plane q_x, q_y the polar coordinates ν, φ, postulating

$$q_x=\nu\cos\varphi,\quad q_y=\nu\sin\varphi,$$

and in the plane, x, y – the polar coordinates r, α, assuming

$$x=r\cos\alpha,\quad y=r\sin\alpha,$$

and inasmuch as $q_xx+q_yy=\nu r\cos(\varphi-\alpha)$, performing the integration over φ (which gives a Bessel function of zero order), we shall obtain

$$\frac{e^{ikR}}{R}=\int\limits_0^\infty\frac{J_0(\nu z)}{\sqrt{\nu^2-k^2}}e^{-z\sqrt{\nu^2-k^2}}\nu d\nu. \tag{16.12a}$$

It is easy to be convinced that in the case $z<0$, closing the integration contour in the lower half-plane, we shall have

$$\frac{e^{ikR}}{R}=\int\limits_0^\infty\frac{J_0(\nu z)}{\sqrt{\nu^2-k^2}}e^{z\sqrt{\nu^2-k^2}}\nu d\nu, \tag{16.12b}$$

provided the square root is taken everywhere with such a sign that there be

$$\mathrm{Re}\sqrt{\nu^2-k^2}>0. \tag{16.13}$$

In the theory of radiowave propagation the representations of the spherical wave in the form (16.12a) or (16.12b) is often utilized.

Therefore, a spherical wave with wavelength $2\pi/k$ may be either represented as a superimposition of ordinary plane waves (16.11), with, however, the inescapable participation of waves with various wavelentths arbitrarily differing from $2\pi/k$, or as a superimposition

(16.11a) admitting the participation of plane waves with complex propagation vector q, but such that each wave correspond to the specific wavelength ($q^2 = q_x^2 + q_y^2 + q_z^2 = k^2$). In reality, if we break down the integration in the relation (16.11a) into two parts, having separated the cases when $q_x^2 + q_y^2 < k^2$, from those when $q_x^2 + q_y^2 > k^2$, and if we choose for each interval between the two forms that for which the subradical expression is positive, we shall obtain (at $z \gtrless 0$)

$$\frac{e^{ikR}}{R} = \frac{i}{2\pi} \iint\limits_{q_x^2+q_y^2<k^2} dq_x\,dq_y \frac{e^{i(q_x x + q_y y \pm z\sqrt{k^2-q_x^2-q_y^2})}}{\sqrt{k^2-q_x^2-q_y^2}} + \tag{16.14}$$
$$+ \frac{1}{2\pi} \iint\limits_{q_x^2+q_y^2>k^2} dq_x\,dq_y \frac{e^{i(q_x x + q_y y) \mp z\sqrt{q_x^2+q_y^2-k^2}}}{\sqrt{q_x^2+q_y^2-k^2}}.$$

In the first term the factor at iz is real; this term therefore represents the combination of usual plane waves with wavelength 2 /k coinciding with the length of the spherical wave. But the second addend consists of waves with a propagation direction lying in the plane x, y (their wavelength in this plane is not equal to 2 /k), but exponentially damping along the axis z when $z \to \pm\infty$. Hence it may be seen that the presence of such damping waves in the spherical wave is necessary even in a nonconducting medium. All this is entirely analogous to the correctness of Formulas (16.9) and (16.9c).

However, just as in Formula (16.9c), every wave with given q_x, q_y entering into the relation (16.14) corresponds in fact to the presence of sources in the plane z = 0, i.e., it is a solution of an *inhomogenous* wave equation.

Evidently, we could have separated in the expansion (16.11) as the direction of damping not the axis z but either the axis x or y, if the integration were performed not over q_z but over q_x or q_y. Hence may be perceived the conditionality of this representation, which, however, is broadly utilized in the theory of radiowave propagation (see §32).

The waves of the type (16.9) are sometimes called *inhomogenous plane waves*.

For plane waves the relationship between electric and magnetic fields is particularly simple. If $\vec{\Pi}$ has a character of a plane wave, $\vec{E}$ and $\vec{H}$ are also described by plane waves in accord with the basic formulas (3.9), (3.10) etc., with the same propagation vector $k = k_0\sqrt{\varepsilon}$. This is why the Maxwellian equations (1.13a) and (1.13b) in free space give

$$k_0 E = -\frac{1}{\sqrt{\varepsilon}}[k_0 H],$$

$$k_0 H = \sqrt{\varepsilon}[k_0 E].$$

§17. ABSORPTION OF A PLANE WAVE

Let us consider a special case, in whic owing to boundary conditions, say, because of conditions' symmetry, the decrease in intensity takes place in the propagation direction. Making coincide the axis x with that direction we shall obtain

$$\Pi = \Pi_0 e^{i(kx - \omega t)}. \qquad (17.1)$$

We shall introduce the denotations for the phase and the amplitude of k which we shall utilize everywhere:

$$k = |k| e^{i\frac{\chi}{2}},$$

where

$$|k| = k_0 \sqrt[4]{\varepsilon'^2 + \left(\frac{4\pi\sigma}{\omega}\right)^2}, \quad \chi = \operatorname{arc\,tg}\frac{4\pi\sigma}{\varepsilon'\omega}. \qquad (17.2)$$

For "long" waves we have $\chi \approx \frac{\pi}{2}$, for "short" waves – $\chi \approx \frac{4\pi\sigma}{\varepsilon\omega} \to 0$. This is why in a dielectric medium $(\sigma = 0)\ \chi = 0$, k is purely real, and the propagation takes place without damping (without losses to heat). (Evidently, for very short waves it will become complex on account of dispersion of ε' itself, and dielectric losses will appear. Then σ will be substituted by $\sigma_{1\omega} = \frac{\omega}{4\pi}\operatorname{Im}\varepsilon$.)

This is why

$$\Pi = \Pi_0 e^{i(|k|\mu\cos\frac{\chi}{2} - \omega t) - |k|\mu\sin\frac{\chi}{2}}. \tag{17.3}$$

Consequently, the phase velocity is

$$v = \frac{\omega}{k_0\sqrt[4]{\varepsilon'^2 + \left(\frac{4\pi\sigma}{\omega}\right)^2}\cos\left(\frac{1}{2}\operatorname{arctg}\frac{4\pi\sigma}{\varepsilon'\omega}\right)}. \tag{17.4}$$

For "long" waves, when $\frac{4\pi\sigma}{\varepsilon'\omega} \gg 1$,

$$v \to \frac{c}{\sqrt{\frac{2\pi\sigma}{\omega}}} = c\sqrt{\frac{c}{\sigma\lambda}}\ \left(v^2 < \frac{2c^2}{\varepsilon'}\right). \tag{17.5}$$

For "short" waves

$$v \to \frac{c}{\sqrt{\varepsilon'}}, \tag{17.6}$$

as this must be for a nonmagnetic dielectric.

The dependence of the propagation velocity on frequency, expressed in these formulas, thus constitutes the first criterion of a dispersing medium.

On passing the characteristic segment

$$l = \frac{1}{k_0|\sqrt{\varepsilon}|\sin\frac{\chi}{2}} = \frac{\lambda_0}{2\pi\sqrt[4]{\varepsilon'^2 + \left(\frac{4\pi\sigma}{\omega}\right)^2}\sin\left(\frac{1}{2}\operatorname{arctg}\frac{4\pi\sigma}{\varepsilon'\omega}\right)}. \tag{17.7}$$

The values of the segment l and of its inverse value $b = 1/l$, having the sense of the absorption coefficient, are shown in Fig. 17.1, as functions of ω for different ε' and σ.

For great conductions and low frequencies $4\pi\sigma \gg \varepsilon'\omega$;

$$l \approx \frac{\lambda_0}{2\pi\sqrt{\frac{2\pi\sigma}{\omega}}} = \frac{1}{2\pi}\sqrt{\frac{\lambda_0 c}{\sigma}}. \tag{17.7a}$$

For low conductions and high frequencies arctan and sin may be replaced by their arguments so that

$$l \approx \frac{\sqrt{\varepsilon'}}{2\pi}\frac{c}{\sigma}. \qquad (17.7b)$$

Therefore, with the increase of frequency the constant l reaches its critical value for the given conduction. Let us draw attention to the

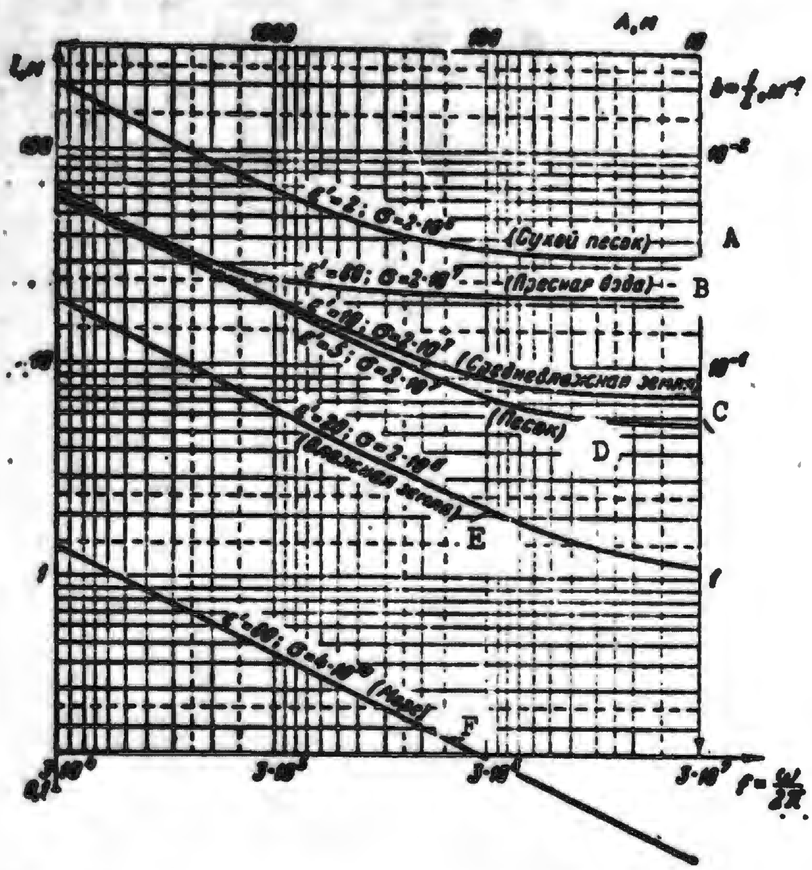

Fig. 17.1. Dependence of the depth of penetration l and of the absorption coefficient b on frequency f (or on the wavelength λ) for various media. A) Dry sand; B) fresh water; C) medium-moist ground; D) sand; E) moist ground; F) sea.

fact that in both extreme cases this constant decreases with the rise of conduction. The greater the medium's conduction, the more rapidly the oscillations damp. The greater the wavelength, the weaker the damping (this regularity is less manifest only for dry soils and very short waves).

In a medium such as the sea the attenuation of radiowaves is exceptionally strong. It renders nearly impossible the radiocommunication between submarines in the submerged state or between them and points in the air. Radiotransmission is possible only at small immersions and even then in very long waves.

On the other hand, the strong dependence of the propagation velocity as well as of the absorption on the character of the soil is the reason of utilization of radiomethods in geologic investigations.

§18. FIELD OF A POINT DIPOLE

In the following we shall be more than once compelled to utilize formulas expressing the field of a point source. We shall compile them here.

According to Formula (6.5), a point electric dipole of moment $\vec{p}$ at the distance R induces a field with a Hertz vector

$$\Pi = \frac{1}{\varepsilon}\frac{p}{R}e^{ikR}. \tag{18.1}$$

We shall direct the polar axis of the spherical system of coordinates (R, ϑ, φ) along the direction of the dipole. Then $\vec{\Pi}$ will be directed everywhere along the axis z:

$$\Pi = \Pi_z. \tag{18.2}$$

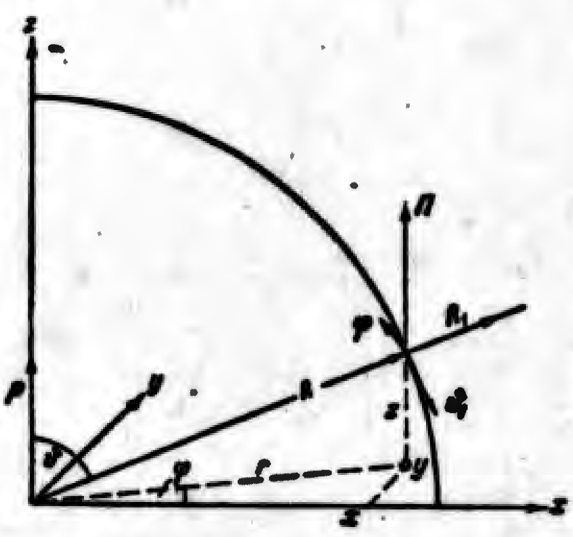

Fig. 18.1. Denotations in the adopted system of spherical coordinates.

Consequently, having directed the axes in the manner shown in Fig. 18,1, we shall obtain

$$\Pi_R = \Pi \cos\vartheta, \quad \Pi_\vartheta = -\Pi \sin\vartheta, \quad \Pi_\varphi = 0.$$

and further,

$$\operatorname{div}\Pi = \frac{\partial \Pi_z}{\partial z} = \frac{1}{\varepsilon}\frac{p}{R} e^{i(kR-\omega t)} \left(ik - \frac{1}{R} \right) \cos\vartheta.$$

Utilizing the expressions for rot and grad in spherical coordinates,

$$\mathrm{grad}_R = \frac{\partial}{\partial R}, \quad \mathrm{grad}_\vartheta = \frac{1}{R}\frac{\partial}{\partial \vartheta}, \quad \mathrm{grad}_\varphi = \frac{1}{R\sin\vartheta}\frac{\partial}{\partial\varphi};$$

$$\mathrm{rot}_R A = \frac{1}{R\sin\vartheta}\left\{ \frac{\partial}{\partial\vartheta}(\sin\vartheta A_\varphi) - \frac{\partial A_\vartheta}{\partial\varphi} \right\},$$

$$\mathrm{rot}_\vartheta A = \frac{1}{R\sin\vartheta}\left\{ \frac{\partial A_R}{\partial\varphi} - \frac{\partial}{\partial R}(R\sin\vartheta A_\varphi) \right\},$$

$$\mathrm{rot}_\varphi A = \frac{1}{R}\left\{ \frac{\partial}{\partial R}(R A_\vartheta) - \frac{\partial A_R}{\partial\vartheta} \right\},$$

and taking into account that, according to Formulas (3.9), (3.10),

$$E = \mathrm{grad}\,\mathrm{div}\,\Pi + \varepsilon k_0^2 \Pi, \quad H = -ik_0\varepsilon\,\mathrm{rot}\,\Pi,$$

we find

$$E_R = \frac{1}{\varepsilon}\frac{p}{R} e^{i(kR-\omega t)} \left\{ \frac{2}{R^2} - \frac{2ik}{R} \right\} \cos\vartheta,$$

$$E_\vartheta = -\frac{1}{\varepsilon}\frac{p}{R} e^{i(kR-\omega t)} \left\{ \varepsilon k_0^2 - \frac{1}{R}\left(-ik + \frac{1}{R} \right) \right\} \sin\vartheta,$$

$$E_\varphi = 0,$$

$$H_R = 0,$$

$$H_\vartheta = 0,$$

$$H_\varphi = -\frac{k_0 p}{R} e^{i(kR-\omega t)} \left(k + \frac{i}{R} \right) \sin\vartheta.$$

Separating the real part from the imaginary, for which we shall postulate $k = k_1 + ik_2 = k_0\sqrt{|\varepsilon|}\, e^{i\frac{\chi}{2}}$ (it should be recalled that $k_1, k_2 > 0$), we obtain

$$E_R = \frac{2k_1 \cos\theta}{|\varepsilon| R^2} p \sqrt{1 + \left(\frac{1}{k_1 R} + \frac{k_2}{k_1}\right)^2} \exp\left\{ i\left[-\omega t + kR - \chi - \operatorname{arctg}\frac{k_1 R}{1 + k_2 R}\right]\right\},$$

$$E_\theta = \frac{k_1^2 \sin\theta}{|\varepsilon| R} p \sqrt{\left(1 - \frac{k_2^2}{k_1^2} - \frac{k_2}{k_1^2 R} - \frac{1}{k_1^2 R^2}\right)^2 + \left(\frac{1}{k_1 R} + \frac{2k_2}{k_1}\right)^2} \times$$

$$\times \exp\left\{ i\left[-\omega t + kR - \pi - \chi + \operatorname{arctg}\frac{k_1 R(1 + 2k_2 R)}{(k_1^2 - k_2^2)R^2 - k_2 R - 1}\right]\right\},$$

$$\left.\begin{aligned} &E_\varphi = 0, \\ &H_R = 0, \qquad H_\varphi = \frac{k_0 k_1}{R} \sin\theta\, p \sqrt{1 + \left(\frac{k_2}{k_1} + \frac{1}{k_1 R}\right)^2} \times \\ &\qquad\qquad \times \exp\left\{ i\left[-\omega t + kR + \pi + \operatorname{arctg}\frac{1 + k_2 R}{k_1 R}\right]\right\}, \\ &H_\theta = 0, \end{aligned}\right|$$

where

$$\chi = \arg \varepsilon = \operatorname{arctg}\frac{4\pi\sigma}{\varepsilon' \omega},$$

$$k_1^2 - k_2^2 = \varepsilon' k_0^2,$$

$$2 k_1 k_2 = \frac{4\pi\sigma}{\omega} k_0^2.$$

For the case of a dipole in the air (vacuum), $k_2 = 0$, $k = k_0$, the expressions are simplified and we obtain

$$\begin{aligned} &E_R = \frac{2k_0 \cos\theta}{R^2} p \sqrt{1 + \frac{1}{k_0^2 R^2}}\, e^{i(k_0 R - \operatorname{arctg} k_0 R - \omega t)}, \\ &E_\theta = \frac{k_0^2 \sin\theta}{R} p \sqrt{1 - \frac{1}{k_0^2 R^2} + \frac{1}{k_0^4 R^4}}\, e^{i\left(k_0 R + \pi + \operatorname{arctg}\frac{k_0 R}{k_0^2 R^2 - 1} - \omega t\right)}, \\ &E_\varphi = 0, \\ &H_R = 0, \\ &H_\theta = 0, \\ &H_\varphi = \frac{k_0^2}{R} p \sin\theta \sqrt{1 + \frac{1}{k_0^2 R^2}}\, e^{i\left(k_0 R + \pi + \operatorname{arctg}\frac{1}{k_0 R} - \omega t\right)}. \end{aligned} \tag{18.4}$$

The region of "small distances" from the dipole, $k_0 R < 1$, forms a quasistatic zone where the field coincides with the quasistationary field of the electric dipole, slowly varying its moment.

The region of "great distances," $k_0 R \gg 1$, forms the so-called wave zone, where E_ϑ and H_φ decrease inversely proportionally to R, while all the remaining components are at least k_0R times smaller than these principal ones.

Namely, in vacuum we have

$$\left.\begin{aligned} E_\vartheta &\approx -k_0^2 p \frac{e^{i(k_0R-\omega t)}}{R}\sin\vartheta = -k_0^2\Pi\sin\vartheta, \\ H_\varphi &\approx -k_0^2 p \frac{e^{i(k_0R-\omega t)}}{R}\sin\vartheta = -k_0^2\Pi\sin\vartheta = E_\vartheta; \end{aligned}\right\} \quad (18.4a)$$

and in the medium -

$$E_\vartheta = -k_0^2 p \frac{e^{i(kR-\omega t)}}{R}\sin\vartheta = \frac{1}{\sqrt{\varepsilon}} H_\varphi. \qquad (18.4b)$$

Here, as well as everywhere, we should see to it that the field satisfy the radiation condition at extraction from ε of the square root in the expression $k = k_0\sqrt{\varepsilon}$; in other words, we shall take the value of the root for which

$$\operatorname{Im} k > 0. \qquad (18.5)$$

The dipole field must at times be known to us in Descartes components. According to (3.25), in the wave zone, in the vacuum, we have

$$\left.\begin{aligned} E_x &= -k_0^2\frac{xz}{R^2}\Pi, \\ E_y &= -k_0^2\frac{yz}{R^2}\Pi, \\ E_z &= -k_0^2\frac{x^2+y^2}{R^2}\Pi, \\ H_x &= k_0^2\frac{y}{R}\Pi, \\ H_y &= -k_0^2\frac{x}{R}\Pi, \\ H_z &= 0. \end{aligned}\right\} \quad (18.4c)$$

Therefore, far off the source the influence of the medium is manifest only by the factor $\sqrt{\varepsilon}$ in H_φ and the complexity of the wave number. It should, however, be noted in the case of a conducting medium a circumstance essential for the emission process of a dipole placed in that medium.

The fact of the matter is that in the direct vicinity of the source the field E is proportional to $1/R^3$. Consequently, the medium-emitted heat, constituting per unit of volume in a unit of time σE^2, will pass to infinity when integrating over the entire volume. Thus, in order to sustain the oscillations of the point dipole with constant amplitude of the moment $\underline{p}$ an infinitely great power should be fed. This absurd result is linked with the fact that we accounted for neither the finiteness of dipole dimensions (on account of which in reality at $R \to 0$, $ER^3 \to 0$ too), nor that the dipole is in fact often placed in a nonconducting cavity.

The consideration of the dipole emission in the cavity is beyond the framework of our problems. This is why we shall limit ourselves to the remark that energy consumptions in a nonwave zone may be neglected only in the case when the cavity encompasses the entire nonwave zone.

In vacuum the energy flux, pulsating in magnitude at every point (but always directed from the dipole), gives the average in time expression (we substitute $\underline{p}$ by $\dot{I}$ and $\underline{h}$ according to Formula (6.4b) $p = \frac{i}{\omega} Ih$, passing to true expressions for the field and taking into account that $\overline{\cos^2(\omega t + \chi)} = \frac{1}{2}$)

$$S = \frac{c}{4\pi}\,\overline{\mathrm{Re}E_\theta\,\mathrm{Re}H_\varphi} = \frac{\pi}{2}\frac{I^2h^2}{\lambda^2}\frac{1}{c}\frac{\sin^2\theta}{R^2}\,\frac{\text{erg}}{\text{cm}^2/\text{sec}} \qquad (18.6)$$

As to the total energy emitted per unit of time, it is obtained upon multiplying by $\sin\theta d\theta d\varphi$ and integrating over the surface of a sphere of radius R. This gives

$$W = \frac{4\pi^2}{3}\frac{I^2}{c}\left(\frac{h}{\lambda}\right)^2\frac{\text{erg}}{\text{sec}} = 40\,\pi^2\left(\frac{h}{\lambda}\right)^2 I^2 \text{ w}, \qquad (18.7)$$

being understood at the same time that in the last term I is expressed in amperes.

If the field strength is expressed in practical units, we shall obtain for the amplitude

$$|E_\theta| = \frac{60\,\pi I l}{R\lambda}\sin\theta\ \text{v/m} \tag{18.8}$$

where the amplitude of the current I is expressed in amperes and all the lengths in meters.

This quantity is often conveniently expressed through the emitted power. Combining Formulas (18.7) and (18.8), we may obtain the formula

$$|E_\theta| = \frac{0.3\sin\theta}{R_{\text{км}}}\sqrt{W_{\text{ком}}}\ \text{v/m} = \frac{3\cdot 10^5\sin\theta}{R_{\text{км}}}\sqrt{W_{\text{ком}}}\ \mu\text{v/m}; \tag{18.9}$$

where R is expressed in kilometers and the power in kilowatts.

In order to estimate how substantially the field attenuation with the distance influences the possibility of reception, it is necessary to bear in mind that, generally speaking, atmospheric and industrial electromagnetic interferences hinder the reception of radiowaves in which the electric field strength is below the level of several microvolts (or even several tens of microvolts) per meter. Only by applying special reception methods (reception by means of space antennas and so forth) this limitation may be diminished and then manifest becomes the limitation superimposed by noises in the tubes and other parts of the receiver (equivalent to less than tenths of microvolt fractions per meter). For special purposes this limitation too may be overcome (by applying diversified modes of noiseproof feature improvement utilized, for example, in radar and radioastronomy). Generally the required excess over the interference level is conditioned by the purpose of reception: for artistic radiobroadcasting the field strength must exceed a great deal the average level of that of interferences.

In real conditions, when the presence of the ground or of the ionosphere exerts influence, the field emitted by an elementary dipole will differ essentially from the field emitted by a dipole in a boundless uniform medium. Obviously, we may always write this in the form

$$\Pi = p\frac{e^{i(kR-\omega t)}}{R}w(x, y, z). \tag{18.10}$$

where w (x, y, z) is a certain function describing the influence of the ground and of the ionosphere.

The problem of the theory of radiowave propagation along the ground amounts in substance to the determination of that *"attenuation factor"* when neglecting the influence of the ionosphere.

We shall maintain the denotation w for that function in the general case. For the attenuation, introduced by a uniform ground surface, it is customary to use the denotation w = y or w = wy (we shall adhere to this denotation, for example, in Chapters 5, 7 and 8), or the denotation f. When accounting for its curvature, the influence of the ground is described by a function denoted by either V or 2W (Chapter 6).

§19. REFLECTION AND REFRACTION ON THE PLANE INTERFACE BETWEEN TWO UNIFORM MEDIA

If two uniform media with penetration factors ε_1 and ε_2 are separated by a plane boundary, coinciding, for example, with the plane z = 0, the propagation of radiowaves in them is described in a broad category of cases by simple formulas, which are called *interference* (or either *reflection* or *Fresnel*). They are valid when the receiver and transmitter are sufficiently removed from the interface, with the result that the emission of a point (and generally concentrated) source may be represented in the form of a plane wave. Here we have to do, in essence, with a standard problem of optics, and the consideration does not really differ from it in any way. The exact criteria of method's applicability have an essential significance. They will be obtained below.

1. Assume that an homogenous plane wave of frequency ω with elec-

tric and magnetic vectors $\vec{E}^{(e)}$ and $\vec{H}^{(e)}$ of field strength and with the propagation vector $k=k_0\sqrt{\varepsilon_1}$, situated in the plane, x, z, is incident from a medium I (z > 0). The total field may always be represented as a set of plane waves (inhomogenous, if required); it is only necessary that such a set ensure the observance of boundary conditions at the interface and that is satisfy the emission conditions. There emerges no additional condition in the source at such a consideration. It is in essence superseded by the admission of presence of only one wave, contradicting the emission condition, which is the plane wave incident from infinity. It is asserted that all these conditions may always be fulfilled provided we add to the incident wave only two waves: that reflected by $\vec{E}^{(r)}$, $\vec{H}^{(r)}$ in the medium 1 and the one refracted by $\vec{E}^{(g)}$ and $\vec{H}^{(g)}$ in the medium 2, having vectors $\vec{k}^{(r)}$ and $\vec{k}^{(g)}$ that lie in the same plane x, z, with, at the same time, $k^{(r)}=k_0\sqrt{\varepsilon_1}$, $k^{(g)}=k_0\sqrt{\varepsilon_2}$. The electric field amplitudes of the reflected and refracted waves differ from $E^{(e)}$ only by the multipliers

$$|E^{(r)}|=f|E^{(e)}|,\quad |E^{(g)}|=g|E^{(e)}|. \qquad (19.1)$$

independent from the coordinates. The factor f may be called the reflection factor. The coefficients f and g depend on the angle of incidence, on the polarization of the incident wave and on constants characterizing both media.

The boundary conditions (4.1), (4.4), (4.5), (4.6) are:

$$\left.\begin{matrix} E_{1x}=E_{2x},\ E_{1y}=E_{2y},\ \varepsilon_1E_{1z}=\varepsilon_2E_{2z} \\ H_{1x}=H_{2x},\ H_{1y}=H_{2y},\ H_{1z}=H_{2z} \end{matrix}\right\} \text{ at } z=0. \qquad (19.2)$$

By assumption

$$E_1=E^{(e)}+E^{(r)},\quad E_2=E^{(g)}. \qquad (19.3)$$

For a plane wave with the propagation vector $\vec{k}$, $\vec{H}$ is determined by $\vec{E}$ according to formula $\operatorname{rot}E=-\frac{1}{c}\frac{\partial H}{\partial t}$, that is, $\vec{x}_1$, $\vec{y}_1$, $\vec{z}_1$ are unitary vectors.

$$k_0 H = [kE] = (k_y E_z - k_z E_y) x_1 + (k_z E_x - k_x E_z) y_1 + (k_x E_y - k_y E_x) z_1. \quad (19.4)$$

We may introduce the angles of incidence α, of reflection α' and of refraction β' (complex in the general case) (Fig. 19.1), and the gliding angles (also called angles of encounter) supplementing them to $\pi/2$, $\Psi = \frac{\pi}{2} - \beta$: $\psi = \frac{\pi}{2} - \alpha$, $\psi' = \frac{\pi}{2} - \alpha'$.)

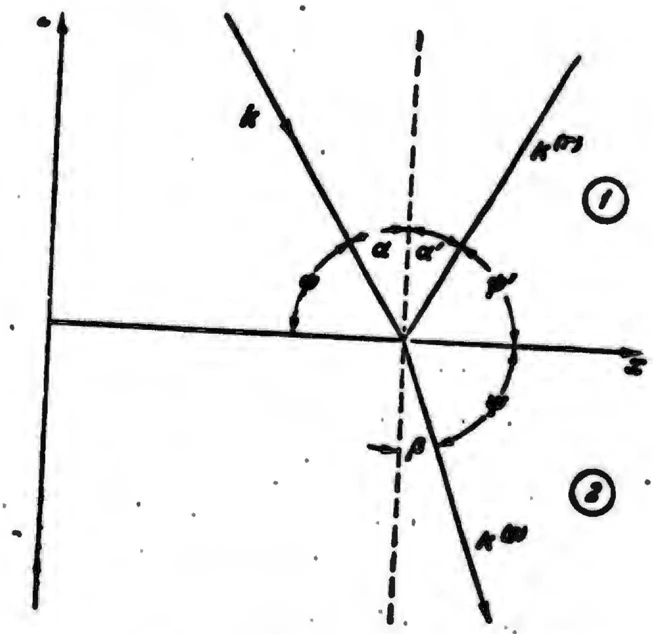

Fig. 19.1. Reflection and refraction of a plane wave.

$$k_x = k_1 \cos\psi, \quad k_x^{(r)} = k_1 \cos\psi', \; k_x^{(g)} = k_2 \cos\Psi; \; k_1 = k_0 \sqrt{\varepsilon_1};$$
$$k_z = -k_1 \sin\psi, \; k_z^{(r)} = k_1 \sin\psi', \; k_z^{(g)} = -k_2 \sin\Psi; \; k_2 = k_0 \sqrt{\varepsilon_2}. \quad (19.5)$$

Let us consider at the outset the incidence of a wave polarized perpendicularly to the incidence plane; omitting the temporal factor, we have

$$E^{(e)} = E_y^{(e)} = E^0 e^{ikr}. \quad (19.6)$$

From the six boundary conditions three are satisfied automatically and the remaining (for E_y, H_x and H_z) give:

$$e^{ik_1 x \cos\psi} + f_\perp e^{ik_1 x \cos\psi'} = g_\perp e^{ik_2 x \cos\Psi},$$
$$-k_1 \sin\psi e^{ik_1 x \cos\psi} + k_1 \sin\psi' f_\perp e^{ik_1 x \cos\psi'} = -k_2 \sin\Psi g_\perp e^{ik_2 x \cos\Psi}, \quad (19.7)$$
$$k_1 \cos\psi e^{ik_1 x \cos\psi} + k_1 \cos\psi' f_\perp e^{ik_1 x \cos\psi'} = k_2 \cos\Psi g_\perp e^{ik_2 x \cos\Psi}.$$

They may be satisfied for all $\underline{x}$ only in the case when $k_1 \cos\psi =$ $= k_1 \cos\psi' = k_2 \cos\Psi$; i.e., firstly, when the "angle of incidence" is equal to the "angle of reflection"; secondly, if the Snellius law is observed

$$\psi = \psi'; \qquad (19.8)$$

$$k_1 \cos\psi = k_2 \cos\Psi, \text{ i.e., } \cos\psi = n\cos\Psi. \qquad (19.9)$$

Introduced here is the complex "coefficient of refraction"

$$n^2 = \frac{\varepsilon_2}{\varepsilon_1}. \qquad (19.10)$$

It is obvious that in the general case Ψ is complex and devoid of ordinary sense. After that the two remaining independent equations (19.7) give

$$1 + f_\perp = g_\perp, \qquad (19.11)$$

$$k_1 \sin\psi (1 - f_\perp) = k_2 \sin\Psi g_\perp, \qquad (19.12)$$

whence *for the horizontal polarization*

$$f_\perp = \frac{E^{(r)}}{E^0} = \frac{k_1 \sin\psi - k_2 \sin\Psi}{k_1 \sin\psi + k_2 \sin\Psi} = \frac{\sin\psi - \sqrt{n^2 - \cos^2\psi}}{\sin\psi + \sqrt{n^2 - \cos^2\psi}}. \qquad (19.13)$$

In this way the field in the 1st and 2nd media are:

$$E_1 = E^0 \left(e^{ik_1 (x\cos\psi - z\sin\psi)} + f e^{ik_1 (x\cos\psi + z\sin\psi)}\right) e^{-i\omega t}, \qquad (19.14)$$

$$E_2 = g_\perp E^0 e^{ik_2 (x\cos\Psi - z\sin\Psi) - i\omega t}. \qquad (19.15)$$

The case of polarization *parallel* to the incidence plane is the object of identical consideration when

$$E_x^{(e)} = E^0 \sin\psi e^{ik_1 (x\cos\psi - z\sin\psi) - i\omega t},$$

$$E_z^{(e)} = E^0 \cos\psi e^{ik_1 (x\cos\psi - z\sin\psi) - i\omega t}, \quad E_y^{(e)} = 0. \qquad (19.16)$$

Alongside with the Snellius law (19.8) and (19.9), we obtain in exactly the same way

$$g_\parallel = \frac{1 + f_\parallel}{n} = \frac{2n\sin\psi}{n^2 \sin\psi + \sqrt{n^2 - \cos^2\psi}}, \qquad (19.17)$$

$$f_\parallel = \frac{n^2 \sin\psi - \sqrt{n^2 - \cos^2\psi}}{n^2 \sin\psi + \sqrt{n^2 - \cos^2\psi}}. \qquad (19.18)$$

Let us recall that in this case the coefficients $g_\parallel$ and $f_\parallel$ give only the electric field amplitudes and not those of its separate components. Thus, contrary to (19.14), (19.15) $E_z^{(r)} \neq f_1 E_z^{(e)}$ etc. However, such a simple relation still takes place for the only magnetic field component that is not zero: $H_y^{(r)} = f_1 H_y^{(e)}$, $H_y^{(t)} = g_1 H_y^{(e)}$.

It remains to see to it that the emission condition is satisfied. To that effect it is necessary that the field do not increase in the medium 2 when $z \to -\infty$. Consequently, when extracting the root from ε_2, we must choose the one which gives

$$\operatorname{Im} k_z^{(t)} < 0, \text{ т. е. } \operatorname{Im}(k_2 \sin \Psi) = k_0 \operatorname{Im}\left(\sqrt{\varepsilon_2^2 - \varepsilon_1^2 \cos^2 \psi}\right) < 0. \quad (19.19)$$

As to the medium 1, only the incident wave may increase at $z \to +\infty$, while the reflected wave must decrease:

$$\operatorname{Im} k_z^{(r)} > 0, \text{ i. e.}, \operatorname{Im} \sqrt{\varepsilon_1} > 0. \qquad (19.20)$$

We agreed that the incident wave is homogenous, that is, in it the directions of amplitude decrease and of phase propagation coincide (this is expressed in that α is real). The reflected wave will have the same property, but, generally speaking, the refracted wave is devoid of it.

Let us consider the incidence of a wave from the air on a plane surface of the soil, $\varepsilon_2 = \varepsilon$, $k_1 = k_0$, $n^2 = \varepsilon$, $\cos \Psi = \frac{1}{n} \cos \psi$. Since $|n| > 1$ (and for a sufficiently well conducting soil $|n| \gg 1$), $|\cos \Psi| < \cos \psi$. The reflected wave has the form

$$E^{(t)} = g_{\perp,\parallel} E^0, \quad E^0 = \exp\left\{ik_0\left[x \cos \psi - \sqrt[4]{(\varepsilon' - \cos^2 \psi)^2 + \left(\frac{4\pi\sigma}{\omega}\right)^2}\, z \cos X(\alpha) + \right.\right.$$

$$\left.\left. + \sqrt[4]{(\varepsilon' - \cos^2 \psi)^2 + \left(\frac{4\pi\sigma}{\omega}\right)^2}\, z \sin X(\alpha)\right]\right\}, \qquad 19.21)$$

$$X(\alpha) = \frac{1}{2} \operatorname{arctg} \frac{4\pi\sigma}{\omega(\varepsilon' - \cos^2 \psi)} \quad (z < 0).$$

The amplitude decrease takes place in the negative direction of the axis $\underline{z}$, and the phase propagation takes place at a certain angle γ to that direction,

$$\operatorname{tg}\gamma = \frac{\cos\psi}{\sqrt[4]{(\varepsilon'-\cos^2\psi)^2+\left(\frac{4\pi\sigma}{\omega}\right)^2}\cos\left\{\frac{1}{2}\operatorname{arctg}\frac{4\pi\sigma}{\omega(\varepsilon'-\cos^2\psi)}\right\}}. \qquad (19.22)$$

The rate of amplitude decrease and also the angle γ depend on the incidence angle α; however, if $\cos^2\psi = \sin^2\alpha \ll \varepsilon'$ or $\sin^2\alpha \ll \frac{4\pi\sigma}{\omega}$, and, consequently, in general, if $\sin^2\alpha \ll |\varepsilon|$, the damping is not dependent on ψ, and the value of γ is very small and much smaller than α. At the same time, the field variation along the axis x does not depend on the medium's properties: in the soil the field is somehow tied up to the field at the surface and is transferred alongside with it along that axis (cf. §14).

Let us now consider the inverse case, i.e., the incidence from the soil (medium 1) to the interface with the air (medium 2) $|\varepsilon| < 1$; in this case we have

$$k_{1x} = k_{2x} = k_0\sqrt{\varepsilon}\cos\psi,\ k_{2z} = -k_0\sin\Psi = -k_0\sqrt{1-\varepsilon\cos^2\psi}.$$

If ε is real, k_{2z} may become purely imaginary. Under these conditions there would be only exponential damping along the axis z in the air, and not propagation. Th is is the well known in optics total reflection from the interface. It thus takes place at

$$\cos\psi = \sin\alpha > \frac{1}{\sqrt{\varepsilon}}. \qquad (19.23)$$

For a complex ε the penetration becomes possible.

Let us consider the reflected wave at incidence from the air at further length.

At horizontal polarization $f_\perp$ (19.13) for a medium with a real n the reflection factor cannot become zero: $f_\perp = 0$ happens only for $n = 1$, but $f_\parallel = 0$ at $n^4\sin^2\psi = n^2 - \cos^2\psi$, that is, at incidence at the Brewster angle $\alpha_B = \frac{\pi}{2} - \psi_B$:

$$\operatorname{ctg}\psi_B = \operatorname{tg}\alpha_B = n = \sqrt{\varepsilon}. \qquad (19.24)$$

However, for complex $|\varepsilon|$ the reflection in that direction of a wave with electric vector in the incidence plane becomes possible, though $f_{\parallel}$ has precisely a minimum here (*pseudo-Brewster* angle). For certain media at wave incidence from the air, the moduli and the phases of the reflection factors are shown in Figs. 19.2 and 19.3, where it is postulated

$$f_{\parallel} = \rho_{\parallel} e^{i\theta_{\parallel}}, \quad f_{\perp} = \rho_{\perp} e^{i\theta_{\perp}} \tag{19.25}$$

For large $|\varepsilon|$ the reflection coefficient $f_{\parallel}$ (19.18) may be written as follows:

$$f_{\parallel} \approx \frac{\sqrt{\varepsilon}\sin\psi - 1}{\sqrt{\varepsilon}\sin\psi + 1}. \tag{19.26}$$

Consequently, at $|\sqrt{\varepsilon}\sin\psi| \gg 1$ we have an ideal reflection: $f_{\parallel} = 1$, $g_{\parallel} = 0$. Note, on the other hand, that at $\psi \to 0$ we have $f_{\perp,\parallel} \to -1$, $g_{\parallel,\perp} \to 0$, that is, total reflection into the 1st medium takes place again, this time, however, already with a phase shift by π. At the same time the admissibility of such a transition still remains obscure if only because this result depends, as can be seen, on the order of transition to the limit $n \to \infty$, $\psi \to 0$.

2. The field substitution by a plane wave is obviously quite admissible when there is question, for instance, of a well collimated ray of decimeter or centimeter waves from a raised source. However, in the general case we are confronted with sources of limited dimensions. The surface region (§12) essential for reflection may be so large that within its bounds the incidence angle of the ray from the emitter may assume the most different values. The introduction for them of a unique reflection factor may result inadmissible. This is why it is necessary to especially ascertain to what extent it is possible to represent, for example, the field of a point vertical dipole in the air in the form of a sum of direct and reflected waves

$$\Pi = \frac{e^{ik_0R}}{R} + f\frac{e^{ik_0R_1}}{R_1},$$
$$R = \sqrt{x^2 + y^2 + (z - z_0)^2}, \quad R_1 = \sqrt{x^2 + y^2 + (z + z_0)^2}. \qquad (19.27)$$

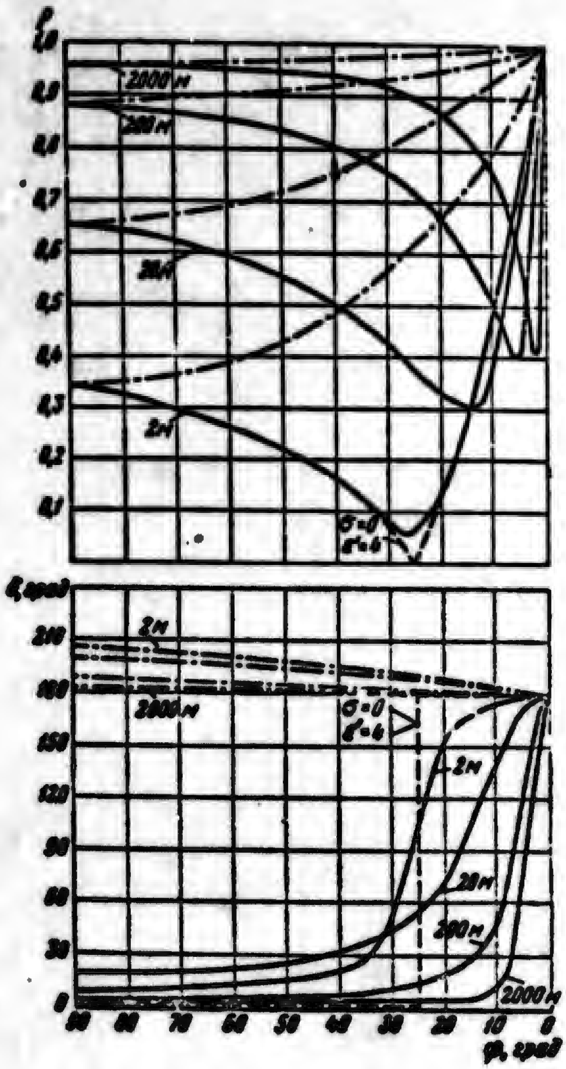

Fig. 19.2. Moduli and phases of reflection factors for a plane wave, of which the electric vector lies in the plane of incidence ($\rho_\parallel$; $\delta_\parallel$: dot-and-dash lines), or is perpendicular to it ($\rho_\perp$; $\delta_\perp$: solid lines at incidence from the air on a plane surface of land $\varepsilon'=4$, $\sigma=9\cdot10^7$ CGSE; the dashed line is the extreme case for $\sigma = 0$.

where z_0 is the height of dipole ascent, R_1 is the distance over the plane to the imaginary source, and f is the reflection factor. The region of approximate formula's applicability may be estimated to a full measure only knowing a more exact solution. However, we shall already be convinced now that Formula (19.27) may be obtained if in the process of deduction specific approximations are made, namely, if 1) we satisfy the boundary conditions not over the entire plane but only within the limits of the essential zone and 2) we admit that f and the amplitude

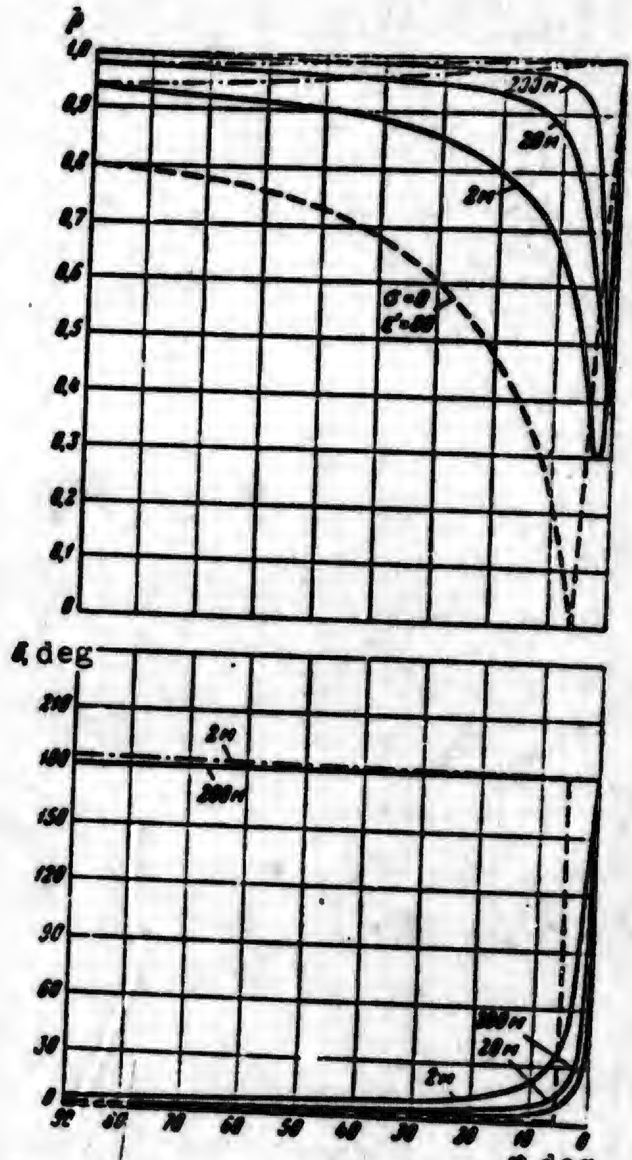

Fig. 19.3. Same as in Fig. 19.2, for incidence from the air on the sea surface $\varepsilon'=80$, $\sigma=3.6\cdot10^{10}$ CGSE

of the passed wave are functions of the point, but varying so slowly that at substitution of the field into the wave equation, their derivatives may be neglected. We will show that the field Π (19.27) and the field of the refracted wave in the soil, Π_1, provide at these admissions a good approximation, and we will obtain a quantitative criterion of this approximation's validity.

We shall expand R and R_1 in series near the correct mirror reflection point $(x_0, 0, 0)$, of which the distance to the point is $R_0 = \sqrt{x_0^2 + z_0^2}$:

$$R \approx R_0 + (x - x_0)\cos\psi + \frac{y^2}{2R_0} - z\sin\psi,$$

$$R_1 \approx R_0 + (x - x_0)\cos\psi \frac{y^2}{2R_0} + z\sin\psi.$$

We shall seek the field in the soil not in the form e^{ikr}, which would be simplest, whereas for obtaining a greater precision we shall add the factor

$$\frac{1}{R_0}\exp\left\{i\left(k_0R_0 - k_x x_0 + k_0\frac{y^2}{2R_0}\right)\right\},$$

little dependent on coordinates. We shall namely assume

$$\Pi_{1z} = \Pi_1 = g\frac{e^{ik_0\left(R_0 - \frac{y^2}{2R_0}\right)}}{R_0}e^{i[k_x(x - x_0) + k_y y - k_z z]}, \qquad (19.28)$$

$$k_x^2 + k_y^2 + k_z^2 = k^2 = k_0^2\varepsilon.$$

The sign minus, before z corresponds to the condition (19.20).

For the function to satisfy the wave equation in the soil we must have (substituting the value of (19.28) into the wave equation and rejecting the terms of the order $1/kR_0$):

$$-k_x^2 - k_y^2 - k_z^2 - k_0^2\frac{y^2}{R_0^2} - 2k_0k_y\frac{y}{R_0} + k^2 = 0. \qquad (19.29)$$

(Moreover, the derivatives from g with respect to each of the axes must be small by comparison respectively with $k_x g$, $k_y g$, and $k_z g$).

The terms entering here and depending on coordinates are very small within the bounds of the essential zone. Indeed, if we neglect them, we shall obtain the usual value $k_x^2 + k_y^2 + k_z^2 = k^2 = \varepsilon k_0^2$. This is why the terms with y and y^2 have, within the zone, i.e., so long as $y^2 < \frac{R_0}{k_0}$, the order

$$\frac{k_y^2 y^2}{R_0^2} < \frac{k_0}{R_0}, \quad 2k_0 k_y \frac{y}{R_0} < 2\sqrt{\varepsilon}\, \frac{k_0^{3/2}}{\sqrt{R_0}}.$$

In this way they differ from the principal multipliers $\left|\frac{1}{k_0 R_0}\right| \ll 1$ and $\left|\frac{1}{\sqrt{\varepsilon k_0 R_0}}\right| \ll 1$. Consequently, within the limits of the essential zone the function (19.28) admitted by us, is sufficiently near the plane wave and it satisfies approximately the wave equation.

With the help of the functions written it is easy to also satisfy within the limits of the essential zone the boundary conditions

$$\Pi_z = \varepsilon \Pi_{1z}, \quad \frac{\partial \Pi_z}{\partial z} = \frac{\partial \Pi_{1z}}{\partial z} \quad \text{at } z = 0. \qquad (19.30)$$

Substituting here the functions (19.27) and (19.28), and neglecting at differentiation the terms of the order $1/k_0 R_0$, we shall obtain

$$(1+f)\frac{e^{ik_0\left(R_0 + (x-x_0)\sin\alpha + \frac{y^2}{2R_0}\right)}}{R_0} = \varepsilon g \frac{e^{ik_0\left(R_0 + \frac{y^2}{2R_0}\right)}}{R_0} e^{i[k_x(x-x_0) + k_y y]}, \qquad (19.31a)$$

$$k_0(1-f)\frac{e^{ik_0\left(R_0 + (x-x_0)\sin\alpha + \frac{y^2}{2R_0}\right)}}{R_0}\cos\beta = \frac{e^{ik_0\left(R_0 + \frac{y^2}{2R_0}\right)}}{R_0} g k_z e^{i[k_x(x-x_0) + k_y y]} \qquad (19.31b)$$

Hence stem the previous formulas for the laws of refraction and reflection (19.9), (19.17) and (19.18). But now we are in a position to evaluate the limits of their applicability, determined by the neglects made at deduction (see [I.9]).

Firstly, we simplified the relation (19.29) estimating that $|\sqrt{\varepsilon k_0 R_0}| \gg 1$. But we consider this condition as always observed.

Secondly, the fields found must satisfy the wave equation. Since $\cos\psi = \frac{x}{R_1}$, then, for example, $\frac{\partial f}{\partial x} \approx -\frac{\sin\psi}{R_1}\frac{\partial f}{\partial \psi}$, $\frac{\partial f}{\partial y} = 0$, $\frac{\partial f}{\partial z} \approx \frac{\cos\psi}{R_1}\frac{\partial f}{\partial \psi}$. Inasmuch as $\frac{\partial f}{\partial \psi}$ and $\frac{\partial g}{\partial \psi}$ are quantities of the order f or g, this means that the wave equation too is approximately satisfied at $kR_1 \gg 1$.

Finally, and this is the most essential, the coefficients f and g in the relations (19.31) were considered constant, while in the result they were obtained dependent on ϕ. Therefore, it is necessary to require that their variation within the bounds of the essential zone, where the boundary conditions must be fulfilled, may be neglected.

In order to account for the variability of f within the bounds of the essential zone in the boundary conditions (19.30), we must utilize not the value $f(\alpha)_0$ of the function $f(\alpha)$ at the center of this zone, but, for example, the expansion in series

$$f(\alpha) = f(\alpha_0) + \Delta\alpha f'(\alpha_0) + \frac{(\Delta\alpha)^2}{2} f''(\alpha_0) + \dots \tag{19.32}$$

As may be seen from Fig. 19.4, the shift along the axis x from the point of correct reflection is

$$\xi = \frac{R\Delta\alpha}{\cos\alpha_0}, \tag{19.33}$$

so that for maximum-remote point of the essential, zone for which [see (12.7)] $\xi \sim \frac{1}{\cos\alpha_0}\sqrt{\frac{R_0}{k_0}}$, we have

$$|\Delta\alpha| \sim \frac{1}{\sqrt{k_0 R_0}}. \tag{19.34}$$

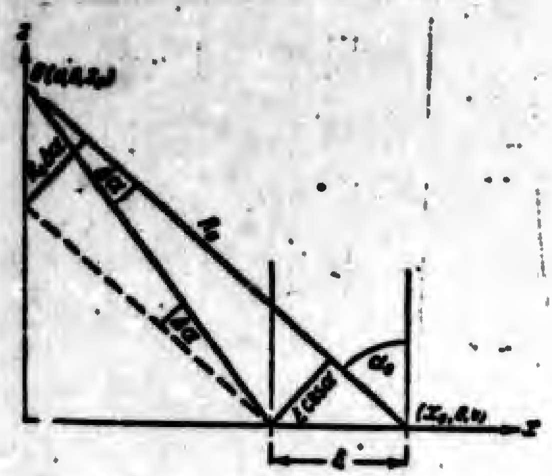

Fig. 19.4. To the deduction of the criterion of validity of interference formulas.

This is why we may neglect the variation of the factor $(1 + f)$ in the relation (19.31a), provided

$$|1 + f(\alpha_0)| \gg \left| f\left(\alpha_0 \pm \frac{1}{\sqrt{k_0 R_0}}\right) - f(\alpha_0)\right|. \tag{19.35}$$

The terms of first order in $\Delta\alpha$ may be disregarded, for having added them we would have obtained on the ground surface a complementary field, providing in the aggregate a zero result at the point of observation, inasmuch as it has in various halves of the zone an amplitude equal in magnitude but inverse by sign. This is why only the terms of second order have to be taken into account, that is, one must require that

$$\left|\frac{1}{2k_0R_0} f''(\alpha_0)\right| \ll |1+f(\alpha_0)|. \qquad (19.36)$$

When computing the right-hand part we shall take into account that disruptions of reflection formulas may be expected only in case of small angles. This is why $\sin\alpha \approx 1$, but $\cos\alpha = \cos\left(\frac{\pi}{2}-\psi\right) \approx \psi$, and

$$f \approx \frac{\varepsilon\psi - \sqrt{\varepsilon-1}}{\varepsilon\psi + \sqrt{\varepsilon-1}}, \qquad (19.37a)$$

$$f' \approx \frac{2\varepsilon\sqrt{\varepsilon-1}}{(\varepsilon\psi+\sqrt{\varepsilon-1})^2}, \quad f'' \approx -\frac{4\varepsilon^2\sqrt{\varepsilon-1}}{(\varepsilon\psi+\sqrt{\varepsilon-1})^3}. \qquad (19.37b)$$

Furthermore, we shall take into account that only the cases $\sqrt{\varepsilon}\psi < 1$, are of interest, for otherwise we would arrive at ideal reflection (cf. (19.25)), and the formulas for the reflection factor are knowingly correct (of which we shall be convinced in §26). This is why $\varepsilon\psi$ in the denominator may be rejected, considering that $\sqrt{\varepsilon-1}$ is in any case a quantity of the order of the unity. With these neglects Correlation (19.36) takes the simple form

$$\left|\frac{2\varepsilon\psi}{\sqrt{\varepsilon-1}}\right| \gg \left|\frac{4\varepsilon^2}{(\varepsilon-1)k_0R_0}\right|,$$

that is, inasmuch as $R_0\psi = z_0$,

$$k_0z_0 \gg \left|\sqrt{\frac{\varepsilon^3}{\varepsilon-1}}\right| \sim |\sqrt{\varepsilon+1}|. \qquad (19.38)$$

The above expounded method is valid only when the essential zone has a symmetrical shape. In the opposite case it is not possible to neglect the terms linear in $\Delta\alpha$. This presumes that z and z_0 are of same order. However, in the following we shall have the opportunity to ascertain that the emitter field is dependent only on the sum of emitter and observation point heights (§26). This is why the condition of applicability of the reflection factor method is the fulfillment of the inequality

$$k(z+z_0) \gg |\sqrt{\varepsilon+1}|. \qquad (19.39)$$

A more rigorous consideration (see (26.28)) leads to the same result. Therefore, the interference ("reflection") formulas are applicable if the source and the point of observation are raised sufficiently high above the interface. In case of long waves, when we have, moreover, $|s| \gg 1$, it becomes difficult to satisfy the criterion (19.39). An entirely different approach is required here, to which a series of subsequent chapters are devoted. At the same time, these formulas are entirely sufficient for the consideration of an enormous majority of problems arising at the study of the propagation of ultrashort and still shorter waves above ground in a uniform atmosphere. Thus, even if ε passes to ε' and becomes a quantity of the order of the unity even for land, it is sufficient that the source be raised by a few wavelengths. Precisely this is always materialized in the indicated band.

3. Let us consider the expression for the field, stemming from interference formulas for the case when the gliding angle is small, with the result that, according to Relations (19.13) and (19.18), the reflection factor is quite near -1. Assume that within the bounds of angles of interest to us the source emits a radiation independent from the angle. At the point of observation the field is composed of the directly arrived wave (along the path $OA = R$) and of the reflected wave, with the reflection factor $f = -1$, thus, in essence, simply with phase shift by π. It may be seen from Fig. 19.5 that the path of the reflected ray from 0 to A is equal to the path R' from the source's image O' in the plane $z = 0$ to A. We shall consider $R \approx x \gg z_0,\ x \gg z_A$. Then

$$R = \sqrt{x^2+(z_A-z_0)^2} \approx \sqrt{x^2+z_A^2+z_0^2} - \frac{z_A z_0}{\sqrt{x^2+z_A^2+z_0^2}} \approx$$

$$\approx \sqrt{x^2+z_A^2+z_0^2} - \frac{z_A z_0}{x},$$

$$R' = \sqrt{x^2+(z_A+z_0)^2} \approx \sqrt{x^2+z_A^2+z_0^2} + \frac{z_A z_0}{x},$$

$$R'_0 - R \approx 2\frac{z_A z_0}{R}. \qquad (19.40)$$

Let us consider at the outset the horizontal polarization of the ray $E = E_y$. Then the field at the point of observation is

$$E = E^0 (e^{ikR} + f_1 e^{ikR'}) = E^0 e^{ikR}(1 - e^{ik\frac{2z_A z_0}{R}}) = E^{(e)}(1 - e^{i\frac{2kz_0 z_A}{R}}). \qquad (19.41)$$

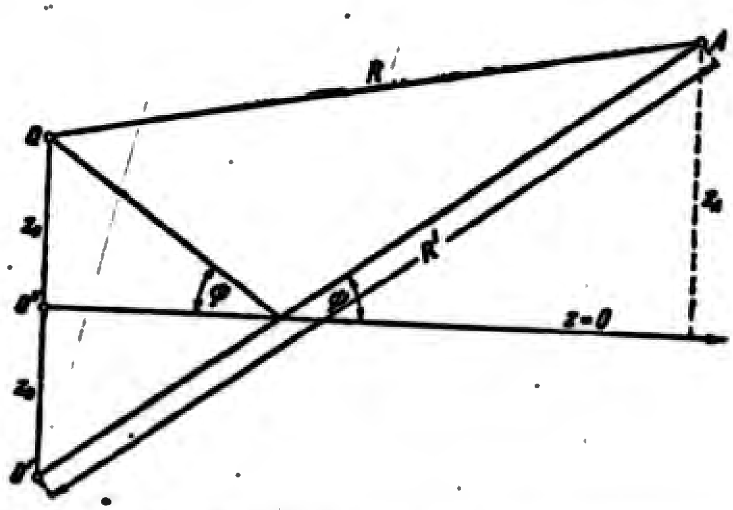

Fig. 19.5. Denotations in the quadratic formula.

For very small gliding angles, introducing the attenuation factor <u>w</u> relative to the incident field $E^{(e)}$, we obtain

$$E = E^{(e)} w, \quad E^{(e)} = E^0 e^{ikR}, \quad w \approx -\frac{2ikz_0 z_A}{R}. \qquad (19.42)$$

If we account for the fact that the field E^0 itself decreases inversely proportionally to the distance, it follows therefrom that for maximum-small angles the field decreases inversely proportionally to the square of the distance. This simple formula, obtained by B.A. Vvedenskiy [6], is one of the basic formulas for the calculation of ultra-short wave fields.

If the gliding angle is not small enough for the sine to be replaced by the argument, it stems from formula

$$w = 1 - e^{2ik\frac{z_A z_0}{R}} = e^{ik\frac{z_A z_0}{R}}(-2i)\sin\frac{kz_A z_0}{R},$$

that is,

$$|E| = 2\left|\sin\frac{kz_A z_0}{R}\right||E^0|, \tag{19.43}$$

that as the mutual disposition of the corresponding points varies, the field variation acquires a quite complex character. Thus, if for given z_0 and R we vary z_A (for example, by shifting along a circumference with center at 0), the field will pass in sequence through maxima and minima, whereas the corresponding pattern has a "petal-shaped" character ("petal-shaped" pattern). The first petal does not encompass the ground surface. Accordingly, if one moves along the horizontal at a specific height, the field variation will have the character of alternating maxima and minima as one drifts away farther, so long as ψ does not become so small that the observation point is found to be located under the maximum of the first petal. After that the field will decrease monotonically (inversely to the square of the distance).

The matter is somewhat more complex with vertical polarization of the ray. In the case of horizontal polarization we could consider $f_\perp \approx -1$, starting from Formula (19.13), for in the region of formula applicability according to Relation (19.38), the least admissible angle ψ has the order $\psi > \frac{|n|}{kR}$, so that for it $\sin\psi \ll \sqrt{n^2 - \cos\psi}$. Meanwhile, in Formula (19.18) we shall obtain for the least admissible ψ $n^2\sin\psi > \frac{|n^3|}{kR}$, and this quantity can be rejected as small by comparison with $\sqrt{n^2 - \cos\psi}$ only if

$$kR \gg |n^2| = |\varepsilon|. \tag{19.44}$$

For ultrashort radiowaves above land (and for centimeter waves always), i.e., in the most important region of applicability of reflection formulas, this is exactly what takes place. That is why we may again con- $f_1 = -1$ and repeat all the previous reasonings leading to the very same petal-shaped pattern.

Only for comparatively long waves, when $|\varepsilon|$ may be measured by great numbers, particularly above the sea, can the inverse situation spring up so that $n^2 \sin\psi$ will be greater than $\sqrt{n^2 - \cos^2\psi}$. Then we shall have $f_1 = +1$ and instead of Formula (19.43) we shall obtain

$$w = 1 + e^{ik\frac{z_0 z_A}{a}}, \quad kR \ll |n^2|. \qquad (19.45)$$

Thus, here the field on the ground surface has a maximum for $z_A = 0$, $w = 2$, and the petal will be pressed against the ground.

For a broad, graphic and numerical material on reflection formulas illustrating their application see, for example, the books [I.2; I.6; I.7; I.8].

§20. INFINITELY CONDUCTING PLANE SURFACE

We have considered at the end of the preceding section the particular case, when the absolute value of the dielectric constant $|\varepsilon|$ is so great that even at small gliding angles for a vertically polarized wave the reflection factor becomes +1, and for a horizontally polarized wave it becomes −1. It is appropriate to pause at that case in more detail. Thus we shall consider the interface between ground and (uniform) atmosphere as plane, and the soil – as ideally conducting.

It is almost obvious that for sufficiently small distances between corresponding points we may neglect the ground curvature. This is indeed admissible for waves longer than 100 m, as will be subsequently shown, at the very least through distances of the order of 100 km.

It seems that in the wave band of interest to us it is also possible to satisfy the second condition, for the conduction is not essential in itself but in the combination $4\pi\sigma/\omega$. Meanwhile, we have seen in

§15 that usually this parameter is great. However, here the position is not that simple. Even for great values of parameter $4\pi\sigma/\omega$ the field at the ground may extremely strongly differ from a field above an ideally conducting surface provided one moves sufficiently far away from the source. We have seen, indeed (see (19.44) and subsequent remarks), that $f_{\parallel}$ becomes +1 only when the absolute value of ε is so great that $\lambda|\varepsilon| \gg 2\pi R$, that is, when the distance to the source is not too large. Only after that, when considering the more general problem – a spherical and not ideally conducting ground, may we obtain more rigorous criteria. They confirm the above-said, and therefore at sufficiently small distances from the source both simplifications are acceptable.

Having admitted these simplifications, we find ourselves confronted with the problem of field determination of the given sources (situated in the upper half-space) at boundary conditions stemming from the conditions (4.1) and (4.5) as the extreme case. Inasmuch as in an ideally conducting soil E = 0 (otherwise an infinitely dense current would be flowing there), the tangential component of the electric field must vanish at the interface:

$$E_t = 0. \qquad (20.1)$$

From equation rot $E = ikH$ follows the fact that the magnetic field too vanishes in the soil. This is why according to Formula (4.5) the normal component of the magnetic field vanishes also at the interface:

$$H_n = 0. \qquad (20.2)$$

The tangential component of the magnetic field does not vanish, for according to Formula (4.6a) it is compensated by the superficial current, just as does the normal component of $\vec{E}$, compensated by pulsating surface charge (cf. (4.3b)).

Introducing a system of rectangular coordinates x, y, z, in which the ground serves as the plane z = 0, we obtain

$$E_x(x, y, 0) = E_y(x, y, 0) = H_z(x, y, 0) = 0. \qquad (20.3)$$

The required solution may be obtained entirely elementarily.

Let us visualize (Fig. 20.1) an element of current in the antenna, divided in two – the vertical j_{0z} and the horizontal one, for example, j_{0x}, and let us consider them separately. If the vertical current element j_{0z} is completed by its reflection $j_{0z}^{(1)}$ in the plane $z = 0$ with the same direction of the current, each of the vertical dipoles will give on the surface $z = 0$ only the horizontal component of the magnetic field, as this is seen from the expressions (18.3b) for a point dipole field, and, therefore satisfy the boundary condition (20.2).

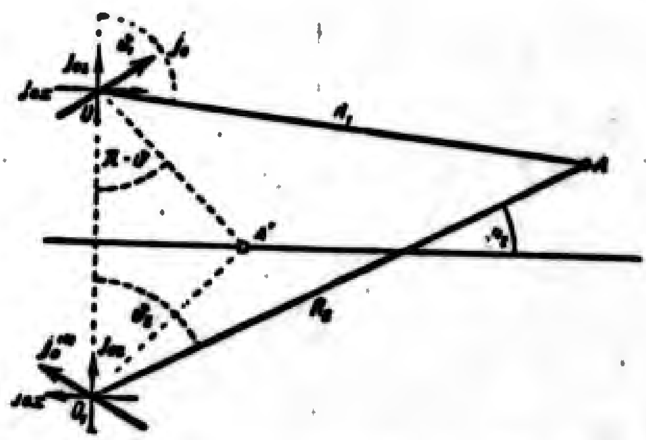

Fig. 20.1. Disposition of the source and of its "image."

On the other hand, the horizontal dipole j_{0x} may also be complemented by its reflection, but we shall change the sign at the reflection. Then the vertical component of the magnetic field intensity at the ground surface (in the system of coordinates utilizing in the formula of §18 this is H_φ) will be exactly quenched by the dipole field $j_{0x}^{(1)} = -j_{0x}$. Consequently, the combination of source $\vec{j}_0$ and $\vec{j}_0^{(1)}$ will given in a boundless medium a field, which satisfies on the surface $z =$ $= 0$ the boundary conditions for a magnetic field above an ideally conducting ground. Moreover, this combination satisfies also the boundary condition for the electric field (20.1). As a matter of fact, utilizing Formulas (18.3), it is easy to be convinced that the tangential components of the electric fields of dipoles j_{0x} and $j_{0x}^{(1)}$ are mutually annihilated at any point of the surface $z = 0$, just as the tangential components of the fields of sources j_{0z} and $j_{0z}^{(1)}$. Effecting a similar construction of reflected sources for all the elements of the antenna, we obtain the complete solution of the problem formulated, inasmuch as the field of such a combination of sources satisfies the wave equation and the boundary conditions.

Therefore, quite rigorously, without any neglects, for example, not only in the wave zone of the emitter, but also in the nearer zone, the field of any source (or any assortment of sources), above an ideally conducting plane, is composed of the field of that source and of the field of its "reflection" in the plane constructed according to a specific rule: to each of the points x', y', z' of the source, in which flows the current with density $j_0(j_{0x}, j_{0y}, j_{0z})$, it is necessary to juxtapose a reflected source at the point x', y', $-z'$ with current density $j_0^{(1)}(-j_{0x}, -j_{0y}, j_{0z})$. We shall designate at times this result as the rigorous theorem on reflection. We shall bring forth, for reference, the explicit expressions for fields of vertical and horizontal electric dipoles above an ideally conducting surface.

Assume that there is disposed at the point O (0, 0, z_0) a *vertical* (index $\underline{v}$) *electric dipole* of moment $\underline{p}$. It would give in a free space (in vacuum) a field with a one-component Hertz vector $\Pi^0 = \Pi_z^0 = p e^{ikR}/R$ (see (18.1)). According to the rigorous theorem on reflection, above an ideally reflecting plane we have

$$\Pi^v = \Pi_z^v = p\left(\frac{e^{ikR_1}}{R_1} + \frac{e^{ikR_2}}{R_2}\right), \quad k = \frac{\omega}{c},$$

$$R_1 = \sqrt{x^2 + y^2 + (z - z_0)^2},$$

$$R_2 = \sqrt{x^2 + y^2 + (z + z_0)^2}. \tag{20.4}$$

This is why, introducing for the vectors $\vec{R}_1$ and $\vec{R}_2$ the polar angles ϑ_1 and ϑ_2 (see Fig. 201.), and the general azimuth φ, and making use of Formulas (3.25) we obtain in the wave zone

$$\begin{aligned}
E_x^v &= -k^2 p\left(\sin\vartheta_1\cos\vartheta_1\frac{e^{ikR_1}}{R_1} + \sin\vartheta_2\cos\vartheta_2\frac{e^{ikR_2}}{R_2}\right)\cos\varphi,\\
E_y^v &= -k^2 p\left(\sin\vartheta_1\cos\vartheta_1\frac{e^{ikR_1}}{R_1} + \sin\vartheta_2\cos\vartheta_2\frac{e^{ikR_2}}{R_2}\right)\sin\varphi,\\
E_z^v &= k^2 p\left(\sin^2\vartheta_1\frac{e^{ikR_1}}{R_1} + \sin^2\vartheta_2\frac{e^{ikR_2}}{R_2}\right).
\end{aligned} \tag{20.4a}$$

$$\left.\begin{aligned}
H_x^v &= k^2 p\left(\sin\vartheta_1\frac{e^{ikR_1}}{R_1} + \sin\vartheta_2\frac{e^{ikR_2}}{R_2}\right)\sin\varphi\\
H_y^v &= -k^2 p\left(\sin\vartheta_1\frac{e^{ikR_1}}{R_1} + \sin\vartheta_2\frac{e^{ikR_2}}{R_2}\right)\cos\varphi
\end{aligned}\right\}, \quad H = H_\varphi;$$

$$H_z^v = 0.$$

It is obvious that for a *horizontal electric dipole* (index h), oriented *along the axis x*, we have in free space $\Pi^0 = \Pi_x^0 = p\frac{e^{ikR}}{R}$. This is why above an ideally conducting plane

$$\Pi^h = \Pi_x^h = p\left(\frac{e^{ikR_1}}{R_1} - \frac{e^{ikR_2}}{R_2}\right);$$

$$\begin{aligned}
E_x^h &= +k^2 p\left((1-\sin^2\vartheta_1\cos^2\varphi)\frac{e^{ikR_1}}{R_1} - (1-\sin^2\vartheta_2\cos^2\varphi)\frac{e^{ikR_2}}{R_2}\right),\\
E_y^h &= -k^2 p\left(\sin^2\vartheta_1\frac{e^{ikR_1}}{R_1} - \sin^2\vartheta_2\frac{e^{ikR_2}}{R_2}\right)\sin\varphi\cos\varphi,\\
E_z^h &= -k^2 p\left(\sin\vartheta_1\cos\vartheta_1\frac{e^{ikR_1}}{R_1} - \sin\vartheta_2\cos\vartheta_2\frac{e^{ikR_2}}{R_2}\right)\cos\varphi.\\
H_x^h &= 0,\\
H_y^h &= k^2 p\left(\cos\vartheta_1\frac{e^{ikR_1}}{R_1} - \cos\vartheta_2\frac{e^{ikR_2}}{R_2}\right),\\
H_z^h &= -k^2 p\left(\sin\vartheta_1\frac{e^{ikR_1}}{R_1} - \sin\vartheta_2\frac{e^{ikR_2}}{R_2}\right)\sin\varphi.
\end{aligned} \tag{20.4b}$$

Finally, for a *horizontal electric dipole* oriented *along the axis y*, we have

$$\Pi^h = \Pi_y^h = p\left(\frac{e^{ikR_1}}{R_1} - \frac{e^{ikR_2}}{R_2}\right).$$

$$
\begin{aligned}
E_x^h &= -k^2 p\left(\sin^2\vartheta_1 \frac{e^{ikR_1}}{R_1} - \sin^2\vartheta_2 \frac{e^{ikR_2}}{R_2}\right)\sin\varphi\cos\varphi,\\
E_y^h &= k^2 p\left((1-\sin^2\vartheta_1\sin^2\varphi)\frac{e^{ikR_1}}{R_1} - (1-\sin^2\vartheta_2\sin^2\varphi)\frac{e^{ikR_2}}{R_2}\right),\\
E_z^h &= -k^2 p\left(\sin\vartheta_1\cos\vartheta_1 \frac{e^{ikR_1}}{R_1} - \sin\vartheta_2\cos\vartheta_2 \frac{e^{ikR_2}}{R_2}\right)\sin\varphi, \qquad (20.4c)\\
H_x^h &= -k^2 p\left(\cos\vartheta_1 \frac{e^{ikR_1}}{R_1} - \cos\vartheta_2 \frac{e^{ikR_2}}{R_2}\right),\\
H_y^h &= 0,\\
H_z^h &= k^2 p\left(\sin\vartheta_1 \frac{e^{ikR_1}}{R_1} - \sin\vartheta_2 \frac{e^{ikR_2}}{R_2}\right)\cos\varphi.
\end{aligned}
$$

In particular, for the points on the plane $z = 0$ we have $R_1 = R_2$, $\vartheta_2 = \pi - \vartheta_1$, $\sin\vartheta_1 = \sin\vartheta_2$, $\cos\vartheta_1 = -\cos\vartheta_2$. This is why, the tangential components of the electric field and the vertical component of the magnetic field vanish as must be, while the vertical component of the electric field and the tangential components of the magnetic field are doubled.

Consequently, a vertical antenna placed on an ideally conducting surface induces the same field as an antenna of doubled length would induce in the free atmosphere (the antenna itself plus its "image").*

This descriptive interpretation allows us to obtain still one more rather general theorem for an ideally conducting surface.

Let us consider a protuberance T on an ideally conducting surface and place above this surface an arbitrary emitter O (Fig. 20.2a). It may be asserted that above the plane $z = 0$ the field will be the same as in the following equivalent problem (Fig. 202.2b).

Assume that in the boundless empty space there is preserved only the protuberance T, of interest to us, completed by its reflection T' in the plane $z = 0$. Let there be further, besides the true source O, its image O', in the sense described above. The field of these two sources in vacuum in the presence of a volume scatterer will precisely provide us with the solution of the problem sought for. In reality, when the field was considered at z = 0, the presence of an ideally conducting half-space was manifest only in that the condition of vanishing

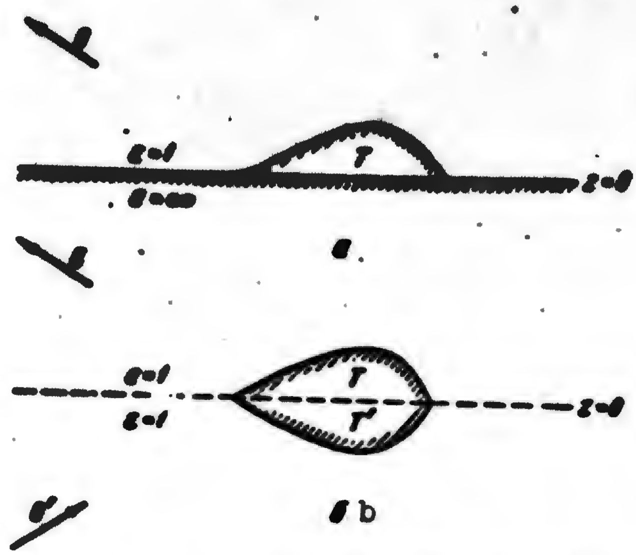

Fig. 20.2. Emitter O in the presence of the protuberance T on an infinitely conducting plane (a) and the equivalent problem: a source O and its image O' in the presence of T and of its reflection T' in the vacuum.

of the electric field's tangential component was superimposed on the plane z = 0. But in the equivalent problem described above the same condition is fulfilled owing to problem's symmetry. In particular, every dipole induced on the surface of the protuberance will emit a field which must also satisfy the indicated condition at z = 0. But this too is assured by symmetry, since the dipole on the surface of protuberance's reflection T' will also emit a field. It will act as a reflected source for the dipole induced on the protuberance.

The simplest application of these considerations is related to the scattering of a plane wave by a plane screen with a rectilinear edge, which is placed on an ideally conducting plane (Fig. 20.3a). Here one may consider the field of two symmetrical waves incident upon the screen in vacuum and completed by its reflection (Fig. 20.3b). We observe at the point of observation A the field of two waves, each of which diffracts on the two edges of the screen; in other words, only four rays arrive: OTA, O'TA, O'T'A and OT'A. It is easy to see that, physically, the question evolves here about rays (Fig. 20.3a), of which

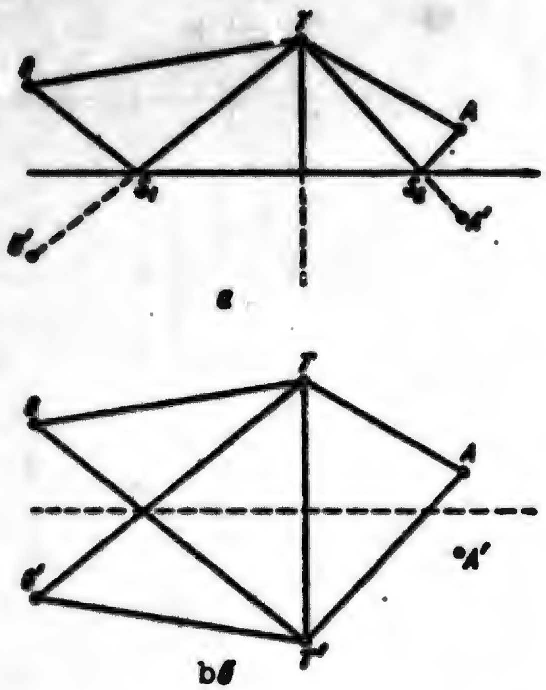

Fig. 20.3. Screen T on an ideally conducting plane (a) and the equivalent problem: screen T and its reflection T' in vacuum (b).

one (OTA) diffracts directly on the rectilinear edge, the second (OS_1-TA) is preliminarily reflected from the plane z = 0 ahead of the screen, the third (OTS_2A) is reflected from the plane z = 0 after diffraction, and the fourth (OS_1TS_2A) is subject to two reflections. Finally, instead of considering in a single observation point A the diffraction of the fields of two sources on a doubled screen, we may consider that we have only one edge of the screen T, but the field of the two sources O and O' is viewed as in the real observation point A, as well as in its image A'; at the same time, all the four fields add up. In such case to the above referred-to four rays correspond the rays OTA, O'TA, OTA', O'TA'.

Such a consideration is particularly often applied precisely for plane waves, when it is possible to speak of reflection at a specific angle of waves from specific portions of the surface S_1 and S_2. Reflection factors may in such a case be introduced for the second, third and fourth rays. Thus, if there is question about a horizontally polarised gliding ray, the reflection factor $f_\perp$ becomes -1 at $\psi \to 0$ in accord with the relation (19, 13). Correspongingly the field of a once reflected ray must be provided with a supplementary phase π.

For the application of this method see, in particular §53.

Note that if the protuberance is not an ideally conducting formation on an ideally conducting plane, such a substitution of one problem by another is also valid. Th is is why, for example, if there is located on the surface an object not ideally conducting and distorting the field, the latter may be searched for as the field of a "doubled" source, scattered by the given object, supplemented by its image and placed in vacuum.

In the particular case when the dimensions are small by comparison with the wavelength, it is possible to find the field according to the Rayleigh formula for light scattered by a small particle. We shall consider the scattering of "small" particles in §48.

Note that this conclusion would also be valid for nonconducting soil with $\varepsilon' = \infty$, had such one existed. In reality, on the plane $z = 0$, $E = E_z$ will take place for it too. Consequently, in certain cases protuberances on the surface of a dielectric may also be studied approximately starting from the representation about images. This refers in particular to the case of the sea ($\varepsilon' = 80$) for very short waves.

The rigorous theorem on reflection may also be formulated somewhat differently.

Assume that we are required to find a function φ satisfying at $z >$ > 0 the wave equation

$$(\nabla^2+k^2)u=-4\pi\rho \qquad (20.5)$$

(where ρ is an arbitrary distribution function of sources in the upper half-space), and one of the two conditions on the plane $z = 0$, either

$$\frac{\partial u(x, y, 0)}{\partial z}=0, \qquad (20.5a)$$

or

$$u(x, y, 0)=0. \qquad (20.5b)$$

In the first case we shall find the solution with the aid of the Green function v_+ (8.6) according to Formula (8.7). Then, because of

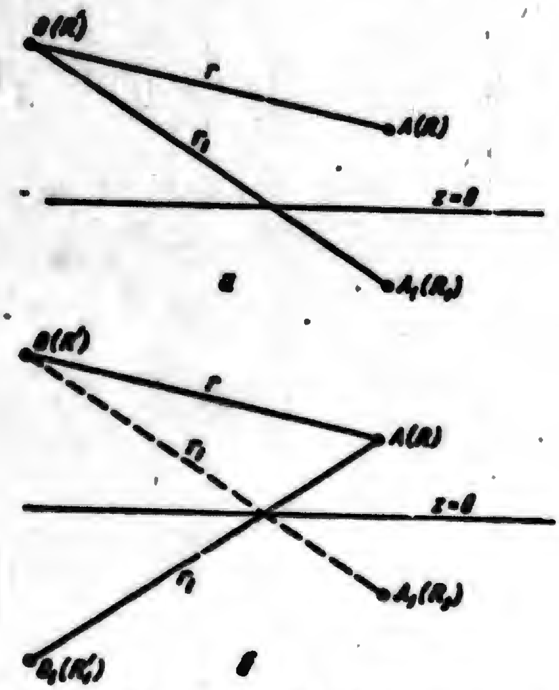

Fig. 20.4. Substitution of the image of the point of observation (a) by the image of the source (b).

the condition (20.5a), the surface integral falls off and

$$u(R)=\int\rho v_+ dV=\int\rho(R')\frac{e^{ik|R-R'|}}{|R-R'|}dR'+\int\rho(R')\frac{e^{ik|R_1-R'|}}{|R_1-R'|}dR'. \qquad (20.6)$$

However (see Fig. 20.4), instead of taking the distance from the source 0 to the reflection $A_1(\vec{R}_1)$ of the point of observation, we may take the distance from the point of observation $A(\vec{R})$ to the reflection of the source $O_1(\vec{R}'_1)$, i.e., assume tnat $|R_1 - R'| = |R - R'_1|$. But then the second of the integrals in Formula (20.6) may be recognized as the field of the reflected source taken exactly as a real one, but placed at the point O_1. Consequently, the solution of Eq. (20.4) with the boundary condition (20.5a) may be formulated as follows: let the solution of Eq. (20.5) for a boundless space be known. We shall denote this solution as follows:

$$u_0(x, y, z; [\rho(x_0, y_0, z_0)]). \tag{20.7}$$

where $[\rho]$ denotes that this solution is obtained for a given distribution of sources $\rho\ (x_0, y_0, z_0)$.

Then in the case of half-space and boundary condition (20.5a), the solution will have the form

$$u = u_0(x, y, z; [\rho(x_0, y_0, z_0)]) + u_0(x, y, z; [\rho(x_0, y_0, -z_0)]), \tag{20.8}$$

where we denoted by $u_0(x, y, z; [\rho(x_0, y_0, -z_0)])$ the field in boundless space induced by the sources reflected in the plane $z = 0$.

But if the boundary condition has the form (20.5b), we shall take the Green function v_- (8.3), as a consequence of which the solution will be expressed by Formula (8.5). The surface integral will vanish again, but the result will differ from the sum (20.6) by the sign at the second addend. Consequently, here the solution will be

$$\begin{aligned} u &= u_0(x, y, z; [\rho(x_0, y_0, z_0)]) - u_0(x, y, z; [\rho(x_0, y_0, -z_0)]) \equiv \\ &\equiv u_0(x, y, z; [\rho(x_0, y_0, z_0)]) + u_0(x, y, z; [-\rho(x_0, y_0, -z_0)]), \end{aligned} \tag{20.9}$$

that is, the superimposing reflected field must be taken for sources of inverse sign (in the last case the fact was utilized, that sign variation of ρ is equivalent to that of the field $\underline{u}$).

We resolved above, in substance, the wave equations for the fields $\vec{E}$ and $\vec{H}$. Since one of them is determined by the other, it is sufficient for example, to seek $\vec{H}$ from the three equations

$$(\nabla^2 + k^2) H = -\frac{4\pi}{c} \operatorname{rot} j_0. \qquad (20.10)$$

The boundary condition for H_z was written [(20.3)], and for H_x and H_y it may be obtained from the conditions for E_x and E_y. Indeed, according to Equality (20.1) and the equation $\operatorname{rot} H = -ik_0E$, for $z = 0$ $\operatorname{rot}_x \vec{H}$ vanishes just as does $\operatorname{rot}_y \vec{H}$. Opening their expressions and taking into account the equality (20.1), we obtain

$$\frac{\partial H_x}{\partial z} = \frac{\partial H_y}{\partial z} = 0. \qquad (20.11)$$

This is why H_x and H_y are determined by Formula (20.8), and H_z – by Formula (20.10). At the same time, for H_x and H_y the sources ρ are

$$\frac{1}{c} \operatorname{rot}_x j_0 = \frac{1}{c}\left(\frac{\partial j_{0z}}{\partial y_0} - \frac{\partial j_{0y}}{\partial z_0}\right),$$
$$\frac{1}{c} \operatorname{rot}_y j_0 = \frac{1}{c}\left(\frac{\partial j_{0x}}{\partial z_0} - \frac{\partial j_{0z}}{\partial x_0}\right),$$

and for H_z –

$$\frac{1}{c} \operatorname{rot}_z j_0 = \frac{1}{c}\left(\frac{\partial j_{0y}}{\partial x_0} - \frac{\partial j_{0x}}{\partial y_0}\right).$$

According to Formula (20.9), in the first two cases these sources must maintain the sign, while in the third (20.10) they could change it to the inverse. This will precisely take place in the case if we substitute j_{0x} and j_{0y} by $-j_{0x}$ and $-j_{0y}$, while j_{0z} is maintained without changes. Here we should only take into account that, inasmuch as in the argument $\vec{j}$, z_0 is simultaneously substituted by $-z_0$, $\partial/\partial z_0$ passes to $-\partial/\partial z_0$. Thus we again obtain the above formulated rule of source's reflection.

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182	The same result would have been valid for an "infinitely dielectric" medium, i.e., for a medium with $\sigma=0, \varepsilon'=\infty$, had such a medium existed. In reality, in boundary conditions ε is present everywhere and for $\varepsilon' \to \infty$ the horizontal component of the electric field in the air vanishes also, just as for $\sigma \to \infty$.

[Transliterated Symbols]

131	мб = mb = millibar = millibar
138	эфф = eff = effektivnyy = effective
160	квт = kvt = kilovatt = kilowatt

Chapter 4

THE FIELD NEAR THE PLANE INTERFACE BETWEEN THE GROUND AND THE ATMOSPHERE

§21. APPROXIMATE BOUNDARY CONDITIONS

1. We considered in Chapter 3 the field of a concentrated emitter, a dipole in an infinite uniform medium, and also the variation introduced in this field by the presence of another uniform medium separated from the first medium by a plane surface. However, the formulas obtained are valid only in the case when the sources are disposed sufficiently far from the interface (19.39). This most simple case from the theoretical viewpoint is insufficient for the study of the propagation of short and longer waves in the direct neighborhood of the ground surface, even when the atmosphere may be considered as uniform. The proximity of the ground surface changes substantially the structure of the field by comparison with that given by the above referred-to formulas. For this case the theory is found to be considerably more complex.

From the physical viewpoint the action of the ground is twofold. The currents excited by the radiowave field in the soil lead to energy losses to Joule heat and thus attenuate the field in the atmosphere. On the other hand, these currents shield the deeper regions of the soil and hinder the energy outflow into the lower hemisphere. This amplifies the field in the atmosphere. A precise accounting for the combined influence of both these cases constitutes, in substance, the object of our consideration.

In the general case the electrodynamic problem consists in the simultaneous consideration of fields in the atmosphere and in the ground during which it is necessary to provide for the satisfaction of the boundary conditions on the ground surface. Such is the path of classical methods (§§31, 32). However, all these methods usually lead to closed and visible results only in the case when we take into account certain specific peculiarities, characteristic for the propagation of radiowaves in terrestrial conditions. We have indeed seen in §15 (see Table 1) that the soils encountered are nearly always such that for them ε is substantially greater than the unity. Only by effecting the corresponding neglects in the general formulas may one be able to obtain practical results. This is why it is natural to account for this peculiarity of terrestrial conditions from the very beginning and correspondingly simplify the formulation of the problem. It was shown in a series of works [I.10; 1; 2; 3; 4; 5], almost simultaneously, that near the surface of a medium with $|\varepsilon| \gg 1$ certain approximate relations are valid for the field components. On the one hand, A.N. Shchukin [I.10] has shown that on the basis of these relations a greater clarify may be imparted to surface processes, and certain essential conclusions of theory may be simply and sufficiently rigorously obtained (for example, the polarization ellipse; see §22), owing to which these relations become useful factors of engineering practice. On the other hand, these relations were formulated, substantiated and applied as boundary conditions of a boundary value problem, at which only a uniform half-space, for example, the atmosphere may be considered without introducing into the consideration the field in the ground at all, taking into account the ground's influence only in the boundary conditions themselves (N.A. Leontovich [2; 3; 4]).

This particular approach to the problem of radiowave propagation has generated the entire direction in radiophysics. It defined as a series of entirely new achievements of the theory of radiowave propagation along the ground the review, the new formulation as well as the new exposition of fundamental old problems of the theory (see §22 and Chapters 5, 6, 7 and 8). Hence, in its turn emerged a clear physical pattern of the entire process of radiowave propagation (§45), helping to discriminate between the variety of practical questions.

The well known shortcoming of all these conclusions was at the outset the fact that they are valid only with a precision to quantities of the order $|\varepsilon|^{-1}$, and since for ultraviolet waves and dry soils $|\varepsilon|$ is reduced to ε', and cannot be always considered very great, the region of method's applicability was limited to the short, medium and longwave bands. Meanwhile, it became possible to attain a greater precision by other methods (true only for a uniform soil) and, by the same token, to encompass all the possible cases in practice. However, subsequently (see below) it was ascertained that if this method were made somewhat more precise, the real criterion of its validity would be, strictly speaking, not the smallness of the quantity $|\varepsilon|^{-1}$ at all, but the condition whereby *the field variation in the horizontal direction has the character of an unperturbed wave, multiplied by a slowly-varying factor*. As is shown by theory, this is what takes place indeed for great $|\varepsilon|$, but it is found that in the *wave zone* the field has the character inherent to almost any $|\varepsilon|$, and here, after a certain refinement, the entire method becomes applicable practically always. In substance, the matter amounts to the possibility of neglecting in the wave zone the wave reaching the source through the soil. This cannot be considered as a serious limitation for all really existing soils.

2. For the deduction of boundary conditions it is sufficient to consider the field over a small portion of the surface, provided only a few wavelengths fit into it. We may consider that in the plane $z = 0$

and near it, the dependence of all field components on x and y is described by functions of the type

$$E \sim we^{i(k_{0x}x+k_{0y}y)}, \tag{21.1}$$

where w is the "attenuation factor" (or the "attenuation function"), which we already introduced in §§18 and 19. Here we shall forego the assumptions that the field depends on z. To the contrary, this dependence must be found. Relative to w one may bring forth one important assertion, in the validity of which, in particular, we shall be convinced subsequently, when final results are obtained. Namely, if may be asserted that *this factor varies little over the segment equal to the "wavelength in the air"* λ_0. We shall adopt this fundamental stand from the very outset, and then we shall determine from the conclusions obtained in which cases it is incorrect. We shall ascertain that it is invalid only in the direct neighborhood of the source or in the region of sharp variation of the properties of the soil (or of the shape of the surface), where one may not neglect the wave arriving through the soil.

Therefore, if the axes x and y are directed along the interface between air and ground (assumed plane), we shall consider that the attenuation function w satisfies the conditions

$$\left|\frac{\lambda_0}{w}\frac{\partial w}{\partial x}\right| \ll 1, \qquad \left|\frac{\lambda_0}{w}\frac{\partial w}{\partial y}\right| \ll 1. \tag{21.2}$$

This is why

$$\frac{\partial E}{\partial x} \approx ik_{0x}E, \quad \frac{\partial E}{\partial y} \approx ik_{0y}E. \tag{21.3}$$

We shall consider at the outset that the axis x is directed in propagation plane of an undistorted wave, that is, in the plane containing the source and the given point of the interface, so that $k_{0y} = 0$.

In this chapter we shall admit that the soil is uniform from the electric point of view, putting off the case of inhomogenous soil till

After these preliminary remarks we shall pass to the consideration of boundary relations for the fields near the plane interface between the uniform ground and the atmosphere.

The following equations are valid respectively in the atmosphere and in the ground

$$\operatorname{rot} H = -ik_0 E; \quad \operatorname{rot} H_1 = -ik_0 \varepsilon E_1; \qquad (21.4)$$

$$\operatorname{div} E = 0; \qquad \operatorname{div} E_1 = 0. \qquad (21.5)$$

Here $\vec{E}$, $\vec{H}$ are the fields in the atmosphere, $\vec{E}_1$, $\vec{H}_1$ the fields in the ground; ε is the ground's complex dielectric constant, not dependent on coordinates. For the air we might not consider $\varepsilon = 1$, but because of this, the result would vary entirely immaterially. First of all we shall define *the law of field decrease with deepening into the soil* for the case when the absolute value of $|\varepsilon|$ is sufficiently great (cf. [7, 1]).

For the field in the soil E_1 Eq. (3.9b) is valid

$$\frac{\partial^2 E_1}{\partial x^2} + \frac{\partial^2 E_1}{\partial y^2} + \frac{\partial^2 E_1}{\partial z^2} + \varepsilon k_0^2 E_1 = 0. \qquad (21.6)$$

According to the condition (21.3) the field on the ground surface varies with the distance nearly strictly periodically. As we saw in §14, for great $|\varepsilon|$ the field is determined at each given point above the surface by the field in the near lying points of the surface. This is why the field in the soil is "tied" to the field on the surface and is carried along with it. The same result was obtained for the refraction of a plane wave (§19). Consequently, one may assert that for the field $\vec{E}_1$ in soil (let us recall that we choose the direction of the axis $\underline{x}$ in such a way that $k_{0y} = 0$)

$$\frac{\partial E_1}{\partial x} \approx ik_{0x} E_1 \text{ for } z = 0. \qquad (21.7)$$

This is why the first two terms in Eq. (21.6) are smaller than the last and also smaller than the third term in the relation $1/|\varepsilon|$.

Neglecting these small terms, we arrive at the equation

$$\frac{\partial^2 E_1}{\partial z^2} + \varepsilon k_0^2 E_1 = 0, \qquad (21.8)$$

which is immediately resolved:

$$E_1(x, y, z) = E_1(x, y, 0)\, e^{-ik_0\sqrt{\varepsilon}z}, \qquad (21.9)$$

whereupon of two possible signs in the exponent we choose the one, with which the field should remain finite at $z \to -\infty$, if $\sqrt{\varepsilon}$ is recognized in the sense

$$\sqrt{\varepsilon} = |\sqrt{\varepsilon}|\, e^{i\frac{\chi}{2}}, \quad \chi = \operatorname{arctg}\frac{4\pi\sigma}{\varepsilon'\omega}. \qquad (21.10)$$

Therefore, knowing the field on the surface z = 0, we may write the field at any point of the soil outright. Note that in §19 we have obtained for the refracted wave in the soil the expression (19.21), coinciding with the expressions (21.9) and (21.10), if $|\varepsilon| \gg \cos^2\psi$. Consequently, in this approximation the field in the soil has the character of a plane wave.

The expression obtained shows that the field decreases rapidly at deepening in the ground, and so much the faster that the real part of the quantity $-ikz$ is greater, i.e.,

$$|k|\sin\frac{\chi}{2} = k_0\sqrt[4]{\varepsilon'^2 + \left(\frac{4\pi\sigma}{\omega}\right)^2}\,\sin\left\{\frac{1}{2}\operatorname{arctg}\frac{4\pi\sigma}{\varepsilon'\omega}\right\}. \qquad (21.11)$$

The very same exponent characterized in §17 the propagation of radiowaves in a uniform absorbing medium. This is why the quantity l shown on the curves of Fig. 17.1 gives the depth of penetration in the ground of radiowaves propagating along its surface. We see that the waves of the radiobroadcast band ($10\ m < \lambda_0 < 6\cdot 10^3\ m$) usually penetrate into the ground to depths of tens of meters. In sea water the penetration depth constitutes centimeters and tens of centimeters.

The attenuation of waves penetrating into the soil are not casually determined by identical parameters as the damping at propagation in a uniform medium. This is closely linked to the fact that the field in the ground under the interface is determined by the conditions only in the nearest portion of the interface. It was shown in §14 that the dimensions of this area may be much smaller than the wavelength in the air and comparable with the "wavelength in the soil". This is why within its bounds the field has one and the same phase. Consequently, each such area, viewed from the point of observation situated in the ground, is represented by an equiphase surface – the plane wave front. *Moving from it into the ground is the wave, behaving in the neighborhood of the point of observation as a plane wave.* It is natural that its attenuation also coincides with that of a plane wave in a uniform medium.

Knowing the law of field attenuation in the soil, we may, with the aid of the boundary conditions (4.9) and (4.10), having in our case the form

$$\frac{\partial E_z}{\partial z} = \frac{\partial E_{1z}}{\partial z} \quad \text{at } z = 0; \tag{21.12}$$

$$E_z = \varepsilon E_{1z} \quad \text{at } z = 0, \tag{21.13}$$

obtain the necessary boundary condition for the field in the air. Substituting E_{1z} from Formula (21.9) into the relations (21.12) and (21.13) and eliminating E_{1z} (x, y, 0), we obtain

$$\frac{\partial E_z}{\partial z} = -\frac{ik_0}{\sqrt{\varepsilon}} E_z \quad \text{at } z = 0. \tag{21.14}$$

This precisely is the boundary condition sought for, which is superimposed on the field in the air and allows, digressing, generally speaking, from the investigation of the field in the soil, to consider the wave equation only in the upper half-space, which is the Leontovich condition. It shows that in the air near the ground the field E_z varies along the vertical much more slowly than along the horizontal (cf. (21.3)).

We may consider the field $\vec{H}$ exactly in the same way. Obviously, for it relations of the type (21.1) are valid (inasmuch as in the air $k_0H = [k_0E]$), the wave equation in the soil of the type (21.6) (substituting $\vec{E}_1$ by $\vec{H}_1$) and the boundary conditions for H_z, stemming from Formulas (4.5a) and (4.6) after having taken into account that div $\vec{H}$ = = div $\vec{H}_1$ = 0:

$$\frac{\partial H_z}{\partial z} = \frac{\partial H_{1z}}{\partial z}, \text{ at } z = 0. \tag{21.12a}$$

$$H_z = H_{1z}, \text{ at } z = 0 \tag{21.13a}$$

(where, contrary to Formula (21.13), the coefficient ε is absent). Repeating for H_z the reasonings made for E_z, we obtain

$$\frac{\partial H_z}{\partial z} = -ik_0\sqrt{\varepsilon}H_z \text{ at } z = 0. \tag{21.14a}$$

3. This condition is valid with a precision to values of the order $|\varepsilon|^{-1}$ (which is visible if only from the above comparison of Formulas (21.9) with the exact solution in case of plane waves (21.9) with the exact solution in case of plane waves (19.21)). For dry soils and short waves such a precision is insufficient. However, as will now be shown, this condition may be refined within the framework of the same fundamental approximation, that is, if we neglect a wave having arrived through the soil, so that the condition of such a form be correct for any $|\varepsilon|$ [7]. Indeed, if it is correct that in the direct neighborhood of the plane z = 0 the variation of the field along the axis x is determined by Formula (21.7), the wave equation in the soil (21.6) will take the form

$$\frac{\partial^2 E_1}{\partial z^2} - k_{0x}^2 E_1 + k^2 E_1 = 0. \tag{21.15}$$

Its solution is obviously the function

$$E_1(x,z) = E_1(x,0)\,e^{\pm i\sqrt{k^2-k_{0x}^2}\,z}. \tag{21.16}$$

In order to obtain a finite expression for $z \to -\infty$, it is necessary to choose the sign minus. Thus, the expression for the field in the ground, more precise than (21.9), has the form:

$$E_1(x,z) = E_1(x,0)\,e^{-ik_0\sqrt{\varepsilon}\sqrt{1-\frac{\cos^2\psi}{\varepsilon}}\,z}, \tag{21.17}$$

where

$$\cos\psi = \frac{k_{0x}}{k_0}. \tag{21.18}$$

This coincides exactly with the expression for a refracted wave in the soil (19.21). Hence, similarly with the way (21.14) was obtained, we may obtain with the aid of (21.12) and (21.13)

$$\frac{\partial E_z}{\partial z} = -\frac{ik_0}{\sqrt{\varepsilon^0}}E_z \quad \text{at} \quad z = 0,$$

$$\varepsilon^0 = \frac{\varepsilon^2}{\varepsilon - \cos^2\psi},$$

$$\frac{1}{\sqrt{\varepsilon^0}} = \frac{1}{\sqrt{\varepsilon}}\left(1 - \frac{\cos^2\psi}{2\varepsilon} - \frac{\cos^4\psi}{8\varepsilon^4} - \frac{\cos^6\psi}{16\varepsilon^6} - \dots\right).$$

Therefore, all the refining amounted to the substitution of ε by a certain ε_{eff}, *which we shall denote here and subsequently by* ε^0. Often when $|\varepsilon|$ is sufficiently great, there is no particular sense in such a refining. This is why we may utilize expressions valid with a precision to one, two or more terms of the series (21.20a). Thus, initially ([3] and so forth), it was considered that

$$\sqrt{\varepsilon^0} = \sqrt{\varepsilon} \quad \left(\text{imprecision in the terms of the order} \; \frac{\cos^2\psi}{2\varepsilon}\right). \tag{21.21a}$$

Then it was shown [I.1], that it is more accurate to consider

$$\sqrt{\varepsilon^0} = \sqrt{\varepsilon + \cos^2\psi} \quad \left(\text{imprecision in the terms of the order} \; \frac{\cos^4\psi}{8\varepsilon^3}\right). \tag{21.21b}$$

Obviously, this precision is nearly always sufficient. However, if in the following we should use the exact expression

$$\sqrt{\varepsilon^0} = \sqrt{\frac{\varepsilon}{1-\frac{\cos^2\psi}{\varepsilon}}}, \qquad (21.21c)$$

this would be not so much on account of quantitative refinement than because this formula admits transition even to the case $\varepsilon = 1$ (absence of ground) and provides in Relation (21.19) a correct result (contrary to entirely senseless expressions obtained if we utilize the approximations (21.21a) and (21.21b). Indeed, for $\varepsilon = 1$ we obtain

$$\frac{\partial E_z}{\partial z} = -ik_0 \sin\psi E_z = -ik_{0z}E_z,$$

as should be for a plane wave incident upon the plane $z = 0$ at the gliding angle ψ,

$$E_z \sim e^{i(k_0\cos\psi\cdot x - k_0\sin\psi\cdot z)}.$$

On the other hand, for quantitative counts the approximation (21.21b) is quite sufficient. It is much more convenient than the exact formula, since by comparison with the entirely simple formula (21.21a), the variation only consists in that the real part of ε increases by $\cos^2\psi$ (or simply by the unity at $\psi \approx 0$). The errors (relative) at utilization of Formulas (21.21a, b) are as follows:

	(21.21a)	(21.21b)
$\lvert\varepsilon\rvert = 2$	$0{,}25 \cos^2\psi$	$0{,}03 \cos^4\psi$
$\lvert\varepsilon\rvert = 5$	$0{,}10 \cos^2\psi$	$0{,}005 \cos^4\psi$
$\lvert\varepsilon\rvert = 10$	$0{,}05 \cos^2\psi$	$0{,}001 \cos^4\psi$

Therefore, these errors are so small that the precision of the approximate formulas cannot be determined by the smallness of $|\varepsilon|$. Under specific conditions the departure from Formula (21.3) may become more substantial (decreasing as distance from the source increases) and so may the influence of the wave reaching through the soil (also dropping with the distance). This last factor will be appraised in §31. For the time being we may note that this condition is knowingly correct not only at $|\varepsilon| \gg 1$, but also at $|\varepsilon - 1| \ll 1$, when the difference between the properties of the media is vanishingly small and this is why w will again vary with the distance much more slowly than $\exp(ik_x x)$. As to the influence of the gliding angle ψ, inasmuch as at conclusion we utilized the relations taking place in a plane wave, (21.29) is correct in any case in the region of applicability of reflection formulas, that is, when $k_0 z_0 \gg \sqrt{|\varepsilon|}$, where z_0 is the height of the source (see (19.38) for $|\varepsilon - 1| \gg 1$). Obviously, in reality it is correct in a wide range of angles.

Let us consider the variation of $|\varepsilon^0|$ at variation of ε:

$$|\varepsilon^0| = \frac{|\varepsilon|^2}{|\varepsilon - \cos^2\psi|} = \frac{\varepsilon'^2 + \left(\frac{4\pi\sigma}{\omega}\right)^2}{\sqrt{(\varepsilon' - \cos^2\psi)^2 + \left(\frac{4\pi\sigma}{\omega}\right)^2}}. \qquad (21.22)$$

When the absolute value of $|\varepsilon|$ is great, ε^0 does not differ from ε. However, for $\sigma \to 0$ and $\varepsilon \to 1$

$$|\varepsilon^0| \to \frac{1}{\sin^2\psi},$$

that is, for $\psi \to 0$ $|\varepsilon^0|$ approaches infinity. The absolute value of ε^0 has a minimum somewhere at $|\varepsilon| \sim 1$. Usually in that region $\frac{4\pi\sigma}{\omega} \ll \varepsilon' - \cos^2\psi$. This is why, postulating $\sigma = 0$, we find that the minimum of $|\varepsilon^0|$ is located at $\varepsilon = 2\cos^2\psi$. It is equal to

$$|\varepsilon^0|_{\min} = 4\cos^2\psi \quad \text{at } \varepsilon = 2\cos^2\psi \ (\sigma = 0), \qquad (21.23)$$

and at $\psi = 0$ (the most important case) $|\varepsilon_0|_{\min} = 4$.

Therefore, we have a boundary condition valid with great precision for all the existing soils and wavelengths, and allowing at consideration of the field in the air to digress from the field in the soil and to limit ourselves to the solution of the wave equation in a uniform space, z > 0. For $\psi = 0$ we have

$$\varepsilon^0 = |\varepsilon^0| e^{i\chi}, \qquad (21.22a)$$

$$|\varepsilon^0| = \frac{\varepsilon'^2 + \left(\frac{4\pi\sigma}{\omega}\right)^2}{\sqrt{(\varepsilon'-1)^2 + \left(\frac{4\pi\sigma}{\omega}\right)^2}}, \qquad (21.22b)$$

$$\chi = 2\,\mathrm{arctg}\,\frac{4\pi\sigma}{\varepsilon'\omega} - \mathrm{arctg}\,\frac{4\pi\sigma}{(\varepsilon'-1)\,\omega}. \qquad (21.22c)$$

As follows from (21.14), the relative variation of the field with the altitude in the air takes place slowly: the field varies notably only at ascent to an altitude of the order $\frac{|\sqrt{\varepsilon^0}|}{k_0} = \lambda_0|\sqrt{\varepsilon^0}|$, which is usually much greater than the wavelength in the air λ_0. The better the soil's conduction (the greater $|\varepsilon|$), the more invariable along the axis $\underline{z}$ the field component directed along it. In other words, on the surface of a well conducting ground the field is almost uniform.

Note that the real part of the derivative is negative,

$$\mathrm{Re}\left(\frac{1}{E_z}\frac{\partial E_z}{\partial z}\right) = \mathrm{Re}\frac{\partial \ln E_z}{\partial z} = -\frac{k_0 \cos\left(\frac{\pi}{2} + \frac{1}{2}\mathrm{arctg}\,\frac{4\pi\sigma}{\varepsilon'\omega}\right)}{\sqrt[4]{\varepsilon'^2 + \left(\frac{4\pi\sigma}{\omega}\right)^2}} < 0, \qquad (21.24a)$$

i.e., at ascent above ground the field decreases. This may seem strange, inasmuch as we know that the ground must exert an attenuating action. However, this deduction is valid only near the ground itself, where the obtained amplification may be computed as a small correction, that is, at distances

$$\Delta z \leq \lambda|\sqrt{\varepsilon}|. \qquad (21.24b)$$

And, indeed, we shall see in the furthest that at great altitudes the field begins again to increase. The decrease noted by us, however, takes place in reality. This means that in order to improve the reception it is necessary to ascend substantially above this attenuation zone, that is, considerably more than by the quantity Δz (21.24b). In the opposite case it is better to remain on the ground.*

Note that for H_z (21.14a) the same reasonings lead to formula

$$\frac{\partial H_z}{\partial x} = -ik_0\sqrt{\varepsilon - \cos^2\psi}\, H_z = -ik_0 \frac{\varepsilon}{\sqrt{\varepsilon'}} H_z \text{ at } z=0, \quad (21.19a)$$

instead of Formula (21.19). The distinction from Formula (21.14a) here is somewhat more complex.

Finally, we shall pass to the establishment of a relationship between the vertical and horizontal components of the fields. If the propagation direction coincides with the axis x, the field derivatives with respect to y may be not equal to zero only owing to wave front deviation from the plane. Consequently, they must be of the order $1/k_0R$ (where R is the distance from the source) relative to derivatives with respect to x and may be dropped. This is why, starting from relation $\operatorname{div} E = 0$ and from Formula (21.19) we have

$$ik_{0x}E_x = \frac{\partial E_x}{\partial x} = -\frac{\partial E_z}{\partial z} = \frac{ik_0}{\sqrt{\varepsilon'}}E_z,$$

i.e.,

$$E_x = \frac{1}{\cos\psi\sqrt{\varepsilon'}}E_z \quad \text{at } z=0. \quad (21.25)$$

Analogously, dropping the derivative with respect to y and utilizing Formulas (21.19a), we obtain

$$H_x = \frac{1}{\cos\psi}\frac{\varepsilon}{\sqrt{\varepsilon'}}H_z = \frac{\sqrt{\varepsilon - \cos^2\psi}}{\cos\psi}H_z \quad \text{at } z=0. \quad (21.25a)$$

Finally, from $\operatorname{rot}_z H = -ik_0E_z$ follows (in the same approximation):

$$-ik_0E_z = \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \approx ik_{0x}H_y,$$

$$H_y = -\frac{1}{\cos\psi}E_z, \quad (21.25b)$$

and from $\operatorname{rot}_z E = ik_0H_z$ we find in exactly the same way

$$E_y = \frac{1}{\cos\psi}H_z. \quad (21.25c)$$

In the general case the relationship between E_y and E_z, H_y and H_z cannot be established. Contrary to the relations considered so far, here we shall be compelled to draw a distinction between the cases of different polarization. Thus, for example, Formula (21.25) shows that on the surface of a good conductor $|E_x| \ll |E_z|$. This is indeed understandable, for on the surface of an ideal conductor the tangential field components generally disappear. However, in case of a horizontally polarized primary radiation on the surface of an ideal conductor *all* the field components are zero, and to derive a conclusion on what kind of E_y we shall have for a finite ε is impossible.

In case of *vertical* polarization, we shall have for reasons of symmetry $E'_y = 0$. Further, taking into account that the derivatives with respect to y drop off, we find from $\operatorname{rot} E = ik_0 H$

$$H_x = 0, \quad H_z' = 0 \quad \text{at } z = 0. \qquad (21.25d)$$

Obviously, if the axis x is directed not in the propagation plane but forms with it an angle φ, we shall obtain

$$\left.\begin{aligned} E_x &= \frac{1}{\cos\psi}\frac{\cos\varphi}{\sqrt{\varepsilon'}}E_z, \\ E_y &= \frac{1}{\cos\psi}\frac{\sin\varphi}{\sqrt{\varepsilon'}}E_z \end{aligned}\right\} \quad \text{at } z = 0. \qquad (21.26)$$

In case of *horizontal* polarization E_y is the main component, just as H_z is for the magnetic field of the wave, whereupon for reasons of symmetry $H_y = 0$. This is shy, according to Formulas (21.25b) and (21.25), we have also $E_x = E_z = 0$, and H_z is determined from E_y according to Formula (21.25c). At arbitrary orientation of the axis x, we have, analogously to Formulas (21.26)

$$\begin{aligned} H_x &= \frac{\cos\varphi}{\cos\psi}\sqrt{\varepsilon - \cos^2\psi}\, H_z = \cos\varphi\sqrt{\varepsilon - \cos^2\psi}\, E_y, \\ H_y' &= \frac{\sin\varphi}{\cos\psi}\sqrt{\varepsilon - \cos^2\psi}\, H_z = \sin\varphi\sqrt{\varepsilon - \cos^2\psi}\, E_y \quad \text{at } z = 0 \end{aligned} \qquad (21.26a)$$

When considering in §19 the reflection factor for plane waves, we distinguished the case of "great" ψ, when $|\sqrt{\varepsilon}\sin\psi| > 1$, from that of "small" ψ, when $|\sqrt{\varepsilon}\sin\psi| < 1$. Since $|\sqrt{\varepsilon}|$ exceeds substantially (and at times extremely strongly) the unity, the region of "large" angles may in reality correspond to ψ, quite small by comparison with the unity. But in any case the region of "small" angles actually corresponds to quite small ψ. This region, essential at a very gliding propagation of radiowaves, presents the greatest difficulties for the theory. We see that in it we may consider $\psi = 0$ and consequently in the boundary condition (21.20) and in formulas giving (at vertical polarization) the relationship between tangential and normal components, we may postulate $\cos\psi = 1$:

$$\frac{\partial E_z}{\partial z} = -\frac{ik_0}{\sqrt{\varepsilon}}\sqrt{1-\frac{1}{\varepsilon}}E_z, \qquad (21.27)$$

$$E_x = \frac{\cos\varphi\sqrt{1-\frac{1}{\varepsilon}}}{\sqrt{\varepsilon}}E_z, \quad E_y = \frac{\sin\varphi\sqrt{1-\frac{1}{\varepsilon}}}{\sqrt{\varepsilon}}E_z \quad \text{at } z=0, \qquad (21.27a)$$

and at the same time it may be considered that $\cos\varphi = \frac{k_{0x}}{k_0}$, $\sin\varphi = \frac{k_{0y}}{k_0}$.

For not too short waves it is possible to neglect the distinction of ε_{eff} from ε and to utilize the approximate formulas [I, 10].

$$E_x = \frac{\cos\varphi}{\sqrt{\varepsilon}}E_z, \quad E_y = \frac{\sin\varphi}{\sqrt{\varepsilon}}E_z \quad \text{at } z=0. \qquad (21.27b)$$

Let us now establish the boundary conditions for *the tangential components of* $\vec{E}$. Taking into account the continuity of these components at transition through the interface, it follows from the equation $\operatorname{rot} E = ik_0H$, that

$$\frac{\partial E_x}{\partial z} = \frac{\partial E_z}{\partial x} + ik_0H_y = ik_0\cos\psi E_z + ik_0H_{1y}.$$

But for $z > 0$ the variation of $\vec{H}_1$ with $\underline{z}$ is described by the usual dependence of the type (21.17), whereupon $\frac{\partial H_1}{\partial y} = 0$. This is why the equation $\operatorname{rot} H_1 = -ik_0\varepsilon E_1$ gives

$$-ik_0\sqrt{\varepsilon}\sqrt{1-\frac{\cos^2\psi}{\varepsilon}}H_{1y} = ik_0\varepsilon E_{1x}.$$

Consequently, the condition for E_x sought for has the form

$$\frac{\partial E_x}{\partial z} = -ik_0\sqrt{\varepsilon^0}E_x + ik_0\cos\psi E_z \text{ at } z=0. \qquad (21.28a)$$

Taking also into account the equality (21.25c), we find in exactly the same way

$$\frac{\partial E_y}{\partial z} = -ik_0\sqrt{\varepsilon^0}\left(1-\frac{\cos^2\psi}{\varepsilon}\right)E_y = -ik_0\sqrt{\varepsilon}\sqrt{1-\frac{\cos^2\psi}{\varepsilon}}E_y \text{ at } z=0. \qquad (21.28b)$$

If the field is emitted by a point source, it is convenient to introduce cylindrical coordinates (r, φ, z). Then the propagation direction will coincide at each point with the axis $\vec{r}$, and these relations may be rewritten in the form

$$\frac{\partial E_r}{\partial z} = -ik_0\sqrt{\varepsilon^0}E_r + ik_0\cos\psi E_z; \qquad (21.28c)$$

$$\frac{\partial E_\varphi}{\partial z} = -ik_0\sqrt{\varepsilon^0}\left(1-\frac{\cos^2\psi}{\varepsilon}\right)E_\varphi \text{ at } z=0. \qquad (21.28d)$$

4. In connection with the obtained boundary conditions, a few remarks should be made.

The first remark refers to the conditions in which it is admissible to perform the refinement linked with the substitution of ε by ε^0. We have seen that it is determined by the representation of the field in the form of a plane wave, feebly modulated in space. Only the solution of the whole problem, realized if only with the aid of approximate boundary conditions, may indicate whether or not this representation is justified. We shall obtain this solution in Chapter 5 and become convinced that it is, generally speaking, in good agreement with the reality; however, at times the coincidence takes place only in the case *when the distance to the transmitter is sufficiently great*. Yet precisely in those cases in which such a refinement generally has sense, that is, for not very great ε (short and ultrashort waves, poor conduction), the refinement found by us is already correct beginning with a distance of a few wavelengths. As the wavelength increases, so does the distance, but then the distinction between ε and ε^0 disappears in general. This is why we shall make use everywhere of the quantity ε^0 instead of ε, allowing for the fact that wherever such a substitution has any sense, it is practically correct in the entire wave zone of the transmitter.

The exact value of ε for a given soil is generally rather indeterminate. The substitution of the real average ε by a somewhat different quantity can never be very material. This is why the meaning of refinement performed should be seen first of all in that, as already ascertained, the whole structure of formulas, field dependence on distance etc., found for great $|\varepsilon|$, is fully maintained for the whortwave region, if only the numerical parameter is changed.

The second remark is related to the form of boundary conditions. The above obtained conditions of first approximation may be written in a different form [3]. Namely, we may start from the fact that in a well conducting soil $(|\varepsilon| \gg 1)$ the field has, as already ascertained, the character of a plane wave; at the same time, the planes of equal phase are perpendicular to the axis z. Consequently, the electric and magnetic vectors are interconnected in it just as they are in a standard plane wave propagating in a medium with $\mu = 1$ and the given value of ε:

$$E_{1x} = -\frac{1}{\sqrt{\varepsilon}} H_{1y}, \quad E_{1y} = \frac{1}{\sqrt{\varepsilon}} H_{1x}. \tag{21.29}$$

But the field components entering here are tangent to the surface $z = 0$ and they thus remain continuous at transition through the interface. Consequently, in the atmosphere too we have

$$E_x = -\frac{1}{\sqrt{\varepsilon}} H_y, \quad E_y = \frac{1}{\sqrt{\varepsilon}} H_x \quad \text{at } z = 0. \tag{21.30}$$

Hence follows again the condition (21.14). Indeed, differentiating the first of these relations with respect to x and the second with respect to y, adding up and utilizing the field equations

$$\operatorname{div} E = 0, \qquad \operatorname{rot} H = -ik_0 E,$$

we obtain

$$\frac{\partial E_z}{\partial z} = -\left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y}\right) = \frac{1}{\sqrt{\varepsilon}}\left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right) = \frac{-ik_0}{\sqrt{\varepsilon}} E_z. \qquad (21.31)$$

But differentiating the first of the correlations (21.30) with respect to y and the second with respect to x, and subtracting one from the other, we obtain the condition (21.14a).

The refinement of this boundary condition, taking into account the terms of following order by $1/\varepsilon$, may also be obtained therefrom on the basis of consecutive expansion by powers $1/\sqrt{\varepsilon}$ [1].

The refined boundary condition (21.29) has the following form [3]:*

$$E_x = -\frac{1}{\sqrt{\varepsilon}}\left\{H_y - \frac{1}{2\varepsilon k_0^2}\left(2\frac{\partial^2 H_x}{\partial x \partial y} - \frac{\partial^2 H_y}{\partial x^2} + \frac{\partial^2 H_y}{\partial y^2}\right)\right\}. \qquad (21.32)$$

For a wave that may be approximately represented as plane, considering $H_x = 0$, $\frac{\partial^2 H_y}{\partial y^2} = 0$, $\frac{\partial^2 H_y}{\partial x^2} = -k_{0x}^2 H_y$, we hence obtain the same result, which is given by Formula (21.29) at substitution of ε by ε^0 and when accounting for the first correction term in Formula (21.20a).

The form of the condition (21.32) differs from that of the condition (21.19) in that it does not include a direct indication on the form of the field dependence on coordinates, though in reality, judging from the conditions of deduction, both forms are equivalent.

In general, so long as we limit ourselves to the consideration of E_z, for example, at vertical polarization of the radiation, the transition from the condition (21.14) to the condition (21.19) (substitution of ε by ε^0) is equivalent to the transition from Relations (21.30) to relations

$$E_x = -\frac{1}{\sqrt{\varepsilon^0}} H_y, \quad E_y = \frac{1}{\sqrt{\varepsilon^0}} H_x \text{ at } z = 0. \qquad (21.32a)$$

Indeed, hence we again obtain, quite analogously to the procedure applied when deriving Formula (21.31), the condition (21.19). However, the same approach will give at consideration of H_z (required in case of horizontal polarization) instead of the correct formula (21.19a) the relation

$$\frac{\partial H_z}{\partial z} = -ik_0\sqrt{\varepsilon^*}\,H_z.$$

For the case $|\varepsilon| \gg 1$, by introducing the vector of external normal to the soil surface $\vec{n}$, we may rewrite the condition (21.29) in the form

$$E = \frac{1}{\sqrt{\varepsilon}}[n,\ H]. \qquad (21.32b)$$

The considered types of boundary conditions bear the designation of ***impedance conditions***, and the quantity $1/\sqrt{\varepsilon}$ or $1/\sqrt{\varepsilon^*}$ that of *surface impedance*, or *superficial impedance*. Designated as *normal impedance* is the ratio of the total values of tangential components to the interface surface $Z = E_t/H_t$. Introducing the impedance as the universal load characteristic of the interface surface between media, we obtain the possibility of considering the field of a given source in one of the media ignoring the field in the other medium. In regard to this method in general, see [10]. If the lower medium is anisotropic, Formula (21.32b) contains the *tensor impedance* (see [VIII, 10], §76).

Finally, note that if instead of electric field strength we assumed for the basis of computation the Hertz vectors, we could reduce the problem for the vertical dipole and a plane surface of uniform ground, as in uniform space, to the search for a one-component vector $\Pi = \Pi_z$ for which the boundary conditions are the same as for E_z:

$$\Pi = \varepsilon\Pi_1,\ \frac{\partial\Pi}{\partial z} = \frac{\partial\Pi_1}{\partial z} \text{ at } z = 0.$$

(see (4.7) and (4.8)). Thus, whatever was said about E_z refers also to Π. Consequently, the approximate boundary condition may be written in the form (21.19), that is,

$$\frac{\partial \Pi}{\partial z} = \frac{-ik_0}{\sqrt{\varepsilon^0}}\Pi. \tag{21.33}$$

We shall utilize this in the following.

5. Let us consider a field induced by a vertical magnetic dipole. In the presence of plane ground it is convenient to describe it by a magnetic Hertz vector with a unique nonzero vertical component $\vec{\Pi}_m$ (§4). All the expressions by means of which we obtained above for $\vec{E}_1$ underground the expression (21.17) may be repeated for the vector $\vec{\Pi}_{m1}$ in the soil. This is why

$$\frac{\partial \Pi_{m1z}}{\partial z} = -ik_0\frac{\varepsilon}{\sqrt{\varepsilon^0}}\Pi_{m1z} \quad \text{at } z = 0. \tag{21.34}$$

Since according to Formulas (4.11), (4.12) $\vec{\Pi}_{mz}$ and its normal derivative are continuous at transition through the interface, we have

$$\frac{\partial \Pi_{mz}}{\partial z} = -ik_0\frac{\varepsilon}{\sqrt{\varepsilon^0}}\Pi_{mz} = -ik_0\sqrt{\varepsilon - \cos^2\psi}\,\Pi_{mz}. \tag{21.35}$$

The above derived boundary conditions (21.19), (21.26) were obtained in the special assumption relative to the form of the field (21.1). Only at $|\varepsilon| \gg 1$ do we have the boundary conditions (21.30), (21.31) free from assumptions about a concrete field form. However, a general form of boundary condition for any $|\varepsilon|$ was indicated, from which stems a consecutive method of approximations leading in some cases to the condition (21.19), in others to the condition (21.31), but allowing to derive also other conclusions. We shall expound the deduction of this condition, following the original work by F.G. Gass [8].

For vertical field components in two media the wave equations (outside the source) are valid. From the exact boundary conditions (21.13), provided we apply to them the operator

$$k_0^2\varepsilon + \Delta,$$

where

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}. \tag{21.36}$$

follows

$$(k_0^2\varepsilon + \Delta) E_{1z} = \frac{1}{\varepsilon}(k_0^2\varepsilon + \Delta) E_z \text{ at } z = 0, \qquad (21.37)$$

whereupon as a consequence of the wave Eq. (21.6)

$$-\left(\frac{\partial^2 E_{1z}}{\partial z^2}\right) = (k_0^2\varepsilon + \Delta) E_{1z} = \frac{1}{\varepsilon}(k_0^2\varepsilon + \Delta) E_z \text{ at } z=0. \qquad (21.38)$$

Let us now consider the function $\frac{\partial E_{1z}}{\partial z}$ in the ground. It obviously satisfies also the wave equation and thus may be expressed by Formula (8.7) through its normal derivative (this will be $\frac{\partial^2 E_{1z}}{\partial z^2}$ on the plane z = = 0. It is essential, however, that the Green function will then be

$$v_+ = v_0(R - R') + v_0(R + R'),$$

where

$$v_0(R - R') = \frac{e^{ik_0\sqrt{\varepsilon}|R-R'|}}{|R - R'|}. \qquad (21.39)$$

Therefore, it contains $\sqrt{\varepsilon}$. in the exponent. This is why, substituting under the integral $\left(\frac{\partial^2 E_{1z}}{\partial z^2}\right)_{z=0}$ with the aid of Formula (21.38), we shall obtain the expression for $\frac{\partial E_{1z}}{\partial z}$ at z = 0 or for the quantity $\frac{\partial E_z}{\partial z}$ equal to it:

$$\frac{\partial E_z(r)}{\partial z} = -\frac{1}{2\pi\varepsilon}\iint\limits_{-\infty}^{+\infty}\frac{e^{ik_0\sqrt{\varepsilon}|r'-r|}}{|r'-r|}(k_0^2\varepsilon+\Delta_{r'}) E_z(r')dr'. \qquad (21.40)$$

Postulating $r' - r = \rho,\ dr' = d\rho,\ \Delta_{r'} = \Delta_r,$ we have

$$\frac{\partial E_z}{\partial z} = -\frac{1}{2\pi}\left(k_0^2 + \frac{1}{\varepsilon}\Delta\right)\iint e^{ik_0\sqrt{\varepsilon}\rho} E_z(r + \rho)\frac{d\rho}{\rho} \text{ at } z = 0. \qquad (21.41)$$

This integral relation is in itself simply one of the possible expressions of the Huygens principle. However, because of the presence in the exponent of $\sqrt{\varepsilon}$, when ε is essentially complex, the integration will be effectively limited by a small neighborhood of the observation point $\vec{r}$. Consequently, Expression (21.4) has the character of a boundary condition linking the field's normal derivative with the field itself in the close vicinity. This condition is thus *nonlocal*. But it is valid for any ε and must be considered as a quite rigorous and general boundary condition for E_z.

In the simplest case, if $|\varepsilon| \gg 1$, so that v_0 varies under the integral sign much more rapidly than E_z, we may take the value of E_z at the point $\rho = 0$ from the integral sign, and inasmuch as

$$\iint e^{ik_0\sqrt{\varepsilon}\rho} \frac{d\rho}{\rho} = -\frac{2\pi}{ik_0\sqrt{\varepsilon}}, \qquad (21.42)$$

we obtain the (local) condition

$$\frac{\partial E_z}{\partial z} = -\frac{ik_0}{\sqrt{\varepsilon}}\left(1 + \frac{1}{k_0^2\varepsilon}\Delta\right)E_z \approx -\frac{ik_0}{\sqrt{\varepsilon}}E_z \text{ at } z = 0. \qquad (21.43)$$

This is the Leontovich condition (21.14). In regard to the field E_z utilized here is only the assumption that it varies little over the "wavelength in the soil," $|\Delta E_z| \ll k_0^2|\varepsilon E_z|$. The concrete form of the field is indifferent.

In order to obtain a consecutive method of approximations, it is convenient to expand in (21.41) $E_z(r+\rho)$ in series by $\vec{\rho}$:

$$E_z(r+\rho) = \sum_{n=0}^{\infty}{}' \frac{1}{n!}(\rho\nabla_r)^n E_z(r).$$

In essence, $\cos\varphi$ enters into this expression, φ being the angle between $\vec{\rho}$ and the gradient of E_z. The odd powers of the integral over φ give zero, and for $n = 2p$

$$\int_0^{2\pi} \cos^{2p}\varphi\, d\varphi = \frac{2\pi}{2^{2p}}\binom{2p}{p}. \qquad (21.44)$$

On the other hand,

$$\int_0^{\infty} e^{ik_0\sqrt{\varepsilon}\rho}\rho^{2p}\, d\rho = -\frac{d^{2p}}{d(ik_0\sqrt{\varepsilon})^{2p}}\frac{1}{ik_0\sqrt{\varepsilon}} = -\left(\frac{1}{ik_0\sqrt{\varepsilon}}\right)^{2p+1}\cdot(2p)! \qquad (21.45)$$

This is why after certain calculations we obtain

$$\frac{\partial E_z(r)}{\partial z} = -\frac{ik_0}{\sqrt{\varepsilon}}\left(1 + \frac{1}{k_0^2\varepsilon}\Delta_r\right)^{\frac{1}{2}} E_z(r). \qquad (21.46)$$

where it is symbolically denoted

$$\left(1+\frac{1}{k_0^2\varepsilon}\Delta_r\right)^{\frac{1}{2}} = \sum_{n=0}^{\infty}\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\ldots\left(\frac{1}{2}-n\right)}{n!}\left(\frac{\Delta_r}{k_0^2\varepsilon}\right)^n. \quad (21.47)$$

In particular, if $\Delta_r E_z = -(k_{0x}^2 + k_{0y}^2)E_z$, as we admitted this in the condition (21.19),

$$\frac{\partial E_z(r)}{\partial z} = -\frac{ik_0}{\sqrt{\varepsilon}}\left(1-\frac{k_{0x}^2+k_{0y}^2}{2k_0^2\varepsilon}\right)E_z = -\frac{ik_0}{\sqrt{\varepsilon}}\left(1-\frac{\cos^2\psi}{2\varepsilon}\right)E_z, \quad (21.48)$$

correspondingly to Formula (21.21b). As we see, this result is correct not only for the plane wave field. In reality, let the field have the character of a package composed of elementary waves E_z^{κ}, being solutions of the wave equation

$$\Delta E_z^{(\kappa)} = \alpha_\kappa^2 E_z^{(\kappa)}, \quad E_z = \int V(\kappa)E_z^{(\kappa)}(r)\,d\kappa, \quad (21.49)$$

where κ numbers the solutions [above it was considered that $\kappa = (k_{0x}, k_{0y})$, $\alpha_\kappa^2 = \kappa^2$]. Substituting E_z into Formula (21.46), we shall obtain

$$\frac{\partial E_z}{\partial z} = -\frac{ik_0}{\sqrt{\varepsilon}}\int V(\kappa)\left(1+\frac{\alpha^2(\kappa)}{k_0^2\varepsilon}\right)^{\frac{1}{2}}E_z^{\kappa}(r)\,d\kappa. \quad (21.50)$$

If the package is sufficiently narrow and κ lies within the limits $\Delta\kappa$ near a certain κ_0, namely, if

$$\left|\frac{1}{2k_0^2\varepsilon}\cdot\frac{d\alpha^2(\kappa_0)}{d\kappa}\cdot\frac{\Delta\kappa}{\left(1+\frac{\alpha^2(\kappa_0)}{k_0^2\varepsilon}\right)^{\frac{1}{2}}}\right| \ll 1, \quad (21.51)$$

then

$$\frac{\partial E_z}{\partial z} \cong -\frac{ik_0}{\sqrt{\varepsilon}}\left(1+\frac{\alpha^2(\kappa_0)}{k_0^2\varepsilon}\right)^{\frac{1}{2}}E_z. \quad (21.52)$$

In particular, this condition takes place also for the expansion by cylindrical functions. When expanding by plane waves of a field having the character of a nearly monochromatic wave (which precisely will take place if the field is described by the product of a plane wave by a slowly varying attenuation function), α^2 is simply substituted by

$\varkappa^2 = k_0^2 \cos^2\psi$, and we again arrive at the result (21.48).

Analogously to Relation (21.46), the boundary conditions (21.28a) and (21.28b) for the tangential components of $\vec{E}$ may be written in a general form, not presuming the field form [8]. To that effect we must utilize the vectorial form of the Green theorem (5.14b). Applying it to the lower half-space, we may obtain

$$[nE(r)] = -\frac{ik_0}{2\pi}\int \frac{e^{ik_0\sqrt{\varepsilon}|r-r'|}}{|r-r'|}\Big\{[n\,[nH(r')]]+ + \frac{1}{k_0^2\varepsilon}[n\nabla](\nabla[nH(r')])\Big\}dr' \text{ for } z=0,$$

and expanding by powers ε^{-1},

$$[nE(r)] = \frac{1}{\sqrt{\varepsilon}}\left(1+\frac{1}{k_0^2\varepsilon}\Delta\right)^{-\frac{1}{2}}\Big\{[n\,[nH]]+\frac{1}{k_0^2\varepsilon}[n\nabla](\nabla[nH])\Big\}.$$

If we utilize the Maxwellian equations, it will follow therefrom

$$\frac{\partial E_{x,y}}{\partial z}+ik_0\sqrt{\varepsilon}\left(1+\frac{1}{k_0^2\varepsilon}\Delta\right)^{\frac{1}{2}}E_{x,y}=\frac{\varepsilon-1}{\varepsilon}\frac{\partial E_z}{\partial x,y} \text{ at } z=0, \tag{21.53}$$

which for $\Delta \to -k_0^2\cos^2\psi$ passes into boundary conditions (21.28a), (21.28b).

§22. POLARIZATION ELLIPSE

By adding to the above obtained relation (21.27) for a wave, gliding along the interface, the temporal multiplier $e^{-i\omega t}$, the former may be represented in the form (the axis x being directed along the propagation direction so that ($k_{0x} = k_0$)

$$E_z = Ae^{-i(\omega t-\varphi)}, \tag{22.1}$$

$$E_x = A\xi e^{-i\left(\omega t-\varphi+\frac{1}{2}\chi_0\right)}, \tag{22.2}$$

where

$$\xi = \left|\frac{1}{\sqrt{\varepsilon}}\sqrt{1-\frac{1}{\varepsilon}}\right| \approx \frac{1}{\sqrt{|\varepsilon+1|}} = \frac{1}{\sqrt[4]{(\varepsilon'+1)^2+\left(\frac{4\pi\sigma}{\omega}\right)^2}},$$

$$\chi_e = \operatorname{arctg}\frac{4\pi\sigma}{\varepsilon'\omega} - \frac{1}{2}\operatorname{arctg}\frac{4\pi\sigma}{(\varepsilon'-1)\omega} \approx \frac{1}{2}\operatorname{arctg}\frac{4\pi\sigma}{(\varepsilon'+1)\omega}.$$

The last expressions correspond to the substitution $\varepsilon^0 \approx \varepsilon + 1 = \varepsilon_{\text{эфф}}$ and, as we know, assure the precision of the order $1/(8|\varepsilon|^2)$.

Passing to real values, we have

$$E_z = A\cos(\omega t - \psi), \quad E_x = \frac{A}{\sqrt{|\varepsilon_{\text{эфф}}|}}\cos\left(\omega t - \psi + \frac{1}{2}\chi_e\right). \quad (22.3)$$

Therefore, E_x and E_z have not only different amplitudes, but are also shifted in phase relative to one another by the quantity $\chi_e/2$. Had such a shift not existed, the resulting field would have always been inclined at a specific small angle α to the vertical

$$\operatorname{tg}\alpha = \left|\frac{E_x}{E_z}\right| = \frac{1}{\sqrt[4]{(\varepsilon'+1)^2 + \left(\frac{4\pi\sigma}{\omega}\right)^2}}, \quad (22.4)$$

pulsating in magnitude with a period equal to $2\pi/\omega$. The presence of phase shift implies that the end of vector $\vec{E}$ describes with a frequency ω an ellipse, whose major axis is inclined to the vertical (Fig. 22.1). In order to obtain the parameters of this ellipse, we must find its equation and then reduce this equation to main axes.

Let us denote $E_x = X$, $E_z = Z$ and reject the immaterial general factor A

$$Z = \cos(\omega t - \psi),$$

$$X = \xi\cos\left(\omega t - \psi + \frac{1}{2}\chi_e\right) =$$

$$= \xi\left\{\cos(\omega t - \psi)\cos\frac{\chi_e}{2} - \sin(\omega t - \psi)\sin\frac{\chi_e}{2}\right\} = \xi\left\{Z\cos\frac{\chi_e}{2} - \sqrt{1-Z^2}\sin\frac{\chi_e}{2}\right\}.$$

Rationalizing in the second relation by raising the difference $X - \xi Z\cos\frac{\chi_e}{2}$ to the power of two, we obtain

$$X^2 + \xi^2 Z^2\cos^2\frac{\chi_e}{2} - 2\xi XZ\cos\frac{\chi_e}{2} = \xi^2(1 - Z^2)\sin^2\frac{\chi_e}{2},$$

or

$$\frac{1}{\xi^2}X^2 + Z^2 - \frac{2}{\xi}XZ\cos\frac{\chi_e}{2} = \sin^2\frac{\chi_e}{2}. \qquad (22.5)$$

Considering this equation as the equation of a curve in rectangular coordinates X, Z, we thus arrive at the equation for the ellipse described by the end of vector $\hat{E}$. Let us now reduce Eq. (22.1) to main

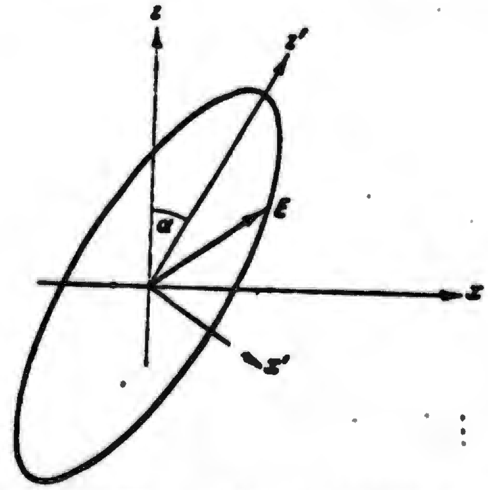

Fig. 22.1. The polarization ellipse.

axes. To that effect we shall rotate the system of coordinates by a certain angle α (see Fig. 22.1).

$$\begin{aligned} X &= X'\cos\alpha + Z'\sin\alpha, \\ Z &= -X'\sin\alpha + Z'\cos\alpha, \end{aligned} \qquad (22.5a)$$

and require that at such a substitution the mixed term disappear. As a result of the substitution, the factor at the mixed term X'Z' will be obtained equal to

$$\frac{1}{\xi^2}2\sin\alpha\cos\alpha - 2\sin\alpha\cos\alpha - \frac{2}{\xi}\cos\frac{\chi_e}{2}(\cos^2\alpha - \sin^2\alpha).$$

Equating it to zero, we obtain the angle α, by which should be rotated the system of coordinates

$$\operatorname{tg} 2\alpha = \frac{\frac{2}{\xi}\cos\frac{\chi_e}{2}}{\frac{1}{\xi^2} - 1} = 2\frac{\sqrt{|e_{xoz}|}}{|e_{xoz}| - 1}\cos\frac{\chi_e}{2}. \qquad (22.6)$$

If the absolute value of $|\varepsilon|$ is sufficiently great, this approximately gives (we also neglect here the distinction of ε_{eff} from ε)

$$\alpha = \frac{\cos\frac{\chi}{2}}{\sqrt{|\varepsilon|}} = \frac{\cos\frac{\chi}{2}}{\sqrt[4]{\varepsilon'^2+\left(\frac{4\pi\sigma}{\omega}\right)^2}}. \tag{22.7}$$

For a well-conducting ground, when $\varepsilon' \ll \frac{4\pi\sigma}{\omega}$, $\chi \approx \frac{\pi}{2}$, we simply have

$$\alpha = \sqrt{\frac{\omega}{8\pi\sigma}} = \sqrt{\frac{f}{4\sigma}}. \tag{22.7a}$$

Here the frequency $f = \frac{\omega}{2\pi}$ is introduced.

For a poorly conducting but "strongly dielectric" ground, when $\varepsilon' \gg \frac{4\pi\sigma}{\omega}$ and $\varepsilon' \gg 1$, we have $\chi = 0$. This is why

$$\alpha \approx \frac{1}{\sqrt{\varepsilon'}}. \tag{22.7b}$$

As already mentioned above, in this case the ellipse degenerates into a straight line, inasmuch as the phase shift disappears.

Note that these results have been obtained in [I, 10] in the assumption that the absolute value of $|\varepsilon|$ is great. Thus they are valid only to the extent that the angle α is obtained small. But in real conditions the absolute value of $|\varepsilon|$ has a range of 2-5 only in exceptional cases (dry sand, short waves), but usually it does not descend below 10. Consequently the angle α on the interface air-ground does not exceed about 20 degrees either.*

Therefore, for a given frequency f, the angle α is unambiguously and simply linked with soil conduction. This circumstance may be utilized for the measurement of soil conduction by the following method.

Having installed a rectilinear receiving antenna approximately vertically, one should rotate it so long as no maximum current value in the receiver is noted. Evidently, at such a position the antenna will be directed along the major axis of the ellipse.

However, in case of good conduction small angles are then obtained. Thus, for a medium-moist soil $\lambda = 300$ м $(f = 10^6)$and for the wavelength $\sigma = 5\cdot 10^7$ CGSE,

$$\alpha = \frac{1}{2}\sqrt{\frac{10^6}{5\cdot 10^7}} \approx \frac{1}{14} \approx 4^\circ.$$

It is difficult to measure such angles with great precision, and the more so since the maximum obtained is still not very sharp.

That is why it was proposed in [9] to measure not α, but the ratio of ellipse's axes b/a = K; namely, to find at one time the position of best audibility, and at another the worst one and to measure the current ratio in the receiver.

This axes' ratio will be obtained if we extract the square root from the ratio of coefficients at X^2 and Z^2 in the equation of the ellipse reduced to main axes. For great $|\varepsilon|$, i.e., for small α the ratio of axes may be obtained also in a simpler fashion. It may be stated that the minor semiaxis of the ellipse is the value of E ootained at the moment of time t when $E_z = 0$, that is, at $\omega t - \psi = \pi/2$. At that moment of time $E \approx E_x = \frac{A}{\sqrt{|\varepsilon|}}\sin\frac{\chi}{2}$. Since the maximum value of E is equal to about A (major semiaxis), the semiaxes' ratio is $\frac{1}{\sqrt{|\varepsilon|}}\sin\frac{\chi}{2}$, that is, according to Formula (22.7), $\alpha\,\mathrm{tg}\,\frac{\chi}{2}$. For a well-conducting ground $\chi = \frac{\pi}{2}$, and therefore

$$K = \frac{b}{a} = \alpha. \tag{22.8}$$

Consequently, the measurement of semiaxes' ratio at great conduction gives the very same quantity α, equal to $\frac{1}{2}\sqrt{\frac{f}{\sigma}}$.

At small conduction K sharply differs from α. Owing to this we may simultaneously determine ε as well as σ by measuring simultaneously K and α.

Note in conclusion that the angle α is positive, that is, the lines of force lean forward. This is indeed understandable: since the magnetic field vector is perpendicular to E_x and E_z, the energy flux vector $\vec{S}$ (Fig. 22.2) will be not horizontal, but inclined, as an average, at an angle α to the surface, which means energy influx into the

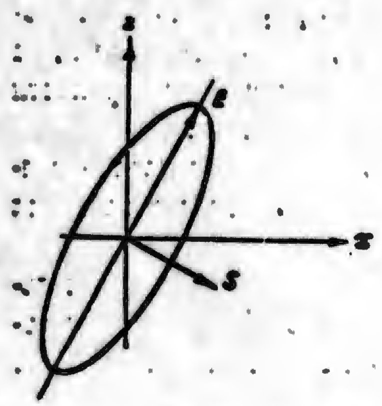

Fig. 22.2. Direction of energy flux on the surface of the interface.

soil (to be more precise, $\vec{S}$ will pulsate in the course of one period in magnitude and somewhat oscillate in direction, so long as the end of vector $\vec{E}$ describes the ellipse. But its value will be precisely least at the moments of time when $\vec{S}$ has an optimum deviation from its average direction).

The energy absorbed per unit of surface in average per unit of time will be obtained by averaging S_z for a single period T. It is (inasmuch as $-H_y \approx E_x$)

$$\frac{1}{T}\int_0^T S_z dt = \frac{c}{4\pi T}\int_0^T \mathrm{Re}\, E_x \mathrm{Re} H_y dt \approx$$
$$\approx -\frac{A^2 c}{4\pi T\sqrt{|\varepsilon|}}\int_0^T \cos(\omega t - \psi)\cos\left(\omega t - \psi + \frac{\chi}{2}\right) dt = \quad (22.9)$$
$$= -\frac{A^2 c}{8\pi\sqrt{|\varepsilon|}}\cos\frac{\chi}{2} \approx -\frac{cA^2}{8\pi}\frac{1}{2}\sqrt{\frac{T}{\varepsilon}} = -\alpha S.$$

It is obvious that these results may be made more precise by accounting for the terms of higher order relative to $1/\varepsilon$ if we utilize Formula (22.6) in the full form, without neglecting the difference be-

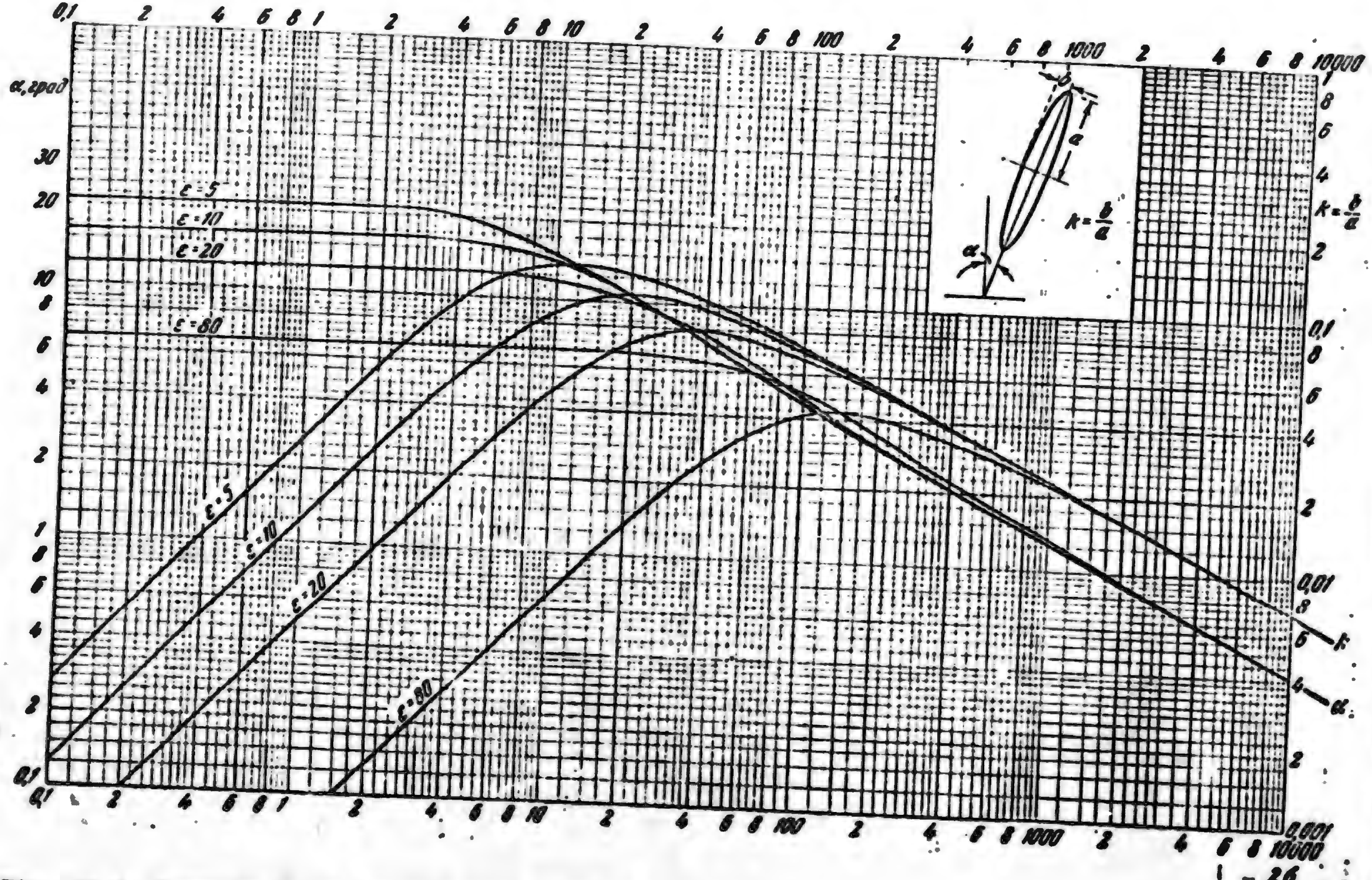

Fig. 22.3. Dependence of the parameters of the polarization ellipse on frequency and on the properties of the soil [V, 5].

tween ε and ε_{eff}. In particular, suostituting $\varepsilon_{эфф} = |\varepsilon + 1|$, we obtain

$$\operatorname{tg} 2\alpha = \sqrt{2}\,\frac{\sqrt{\sqrt{(\varepsilon' + 1)^2 + \eta^2} + (\varepsilon' + 1)}}{\sqrt{(\varepsilon' + 1)^2 + \eta^2} - 1}, \qquad (22.10)$$

where

$$\eta = \frac{4\pi\sigma}{\omega}.$$

Plotted in Fig. 22.4 are the curves for the angle α, computed by Formula (22.10) for various soils and wavelengths and borrowed from the

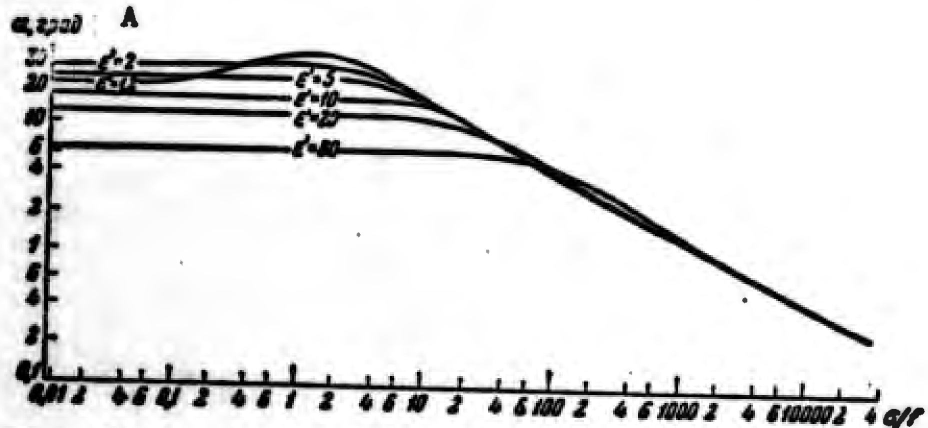

Fig. 22.4. Inclination angle α of the major semiaxis of the polarization ellipse for small values of ε. A) Degrees.

works [V, 5] and [7]. According to a more precise formula (21.21c), the calculation of the angle α provides curves for the values of ε' very close to the unity and σ = 0 (see [7]).

Manuscript Page No.	[Footnotes]
201	Evidently, at times the improvement takes place because at ascent the perturbing action of surrounding objects is eliminated. Our reasonings are related to the case of ideally plane and uniform ground surface.
207	In the work [3] Formula (14) contains a few misprints.
216	Within the framework of this conclusion, no absurd result may

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be obtained, which would correspond to the incorrect theory of Zenneck [V.1], and according to which on the interface between two identical media the angle α becomes equal to 45°. In reality, if we postulate in the formula (22.4), understood literally, $\varepsilon_{eff} = 1$ (it was considered in Zenneck's work that it has an exact value for any α), we would have $\alpha = \pi/4$. But in reality, if we postulate $\varepsilon \to 1$, it becomes necessary to utilize for ε^0 the more complete expression (21.21c), and we shall obtain that $\varepsilon_{eff} \to \infty$ (as for a metallic, reflecting surface), $\alpha \to 0$. At variation of ε from ∞ to 1, the angle α increases at the outset to a certain value α_{max}, and then decreases again to zero. This value α_{max} may be found only after cumbersome operations.

[Transliterated Symbols]

198 эфф = eff = effektivnyy = effective

200 мин = min = minimal'nyy = minimum

Chapter 5

DIPOLE NEAR A PLANE GROUND SURFACE

§23. PRELIMINARY REMARKS

We pass to the consideration of one of the fundamental and "classical" questions of theory of radiowave propagation along the ground. It is the problem of radiowave propagation in a uniform atmosphere along the ground, idealized in the form of uniform medium with the plane surface, when the source is located at the interface between media, or near it, so near that the interference formulas of §19 are insufficient. Formulas which will be obtained may be applied only to radio transmission over rather small distances, so long one may neglect the surface curvature. The quantitative criterion determining these distances may be obtained from the following considerations.

On the one hand, the deviation of the observation point from a plane surface does not manifest itself if the rise Δz is small by comparison with $\Delta z_1 = \lambda\sqrt{|\varepsilon|} = \frac{|\sqrt{\varepsilon}|}{k}$ (see (21.24)); on the other hand, the field does not vary during ascent, even for an ideally conducting surface for which Δz_1 becomes infinite, provided only Δz_1 is small by comparison with the dimensions of the Fresnel zone constructed in the vertical plane $\Delta z_2 = \sqrt{R\lambda}$, where R is the distance to the source. Let us denote by <u>a</u> the radius of the ground surface. The range from the source counted along the ground surface and the distance from the observation point to the plane tangent to the ground at the point of the source, Δz, are linked by the relation

$$\Delta z \approx R \sin\alpha \approx \frac{R^2}{a},$$

where $\alpha = R/a$ is the angular distance between the source and the point of observation. Obviously, for the possibility of substituting the ground surface by the plane it is necessary that Δz be small by comparison with Δz_1, as well as with Δz_2:

$$\Delta z \ll \Delta z_1, \text{ i.e. } R \ll \sqrt{a\lambda\sqrt{|\varepsilon|}} \approx \sqrt[4]{a^2\lambda^2\frac{4\pi\sigma}{\omega}},$$

$$\Delta z \ll \Delta z_2, \text{ i.e. } R \ll \sqrt[3]{a^2\lambda}.$$

We shall postpone to Chapter 6 a more detailed analysis of the conditions in which the ground may be considered as plane.

Despite the limited region of application, the problem of radio-wave propagation along the ground surface occupies in theory a particularly important place as a consequence of its fundamental significance.

In order to estimate the delicacy of the question, we shall consider one method of treatment that appears to be quite natural and nonetheless profoundly incorrect. It was actually applied about half a century ago, when the problem was formulated at first and the erronecus conclusions obtained on that basis continued to exert in the course of decades their influence in scientific investigations as well as in the practice of radio communication.

Considering a region of the ground surface situated not too close to the emitter, one may attempt to assume that here, where the wave front curvature is not great, the field must have the form of a plane wave, gliding along the interface. We may namely assume that if the axis $\underline{x}$ is directed along the surface of the interface in the propagation direction, and the axis $\underline{z}$ vertically upward, we must have in the atmosphere*

$$\Pi = \Pi_{1z} = \Pi_1 = A_1 e^{ik_1(\alpha_1 x + \beta_1 z)}, \qquad (23.1a)$$

in the soil

$$\left.\begin{aligned}\Pi = \Pi_{0z} = \Pi_2 = A_2 e^{ik_2(\alpha_2 x + \beta_2 z)},\\ k_1 = k_0 = \frac{\omega}{c},\ k_2 = \sqrt{\varepsilon}\, k_0,\end{aligned}\right\} \qquad (23.1b)$$

where on the strength of wave equations for Π we must have

$$\alpha_1^2 + \beta_1^2 = \alpha_2^2 + \beta_2^2 = 1. \qquad (23.2)$$

Therefore, α_1, α_2, β_1 and β_2 may be interpreted as the direction cosines of the propagation vectors $\vec{K}_1$ and $\vec{K}_2$ (complex in the general case, cf. §16). Because of βz addends, the gliding wave propagates also along the vertical.

The proposed form of solution seems rather natural, for under condition (23.2) this solution constitutes the exact solution of wave equations beyond the region of the source, whereas the errors, stemming from the substitution of a cylindrical wave by a plane wave, must decrease as the distance from the source increases, and they must not be great at large distances. However, in reality two arbitrary admissions are made here. First, it is implied that the wave front inclination, determined by the ratio of the constant numbers β_1 and α_1 in the air, does not depend on $\underline{z}$; second, it is understood that the field variation with $\underline{x}$ has a purely exponential character. Both these assumptions are erroneous as we shall see subsequently from the exact solution.

Besides the two conditions (23.2) imposed on the coefficients α_1, ..., β_2, there still are for $z = 0$ two boundary conditions (4.7) and (4.8)

$$\Pi_1 = \varepsilon \Pi_2, \qquad (23.3a)$$

$$\frac{\partial \Pi_1}{\partial z} = \frac{\partial \Pi_2}{\partial z}. \qquad (23.3b)$$

Hence in the first place from condition (23.3a) we obtain

$$A_1 e^{ik_1 \alpha_1 x} = \varepsilon A_2 e^{ik_2 \alpha_2 x}. \qquad (23.4)$$

Equating the exponents and the coefficients, we find outright two conditions imposed on the "direction cosines" and the amplitudes:

$$\frac{\alpha_1}{\alpha_2} = \frac{k_2}{k_1} = \sqrt{\varepsilon}, \quad \alpha_1^2 = \varepsilon\alpha_2^2; \tag{23.5a}$$

$$A_1 = \varepsilon A_2. \tag{23.5b}$$

Second, from condition (23.3b) we obtain

$$\beta_1 k_1 A_1 = \beta_2 k_2 A_2, \tag{23.6a}$$

which alongside with conditions (23.2) and (23.5) gives on raising to the second power

$$\varepsilon^2 (1 - \alpha_1^2) k_1^2 = (1 - \alpha_2^2) k_2^2. \tag{23.6b}$$

Hence, and from condition (23.5a) we find

$$\left.\begin{aligned} \alpha_1 &= \frac{\sqrt{\varepsilon}}{\sqrt{\varepsilon+1}}, & \alpha_2 &= \frac{1}{\sqrt{\varepsilon+1}}; \\ \beta_1 &= \frac{1}{\sqrt{\varepsilon+1}} = \alpha_2, & \beta_2 &= \frac{\sqrt{\varepsilon}}{\sqrt{\varepsilon+1}} = \alpha_1. \end{aligned}\right\} \tag{23.7}$$

Now the solution (23.1) is seemingly fully defined: it satisfies the wave equation and the boundary conditions at z = 0. However, from a formal point of view this is not so: the conditions at the location of the source were not taken into account, for example, the required character of field singularity in the case of a point dipole.

In §19 we also began with the consideration of a plane wave. However, there the question of field attenuation with the distance from the source was not raised, whereas mathematically, aside from the incident and refracted wave, the reflected wave was also taken into account. This is why no incorrect deductions have been obtained there.

Let us analyze the result obtained here. If the properties of the second medium (soil) approach the properties of the first medium, which means that the plane z = 0 becomes the imaginary interface in an infinitely extended atmosphere, that is, if $\varepsilon \to 1$, it follows from formula (23.7) that

$$\alpha_1 = \frac{1}{\sqrt{2}} = \beta_2; \quad \alpha_2 = \frac{1}{\sqrt{2}} = \beta_1. \tag{23.8}$$

Therefore, it appears that in these conditions the wave propagates at an angle of 45° to the imaginary interface. Meanwhile, from considerations of symmetry it follows that the field of a vertical dipole placed on the plane $z = 0$ must then pass into a type e^{ikx} plane wave propagating along the axis $\underline{x}$. This is why the result obtained is clearly absurd. Consequently, the solution cannot have the proposed form and the entire treatment is incorrect.

This circumstance releases us from the requirement of discussing the further conclusions in detail. It should however be noted that in another extreme case, $\varepsilon \to \infty$, a result compatible with the rigorous theorem on reflection is obtained. Namely, as $\varepsilon \to \infty$

$$\alpha_1 \to 1, \ \alpha_2 \to 0, \ A_2 = \frac{A_1}{\varepsilon} \to 0. \tag{23.9}$$

Thus, a plane wave is obtained, which propagates with the wave number k_0, i.e., with the speed of light along the interface, while the field in the ground vanishes. Evidently, the soundness of the solution at limit $\varepsilon \to \infty$ suggested at the time that it has a physical sense if only for great $|\varepsilon|$. If in such a case we postulate

$$\alpha_1 \approx 1 - \frac{1}{2\varepsilon}, \quad \beta \approx \frac{1}{\pm\sqrt{\varepsilon}},$$

the solution in the air will have the form

$$\Pi_1 = \Pi_{1s} = A_1 \exp\left\{ik_0\left[\left(1 - \frac{\cos\chi}{2|\varepsilon|}\right)x - \frac{\cos\frac{\chi}{2}}{\sqrt{|\varepsilon|}}z\right] - \right.$$

$$\left. - \frac{k_0 \sin\chi}{2|\varepsilon|}x - \frac{k_0 \sin\frac{\chi}{2}}{\sqrt{|\varepsilon|}}z\right\}, \tag{23.10}$$

whereupon when extracting the root we select of the two possible signs for β_1 the one that does not lead to exponential increase with the rise of z.

Thus, the speed of radiowave propagation is found to be dependent on the properties of the soil: for the propagation along the axis x it is greater than the speed of light and is equal to

$$c' = \frac{\omega}{k_0\left(1 - \frac{\cos\chi}{2|\varepsilon|}\right)} > c. \qquad (23.11)$$

Moreover, in propagation the wave attenuates, and this attenuation is described by the exponential factor

$$e^{-\frac{k_0 \sin\chi}{2|\varepsilon|}x}. \qquad (23.12)$$

On the whole it may be said that in this incorrect solution, as we have seen, the influence of the soil is manifest at $z = 0$ in the appearance of the factor $y = e^{-sx} = e^{-\rho}$,

$$\rho = sx = i\frac{k_0 x}{2\varepsilon} = i\frac{k_0 x}{2|\varepsilon|}(\cos\chi - i\sin\chi). \qquad (23.13)$$

A real significance was given for a long time to the indicated difference between the propagation velocity and the speed of light c, and also to the exponential decrease with the distance. However, in reality these conclusions are erroneous, just as is the whole described solution. The field on the ground surface cannot be described by a plane was of the type (23.1). As we shall see, in reality, though the wave attenuation (slow for great $|\varepsilon|$) takes place over a characteristic distance $\frac{1}{|s|} \gg \lambda_0$, it does not have an exponential character. The wave front bends ascent above the surface, whereupon this change in inclination is performed in ascent to a height, much smaller by order of magnitude than the distance to the source. This is why, substituting the real wave by a plane one, we commit an error considerably greater than that resulting from the substitution of a cylindrical wave by a plane one. As is shown by the complete solution (Sommerfeld [2], Weyl [3] and others; for details see below), the error is manifest precisely in the terms containing the very essence of the solution.

In this and in certain subsequent chapters we laid at the basis of the consideration a method allowing to investigate comparatively simply and on a single foundation the "classical" problems, resolved long ago (such as the question of uniform plane or spherical ground), as well as more complex questions (inhomogeneous soil, etc., for the solution of which it was, strictly speaking, precisely worked out). The idea of this method stems from two cases, always taking place in practically the most interesting problems: First, the distance between the source and the point of observation is large by comparison with the wave-length; second, the soil conduction is always significant. In traditional methods (Sommerfeld, Weyl, van der Pol and others) the problem is generally formulated quite rigorously, and only at a certain stage of the solution, usually almost at the very end, the indicated cases are utilized for obtaining approximate practical conclusions, otherwise unattainable.

Another method, to which it is referred, utilizes these cases from the very beginning. The following four points are at the foundation of that method.

First of all, instead of seeking the field, say E, it is reasonable to write from the very beginning

$$E(R)=A\frac{e^{ikR}}{R}w(R), \qquad (23.14)$$

where R is the distance from the source (assumed to be a point source), and w is the attenuation function, which reverts to the unity for R = = 0. The latter is precisely that offering interest. Characteristic for it is the case, whereby it varies little over the wavelength, that is, it modulates smoothly in space the wave field exp(ikR). In traditional methods the total field is sought for, and the function w is separated only at the end. Meanwhile it is appropriate to take into account the relative slowness of its variation from the very beginning. We may then pass to the approximate equation for w (which is then found to be a parabolic type equation, see §38) [IV, 4; VI, 6], either by substituting expression (23.14) into the wave equation, or to obtain a practical integral equation for w by applying to the field the Green formula [VII, 2; VII, 3].

In the second place, if $|\varepsilon| \gg 1$, we may utilize the Leontovich boundary condition (21.14) [IV, 3] (but if the absolute value $|\varepsilon|$ is even not great, whereas w still varies slowly, we may use condition (21.19) generalizing it), which is an impedance-type condition, thus allowing to seek the solution in one half-space (in the air), not including into the consideration the field in the other half-space. In case of inhomogeneous soil it is necessary (and possible) to correspondingly generalize this condition (§40).

Third, utilizing the integral equation method (i.e., the Huygens principle, in substance), we may reduce the integration over the ground surface to integration over a line. This is possible, for on the strength of the condition $kR \gg 1$, not all the integration surface plays an identically substantial role, the most essential being the first Fresnel zone (§12), which, the other conditions being equal, may be conditionally considered as the "radiowave course." Therefore, we obtain for the determination of w an integral equation with one independent variable [VII, 2; VII, 3].

Finally (and this is essential in problems of inhomogeneous soil), formulating the Green theorem in place of the Green function $(1/R)\exp(ikR)$, physically corresponding to a dipole field in vacuum or above an ideally conducting plane, is appropriate to take the dipole field $(1/R)\exp(ikr)\cdot w_0(R)$ above a uniform surface of the same form as in the given concrete problem (plane or sphere), but with a certain arbitrary characteristic ε^0 (see remarks at the end of §8, and also §41) [VII, 2; VII, 3]. Disposing of the value of the arbitrary parameter ε^0, we may in a series of practically important cases pass at once from the integral equation for $\underline{w}$ to its expression in quadratures.

It should be stressed that all these simplifications and neglects correspond to standard ones for traditional methods, so that no decrease in the precision of the result then occurs (which is confirmed by comparison of the conclusions obtained by different methods in those cases when it is possible).

In the following two sections we shall give the solution of the problem of a vertical dipole field above a plane surface of the interface between uniform ground and uniform atmosphere on the basis of the method described. The method of integral equation will be formulated at the outset (§24) (which will be subsequently generalized in §41 for the case of inhomogenous soil, and will lie at the foundation of theory of radiowave propagation along a nonuniform and uneven surface).

The solution of this integral equation will be obtained in §25. Then, in §26 the solution will be obtained in a more general form by the method which we designate as the method of reflected source. The solution is analyzed in §27. In §28 it is generalized for the case of a vertical magnetic dipole, and in §29 – for horizontal electric and magnetic dipoles. The field of an underground emitter is considered in §30. Other methods are brought out in §§31 and 32; these methods are necessary to us in particular, and also because either of them admit generalizations to certain cases of disruption of the indicated ideal conditions (stratified atmosphere, see Chapter 9), and also allows us to estimate more accurately the error of the method of approximate boundary conditions.

§24. INTEGRAL EQUATION FOR THE ATTENUATION FUNCTION

First of all we shall obtain the integral equation for the attenuation function $\underline{w}$ in the case when the source is a vertical dipole located on the ground or at a certain altitude z_0 above it, and the point of observation is situated in the plane $z = 0$ of the interface. In this case the field may be described by the Hertz vector with the unique nonzero component $\Pi = \Pi_z$. If the dielectric constant ε were infinitely great, the field would have been, as we know, twice as great as that induced by the very same dipole in the absence of ground. It would have the form $\text{const}\cdot(e^{ikR}/R)$. Let us select a constant factor, such that in case of vacuum (we then limit ourselves to the study of the wave zone everywhere)

$$\Pi(x, y, 0) = \frac{1}{2}\cdot\frac{e^{ikR}}{R}. \tag{24.0}$$

Then, in the presence of ground with $\varepsilon = \infty$ we would have

$$\Pi = \Pi^0(x, y, 0) = \frac{e^{ikR}}{R}, \quad R = \sqrt{x^2 + y^2 + z_0^2}. \tag{24.1}$$

With $\varepsilon \neq \infty$ the real field would differ by a certain "attenuation factor" $\underline{w}$ which we must find:

$$\Pi(x, y, 0) = w(x, y, z_0)\frac{e^{ikR}}{R}. \tag{24.2}$$

We shall express the field at an arbitrary point of the plane with the aid of the Green function v_+ according to formula (8.7). In the given case $z' = 0$, $\bar{R}$ and $\bar{R}_1$ coincide. Without any limitation of generality we may consider that at the point of observation $y = 0$. This is why

$$\Pi(x, 0, 0) = -\frac{1}{2\pi}\int U_z\frac{e^{ik\rho}}{\rho}dV' + \frac{1}{2\pi}\int\frac{\partial\Pi}{\partial n}\frac{e^{ik\rho}}{\rho}dS',$$

$$\rho = \sqrt{(x-x')^2 + y'^2}, \quad dS' = dx'dy'. \tag{24.3}$$

Here $\frac{\partial}{\partial n} = -\frac{\partial}{\partial z'}$. According to boundary condition (21.34),

$$\frac{\partial \Pi}{\partial z} = -\frac{ik}{\sqrt{\varepsilon^0}}\Pi \quad \text{at } z = 0, \tag{24.4}$$

we would have in case of ideally conducting ground $\frac{\partial \Pi}{\partial n} = 0$, so that the integral over the surface would vanish. As to the volume integral, it is independent of ε and, by way of consequence, it gives Π^0 (24.1). This is why

$$\Pi(x, y, 0) = \frac{e^{ikR}}{R} + \frac{ik}{2\pi\sqrt{\varepsilon^0}}\int \frac{e^{ik\rho}}{\rho}\Pi dx'dy'. \tag{24.5}$$

The surface integral shows to what extent the departure of soil conduction from ideal attenuates the field and generally distorts it. Owing to relation (24.4) (valid of w is a sufficiently slowly varying function, which must be subsequently justified by the solution obtained; this takes place in the wave zone), we obtained the integral equation for one unknown function of Π. This method, allowing us to avoid the inclusion into the count the field in the soil, was first utilized by Leontovich [Chapter 4, 4] and also by Greenberg [Chapter 7, 1]. We shall be convinced of its fruitfulness in the following.

Substituting the value of (24.2) into Eq. (24.5), we obtain the equation

$$\begin{gathered} w(D; z_0) = 1 + \frac{ikD}{2\pi\sqrt{\varepsilon^0}}\int e^{ik(R+\rho-D)}\frac{dS'}{r\rho}w(x', y'; z_0), \\ R = \sqrt{x'^2 + y'^2 + z_0^2},\ \rho = \sqrt{(x-x')^2 + y'^2},\ D = \sqrt{x^2 + y^2} = x. \end{gathered} \tag{24.6}$$

It may be seen from the meaning of formula (24.4) deduction that it is also correct for a dielectric constant ε dependent on the point, i.e., for a nonuniform ground. Detailed consideration shows that it is only necessary for the quantity ε to vary not too rapidly, namely that its relative variation over a segment equal to "wavelength in the soil" be small (for details see Chapter 7; as to the possibility of neglecting the nonwave zone, see below):

$$\left|\frac{\operatorname{grad} \varepsilon^0}{\varepsilon^0}\right| \ll k\left|\sqrt{\varepsilon^0}\right|. \qquad (24.4a)$$

This is an easily fulfillable condition, as a consequence of which ε^0 may be maintained under the integral, and it may be considered that $\varepsilon^0 = \varepsilon^0(\vec{r})$. However, in the present chapter we consider the soil as uniform: $\varepsilon^0 = \text{const}$.

The integral, standing in the right-hand part of the equation, belongs to a type more than once encountered by us. The rapidly oscillating function $\exp\{ik(R + \rho - D)\}$ is multiplied by comparatively slowly varying $\underline{w}$ and $1/r\rho$. It outlines on the surface an essential region which we already considered in §12; if both corresponding points are situated in the plane $z = 0$, such a region has the shape of an ellipse encompassing these points (see Fig. 12.3, and also Fig. 24.1 and formulas (12.14), (12.15); if one of the points is raised, it has the shape of an ellipse, in one of whose foci [or focuses] is located the point remaining on the ground (see Fig. 12.2 and formulas (12.11), (12.11a)).

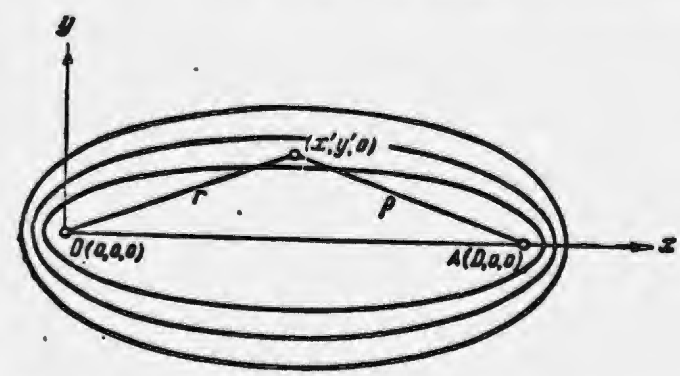

Fig. 24.1. Denotations meant for deriving the integral equation for the attenuation function. In the cases of practical interest the ellipses are considerably more elongated than those shown in the sketch.

Let us consider at the outset the first case. Assume that the source is situated on the ground, $z_0 = 0$.

For $x < 0$ and $x > D$, that is, outside the region that may be considered as "radiowave course," the ellipses separating various Fresnel zones, converge very close – to distances of the order λ (for wave-lengths and distances, really encountered in practice, these ellipses are considerably narrower than those shown in Fig. 24.1). According to formula (12.15a), the ratio of the first zone's width, $b \approx \frac{1}{2}\sqrt{\lambda D}$, to its length D,

$$\frac{b}{D} \sim \sqrt{\frac{\lambda}{D}} \ll 1, \qquad (24.7)$$

is quite small at all times. Thus, in the essential region of values x' and y' we have

$$y' \ll x', \qquad y' \ll D - x'.$$

The exception is constituted by regions with dimensions of the order λ each, adjacent to the points O and A. However, the influence of these regions may generally be neglected as this will be seen from the following. On the other hand, the attenuation function w depends comparatively feebly on the distance, and this dependence on y' is particularly feeble for a constant x' within the bounds of the first zones. Consequently, similarly to what was done in §§ 10-12, we may materialize a series of useful simplifications, as a result of which the integral equation for the function of two variables will be transformed into an equation for a single variable function, so that the integral in the right-hand part will be extended not over the plane, but over the line. First, we may expand the exponent in series by y' and limit ourselves to first nonvanishing terms:

$$R \to r = \sqrt{x'^2 + y'^2} \approx x' + \frac{y'^2}{2x'},$$

$$\rho = \sqrt{(D-x')^2 + y'^2} \approx D - x' + \frac{y'^2}{2(D-x')}.$$

Besides, it may be considered that

$$w(r) = w(x'). \qquad (24.8)$$

We may substitute in the denominator r by x', ρ by x – x'.

Finally, we may perform the integration over x' itself not from $-\infty$ to $+\infty$, but, say, from 0 to D. The exact position of these limits is immaterial. We might have changed it by a quantity of the order λ to either side, introducing by the same token, as may be shown, an error of the order $\sqrt{s\lambda} \ll 1$ (see (24.11)).

Thus we transform Eq. (24.6) for the case $z_0 = 0$ into the following approximate equation:

$$w(D) = 1 + \frac{ikD}{2\pi\sqrt{\varepsilon^0}} \int_0^D \frac{dx'}{x'} \frac{w(x')}{D-x'} \int_{-\infty}^{+\infty} e^{\frac{iky'^2}{2}\left(\frac{1}{x'}+\frac{1}{D-x'}\right)} dy'. \qquad (24.9)$$

The integral over y' is the Fresnel integral (10.13); it is equal to

$$\sqrt{\frac{2\pi i}{kD}}\sqrt{x'(D-x')}, \quad \sqrt{i} = e^{i\frac{\pi}{4}},$$

so that

$$w(D) = 1 - \sqrt{\frac{ikD}{2\pi\varepsilon^0}} \int_0^D \frac{w(x')}{\sqrt{x'(D-x')}}\, dx'. \qquad (24.10)$$

Let us introduce the quantity s, which is extremely important for the following:

$$i\frac{k}{2\varepsilon^0} = s = \frac{ik(\varepsilon-1)}{2\varepsilon^2}. \qquad (24.11)$$

Then the equation will take the form

$$w(D) = 1 + i\sqrt{\frac{sD}{\pi}} \int_0^D \frac{w(x')}{\sqrt{x'(D-x')}}\, dx', \quad z_0 = 0. \qquad (24.12)$$

This is the final integral equation for the attenuation function w searched for in the case $z_0 = 0$. Hence it may be seen at once that it is dependent on D and on electrical characteristics of the soil only as a function of the product sD:

$$w(D; 0) = w(sD). \qquad (24.13)$$

The result expressed by this formula is very substantial. Indeed, since the least value $|\varepsilon^0| = \frac{|\varepsilon|-1}{|\varepsilon|^2}$ is equal to 4 (and is attained for

$|\varepsilon| = 2$, see (21.23)), but usually much greater than four, we have

$$|s| < \frac{k}{8}. \tag{24.14}$$

Consequently, the absolute value $|s|$ is always substantially smaller than $\underline{k}$ and this is why the function $\underline{w}$ varies over a segment equal to $\lambda = k^{-1}$ very little (in reality, the situation is still better than indicated by formula (24.14), see (26.31)). This is why, so long as one may neglect the derivatives of $1/r$ in formula (24.2), relation (21.2), lying at the basis of the method of approximate boundary conditions, is always valid. This is known to be possible in the wave zone. Consequently, the method is correct in the entire wave zone.

Let us now turn to the case $z_0 \neq 0$. Then we shall consider $\frac{z_0}{x} = \sin\psi \ll 1$. As previously, here the essential zone is relatively narrow, $y' \ll \sqrt{x'^2 + z_0^2}$ (see Fig. 12.2), and this is why

$$R \approx \sqrt{x'^2 + z_0^2} + \frac{y'^2}{2\sqrt{x'^2 + z_0^2}}, \quad \rho = x - x' + \frac{y'^2}{2(x - x')},$$

$$w(x', y'; z_0) \to w(x'; z_0).$$

Substituting these expressions into Eq. (24.6), performing the integration over y' and neglecting in pre-exponential factors the quantity z_0^2, which is small by comparison with x^2, we obtain

$$w(x; z_0) = 1 + i\sqrt{\frac{\varkappa}{\pi}} \int_0^x \frac{e^{ik\left(x - \sqrt{x^2 + z_0^2} - x' + \sqrt{x'^2 + z_0^2}\right)}}{\sqrt{(x - x')x'}}\, w(x'; z_0)\, dx'. \tag{24.15}$$

$$\left(\frac{z_0}{x} = \sin\psi \ll 1\right).$$

§25. DERIVATION OF THE ATTENUATION FUNCTION FOR A VERTICAL DIPOLE BY THE INTEGRAL EQUATION METHOD

Let us consider to begin with Eq. (24.12) (both the source and the point of observation are located in the plane z = 0).

We may obtain outright the expansion in series of the function $\underline{w}$

practically valid for distances at which the difference between w and 1 is still not great. To that effect it is sufficient to postulate in the first place w = 1 under the integral. Since

$$\int_0^D \frac{dx'}{\sqrt{x'(D-x')}} = \pi,$$

we have

$$w(D) \approx 1 + i\sqrt{\pi sD}, \qquad (25.1)$$

$$sD \doteq i\frac{kD}{2\varepsilon^0} = \frac{ikD(\varepsilon-1)}{2\varepsilon^2} = \frac{ikD}{2\left(\varepsilon+1+\frac{1}{\varepsilon}+\dots\right)}. \qquad (25.2)$$

The quantity sD is called <u>numerical distance</u> corresponding to the real distance D. It is a complex dimensionless quantity, so much the smaller for a given D that ε^0 is greater (i.e., for example, that the soil conducts better) and that the longer is the wavelength. The dependence on the wavelength is particularly sharp, for it is taken into account twice: through <u>k</u> and through ε. If we neglect the shifting currents, that is, if we consider $|\varepsilon| \gg 1$, $\varepsilon = \frac{4\pi\sigma}{\omega} i$, the numerical distance becomes material:

$$sD = \frac{\omega k}{8\pi\sigma} D = \frac{\pi}{2}\,\frac{c}{\sigma\lambda^2} D. \qquad (25.2a)$$

According to formula (25.1), the smaller the numerical distance for a given D, the closer the field to the field above an ideally conducting surface. Thus the smallness of sD is the measure of the extent the departure of soil conduction from ideal is essential for the given D. For a wavelength λ = 1000 m and σ = $2\cdot10^8$ we find from formula (25.2a)

$$s = \frac{\pi}{2}\,\frac{3\cdot10^{10}}{2\cdot10^8(10^5)^2} \approx 2\cdot10^{-8}\,\text{см}^{-1} = \frac{1}{500\ \text{км}}.$$

Thus, for example, at the distance D = 5 km, sD will be equal to only 0.01 and the soil may still be considered as ideally conducting. For λ = 100 m the same distance of 5 km will correspond to the numerical distance sD = 1. Generally, expressing λ in meters and σ in units CGSE, we shall obtain in reciprocal kilometers

$$|s| = \frac{1}{2{,}12\sigma\lambda^2 10^{-6}\,км}. \qquad (25.3)$$

For the determination of $\underline{s}$ in the general case, one should utilize (at $\varepsilon^0 = \frac{\varepsilon^0}{\varepsilon - 1}$) formula

$$s = |s| e^{ib}, \qquad (25.4)$$

where

$$|s| = \frac{k}{2|\varepsilon^0|} = \frac{\pi}{\lambda} \frac{\sqrt{(\varepsilon' - 1)^2 + \left(\frac{4\pi\sigma}{\omega}\right)^2}}{\varepsilon'^2 + \left(\frac{4\pi\sigma}{\omega}\right)^2}, \qquad 25.5a)$$

$$b = \arg s = \frac{\pi}{2} - \chi = \frac{\pi}{2} - 2\,\mathrm{arctg}\,\frac{4\pi\sigma}{\omega\varepsilon'} + \mathrm{arctg}\,\frac{4\pi\sigma}{(\varepsilon' - 1)\omega}. \qquad (25.5b)$$

For $\frac{4\pi\sigma}{\varepsilon'\omega} \gg 1$ we may consider $b \approx \frac{\pi}{2} - \mathrm{arctg}\,\frac{4\pi\sigma}{\omega\varepsilon'} \ll 1$. For $\frac{4\pi\sigma}{\varepsilon'\omega} \ll 1$ we may substitute arc tg by its argument, and then

$$b \approx \frac{\pi}{2} - \frac{4\pi\sigma}{\omega} \frac{\varepsilon' - 2}{\varepsilon'(\varepsilon' - 1)} \approx \frac{\pi}{2}. \qquad (25.5c)$$

The values $|s|$ in km^{-1} for various values of λ, ε' and σ are compiled in Table 2.

The approximation following (25.1) may be obtained by substituting again the value (25.1) found into the integral in Eq. (24.12). This gives

$$w(D) = 1 + i\sqrt{\frac{sD}{\pi}} \int_0^D \frac{1 + i\sqrt{\pi s x'}}{\sqrt{x'(D - x')}}\,dx' = 1 + i\sqrt{\pi s D} - 2sD. \quad (25.6)$$

We may proceed further in the same manner. However, the series then obtained will converge well only for small sD, that is, so long as the numerical distance between the points 0 and A is small. Precisely in this case the departure from ideal conduction is insignificant and the result is of little practical interest.

The total solution of the integral equation (24.12), valid for any distances, may be sought for either by direct method of summing up the

TABLE 2

$|\delta|$, км$^{-1}$

λ, м	σ = 2·10^6			σ = 2·10^7				
	ε' = 2	5	10	ε' = 2	5	10	20	80
1,0								
2,0								$5,16 \cdot 10^{-2}$
5,0						$1,77 \cdot 10^{-2}$	$3,34 \cdot 10^{-2}$	$1,29 \cdot 10^{-1}$
10	$1,27 \cdot 10^{-2}$	$1,99 \cdot 10^{-2}$	$3,54 \cdot 10^{-2}$	$1,10 \cdot 10^{-2}$	$2,02 \cdot 10^{-2}$	$3,56 \cdot 10^{-2}$	$6,71 \cdot 10^{-2}$	$2,58 \cdot 10^{-1}$
20	$2,50 \cdot 10^{-2}$	$3,98 \cdot 10^{-2}$	$7,08 \cdot 10^{-2}$	$2,48 \cdot 10^{-2}$	$4,25 \cdot 10^{-2}$	$7,26 \cdot 10^{-2}$	$1,35 \cdot 10^{-1}$	$5,16 \cdot 10^{-1}$
50	$5,89 \cdot 10^{-2}$	$9,95 \cdot 10^{-2}$	$1,77 \cdot 10^{-1}$	$1,14 \cdot 10^{-1}$	$1,42 \cdot 10^{-1}$	$1,94 \cdot 10^{-1}$	$3,51 \cdot 10^{-1}$	1,29
100	$1,11 \cdot 10^{-1}$	$2,02 \cdot 10^{-1}$	$3,57 \cdot 10^{-1}$	$4,33 \cdot 10^{-1}$	$4,64 \cdot 10^{-1}$	$5,50 \cdot 10^{-1}$	$7,92 \cdot 10^{-1}$	2,61
200	$2,48 \cdot 10^{-1}$	$4,25 \cdot 10^{-1}$	$7,26 \cdot 10^{-1}$	1,71	1,74	1,84	2,16	5,43
500	1,14	1,42	2,04	$1,06 \cdot 10$	$1,07 \cdot 10$	$1,07 \cdot 10$	$1,11 \cdot 10$	$1,65 \cdot 10$
1000	4,32	4,65	5,51	$4,27 \cdot 10$	$4,28 \cdot 10$	$4,28 \cdot 10$	$4,29 \cdot 10$	$4,96 \cdot 10$
2000	$1,71 \cdot 10$	$1,74 \cdot 10$	$1,84 \cdot 10$	$1,69 \cdot 10^{2}$	$1,60 \cdot 10^{2}$	$1,69 \cdot 10^{2}$	$1,70 \cdot 10^{2}$	$1,78 \cdot 10^{2}$
5000	$1,06 \cdot 10^{2}$	$1,07 \cdot 10^{2}$	$1,08 \cdot 10^{2}$	$1,06 \cdot 10^{3}$	$1,06 \cdot 10^{3}$	$1,06 \cdot 10^{3}$	$1,08 \cdot 10^{3}$	$1,07 \cdot 10^{3}$

λ, м	σ = 2·10^9					σ = 4·10^10		
	ε' = 2	5	10	20	80	ε' = 20	40	80
1,0					$2,58 \cdot 10^{-2}$	$8,52 \cdot 10^{-2}$	$8,59 \cdot 10^{-2}$	$8,87 \cdot 10^{-2}$
2,0				$1,35 \cdot 10^{-2}$	$5,18 \cdot 10^{-2}$	$3,40 \cdot 10^{-1}$	$3,41 \cdot 10^{-1}$	$3,43 \cdot 10^{-1}$
5,0	$1,14 \cdot 10^{-2}$	$1,42 \cdot 10^{-2}$	$2,05 \cdot 10^{-2}$	$3,51 \cdot 10^{-2}$	$1,30 \cdot 10^{-1}$	$2,11 \cdot 10^{-1}$	2,11	2,11
10	$4,32 \cdot 10^{-2}$	$4,64 \cdot 10^{-2}$	$5,47 \cdot 10^{-2}$	$7,92 \cdot 10^{-2}$	$2,62 \cdot 10^{-1}$	8,49	8,49	8,49
20	$1,71 \cdot 10^{-1}$	$1,74 \cdot 10^{-1}$	$1,84 \cdot 10^{-1}$	$2,15 \cdot 10^{-1}$	$5,43 \cdot 10^{-1}$	$3,40 \cdot 10$	$3,40 \cdot 10$	$3,40 \cdot 10$
50	1,06	1,06	1,07	1,12	1,67	$2,11 \cdot 10^{2}$	$2,11 \cdot 10^{2}$	$2,11 \cdot 10^{2}$
100	4,26	4,26	4,28	4,32	4,96	$8,49 \cdot 10^{2}$	$8,49 \cdot 10^{2}$	$8,49 \cdot 10^{2}$
200	$1,70 \cdot 10$	$1,70 \cdot 10$	$1,70 \cdot 10$	$1,72 \cdot 10$	$1,78 \cdot 10$	$3,40 \cdot 10^{3}$	$3,40 \cdot 10^{3}$	$3,40 \cdot 10^{3}$
500	$1,07 \cdot 10^{2}$	$1,07 \cdot 10^{2}$	$1,07 \cdot 10^{2}$	$1,07 \cdot 10^{2}$	$1,07 \cdot 10^{2}$	$2,11 \cdot 10^{4}$	$2,11 \cdot 10^{4}$	$2,11 \cdot 10^{4}$
1000	$4,26 \cdot 10^{2}$	$4,26 \cdot 10^{2}$	$4,26 \cdot 10^{2}$	$4,26 \cdot 10^{2}$	$4,26 \cdot 10^{2}$			
2000	$1,70 \cdot 10^{3}$	$1,70 \cdot 10^{3}$	$1,70 \cdot 10^{3}$	$1,70 \cdot 10^{3}$	$1,70 \cdot 10^{3}$			
5000	$1,06 \cdot 10^{4}$	$1,06 \cdot 10^{4}$	$1,06 \cdot 10^{4}$	$1,06 \cdot 10^{4}$	$1,06 \cdot 10^{4}$			

series referred to, for example, by substituting the resolvent [Chapter 7, 3d], which is simple but very cumbersome, or by applying the Laplace transformation (see below §36), or, simplest of all, by passing from integral to differential equation.

With this in view we shall transform Eq. (24.12) from the beginning, again substituting under the integral in place of w(x') the right-hand part of this equation:

$$w(D) = 1 + i\sqrt{\frac{s\omega}{\pi}} \int_0^D \left\{ \frac{1}{\sqrt{x'(D-x')}} + \frac{i}{\sqrt{x'(D-x')}} \sqrt{\frac{sx'}{\pi}} \int_0^{x'} \frac{w(x'')\,dx''}{\sqrt{x''(x'-x'')}} \right\} dx'. \qquad (25.7)$$

The first integral is equal to π; changing the order of integration in the second, we shall obtain

$$\int_0^D \frac{dx'}{\sqrt{D-x'}} \int_0^{x'} \frac{w(x'')\,dx''}{\sqrt{x''(x'-x'')}} = \int_0^D \frac{w(x'')\,dx''}{\sqrt{x''}} \int_{x''}^D \frac{dx'}{\sqrt{(D-x')(x'-x'')}} = \pi \int_0^D \frac{dx''w(x'')}{\sqrt{x''}}. \qquad (25.8)$$

This is why Eq. (24.12) acquires another, fully equivalent form

$$w(D) = 1 + i\sqrt{s\omega\pi} - s\sqrt{D} \int_0^D \frac{w(x')}{\sqrt{x'}}\,dx'. \qquad (25.9)$$

Introducing a new function ω, Eq. (25.9) may be rewritten as follows:

$$\omega(D) = \frac{w(D)-1}{\sqrt{s\omega}} = i\sqrt{\pi} - \sqrt{s} \int_0^D \frac{w(x')}{\sqrt{x'}}\,dx'. \qquad (25.10)$$

Differentiating this equation with respect to D and substituting w by ω, we shall obtain

$$\omega'(D) + s\omega(D) + \sqrt{\frac{s}{D}} = 0. \qquad (25.11)$$

This equation is integrated by the substitution

$$w(D) = e^{-sD}\eta(D), \tag{25.12}$$

whence

$$\eta'(D) = -\sqrt{\frac{s}{D}}\, e^{sD}, \tag{25.13}$$

$$\eta(D) = \sqrt{s}\int_D^C e^{sD}\frac{dD}{\sqrt{D}} = 2\sqrt{s}\int_{\sqrt{D}}^{\sqrt{C}} e^{su^2}du. \tag{25.14}$$

Here C is an integration constant. Therefore

$$w(D) = \sqrt{sD}\,\omega(D) + 1 = 1 + 2\sqrt{D}\, se^{-sD}\int_{\sqrt{D}}^{\sqrt{C}} e^{su^2}du. \tag{25.15}$$

Breaking the integral in two parts – from 0 to $\sqrt{D}$ and from 0 to $\sqrt{C}$, Eq. (25.15) may be rewritten as follows:

$$w(D) = 1 + 2\sqrt{D}\, se^{-sD}\int_0^{\sqrt{C}} e^{su^2}du - 2\sqrt{D}\, se^{-sD}\int_0^{\sqrt{D}} e^{su^2}du. \tag{25.16}$$

Since according to Eq. (25.9) (or (25.1)), as $D \to 0$ the solution must pass into $1 + i\sqrt{sD\pi}$ (the last addend in (25.16) gives a term of higher order relative to D), we must have

$$2\sqrt{s}\int_0^{\sqrt{C}} e^{su^2}du = i\sqrt{\pi}\,. \tag{25.16a}$$

Consequently, if C may be so chosen that this condition is observed, the attenuation function will have the form

$$w(x) = 1 + i\sqrt{\pi sx}\, e^{-sx} - 2s\sqrt{x}\, e^{-sx}\int_0^{\sqrt{x}} e^{su^2}du. \tag{25.17}$$

for any $\underline{s}$ (and for $z_0 = z_A = 0$).

But C may be so chosen. Indeed, let us postulate

$$\sqrt{s}\,u = iv = e^{i\frac{\pi}{2}}v.$$

Then condition (25.16a) will pass into

$$\frac{2}{\sqrt{\pi}}\int_0^{e^{-i\frac{\pi}{2}}\sqrt{sC}} e^{-v^2}dv = 1. \tag{25.16b}$$

This equality will be observed, in particular, in the case when the upper limit is $+\infty$:

$$e^{-i\frac{\pi}{4}}\sqrt{s\zeta} = \infty,$$
$$|\zeta| = \infty, \qquad (25.16c)$$
$$\arg\sqrt{\zeta} = \frac{\pi}{8} - \frac{1}{2}\arg s.$$

Then according to formula (25.5b)

$$\arg\sqrt{\zeta} = \frac{\pi}{8} + \frac{\chi}{2}.$$

Obviously, condition (25.16b) will be fulfilled not only for that value of C, but also in the case when the upper limit's argument differs from zero in any direction by a quantity not exceeding $\pi/4$. We may then again reduce the integral to the integral over the material axis, by distorting the contour. This is why the value of arg $\sqrt{c}$ in the interval

$$\frac{\chi}{2} < \arg\sqrt{\zeta} < \frac{\pi}{2} + \frac{\chi}{2}. \qquad (25.16d)$$

may be taken ad libitum.

Obviously, this is always possible. In particular, inasmuch as χ lies between zero (for $\sigma = 0$) and $\pi/2$ (for $\sigma = \infty$), we may postulate arg $\sqrt{C} = \pi/2$, i.e., $\sqrt{C} = i\infty$.

Consequently, function (25.17) is a solution for any soils. The integral entering here is not taken in elementary functions. The formula obtained thus is the final expression for the attenuation function of the field of radiowaves emitted by a vertical dipole located on the surface of a plane and uniform ground, and observed at the distance $\underline{x}$ from it also on that surface. This formula is usually written somewhat differently, denoting $\underline{w}$ by y(sx) and substituting in the integral $su^2 = v^2$. The integral is then taken over a certain line on the complex plane between the points 0 and $\sqrt{sx}$:

$$w(x) = y(sx) = 1 + i\sqrt{\pi sx}\, e^{-sx} - 2\sqrt{sx}\, e^{-sx} \int\limits_0^{\sqrt{sx}} e^{v^2}\, dv. \qquad (25.18)$$

It is seen here that the function's argument is the numerical distance $\rho = sx$. The attenuation function (25.18), usually denoted by the letter $\underline{V}$, plays a fundamental role in the problem of radiowave propagation. It is often called the Sommerfeld function (Sommerfeld himself obtained it with imprecision: the incorrect sign of the second term). We shall refer to it as the <u>normal attenuation function</u>. We shall obtain the value Π of the field itself by multiplying $w(x) = y(sx)$ by e^{ikx}/x. If in formula (21.15) it is already postulated $su^2 = v^2$, $\sqrt{C} = i\infty$, $x = D$, we shall obtain

$$y(sx) = 1 - 2\sqrt{sx}\, e^{-sx} \int\limits_{i\infty}^{\sqrt{sx}} e^{v^2}\, dv. \qquad (25.18a)$$

For small ρ we may expand the integral and the exponential factors in formula (25.18) in series, which will again give formula (25.6).

On the other hand, for great $|\rho|$ we may take advantage of the form of (25.18a) and integrate by parts, as in computation of Fresnel integrals:

$$\int\limits_{\sqrt{sx}}^{i\infty} e^{v^2}\, dv = \frac{e^{v^2}}{2v}\Bigg|_{\sqrt{sx}}^{i\infty} + \int\limits_{\sqrt{sx}}^{i\infty} \frac{e^{v^2}\, dv}{2v^2} = \left[\frac{e^{v^2}}{2v} + \frac{e^{v^2}}{4v^3}\right]_{\sqrt{sx}}^{i\infty} + \frac{3}{4}\int\limits_{\sqrt{sx}}^{i\infty} \frac{e^{v^2}}{v^4}\, dv,$$

that is,

$$e^{-sx} \int\limits_{\sqrt{sx}}^{i\infty} e^{v^2}\, dv = -\frac{1}{2\sqrt{sx}} - \frac{1}{4(sx)^{3/2}} + M, \qquad (25.19)$$

where

$$|M| < \left|\frac{3e^{-sx}}{4(sx)^{5/2}} \int\limits_{\sqrt{sx}}^{i\infty} e^{v^2}\, v\, dv\right| = \left|\frac{3}{8}\,\frac{1}{(sx)^{5/2}}\right|.$$

Substituting expansion (25.19) into formula (25.18a), we see that if $|sx| \gg 1$, it may be postulated

$$y(sx) \approx -\frac{1}{2sx} + 0\left(\frac{1}{|sx|^2}\right). \qquad (25.20)$$

A more complete expansion for y(sx) has the form

$$y(\rho) = -\frac{1}{2\rho} - \frac{3}{4\rho^2} - \frac{15}{8\rho^3} \dots . \qquad (25.20a)$$

Thus, at very great distances x from the source, so great that the numerical distance exceeds in absolute value the unity, the only nonzero vertical component of the Hertz vector is

$$\Pi(x, 0, 0) \approx -\frac{1}{2sx}\frac{e^{ikx}}{x} = \frac{1}{2x|s|}\frac{e^{i\left(kx+\frac{\pi}{2}+\chi\right)}}{x}. \qquad (25.21)$$

Here formula (25.5b) was taken into account.

Therefore, at great distances the field amplitude begins to decrease as $1/x^2$ and not as $1/x$ (case of free atmosphere or ideal conduction), whereas the phase shifts by a certain constant quantity

$$\Delta\varphi = \frac{\pi}{2} + \chi, \qquad (25.22)$$

equal to π for a well conducting soil. Consequently the propagation velocity remains equal to c. Hence may be seen in particular the fallibility of the attenuation function (23.10).

In the intermediate region of values $|sx|$ the computation of the attenuation function may be conducted only by way of numerical integration. Plotted in Figs. 25.1 and 25.2 are the absolute value and the phase of the attenuation function $y(\rho)$, computed in the works [4] and [5] in the approximation $\varepsilon^0 = \varepsilon + 1$.

2. Let us now pass to the spatial case, $z_0 \neq 0$, and resolve Eq. (24.15). Acting exactly as with Eq. (24.12), we shall substitute the entire right-hand part of this equation instead of w under the integral. We shall encounter first of all the integral

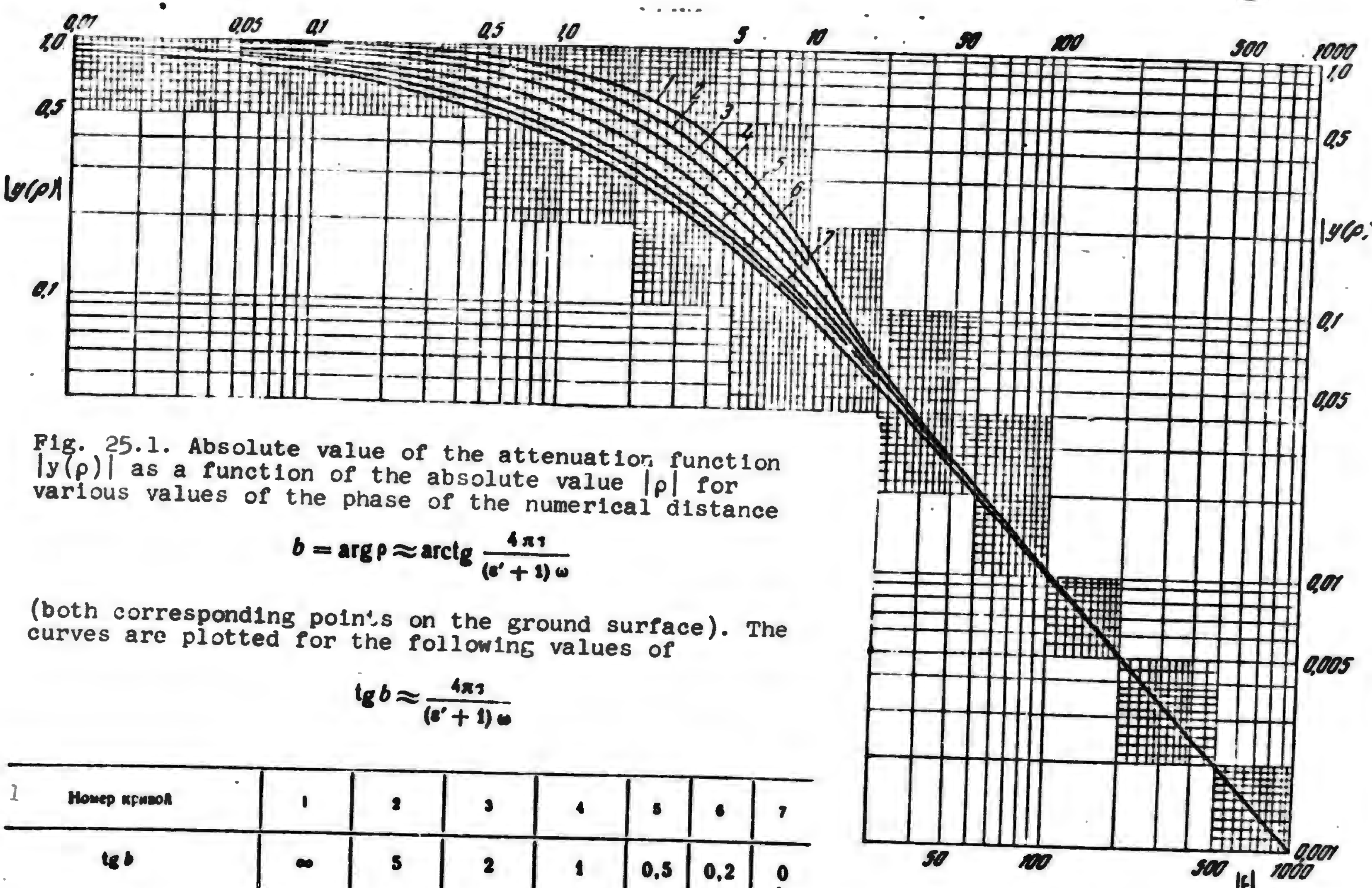

Fig. 25.1. Absolute value of the attenuation function $|y(\rho)|$ as a function of the absolute value $|\rho|$ for various values of the phase of the numerical distance

$$b = \arg \rho \approx \operatorname{arctg} \frac{4\pi\sigma}{(\varepsilon' + 1)\omega}$$

(both corresponding points on the ground surface). The curves are plotted for the following values of

$$\operatorname{tg} b \approx \frac{4\pi\sigma}{(\varepsilon' + 1)\omega}$$

Номер кривой [1]	1	2	3	4	5	6	7
tg b	∞	5	2	1	0,5	0,2	0

1) Number of the curve.

$$\int_0^x \frac{e^{ik\left(\sqrt{x'^2+z_0^2}-x'\right)}}{\sqrt{(x-x')x'}}dx' \approx \int_0^1 \frac{e^{i\frac{a}{x\xi}}}{\sqrt{\xi(1-\xi)}}d\xi =$$

$$= e^{i\frac{a}{x}}\int_0^\infty \frac{e^{i\frac{a}{x}v^2}dv}{1+v^2} = \pi\left[1-\Phi\left(\sqrt{-i\frac{a}{x}}\right)\right]. \quad (25.23)$$

Here we have expanded first of all the exponent by powers z_0^2/x'^2. This is admissible, inasmuch as the expansion might have introduced errors for small x', $(z_0/x) \sim 1$, but this region constitutes a small part of the entire integration interval $0 < x' < x$, for z_0^2/x^2. Second, there was the denotation $a = (1/2)kz_0^2$ and, third, upon substitution of the integration variable $x' = x\xi = xv^{-2}$, the integral was reduced to the integral of errors

$$\Phi(z) = \frac{2}{\sqrt{\pi}}\int_0^z e^{-x^2}dx. \quad (25.23a)$$

With $z_0 \sim a^{1/2} \to 0$ we have $\Phi(0) = 0$ and we shall obtain the second term of the right-hand part in (25.9).

Further, with the substitution of w into the right-hand part of Eq. (24.15), there arises a double integral, as in formula (25.7). We shall intervert in it the order of integration, as in formula (25.8), and we shall perform one of the integrations. As a result, instead of Eq. (24.15) we shall obtain the equation

$$w = 1 + is\sqrt{\pi x}\,e^{-i\frac{a}{x}}\left[1-\Phi\left(\sqrt{-i\frac{a}{x}}\right)\right] -$$

$$- s\sqrt{x}\,e^{-i\frac{a}{x}}\int_0^x \frac{w(x')e^{i\frac{a}{x'}}}{\sqrt{x'}}dx'; \quad a = \frac{kz_0^2}{2}. \quad (25.24)$$

Inasmuch as here $z_0^2 \ll x^2$, we have expanded the exponents everywhere, retaining the first term of each expansion.

We form the function

$$\Omega = \frac{w-1}{\sqrt{\pi x}}e^{i\frac{a}{x}} = i\sqrt{\pi s}\left[1-\Phi\left(\sqrt{-i\frac{a}{x}}\right)\right] - \sqrt{s}\int_0^x \frac{w(x')e^{i\frac{a}{x'}}}{\sqrt{x'}}dx'. \quad (25.25)$$

Differentiating this equation with respect to x and substituting again w by Ω, we obtain the equation for Ω:

$$\Omega' + s\Omega - e^{i\frac{a}{x}}\left(\frac{\sqrt{ia}}{x^{3/2}} - \frac{\sqrt{s}}{x^{1/2}}\right) = 0. \tag{25.26}$$

Effecting the substitution $\Omega = e^{-sx}\zeta$, we find

$$\zeta' = e^{sx+i\frac{a}{x}}\left(\frac{\sqrt{ia}}{x^{3/2}} - \frac{\sqrt{s}}{x^{1/2}}\right) \equiv e^{-2\sqrt{ias}}\, e^{\left(\sqrt{sx}+\sqrt{\frac{ia}{x}}\right)^2}\left(\frac{\sqrt{ia}}{x} - \sqrt{s}\right)\frac{1}{x^{1/2}}. \tag{25.27}$$

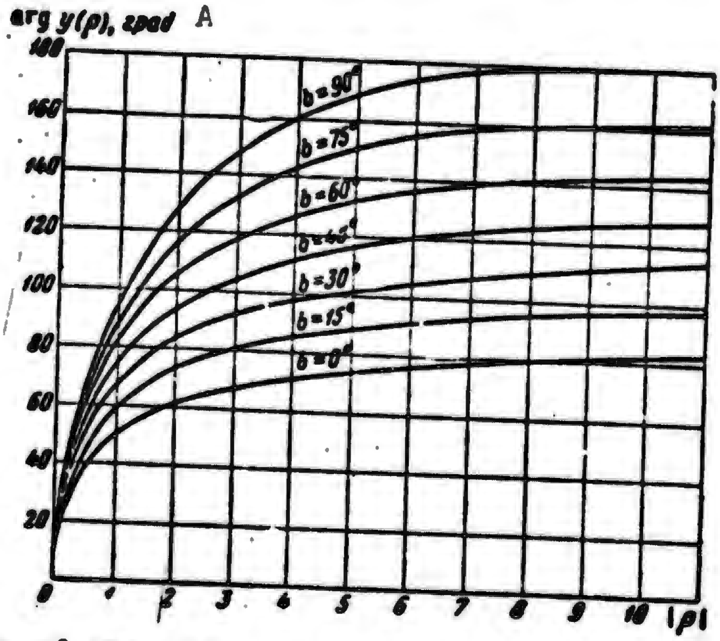

Fig. 25.2. Phase of the attenuation function, arg $y(\rho)$ as a function of the absolute value of the numerical distance $|\rho|$ for various b. A) deg.

whence (introducing into the integral a new integration variable $\sqrt{sx} + \sqrt{\frac{ia}{x}} = v$)

$$\zeta = e^{-2\sqrt{ias}}\int_0^x e^{\left(\sqrt{sx}+\sqrt{\frac{ia}{x}}\right)^2}\left(\frac{\sqrt{ia}}{x} - \sqrt{s}\right)\frac{dx}{x^{1/2}} =$$

$$= -2e^{\sqrt{ias}}\int_{\infty}^{\sqrt{sx}\left(1+\sqrt{\frac{ia}{sx^2}}\right)} e^{v^2}\,dv. \tag{25.28}$$

It is easy to see that

$$\sqrt{\frac{ia}{s}} = \sqrt{\varepsilon'}\, z_0 = \sqrt{\varepsilon'}\, x\sin\psi, \tag{25.29}$$

whereas B' is determined in the same manner as C in Eq. (25.15). We finally obtain

$$w(x; z_0) \equiv y(ax; z_0) = 1 + 2\sqrt{ax}\, e^{-ax(1+\sqrt{\varepsilon'}\sin\psi)^2} \int_{\infty}^{\sqrt{ax}(1+\sqrt{\varepsilon'}\sin\psi)} e^{v^2}\, dv. \quad (25.30)$$

The attenuation function found does not constitute the most general solution of the dipole problem above a uniform plane surface in that it does not encompass the cases when the source, as well as the point of observation, are raised above ground. It is true, however, that function <u>w</u> depends on the height z_0 of the source only in the combination $\sin\psi = \frac{z_0}{x}$. This conveys the hint that, generally speaking, only the reflection angle ψ is material, so that for the generalization of the formula this angle should only be replaced by its value for the case when both z_0 and $z_A \equiv z$ are not zero. According to Fig. 19.5

$$\sin\psi = \frac{z_0 + z_A}{R'} \approx \frac{z_0 + z_A}{x} \equiv \frac{z_0 + z}{x}. \quad (25.31)$$

If this is the way we understand the meaning of sin ψ in formula (25.30), <u>w</u> will provide the field in the most general case. We shall not rigorously demonstrate this result by the method adopted in this section. It will be obtained below by a different method (§26).

3. The entire derivation of the attenuation function might be conducted not for Π, but for E_z. This is particularly clear for the case $z_0 = 0$ (dipole and point of observation located on the ground), inasmuch as here Π^0 and E_z^0 coincide with a precision to the factor $-k^2$. If, in particular, we utilized in the integral equation (24.12) the condition

$$\frac{\partial E_z}{\partial z} = -\frac{ik}{\sqrt{\varepsilon'}} E_z,$$

Eq. (24.12) would have exactly the same form.

The integral equations (24.12) and (24.15) will subsequently play a great role in our consideration. This is why it is necessary to analyze attentively the admissions made during their derivation. The fundamental one of them is concealed in the substitution

$$\frac{\partial \Pi}{\partial z} = -\frac{ik}{\sqrt{\varepsilon^{\circ}}}\Pi \quad \text{or} \quad \frac{\partial E_z}{\partial z} = -\frac{ik}{\sqrt{\varepsilon^{\circ}}}E_z,$$

valid, as we are aware, only in the wave zone. Meanwhile the integration encompasses the entire plane z = 0, including also the region where E_z is varying as $1/r^3$ (in the immediate vicinity of the dipole). However, our inference is correct. Indeed, the quantity $\frac{\partial E_z}{\partial z}$ near the vertical dipole becomes exactly zero at z = 0 (see below). This is why in the rigorous initial integral equation (24.3) the region of small r has no part. Substituting in this region $\frac{\partial E_z}{\partial z}$ with the help of the small (and inasmuch as we substitute $E_z = w(r)\frac{e^{ikr}}{r}$) integrable quantity $\frac{-ik}{\sqrt{\varepsilon^{\circ}}}E_z$, we cannot obtain an error.

An analogous integral equation for the horizontal dipole may also be considered. Here a well-known caution must be exercised.

We shall consider the field of a static charge and a horizontal dipole, placed on the surface of a dielectric. The charge potential e at the point R(x, y, z) will be equal to

$$\varphi = \frac{2e}{\varepsilon + 1}\frac{1}{R}, \qquad (25.32)$$

so that at z = 0

$$E_x = \frac{2ex}{(1+\varepsilon)R^3}, \quad E_y = \frac{2ey}{(1+\varepsilon)R^3}, \quad E_z = 0, \quad \frac{\partial E_z}{\partial z} = \frac{2e}{(\varepsilon+1)R^3}. \qquad (25.33)$$

For a vertical dipole of moment p, disposed at z = 0, the field, also observed at z = 0, has the form

$$E_x = 0, \quad E_y = 0, \quad E_z = -\frac{2p\varepsilon}{\varepsilon+1}\frac{1}{R^3}, \quad \frac{\partial E_z}{\partial z} = 0. \qquad (25.34)$$

As for the dipole of moment p, oriented along the axis x,

$$E_x = \frac{2p}{1+\varepsilon}\frac{3x^2 - R^2}{R^5}, \quad E_y = \frac{6p}{1+\varepsilon}\frac{xy}{R^5}, \quad E_z = 0, \quad \frac{\partial E_z}{\partial z} = \frac{6px}{(\varepsilon+1)R^5}. \qquad (25.35)$$

If we utilized the Green function v_+ for the horizontal dipole, just as for the vertical, we would obtain the equation

$$E_z = E_z^0 - \frac{1}{2\pi}\int \frac{\partial E_z}{\partial z}\frac{e^{ik\rho}}{\rho}dx'\,dy'. \tag{25.36}$$

Though in the wave zone we have, as previously, $\frac{\partial E_z}{\partial z} = \frac{-ik}{\sqrt{\varepsilon^0}}E_z$, we could not effect a similar substitution in the integral, for by the same token we would have neglected the role of the nonwave zone, while here $\frac{\partial E_z}{\partial z}$ is particularly great, and E_z is small.

This is why it is more practical to utilize here for the vertical component the Green function v_-. for E_z will enter into the corresponding equation

$$E_z = \frac{i}{\omega\varepsilon}\int(\mathrm{grad}_z\,\mathrm{div}\,j_0 + k^2 j_{0z})v_-\,dV' + \frac{1}{2\pi}\int E_z\frac{\partial}{\partial z'}\left(\frac{e^{ik\rho}}{\rho}\right)dx'\,dy'. \tag{25.37}$$

and the role of the region near the source may be neglected, inasmuch as here $E_z = 0$. Consequently, we consider that under the integral $E_z = \frac{e^{ikr}}{r}w(r)$, where w(r) is a slowly varying function. For the same reasons v_- cannot be chosen for E_x and E_y, for here the nonwave zone may prove to be essential.

On the other hand, when computing E_z for the vertical dipole, the use of the Green function v_- is improper.

4. We utilized in all formulas for the complex dielectric constant ε the effective expression ε^0. It is evident, however, that the attenuation function then found is by no means valid for all ε. In reality, as $\varepsilon \to 1$, we have $\varepsilon^0 \to \infty$, $s \to 0$, and the formulas derived above give $y \to 1$, in exactly the same way as when $\varepsilon \to \infty$, that is, just as for an ideal conductor. Meanwhile $\varepsilon \to 1$ implies the transition to free space, and we should have obtained $y \to 1/2$. The erroneous result is linked with the fact that as ε decreases, the region of validity of the approximate boundary condition with $\varepsilon^0 = \varepsilon^2/(\varepsilon - 1)$ (24.4) shifts farther and farther from the source, for farther reaches the field propagating

through the soil. At limit, for $\varepsilon = 1$, this condition is valid only at infinity. Meantime, when deriving the integral equation (24.12), it was specifically assumed that the boundary condition is valid for the greater part of the course. Therefore, as is already clear from physical considerations, we in fact assumed everywhere a departure of ε from the unity sufficiently great for the wave, propagating through the soil, to damp over those effective distances r_{eff}, where the boundary condition is utilized. For $z_0 = z_A = 0$, the essential zone covers the entire course, $r_{eff} \sim D$, and we must have

$$\mathrm{Im}\sqrt{\varepsilon}\cdot kD \gg 1. \tag{25.38}$$

For a raised point of observation, when the essential zone is pressed against the source, $r_{eff} \sim D\cdot\sin\psi \ll D$, the condition becomes more rigid. Only with fulfillment of this condition will the attenuation function containing ε^0 be correct (this condition will be rigorously obtained in §31). However, in practice this limitation is not very serious in propagation along the ground.

The above-said refers also to the subsequent chapters, where attenuation functions for other cases (spherical ground, nonuniform surface and others) will be obtained. Let us not once more (see §21), that it is sufficient to recognize $\varepsilon + 1$ for ε^0 almost everywhere.

§26. METHOD OF REFLECTED SOURCE

1. A very simple method of deriving the normal attenuation function may be based upon the combination of the approximate boundary conditions (21.19) with the representation of an imaginary source dispatching the ground surface-reflected wave, which is customary, for example, in case of ideally reflecting surface. In connection with this let us turn to §20.

We have seen that when the question arises about the solution of the wave equation for a certain scalar function u, satisfying the condition that on the plane $z = 0$ either the normal derivative function (20.5a), or function (20.5b) itself vanish, the solution is expressed through the sum of the fields of the real source in boundless space and of its reflection, constructed according to a specific rule (formulas (20.8) and (20.9) respectively).

As was shown by G.D. Malyuzhinets [6], this consideration may also be generalized to the case when the boundary condition for the function u has the form

$$\frac{\partial u}{\partial z} = -\frac{ik}{\sqrt{\varepsilon'}} u \text{ at } z = 0. \tag{26.1}$$

It is valid, as we know, for the vertical component of the electric field ($u = E_z$), and in the case of vertical electric dipole it is also valid for the unique nonzero component of the Hertz vector ($u = \Pi = \Pi_z$).

We must find the solution of the wave equation

$$\nabla^2 u + k^2 u = -4\pi\rho, \tag{26.2}$$

where ρ is a certain source.

We shall seek it in the form

$$u = u_0 + u_1, \tag{26.3}$$

where u_0 is a field emitted by the given source and located in boundless atmosphere, for example, at the point $(0, 0, z_0)$,

$$u_0 = \frac{1}{2}\frac{e^{ikR}}{R}, \quad R = \sqrt{x^2 + y^2 + (z - z_0)^2}. \tag{26.4}$$

The complementary field u_1 may be called the reflected field. It satisfies the wave equation (26.2) everywhere for $z > 0$. Had the boundary condition had the form

$$u = 0 \text{ at } z = 0, \tag{26.5}$$

the reflected field would have been, as we know (see (20.9)), the field of reflected sources, taken with inverse sign:

$$u_1 = -u_0(x, y, z; [\rho(x_0, y_0, -z_0)]). \tag{26.6}$$

Further, since z and z_0 are part of u_0 (the field of a source in boundless atmosphere) only in the combination $(z - z_0)^2$, the sign variation at z_0 is equivalent to sign variation of z, i.e., it may be written:

$$u_1 = -u_0(x, y, -z; [\rho(x_0, y_0, z_0)]) = -u_0(x, y, -z). \tag{26.6a}$$

Therefore, for the boundary condition (26.5) the solution is sought very simply.

Let us bring to this scheme our problem with boundary condition (26.1). Let us consider to that effect the function v, obtained from u by way of such a transformation

$$v = v(u), \tag{26.7}$$

that a type (26.5) boundary condition be obtained for v, and that at the same time v satisfy Eq. (26.2). This is found to be feasible. Namely, assume

$$v = \frac{\partial u}{\partial z} + \frac{m}{\sqrt{\sigma^2}} u = e^{-\frac{m}{\sqrt{\sigma^2}} z} \frac{\partial}{\partial z}\left(u e^{\frac{m}{\sqrt{\sigma^2}} z}\right). \tag{26.8}$$

Since every addend in v must satisfy the wave equation, v must also be subordinated to it. In particular, to the source in boundless space corresponds $v = v_0$, where

$$v_0 = e^{-\frac{m}{\sqrt{\sigma^2}} z} \frac{\partial}{\partial z}\left(u_0 e^{\frac{m}{\sqrt{\sigma^2}} z}\right). \tag{26.9}$$

It is further evident that on the strength of condition (26.1), the boundary condition

$$v = 0 \text{ at } z = 0. \tag{26.10}$$

is superimposed on v.

Therefore, we are concerned with the finding of a function v, satisfying the wave equation and the boundary condition (26.10) (and afterwards finding from it the function u required by us). But the solution of this problem is given by formulas (26.3), (26.6a), after which we shall utilize formula (26.8):

$$v = v_0 + v_1, \tag{26.11}$$

$$v_1 = -v_0(x, y, -z) = -e^{\frac{k}{\sqrt{\varepsilon}}z} \frac{\partial}{\partial(-z)}\left(u_0(x, y, -z)e^{-\frac{k}{\sqrt{\varepsilon}}z}\right), \tag{26.11a}$$

i.e.,

$$v = v_0(x, y, z) - v_0(x, y, -z). \tag{26.11b}$$

Having determined v, we may find u too. In reality, it follows from formula (26.8) that

$$u = e^{-\frac{k}{\sqrt{\varepsilon}}z}\int_C^z e^{\frac{k}{\sqrt{\varepsilon}}\zeta} v(x, y, \zeta)\,d\zeta. \tag{26.12}$$

The lower limit of the integral will be chosen subsequently. In any case one must so choose it that the integral converge; consequently, at lower limit the integrand must vanish. It will be shown below that there must be C = i∞. We shall substitute this value at once. This is why, substituting expressions (26.11a) and (26.11b) into formula (26.12) and integrating by parts, we obtain

$$u_1 = u_0(x, y, -z) - \frac{2ik}{\sqrt{\varepsilon}}\int_{i\infty}^z e^{\frac{k}{\sqrt{\varepsilon}}(\zeta - z)} u_0(x, y, -\zeta)\,d\zeta. \tag{26.13}$$

Therefore the total field is

$$u = u_0(x, y, z) + u_0(x, y, -z) - \frac{2ik}{\sqrt{\varepsilon}}\int_{i\infty}^z e^{\frac{k}{\sqrt{\varepsilon}}(\zeta - z)} u_0(x, y, -\zeta)\,d\zeta. \tag{26.14}$$

In the case when the source is located at the point (0, 0, z_0) and is a point dipole, this gives

$$\Pi = \frac{1}{2}\frac{e^{ikR_1}}{R_1} + \frac{1}{2}\frac{e^{ikR_2}}{R_2} + \frac{ik}{\sqrt{\varepsilon}}\int_z^{\infty} e^{ik\left[R_2' + \frac{\zeta - z}{\sqrt{\varepsilon}}\right]}\frac{d\zeta}{R_2'}, \tag{26.15}$$

where $R_1 = \sqrt{x^2 + y^2 + (z - z_0)^2}$ is the distance from the point of observation to the source, $R_2 = \sqrt{x^2 + y^2 + (z + z_0)^2}$ is the distance to its reflection in the plane $z=0$; $R_1' = \sqrt{x^2+y^2+(\zeta+z_0)^2}$. The integral expresses the correction to a field that would be obtained in the case of an ideally conducting surface.

This formula, at times called Weyl formula (or Weyl lemma), was obtained earlier from a rigorous cumbersome theory [2b], [3], [7] only after complex computations and also neglects equivalent to the admission that the absolute value $|\varepsilon|$ is sufficiently great. Its deduction, brought up here [6], is unquestionably simple, and, inasmuch as it rests only upon an approximate boundary condition, it stems from the same admission as does the whole theory; this is why it is in any case valid in the wave zone of the emitter for all the soils encountered in nature. The generalization of the method described, allowing to consider in case of impedance boundary condition (26.1) a broader range of problems, was given by M.D. Khaskind [18]. The propagation of electromagnetic waves above a plane surface of an anisotropic medium was investigated by the method described above in the work [19].

We shall now demonstrate that hence may be very simply obtained the expression for a normal attenuation function.

Let us consider the integral in formula (26.15). Inasmuch as $\mathrm{Re}\sqrt{\varepsilon'}>0$, the integrand will rapidly (exponentially) decrease as ζ varies from $\underline{z}$ along the direction toward $\zeta = C$. This is why at integration the main role is played by the region of values of ζ, directly adjacent to the point $\zeta = z$. Consequently, we may transform R_z', taking into account that $|\zeta - z|$ is small.

We shall introduce a new variable

$$\xi = \zeta - z.$$

Then

$$R_1' = \sqrt{r^2+(\zeta+z_0)^2} = \sqrt{r^2+(z_0+z)^2+2\xi(z_0+z)+\xi^2}. \quad (26.16)$$

From Fig. 26.1 we may see that $r^2+(z+z_0)^2 = R_1^2$, $z+z_0 = R_1 \sin\psi$. This is why, expanding by powers ξ, we shall obtain

$$R_2' \approx R_2 + \xi \sin\psi + \frac{\xi^2}{2R_2}\cos^2\psi. \qquad (26.17)$$

We shall limit ourselves to the terms written. The terms of second order in ξ must, generally speaking, be retained, inasmuch as the points of observation, for which $\psi = 0$, may be of interest to us (points at ground level, when the source is also on the ground), and the term of first order in ψ drops out.

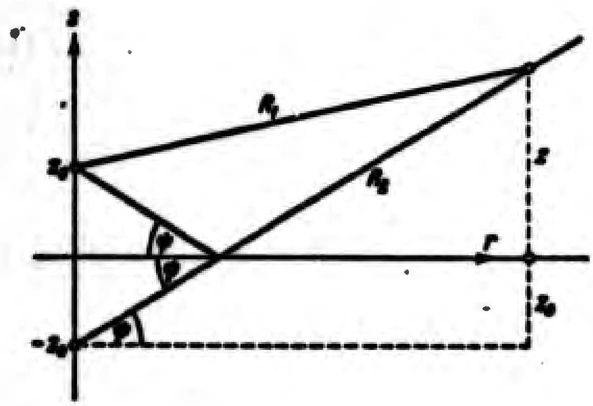

Fig. 26.1. Denotations for formula (26.20).

In such a case there appears under the integral in formula (26.15) an exponential factor with an exponent which may be transformed as follows:

$$k\left[R_2' + \frac{\xi - z}{\sqrt{\varepsilon^0}}\right] = k\left[R_2 - \frac{R_2(1+\sqrt{\varepsilon^0}\sin\psi)^2}{2\varepsilon^0\cos^2\psi} + \right.$$
$$\left. + \left(\frac{\xi\cos\psi}{\sqrt{2R_2}} + \sqrt{\frac{R_2}{2\varepsilon^0}}\,\frac{1+\sqrt{\varepsilon^0}\sin\psi}{\cos\psi}\right)^2\right]. \qquad (26.18)$$

Substituting R_2' by R_2 in the pre-exponential factor and introducing into the integral a new integration variable

$$t = \sqrt{ik}\left(\frac{\cos\psi}{\sqrt{2R_2}}\,\xi + \sqrt{\frac{R_2}{2\varepsilon^0}}\,\frac{1+\sqrt{\varepsilon^0}\sin\psi}{\cos\psi}\right), \qquad (26.18a)$$

we obtain

$$\frac{ik}{\sqrt{\varepsilon^0}}\int_0^{\infty} e^{ik\left(R_2'+\frac{\xi}{\sqrt{\varepsilon^0}}\right)}\frac{d\xi}{R_2'}=\frac{2\sqrt{\rho}}{\cos\psi}e^{-\rho(1+\sqrt{\varepsilon^0}\sin\psi)^2\sec^2\psi}\int_{\sqrt{\rho}(1+\sqrt{\varepsilon^0}\sin\psi)\sec\psi}^{i\infty} e^{t^2}dt=$$

$$=\frac{i\sqrt{\rho\pi}}{\cos\psi}e^{-\rho(1+\sqrt{\varepsilon^0}\sin\psi)^2\sec^2\psi}\left(1+\frac{2}{\sqrt{\pi}}\int_0^{i\sqrt{\rho}(1+\sqrt{\varepsilon^0}\sin\psi)\sec\psi} e^{-\tau^2}d\tau\right)= \tag{26.19}$$

$$=\frac{i\sqrt{\rho\pi}}{\cos\psi}e^{-\rho(1+\sqrt{\varepsilon^0}\sin\psi)^2\sec^2\psi}\left(1+\Phi\left(i\sqrt{\rho}(1+\sqrt{\varepsilon^0}\sin\psi)\sec\psi\right)\right),$$

where $\Phi(x)$ is the integral of errors, in the given case from a complex argument. Detailed tables exist for it [8]. Consequently, if we denote by $(f - 1)/2$ the entire expression (26.19), we shall have

$$\Pi=\frac{1}{2}\left\{\frac{e^{ikR_1}}{R_1}+\frac{e^{ikR_2}}{R_2}+(f-1)\frac{e^{ikR_2}}{R_2}\right\}. \tag{26.20}$$

$$f-1=\frac{4\sqrt{\rho}}{\cos\psi}e^{-\rho(1+\sqrt{\varepsilon^0}\sin\psi)^2\sec^2\psi}\int_{\sqrt{\rho}(1+\sqrt{\varepsilon^0}\sin\psi)\sec\psi}^{i\infty} e^{t^2}dt, \tag{26.20a}$$

$$\rho=\frac{ikR_2}{2\varepsilon^0},\ \sin\psi=\frac{z+z_0}{R_2},\ R_1=\sqrt{x^2+y^2+(z-z_0)^2},$$
$$R_2=\sqrt{x^2+y^2+(z+z_0)^2}. \tag{26.20b}$$

The first term in parentheses in formula (26.20) gives the field, which given source would have induced in a boundless free space. The second term represents the field of an identical (imaginary) source at the point $(0, 0, -z_0)$. Together, these two terms consequently give the field of a vertical dipole above an ideally conducting surface. Indeed, as $\varepsilon \to \infty$ when $\varepsilon^0 \to \infty$ and $\rho \to 0$, we have $f - 1 \to 0$, and the last term in parentheses in formula (26.20) disappears. It provides the correction determined by the "nonideality" of soil conduction. Together, the second and the third terms give the field reflected from such a real soil,

$$\Pi=\frac{1}{2}\left(\frac{e^{ikR_1}}{R_1}+f\frac{e^{ikR_2}}{R_2}\right). \tag{26.20c}$$

and, consequently, <u>f</u> is the reflection factor. In order for this formula to provide the earlier found attenuation function (25.18a) at $z=z_0=0,\ R_1=R_2=x,\ (\cos\psi=1,\ \sin\psi=0)$, it is necessary that

$$f = 2y - 1, \qquad (26.20d)$$

the upper limit of the integral in formula (26.20a) being postulated equal to $i\infty$ (which has already been done). This is precisely what determines the choice of the constant C (in essence, it was simply taken into account here that at $z = z_0 = 0$ and $x \to 0$, expansion (25.1) must be obtained.

In this formula enters the quantity ε^0, which includes $\cos\psi$, if the absolute value $|\varepsilon|$ is not great. In the process of the deduction we neglected the corresponding additional dependence on coordinates. This is indeed admissible with good precision. In reality, the quantity ε^0 emerged from the boundary condition (26.1). The latter must be satisfied, first of all, within the limits of the essential region (see the corresponding reasonings in §19). This is why we recognize for the angle ψ, entering into ε^0 a constant value, coinciding with the angle ψ, which figures in other expressions in formula (26.20). The substitution of $\cos^2\psi$ in ε^0 by a constant may seem to be incorrect only at $z = z_0 = 0$, when the essential region encompasses the entire course. However, in this case $\cos^2\psi$ becomes equal to the unity precisely for all the points.

The obtained formula (26.20) is the most general for the chosen formulation of the question.

Its quite important property is the fact that the heights of the point of observation and of the source enter into the reflection factor f only in the combination $z + z_0$ (through $\sin\psi = (z + z_0)/R_2$). This is why the reflection factor will not vary if one of the corresponding points is raised and the other lowered by one and the same quantity.

For a source disposed on the ground ($R_1 = R_2$), and not too great ψ ($\cos\psi \approx \cos^2\psi \approx 1$) formula (26.20) may be written in the form (for the attenuation function we introduce the denotation $y(r, z; z_0)$, where the arguments denote r as the horizontal distance from the source, z as the altitude of the point of observation above ground, z_0 as the height of the source):

$$\Pi = \frac{e^{ikR}}{R} y(r, z; 0), \tag{26.21}$$

$$y(r, z; 0) = 1 + 2\sqrt{sr} \int\limits_{\sqrt{sr}(1+\sqrt{\varepsilon^0}\sin\psi)}^{i\infty} e^{v^2 - sr(1+\sqrt{\varepsilon^0}\sin\psi)^2} dv, \tag{26.22}$$

$$2y(r, z) - 1 = f(r, z). \tag{26.22a}$$

2. Let us consider f for the case when the numerical distance is great:

$$\frac{|\sqrt{sr}(1+\sqrt{\varepsilon^0}\sin\psi)|}{\cos\psi} \gg 1. \tag{26.23}$$

Integrating by parts in relation (26.20a), as in formula (25.19), rejecting the small terms, we obtain

$$f = 2w - 1 = 1 - \frac{2}{\sqrt{\varepsilon^0}\sin\psi + 1} - \frac{\cos\psi}{sr(1+\sqrt{\varepsilon^0}\sin\psi)^3} - \cdots \tag{26.24}$$

Consequently, when the numerical distance is great, by retaining the first two terms and utilizing formula (21.20) we have

$$f \approx \frac{\sqrt{\varepsilon^0}\sin\psi - 1}{\sqrt{\varepsilon^0}\sin\psi + 1} = \frac{\varepsilon\sin\psi - \sqrt{\varepsilon - \cos^2\psi}}{\varepsilon\sin\psi + \sqrt{\varepsilon - \cos^2\psi}} = f_{\parallel}, \tag{26.25}$$

as must be for the reflection of a wave polarized in the incidence plane (cf. (19.18)). This obviously does not mean that at great numerical distances one generally may utilize instead of the attenuation function (26.20) the addition of plane waves with the reflection factor (19.18). For very small ψ and great $|sx|$ formula (26.22a) together with (26.24) gives

$$y(r, z; 0) \approx \frac{\sqrt{\varepsilon^0}\sin\psi}{1+\sqrt{\varepsilon^0}\sin\psi} - \frac{1}{2sr(1+\sqrt{\varepsilon^0}\sin\psi)^3} \to -\frac{1}{2sr}. \tag{26.26}$$

This result is impossible to obtain from the interference formulas (giving here a zero field if $n \neq \infty$). The latter may obviously be utilized only in the case when the third term in the right-hand part of formula (26.24) is small by comparison with the sum of the first two, i.e., if

$$\sqrt{\varepsilon'}\sin\psi\cdot ax(1+\sqrt{\varepsilon'}\sin\psi)^2 \gg 1,$$

or

$$k(z+z_0)(1+\sqrt{\varepsilon'}\sin\psi)^2 \gg \sqrt{\varepsilon'}. \qquad (26.27)$$

Since at $|\sqrt{\varepsilon'}\sin\psi| \gg 1$ we arrive at the ideal reflection (when the interference formulas are knowingly correct), for the estimate in the region of small angles we are interested in the case $\sqrt{\varepsilon'}\sin\psi < 1$ and $\sqrt{\varepsilon'}\sin\psi$ in parentheses may be rejected. As a result we obtain the criterion of applicability of interference formulas

$$k(z+z_0) \gg |\sqrt{\varepsilon'}| = \left|\sqrt{\frac{\varepsilon'}{\varepsilon-1}}\right|, \qquad (26.28)$$

coinciding with the criterion (19.38) obtained earlier from other considerations.

Now it is pertinent to analyze the question about the physical sense of the condition allowing to pass in formula (19.18) (or (26.24)), as well as in formula (19.13) to the extreme case of reflection $(f_\parallel = 1,\ f_\perp = -1)$. As may be seen, this transition takes place for

$$|\sqrt{\varepsilon'}\sin\psi| \gg 1. \qquad (26.29)$$

At the raised point of observation (see Fig. 26.2), the field is formed by addition of the direct field of the source and of field of secondary emitters, fundamentally those which are created within the bounds of the essential region of the surface z = 0 (shaded ellipse in the figure). It is clear that if within the limits of this region the primary field, exciting the secondary emitters, is not yet attenuated and is the same as above an ideally reflecting surface, the secondary fields too, and consequently also the aggregate field at the point of observation A will be the same as above an ideally reflecting surface. We

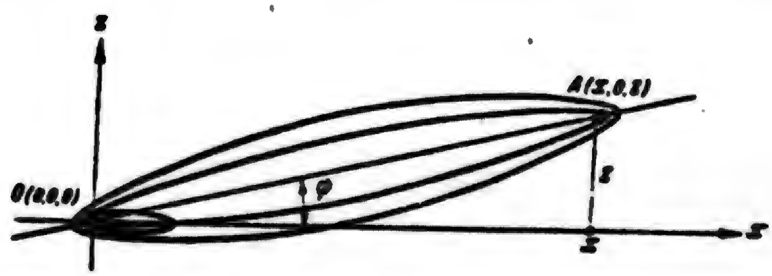

Fig. 26.2. Formation of the field at the raised point of observation. The cross-hatched ellipse, situated near the source in the plane z = 0, defines the essential region.

have determined earlier the dimensions of the essential zone: according to formula (12.11) the length of the crosshatched region is equal to $2a_1 = \frac{\pi}{k\sin^2\psi}$. At that distance the field will be the same as above an ideally reflecting surface provided the corresponding numerical distance is much less than the unity, that is, if (see (25.1))

$$\sqrt{|\varkappa s\cdot 2a_1|}=\sqrt{\left|\frac{\pi}{k\sin^2\psi}\cdot\frac{k}{2\varepsilon'}\right|}\ll 1,$$

or

$$|\sqrt{\varepsilon'}\sin\psi|\gg\frac{\pi}{\sqrt{2}}. \qquad (26.30)$$

But this is practically the same as condition (26.29).

Thus, at the raised point the field coincides with the field above an ideal reflector, provided it is raised sufficiently high for the essential region, pressed against the emitter (located on the ground), to become sufficiently small, so small that within its bounds the field sliding along the ground, still remain unaffected by the departure of $|\varepsilon|$ from ∞.

Therefore, the problem of field determination above the interface may be considered as resolved. The attenuation function of the field of a dipole placed on the ground surface decreases, as the source drifts farther along the interface, from the unity, beginning with distances for which the numerical distance $|\rho|$ is of the order of the unity, whereupon, beginning from $|\rho| \approx 5$, this decrease is conveyed by the factor $-1/2\rho$. Consequently, at $|\rho| \gg 1$, the derivative dw/dr is a quantity of the order $|w/r|$. This is why, provided we limit ourselves to the wave zone $k_0 r \gg 1$, this decrease takes place slowly over an interval of the order of the wavelength, and the assumption of slowness of w variation (see (21.2)), which we lie at the basis of the whole consideration, is found to be valid. A doubt may be aroused only by the region $|\rho| \lesssim 1$. In this region w and w' are of the order of the unity, or, to be more precise, as is shown by the numerical calculations, $\left|\frac{dw}{d\rho}\right| \sim 0{,}2$ at $|\rho| \sim 1$:

$$\left|\frac{dw}{dr}\right| \approx \left|\frac{dw}{d\rho}\frac{d\rho}{dr}\right| \sim 0{,}2\left|\frac{d\rho}{dr}\right| \sim \frac{k}{10|\varepsilon^0|}. \qquad (26.31)$$

Consequently, condition (21.2) is satisfied if the value $10|\varepsilon^0|$ is sufficiently great. But this is not always so.

For very short waves and dry soils, we practically always find ourselves in the region $|\rho| \gg 1$, in which it is sufficient to utilize the asymptotic formula $y(\rho) = -(1/2\rho)$. The attenuation is found to be quite strong (the field amplitude decreases inversely proportionally to the square of the distance). This is why transmission by spatial radio-wave (reflected from the ionosphere, or raising the corresponding points above ground so as to attenuate its influence (for waves shorter than about 10 m, not reflecting from the ionosphere) are here more often applied.

§27. EXPRESSIONS FOR THE INTENSITIES OF THE FIELDS OF VERTICAL ELECTRICAL DIPOLE

The expression for Π, obtained in §26 allows to obtain by way of a somewhat cumbersome differentiation the components $\mathbf{E}$ and $\mathbf{H}$ of the fields. They are usually computed by rejecting the terms of the order

$(1/\varepsilon^0)^2$ relative to the principal ones, but retaining the terms of the order $1/\varepsilon^0$. However, it may at times become necessary to account also for the nonwave part of the field, i.e., the terms of the order $1/kr$ relative to principal ones, for at distance from the source their ratio to the term containing $y(sr)$, influences the value of E_r in a correction of the order $1/\varepsilon^0$.

The resulting expressions again contain the function $y(r, z; z_0)$, not reducible to elementary ones. The integral, figuring in it, is nothing but the integral of errors from complex argument. As already mentioned, sufficiently complete tables for its phase and amplitude are available [8]. Moreover, special numerical calculations were completed in older works on the theory of radiowave propagation; tables and graphs were given for certain special cases. It is quite often sufficient to utilize asymptotic expansions for small and great values of the argument (see §10, formula (10.13) in particular, and also formulas (25.6), (25.20) and (26.26)).

In our formulas for the attenuation function $y(r, z; z_0)$ figures not ε, but ε_{eff} or ε^0. The substitution of ε by ε^0 is essential to us first of all not because at real computations of the field we must substitute in place of ε, say $\varepsilon + 1$ (which is equivalent to substituting the real dielectric constant ε' by $\varepsilon' + 1$) or, still more precisely, $\varepsilon^2/(\varepsilon - \cos^2\psi)$. Practically, this refinement contributes little, inasmuch as the values of ε' for soils are generally rather indeterminate quantities and are most often measured precisely by radiowave attenuation, or by inclination of the lines of force on the surface, or by any other method utilizing the theory of radiowave propagation, that is, measured in fact is ε_{eff}.

The value of the refinement performed consists first of all in that the form of the solution is preserved in terms of the order $1/\varepsilon$

and still further, as we have been convinced. In reality it might have occurred that the solution's refinement would lead to the substitution of function $y(\rho)$ by another one, which, as we shall see, does take place if we pursue the refinement of the result (see §31).

This is why we may be satisfied by this general conclusion, and utilise at field computation the attenuation function, neglecting the terms of the order $1/\varepsilon$, i.e., neglecting the difference between ε and ε_{eff}. More complete expressions for the derivatives of the function y and for field intensity components, valid also in the nonwave zone and accounting for the terms of the order $1/\varepsilon^0$, may be found, for example, in the works by K. Norton [5, 9] and P.A. Ryazina [4].* We limit ourselves here (when computing the field intensity) to the case, when the vertical electric dipole is located on the ground surface.

Differentiating formula (26.22) with the indicated precision, we obtain:

$$\frac{\partial y}{\partial \alpha}=\frac{-ik}{\sqrt{\varepsilon^0}}(1+\sqrt{\varepsilon^0}\sin\psi)y+ik\sin\psi, \tag{27.1a}$$

$$\frac{\partial y}{\partial r}=s\{(\varepsilon^0\sin^2\psi-1)y-\sqrt{\varepsilon^0}\sin\psi\,(\sqrt{\varepsilon^0}\sin\psi-1)\}, \tag{27.1b}$$

$$\frac{\partial^2 y}{\partial\alpha\,\partial r}=\frac{k^2}{2(\varepsilon^0)^{3/2}}(\varepsilon^0\sin^2\psi-1)\{(1+\sqrt{\varepsilon^0}\sin\psi)y+\sqrt{\varepsilon^0}\sin\psi\}, \tag{27.1c}$$

$$\frac{\partial^2 y}{\partial\alpha^2}=-\frac{k^2}{\varepsilon^0}\{(1+\sqrt{\varepsilon^0}\sin\psi)^2y-\sqrt{\varepsilon^0}\sin\psi\,(1+\sqrt{\varepsilon^0}\sin\psi)\}. \tag{27.1d}$$

Let us note that according to formula (27.1b), we have in reality

$$\left|\frac{\partial y}{\partial r}\right|\ll k|y|, \tag{27.1e}$$

as required when deriving the boundary conditions in correspondence with relation (21.2). This is also correct even for $|\sqrt{\varepsilon^0}\sin\psi|\gg 1$, when

$$\frac{\partial y}{\partial r}\approx\frac{ik}{2}\sin^2\psi\,(y-1),$$

for in this case, according to formulas (26.24) and (26.25),

$$y-1\equiv w-1=\frac{t-1}{2}-1\approx\frac{-1}{\sqrt{\varepsilon'}\sin\psi},$$

and relation (27.1e) is again valid.

Note further that at z = 0 we obtain

$$\frac{\partial y}{\partial z}=\frac{-ik}{\sqrt{\varepsilon'}}y, \tag{27.2}$$

i.e., y obviously satisfies the boundary condition imposed on Π.

Passing to the calculation of the fields with the aid of relations

$$E_z=\frac{\partial^2\Pi}{\partial z^2}+k^2\Pi,\ E_r=\frac{\partial^2\Pi}{\partial z\,\partial r},\ \Pi=\frac{e^{ikR}}{R}y(r,z;0), \tag{27.3}$$

we obtain with the same precision, rejecting the terms of the order or higher than $1/\varepsilon'$, $1/kr$ and $\sin^2\psi\approx\psi^2$:

$$E_z=k^2\frac{e^{ikR}}{R}\left\{y(r,z;0)-\sin^2\psi+\frac{\sin\psi}{\sqrt{\varepsilon'}}\right\}, \tag{27.4}$$

$$E_r=k^2\frac{e^{ikR}}{R}\left\{\frac{1}{\sqrt{\varepsilon'}}y(r,z;0)-\sin\psi\right\}. \tag{27.5}$$

An elegant form may be imparted to the expressions for the field [9] by representing the field $\vec{E}$ in the form of superimposition of two waves: the "surface" and the "spatial" waves.

With this in view, we shall introduce instead of our attenuation function (26.22) the function

$$y^*(w)=1-2\sqrt{w}\,e^{-w}\int_{i\infty}^{\sqrt{w}}e^{u^2}du, \tag{27.6}$$

$$w=sr(1+\sqrt{\varepsilon'}\sin\psi)^2. \tag{27.6a}$$

The function y*(w) differs from y(r, z; 0) by the factor $1+\sqrt{\varepsilon'}\sin\psi$ before the integral. This is why it coincides with the function y(r, 0; 0) in which r is substituted by $r(1+\sqrt{\varepsilon'}\sin\psi)^2$:

$$y^*(w)-1=(1+\sqrt{\varepsilon'}\sin\psi)(y-1). \tag{27.7}$$

Besides, we shall utilize the formula for the Fresnel reflection factor of the wave, of which the electric vector lies in the incidence plane (26.25):

$$f_{\parallel} = \frac{\sqrt{\varepsilon^0}\sin\psi - 1}{\sqrt{\varepsilon^0}\sin\psi + 1}, \quad 1 - f_{\parallel} = \frac{2}{1+\sqrt{\varepsilon^0}\sin\psi} = \frac{1+f_{\parallel}}{\sqrt{\varepsilon^0}\sin\psi}. \quad (27.8)$$

Then we have

$$y(r, z; 0) = \frac{1-f_{\parallel}}{2} y^*(w) + \frac{1+f_{\parallel}}{2}; \qquad (27.9)$$

the attenuation function broke up into two parts, of which the second on the ground surface ($\psi = 0, f_{\parallel} = -1$) disappears, and the first is, to the contrary, maximum.

Now formulas (27.4) and (27.5) will take the form (we again limit ourselves to the terms of lowest order relative to $1/\varepsilon^0$)

$$E_z = k^2 \frac{e^{ikR}}{R}\left\{\frac{1-f_{\parallel}}{2} y^* + \frac{1+f_{\parallel}}{2}\right\}, \qquad (27.10a)$$

$$E_r = -k^2 \frac{e^{ikR}}{R}\left\{\frac{1-f_{\parallel}}{2\sqrt{\varepsilon^0}} y^* - \frac{1+f_{\parallel}}{2}\sin\psi\right\}. \qquad (27.10b)$$

Let us introduce the vectors $\vec{\psi}_I$, $\vec{z}_I$, $\vec{r}_I$, directed along the axes ψ, $\underline{z}$ and $\underline{r}$. There exists between them the relation

$$\psi_1 = z_1\cos\psi - r_1\sin\psi \approx z_1 - r_1\sin\psi. \qquad (27.11)$$

Utilizing it, the field components may be so grouped, that two vectors be forming, of which one is proportional to $y^*(w)$, dependent on the numerical distance, decreases with the height of the point of observation and thus may be called "surface" wave. The other vector does not contain $y^*(w)$, vanishes on the plane $z = 0$, is directed perpendicularly to the radius-vector, and thus may be called "spatial" wave:

$$E_{\text{пов}} = \frac{1-f_{\parallel}}{2} y^*(w)\left(z_1 + \frac{1}{\sqrt{\varepsilon^0}} r_1\right) k^2 \frac{e^{ikR}}{R}, \qquad (27.12a)$$

$$E_{\text{простр}} = \frac{1+f_{\parallel}}{2} \psi_1 k^2 \frac{e^{ikR}}{R}. \qquad (27.12b)$$

Indeed, at $\psi = 0$, we have $f_{\parallel} = -1$ and $\vec{E}_{prostr} = 0$. $\vec{E}_{prostr}$ may be interpreted as a superimposition of two waves, the directed and the reflected wave, the latter with reflection factor $f_{\parallel}$. As ψ rises, the reflection factor approaches the unity, and $\vec{E}_{pov}$ correspondingly to that vanishes.

The electric field components may be computed by the Hertz vector (26.21) with greater precision too, taking into account the terms of the following order relative to $1/\varepsilon^0$ and sin and of the form [9]

$$E_{\text{верт}} = k^2 \frac{e^{ikR}}{R} \frac{1-f_{\parallel}}{2} y^0(w) \left[z_1 + r_1 \cos\psi \left(1 + \frac{\sin^2\psi}{2}\right) \frac{1}{\sqrt{\varepsilon}} \right]. \tag{27.13a}$$

$$E_{\text{перп}} = k^2 \frac{e^{ikR}}{R} \cos\psi \, \frac{1+f_{\parallel}}{2} \, \varphi_1, \tag{27.13b}$$

whereupon here by $f_{\parallel}$ we should understand the expression (26.25):

$$f_{\parallel} = \frac{\varepsilon \sin\psi - \sqrt{\varepsilon - \cos^2\psi}}{\varepsilon \sin\psi + \sqrt{\varepsilon - \cos^2\psi}}. \tag{27.13c}$$

This breakdown into the "spatial" and "surface" wave is already included in the breakdown of $\underline{y}$ (27.9).

All these formulas refer to the case when the source is located on the ground surface. A generalization to the case of arbitrary z_0 may be obtained if we take into account that in the general formula (26.20) $f_{\parallel}$ is a function of only the sum $z + z_0$. This is why $\underline{f}$ must be considered as a function of $\sin\psi = \frac{z+z_0}{R_2}$ and, moreover, distinguish the angle ψ_1 from ψ, $\sin\psi_1 = \frac{z-z_0}{R_1}$ $\left(\cos\psi\cos\varphi = \frac{x-x_0}{R_2}, \cos\psi_1\cos\varphi = \frac{x-x_0}{R_1}\right)$. Therefore, if we reject the terms of the order $1/R_1^2$, $1/R_2^2$, $\sin^2\psi$, $\sin^2\psi_1$ and $1/\varepsilon$ relative to the abandoned ones, we have

$$E_z^0(x, y, z;\ 0, 0, z_0) = \frac{k^2}{2} \left\{ \cos^2\psi_1 \frac{e^{ikR_1}}{R_1} + f_{\parallel} \cos^2\psi \frac{e^{ikR_2}}{R_2} + \right.$$
$$\left. + (1 - f_{\parallel}) y^0(w) \frac{e^{ikR_2}}{R_2} \right\}, \tag{27.14a}$$

$$E_r^0(x, y, z;\ 0, 0, z_0) = -\frac{k_0^2}{2} \left\{ \sin\psi_1 \cos\psi_1 \frac{e^{ikR_1}}{R_1} + f_{\parallel} \sin\psi \cos\psi \frac{e^{ikR_2}}{R_2} - \right.$$
$$\left. - \frac{\cos\psi}{\sqrt{\varepsilon^0}} (1 - f_{\parallel}) y^0(w) \frac{e^{ikR_2}}{R_2} \right\}, \tag{27.14b}$$

$$E_\varphi^0 = 0 \tag{27.14c}$$

$(E_x^0 = \cos\varphi\, E_r^0,\ E_y^0 = \sin\varphi E_r^0)$ where, bearing in mind the subsequent cases, we have denoted by the index v that reference is made to the vertical electric dipole.

Lately it has been customary in Soviet literature on radio to designate by spatial wave the field of waves reflected by the ionosphere, and by surface wave the field settling above ground in the absence of influence of the ionosphere [I, 14]. With such a terminology the "spatial" and the "surface" waves (in quotes) of this section form together the surface wave.

The space amplitude and phase dependence of the vertical dipole field was represented in the form of graphs [4] with the help of specially constructed convenient series for y*(w). These calculations were moreover not limited to the wave zone only. As a rule, ε^0 was postulated to be equal to either $\varepsilon + 1$, or $\varepsilon + \cos^2\psi$, which, according to the above considerations, assures a sufficient precision.

Reproduced in Figs. 27.1, 27.2 and 27.3 are certain results of these calculations in the form of polar diagrams of electric field strength. Each curve shows the amplitude variation of the normal to the radius-vector of the component of electric field strength $|E_\psi|$ at shift along a circle of radius R, described in the vertical plane around the source taken for the center. The curves are plotted for various distances from the source R and for various soils and corresponding them different numerical distances $\rho = \left|\frac{kR}{2(\varepsilon + 1)}\right|$. Taken for the unity is the intensity that would be obtained in boundless space. The curves for long waves (great $|\varepsilon|$, Fig. 27.1) show that the field on the ground is two times greater than in the absence of ground. As the numerical distance increases, the field on the ground declines. For an ideal conductor the diagram would have the shape of a semicircumference of radius equal to the unity and with its diameter lying on the horizontal axis.

From the curves it may be seen that all the curves become such a circumference already for small ψ. This corresponds precisely to the condition that for $|\sqrt{\varepsilon+1}\,\sin\psi|\gg 1$ the attenuation function and the reflection factor become equal to the unity.

Because of that, in practice, as we underscored more than once, it is sufficient to study the attenuation function for small ψ.

It is further seen from the curves that with the increase of the altitude, the field begins at first to decrease correspondingly to the attenuation of the surface wave (as follows, in particular, from the boundary condition (see (21.24)), reaches the minimum at a certain altitude, and then, when the spatial wave accrues, it begins to increase rapidly.

Besides, we shall bring forth formulas for the magnetic field.

$$H_{\text{пов}} = -k^2\frac{e^{ikR}}{R}\frac{1-f_{\parallel}}{2}\cos\psi\left(1+\frac{\sin^2\psi}{2}\right)y^*(w)\,\varphi_1, \qquad (27.15a)$$

$$H_{\text{простр}} = -k^2\frac{e^{ikR}}{R}\frac{1+f_{\parallel}}{2}\cos\psi\,\varphi_1. \qquad (27.15b)$$

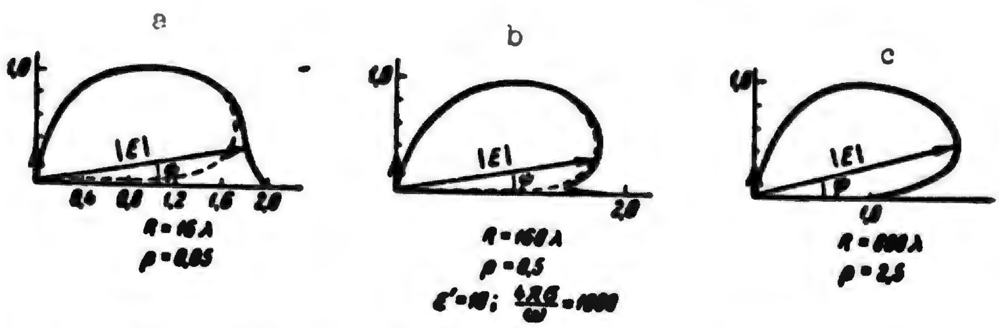

Fig. 27.1. Polar diagram of electric field amplitude in the vertical plane at various distances from the source in case of prevalence of conduction currents.

where $\vec{\varphi}_1$ is a unitary vector oriented along the direction of increase of the azimuthal angle φ.

According to the results obtained, the conditions of propagation

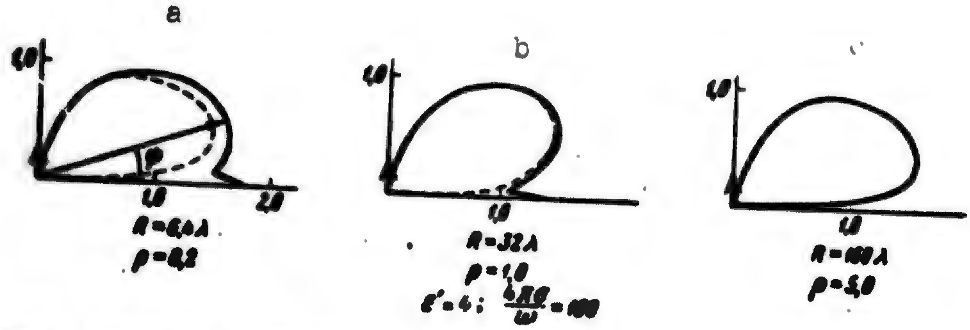

Fig. 27.2. Same as in Fig. 27.1, for a smaller $|\varepsilon|$.

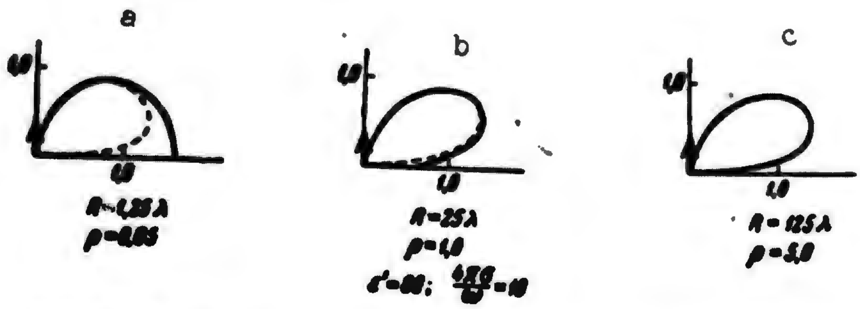

Fig. 27.3. Same as in Fig. 27.1, but for a smaller $|\varepsilon|$ and with prevalence of displacement currents.

are essentially different for the various frequencies, i.e., dispersion takes place. Thus, according to formula (26.26), at great distances the attenuation is proportional to the square of frequency ($s \sim \omega^2$) for long waves. The frequency bands used in practice are so narrow that this does not provoke the distortion of signals' shape. However, attention is more and more drawn by the question of pulse propagation and of their front distortion caused by the indicated dispersion (emission of thunderstorm discharges, and so forth), (see [16; 17]).

The experimental verification confirmed the correctness of the theoretical results. The checking of forecasts relative to phase variation of radiowaves with the distance had a particular significance, for this variation influences the observed phase velocity of radiowaves, which becomes in this way a function of distance to the emitter through distances of $|sr| \sim 10$ and greater (for dry soils). However, as emphasized in connection with formula (25.22), at great distances it attains the value of c.

Careful measurements gave a perfect agreement with the theory. In particular, the value of c obtained from measurements by L.M. Mandel'shtam, N.D. Papaleksi, E.Ya. Shchegolev, Ya.L. Al'pert and V.V. Migulin [10; 11] had a precision close to that of optical measurements, and differing from the figure obtained by optical methods by Michelson by the fourth decimal [10] (for details see [1, 2].

§28. VERTICAL MAGNETIC DIPOLE

Knowing the field of a vertical electric dipole, one may easily write also the expressions for a vertical magnetic dipole. We shall describe it by its magnetic Hertz vector $\vec{\Pi}_m = \Pi_{mz}$. The departure of the vertical electric dipole field from the field Π arises only because for $\vec{\Pi}_m$ at z = 0 (21.35) $\sqrt{\varepsilon - \cos^2\psi}$ enters into the boundary conditions instead of $1/\sqrt{\varepsilon}$ (21.19). This is why all the reasonings of §§ 24-26 may be repeated, and the expression for Π_m will differ from Π by $(\varepsilon - \cos^2\psi)^{-1}$, in particular, by the substitution of the numerical distance by the "magnetic numerical distance" (for the horizontal distance, $\psi = 0$)

$$s_m r = \frac{ik}{2}(\varepsilon - 1)r. \qquad (28.1)$$

It is obvious this "magnetic numerical distance" is always large in the entire wave zone of the emitter. According to formula (26.20), we therefore obtain for a dipole, giving in vacuum the field $\Pi_m = \frac{1}{2}\frac{e^{ikR}}{R}$ that is, according to (6.7), having a moment m = 1/2,

$$\Pi_m = \frac{1}{2}\left\{\frac{e^{ikR_1}}{R_1} + f_m\frac{e^{ikR_2}}{R_2}\right\}, \qquad (28.2)$$

$$f_m = 1 - \frac{4\sqrt{s_m r}}{\cos\psi}\, e^{-s_m r\left(1+\frac{1}{\sqrt{\varepsilon-\cos^2\psi}}\sin\psi\right)^2} \int\limits_{i\infty}^{\sqrt{s_m r}\left(1+\frac{1}{\sqrt{\varepsilon-\cos^2\psi}}\sin\psi\right)} e^{v^2}\, dv \approx$$

$$\approx 1 - \frac{2}{1+\frac{\sin\psi}{\sqrt{\varepsilon-\cos^2\psi}}} - \frac{\cos\psi}{s_m r\left(1+\frac{\sin\psi}{\sqrt{\varepsilon-\cos^2\psi}}\right)^2} + \ldots, \tag{28.3}$$

i.e.,

$$f_m \approx f_\perp - \frac{\cos\psi}{s_m r\left(1+\frac{\sin\psi}{\sqrt{\varepsilon-\cos^2\psi}}\right)^2} \tag{28.4}$$

(see (19.13)). But $\frac{\sin\psi}{\sqrt{\varepsilon-\cos^2\psi}}$ is always rather small, while $|s_m r| \gg 1$. This is why, abandoning the first two terms, we have

$$f_m = \frac{\sin\psi - \sqrt{\varepsilon-\cos^2\psi}}{\sin\psi + \sqrt{\varepsilon-\cos^2\psi}} = f_\perp, \tag{28.5}$$

which coincides with the expression for the reflection factor of a horizontally polarized wave (19.13). This, obviously, must be, for in the case of a vertical magnetic dipole only the horizontal component of the electric field (perpendicular to the incidence plane, see below (28.6)) is not zero. Thus, the singularity of the considered case by comparison with that of an electric dipole consists in that here the numerical distance is always great, if only ε exceeds the unity by several times. This is why in case of moist soils, for a low-disposed magnetic dipole we always find ourselves in a region where $g \sim (-2s_m r)^{-1}$, i.e., the field is subject to quadratic decrease with the distance. Only at greater heights of the point of observations and/or the source, when the "spatial wave" begins to manifest itself, the interference formulas become applicable and the field may decrease inversely to the first power of distance.

Doubt may arise in the applicability of approximate boundary conditions (21.35) to the case under consideration, inasmuch as in the near-by zone the decrease will be rapid and condition (21.2) will not be fulfilled. However, this region encompasses only a small part of the zone, essential for propagation, and this is why it cannot affect the final result. As to the remaining region,

$$y = -\frac{1}{2s_m r} \quad \text{and} \quad \frac{\partial y}{\partial r} \sim \frac{y}{r} < ky.$$

Let us pass to the expressions for the field. According to formulas (3.25) and (3.25a), utilizing formulas (27.15), it is easy to write the expressions for the electric field of a vertical magnetic dipole $\vec{E}_m^v$, which is equal to ik rot $\vec{\Pi}_m$. To that effect it is sufficient to substitute in $\vec{H}^v$ (27.15) (equal to $-ik\varepsilon$ rot$\vec{\Pi}$, where ε is related to the air and thus is equal to 1) $\underline{f}$ by $f_m = f_\perp$ according to either formula (28.3) or (28.5) and change the common sign:

$$E^v_{m_\text{вол}} = k^3 \frac{e^{ikR}}{R} \frac{1-f_\perp}{2} \cos\psi \left(1 + \frac{\sin^2\psi}{2}\right) y^v(w_m)\varphi_1, \quad (28.6a)$$

$$E^v_{m_\text{простр}} = k^3 \frac{e^{ikR}}{R} \frac{1+f_\perp}{2} \cos\psi\varphi_1, \quad (28.6b)$$

$$w_m = s_m r\left(1 + \frac{\sin\psi}{\sqrt{\varepsilon - \cos^2\psi}}\right), \quad R \equiv R_1 \equiv R_2. \quad (28.6c)$$

We practically always have $|s_m r| \gg 1$, $\sin\psi \ll |\sqrt{\varepsilon - \cos^2\psi}|$ and this is why, according to (27.7),

$$y^v(w_m) = -\frac{1}{2s_m r} - \frac{\sin\psi}{\sqrt{\varepsilon - \cos^2\psi}}. \quad (28.7)$$

As to the magnetic field, and inasmuch as it is expressed by $\vec{\Pi}_m$ in exactly the same way $\vec{E}$ of the electric dipole is expressed by $\vec{\Pi}$, we obtain from formula (27.12)

$$H^v_{m_\text{вол}} = k^3 \frac{e^{ikR}}{R} \frac{1-f_\perp}{2} y^v(w_m)\left(z_1 + \sqrt{\varepsilon - \cos^2\psi}\, r_1\right), \quad (28.8a)$$

$$H^v_{m_\text{простр}} = k^3 \frac{e^{ikR}}{R} \frac{1+f_\perp}{2} \psi_1. \quad (28.8b)$$

§29. HORIZONTAL ELECTRIC AND MAGNETIC DIPOLES

In certain respects the field of a horizontal electric dipole is more complex than that of a vertical dipole. Here the axial symmetry of the problem is disrupted, and this is why, in particular, the Hertz vector cannot be reduced to a single component, and at least two of its components have to be taken into account. The wave equations may be resolved in this case also [2]. However, we shall take another path and, utilizing the already found equations for the vertical electric and magnetic dipoles and the reciprocity theorem, we shall obtain all the data required by us almost without computations.

Any horizontal dipole may be decomposed into two dipoles, of which one is oriented along the direction toward the point of observation, and the other across it. We shall study their fields separately.

Assume that the dipole is located at the point $(0, 0, z_0)$, i.e., it is raised to the altitude z_0, while observation is conducted in the plane xz. Thus, the propagation direction coincides with the axis $\underline{x}$.

Let us consider at the outset the field E^{hx} of a dipole directed along the axis x.

Let us find, first of all, the vertical z-component of its electric field at the point (x, y, z)

$$E_z^{hx}(x, y, z; x_0, y_0, z_0).$$

According to the reciprocity theorem, it is equal to the x-component of the field at the point x_0, y_0, z_0, which induces an identical dipole, oriented vertically (index $\underline{v}$) and placed at the point (x, y, z):

$$E_z^{hx}(x, y, z; x_0, y_0, z_0) = E_x^{v}(x_0, y_0, z_0; x, y, z). \qquad (29.1)$$

According to formula (27.14b), we obtain for y = 0 (it must be taken into account that when substituting xyz by $x_0y_0z_0$ the quantity $\sin\psi_1$ changes sign)

$$E_z^{hx}(x, 0, z; 0, 0, z_0) = \frac{k^2}{2}\left[\frac{e^{ikR_1}}{R_1}\sin\psi_1\cos\psi_1 - f_{||}\frac{e^{ikR_2}}{R_2}\sin\psi\cos\psi + \right.$$

$$+\frac{e^{ikR_2}}{R_2}(1-f_{\parallel})g^{e}(w)\frac{\cos\psi}{\sqrt{\varepsilon^{0}}}\Big], \tag{29.2}$$

$$\left.\begin{aligned} &R_{1,2}=\sqrt{(x-x_0)^2+(y-y_0)^2+(z\pm z_0)^2},\\ &\sin\psi_1=\frac{z-z_0}{R_1},\quad \sin\psi=\frac{z+z_0}{R_2},\\ &w=sr(1+\sqrt{\varepsilon^{0}}\sin\psi)^2. \end{aligned}\right\} \tag{29.3}$$

This formula allows us to write, in particular, the value of E_z^{hx} also in the plane z = 0, and by virtue of the link between the field components, expressed by formula (21.25), the other components may be written too (it should be remembered that at $z=0$, $\sin\psi_1=-\sin\psi$ and, moreover, $R_1=R_2=R=\sqrt{(x-x_0)^2+(y-y_0)^2+z_0^2}$):

$$E_z^{hx}(x,0,0;\ 0,0,z_0)=\frac{k^2}{2}\Big[-(1+f_{\parallel})\sin\psi+(1-f_{\parallel})\frac{g^{e}(w)}{\sqrt{\varepsilon^{0}}}\Big]\frac{e^{ikR}}{R}\cos\psi, \tag{29.4a}$$

$$E_x^{hx}(x,0,0;\ 0,0,z_0)=\frac{k^2}{2}\Big[-\frac{(1-f_{\parallel})\sin\psi}{\sqrt{\varepsilon^{0}}}+(1-f_{\parallel})\frac{g^{e}(w)}{\varepsilon^{0}}\Big]\frac{e^{ikR}}{R}, \tag{29.4b}$$

$$E_y^{hx}(x,0,0;\ 0,0,z_0)=0. \tag{29.4c}$$

In the case when the source is located on the ground, $z_0 = 0$, while the observation point is raised, the expression for E_z^{hx} stems from formula (29.2), where we must simply consider $\psi = \psi_1$, $R_1 = R_2 = R$, and the expression for E_x^{hx} coincides with formula (29.4b), as follows from the reciprocity theorem. If in this last formula we still take into account that $1+f_{\parallel}=\sqrt{\varepsilon^{0}}\sin\psi(1-f_{\parallel})$, we obtain

$$E_z^{hx}(x,0,z;\ 0,0,0)=\frac{k^2}{2}(1-f_{\parallel})\Big[\sin\psi+\frac{g^{e}(w)}{\sqrt{\varepsilon^{0}}}\Big]\frac{e^{ikR}}{R}\cos\psi, \tag{29.5a}$$

$$E_x^{hx}(x,0,z;\ 0,0,0)=\frac{k^2}{2}(1-f_{\parallel})\Big[-\sin^2\psi+\frac{g^{e}(w)}{\varepsilon^{0}}\Big]\frac{e^{ikR}}{R}, \tag{29.5b}$$

$$E_y^{hx}(x,0,z;\ 0,0,0)=0. \tag{29.5c}$$

In particular, on the ground surface (ψ = 0) the field is present only owing to the inequality of ε^0 to ∞, whereupon its main component is directed along the axis z, though the dipole is devoid of the z-component. This means that the dipole excites in the ground vertical currents, which induce namely the observed field.

Let us now obtain the field E^{hy} of the horizontal dipole, raised to the altitude z_0, and directed along the axis y. We shall take advantage to that effect of the already found vertical magnetic dipole field (§28). Such a magnetic dipole may be represented without any loss of its general character in the form of a square frame, of which two sides (a and c) are parallel to the axis x, and the two others (b, d) to the axis y, whereupon they all have, say a length l (small by comparison with the wavelength) and each is displaced from the axis z by $l/2$ (Fig. 29.1). Its field is a simple superimposition of fields of four horizontal dipoles, which are the four sides of the frame. Inasmuch as $l \ll \lambda$, their currents are cophasal. Each of these fields has at the point A(x, 0, 0) the form

$$E^{(i)} = \pm w^{(i)}(R_i)\frac{e^{ikR_i}}{R_i}, \quad i = a, b, c, d, \tag{29.6}$$

where $R_{a,c} = \sqrt{x^2 + \frac{l^2}{4} + z_0^2}$, $R_{b,d} = \sqrt{\left(x \pm \frac{l}{2}\right) + z_0^2}$; the attenuation function $w^{(i)}$ is a very slow function of coordinates and it may be replaced by $w(R)$, $R = \sqrt{x^2 + z_0^2}$. The signs before w are taken different for a and c, and also for b and d, inasmuch as in them the currents have opposite directions. Consequently, the field of each pair is obtained by differentiation with respect to the coordinates of the source of the electric dipole field.

It is obvious that the dipoles a and c are mutually annihilated, since

$$E^{(a)} + E^{(c)} \approx w^{(a)}(R_a)\, l\, \frac{\partial}{\partial y_0}\frac{e^{ikR}}{R} \approx w^{(a)}(R_a)\, ikl\, \frac{y_0}{R_0}\,\frac{e^{ikR}}{R} = 0.$$

On the other hand $\left(\frac{\partial}{\partial x_0} = -\frac{\partial}{\partial x}\right)$.

$$E^{(b)} + E^{(d)} \approx w^{(d)}(R_0)\, l \frac{\partial}{\partial x_0}\frac{e^{ikR}}{R} \approx - w^{(d)}(R_0)\, ikl \frac{x}{R_0}\frac{e^{ikR}}{R} = - ikl \cos\psi\, E^{(d)}.$$

Consequently, this expression is precisely equal to E_m^v.

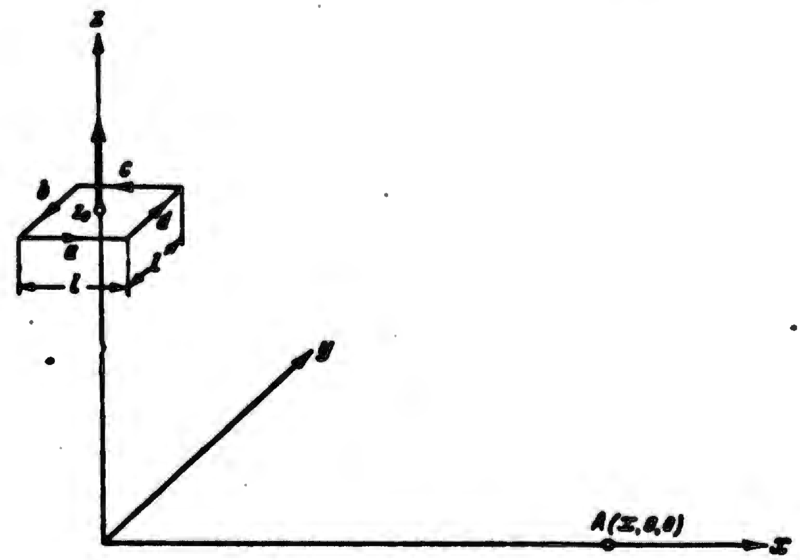

Fig. 29.1. Magnetic dipole and its equivalent frame.

Therefore, the field of a vertical magnetic dipole differs from the field $E^{(d)}$ of the horizontal electric dipole, of interest to us, and directed along the axis y, by the factor $-ikl\cos\psi$. But the magnetic moment of the frame is $m = \frac{IS}{c} = \frac{I}{c} l^2$, where I is the current in the frame, S is its area, and, according to formula (6.4b), it may be expressed by the dipole moments $p = \frac{Il}{-i\omega}$ of each of the frame's sides, $m = \frac{-i\omega}{c}\, lp = -iklp.$

Consequently,

$$E^{(d)} = \frac{1}{-ikl\cos\psi} E_m^v = \frac{p}{m}\frac{1}{\cos\psi} E_m^v. \qquad (29.7)$$

If for E_m^v we substitute the formulas of §28, written for $m = 1/2$, we shall obtain the field of a horizontal electric dipole with $p = 1/2$. As a result, according to formulas (28.6), rejecting the terms containing $y^*(w_m) \approx -(2s_m r)^{-1}$, as being proportional to r^{-2}, we obtain on the axis x the field of a horizontal electric dipole directed along the axis

y (the axis φ coinciding with it)

$$E_z^{hy} = 0, \tag{29.8a}$$

$$E_x^{hy} = 0, \tag{29.8b}$$

$$E_y^{hy}(x, 0, 0;\ 0, 0, z_0) \equiv E_y^{hy}(x, 0, z;\ 0, 0, 0) = \frac{k^2}{2}(1 + f_\perp)\frac{e^{ikR}}{R}. \tag{29.8c}$$

In this way the field of the horizontal dipole in the plane perpendicular to the dipole axis is reduced to the field that gives reflection by the interference formulas.

Now we may write the field of an arbitrarily oriented horizontal dipole. If it forms an angle φ with the axis x, we may, decomposing it into two dipoles, one along the axis x, and the other along the axis y, with moments respectively proportional to cos φ and sin φ, combine the fields of these dipoles. To that effect we shall multiply the right-hand parts of formulas (29.5a)-(29.5c) by cos φ, and the right-hand parts of formulas (29.8a)-(29.8c) by sin φ and add up.

Instead of considering in the formulas obtained the dipole direction as variable (angle φ), and the point of observation as fixed (on the axis x), we shall take for the axis x the direction of the dipole, while the point of observation will be taken in cylindrical coordinates r, φ, z. In such a case we must simply substitute φ by −φ, after which E_x^h will give E_r^h, and $E_y^h \rightarrow E_\varphi^h$. In this way,

$$E_z^h(r, \varphi, z;\ 0, 0) = \frac{k^2}{2}(1 - f_\parallel)\left[\frac{\eta^*(w)}{\sqrt{\varepsilon^*}} + \sin\psi\right]\frac{e^{ikR}}{R}\cos\psi\cos\varphi, \tag{29.9a}$$

$$E_r^h(r, \varphi, z;\ 0, 0) = \frac{k^2}{2}(1 - f_\parallel)\left[\frac{\eta^*(w)}{\varepsilon^*} - \sin^2\psi\right]\frac{e^{ikR}}{R}\cos\varphi, \tag{29.9b}$$

$$E_\varphi^h(r, \varphi, z;\ 0, 0) = \frac{k^2}{2}(1 + f_\perp)\frac{e^{ikR}}{R}\sin\varphi. \tag{29.9c}$$

The structure of these formulas clearly shows how they must be modified if the source is also raised above ground. Indeed, the addends

$(1-f_{\parallel})\sin\psi\cos\psi$ in formula (29.9a), $(1-f_{\parallel})\sin^2\psi$ in formula (29.9b) and expression (29.9c) clearly give the field of the incident wave, reflected with a reflection factor $f_{\parallel}$ for the component lying in the incidense plane, and $f_{\perp}$ for the perpendicular component. Of this, in particular, it is easy to be convinced by comparing the enumerated formulas with what is given by formulas (20.4a), (20.4b) and (20.4c) in the case of ideal reflection. This is why when $z_0 \neq 0$, they must have the structure of these last formulas, but with reflection factors generally speaking not equaling ± 1 (it may be said that they must form the solution of the wave equation passing into the respective addends in (29.9a)-(29.9c) as $z \to 0$. As to the remaining addends containing $y^*(w)$, it is sufficient to recognize $(z + z_0)/R_2$ for $\sin\psi$. Therefore, at any arbitrary point (r, φ, z) the field of a horizontal dipole, placed at the point $(0, z_0)$ and oriented along the axis $\underline{x}$, is

$$E_z^h = \frac{k^2}{2}\cos\varphi\left\{\frac{(1-f_{\parallel})\cos\psi}{\sqrt{\varepsilon^*}}\, y^*(w)\frac{e^{ikR_2}}{R_2} + \sin\psi_1\cos\psi_1\frac{e^{ikR_1}}{R_1} - f_{\parallel}\sin\psi\cos\psi\frac{e^{ikR_2}}{R_2}\right\},$$

$$E_r^h = \frac{k^2}{2}\cos\varphi\left\{\frac{(1-f_{\parallel})}{\varepsilon^*}\, y^*(w)\frac{e^{ikR_2}}{R_2} - \sin^2\psi_1\frac{e^{ikR_1}}{R_1} + f_{\parallel}\sin^2\psi\frac{e^{ikR_2}}{R_2}\right\}, \quad (29.10)$$

$$E_y^h = \frac{k^2}{2}\sin\varphi\left\{\frac{e^{ikR_1}}{R_1} + f_{\perp}\frac{e^{ikR_2}}{R_2}\right\},$$

with the denotations of (29.3) $\left(\cos\psi_1\cos\varphi = \frac{r}{R_1},\ \cos\psi\cos\varphi = \frac{r}{R_2}\right)$.

Similarly to formulas (27.12), grouping the components so as to obtain the "surface" and spatial" waves, we may write instead of formulas (29.9a)-(29.9c) (with a somewhat greater precision) [9]

$$E^h_{\text{пов}} = \frac{k^2}{2}\frac{e^{ikR}}{R}\left\{\cos\psi\left(1+\frac{\sin^2\psi}{2}\right)z_1 + \frac{1}{\sqrt{\varepsilon^*}}r_1\right\}\frac{1-f_{\parallel}}{\sqrt{\varepsilon^*}}\, y^*(w)\cos\varphi, \quad (29.11a)$$

$$E^h_{\text{простр}} = \frac{k^2}{2}\frac{e^{ikR}}{R}\left(\cos\varphi\sin\psi(1-f_{\parallel})\psi_1 + \sin\varphi(1+f_{\perp})\varphi_1\right) \quad (29.11b)$$

(for a source located on the ground).

Let us now pass to the horizontal magnetic dipole with axes oriented along the axis y. We shall replace it by a square frame in the plane xOz with a side l consisting of two vertical and two horizontal electric dipoles, the latter lying along the axis x. Differentiating formulas (27.12) and (29.10) for their fields, it is easy to obtain the field of the frame. At the same time, we should take into account that if the magnetic moment is oriented along the axis y, the current will be positive in the horizontal dipole closer to the observation point (upper dipole) and negative in the closer vertical dipole. This is why, differentiating as usual only the factor $\exp(ikR)$, we have

$$E^h_m = l\cos\varphi\frac{\partial}{\partial R}E^v + l\frac{\partial}{\partial z_0}E^h.$$

Besides, when differentiating over z_0 it must be taken into account that as the source shifts upward along the axis z, its reflection is displaced downward. This is why $\frac{\partial}{\partial z_0} = -\frac{\partial}{\partial z} = -ik\sin\psi$, must be taken for the incident field, while for the reflected field (containing $f_\perp$) we must take $f_\perp)-\frac{\partial}{\partial z_0} = +\frac{\partial}{\partial z} = ik\sin\psi$. As a result of the substitution, rejecting the terms of the order $\sin^2\psi$ and $\sin\psi/\sqrt{\varepsilon^0}$, and dropping the factor $ikl = -m/p$, we obtain

$$E^h_{m\,\text{зем}} = -\frac{k^2}{2}\frac{e^{ikR}}{R}(1-f_{\parallel})y^0(w)\cos\varphi\left\{z_1+\frac{1}{\sqrt{\varepsilon^0}}r_1\right\}. \qquad (29.12a)$$

$$E^h_{m\,\text{простр}} = -\frac{k^2}{2}\frac{e^{ikR}}{R}\{\cos\varphi(1+f_{\parallel})\phi_1-\sin\varphi\sin\psi(1-f_\perp)\varphi_1\}. \qquad (29.12b)$$

It is obvious that in certain cases, when the fields written in the formulas of the present section vanish, the terms of second order relative to $1/r$ may be material. More detailed formulas may be found in the work under reference [5].

§30. SOURCE IN THE SOIL

In this section we shall investigate the field of an emitter, an electric dipole, placed into a medium with a sufficiently great $|\varepsilon|$ $(z < 0)$, separated by a plane interface from the medium with $\varepsilon = 1$ $(z > 0)$. This case offers a substantial interest, inasmuch as it encompasses such phenomena as transmission through an underwater antenna, etc. It may be studied without difficulty with the help of solutions for a source in the air, already obtained by us, provided we utilize in the first place the reciprocity theorem (§9), second, the fact of exponential field variation in the ground under the surface, expressed by formula (21.17) (here we limit ourselves to the precision of representation of ε^0, corresponding to formula (21.26b), and third, the boundary conditions (4.1), (21.13) and (21.26).

a) <u>Vertical dipole at the point</u> $(0, 0, -z_0)$.

We shall consider the vertical component of the field of this dipole in the air at $z = 0$ at the point $(x, 0, +0)$.* According to the reciprocity theorem (9.11), it is equal to the vertical component of the field, that an identical dipole, placed at the point $(x, 0, +0)$, would have induced at the point $(0, 0, -z_0)$. But this last field differs from the field induced by the same dipole (placed in the air) at the point $(0, 0, -0)$ only by the factor $e^{ik\sqrt{\varepsilon-1}z_0}$ (for a gliding wave $\psi = 0$); this field differs in its turn from its field at the point $(0, 0, +0)$ by the factor $1/\varepsilon$; this last field is equal (at specific normalization of the power of the dipole) to $\frac{e^{ikx}}{x}y(sx)$. Assembling all these factors together, we obtain for the vertical field component of the dipole placed at the point $(0, 0, -z_0)$ (and having such a production of current's force on the acting height that in a boundless atmosphere its field would be $\frac{1}{2}\frac{e^{ikR}}{R}$).

$$E_z(x, 0, +0) = \frac{1}{\varepsilon} e^{ik\sqrt{\varepsilon-1}\, z_0} \frac{e^{ikx}}{x} y(sx). \qquad (30.1)$$

Since

$$\sqrt{\varepsilon - 1} = \sqrt[4]{(\varepsilon' - 1)^2 + \left(\frac{4\pi\sigma}{\omega}\right)^2}\, e^{\frac{i}{2} \operatorname{arctg} \frac{4\pi\sigma}{(\varepsilon'-1)\omega}},$$

the dependence on z_0 contains the exponential decrease:

$$\exp\left\{-kz_0 \sqrt{|\varepsilon - 1|} \sin\frac{\chi'}{2}\right\}, \quad \chi' = \operatorname{arctg} \frac{4\pi\sigma}{(\varepsilon' - 1)\omega}.$$

Therefore, the sinking of the vertical dipole in the soil to the depth z_0 for the field, observed in the air, leads to the same result as the variation of the dipole moment by a factor of $\varepsilon^{-1}\exp(ik\sqrt{\varepsilon - 1}\, z_0)$. This is correct for the point A(x, 0, +0) of the surface itself z = 0, and since the field in the space z > 0 is determined exclusively by the field in the plane z = 0, this is valid also at any point z > 0. In other words, to obtain the field in space, it is sufficient to add in formulas (27.13) the factor $\varepsilon^{-1}\exp(ik\sqrt{\varepsilon - 1}\, z_0)$ only.

Let us consider now the field under the surface z = 0, at the point Z(x, 0, –0). According to formula (21.13), it differs from the field in the air above the surface by still another factor ε^{-1}:

$$E_z(x, 0, -0) = \frac{1}{\varepsilon^2} e^{ik\sqrt{\varepsilon-1}\, z_0} \frac{e^{ikx}}{x} y(sx), \qquad (30.2)$$

and for the horizontal component, we obtain according to formula (21.26)

$$E_x(x, 0, -0) = E_x(x, 0, +0) = = \frac{1}{\sqrt{\varepsilon + 1}} E_z(x, 0, +0) = \frac{1}{\varepsilon\sqrt{\varepsilon^2}} e^{ik(x+\sqrt{\varepsilon-1}\, z_0)} \frac{y(sx)}{x}. \qquad (30.3)$$

Finally, the field at a depth $-|z|$ under the surface will be expressed, according to (21.17), by the formulas

$$E_z(x, 0, -|z|) = \frac{1}{\varepsilon^3} e^{ik(x+\sqrt{\varepsilon-1}(z_0+|z|))} \frac{y(sx)}{x}, \qquad (30.4)$$

$$E_x(x, 0, -|z|) = \frac{1}{\varepsilon\sqrt{\varepsilon}} e^{ik(x+\sqrt{\varepsilon-1}(z_0+|z|))} \frac{y(sx)}{x}, \qquad (30.5)$$

$$E_y(x, 0, -|z|) = 0. \qquad (30.6)$$

Therefore, in the soil the horizontal component is greater than the vertical by $\sqrt{\varepsilon}$ times at all points. The method of deduction itself and the structure of the formulas obtained suggest the physical treatment of the result: in view of the fact that while propagating in the thickness of the ground the waves damp exponentially, the effective path of radiowave passage is the following. From the immersed source a wave propagates upward, attenuating as it progresses. The direction of its propagation does not quite coincide with the vertical. In order to find it we must separate the real part of wave's phase, and then determine the direction of the normal to the plane of equal phases. We considered this question in §14 and found that the effective region of the interface, that is, the region where the radiation basically originates, is somewhat displaced from the vertical toward the side of the corresponding point. This region approaches the vertical only as $\varepsilon \to \infty$. In that region the wave induces virtual dipoles which constitute the source of radiowaves then propagating in the air (factor e^{ihx}/x) and attenuating in the way the waves in the air should damp above a plane conducting soil (factor $y(sx)$). There appear at each point of the surface $z = 0$ virtual dipoles induced by that wave. Up until now we considered the field of these dipoles (in fact the currents in the surface layer) only in the air. Now we see that they dispatch the waves also in the depth of the soil. At the point of observation $z = -|z|$ the field is $\exp(ik\sqrt{\varepsilon - 1}|z|)$ times smaller than the field which settles at the point of the surface situated above the point of observation (A.I. Berg, 1928).

This result may also be obtained from formulas (30.4)-(30.6). In reality, we shall find the propagation direction of the wave arriving at the point $A(x, 0, -|z|)$. For the sake of simplicity we shall consider the distance $\underline{x}$ so great that $|sx| \gg 1$, and, therefore $y(sx) \approx -(2sx)^{-1}$. This means that the wave phase contains the terms dependent on $\underline{x}$ and $\underline{y}$ only in the form $i\varphi = ik(x + \sqrt{\varepsilon - 1}\,|z|)$. The real part of φ is

$$\operatorname{Re}\varphi = kx + k\sqrt[4]{(\varepsilon' - 1)^2 + \left(\frac{4\pi\sigma}{\omega}\right)^2}\cos\frac{\chi'}{2}|z| =$$

$$= k\sqrt{1 + \sqrt{(\varepsilon' - 1)^2 + \left(\frac{4\pi\sigma}{\omega}\right)^2}}\,(x\sin\theta + |z|\cos\theta),$$

where θ is the angle formed by the direction of propagation with the vertical. Hence

$$\operatorname{ctg}\theta = \sqrt[4]{(\varepsilon' - 1)^2 + \left(\frac{4\pi\sigma}{\omega}\right)^2}\cos\left(\frac{1}{2}\operatorname{arctg}\frac{4\pi\sigma}{(\varepsilon' - 1)\omega}\right). \quad (30.7)$$

For great $|z|$ the angle θ is really very small. The amplitude of the wave decreases exponentially in another direction (see §16).

If we take the horizontal numerical distance smaller, $|sx| \sim 1$, the argument $y(sx)$ will contribute to the phase the part dependent on the distance, and the formula for θ will be somewhat changed.

Evidently, the propagation path of electromagnetic energy is such only in the case when the damping over vertical segments is less than the attenuation of a direct wave in the soil over the path OA (Fig. 30.1). But since the radiation of the immersed source is worth considering only when the depths z_0 and $|z|$ are not too great by comparison with $|\sqrt{\varepsilon}k|^{-1}$, and the horizontal segment of the path $\underline{x}$ is considered by us as great by comparison with the "wavelength" in the air $1/k$, the relation $x \gg z_0 + |z|$ is known beforehand to be fulfilled.

In connection with formulas (30.1)-(30.6) it should be recalled what was said after formula (18.5). Following the reciprocity theorem,

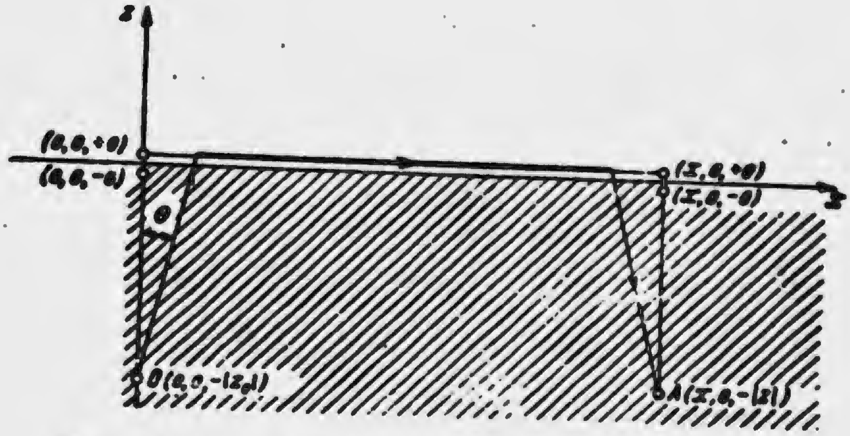

Fig. 30.1. To the computation of the field induced by a source immersed in the ground. A) Point of observation.

we required that the amplitude of the current I and the acting height h of the dipole, immersed in the soil, be sustained the same as those in a uniform atmosphere assuring at the distance R from the dipole a field $(1/2R)e^{ikR}$. It is, however, not necessary to believe that the power expended in such an immersed dipole will coincide with the expended power of an identical dipole in the air. Indeed, in the air the consumption of power is determined by the radiation only, and is given by the expression for the field in the wave zone. Contrary to that, in a conducting medium, aside from power expenditure in the wave zone (by virtue of damping it will be determined by the integral over the entire space of expenditures on Joule heat), there exists a significant energy expenditure on the nonwave zone, where the intensity of the field, and, consequently, the current density vary inversely to the cube of the distance to the dipole. This is why in practice one must take into account the finiteness of dipole dimensions, or to place the dipole in a nonconducting cavity. Only in the case when this cavity encompasses the entire nonwave zone so that there are no losses in it, will the power coincide with that emitted by a dipole of same moment in the air. In the opposite case, maintaining the necessary variable moment in the dipole, it is necessary by the same token, to sustain the quasistationary currents in the conducting portions of the nonwave zone. Since in the formulation of the reciprocity theorem the constants of the medium are not encountered, the placement of the dipole in the cavity does not affect the formulas obtained and only the relation between the consumed power and the dipole moment will vary.

b) Horizontal dipole at the point $(0, 0, -z_0)$.

According to formula (27.5), a vertical dipole, placed at the point $(x, 0, +0)$, induces at the point $(0, 0, +0)$ the horizontal component of the field, oriented along the negative direction of the axis x and provided the dipole moment is the same as in the previous part of this section, is equal to

$$\frac{1}{\sqrt{\varepsilon^0}}\frac{e^{ikx}}{x}g(sx).$$

The component oriented along the axis y disappears. Under the surface, at the point $(0, 0, -0)$ the horizontal component is identical, whereas at the depth z_0 at the point $(0, 0, -z_0)$, it differs by an exponential factor of the standard type. Treating the reasonings with the help of the reciprocity theorem, we obtain the following expression for the vertical field component of the dipole directed along the positive orientation of the axis x and placed at the point $(0, 0, -z_0)$:

$$E_z^h(x, 0, +0) = -\frac{1}{\sqrt{\varepsilon^0}}e^{ik(x+\sqrt{\varepsilon-1}z_0)}\frac{g(sx)}{x}. \qquad (30.8)$$

At that point the horizontal field component is still $\sqrt{\varepsilon^0}$ times smaller:

$$E_x^h(x, 0, +0) = -\frac{1}{\varepsilon^0}e^{ik(x+\sqrt{\varepsilon-1}z_0)}\frac{g(sx)}{x}. \qquad (30.9)$$

At the point $(x, 0, -0)$ the vertical component is smaller than $E_z^h(x, 0,$

+0) by a factor of ε (30.8), whereas the horizontal component (oriented along the axis $\underline{x}$) coincides with the component (30.9). In depth, at the point $(x, 0, z) = (x, 0, |z|)$ the factor $\exp(ik\sqrt{\varepsilon-1}\,|z|)$ adds up to these expressions. The dipole oriented along the axis $\underline{y}$, at points disposed on the axis $\underline{x}$, induces no field. This is why if the horizontal dipole forms with the axis $\underline{x}$ a certain angle φ, by decomposing it along the axes $\underline{x}$ and $\underline{y}$ into dipoles with moments proportional to $\cos\varphi$ and $\sin\varphi$, and taking into account that the field of the second of these moments may be neglected, we arrive at the following final expressions for the field of the horizontal dipole:

$$E_z^h(x, 0, -|z|) = -\frac{1}{\varepsilon\sqrt{\varepsilon^0}}e^{ik(x+\sqrt{\varepsilon-1}(z_0+|z|))}\frac{y(sx)}{x}\cos\varphi, \tag{30.10}$$

$$E_x^h(x, 0, -|z|) = -\frac{1}{\varepsilon^0}e^{ik(x+\sqrt{\varepsilon-1}(z_0+|z|))}\frac{y(sx)}{x}\cos\varphi, \tag{30.11}$$

$$E_y^h(x, 0, -|z|) = 0. \tag{30.12}$$

All the remarks concerning the energy transfer and the implied value of the dipole moment, made in connection with the formulas for the vertical immersed dipole, obviously refer here too.

Comparison of formulas (30.1) and (30.8) shows that for equal dipole moments the horizontal immersed antenna induces on the ground a field $\varepsilon/\sqrt{\varepsilon^0}$ times greater than the vertical antenna. Further, from formulas (30.4) and (30.5), and also from (30.10) and (30.11) it may be seen that the reception underground of immersed antenna radiation is better realized on a horizontal antenna. The difference is given by the factor $\varepsilon/\sqrt{\varepsilon^0}$. Obviously, on the ground surface the vertical component remains greater than the horizontal.

Similarly to the case of vertical dipole, the field in the air above ground is obtained from formulas for a ground horizontal dipole by diminishing the dipole moment. However, this change is given by the factor $\exp(ik\sqrt{\varepsilon-1}\,z_0)$, which is ε times greater than in the case of vertical dipole and this is why here the situation is more favorable.

§31. SOLUTION OF THE PROBLEM OF VERTICAL DIPOLE BY SEPARATION OF VARIABLES IN CYLINDRICAL COORDINATES

1. Expounded in this and the following sections will be two methods of treatment of the problem about a vertical electric dipole on the ground surface, which are different from the method utilized earlier. The first of them was historically the first rigorous method applied to the indicated problem. It belongs to Sommerfeld [2] and was developed by Van der Pol [7]. The second method is that of plane waves, envisioned by Sommerfeld and applied by Weyl [3].

The description of these classical methods is necessary not only with the view of completeness, but also because the results obtained here admit the generalization to cases (stratified atmosphere and waveguide propagation, Chapter 9), which cannot be considered within the framework of the method utilized above. Besides, we shall obtain certain results which will help in the refining of the estimate of neglects made while utilizing the method of approximate boundary conditions.

On the one hand, it is possible to start from the fact that the field of a vertical dipole on plane ground has a cylindrical symmetry, and to resolve the wave equation by the method of variable separation in cylindrical coordinates, which is standard for boundary value problems (Sommerfeld, [2]). The solution has the form of a specific integral (see below (31.9)). However, the evaluation of this integral, realized by a rather complex method of deductions, is feasible only for the case of not too small $|\varepsilon|$ (which is, as we know, the most important in practice).

At first, the solution of the problem considered by us was obtained precisely that way. True, this solution contained in the final result a small error, that went unnoticed for a long time.

Subsequently, the integral expression (31.9) was the starting point of various computations ([12; 4; 5; 9] and others), with the view of again obtaining the final formula for the field.

A treatment of another kind (Weyl [3]) starts from the representation of the solution of the wave equation in the form of superimposition of partial solutions of an entirely different type, i.e., that of plane waves. Satisfying the boundary condition for each of the plane waves, which is simple, the superimposition of solutions is then matched in a way that would guarantee the required behavior near the source (§32). The formula obtained, which is precisely called by us the normal attenuation function, was also initially limited to the condition [illegible] [3]. It did not include the error that appeared in the final formula of the work [2a]. Both conclusions were so complex that we attempted to resolve the question as to which expression is correct by experimental means (and obviously resolved it in favor of the formula of [3]). A substantial share of reasons of such an unsatisfactory situation consisted in that in the Sommerfeld's solution an exponentially damping surface wave was included as one of the addends (§23), which was considered as nearly indispensable through erroneous concepts. However, on the one hand, Van der Pol, having conducted this investigation by a somewhat different method in cylindrical coordinates [7], has obtained here an attenuation function in correct form also. On the other hand, when editing the Russian translation of the Sommerfeld's work, V.A. Fok [2b] revealed its "nonrigorism," corrected the conclusion, and obtained a correct result.* In the following the method of plane waves was applied in a series of works, in particular in the works by L.M. Brekhovskikh [Chapter 1, 9; Chapter 9, 3], having generalized the solution to certain problems (stratified medium and others; see Chapter 9). It should be noted that for the validity of the solution then obtained, it was not required, strictly speaking, that the absolute value of $|\varepsilon|$ be really very great. The solution takes place at great kr, but of ε it is only required that a sufficiently rapid damping of the wave directly propagating through the soil be assured (see below). Finally, the method of approximate boundary conditions was applied for obtaining the normal attenuation function with the aid of the differential (and not integral, as in §25) field equation by M.A. Leontovich [Chapter 4, 4]. We do not bring forth this solution, for subsequently, the same method was applied for deriving the attenuation function above a spherical ground, and we give this solution in §38.

Assume, as usual, that a vertical dipole is placed at the point (0, 0, +0) and that it is required to find the expression for the Hertz vector (unique nonzero component $\vec{\Pi} = \Pi_z$) at boundary conditions (4.7), (4.8):

$$\Pi(r, \varphi, 0) = \varepsilon\Pi_1(r, \varphi, 0), \qquad (31.1)$$

$$\frac{\partial\Pi(r, \varphi, 0)}{\partial z} = \frac{\partial\Pi_1(r, \varphi, 0)}{\partial z}, \qquad (31.2)$$

where Π is related to the atmosphere (upper half-space), and Π_1 to the soil. The equations for Π and Π_1 in variables r, φ, z will have the form

$$\nabla^2\Pi + k_0^2\Pi = 0,$$

$$\nabla^2\Pi_1 + k^2\Pi_1 = 0, \quad k^2 = \varepsilon k_0^2,$$

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2} + \frac{\partial^2}{\partial z^2}. \qquad (31.3)$$

These equations are invalid at the point r = z = 0, and the solution must have a singularity determined by the structure of the source (in our case a point dipole).

We shall seek the solution by separating the variables, in the form of superimposition of the partial solutions

$$J_0(vr)e^{-z\sqrt{v^2-k_0^2}},\ z>0, \tag{31.4a}$$

$$J_0(vr)e^{+z\sqrt{v^2-k^2}},\ z<0, \tag{31.4b}$$

where J_0 is a zero order Bessel function from the argument vr, where v is the separation constant. It is possible to be convinced that these expressions satisfy it, i.e., that they are really partial solutions, by direct substitution into the wave equation.

For the solution to satisfy also the radiation condition, they must in any case remain bounded at infinity. This is why the square roots must be recognized in a sence, whereby the observation of the requirement

$$\mathrm{Re}\sqrt{v^2-k_0^2}>0,\ \mathrm{Re}\sqrt{v^2-k^2}>0. \tag{31.5}$$

is assured. Then as $z\to+\infty$ the upper solution, and as $z\to-\infty$ the lower solution will remain bounded.

The superimposition of expressions of the form (31.4) for various v must moreover satisfy specific requirements near the dipole. Since while r and z approach zero simultaneously the field, directly emitted by the dipole, acquires a prevalent value, it may be required that as r, z → 0 the solution has the character of the field of a solitary dipole $\Pi\approx\frac{1}{\sqrt{r^2+z^2}}=\frac{1}{R}$. Instead of that, it could be required that as ε → → 1 the obtained solution pass to a dipole field in free atmosphere. We shall see that the solution obtained at the condition $\lim\limits_{R\to 0}(R\Pi)=\mathrm{const}$, will also satisfy this second condition. Under both assumptions it is appropriate to separate from the total solution the type of dipole field in free atmosphere. In order that besides, this term furnish the total solution as ε → 1, it may be provided with corresponding factors. This is why, summing up the partial solutions with the arbitrary factors f(v) and $f_1(v)$,* we arrive at the following form of the solution searched for:

$$\Pi = \frac{\varepsilon}{\varepsilon+1}\frac{e^{ikR}}{R} + \int_0^\infty f(\nu) J_0(\nu r) e^{-z\sqrt{\nu^2-k_0^2}}\,d\nu, \qquad (3.16a)$$

$$\Pi_1 = \frac{1}{\varepsilon+1}\frac{e^{ikR}}{R} + \int_0^\infty f_1(\nu) J_0(\nu r) e^{+z\sqrt{\nu^2-k^2}}\,d\nu \qquad (31.6b)$$

(the addition of the integral over negative ν contributes nothing new on account of the parity of J_0 and of the entire partial solution).

The separated terms also satisfy the wave equations in both solutions. It is now necessary to satisfy the boundary conditions (31.1), (31.2) by way of assortment of correct functions f and f_1. To that effect it is convenient to expand the first addends in the solutions (31.6) into an integral over zero order Bessel functions, making use of formula (16.12a, b):

$$\frac{e^{ikR}}{R} = \int_0^\infty J_0(\nu r)\, e^{\mp z\sqrt{\nu^2-k^2}}\,\frac{\nu\,d\nu}{\sqrt{\nu^2-k^2}}.$$

At the same time we may combine both integrals together, so as to obtain

$$\Pi = \int_0^\infty \left\{\frac{\varepsilon}{\varepsilon+1}\frac{\nu}{\sqrt{\nu^2-k_0^2}} + f(\nu)\right\} J_0(\nu r) e^{-z\sqrt{\nu^2-k_0^2}}\,d\nu, \qquad (31.7a)$$

$$\Pi_1 = \int_0^\infty \left\{\frac{1}{\varepsilon+1}\frac{\nu}{\sqrt{\nu^2-k^2}} + f_1(\nu)\right\} J_0(\nu r) e^{+z\sqrt{\nu^2-k^2}}\,d\nu. \qquad (31.7b)$$

Substituting these expressions into formulas (31.1) and (31.2) and taking into account the orthogonality of Bessel functions related to various ν, we obtain two equations for the determination of two functions f and f_1 from which it follows

$$f(\nu) = \frac{\varepsilon}{\varepsilon+1}\frac{\nu}{\sqrt{\nu^2-k_0^2}}\,\frac{\sqrt{\nu^2-k_0^2}-\sqrt{\nu^2-k^2}}{\varepsilon\sqrt{\nu^2-k_0^2}+\sqrt{\nu^2-k^2}}, \qquad (31.8a)$$

$$f_1(\nu) = -\frac{\varepsilon}{\varepsilon+1}\frac{\nu}{\sqrt{\nu^2-k^2}}\,\frac{\sqrt{\nu^2-k_0^2}-\sqrt{\nu^2-k^2}}{\varepsilon\sqrt{\nu^2-k_0^2}+\sqrt{\nu^2-k^2}}. \qquad (31.8b)$$

and finally

$$\Pi = \int_0^\infty \frac{\varepsilon}{N} J_0(\nu r) e^{-z\sqrt{\nu^2-k_0^2}} \nu d\nu, \tag{31.9a}$$

$$\Pi_1 = \int_0^\infty \frac{1}{N} J_0(\nu r) e^{z\sqrt{\nu^2-k^2}} \nu d\nu, \tag{31.9b}$$

$$N = \varepsilon\sqrt{\nu^2-k_0^2} + \sqrt{\nu^2-k^2}. \tag{31.9c}$$

In the extreme case $k = k_0$, $\varepsilon = 1$, we have $N = 2\sqrt{\nu^2-k_0^2}$ and, according to relation (16.12), together both formulas give

$$\Pi = \Pi_1 = \frac{1}{2}\frac{e^{ik_0R}}{R}, \quad R = \sqrt{r^2+z^2}, \tag{31.10}$$

i.e., the dipole field in boundless atmosphere. On the other hand, as $\varepsilon \to \infty$, we have $1/N \to 0$ and $\frac{\varepsilon}{N} \to \frac{1}{\sqrt{\nu^2-k_0^2}}$, so that

$$\begin{aligned} \Pi &= \frac{e^{ik_0R}}{R}, \quad z>0, \\ \Pi_1 &= 0, \quad z<0, \end{aligned} \tag{31.11}$$

as must be for a dipole field above an ideally reflecting surface.

Therefore, in principle the solution is obtained for any ε and expressed by formulas (31.9). However, in reality this result is for the time being practically useless. Moreover, the exterior view of this (quite rigorously obtained) solution does not even suggest the required solution: the product of the "unperturbed" solution (31.11) by a slowly-varying attenuation function. The obtaining of this attenuation still remains a fairly complex problem.

3. Taking advantage of the parity of the integrand relative to ν, it is possible to substitute $\int_0^\infty$ by $\frac{1}{2}\int_{-\infty}^{+\infty}$. After that, having expressed the Bessel function by the half-sum of Hankel functions of first and second order, it is possible to close the contour by a semicircumference and apply the subtraction method. It is then necessary to carefully bypass the two points of branching (at $\nu = k_0$, and $\nu = k$) and take

into account the pole N = 0. It is found to be material that one of the branching points lies near the pole (the neglecting of this case was precisely the cause of the error in the work [2a]). Utilizing, besides, the saddle point method and neglecting the terms of the order $1/\varepsilon$, we succeed in obtaining a closed formula (for details see [2b; Chapter 1, 9].

However, it is still much simpler to transform the integral (31.9) toward the expression of interest to us by another method [12], which allows, moreover, to account very simply for the terms of the order $1/\varepsilon$ and generally estimate the error committed [5].

We shall limit ourselves here to the case z = 0, i.e., we shall obtain the attenuation function for points on the ground surface.

We namely shall take into account that the identity

$$\frac{1}{N}=\frac{\sqrt{\varepsilon}}{1-\sigma^2}\int\limits_{u=\frac{1}{\sigma}}^{u=\frac{\sqrt{\varepsilon}}{\sigma}}\frac{1}{\sqrt{v^2-k_0^2\sigma^2u^2}}\,d\frac{1}{\sqrt{u^2-1}}, \qquad (31.12)$$

takes place; here $\frac{1}{\sigma^2}=1+\frac{1}{\varepsilon}$, of the validity of which one may be easily convinced by direct integration with the help of the substitution of $\frac{1}{\sqrt{u^2-1}}=v$. Owing to this identity, we may write instead of formula (31.9a)

$$\Pi=\frac{\varepsilon\, h}{1-\sigma^2}\int\limits_{u=\frac{1}{\sigma}}^{u=\frac{\sqrt{\varepsilon}}{\sigma}}d\frac{1}{\sqrt{u^2-1}}\int\limits_0^{\infty}\frac{v\,dv}{\sqrt{v^2-k_0^2\sigma^2u^2}}J_0(vr). \qquad (31.13)$$

Changing the order of integration, and by virtue of formula (16.12a), giving the expansion of the spherical wave by Bessel functions we obtain

$$\Pi=\frac{\varepsilon^{0}/_{h}}{1-\sigma^2}\int\limits_{u=\frac{1}{\sigma}=\sqrt{1+\frac{1}{\varepsilon}}}^{u=\frac{\sqrt{\varepsilon}}{\sigma}=\sqrt{1+\varepsilon}}\frac{1}{r}e^{ik_0\sigma r u}\,d\frac{1}{\sqrt{u^2-1}}. \qquad (31.14)$$

No neglects and assumptions relative to ε are yet made in this expression. It is quite precise. Since the quantity σ is complex in the general case, the integration is performed over a certain path in the complex plane of the variable u. For $|\varepsilon| \gg 1$ we have

$$\sigma = \frac{1}{\sqrt{1+\frac{1}{\varepsilon}}} = 1 - \frac{1}{2\varepsilon} + \frac{3}{8\varepsilon^2} - \ldots \approx 1 - \frac{1}{2\varepsilon} \qquad (31.15)$$

and consequently, the value of σ is close to unity. The lower limit of the integral is then also located near the point $u = 1$. The upper limit is, to the contrary, situated in the region of great $|u|$.

Let us break the integral into two parts (Fig. 31.1):

$$\Pi = \frac{\varepsilon^{3/2}}{1-\varepsilon^2}(I_1 - I_2), \qquad (31.16a)$$

$$I_1 = \int_{u=\frac{1}{\sigma}}^{u=i\infty} \frac{1}{r} e^{ik_0 r \sigma u} d\frac{1}{\sqrt{u^2-1}}, \qquad (31.16b)$$

$$I_2 = \int_{u=\sqrt{1+\varepsilon}}^{u=i\infty} \frac{1}{r} e^{ik_0 r \sigma u} d\frac{1}{\sqrt{u^2-1}} \qquad (31.16c)$$

and consider them separately, utilizing the above referred-to property of limits and also the fact that we may always consider $k_0 r \gg 1$. On the strength of this last condition the integrand contains the rapidly-varying function $\exp(ik_0 r\sigma u)$, which decreases exponentially as u varies with its approaching the limit $u = i\infty$.

Let us consider I_1. Here the principal region of values is the region of u, close to the lower limit, which is itself close to unity. This is why we may postulate

$$u = 1 - iv^2 \qquad (31.17a)$$

and consider the regions of small v as effective. At lower limit with $u = u_1 = \frac{1}{\sigma}$, we have (taking into account (31.15))

$$v = v_1 = e^{i\frac{\pi}{4}}\sqrt{\frac{1-\sigma}{\sigma}} \approx e^{i\frac{\pi}{4}}\sqrt{\frac{1}{2\left(\varepsilon+\frac{1}{4}\right)}}, \quad |v_1| \ll 1. \tag{31.17b}$$

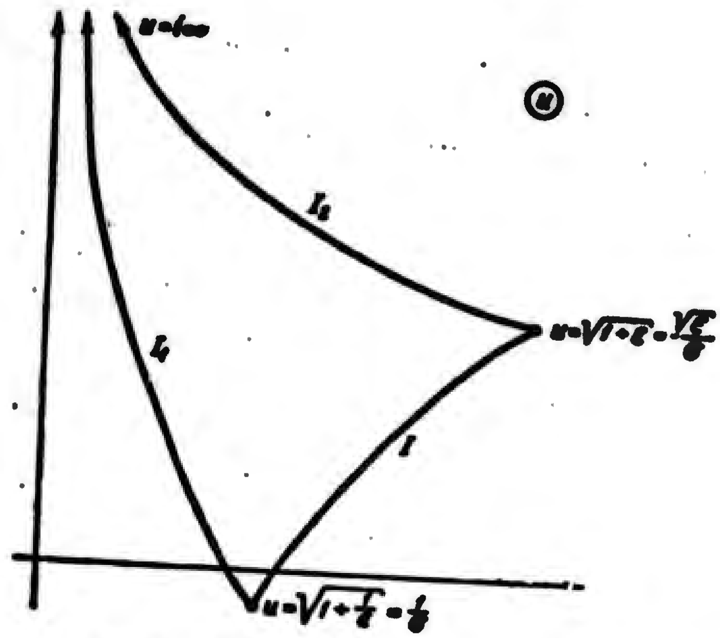

Fig. 31.1. Breaking up the path of integration.

At the upper limit $v \to i\infty$. Let us expand the pre-exponential function in series by $\underline{v}$. This gives

$$d\frac{1}{\sqrt{u^2-1}} = \frac{e^{i\frac{\pi}{4}}}{\sqrt{2}}\, d\left\{\frac{1}{v}+\frac{iv}{4}-\frac{3}{32}v^3+\dots\right\} =$$

$$= \frac{e^{i\frac{\pi}{4}}}{\sqrt{2}}\left\{-\frac{1}{v^2}+\frac{i}{4}-\frac{9}{32}v^2+\dots\right\}dv. \tag{31.17c}$$

Therefore,

$$I_1 = \frac{e^{i\frac{\pi}{4}+ik_0r\sigma}}{r\sqrt{2}}\int_{v_1}^{i\infty} e^{k_0r\sigma v^2}\left\{-\frac{1}{v^2}+\frac{i}{4}-\frac{9}{32}v^2+\dots\right\}dv =$$

$$= \frac{e^{i\frac{\pi}{4}}}{r\sqrt{2}}e^{ik_0r\sigma}\sqrt{k_0r\sigma}\cdot\int_{\sqrt{p}}^{i\infty} e^{w^2}dw\left\{-\frac{1}{w^2}+\frac{i}{4k_0r\sigma}-\frac{9}{32}\frac{w^2}{(k_0r\sigma)^2}+\dots\right\}, \tag{31.18}$$

$$p = v_1^2k_0r\sigma = ik_0r(1-\sigma). \tag{3.19}$$

Let us introduce the already well known to us denotation

$$y(\rho) = 1 - 2\sqrt{\rho}\, e^{-\rho} \int_{i\infty}^{\sqrt{\rho}} e^{w^2}\, dw. \tag{31.20a}$$

Below we shall return to the question, with what precision the ρ figuring here coincides with our former numerical distance. Now, by integration by parts, all the addends in I_1 may be reduced to integrals containing $y(\rho)$. Thus, for example,

$$\int_{i\infty}^{\sqrt{\rho}} e^{w^2}\, dw = (1 - y(\rho))\, \frac{e^{\rho}}{2\sqrt{\rho}}, \tag{31.20b}$$

$$\int_{i\infty}^{\sqrt{\rho}} e^{w^2} \frac{dw}{w^2} = -\frac{y(\rho)}{\sqrt{\rho}} e^{\rho} \tag{31.20c}$$

etc. This is why I_1 takes the form

$$I_1 = -\frac{e^{ik_0 r}}{r} \sqrt{\frac{\varepsilon}{2(1-\varepsilon)}} \left\{ y(\rho) + \frac{1}{8}\frac{1-\varepsilon}{\varepsilon}\frac{y(\rho)-1}{\rho} + \frac{9}{64}\left(\frac{1-\varepsilon}{\varepsilon}\right)^2 \frac{1}{\rho} + \dots \right\}. \tag{31.21}$$

Since, according to formula (31.15) the ratio $\frac{1-\varepsilon}{\varepsilon} \approx \frac{1}{2\varepsilon}$, i.e., it is small, this expansion contains consecutively decreasing terms. Preserving only the first of them, we commit a relative error of the order

$$\frac{1}{8}\frac{1}{2\varepsilon}\frac{y(\rho)-1}{y(\rho)}\frac{1}{\rho}. \tag{31.22a}$$

As we know, for small ρ $y(\rho) \approx 1 + i\sqrt{\pi\rho}$, consequently, this is a quantity of the order

$$\frac{1}{16\varepsilon}\sqrt{\frac{\pi}{\rho}} \sim \sqrt{\frac{\pi}{2}}\frac{1}{8\sqrt{k_0 r \varepsilon}}. \tag{31.22b}$$

At great ρ we may consider $y(\rho) \approx -1/2\rho$; consequently, the error has the order

$$\frac{1}{16\varepsilon} 2 = \frac{1}{8\varepsilon}. \tag{31.22c}$$

Therefore, assuming that $|\varepsilon| > 2$, $k_0 r > 4$, and preserving only the first term of the expansion, we shall be committing, even for waves as short as desirable and the driest soils, an error in no way exceeding ~5%. Preserving the second term, we may improve the precision still further.

Being satisfied with the precision indicated, we may consider

$$\left.\begin{aligned} I_1 &\approx -\frac{e^{ik_0 r}}{r}\sqrt{\frac{\varepsilon}{2(1-\varepsilon)}}\,\psi(p) = -\frac{e^{ik_0 r}}{r\sqrt{2}}\frac{\psi(p)}{\sqrt{\sqrt{1+\frac{1}{\varepsilon}}-1}}, \\ p &= ik_0 r\left(1-\frac{1}{\sqrt{1+\frac{1}{\varepsilon}}}\right). \end{aligned}\right\} \quad (31.23)$$

Let us pass to I_2. As we shall be convinced, this integral will contribute a small correction to the result, required in essence only for the estimate of the error committed at its rejection. This is why we may limit ourselves to the first approximation.

Taking into account, namely, that here too the value of the integrand function is material only near the lower limit, we may expand the pre-exponential factor in series and preserve only the first addend, taking its value at $u = \frac{\sqrt{\varepsilon}}{\varepsilon}$ (there is no difficulty in taking into account the furthest terms of the series also, these series being by powers $1/\varepsilon k_0 r$ and converging well):

$$d\frac{1}{\sqrt{u^2-1}} = -\frac{u\,du}{(u^2-1)^{3/2}} \approx \frac{u_1\,du}{(u_1^2-1)^{3/2}} = -\frac{\sqrt{\varepsilon}\,\varepsilon^2}{(\varepsilon-\varepsilon^2)^{3/2}}\,du.$$

This is why

$$I_2 = -\frac{\varepsilon^2\sqrt{\varepsilon}}{(\varepsilon-\varepsilon^2)^{3/2}}\int\limits_{u=\frac{\sqrt{\varepsilon}}{\varepsilon}}^{i\infty}\frac{e^{ik_0 r u}}{r}\,du = -i\,\frac{1+\varepsilon}{\varepsilon^2}\,\frac{e^{ik_0 r\sqrt{\varepsilon}}}{k_0 r^2}, \quad (31.24)$$

and consequently,

$$\Pi = \frac{\varepsilon^{3/2}}{\varepsilon^2-1}\frac{e^{ik_0 r}}{r}\left\{\frac{1}{\sqrt{2}}\frac{\psi(p)}{\sqrt{\sqrt{1+\frac{1}{\varepsilon}}-1}} - \frac{e^{ik_0 r(\sqrt{\varepsilon}-1)}}{ik_0 r}\,\frac{\varepsilon+1}{\varepsilon^2}\right\}. \quad (31.25)$$

Leaving in the pre-exponential factors only the terms of the order $1/\varepsilon$ relative to the principal ones, we obtain

$$\Pi = \left(1+\frac{1}{8\varepsilon}\right)\frac{e^{ik_0 r}}{r}\left\{\psi(p) + i\,\frac{1+\varepsilon}{\varepsilon^{3/2}}\,\frac{e^{i(k-k_0)r}}{k_0 r}\right\}. \quad (31.26)$$

where $k-k_0 = k_0(\sqrt{\varepsilon}-1)$ is the difference of wave numbers in the ground and in the air.

Therefore, we obtained the final expression for the field in the air in the form of two terms. One of them, the principal, has a familiar form and contains the normal attenuation function. The other describes a wave, propagating from the source to the observation point with a wave number $k = \sqrt{\varepsilon}k_0$, characterizing the propagation in the soil. This is the wave, reaching the observation point through the ground.

The found expression (31.25) is valid with a precision to terms $1/8\varepsilon$ in the region of large numerical distances ρ (31.22c) or to terms $1/8\sqrt{k_0 r \varepsilon}$ (in the region of small numerical distances (31.22b)). The transition to formula (31.26) is linked with additional (as may be easily seen, little substantial) neglect in pre-exponential factors, namely with the neglect in terms of the order $1/\varepsilon^2$. Obviously, this may be also omitted.

Let us pass now to the comparison of the result obtained with our previous expression for the attenuation function (25.18).

It may be seen first of all that the term, describing the passage of radiowaves through the ground, was absent in our earlier considerations. This term describes a wave, investigated in detail in a series of works ([13; 14; 15]; see Chapter 1, 9) for the case when the transmitter and the receiver are located in a medium with a great $|\varepsilon|$ (for example, in the ground). It was proposed to call it the "lateral wave."

In particular, the solution, found by us above for a source in the ground (§30), is reduced entirely to a wave mainly propagating outside the medium in which the emitter is located. Consequently, according to this terminology it is reduced to a lateral wave in the atmosphere. Here the wave corresponding to the term I_1, that is, the directly arriving, has time to damp. In case of an emitter situated in the air, provided the ground is sufficiently conducting, this term will be small because of the usual attenuation taking place during wave propagation in an absorbing medium, i.e., it will be exponentially small. Its damping is described by the exponent

$$e^{-k_0 r\sqrt{|\varepsilon|}\sin\frac{4\pi\sigma}{\varepsilon'\omega}}.$$

In certain cases the damping may result insignificant. Thus, if $\frac{4\pi\sigma}{\varepsilon'\omega} \ll 1$, the attenuation factor obtained will be of the order

$$e^{-k_0 r\sqrt{\varepsilon'}\frac{4\pi\sigma}{\varepsilon'\omega}} = e^{-\frac{4\pi\sigma r}{c\sqrt{\varepsilon'}}}, \qquad (31.27)$$

which, for the driest soils, $\sigma \approx 10^{6}$ CGSE, $\varepsilon \approx 2$, gives a damping by e times over a path of the order of 40 m. True, in reality dispersion begins to manifest itself precisely in the region of superhigh frequencies, σ rises sharply, and the true damping becomes still greater. The additional factor $1/k_0 r$ in this term does not affect the situation, for the first addend $y(\rho)$ for small $|\varepsilon|$, reverts to $-\frac{1}{2\rho} \approx \frac{i\varepsilon}{k_0 r}$ already for rather small values. However, the ratio of the second term to the principal still has the order $e^{\frac{r}{\sqrt{\varepsilon}\cdot 16.4}} \frac{1+\varepsilon}{\varepsilon^{1/2}}$ and for $\varepsilon = 2$ this ratio will be nearly 15% already with r 20 m. For $\varepsilon = 3$ it is about 6% at the same distance.

In the case of all such difficulties we may utilize the more complete formulas (31.26) or (31.25), clearly taking into account all the required terms.

In our previous solution the absence of a wave passing through the ground was quite legitimate: when deriving the boundary conditions lying at the basis of the entire solution, we started in Chapter 4 from the fact that this wave may be neglected and that near the surface the field variation is described by the function $e^{ik_0 x}w(x)$ with the slowly varying factor w(x). The wave passing through the ground does not satisfy this condition. However, this, as we have seen, may be material only near the source. In this region, therefore, the boundary condition (21.20) does not pretend to describe the field.

Let us now turn to the question as to what extent the numerical distance (31.19) coincides with the numerical distance (25.2) which we heretofore considered as being the argument of the attenuation function. From formula (31.19) we have

$$\rho = \rho_1 = ik_0 r\left(1 - \frac{1}{\sqrt{1+\frac{1}{\varepsilon}}}\right). \qquad (31.28a)$$

Expanding ρ_1 in powers $1/\varepsilon$ and limiting ourselves to the terms of first order, we shall obtain

$$\rho_1 \approx ik_0 r\,\frac{1}{2\varepsilon}\left(1 - \frac{3}{4\varepsilon}\right) \approx \frac{ik_0 r}{2(\varepsilon + \;/4)}. \qquad (31.28b)$$

Meanwhile, we considered heretofore that

$$\rho = \rho_2 = \frac{ik_0 r}{2\varepsilon^0} = \frac{ik_0 r}{2\left(\varepsilon + 1 + \frac{1}{\varepsilon} + \ldots\right)}. \qquad (31.29)$$

From a practical viewpoint the departure is insignificant. It is so much the more immaterial that it could manifest itself only for small $|\varepsilon|$, when the attenuation function may be replaced by its asymptotic expression $-1/2\rho$ already at a distance of one wavelength from the source, so that (if we neglect the wave propagating through the ground) formula (31.26) will give

$$\Pi \approx \frac{e^{ik_0 r}}{r}\left(1 + \frac{1}{8\varepsilon}\right)\frac{\varepsilon + \frac{3}{4}}{-ik_0 r} \approx \frac{e^{ik_0 r}}{r}\,\frac{i\varepsilon}{k_0 r}\left(1 + \frac{7}{8\varepsilon}\right), \qquad (31.30)$$

while our previous solution (24.22), (25.20) is

$$\Pi \approx \frac{e^{ik_0 r}}{r}\,\frac{i\varepsilon}{k_0 r}\left(1 + \frac{1}{\varepsilon}\right). \qquad (31.31)$$

Consequently, the relative error is equal to $1/8\varepsilon$, which means the introduction of a correction of the order of 6%, independent of the distance, even for the driest soils.

Therefore, the neglect of this distinction will be manifest in the same quantities, as the neglect of the terms of the order $1/\varepsilon^2$.

It should be noted in general that expressions, somewhat differing by the terms of higher order relative to $1/\varepsilon$, are recognized for the numerical distance in various works. Thus in the works [2; 4], the product of ik_0r by the factor

$$\frac{k_0^2(k^2-k_0^2)}{2k^4}=\frac{\varepsilon-1}{2\varepsilon^2}=\frac{1}{2\varepsilon^2}\approx\frac{1}{2\varepsilon\left(1+\frac{1}{\varepsilon}+\frac{1}{\varepsilon^2}+\dots\right)}. \qquad (31.32)$$

is recognized for the numerical distance. As may be seen, this coincides with what was given by formula (31.29), which we obtained in the solution based on an approximate boundary condition and which we utilized everywhere.

In the exact solution, brought up in the present section ([12; 5;* 13]), formula (31.28a) is obtained instead of that which, as we have seen, is practically coinciding with formulas (31.29) and (31.32).

Finally, in the work [2b] instead of the factor (31.32) figures

$$\frac{k}{\sqrt{k_0^2+k^2}}\frac{k_0^2}{2k^2}=\frac{1}{2\varepsilon}\frac{1}{\sqrt{1+\frac{1}{\varepsilon}}}\approx\frac{1}{2\left(\varepsilon+\frac{1}{2}\right)}. \qquad (31.33)$$

As already stressed more than once, inasmuch as the value of ε is established from comparison of experimental data with the deductions of theory, the really measured quantity is a certain effective value ε_{eff} (that is, ε^0, $\varepsilon + 3/4$, $\varepsilon + 1/2$ or $\varepsilon + 1$, depending upon which theoretical formula is used). The refinement of the solution obtained by way of accounting for these small additions has only the sense whereby it proves the applicability of the normal attenuation function down to the driest soils and shortest waves.

We do not bring up the corresponding conclusion for a field in space or for a source raised above ground [5] (see also §58.2). It is obvious that the field in space is determined by the field on the surface. This is why the coincidence of solutions for z = 0 predetermines their coincidence at $z \neq 0$ also.

§32. METHOD OF INHOMOGENEOUS PLANE WAVES

The basic problem of theory of radiowave propagation may be also resolved in the case when the partial solutions of the wave equation are taken in the form of plane waves [2; 3]. As is shown in §16, the field of a point emitter in a uniform medium may be represented by assortment of plane waves only in the case when waves with complex propagation vector are present in it, or, which is the same, with complex direction cosines (16.11a):

$$\Pi_0 = \frac{1}{2}\frac{e^{ik_0R}}{R} = \frac{i}{2}\frac{1}{2\pi}\iint_{-\infty}^{+\infty} e^{i\left(q_x x + q_y y + z\sqrt{k_0^2 - q_x^2 - q_y^2}\right)} \frac{dq_x dq_y}{\sqrt{k_0^2 - q_x^2 - q_y^2}}. \quad (32.1)$$

This is why the presence of such waves is inescapable even in the complete solution for an emitter above ground.

Assuming

$$q_x = k_0 \sin\alpha\cos\varphi,\ q_y = k_0 \sin\alpha\sin\varphi,\ q_z = k_0\cos\alpha,$$

we shall pass to integration variables (α, φ). As is easy to verify, the Jacobian of this transformation is equal to $k_0^2 \sin\alpha\cos\alpha$. Considering that φ varies from 0 to 2π, we shall obtain

$$\Pi_0 = \frac{1}{2}\frac{ik_0}{2\pi}\int_\Gamma \sin\alpha d\alpha \int_0^{2\pi} d\varphi e^{ik_0(x\sin\alpha\cos\varphi + y\sin\alpha\sin\varphi + z\cos\alpha)}. \quad (32.2a)$$

The question of limit of the integral over α or, to be more precise, of the integration path Γ in the complex plane α is resolved by breaking up the integral over q_x, q_y in two parts. It is materialized in formula (16.14). So long as $q_x^2 + q_y^2 = k_0^2\sin^2\alpha < k_0^2$, it may be considered that α covers all the values from 0 to $\pi/2$. In the region $q_x^2 + q_y^2 > k_0^2$ we must consider $\sin\alpha > 1$. Consequently, in order to describe the integration over the entire plane of variables (q_x, q_y), we must add to the integral over α, taken from 0 to $\pi/2$, the integral over the path from $\pi/2$ to $\frac{\pi}{2} \pm i\infty$. The requirement of finiteness as $z \to \infty$ compels us to choose the sign minus. We thus finally obtain that the contour Γ begins at the point $\alpha = 0$ and drifts toward $\alpha = \frac{\pi}{2} - i\infty$. In view of the absence of poles in the integrand it may be deformed in the plane α in an arbitrary fashion, provided the terminal remains fixed at $\alpha = 0$.

Assume now that the source is located not at the point $x = y = z = 0$, but is raised to the altitude z_0. Then we must distinguish the cases $z > z_0$ and $z < z_0$ even in the region of only positive z. In the former we simply shift the origin of the count. But in the region $z < z_0$, the sign at z should be reversed in agreement with what was said in §16. This is why, instead of formula (31.2a) we shall have

$$\Pi_0 = \frac{1}{2}\frac{ik_0}{2\pi}\int_\Gamma \sin\alpha d\alpha \int_0^{2\pi} d\varphi e^{ik_0(x\sin\alpha\cos\varphi + y\sin\alpha\sin\varphi \pm (z - z_0)\cos\alpha)} \quad (z \gtrless z_0). \quad (32.2b)$$

The method consists, therefore, in replacing the field of the source by a group of plane waves. Considering the reflection and refraction of each of them on the surface of medium interface, we may obtain the total field in the presence of ground.

Each of the plane waves, incident upon the interface, generates a reflected plane wave, differing from the incident wave by a complex factor f and change of the sign at z in the exponent, inasmuch as a reflected wave propagates from the ground. Every refracted wave is distinguished by the complex factor $g_{||}$, by the substitution of k_0 by $k = k_0\sqrt{\varepsilon}$ (inasmuch as it must satisfy the wave equation in the lower medium) and by substitution of α, φ by certain α', φ', which are functions of the first. Consequently, it is described by the function

$$g_{||} e^{ik_0\sqrt{\varepsilon}(x\sin\alpha'\cos\varphi' + y\sin\alpha'\sin\varphi' - z\cos\alpha')} \quad (32.3)$$

Each set of three such waves must satisfy the boundary conditions (31.1), (31.2) at z = 0. We already considered this question earlier (see §19.1) and we found that hence follows the standard law of refraction, valid, as is easily seen, also for "complex incident angles" of the plane wave. For $f \equiv f_1$ and $g \equiv g_1$, and also for α they are expressed by Fresnel formulas (19.9), (19.17) and (19.18). Inasmuch as ψ and Ψ entering in them are expressed by α and α', $\psi = \frac{\pi}{2} - \alpha$, $\Psi = \frac{\pi}{2} - \alpha'$, we have

$$\frac{\sin\alpha'}{\sin\alpha} = \frac{1}{\sqrt{\varepsilon}}, \tag{32.4}$$

$$f_1 = \frac{\varepsilon\cos\alpha - \sqrt{\varepsilon - \sin^2\alpha}}{\varepsilon\cos\alpha + \sqrt{\varepsilon - \sin^2\alpha}}, \tag{32.5}$$

$$g_1 = \frac{1 + f_1}{\varepsilon} = \frac{2\cos\alpha}{\varepsilon\cos\alpha + \sqrt{\varepsilon - \sin^2\alpha}} \tag{32.6}$$

Adding the incident and the reflected waves for every α and then integrating over α, we find the field in the upper medium for $z > z_0$ and, in particular, postulating $z_0 = 0$ for $z > 0$:

$$\Pi = \frac{1}{2}\frac{ik_0}{2\pi}\int_\Gamma \sin\alpha\, d\alpha \int_0^{2\pi} d\varphi\,(1 + f_1(\alpha))\, e^{ik_0(x\sin\alpha\cos\varphi + y\sin\alpha\sin\varphi + z\cos\alpha)}. \tag{32.7}$$

Integrating over α the expressions for the refracted wave, we obtain, however, for the field in the ground

$$\Pi_1 = \frac{1}{2}\frac{ik_0}{2\pi}\int_\Gamma \sin\alpha\, d\alpha \int_0^{2\pi} d\varphi\, g_1(\alpha)\, e^{ik_0(x\sin\alpha\cos\varphi + y\sin\alpha\sin\varphi + z\sqrt{\varepsilon - \sin^2\alpha})}, \tag{32.8}$$

where by virtue of formula (32.4) $\sqrt{\varepsilon}\sin\alpha'$ and $\sqrt{\varepsilon}\cos\alpha'$ in the exponent are respectively substituted by $\sin\alpha$ and $\sqrt{\varepsilon - \sin^2\alpha}$.

Subsequently, we may proceed in two ways.

First of all, the formulas obtained may be reduced to an already known expansion by Bessel functions (31.9). To that effect we shall postulate

$$x = r\cos\gamma, \quad y = r\sin\gamma,$$

so that we shall have in the exponents of formulas (32.7) and (32.8)

$$x \sin\alpha\cos\varphi + y\sin\alpha\sin\varphi = r\sin\alpha\cos(\varphi-\gamma).$$

Now performing the integration over φ, we have

$$\int_0^{2\pi} e^{ik_0 r \sin\alpha\cos(\varphi-\gamma)}\,d\varphi = 2\pi J_0(k_0 r\sin\alpha), \tag{32.9}$$

and then postulating

$$k_0\sin\alpha = \nu,\ k_0\ \sin\alpha d\alpha = \frac{\nu d\nu}{\sqrt{k_0^2-\nu^2}},$$

we obtain

$$\Pi = \frac{1}{2}\int_0^\infty \frac{\nu d\nu}{\sqrt{\nu^2-k_0^2}}(1+f_1)J_0(r\nu)\,e^{-z\sqrt{\nu^2-k_0^2}}. \tag{32.10}$$

But (cf. (31.9c))

$$\frac{1+f_1}{2\sqrt{\nu^2-k_0^2}} = \frac{\varepsilon}{\varepsilon\sqrt{\nu^2-k_0^2}+\sqrt{\nu^2-k^2}} = \frac{\varepsilon}{N}.$$

Therefore, the result coincides indeed with formula (31.9a). Analogously it may be shown that formulas (32.8) and (31.9b) coincide also.

However, instead of this it is possible to perform the integration directly in variables α, φ. This method was applied also to other problems, maintaining the problem's symmetry and in the first place to the problem of a stratified-nonuniform medium, important in particular for the study of propagation of very short waves in a nonuniform atmosphere, in which the properties of the medium are considered to be dependent only on a single coordinate $\underline{z}$ (§55).

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223	In the following we shall reject the time factor $e^{-i\omega t}$ everywhere.
264	Then it should be borne in mind that in formulas, figuring in [5], expressions of the type $\frac{1}{\sqrt{\varepsilon^0}} = \frac{1}{\sqrt{\varepsilon}}\sqrt{1-\frac{\cos^2\psi}{\varepsilon}}$ are encoun-

tered, which may be substituted by $1/\sqrt{\varepsilon+1}$ (inasmuch as all the calculations were performed with a precision to the terms of the order $1/\varepsilon$ inclusive) and expressions containing the terms of the order $1/\varepsilon^2$ which in such a case may be dropped.

281 The argument +0 indicates that we consider the field in the plane $z = 0$ in the air, whereas the argument -0 denotes the consideration of a point directly under the plane $z = 0$ in the soil.

289 In the text of the Russian edition of [2b], page 954, a reference present in the German edition was inadvertently maintained, in which it is stated that the result would coincide with that obtained by Sommerfeld in 1909.

291 They should not be mixed up with the reflection factors of other sections.

302 In the text of the work [5], with which the conclusion coincides in essence with that of this section, formula (31.28a) is utilized. However, passing to the numerical data [5, page 1385], the author utilizes formula (31.29), without pausing at the immaterial change then made.

[Transliterated Symbols]

251 эфф = eff = effektivnyy = effective

266 пов = pov = poverkhnostnyy = surface

266 простр = prostr = prostranstvennyy = spatial

Chapter 6

DIPOLE NEAR A SPHERICAL GROUND SURFACE

§33. INTRODUCTORY REMARKS

The spherical shape of the earth must be taken into account in the most general case of the theory of radio-wave propagation above the ground in an electrically homogeneous atmosphere. The history of the problem can be broken down into two periods as regards the objectives with which the theory was confronted.

In the very first years of the development of radio, in the middle of the first decade of the XX Century, when it unexpectedly became possible to establish radio communications between Europe and America, the question arose as to whether radio waves emitted by a source on the ground could follow the curvature of the earth's surface. The optical analogies suggested that such significant diffractive penetration of the radiation in the region of the geometrical shadow was impossible (if the dimensions are changed proportionally, radio waves with a length of the order of 1 km would no more be capable of encircling the globe than a light on the surface of a cherrystone could illuminate its opposite side). This immediately gave rise to the hypothesis of the existence of an ionized layer in the upper regions of the atmosphere – a layer which, reflecting the radio waves, channels them through a relatively thin layer near the earth.

The boldness of this hypothesis and the lack of a confirmation for it gave rise to many attempts over the years to find an exact mathematical solution of the diffraction problem. The fact that a cylindrical wire serves as an axis along which a wave can propagate on the one hand, and, on the other, the irregular Zenneck solution for the plane wave, which apparently would confirm the hypothesis that such a surface wave does exist at the plane (see §23), gave rise to the erroneous idea that part of the field radiated in the form of a wave "clinging to the surface of the globe might follow this surface over a considerable part of its circumference.

A rather long time elapsed before this concept was refuted and it was ascertained that the field diminishes exponentially with distance over long distances. Since this decrease, like any other diffractive attenuation of a field in shadow, is less noticeable for long waves (see §39 below), interest was concentrated on the problem of long waves up to 10-20 km) and considerable angular distances (considerable fractions of the earth's circumference.

When it was found by experiment that the true mechanism of radio wave propagation around the world is precisely reflection from the ionosphere, and radio communications over long distances began to follow a trend toward shorter wavelengths, propagation of the surface or ground wave ("daytime field") with consideration of the curvature came to attract special interest for precisely the short waves and, with the development of radar, for ultrashort waves. The distances that had come to play the principal role in theory were accordingly shortened, and it may be considered at the present time that in the problem of the homogeneous atmosphere, as long as we disregard the beam reflected from the ionosphere, we are interested basically only in distances between corresponding points that are very short as compared with the earth's radius.

In one of the early papers [1] on diffraction of radio waves around the earth, Watson used an ordinary expansion in orthogonal (spherical) functions to obtain a solution in the form of a series, which, however, showed very poor convergence. The only result found from it was an asymptotic expression for the field of very long waves at the earth's surface over very long distances from the source. It brought out the exponential nature of the decrease in field amplitude, and it was thus proven that we may not explain the propagation of radio waves around the world by diffraction alone.

This series was transformed to practically useful formulas at first by substituting various asymptotic expressions for certain of the functions that appeared in it. These substitutions were frequently made without adequate justification, were mutually contradictory and did not permit estimation of the error incurred.

A sequential formula that permitted approximate calculation of the attenuation function for ground of finite conductivity with diffraction taken into account was obtained by B. A. Vvedenskiy [2] in 1935-1937. Diagrams constructed on the basis of this formula were used extensively, and their role in practice was very important. Then van der Pol and Brummer [3; I, 11] used more exact asymptotic expressions for the substituted functions and obtained a general regular formula (see (39.20) below). They presented it in the form of diagrams for a large number of practical cases.

The important problem of the transitional region near the geometrical boundary of the shadow was not adequately analyzed in these studies, and the range of admissibility of the approximations adopted remained unclear. These problems were investigated by V.A. Fok [4], in which he submitted a very complete solution to the problem. The basic final formulas obtained by Fok are in agreement with the van der Pol and Bremmer formulas.

It was every important fact for practical applications that Fok reduced the solution in series form (which converges poorly near the boundary of the shadow) to an integral (see Formula (39.2)). Together with the results of subsequent tabulation of the functions appearing in the solution [4; 5], these results may be regarded as a firm basis for the necessary numerical calculations and as the culmination of the entire problem. Soon afterwards, the identical solution was obtained by M.A. Leontovich and V.A. Fok [6] by a totally different and much simpler method that used the Leontovich approximate boundary condition (§21) as a starting point.

At the present time, therefore, the problem of radio-wave propagation in a homogeneous atmosphere near an electrically homogeneous and smooth spherical earth's surface may be regarded as solved.

However, before setting forth its solution, we shall present a semiquantitative analysis in order to ascertain the relative importance of the various factors that enter into the problem and to obtain something of a vantage point in our treatment of the process.

§34. ESTIMATES AND DETERMINATION OF THE BASIC PARAMETERS

The basic fact responsible for the possibility of various simplifications in the problem under study is the shortness of the wavelengths of all of the radio waves of interest to us as compared with the radius of curvature of the earth's surface. This radius, which we shall denote by the letter $\underline{a}$, may be regarded as equal on the average to 6370 km $\approx 6.4 \cdot 10^8$ cm. The quantity ka is the large new geometrical parameter of the problem, which is added to the electrical (usually large) parameter ε that we already have.

The presence of two large parameters complicates classification of the various particular cases to some degree, because both their ratios and the ratios of the various powers of these parameters may appear in the solution.

It was stated in §33 that we are usually interested in segments of the earth's surface whose linear dimensions are small as compared with the earth's radius. Hence if the central angle (Fig. 34.1) is taken as the coordinate of a point on the surface, this angle may everywhere be considered small and we may accordingly replace sin ϑ by ϑ or, in the extreme case, $\vartheta - (\vartheta^3/6$, and cos ϑ by 1 or by $1 - (\vartheta^2/2)$.

If we pass a plane xy tangent to a certain point on the surface of the earth and direct the vertical axis z along a radius into the atmosphere (Fig. 34.1), the equation of the surface may be written in approximation in rectangular coordinates as follows:

$$z = -\frac{x^2 + y^2}{2a}. \tag{34.1}$$

(Actually, the exact equation would be $a^2 = x^2 + y^2 + (a + z)^2$, i.e., $z = -\frac{x^2+y^2+z^2}{2a}$. But $x^2 + y^2 \approx a^2\vartheta_1^2$, while $z^2 \approx \frac{1}{2}a^2\vartheta_1^4$, and hence we may drop z^2 for small ϑ_1.)

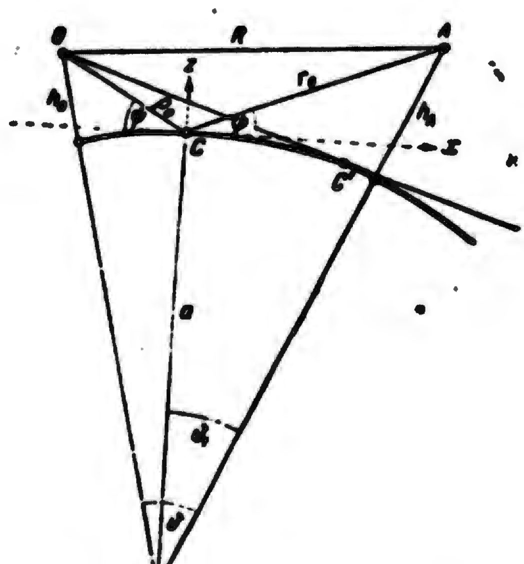

Fig. 34.1. Symbols to §34.

Just as in analysis of the field near a flat round surface, we may limit ourselves to consideration of the field outside the earth if we introduce the Leontovich approximate boundary condition for the field at the surface. The admissibility of using this boundary condition in a nonplanar case is explained by the fact that the radius of curvature of the surface is very large as compared with the depth of penetration of the field into the ground. Here also, therefore, the field in the ground is determined near its surface by the field at the surface in the vicinity of the observation point, on a segment that may be regarded as flat.

The applicability criterion for a nonplanar ground can be determined more exactly on the basis of the following considerations.

We saw in §14 that a plane wave is propagated from the surface into the interior in a ground with a high specific inductive capacitance ε. It is essential that, at depths of the order of the depth of penetration $\frac{\lambda}{\sqrt{\varepsilon}}$, its front is created by in-phase oscillations of the field at the ground surface. To retain this picture in the case of a curved surface, it is important that within the limits of a segment whose length is at least of the order of the penetration depth, the surface fall away from the plane by an amount that is still small as compared with the "wavelength" in the soil $\frac{\lambda}{\sqrt{|\varepsilon|}}$.

Let us place the origin of the rectangular coordinate system xyz in coincidence with the middle of the segment under consideration in such a way that Eq. (34.1) will be the equation of the surface. Then, according to the above, it is necessary that for $x^2+y^2=\frac{\lambda^2}{|\varepsilon|}$ the subsidence $\underline{z}$ of the earth's surface from the plane be very small as compared with $\lambda/\sqrt{|\varepsilon|}$, i.e., that

$$2ka\sqrt{|\varepsilon|} \gg 1. \tag{34.2}$$

It is readily seen from this that if we disregard the curvature, the error in the boundary condition will be of the order of $\frac{1}{ka\sqrt{|\varepsilon|}}$ ([IV, 3], see (40.13) below). This condition is always more than satisfied.

Below we shall consider the field of electrical and magnetic vertical dipoles. In both of these cases, the field in spherical coordinates can be described by the Hertzian single-component vector Π or Π_m or the related Debye function u(r) and v(r), in accordance with Formulas (3.29) and (3.30) and the wave equations (3.28a) and (3.31). The components of the electric and magnetic fields are determined by the Debye functions from Formulas (3.32) and (3.33).

To the extent that the approximate boundary conditions (21.33) and (21.35) apply (criterion (34.2) is satisfied with an enormous margin), we may simply rewrite them, substituting Π_z by ru and Π_{mz} by rv:

$$\frac{\partial (ru)}{\partial r} = -\frac{ik}{\sqrt{\varepsilon'}} ru \quad \text{for } r = a; \tag{34.3}$$

$$\frac{\partial (rv)}{\partial r} = -ik\sqrt{\varepsilon - \cos^2\psi}\; rv \text{ for } r = a. \tag{34.4}$$

However, at the earth's surface, r = a, taking advantage of the fact that up to $\lambda \sim 10$ km even over the ocean surface, and for even longer wavelengths over land

$$ka \gg |\sqrt{\varepsilon'}|,$$

we may use the boundary conditions for u and v in the simpler form

$$\frac{\partial u}{\partial r} = -\frac{ik}{\sqrt{\varepsilon'}} u \quad \text{for } r = a, \tag{34.5}$$

$$\frac{\partial v}{\partial r} = -ik\sqrt{\varepsilon - \cos^2\psi}\; v \text{ for } r = a. \tag{34.5a}$$

We can combine these two cases, considering a function u that satisfies the wave equation and a boundary condition

$$\frac{\partial u}{\partial r} = -ik\eta u \text{ for } r = a, \tag{34.6}$$

$$\eta = \frac{1}{\sqrt{\varepsilon'}} \text{ for a vertical electric dipole,} \tag{34.6a}$$

$$\eta = \sqrt{\varepsilon - \cos^2\psi} \text{ for a vertical magnetic dipole.} \tag{34.6b}$$

The field of sources at the surface must be defined in such a way that it will merge at short distances from the sources into the field found for the plane wave. Hence we need not write out the sources in explicit form.

Let us imagine a source at a certain point above the surface. This case is illustrated in Fig. 34.1 with a distinct distortion of the scales (the elevation of the source and of the observation point almost never exceed fractions of a per cent of the earth's radius; exceptions are radio communications with satellites and comets and radio astronomy in general). The difference between the radial coordinate r of the point and the earth's radius a will be denoted by h_0 for the source, and by h_A or simply by h for the observation point:

$$h_0 = r_0 - a,$$
$$h = h_A = r - a \ll a.$$

The angular distance between the source O and the observation point A will be denoted by ϑ, and the linear distance by R. The distance along the arc of the great circle between the projections of these points on the surface is $D = a\vartheta$.

If we suppose that the source is emitting rays that propagate in accordance with the laws of geometrical optics, it is obvious that the horizon plane OC', i.e., the plane passing through the source O tangentially to the sphere, plays a fundamental role. In the approximation of geometrical optics, the surface of the sphere is illuminated to the left of point C', while the rest of the surface (to the right of C') is in shadow, below the horizon. This, however, may occur only for infinitesimally short wavelengths. For finite λ, diffraction will blur the boundary of the shadow. If the boundary were formed by the sharp edge of a screen, as was considered in §11, the field strength would diminish as we approached the observation point below the plane of the horizon, following the curve of Fresnel diffraction from the edge of the screen shown in Fig. 11.3, while the attenuation function would vary in accordance with Formula (11.7a), i.e., in inverse proportion to the distance to the plane of the horizon (see (11.8c); $\frac{h'_A}{r_1} \approx \sin\psi$):

$$w \approx \frac{e^{i\frac{\pi}{2}\xi^2}}{\pi i\xi}, \quad \xi = -\sqrt{\frac{k}{\pi}\left(\frac{1}{r_1}+\frac{1}{\rho_1}\right)}\,\frac{h'_A}{1+\frac{r_1}{\rho_1}}, \tag{34.7}$$

where (Fig. 34.2) h'_A is the distance to the "plane of the horizon," ρ_1 = OC and r_1 = AC are the distances to the edge of the screen. Thus, we should have (for $r_1 \sim \rho_1$)

$$|w| \sim \frac{1}{h'_A}\sqrt{\frac{r_1}{k}}, \tag{34.7a}$$

and the attenuation would be measured in terms of the number of Fresnel zones in the vertical plane that can be fitted between the observation point and the horizon.

As was shown by V.A. Fok [7], this actually is the case, but only in the immediate vicinity of the horizon, where h' is small and represents a small fraction of the distance from the horizon plane to the ground surface at the place in question (for greater detail, see §39, subsection 3). As h'_A increases further, the field diminishes much more rapidly with increasing $|h'_A|$ (exponentially, see §39), and this is essentially due to the fact that the Kirchhoff method is inapplicable to the present case. That is to say, integration in the Kirchhoff method would be carried out over the "plane of the hole," i.e., over plane CD above point C, and the field would be assumed equal to zero on the rest of the surface around the observation point (the "shadow side of the screen"). At the same time, due to the slight curvature of the earth, the field penetrates along the earth's surface considerably beyond point C, and for this reason the integral over the "shadow part" of the ground surface cannot be dropped, as it was for the screen. Hence it will be correct to substitute a sharp screen edge for the spherical earth in the calculation only in certain cases.

Let us now turn to the question of the extent to which the field on the illuminated part of the sphere may be regarded as undisturbed.

To obtain the bounds, we shall start from the same integral rela-

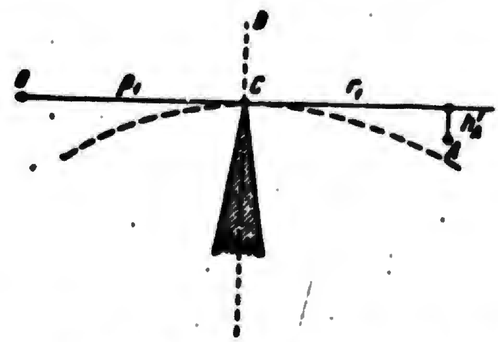

Fig. 34.2. Substitution of wedge for spherical surface.

tionship (5.14) that served us for analysis of the field above a plane. Assuming that the surface S coincides with the surface of the earth and using r' to denote the distance of the observation point from the current point on the surface, we rewrite this relationship for u (all conclusions are, of course, equally applicable for v) in the form

$$u = u_0 + \frac{1}{4\pi}\int_S \left\{\frac{e^{ikr'}}{r'}\frac{\partial u}{\partial n} - u\frac{\partial}{\partial n}\frac{e^{ikr'}}{r'}\right\} dS', \qquad (34.8)$$

where differentiation is with respect to the outer normal, i.e., $\partial/\partial n = -(\partial/\partial r)$. Substituting $\partial u/\partial n$ from the boundary condition (34.5), we have

$$u = u_0 + \frac{1}{4\pi}\int_S u\frac{e^{ikr'}}{r'}\left\{\frac{ik}{\sqrt{\varepsilon'}} - ik\left(1 - \frac{1}{ikr'}\right)\frac{\partial r'}{\partial n}\right\} dS'. \qquad (34.9)$$

The terms in braces are relatively slowly varying functions of the present coordinates, and u may be written in the form

$$u = 2u_0\ w(r, \vartheta), \qquad (34.10)$$

where w is a relatively slowly varying attenuation function and u_0 is the field that would be created by the source in question in the absence of the earth. Selecting a source power such that in free space

$$u_0 = \frac{e^{ikR}}{R}, \tag{34.11}$$

where R is the distance to the source (for the present points on the surface, we denote $R = \rho'$), we see that even here the relative importance of the various integration segments is determined primarily by the multiplier $\exp(ik(r' + \rho'))$, which delineates zones on the ground that are nonplanar in the present case. Obviously, the influence of the curvature is determined by the shape distortion of the first few zones as compared with those that we studied in the case of the plane. We note that the value of u_0 that we have selected (34.11) corresponds to $\Pi_0 = ru_0 \approx au_0$, which differs by the factor 2a from that used in Chapter 5, see (24.0).

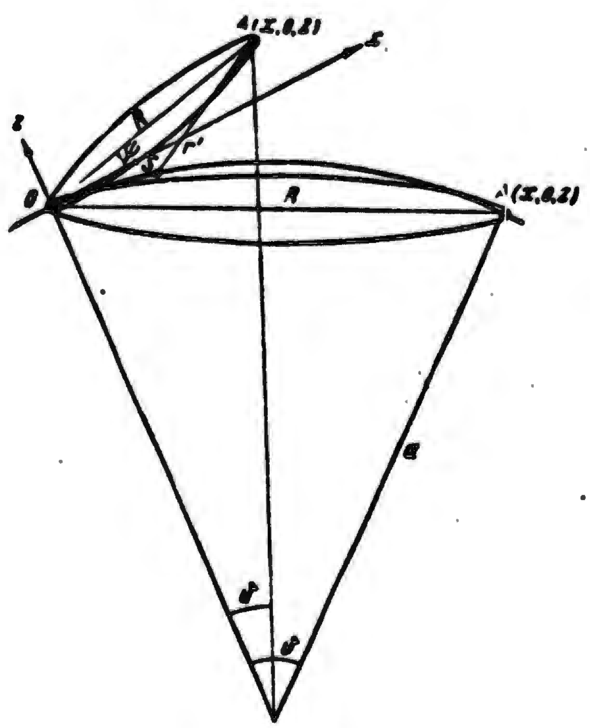

Fig. 34.3. Role of surface curvature within essential zone.

Suppose, for example, that the source is on the ground and that for this reason the plane of the horizon coincides with the tangent plane at the point of the source. As in Fig. 34.1, we shall introduce the Cartesian coordinates x, y for this plane and z along the normal to it (Fig. 34.3). Equation (34.1) will again be the equation of the earth's surface.

Let us consider two cases separately.

Suppose at first that the observation point A is elevated above the horizon. If the earth could be regarded as flat, then, as we know (Formula (12.11), Fig. 12.2 or 26.2), the essential point would be values of x' such as would fit into the first zone, which is cut out on the plane by an ellipsoid. That is to say, at significant heights z, the observation points

$$x' < 2a_1 = \frac{\pi}{k \sin^2 \psi}, \qquad (34.12)$$

where ψ is the elevation angle. For small ψ, the zone covers all distances, $x' \sim x$. This occurs for

$$\sin\psi \sim \frac{\sqrt{\pi}}{\sqrt{kx}} \sim \frac{2}{\sqrt{kx}}. \qquad (34.13)$$

The interference (reflection) formulas are, in particular, valid for large enough ψ. Obviously, the curvature of the earth is nonessential and will not affect the result obtained for a flat earth if within the limits of this essential zone the surface drops away from the plane insignificantly, i.e., by so much that substitution of a curved zone for the plane zone will introduce, even at its end, a small phase lead Φ on a path OSA, $\Delta\Phi \ll \pi$. But for a wave passing at an angle ψ to the horizon, i.e., with a wave number $k_z = k \sin\psi$ on the z-axis, this additional phase lead amounts to

$$\Delta\Phi = k_z z' = kz' \sin\psi, \qquad (34.14)$$

where z' is the distance of the point on the ground surface from the xy-plane at the end of the zone. According to Formulas (34.1) and (34.12),

$$z' \sim \frac{r'^2}{2a} \sim \frac{\pi^2}{2k^2 a \sin^4 \psi}, \qquad (34.15)$$

and the condition assumes the form

$$\Delta\Phi = \frac{\pi^2}{2ka \sin^3 \psi} \ll \pi, \qquad (34.16)$$

or

$$\sin \psi > \frac{1}{\sqrt[3]{ka}} = \frac{1}{p_1}. \qquad (34.17)$$

Here p_1 denotes the large dimensionless parameter $\sqrt[3]{ka}$, which is of importance for the entire theory and appears here for the first time. Since these considerations have applied to the case of sufficiently high observation points, i.e., in any event to the region

$$\sin \psi > \frac{2}{\sqrt{kx}}, \qquad (34.17a)$$

it will not make sense to speak of Condition (34.17) at all unless it proves to be a new limitation, i.e., if

$$\frac{2}{\sqrt{kx}} < \frac{1}{\sqrt[3]{ka}}$$

or, in other words, unless the following condition is satisfied for a new dimensionless quantity p_2:

$$p_2 = \frac{kx^3}{8a^2} > 1. \qquad (34.18)$$

This parameter p_2 is also of great importance for theory. Expressing distance in kilometers, x_{km}, and wavelength in meters, λ_m, this parameter, assuming $a = 6.4 \cdot 10^3$ km, is conveniently expressed as follows:

$$p_2 \approx 2 \cdot 10^{-6} \frac{x_{km}^3}{\lambda_m}. \qquad (34.18a)$$

The derivation of (34.16) may be set forth in greater detail. For this purpose, let us consider the sum of the paths $\rho' + r'$ for a current point, firstly on the plane xy and secondly on the surface of the earth.

For a point on the plane, we have $x' \ll x$ for the case of considerable $\underline{z}$ and

$$r'+\rho'=\sqrt{(x-x')^2+y'^2+z^2}+\sqrt{x'^2+y'^2}\approx x+\frac{y'^2}{2}\frac{x}{x'(x-x')}+\frac{z^2}{2(x-x')}. \quad (34.19)$$

For a point on the sphere

$$r'+\rho'=\sqrt{(x-x')^2+y'^2+\left(z+\frac{x'^2+y'^2}{2a}\right)^2}+$$
$$+\sqrt{x'^2+y'^2+\left(\frac{x'^2+y'^2}{2a}\right)^2}\approx \quad (34.19a)$$
$$\approx x+\frac{1}{2}y'^2\frac{x}{x'(x-x')}+\frac{z^2+\frac{x'^2z}{a}}{2(x-x')}+\frac{x'^3x}{8a^4(x-x')},$$

where we have dropped higher-order terms (we are assuming that $y' \ll \ll x'$). Consequently, the phase excess obtained by multiplying the difference between these two expressions by $\underline{k}$ is, as assumed in Formula (34.16) (the last term in (34.19a) is small and x' is substituted from Formula (34.12) equal to:

$$\Delta\Phi\approx\frac{kx'^2z}{2a(x-x')}\sim\frac{kx'^2z}{2ax}\sim\frac{\pi^2kz}{2k^2ax\sin^4\psi}=\frac{\pi^2}{2ka\sin^3\psi}.$$

Let us now consider the case of small heights, in the vicinity of $z = 0$, i.e., the case in which the observation point is in the plane of the horizon. Here we obtain instead of Formula (34.19a)

$$r'+\rho'\approx x+\frac{y'^2}{2}\frac{x}{x'(x-x')}+\frac{1}{8a^2}\frac{x}{x'(x-x')}(x'^2+y'^2)^2. \quad (34.19b)$$

Assuming that the first zone now covers the region of all x' up to $x' \sim x$, with all of them generally playing the same role, we put

$$x'\sim x\sim x-x'.$$

This gives

$$\Delta\Phi\sim k\frac{x^3}{8a^2}=p_2.$$

Consequently, for the curvature to exert no influence we must have

$$p_2 = \frac{kx^3}{8a^2} \ll 1. \qquad (34.20)$$

Thus, summarizing Conditions (34.17), (34.18) and (34.20), we may make the following statement.

If the dimensionless parameter $p_2 = kx^3/8a^2$, where x, the projection of the distance to the observation point onto the plane of the horizon, is small, the curvature will be nonessential even for observation points lying in this plane. If it is large, the curvature will not make itself felt for observation points elevated to a height such that the glancing angle ψ satisfies Condition (34.17).

Using the reciprocity theorem, we may also make the following statement. If the source is located above the plane of the horizon or on it, then the field that it creates at the surface of the earth will not differ from the field that would be created here if this point were on the plane of the horizon, which coincides with the tangent (at the observation point) plane provided either that $p_2 \ll 1$ or, if $p_2 \gtrsim 1$, that the glancing angle of the incident rays, ψ, exceeds $\frac{1}{\sqrt[3]{ka}}$.

Thus, the range of distances x (from a source at the surface) in the tangent plane that satisfy the condition $\frac{kx^3}{8a^2} = p_2 \ll 1$ may be regarded as undisturbed. The result for an elevated observation point can therefore be obtained from the following simple considerations: for curvature not to be a factor at an elevated point, it is necessary that the region of the surface $x' \lesssim \frac{2}{k \sin^3 \psi}$ essential for it (compare (34.12)) fit into the undisturbed zone $x' < \sqrt[3]{\frac{8a^2}{k}}$. Condition (34.17) follows directly from this.

Up to this point, we have spoken only of the geometrical parameters. But it is also obvious that the curvature will manifest to a greater or lesser degree depending on how rapidly the field is attenuated with increasing distance due to the poor electrical properties of the soil. That is to say, it is essential which of the two attenuations – geometrical or electrical – comes into evidence first as we increase our distance on the actual surface from a source on the surface of the earth. This may be formulated somewhat differently as follows: has ε begun to differ from infinity when the curvature begins to appear, i.e., at distances $x\sim\sqrt[3]{\frac{8a^2}{k}}$? To answer this question, it is necessary to determine whether the corresponding numerical distance is greater or less than unity. The situation will, of course, be different for electrical and magnetic dipoles, since the numerical distances $\rho=\frac{ikx}{2\varepsilon^0}$ (25.2) and $\rho_m=\frac{ik}{2}(\varepsilon-1)x$ (28.1) for these will be related differently to the geometrical distance $\underline{x}$. We denote for the vertical electric dipole

$$\delta=i\frac{\sqrt{\varepsilon^0}}{\sqrt[3]{ka}}=i\frac{\varepsilon}{\sqrt[3]{ka}\sqrt{\varepsilon-1}}=\frac{-1}{\sqrt[3]{2}\,q} \qquad (34.21)$$

(the parameter δ figures in Bremmer [I, 11], while $\underline{q}$ figures in V.A. Fok [IX, 18]; in their essentials, they are also encountered in earlier papers on diffraction of radio waves around the earth, for example, in B.A. Vvedenskiy).

The numerical distance of interest to us is

$$sx_0=\frac{ik}{2\varepsilon^0}\sqrt[3]{\frac{8a^2}{k}}=i\frac{\sqrt[3]{k^2a^2}}{\varepsilon^0}=\frac{-i}{\delta^2}=-i\cdot 2^{2/3}q^2. \qquad (34.22)$$

Thus, if the parameter δ is large (wavelengths, sea surface, the geometrical decrease will be the principal factor. The parameter δ is small, i.e., the corresponding numerical distance is large, for short and ultrashort waves over dry land. The parameter δ is also a very important factor in theory. Let us assume that $\frac{4\pi\sigma}{\omega}\gg\varepsilon'$ (although this is not usually true for ultrashort waves). In such a case

$$\frac{1}{|\delta|}\approx\sqrt{\frac{k\cdot a\,c^2}{(4\pi)^2\sigma^2}}=\frac{A}{\sigma^{1/2}\lambda^{5/6}}. \qquad (34.23)$$

Often, in computing this parameter, σ is expressed in electromagnetic units and λ in kilometers. Then

$$A \approx 0{,}44\cdot 10^{-6},$$

or

$$|\sqrt{sx_0}|^{-1} = 0{,}23\cdot 10^{7}\,\sigma_{CGSM}^{1/2}\lambda_{km}^{5/3}, \qquad (34.24)$$

where x_0 is taken from the condition $\frac{kr_0^3}{8a^2} = 1$.

As the complete investigation shows, cases with $|\sqrt{sx_0}| \gtrless 1$ actually differ sharply as regards the form of the solution (compare [I, 6, page 199], [I, 11, page 47]).

For a vertical magnetic dipole, on the other hand, the action of the electrical parameters of the ground asserts itself at much shorter distances. Accordingly, we introduce

$$\delta_m = \frac{1}{\varepsilon}\delta = \frac{i}{\sqrt[3]{ka}\,\sqrt{\varepsilon - 1}} = -\frac{1}{\sqrt[3]{2}\,q_m}. \qquad (34.25)$$

The numerical distance is

$$s_m x_0 = \frac{ik}{2}(\varepsilon - 1)\sqrt[3]{\frac{8a^2}{k}} = -\frac{i}{\delta_m^2} = -i2^{2/3}q_m^2. \qquad (34.26)$$

If the conductivity is so that that $4\pi\sigma \gg \varepsilon\omega$, then

$$|\delta_m| = \frac{\sqrt{\omega}}{\sqrt[3]{ka}\,\sqrt{4\pi\sigma}} = \frac{B}{\lambda^{1/6}\sigma^{1/2}},\ B = \sqrt[6]{\frac{c^3}{32\pi^2 a^2}}, \qquad (34.27)$$

or

$$|\sqrt{s_m x_0}| = \frac{1}{B}\sigma_{CGSM}^{1/2}\lambda_{km}^{1/6}, \quad \frac{1}{B} \approx 2{,}4\cdot 10^{9}.$$

§35. INTERFERENCE (REFLECTION) FORMULAS AND THE DIVERGENCE FACTOR

Let us turn to determination of the field in the illuminated part of space.

As was already ascertained in §34, we may discuss observation points on the ground surface here only when the source is elevated, since otherwise the entire surface of the earth would be in the penumbra or umbra.

For large enough glancing angles ψ (34.17), we may consider the field at any point on the surface equal to that which would obtain if the surface coincided with the tangential plane in the neighborhood of this point. But in this case the field would be composed of the incident field and a reflected field with a Fresnel reflection coefficient $f = f_1$. Introducing the Debye function u (§34), we shall have around point C (Fig. 34.1)

$$u = u_0 + u_1 = (1 + f)u_0, \tag{35.1}$$

where (see (34.11))

$$u_0 = \frac{e^{ik\rho}}{\rho}. \tag{35.2}$$

At a certain elevated observation point A, the field will in the general case differ from that which would be obtained on reflection from a plane tangential to a sphere at point C. This is accounted for by the fact that the zone essential for the field on the surface of the sphere may become so large that the curvature may assert itself on it.

For the calculations, we return to Eq. (34.9). In this case

$$\frac{\partial r'}{\partial n} = \sin\psi, \tag{35.3}$$

due to which, on substitution of Expressions (35.1) and (35.2), we obtain (here, in accordance with the principles of the reflection-formula method, we consider ψ to be constant for the entire essential region of integration and take all expressions containing ψ out of the integrand, considering that $kr_0 \gg 1$):

$$u = u_0 + \frac{ik}{4\pi}\left(\frac{1}{\sqrt{\varepsilon'}} - \sin\psi\right)(1 + f)I, \tag{35.4a}$$

$$I = \frac{1}{2}\int \frac{dS}{r'\rho'} e^{ik(r'+\rho')}. \tag{35.4b}$$

The integral is extended over the surface of the sphere.

Let us use a rectangular coordinate system x', y', z' (with its origin at the reflection point C), in which the source and the observation point have the coordinates x_0', 0, z_0' and x_A', 0, z_A'. Expanding $r' + \rho'$ in powers of x', y' and z' near point C, we have

$$\rho' + r' \approx \rho_0 + r_0 + \frac{1}{2} y'^2 \left(\frac{1}{r_0} + \frac{1}{\rho_0}\right) + z' \sin\psi + x'^2 \frac{\sin^2\psi}{2}\left(\frac{1}{r_0} + \frac{1}{\rho_0}\right). \quad (35.5)$$

Replacing the coordinate z' by its value on the surface, $z' = -[(x'^2 + y'^2)/2a]$, and replacing $r'\rho'$ in the multiplier before the exponential function in I by $r_0\rho_0$, we arrive at two Fresnel integrals (over x' and over y'). Using the formula for these integrals, (10.13), we obtain

$$u = \frac{e^{ikR}}{R} + f\frac{e^{ikR'}}{R'}\sqrt{\frac{aR' \sin\psi}{r_0\rho_0 + aR' \sin\psi}}, \quad (35.6)$$

where $R' = r_0 + \rho_0$ is the total path traversed by the reflected wave.

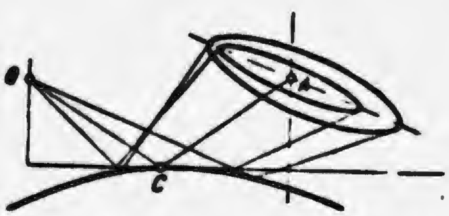

Fig. 35.1. Origin of divergence factor.

On reflection from a flat surface, $a = \infty$, this last multiplier (the square root) becomes unity. In this case, the formula has the usual clear significance: In addition to the direct wave, a reflected wave arrives at the observation point, attenuated by the reflection coefficient f and with a total phase lead kR' corresponding to the distance traveled. The difference between the denominators R and R' of the two terms also has a simple significance. Any bundle of rays emitted by a source diverges as a cone and its energy is distributed over an area proportional to the square of the path R traversed. Consequently, the field amplitude (which is proportional to the square root of energy) should, in view of this purely geometrical effect, diminish in inverse proportion to distance. The reflected beam traverses a distance R' greater than the path of the direct beam (see Fig. 35.1). This is what accounts for the factor R/R', which might be called the divergence factor for reflection from a plane. In the case of a sphere, the divergence will obviously be greater. Hence the last factor in Formula (35.6), which is smaller than unity and is also commonly known as the factor (or coefficient) of divergence for a sphere.

In the case of the field of a horizontal electric or a vertical magnetic dipole, these purely geometrical considerations retain their force. Consequently, without repeating the entire derivation, we may state that in this case also the formulas for reflection from a plane may be used on addition of the same divergence factor to the reflected wave. Actually, the divergence coefficient appeared in our work when we integrated over the surface, since a new term, $z' \sin\psi = -\frac{x'^2}{2a}\sin\psi$, not encountered in the case of the plane appeared in the phase expansion plan (35.5), and its effect appears in the Fresnel integral over x'. It is obvious that in calculation with the integral formula (34.8), this result will be obtained for any source and any field component. It is connected exclusively with the curvature of the surface on which the zones are drawn.

For a detailed exposition of particular cases and the technique of using the reflection formulas, see the books [I, 6], [I, 7] and [I, 8].

The divergence coefficient

$$\alpha = \sqrt{\frac{aR \sin\psi}{r_0\rho_0 + aR' \sin\psi}} \qquad (35.7)$$

differs substantially from unity only for very small ψ,

$$\psi \leqslant \frac{r_0}{R}\frac{\rho_0}{a} < \frac{R}{a} \ll 1.$$

For practical use, the formulas presented above must be rewritten in such a way that ψ, r_0, ρ_0, and R' do not appear in them, but are replaced by the quantities h_0, h_A and ϑ, which are assigned directly by practical conditions. Hence we shall take into account the fact that

$$\rho_0 = \frac{h_0}{\sin\psi}, \quad r_0 = \frac{h_A}{\sin\psi}, \tag{35.8a}$$

$$a\vartheta = \rho_0\cos\psi + r_0\cos\psi = R'\cos\psi. \tag{35.8b}$$

Consequently,

$$\operatorname{tg}\psi = \frac{h_0+h_A}{a\vartheta}. \tag{35.9}$$

It also follows from this that

$$R' = \frac{a\vartheta}{\cos\psi} = a\vartheta\sqrt{1+\left(\frac{h_0+h_A}{a\vartheta}\right)^2} \approx a\vartheta + \frac{1}{2}\frac{(h_0+h_A)^2}{a\vartheta}, \tag{35.10a}$$

while

$$R = a\vartheta + \frac{1}{2}\frac{(h_0+h_A)^2}{a\vartheta}. \tag{35.10b}$$

Consequently,

$$R' - R = \frac{2h_0 h_A}{a\vartheta} \approx \frac{2h_0 h_A}{R}, \tag{35.10c}$$

and the divergence coefficient may be written as follows (assuming here that $\cos\psi \approx 1$):

$$\alpha = \sqrt{\frac{1}{1+\frac{h_0 h_A R^2}{a(h_0+h_A)^3}}}. \tag{35.11}$$

§36. FIELD AT THE GROUND SURFACE IN THE CASE OF A SMALL CURVATURE EFFECT

Under this heading, we shall consider the field created at the ground by a vertical dipole that is also on this surface, with special interest in source distances so small that the influence of curvature enters as a small correction. The result obtained will enable us to determine the range of applicability of the theory in which the earth is regarded as flat.

We have already retained the corresponding criterion (see (34.20)) from very simple considerations. In the present paragraph, we shall obtain a more exact evaluation that will enable us to estimate the error.

Let us consider Eq. (34.9) for an observation point on the ground surface. Here a certain amount of caution is in order. This is because in the second term in the integral

$$\int u \frac{\partial}{\partial n} \frac{e^{ikr'}}{r'} dS' = \int u \left(ik - \frac{1}{r'} \right) \frac{e^{ikr'}}{r'} \frac{\partial r'}{\partial n} dS' \qquad (36.1)$$

there are terms that appear to diverge for an observation point on the sphere $h_A = 0$. We shall transform them, assuming $h_A \neq 0$, and then go to the limit $h_A = 0$.

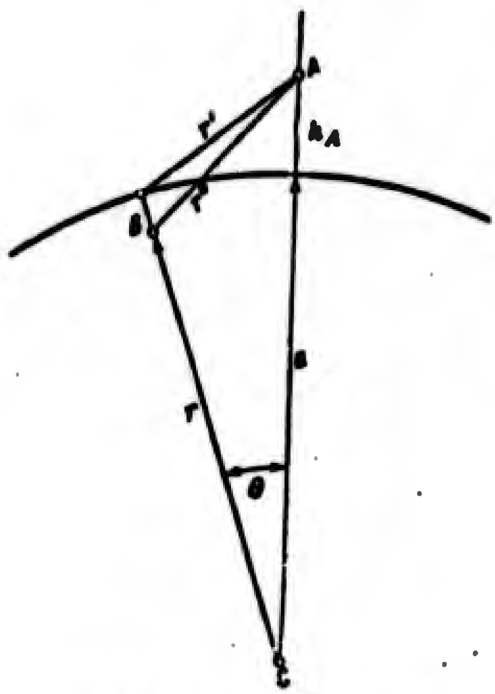

Fig. 36.1. Symbols to §36.

We introduce for the time being a spherical coordinate system passing through the observation point A, with the polar angle θ (Fig. 36.1). From triangle ABC (point B is a variable point at a distance r from C), we have for the distance r" from A to point B

$$r''^2 = (a + h_A)^2 + r^2 - 2r(a + h_A)\cos\theta,$$

$$\frac{\partial r'}{\partial n} = -\left(\frac{\partial r''}{\partial r}\right)_{r=a} = -\frac{a - (a + h_A)\cos\theta}{r'} \approx \qquad (36.2)$$

$$\approx -\frac{a\theta^2 - 2h_A}{2r'} = \left(\frac{\partial r'}{\partial n}\right)_0 - \frac{h_A}{r'},$$

where $(\partial r'/\partial n)_0$ denotes the isolated part of $\partial r'/\partial n$ that does not vanish as $h_A \to 0$ and does not depend on h_A. As we see,

$$\left(\frac{\partial r'}{\partial n}\right)_0 = -\frac{(a\theta)^2}{2ar'} \approx -\frac{\theta}{2}. \qquad (36.3)$$

We shall show that the integral remaining after this separation is equal in the limit to $-2\pi u$. Indeed, $dS' = 2\pi a^2\theta d\theta$, and since $r'^2 = (r''^2)_{r=a} = h_A^2 + 2a(a + h_A)(1 - \cos\theta)$, then $r'dr' \cong a^2\theta d\theta$, and hence this remaining integral is equal to

$$\lim_{h_A\to 0}\int u\left(ik-\frac{1}{r'}\right)\frac{e^{ikr'}}{r'}\left(-\frac{h_A}{r'}\right)2\pi r'dr'=$$
$$=\lim_{h_A\to 0}(-2\pi h_A)\int_{h_A}^{\infty}u\left(ik-\frac{1}{r'}\right)\frac{e^{ikr'}}{r'}dr'. \qquad (36.4)$$

Since

$$-\int_{h_A}^{\infty}\frac{u}{r'^2}e^{ikr'}dr'=\frac{e^{ikr'}}{r'}u\Big|_{h_A}^{\infty}-\int_{h_A}^{\infty}\left(\frac{\partial u}{\partial r'}+iku\right)\frac{e^{ikr'}}{r'}dr', \qquad (36.5)$$

then, considering that the quantity $\partial u/\partial r'$ is bounded and that $\underline{u}$ vanishes on the substitution $r'=\infty$, introducing this result into Integral (36.4) and letting h_A go to zero, we obtain

$$\int u\frac{\partial}{\partial n}\frac{e^{ikr'}}{r'}dS=\int u\left(ik-\frac{1}{r'}\right)\frac{e^{ikr'}}{r'}\left(\frac{\partial r'}{\partial n}\right)_0 dS'-2\pi u. \qquad (36.6)$$

We may set $k \gg 1/r'$ in the remaining integral. Actually, the term $1/r'$ does not cause divergence as $r'\to 0$, since, according to Formula (36.3), as $r'\to 0$ we shall have $\left(\frac{\partial r'}{\partial n}\right)_0\sim-\frac{r'}{2a}$, $dS'=2\pi r'dr'$ and r' will cancel. At the same time, the first term in the parentheses in Integral (36.6) will contain kr'. Hence the main contribution will be made by precisely this term, which increases with r'. Keeping only this term, we introduce the value of (36.6) into Formula (34.9), substitute $\frac{\partial u}{\partial n}=-\frac{\partial u}{\partial r}$ with the aid of Formula (34.3) and obtain (it is also necessary to transfer $\frac{1}{2}u$ to the left member and multiply the entire equation by 2) for observation points on the surface of the sphere in the case of a vertical electric dipole

$$u=2u_0+\frac{ik}{2\pi}\int u\frac{e^{ikr'}}{r'}\left(\frac{1}{\sqrt{\varepsilon'}}-\frac{\partial r'}{\partial n}\right)dS'. \qquad (36.7)$$

We have dropped the subscript zero on $\partial r'/\partial n$, since it is now clear that r' connects two points on the sphere: the present point and the point of observation. For a magnetic vertical dipole $1/\sqrt{\varepsilon'}$ is replaced by $\sqrt{\varepsilon-1}$. Here the integral extends over the surface of the ground.

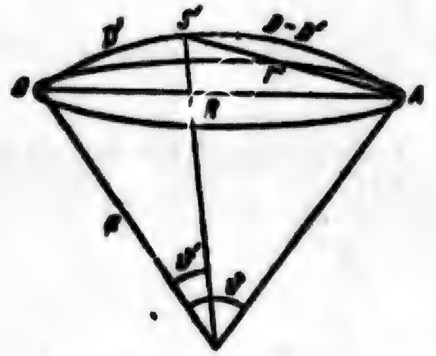

Fig. 36.2. Passage to a distance reckoned along the arc of a great circle.

The term u_0 expresses the field that the source in question would set up in the absence of the ground. Let us set (which agrees with (34.11) to within a factor 1/2)

$$u_0 = \frac{1}{2}\frac{e^{ikR}}{R}. \qquad (36.8)$$

It is essential that the distance R be reckoned along the straight line OA connecting the source and receiver points, i.e., passing in this case through the ground. The same applies for the distance r' between point A and the present point S' (Fig. 36.2).

In practice, however, the distance is reckoned along the surface of the earth, i.e., along an arc. The difference between these two distance measures is highly essential. We introduce the distance AO reckoned along the arc

$$D = a\vartheta, \qquad (36.9)$$

where ϑ is the central angle. For the present point S' on arc OA, we have

$$D' = a\vartheta'. \qquad (36.10)$$

Since

$$R = 2a\sin\frac{\vartheta}{2} \approx a\vartheta - \frac{1}{3}\frac{a\vartheta^3}{8} = D - \frac{D^3}{24a^2} \qquad (36.11)$$

and by analogy for any point on arc OA

$$r' \to r'_0 = 2a\sin\frac{\vartheta - \vartheta'}{2} \approx D - D' - \frac{(D-D')^3}{24a^2}, \qquad (36.11a)$$

where the second terms are always small for small ϑ, then, solving these equations approximately for D and D − D' we obtain

$$D \approx R + \frac{R^3}{24a^2}, \qquad (36.12)$$

$$D - D' \approx r'_0 + \frac{r'^3_0}{24a^2}. \qquad (36.13)$$

Further, according to Formula (36.3),

$$\frac{\partial r'}{\partial n} = -\frac{\psi - \psi'}{2} = -\frac{D - D'}{2a}. \qquad (36.14)$$

We introduce, as usual, the attenuation function (<u>with respect to distance reckoned along the arc</u>)

$$u = \frac{e^{ikD}}{D} w(D), \quad u(P') = \frac{e^{ikD_1}}{D_1} w(D_1). \qquad (36.15)$$

where D_1 is the distance along the arc of the great circle from the source to the present point. We substitute all these expressions into Eq. (36.7) for <u>u</u>, which then becomes the following equation for <u>w</u>:

$$w = \frac{e^{ik(R-D)}}{\frac{R}{D}} + \frac{ik}{2\pi} D \int \left(\frac{1}{\sqrt{\eta}} + \frac{D - D_1}{2a} \right) e^{ik(D_1 + r' - D)} \frac{w(D_1)\, dS'}{r' D_1}. \qquad (36.16)$$

For integration over the surface we introduce the following somewhat unconventional orthogonal coordinates. Considering the great circle of which the arc OA is a part as the equator, we draw latitude circles parallel to it and then a corresponding grid of meridians, which pass perpendicular to arc OA at each of its points. Then in a narrow region near arc OA (only this region is effective on integration), we obtain an almost rectilinear coordinate grid (D', y'), where y' measures the distance of the parallels from the "equator," the arc OA. In this coordinate system we shall have

$$dS' = dD'\, dy',$$

$$r' = \sqrt{r_0'^2 + y'^2} \approx r_0' + \frac{y'^2}{2r_0'} \approx D - D' - \frac{(D - D')^3}{24a^2} + \frac{y'^2}{2(D - D')}, \qquad (36.17)$$

$$D_1 \approx \sqrt{D'^2 + y'^2} \approx D' + \frac{y'^2}{2D'}.$$

Dropping the quantity y'^2 in the multipliers of the exponential functions and in the argument of $w(D_1)$, since it is small as compared with D'^2 and with $(D - D')^2$, and also dropping the quantity $D^2/24a^2$, as small by comparison with unity, we reduce Eq. (36.16) to the form

$$w(D) = e^{-i\alpha\frac{D^3}{24a^2}} + \frac{ikD}{2\pi}\int\left(\frac{1}{\sqrt{s^3}} + \frac{D-D'}{2a}\right)\frac{w(D')\,dy'\,dD'}{D'(D-D')}\, e^{-i\alpha\frac{(D-D')^3}{24a^2} + \frac{ikD}{2D'(D-D')}y'^2} \tag{36.16a}$$

Integrating as usual over y'^2, we obtain the final form of the equation:

$$w(D) = e^{-i\alpha\frac{D^3}{24a^2}}\left\{1 + i\sqrt{\frac{kD}{2\pi}}\int_0^D \frac{w(D')\,e^{i\frac{k}{24a^2}[D^3-(D-D')^3]}}{\sqrt{D'(D-D')}} \times \left(1 + \sqrt{s^3}\,\frac{D-D'}{i2a}\right)dD'\right\}, \tag{36.18}$$

where $\underline{s}$ has its usual significance: $s = (ik/2a^2)$.

This equation differs from the equation for a flat berth (24.12) not only in the substitution of distance reckoned along an arc for horizontal distance, but also in two multipliers under the integral sign: one of them - the exponential - is purely geometrical in nature. It indicates a region of integration over D' that is effective in a certain sense, as it is given by the ellipsoid corresponding to the first zone. It is shown in Fig. 36.2 for the case of large $p_2 = (kD^3/8a^2)$, when the ellipsoid is extremely prolate. It will be seen from the drawing, there are two "effective" regions: at point O and at point A,

$$D^3-(D-D')^3 < \frac{24a^2}{k}.$$

However, the situation is far from being so simple as in the case of a flat earth. On the one hand, the influence of the segment around the observation point is suppressed by the presence of the multiplier w(D'). On a flat earth, it varies in the least favorable case only in inverse proportion to distance. Here, however, if the point A is far into the umbra, it is exponentially small. On the other hand, especially if $|s^3|$ is large, the multiplier D - D' stresses the role of segments remote from the source. Hence separation of these "effective" regions is far from always of decisive significance.

In the case of small p_2, this ellipsoid covers the entire arc OA.

Then the exponential multiplier is near unity for all D' and curvature can make itself felt only by virtue of the second multiplier in the integrand: $1+\sqrt{\varepsilon'}\,\frac{D-D'}{2a}$. This factor is particularly essential for relatively large $|\varepsilon'|$. In the limit, therefore, for an infinitely conductive earth, we obtain, setting D = Dξ,

$$w(D) = e^{-ik\frac{D^3}{24a^2}}\left\{1+i\sqrt{\frac{kD}{8\pi a^2}}\int_0^1 w(D\xi)\sqrt{\frac{1-\xi}{\xi}}\,e^{i\frac{kD^3}{24a^2}[1-(1-\xi)^3]}d\xi\right\}. \quad (36.19)$$

From this it is clear that in the case of an ideally conducting earth, when all of the attenuation results solely from geometrical distortion of the Fresnel zones, the attenuation function will depend on distance only in the combination $p_2 = (kD^3/8a^2)$ and will tend to unity when p_2 tends to zero.

Equation (36.19) can be solved by expansion of w in series in powers of the square root of p_2, this series converging well for not very large p_2, $p_2 < 2$. In the case of a poorly conducting earth, it may be found that w(D') under the integral diminishes with distance more rapidly than the oscillations of the exponential factor and that the multiplier w(D') makes itself felt even within the first Fresnel zone. For this to happen, as we know (see §34), the parameter

$$\delta = i\frac{\sqrt{\varepsilon'}}{\sqrt[3]{ka}} \quad (36.20)$$

must be small.

This case is of particular interest for ultrashort waves.

In deriving Eqs. (36.18) and (36.19), we have not imposed any limitations on the size of the distance D (apart from smallness by comparison with a) and, in this sense, the equations obtained are exact (in the wave zone). Now we shall pass to examination of the question as to how exactly the formulas derived for a flat earth are valid. For this purpose, we shall consider the geometrical distortions to be small, setting accordingly

$$p_2 = \frac{kD^3}{8a^3} \ll 1. \qquad (36.21)$$

The difference of the exponential multipliers in Eq. (36.18) from unity can produce errors only of the order of p_2. On the other hand, the term $\sqrt{s^0}\,\frac{D-D'}{2a}$, as we shall see below, gives a larger correction. Hence we may substitute unity for e^{ip_2} and examine the remaining simpler equation

$$w(D) = 1 + i\sqrt{\frac{sD}{\pi}}\int_0^D w(D')\,\frac{1+\alpha(D-D')}{\sqrt{D'(D-D')}}\,dD', \quad \alpha = \sqrt{s^0}\,\frac{1}{2a}. \qquad (36.22)$$

We divide the entire equation by $\sqrt{D}$ and apply the Laplace transformation to it, i.e., we multiply the equation by e^{-pD} and integrate over D from 0 to ∞. In the right member, the integrations over D' and over D are interchangeable. Then, applying

$$\int_0^\infty e^{-p\xi}\,\frac{d\xi}{\sqrt{\xi}} = \sqrt{\frac{\pi}{p}}, \qquad (36.23)$$

we obtain for the Laplace-transformed function

$$\Omega(p) = \int_0^\infty \frac{w(D)}{\sqrt{D}}\,e^{-pD}\,dD \qquad (36.24)$$

The simple algebraic equation

$$\Omega(p) = \sqrt{\frac{\pi}{p}} + i\sqrt{\frac{s}{\pi}}\left\{\sqrt{\frac{\pi}{p}} + \frac{1}{2}\,\frac{\sqrt{\pi}}{p^{3/2}}\,\alpha\right\}\Omega(p). \qquad (36.25)$$

From this,

$$\Omega(p) = \frac{\sqrt{\pi p}}{p^{3/2} - i\sqrt{sp} - \frac{i}{2}\,\alpha\sqrt{s}}. \qquad (36.26)$$

The "primorial" function of interest to us is obtained, as we know, after the inverse transformation

$$\frac{w(D)}{\sqrt{D}} = \frac{1}{2\pi i}\int_{\gamma'-i\infty}^{\gamma'+i\infty} \Omega(p)\,e^{pD}\,dp, \quad \gamma' > 0, \qquad (36.27)$$

where integration is performed over a straight line passing along the axis of imaginaries at a small distance γ' from it.

We shall evaluate this integral only for large numerical distances ($|sD| \gg 1$) and, moreover, considering the smallness of αD. That is to say, setting $pD = p'$, $sD = \rho$ and $\alpha D = \alpha'$ and dropping the prime, we obtain

$$w(D) = \frac{\sqrt{\pi}}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{p e^p dp}{p^{3/2} - i\sqrt{\rho}\left(p + \frac{\alpha}{2}\right)}. \tag{36.28}$$

For $\alpha = 0$, we obtain the attenuation function for a flat earth $y(sD)$ (25.17) (applying Formula (2.9) from the handbook [8] or integrating independently).

We expand the integrand in powers of $\frac{1}{\sqrt{\rho}}$:

$$w(D) = \frac{\sqrt{\pi}}{2\pi i} \frac{i}{\sqrt{\rho}} \left\{ \int \frac{p e^p dp}{p + \frac{\alpha}{2}} - \frac{1}{\sqrt{\rho}} \int \frac{i p^{5/2} e^p dp}{\left(p + \frac{\alpha}{2}\right)^2} + \dots \right\} = \tag{36.29}$$

$$= \frac{1}{2\pi} \sqrt{\frac{\pi}{\rho}} \left\{ I_1 - \frac{i}{\sqrt{\rho}} I_2 + \dots \right\}.$$

The first integral is simply defined if we close the contour on the left with a semicircle of infinite radius, and is expressed in terms of a residue at the point $p = -\frac{\alpha}{2}$:

$$I_1 = 2\pi i \left(-\frac{\alpha}{2}\right) e^{-\frac{\alpha}{2}} \approx -i\pi\alpha. \tag{36.30}$$

The second integral contains the branch point $p = 0$. However, since we are interested only in the result for small α, we may expand I_2 in powers of $\alpha/2$:

$$I_2 = I_2(0) + \left(\frac{dI_2}{d\frac{\alpha}{2}}\right)_{\alpha=0} \frac{\alpha}{2} + \dots \tag{36.31}$$

and limit ourselves to the first term, which is independent of α (since the term in (36.30) is manifested more strongly in the expansion of (36.29))

$$I_2(0) = \int_{\gamma-i\infty}^{\gamma+i\infty} \sqrt{p}\, e^p\, dp. \tag{36.32}$$

It is defined by the substitution $p = i\xi^2$:

$$I_2 = -i\sqrt{\pi}. \tag{36.33}$$

Consequently,

$$w(D) \approx -\frac{1}{2\rho}\left(1 + i\alpha\sqrt{\pi\rho}\right), \tag{36.34}$$

where we have dropped terms that have the relative principal order $1/\rho$, the order α and the order p_2.

Substituting the values of α and ρ, we obtain

$$\left.\begin{aligned} w(D) &\approx -\frac{1}{2\rho}\left(1 + i\sqrt{i\pi p_2}\right), \\ |w(D)| &\approx \frac{1}{2|\rho|}\left(1 - \sqrt{\frac{\pi p_2}{2}}\right). \end{aligned}\right\} \tag{36.35}$$

Thus, it is found that the correction appears in the term of order $\sqrt{p_2}$.

In summary, we may state that the normal attenuation function calculated for a flat earth, $y(\rho)$, remains correct for the earth's surface with the single restriction:

$$\sqrt{\frac{\pi p_2}{2}} = \sqrt{\frac{\pi k D^3}{4a^2}} \ll 1. \tag{36.36}$$

If we can satisfy ourselves with an error of, let us say, 10%, this means that the formula is valid up to a distance

$$D < \sqrt[3]{\frac{4a^2}{100\pi k}} \approx (7\lambda_m^{1/3})\ \text{km}. \tag{36.37}$$

The distance at which $p_2 = 1$ may be written as follows (compare (34.18a)):

$$D \approx (37\,\lambda_m^{1/3})\ \text{km}. \tag{36.38}$$

It is assumed in (36.37) and (36.38) that the wavelength λ is expressed in meters.

§37. FIELD NEAR THE HORIZON PLANE AT GREAT DISTANCES

For short wave and ultrashort wave practice, the case in which the distance from the source is great, i.e., $\rho_2 = \frac{kr^2}{2a^2} \gg 1$ is highly important (this means that $x_{km} > 37\lambda_m^{1/3}$), but the observation point is near the

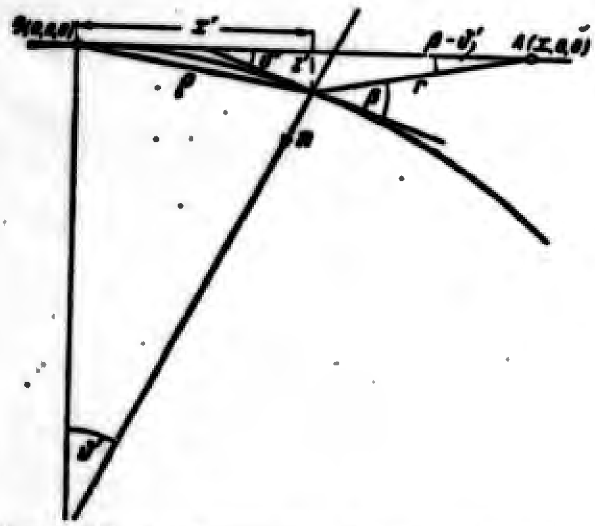

Fig. 37.1. Symbols to derivation of Eq. (37.3).

plane of the horizon, in the region of the penumbra, where the field attenuation due to the curvature of the earth is still not very great. In this section, we shall content ourselves with derivation of the attenuation function for this case. Moreover, with special interest in the short-wave region we shall assume that $|\delta| \doteq \left|\frac{\sqrt{\varepsilon^0}}{\sqrt[3]{ka}}\right| \ll 1$ (see Formulas (34.21) and (34.25)).

Let the source be on the surface of the earth. We may again use Eq. (34.9) for $\underline{u}$ and u_0 given by Formulas (34.10) and (34.11), where, since the height of the observation point above the ground surface is always very great (as compared with the wavelength) under the conditions now of interest to us, the term $1/ik_0r'$ can be disregarded. Consequently, the equation takes the form (for the symbols, see Fig. 37.1)

$$w = \frac{1}{2} + \frac{ik}{4\pi} R \int w(\rho)\, e^{ik(r+\rho-R)} \frac{dS}{r\rho}\left(\frac{1}{\sqrt{\varepsilon^0}} - \frac{\partial r}{\partial n}\right). \qquad (37.1)$$

Using rectangular coordinates with the (x, y) plane tangent to the sphere at the location of the source, and setting $z' \approx -\frac{x'^2}{2a}$ for the present point, we obtain in approximation

$$r+\rho-R=\sqrt{(x-x')^2+y'^2+(z-z')^2}+\sqrt{x'^2+y'^2+z'^2}-$$
$$-\sqrt{x^2+z^2}\approx\frac{1}{2}y'^2\frac{x}{x'(x-x')}+\frac{z^2}{2x}\frac{x'}{x-x'}+z\frac{x'^2}{2a(x-x')}+\frac{xx'^3}{8a^2(x-x')}. \quad (37.2)$$

Further, the factor ∂r/∂n before the exponential function can be determined in approximation, e.g., for a point lying in the plane of the horizon, z = 0. In this case, it follows from Fig. 37.1 that

$$(x-x')\,\mathrm{tg}(\beta-\vartheta')=-z'=\frac{x'^2}{2a},$$

or, replacing the sine and tangent by the argument,

$$\frac{\partial r}{\partial n}\approx\beta\approx\vartheta'+\frac{x'^2}{2a(x-x')}\approx\frac{x'}{a}\left(1+\frac{x'}{2(x-x')}\right). \quad (37.2a)$$

After the usual integration over y', we obtain

$$w=\frac{1}{2}+\frac{i}{2}\sqrt{\frac{kx}{\pi}}\int\limits_0^x\frac{dx'\,w(x')}{\sqrt{x'(x-x')}}\times$$
$$\times\left(1-\frac{x'}{a}\sqrt{\varepsilon^0}-\frac{\sqrt{\varepsilon^0}x'^2}{2a(x-x')}\right)e^{i\left\{p_1\frac{x'^3}{x^2(x-x')}+\frac{kzx'^2}{2a(x-x')}+\frac{kz^2x'}{2x(x-x')}\right\}}. \quad (37.3)$$

This equation is as yet quite exact (here we have assumed only that z << x and that other similar quite easy conditions are satisfied).

If we now assume that the distance from the source is very great,

$$p_1=\frac{kx^3}{8a^2}\gg 1, \quad (37.4)$$

the exponential factor containing this parameter will effectively limit integration to the range of very small x', specifically those such that

$$\frac{kx'^3}{8a^2}\lesssim 1, \quad (37.4a)$$

i.e.,

$$\frac{x'}{x}<\frac{1}{\sqrt[3]{p_1}}\ll 1. \quad (37.4b)$$

where, in virtue of the very frequent oscillations as x' increases above this value, the importance of the larger x' drops sharply. It is easily seen that this speed of attenuation in regions of x' remote from the source is due to the fact that the earth's surface, dropping away from the plane of the horizon, very suddenly leaves the Fresnel zone constructed in the vertical plane. Hence x' that are small by comparison with x may be dropped everywhere in the expressions x - x'. Moreover, it follows from Condition (37.4b) that the second (and, even more so, the third) terms in the parentheses under the integral sign in Eq. (37.3), which differ from the first by a factor

$$\frac{x'}{a}\sqrt{\varepsilon'} < \frac{x}{a}\frac{\sqrt{\varepsilon'}}{s\sqrt{p_1}} = \frac{2\sqrt{\varepsilon'}}{s\sqrt{ka}},$$

are small by comparison with the first, so that the first term can be left standing alone.

Finally, and this is the most important, Condition (37.4a) indicates that according to Formula (36.35), the curvature does not yet make itself felt in the range of integration. In the integrand, therefore, w(') may be replaced by its value as known for a flat earth:

$$w(x') \approx g(sx'). \qquad (37.5)$$

Thus, the calculation of the attenuation function is reduced to a quadrature. We shall limit ourselves at first to observation points situated strictly on the plane of the horizon, z = 0. Then

$$w = \frac{1}{2} + \frac{i}{2}\sqrt{\frac{s}{\pi}}\int_0^{x}\frac{g(sx')}{\sqrt{x'}}\,e^{im\frac{x'^3}{x^2}}\,dx'. \qquad (37.6)$$

Substituting the value from (25.18a), we may readily satisfy ourselves of the existence of the identity

$$\sqrt{s}\,\frac{g(sx)}{\sqrt{x}} \equiv \frac{d}{dx}\frac{1-g(sx)}{\sqrt{sx}}. \qquad (37.7)$$

Actually, the right member is equal to

$$\frac{d}{dx}2e^{-sx}\int\limits_{\infty}^{\sqrt{sx}} e^{v^2}dv = -2se^{-sx}\int\limits_{\infty}^{\sqrt{sx}} e^{v^2}dv + \frac{\sqrt{s}}{\sqrt{x}} =$$

$$= -s\frac{1-y(sx)}{\sqrt{sx}} + \sqrt{\frac{s}{x}} - \sqrt{\frac{s}{x}}\,y(sx).$$

Hence, integrating by parts in (37.6), we obtain

$$w = \frac{1}{2} + \frac{i}{2\sqrt{\pi}}\left\{\left[\frac{1-y(sx')}{\sqrt{sx'}}e^{ip_2\frac{x'^3}{x^3}}\right]_0^x - \frac{3i}{x^3}p_2\int\frac{1-y(sx')}{\sqrt{sx'}}x'^2e^{ip_2\frac{x'^3}{x^3}}dx'\right\}. \quad (37.6a)$$

On substitution of the limits, we take into account that (see Formula (25.1)) near x' = 0

$$y(sx') \approx 1 + i\sqrt{\pi s x'},$$

and extend the remaining integral to infinity. Now in the integrand we may drop the quantity y(sx'), which is small by comparison with unity (since now the substitution of y(sx') by –1/2sx' does not result in divergences for x' = 0). Following this with another substitution of variable $p_2\frac{x'^3}{x^3} = \xi^3$ and disregarding terms of the order $1/\sqrt{sx}$, in the result, we obtain

$$u = \frac{e^{ikR}}{R}w_0, \quad w_0 = \frac{3C}{2\sqrt{\pi}}\frac{\sqrt{\varepsilon'}}{\sqrt[3]{p_2}}e^{-i\frac{\pi}{4}}, \quad (37.8a)$$

where, setting $i\xi^3 = t$, we have

$$C = \int\limits_0^\infty \xi^{3/2}e^{i\xi^3}d\xi = \frac{1}{3}e^{i\frac{5\pi}{12}}\Gamma\left(\frac{5}{6}\right) \approx 0{,}38\,e^{i\frac{5\pi}{12}}. \quad (37.8b)$$

C is a constant that depends neither on the properties of the soil nor on wavelength.

Using the nomenclature of (34.21), we might also write

$$w_0 = \frac{3C}{2\sqrt{\pi}}e^{-i\frac{3\pi}{4}}\delta = \frac{0{,}57}{\sqrt{\pi}}e^{-i\frac{\pi}{3}}\delta. \quad (37.8c)$$

Thus, we have arrived at an important result: if we move along the plane of the horizon, then, beginning at distances for which $p_2 = kx^3/8a^2 > 1$, i.e., for practical purposes with $x_{km} > 37\lambda_m^{1/3}$, the attenuation function does not depend on distance at all, the field strength diminishes as in free space (as $1/x$), but is weakened substantially by the factor $\sim\delta$, whose absolute value is assumed small by the conditions of the derivation. In actuality, as is shown by comparison with the results of the more complete theory (see below, Figs. 39.7a-39.8b), this relationship is established already at distances that are only fractionally as large. This is explained by the following. We required the condition $p_2 \gg 1$ only in order to isolate a region $x' \ll x$ as the most essential in the integral of (37.3). But this region is essentially separated by the multiplier $w(x') \approx y(sx')$, itself, which makes itself felt earlier for $|\delta| \ll 1$ than does the exponential factor.

To obtain the field near the plane of the horizon, we may expand the right member of Eq. (37.3) in series in $\underline{z}$. That is to say (with the same simplifications as in (37.6)),

$$\left(\frac{\partial w}{\partial z}\right)_{z=0} = \frac{i}{2}\sqrt{\frac{s}{\pi}}\,\frac{ik}{2\alpha x}\int y(sx')\,x'^{3/2}\,e^{i\alpha\frac{x'^3}{x^3}}\,dx', \tag{37.9a}$$

$$\left(\frac{\partial^2 w}{\partial z^2}\right)_{z=0} = \frac{i}{2}\sqrt{\frac{s}{\pi}}\int y(sx')\left\{-\frac{k^2}{4\alpha^2 x^2}x'^{\frac{7}{2}} + i\frac{k}{x}\sqrt{x'}\right\}\cdot e^{i\alpha\frac{x'^3}{x^3}}\,dx', \tag{37.9b}$$

etc.

In these integrals, we may at once set $y(sx') = -\frac{1}{2sx'}$, which gives

$$\left(\frac{\partial w}{\partial z}\right)_{z=0} = \frac{\sqrt{\varepsilon^3}}{4x}, \tag{37.10a}$$

$$\left(\frac{\partial^2 w}{\partial z^2}\right)_{z=0} = e^{-i\frac{\pi}{4}}\frac{1}{2}\sqrt{\frac{\varepsilon^3}{\pi}}\,\frac{\sqrt[3]{k\alpha}}{x^2}(2iC_1 + C_2), \tag{37.10b}$$

where, setting $i\xi^3 = t$, we have

$$C_1 = \int_0^\infty \xi^{5/2} e^{i\xi^3}\,d\xi = \frac{1}{3}\Gamma\left(\frac{7}{6}\right)e^{i\frac{7\pi}{12}},$$

$$C_2 = \int_0^\infty \frac{1}{\sqrt{\xi}}e^{i\xi^3}\,d\xi = 2\Gamma\left(\frac{7}{6}\right)e^{i\frac{\pi}{12}}.$$

$$2iC_1 + C_2 \approx 1{,}24\, e^{i\frac{\pi}{18}} \tag{37.10c}$$

– constants with absolute values of the order of unity.

Thus, we may write for the attenuation function near the plane of the horizon z = 0

$$w \approx w_0 \left\{ 1 + 0{,}52\, e^{-i\frac{\pi}{6}} \sqrt[3]{ka}\,\frac{z}{x} + 1{,}09\, e^{-i\frac{\pi}{3}} \left(\sqrt[3]{ka}\,\frac{z}{x} \right)^2 + \ldots \right\}. \tag{37.11}$$

Consequently, the attenuation function is expanded in powers of the ratio $\frac{z}{x}\sqrt[3]{ka} \approx \sqrt[3]{ka}\sin\psi$, where ψ is the elevation angle of the observation point and depends on the coordinates x and z only in this combination. This approximate expansion can be used as long as the parameter of the expansion is small. It will be recalled that the reflection formulas apply for large values of this parameter (see §35 and Condition (34.17)). Thus, the results obtained in the present paragraph represent a supplement to the reflection formulas.

In the more general case, $\sqrt[3]{k_0 a}\sin\psi \sim 1$, we may write

$$w = \frac{1}{2\sqrt{\pi}} \int_0^\infty \frac{1}{\sqrt{x'^3}} \left\{ \frac{3\rho_0 x'^2}{x^3} + \frac{kzx'}{ax} + \frac{kz^2}{2x^2} \right\} e^{i\left\{ \rho_0 \frac{x'^3}{x^3} + \frac{kx'^2}{2ax} z + \frac{kx'}{2x^2} z^2 \right\}} \tag{37.12}$$

in Eq. (37.3), as in the transition to Eq. (37.6a), after dropping the small terms and integrating by parts.

This integral can, of course, be evaluated numerically for arbitrary relationships among k, z and x. However, for very large negative ψ (deep in the shadow), the method becomes unusable inasmuch as the effective region in the integral increases as we move deeper into the shadow and x' comes to be of the order of x.

Now we can also analyze the case converse to that considered, i.e., that in which the source is elevated so high that its radiation is closely approximated by a plane wave, while the observation point is situated on the surface. In the illuminated region remote from the boundary of the shadow, this field is, of course, described by the interference (reflection) formulas. Only the problem of the field near the shadow boundary is complex. It is this problem that we shall consider now. As before, we shall assume that $|\delta| \ll 1$.

Let a plane wave arrive in the direction tangential to the earth's sphere at point B (Fig. 37.2). According to Formulas (3.32), which link the Debye function u with the radial component of the electric field E_r, the function u is equal to E_r to within a constant factor k^2r since it is definite that $\frac{\partial u}{\partial r} \ll ku$. Hence the reciprocity theorem may be applied to u. It can be affirmed that, as seen at point B, the field of a radially directed electric dipole situated at point O and emitting the plane wave that we are studying is equal to the field at point O that would be created by the same dipole situated at B. This, however, we know; it is given by Formula (37.8) or (37.11) for z = 0:

$$u = \frac{e^{ikR}}{R} w \to Ae^{ikx} w_0, \quad w_0 = \frac{0{,}57}{\sqrt{\pi}} e^{i\frac{\pi}{3}} \delta. \qquad (37.13)$$

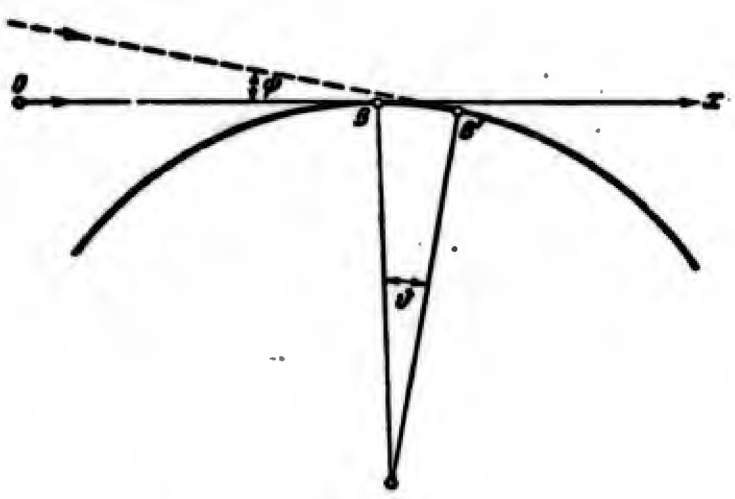

Fig. 37.2. Illustrating calculation of field near the plane of the horizon in the case of incidence of a plane wave.

Here the dipole field has been replaced by the plane wave and the coordinate x is reckoned along the line OB. Thus, even at the point at which the geometric boundary of the shadow is situated, the field is substantially reduced, the more so the smaller $|\varepsilon'|$ and λ. Let us now consider an observation point B' that is shifted away from B. According to the same reciprocity theorem, the field at this point is equal to the field that would be created at point O by a dipole situated at this point B'. But we have calculated this field before (for small angles $\psi = 0$). Here we are speaking of negative z, and hence, substituting $-\vartheta$ for ψ, we have, according to Formula (37.11),

$$w(B') \approx w_0 \left(1 - 0{,}52\, e^{-i\frac{\pi}{6}} \sqrt[3]{\kappa a} \sin\vartheta + 1{,}09\, e^{-i\frac{\pi}{3}} (\sqrt[3]{\kappa a} \sin\vartheta)^2 + \ldots\right). \quad (37.14)$$

This expansion is applicable as long as $\sqrt[3]{\kappa a} \sin\vartheta < 1$, i.e., in the exact region in which the reflection formulas are incorrect.

Needless to say, this formula is also valid for points B' to the left of point B, but here it is necessary to set $\vartheta < 0$. We have seen that the deviation of the attenuation function from its valud at point B is small as long as

$$\frac{1}{2} \sqrt[3]{\kappa a} \sin\vartheta \sim \sqrt[3]{\frac{kD^3}{8a^2}} \sim \sqrt[3]{p_s} \ll 1, \quad (37.15)$$

where D = aϑ is the distance to the earth's surface from the boundary of the shadow. This length is a measure of that distance on the earth's surface $D \sim (37 \lambda_m^{\frac{1}{3}})$ km spanning the penumbra, the transition from the illuminated region far to the left of point B to the full umbra far to the right of B.

§38. COMPLETE SOLUTION BY THE PARABOLIC-EQUATION METHOD

Now that we have obtained solutions for certain limiting cases and identified the fundamental parameters of our problem, let us turn to an exposition of the complete solution. We shall follow [6], retaining the nomenclature used there.

The analysis is based on the assumption that it is permissible to use boundary condition (34.6). We shall assume that the source, which is a vertical dipole, is situated on the surface of the earth. Instead of solving the wave equation for the function u, we shall pass at once to the differential equation for the attenuation function W (compare Formula (36.8) and the remark accompanying it):

$$u = \frac{e^{ikR}}{R} W. \qquad (38.1)$$

Here R is the straight-line distance from the dipole to the observation point. We draw attention to the fact that in this notation, the function W for a flat ideally conductive earth is equal to 2 and not 1, which we used earlier (W = 2w).

The equation obtained on substitution of Eq. (38.1) into the wave equation takes the form

$$\nabla^2 W + 2\left(ik - \frac{1}{R}\right)\frac{(R, \nabla W)}{R} = 0. \qquad (38.2)$$

It can be simplified substantially if we take account of certain peculiarities in the behavior of the function W.

Investigation of the flat-earth case showed that the attenuation function varies along the vertical at its surface much more rapidly than it does along the horizontal. Indeed, the derivative of w is of the order of sw' along the horizontal if we are at small numerical distances, and of $\frac{1}{x}w$ if we are at great distances (where $w \approx -\frac{1}{2sx}$). At the same time, we have for the derivative with respect to the vertical $\left|\frac{\partial w}{\partial z}\right| \approx \left|\frac{k}{\sqrt{\varepsilon^0}} w\right|$. Thus, the ratio of the derivatives with respect to the horizontal and vertical is of the order of $\frac{1}{2\sqrt{\varepsilon^0}}$ at short distances $(w' \sim w)$ and of the order of $\frac{1}{2\sqrt{\varepsilon^0}}\frac{1}{sx}$ at long distances. In the space where the reflection formulas apply, the variation of w is manifest on variation of $\sqrt{\varepsilon^0}\sin\psi$ by an amount of the order of unity, i.e., for

$\Delta z \sim \frac{1}{\sqrt{\varepsilon^0}}\Delta r$, from which again $\left|\frac{\partial w}{\partial r}\right| \sim \left|\frac{1}{\sqrt{\varepsilon^0}}\frac{\partial w}{\partial z}\right|$.

The secondary distortion that appears when the sphericity of the earth is very strongly in evidence has similar properties. Near the plane of the horizon at great distances from the source, where $p_2 \gg 1$ (see (37.11)), w changes significantly as $\frac{2}{\sqrt[3]{ka_3}}\sin\psi$ increases by an amount of the order of unity, i.e., when $\Delta r \sim \sqrt{ka}\,\Delta z \gg \Delta z$. Consequently, $\left|\frac{\partial w}{\partial r}\right| \sim \frac{1}{\sqrt[3]{ka}}\left|\frac{\partial w}{\partial z}\right|$.

Thus, the attenuation function always changes more slowly along the horizontal than along the vertical. This may be written as follows.

Instead of the coordinates r, ϑ, we introduce the dimensionless coordinates

$$x = aN\vartheta, \quad y = (r - a)\cdot 2MN = h\cdot 2MN, \qquad (38.3)$$

and define the scale factors M and N accordingly. The attenuation function will now be written as W(x, y) and the condition that the variation of W take place more slowly along the horizontal than along the vertical,

$$\left|\frac{\partial W}{\partial (a\vartheta)}\right| \ll \left|\frac{\partial W}{\partial h}\right|, \qquad (38.4a)$$

assumes the form

$$N\left|\frac{\partial W}{\partial x}\right| \ll 2MN\left|\frac{\partial W}{\partial y}\right|, \qquad (38.4b)$$

i.e., if we regard $\partial W/\partial x$ and $\partial W/\partial y$ as quantities of the order of W, we must have

$$2M \gg 1. \qquad (38.4)$$

Thus M has the significance of the ratio of the derivatives of W with respect to the vertical and horizontal. Hence in the case of short distances, where the curvature is not a factor, M may be taken of the order of $\sqrt{\varepsilon^0}$. For $p_2 \gg 1$ near the plane of the horizon $M \sim \sqrt[3]{ka}$. The quantity N remains perfectly arbitrary. For convenience, we shall set it equal to N = M/a.

Now we can transform the equation for W. Passing to the new variables, we expand R in series in powers of ϑ and $\underline{h}$, limiting ourselves to the terms written below:

$$R \approx \frac{ax}{M}\left\{1+\frac{1}{4M^2}\left(y+\frac{y^2}{2x^2}-\frac{x^2}{6}\right)\right\}. \qquad (38.5)$$

Here Eq. (38.2) is expanded in powers of 1/M. Limiting it to the lowest powers, we obtain the following equation (terms of the order of $1/M^2$ have been dropped):

$$\frac{\partial^2 W}{\partial y^2}+\frac{ika}{2M}\left\{\left(x+\frac{y}{x}\right)\frac{\partial W}{\partial y}+\frac{\partial W}{\partial x}\right\}=0. \qquad (38.6)$$

The sense of the above approximation is that the second derivative with respect to $\underline{x}$, which is already quite small, has been dropped from the equation, so that we must solve an equation of the parabolic type instead of an equation of the elliptical type. This transformation, which is connected to the transition to the attenuation function, was first made by Leontovich for a flat earth, and the entire method has come to be known as the method of the parabolic equation.

Now we can select the length scale M. The choice still remains arbitrary to a considerable degree. All that is necessary is that M >> 1. However, it is expedient to select it such that the dimensional quantities will drop out of Eq. (38.6). This can be achieved by putting

$$M=\sqrt[3]{\frac{ka}{2}}. \qquad (38.7)$$

In this case, the variables $\underline{x}$ and $\underline{y}$ have the following sense (see Fig. 38.1):

$$x=M\vartheta=M\frac{D}{a}=\sqrt[3]{\frac{ka}{2}}\,\vartheta=\sqrt[3]{4\rho_2}, \qquad (38.8)$$

$$y=\frac{h}{a}\cdot 2\sqrt[3]{\frac{k^2a^2}{4}}=\sqrt[3]{2(ka)^2}\,\frac{h}{a}=\sqrt[3]{2}\,\rho_1^3\frac{h}{a}. \qquad (38.9)$$

We see that the length scales introduced here are simply related to the characteristic parameters p_1 and p_2 that we introduced in the earlier sections.

The convenience of the new units is obvious, in particular, from the fact that the equation of the horizon plane

$$r \approx a + \frac{D^2}{2a} = a + \frac{a\theta^2}{2} \qquad (38.10)$$

will be written as follows in these units:

$$y = x^2. \qquad (38.10a)$$

To agree with the equation, the boundary condition (34.6) must also be written in the new variables. Since $\frac{\partial}{\partial r} = \frac{2M^2}{a}\frac{\partial}{\partial y}$, this condition assumes the form

$$\frac{\partial}{\partial y}\left(W\frac{e^{ikR}}{R}\right) = -\frac{ik}{\sqrt{\eta}}\frac{a}{2M^2}W\frac{e^{ikR}}{R} = -\frac{iM}{\sqrt{\eta}}We^{ikR}, \qquad (38.11)$$

or, dropping terms of the order of $1/kR$ (in differentiation it is necessary to use Formula (38.5)),

$$\frac{\partial W}{\partial y} = -\left(q + \frac{ix}{2}\right)W. \qquad (38.12)$$

Here q denotes the quantity $i\frac{M}{\sqrt{\eta}}$, which we have already encountered. That is to say, according to Formulas (34.21) and (38.7),

$$i\frac{M}{\sqrt{\eta}} = i\sqrt[3]{\frac{ka}{2}}\frac{1}{\sqrt{\eta}} = -\frac{1}{\sqrt[3]{2\delta}} = q. \qquad (38.12a)$$

This quantity is small in absolute value for long waves and soils that conduct well. As we know, its absolute value indicates which attenuation asserts itself first – that due to the earth's curvature or that resulting from the nonideal nature of conduction.

Finally, to be explicit in setting up the problem in the case of the flat earth, we have taken into consideration the behavior of the attenuation function at the source. Here this would mean that it is necessary to assign a behavior to the attenuation function at the point

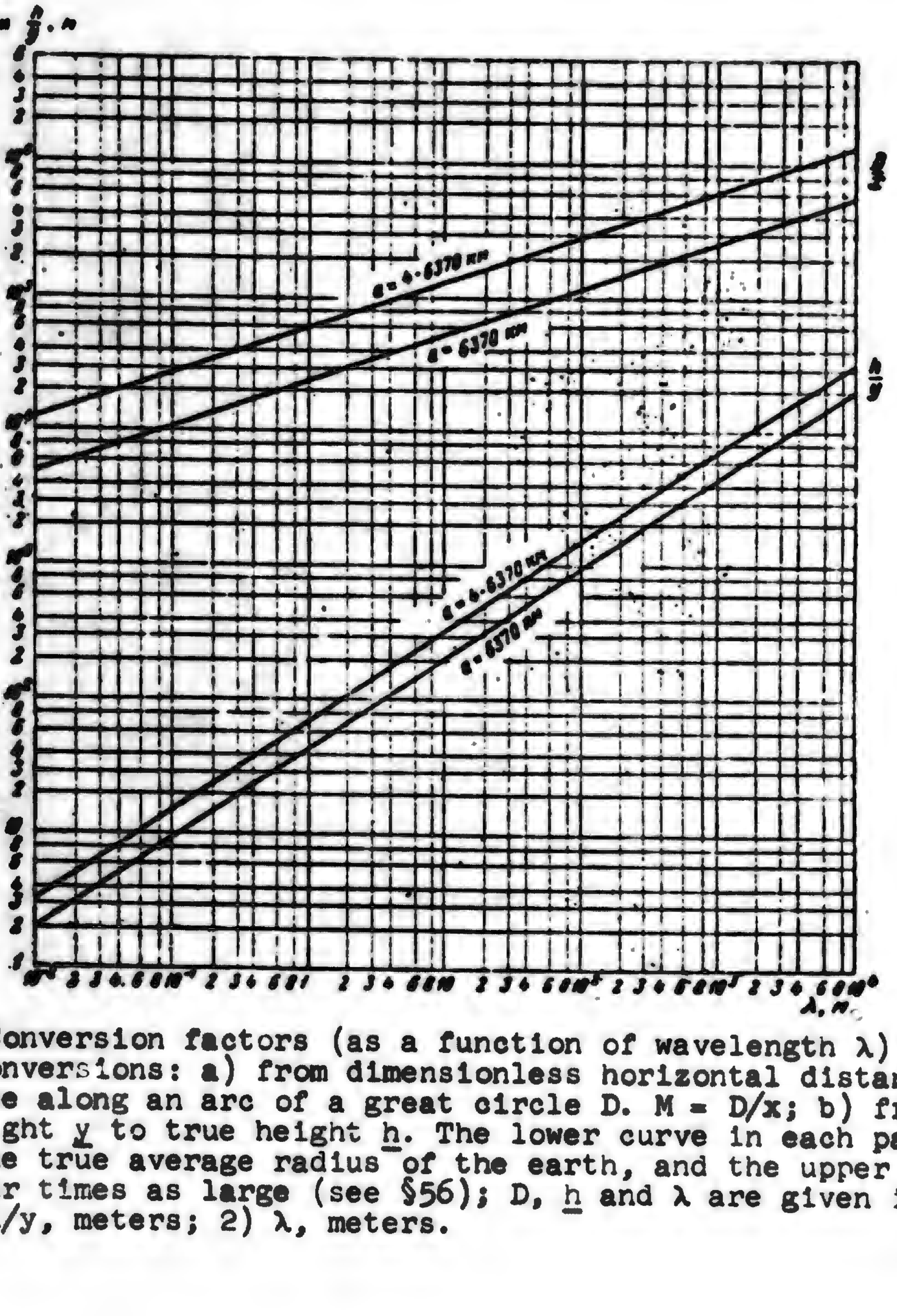

Fig. 38.1. Conversion factors (as a function of wavelength λ) for the following conversions: a) from dimensionless horizontal distance x to true distance along an arc of a great circle D. $M = D/x$; b) from dimensionless height y to true height h. The lower curve in each pair applies for the true average radius of the earth, and the upper curve to a radius four times as large (see §56); D, h and λ are given in meters. 1) D/x and h/y, meters; 2) λ, meters.

x = y = 0 at which the source is situated, or to take into consideration that at short enough distances the field must be the same as that above a flat earth, i.e., according to Formula (25.1),

$$\lim_{D\to 0}\frac{W-2}{\sqrt{D}} = 2i\sqrt{\pi s}\ \text{for}\ y=0. \tag{38.13}$$

In the present problem, however, this is not enough. Unlike the wave equation, the parabolic equation requires that the condition be assigned not at a point but, for example, over the entire axial line x = 0, i.e., for all y (in much the same way as it is necessary in the case of the thermal conduction equation to assign a temperature distribution over the entire space at the initial point in time). But at x = 0 and arbitrary heights y, the earth's curvature may be disregarded. We shall be at all times in the region of applicability of the reflection formulas. In this case, approaching infinitesimally close to this line, we may use Formula (26.20d), setting $\lim_{D\to 0} W(D) = 1+f$. A more detailed analysis indicates [6] that even this condition is inadequate for uniqueness of the solution. Uniqueness is obtained only if we impose the more rigid condition

$$\lim\frac{W-(1+f)}{\sqrt{D}} = 0,\quad y>0. \tag{38.14}$$

Here f is everywhere near unity and we may assume that 1 + f = 2. If this substitution is not made, then b, as determined by Formula (38.36) and, consequently, the entire solution (38.33) are simply multiplied by a factor near unity.

Consequently, the limit condition must be written in the form

$$\lim_{D\to 0}\frac{W-2}{\sqrt{D}} = 0\ \text{for}\ \sin\psi \gg \frac{1}{|\sqrt{\varepsilon}|}. \tag{38.15}$$

It is this condition that is adopted below.

We replace the unknown attenuation function W by the function

$$V = e^{[\text{illegible}]}W, \tag{38.16}$$

$$\omega_0 = \frac{y^2}{4x} + \frac{xy}{2} - \frac{x^3}{12}, \tag{38.17}$$

where, according to Formula (38.5),

$$\omega_0 = \left(\frac{M}{a}\frac{R}{x} - 1\right) 2M^2 x = kR - ka\theta \tag{38.17a}$$

is the difference between the distances along the straight line and the arc from the source to the projection of the observation point onto the sphere.

Substituting the function (38.16) into the differential equation for W (38.6), and using the value of ω_0 (38.17), we obtain for V

$$\frac{\partial^2 V}{\partial y^2} + i\frac{\partial V}{\partial x} + \left(y - \frac{i}{2x}\right)V = 0, \tag{38.18}$$

where now

$$u = \frac{e^{ika\theta}}{R}V. \tag{38.19}$$

Now the sense of the conversion from W to V is clear; the attenuation function W is such that its phase measures the shift relative to the phase lead of the undisturbed wave on the straight-line distance from the source to the observation point, while V is an attenuation function such that the phase shift is reckoned with respect to the phase lead along an arc of a great circle. Setting

$$V = \sqrt{x}\,W_1, \tag{38.20}$$

we have instead of Eq. (38.18), the boundary condition (38.12) and the limit condition (38.15)

$$\frac{\partial^2 W_1}{\partial y^2} + i\frac{\partial W_1}{\partial x} + yW_1 = 0; \tag{38.21}$$

$$\frac{\partial W_1}{\partial y} = -qW_1 \text{ for } y = 0; \tag{38.22}$$

$$\lim_{x\to 0}\left\{W_1 - \frac{2}{\sqrt{x}}e^{\frac{y^2}{4x}}\right\} = 0 \text{ for } \frac{y}{x} \gg |q|. \tag{38.23}$$

The problem formulated by these three relationships is solved by separation of variables; we set

$$W_1 = X(x)Y(y). \tag{38.24}$$

On substitution into (38.21), we obtain two equations: first, an equation for X(x), which is solved in an elementary fashion:

$$X(x) = e^{itx}, \tag{38.25}$$

where t is the separation constant (and not time!); secondly, an equation for Y:

$$Y'' + (y - t)Y = 0. \tag{38.26}$$

Substituting the variable, we may write

$$Y = w(t - y), \tag{38.26a}$$

where w(t) is a function that is a solution to the equation*

$$w''(t) - tw(t) = 0. \tag{38.27}$$

This equation can be solved with the aid of cylindrical functions of order 1/3. That is to say, for negative t, a solution that decreases with increasing $|t|$ is obtained if we select from among all possible cylindrical functions a Hankel function of the first kind:

$$w(t) = e^{i\frac{2\pi}{3}}\sqrt{\frac{\pi}{3}}(-t)^{1/2}H^{(1)}_{1/3}\left(\frac{2}{3}(-t)^{3/2}\right). \tag{38.28}$$

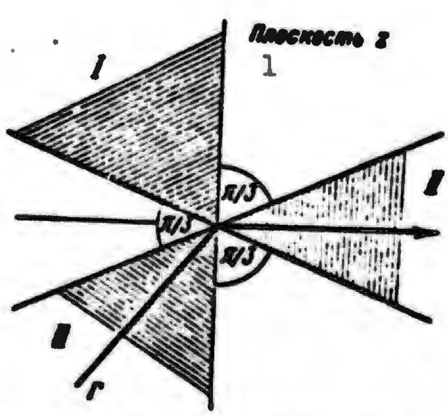

Fig. 38.2. Contour of integration in Formula (38.30). 1) z-plane.

In the general case, however, it is more convenient to write the solution in the form of an integral in the complex plane:

$$w(t) = \frac{1}{\sqrt{\pi}}\int_\Gamma e^{zt - \frac{1}{3}z^3}\,dz. \tag{38.29}$$

Indeed, substituting it into Eq. (38.27) and performing the differentiation in the integrand, we get

$$\frac{1}{\sqrt{\pi}}\int_\Gamma (z^2 - t)e^{zt - \frac{1}{3}z^3}\,dz = \frac{1}{\sqrt{\pi}}\int_\Gamma -\,de^{zt - \frac{1}{3}z^3} = 0. \tag{38.30}$$

We obtain zero, of course, only if the contour is closed or if it is so selected that the primordial function $\exp\left(zt-\frac{1}{3}z^3\right)$ vanishes at its ends. The contour shown in Fig. 38.1 may be one such contour. All possible nonclosed contours are determined in general by the condition that the exponent go to $-\infty$ as we move onto the ends of the contour. For this it is necessary that

$$\mathrm{Re}\, z^3 > 0.$$

This is satisfied by z lying in the sectors shaded on Fig. 38.2. Hence we might take instead of Γ, for example, a contour going from region III to region I. However, the selection Γ shown on the figure (the heavy broken line) ensures that the function will go to infinity when $|t| \to \infty$. Why this is necessary will be clear from what follows.

Finally, the asymptotic behavior of the function w for large |t| can be obtained directly from Eq. (38.27). Actually, it is obvious that the approximate solution is

$$w = N_0 e^{\pm\frac{2}{3}t^{3/2}}, \qquad (38.29a)$$

where N_0 is a constant. We may easily satisfy ourselves of this by substituting the solution (38.29a) into Eq. (38.27) and dropping terms of the order of 1/t as small by comparison with the principal terms. A solution that goes to infinity as $|t| \to \infty$ (which corresponds to our selection of the contour Γ, i.e., the function (38.29)) will be obtained if we take different signs in the exponent in different sectors of the plane of the complex variable t. A more exact asymptotic expression that proceeds from Formula (38.28) for negative t is given by the familiar asymptotic expressions for Hankel functions of the first kind:

$$w = e^{i\frac{\pi}{4}}(-t)^{-\frac{1}{4}}e^{i\frac{2}{3}(-t)^{3/2}}. \qquad (38.29b)$$

However, on other rays drawn in the t-plane from the origin (with arguments of t different from that selected here $\arg t = \pi$), they have different expressions. The functions w are of fundamental importance for the problem under consideration. They were investigated in detail by Fok [4] and are known as Airy functions.

The separation constant is determined from the boundary condition (38.22). Obviously, the function w must satisfy it. Since $\frac{dw(t-y)}{dy} = -\frac{dw}{dt}$,

$$\frac{dw}{dt} - qw(t) = 0. \tag{38.31}$$

Generally speaking, this equation can be satisfied only for certain values of $t = t_s$, which it determines. The particular solution therefore takes the form

$$W_1 = e^{i\alpha t_s} w(t_s - y). \tag{38.32}$$

However, we have not yet taken Condition (38.23) into account, and hence this function may not yet be regarded as the solution. At the same time, this condition is extremely important. It will be recalled, for example, that the error in the Zenneck solution for the vertical dipole on a flat earth (§23) consisted in the fact that this solution, which satisfied a differential equation and boundary conditions at the earth-atmosphere interface did not have the necessary singularity at the position of the source. Condition (38.23) requires that the function W_1 go to infinity at $x = 0$ and do so in a definite manner. Obviously, Function (38.32) can by no means satisfy this condition. Moreover, any sum of such solutions for finite t_s remains bounded everywhere. However, the contour Γ that we selected in Formula (38.29) (see Fig. 38.2) differs from other possible contours precisely in that it ensures the function w going to infinity as $|t| \to \infty$. Hence this contour selection also permits us to obtain the singularity at $x = 0$.

Consequently, the solution must be composed of particular solutions of the form (38.32) with participation of $\pm$ that assume indefinitely large values. It can be shown that such roots of Eq. (38.31) are encountered. Thus,

$$W_1 = \sum_{s=1}^{\infty} b_s e^{ixt_s} w(t_s - y). \tag{38.33}$$

where the subscript s numbers the roots of Eq. (38.31). It is expedient to write this sum differently, in the form of a contour integral, as follows. We draw a contour C that encompasses all roots t_s (a more detailed analysis would show that all of these roots lie in the first quadrant of the plane of the variable t) and take the contour integral

$$I = b\int_C \frac{e^{ixt} w(t-y)}{w'(t) - qw(t)}\, dt, \tag{38.33a}$$

where b is a certain constant. The integrand has singularity where the denominator vanishes, i.e., according to Eq. (38.31), precisely at the points $t = t_s$. Since the numerator is finite at these points, the integral is equal to the sum of the residues at these points multiplied by $2\pi i$.

To calculate these residues, we expand the denominator in series, for example, near the pole $t = t_s$, at which, therefore,

$$w'(t_s) - qw(t_s) = 0. \tag{38.33b}$$

We fet

$$w'(t) - qw(t) = (w'(t) - qw(t))_{t=t_s} + (w''(t) - qw'(t))_{t=t_s} \cdot (t - t_s) + \ldots,$$

i.e.,

$$w'(t) - qw(t) \approx (w''(t_s) - qw'(t_s))(t - t_s).$$

But, applying Formula (38.33b) and the differential equation (38.27) for the function w, we write

$$w''(t_s) - qw'(t_s) = t_s w(t_s) - qw'(t_s) = (t_s - q^2)\, w(t_s).$$

Hence the sum of the residues is equal to

$$I = 2\pi i b \sum_s \frac{e^{ixt_s}}{t_s - q^2} \frac{w(t_s - y)}{w(t_s)}. \tag{38.34}$$

Thus, Integral (38.33a) is Solution (38.33) with a specific value of b_s, i.e., $b_s = \frac{2\pi i b}{(t_s - q^2)\, w(t_s)}$. It is found that such selection of coefficients b_s is sufficient for the expression for W_1 (38.33) to satisfy the condition at the coordinate origin. This is demonstrated by investigation of Integral (38.33a), which we shall describe briefly (for details see [6]).

As we have already noted, a singularity at the location of the source can appear only through terms with very large $\underline{t}$. Hence to analyze the behavior of the function W_1 as $x \to 0$, it is sufficient in Integral (38.33a) to consider segments of contour C that lie near infinitely large $\underline{t}$. On these segments, the function $\underset{\cdot}{w}$ can be replaced by its asymptotic expression (38.29a). Here it is necessary to know expressions for $\underset{\cdot}{w}$ for arg $\underline{t}$ other than π. Namely, in order to cover all t_s poles, which, as we noted, lie in the first quadrant, we may select a contour C originating from $t = i\infty$, passing along the imaginary axis to $t = 0$, and then along the real axis to $t = \infty$. Taking signs that correspond to the sector in question in the exponent of Solution (38.29), expanding the exponent $(t - y)^{3/2}$ in powers of $\underline{t}$, and leaving the principal terms, we obtain

$$\frac{w(t-y)}{w'(t)-qw(t)} = \begin{cases} \frac{1}{\sqrt{t}} e^{-y\sqrt{t}} \text{ for } -\frac{2\pi}{3} < \arg t < \frac{\pi}{3}, \\ -\frac{1}{\sqrt{t}} e^{y\sqrt{t}} \text{ for } \frac{\pi}{3} < \arg t < \frac{4\pi}{3}. \end{cases} \qquad (38.35)$$

Now we can calculate the asymptotic value of Integral (38.33a). On the part of contour C that coincides with the imaginary semiaxis, we substitute the lower of the values of (38.35), while on the segment coinciding with the real axis, we use the upper values. Thus, the integrands in the integrals over the half-lines are different. Then we may bring the second half-line into coincidence with the first (it will be recalled that the integrands now no longer contain poles) and bring the integral to the following form by the substitution $t = ip^2$:

$$W_1 \sim b\cdot 2e^{i\frac{\pi}{4}}\int_{-\infty}^{+\infty} e^{-xp^2+\sqrt{i}\,yp}\,dp = 2e^{i\frac{\pi}{4}}b\sqrt{\frac{\pi}{x}}\,e^{i\frac{y^2}{4x}}. \qquad (38.35a)$$

Consequently, taking a constant b equal to

$$b = \frac{1}{\sqrt{\pi}}e^{-i\frac{\pi}{4}}, \qquad (38.36)$$

we obtain

$$W_1 \sim \frac{2}{\sqrt{x}}e^{i\frac{y^2}{4x}}, \qquad (38.37)$$

and, consequently, for this value of b, the solution W_1 satisfies not only the differential equation and the boundary conditions, but also the requirement of (38.23).

§39. ANALYSIS OF THE COMPLETE SOLUTION

Collecting Formulas (38.19), (38.20), (38.33), (38.34) and (38.36) we have

$$u = \frac{e^{ika\theta}}{R}V = \frac{e^{ikR}}{R}W, \qquad (39.1)$$

$$V = 2\sqrt{i\pi x}\sum_s e^{ixt_s}\frac{w(t_s-y)}{(t_s-q^2)\,w(t_s)} = \sqrt{\frac{x}{i\pi}}\int_C \frac{e^{ixt}\,w(t-y)}{w'(t)-q\,w(t)}\,dt, \qquad (39.2)$$

where

$$x = \sqrt[3]{4p_s} = \sqrt[3]{\frac{k}{2a^2}}\,\vartheta a, \quad y = \sqrt[3]{\frac{2}{ka}}\,kh, \quad h = r-a, \quad q = \frac{i}{\sqrt{\varepsilon'}}\sqrt[3]{\frac{ka}{2}}, \qquad (39.2a)$$

and w is the function (38.29) or (38.28), satisfying Eq. (38.27): t_s are the roots of Eq. (38.31).

This formula was obtained for the vertical electric dipole and, accordingly, u is the Debye function, while $\Pi = ru$ is the absolute value of the Hertzian vector for the electric dipole. But it is obvious that this same formula is also valid for the magnetic Debye function v (when the Hertzian magnetic vector is $\Pi_m = rv$), the only difference being that q must be replaced by

$$q_m = \sqrt{\varepsilon^{0}}\sqrt{\varepsilon - 1}\, q = i\sqrt{\varepsilon - 1}\,\sqrt[3]{\frac{ka}{2}} \qquad (39.2b)$$

(see §28 and the boundary conditions (34.5)-(34.6)).

These expressions represent the final general form of the solution sought in the case in which one of the corresponding points is elevated to an arbitrary height above the surface of the earth. It can be shown by further investigation that it becomes the reflection formula for a sufficient elevation above the horizon line (this will be the case, as we know, for $\psi \gg \frac{1}{\sqrt[3]{ka}}$, or, in the nomenclature of §38 – see Formulas (38.8) and (38.9) – for $y - x^2 \gg x$), while for small enough $\underline{x}$ at y = 0 it becomes the formula for the flat earth (as we know, this will be Formula (36.35) for $\sqrt{p_3} \ll 1$, i.e., according to Formula (38.8), for $x^{3/2} \ll 1$).

To use the solution obtained, it is necessary either to sum the series (39.1), for which it is first necessary to determine all values t_s of roots of the equation $w'(t) = qw(t)$ for the wavelength of interest to us and the properties of the soil (since $\underline{q}$ depends on them), or to perform integration numerically in Formula (39.2).

Let us dwell in greater detail on the use of the solution in the form of Series (39.1).

First of all, two limiting cases are important: $q = \infty,\ \delta = 0$ (short waves) and $q = 0,\ \delta = \infty$ (long waves). In the former case, the values of t_s will be denoted by t_s^0. Consequently, they are roots of the equation

$$w(t_s^0) = 0. \qquad (39.3)$$

In the latter case, we denote the roots by t_s'. They satisfy the equation

$$w'(t_s') = 0. \qquad (39.4)$$

The complex numbers t_s^0 and t_s' may be found as zeroes of the Hankel function (38.28) and its derivatives (see [4] and [I, 11]; it must be remembered that the parameter τ_s, which differs from the t_s in Fok's papers [4], $t_s = \sqrt[3]{2}\tau_s$, figures in the second of these books. The first roots are as follows:

"Short waves," $q = \infty$	"Long waves," $q = 0$
$t_1^0 = 2{,}33811e^{i\frac{\pi}{3}}$	$t_1' = 1{,}01879e^{i\frac{\pi}{3}}$
$t_2^0 = 4{,}08795e^{i\frac{\pi}{3}}$	$t_2' = 3{,}24820e^{i\frac{\pi}{3}}$
$t_3^0 = 5{,}52056e^{i\frac{\pi}{3}}$	$t_3' = 4{,}82010e^{i\frac{\pi}{3}}$
$t_4^0 = 6{,}78671e^{i\frac{\pi}{3}}$	$t_4' = 6{,}16331e^{i\frac{\pi}{3}}$
$t_5^0 = 7{,}94417e^{i\frac{\pi}{3}}$	$t_5' = 7{,}37218e^{i\frac{\pi}{3}}$

To calculate the roots for small q (long waves), it is convenient to use a differential equation, which we obtain by differentiating Eq. (38.31) with respect to q:

$$w''\frac{dt}{dq} - w - qw'\frac{dt}{dq} = 0,$$

and substituting w'' by tw (38.27) and w' by qw (38.31). From this we find that

$$\frac{dt}{dq} = \frac{1}{t - q^2}. \qquad (39.5)$$

Now we may determine, one by one, all coefficients of the series giving the expansion of t in powers of q. For large q (small δ, short waves), we rewrite this equation as follows:

$$\frac{dt}{d\frac{1}{q}} = \frac{1}{1 - \frac{t}{q^2}}. \qquad (39.6)$$

From this it is easy to obtain a series for t in powers of the small quantity 1/q.

Substituting the values of t_s^0 or t_s' from the table given above into the exponents in Series (39.2), we see that the exponents of all exponential factors contain a negative real part. As we have already noted, all t_s lie in the first quadrant to begin with, i.e., they have positive imaginary parts. Hence the exponential decrease of each term in the series with increasing $\underline{x}$ is a general rule. With increasing term number $\underline{s}$ in both extreme cases and, consequently, also in any intermediate case, the absolute value of the exponent increases sharply. Hence for significant $\underline{x}$ (i.e., for $\sqrt[3]{\rho_2} \gg 1$), the first few terms of the series will play the principal role if the other cofactors do not increase concurrently. It is easily seen that this will be the case in any event for y = 0. That is to say, we obtain on the earth's surface

$$u = 2\frac{e^{ika\theta}}{a\theta}\sqrt{i\pi x}\sum_s \frac{e^{ixt_s}}{t_s - q^2}. \tag{39.7}$$

The larger $\underline{x}$, the smaller the number of terms of the series that must be taken. Limiting ourselves to the first term, we obtain

$$u = 2\frac{e^{ika\theta}}{a\theta}\sqrt{i\pi x}\,\frac{e^{ixt_1}}{t_1 - q^2}. \tag{39.7a}$$

In the case of an ideally conducting earth, q = 0, we obtain

$$u = C\frac{e^{ikD}}{\sqrt{D}}e^{-\alpha\sqrt[3]{ka}\,\theta}, \tag{39.8}$$

$$\sqrt[3]{ka}\,\theta = \sqrt[3]{\frac{kD^3}{a^2}} = \sqrt[3]{8\rho_2}, \quad \alpha = \frac{1{,}019\cdot\sqrt{3}}{2\sqrt[3]{2}}\left(1 - \frac{i}{\sqrt{3}}\right) \approx 0{,}70\left(1 - \frac{i}{\sqrt{3}}\right),$$

$$C = \frac{4\sqrt{\pi i}}{(1 + i\sqrt{3})\,1{,}019}\sqrt[6]{\frac{k}{2a^4}}.$$

In the case of q = ∞ (infinitesimally short waves), of course, we obtain a zero, since this corresponds to geometrical optics, and penetration of the radiation into the shadow region is excluded.

Thus, the field diminishes exponentially at great distances if the observation point is on the surface of the earth, as was first shown by Watson [1]. This circumstance would exclude transmission over great distances if there were no ionosphere.

At a distance D from the source, an observation point on the ground is situated below the plane of the horizon by a distance

$|h_A| = \frac{D^2}{2a}$, so that the exponential decrease with distance is simultaneously an exponential $\left(\sim\exp\left\{-\alpha\sqrt[3]{\frac{k^2}{a}}\sqrt{|h_A|}\right\}\right)$ decrease with descent below the plane of the horizon. This is something completely different from the diffraction decrease in the shadow of a flat screen, of which we spoke in §34.

As will be seen from the table given for them, the coefficients t_s depend rather weakly on wavelength. Even as we pass from $\lambda = 0$ to $\lambda = \infty$, only t_1 and t_2 change greatly. Nevertheless, the absolute value $|t|$ diminishes with increasing λ. On the other hand, $x \sim \lambda^{1/3}$. Hence we can satisfy ourselves that with increasing λ, the penetration of radio waves is intensified (as, of course, it should).

At short distances $(p_2 < 1)$, the series converges slowly. It cannot be used at all for $p_2 \ll 1$, when it would be necessary to calculate an enormous number of terms. However, Formula (36.35) can be used here, if necessary continuing its expansion (compare [I, 11], Chapter 4, §4).

Let us now examine the field in space, $y \neq 0$. It is essential that a difference from the field at the ground appears only because of the presence of the "altitude factor" in each term of the series:

$$f(h) = \frac{w(t_s - y)}{w(t_s)}. \qquad (39.9)$$

If the distance $\underline{x}$ from the source is so great $(p_2 > 1)$ that one term of the series is sufficient, this means that the entire field increases with increasing $\underline{h}$ as does this "altitude factor." True, a decrease will occur at first for very small heights since, according to Formula (38.31),

$$w(t_s - y) \approx w(t_s) - yw'(t_s) = w(t_s)(1 - qy) \qquad (39.10)$$

and the real part of $\underline{q}$ is positive. This is the same decrease that we encountered in the case of the flat earth. It is simply a consequence of the boundary conditions (see (21.24)) and plays an insignificant role. With a further increase in h, in view of the fact that t_s is always markedly larger than unity, we can estimate the curve of the altitude factors from the asymptotic behavior of w (38.29a). From this it is evident that the field will increase exponentially.

The case of large y, when the diffraction phenomena are sharply in evidence, as occurs, for example, for short and ultrashort waves at considerable distances, is of particular interest. In this case, expanding the exponent in Formula (38.29b) in powers of t/y,

$$w(t-y)\approx\frac{\sqrt{\pi}}{\sqrt[4]{y}}e^{i\frac{2}{3}y^{3/2}-i\sqrt{y}\,t+\cdots}, \tag{39.11}$$

and limiting ourselves to the terms given above, we obtain on substitution of the value of (39.11) into Formula (39.2),

$$V=2\frac{i\sqrt{\pi x}}{\sqrt[4]{y}}e^{i\frac{2}{3}y^{3/2}}\sum_s\frac{e^{i(x-\sqrt{y})t_s}}{(t_s-q^2)\,w(t_s)}=\sqrt{\frac{x}{\pi\sqrt{y}}}e^{i\frac{2}{3}y^{3/2}}\int_C\frac{e^{i(x-\sqrt{y})t}}{w'(t)-qw(t)}dt. \tag{39.12}$$

In particular, in the actual horizon plane, $y = x^2$,

$$V=2i\sqrt{\pi}e^{i\frac{2}{3}x^3}\sum_s\frac{1}{(t_s-q^2)\,w(t_s)}. \tag{39.13}$$

and since, according to Formula (38.5), for $y = x^2$

$$R=a\vartheta\left(1+\frac{2}{3}\frac{h}{a}\right),\ \text{where}\ h\approx\frac{(a\vartheta)^2}{2a},$$

then

$$u=\frac{e^{ika\vartheta}}{a\vartheta}V=\frac{e^{ik(a\vartheta-R)}}{R}e^{ikR}V=\frac{e^{ikR}}{R}Ve^{-i\frac{2}{3}x^3}; \tag{39.14}$$

consequently,

$$u=\frac{e^{ikR}}{R}2i\sqrt{\pi}\sum_s\frac{1}{(t_s-q^2)\,w(t_s)}. \tag{39.15}$$

Thus, as we found earlier (37.8), the attenuation function (referred to the undisturbed field $\frac{1}{R}\exp(ikR)$) does not depend on distance in the plane of the horizon. In general, however, for $x-\sqrt{y}\neq 0$, the attenuation function decomposes into factors of which one is of purely oscillatory nature $\left(\exp\left(i\frac{2}{3}y^{3/2}\right)\right)$, while the other depends on the coordinates in a definite combination, which may be denoted by z:

$$z = x - \sqrt{y}. \tag{39.16}$$

Above the plane of the horizon, z is negative, while in the shadow z > > 0. If $(x/\sqrt{y})^{1/2}$ in the multiplier before the exponential is replaced by its value in the horizon plane, i.e., by unity,

$$V = e^{i\frac{2}{3}y^{3/2}} V_1(z, q). \tag{39.17}$$

$$V_1(z, q) = \frac{1}{\sqrt{\pi}} \int_\Gamma \frac{e^{izt}dt}{w'(t) - qw(t)} = 2i\sqrt{\pi} \sum_s \frac{e^{izt_s}}{(t_s - q^2)\, w(t_s)}. \tag{39.18}$$

It must be remembered that the representation in series form in Formula (39.18) is possible only for z positive, i.e., in the shadow, since with negative z, the successive terms of the series increase exponentially because t_s contains a positive imaginary part $(t_s \sim e^{i\frac{\pi}{3}})$. But even for positive z, it is convenient to use the series only when z is not very small. Thus, in the immediate vicinity of the horizon, it is necessary to resort to an integral evaluated by numerical methods. We note, however, that if the asymptotic expression is not used, and the series given the complete solution (39.2) is applied at once, then even though it will not converge very well for points in the plane of the horizon, the direct calculation is possible, since the values of t_s and $w(t_s)$ are tabulated for the various cases. It is, of course, then necessary to take many terms of the series [5].

Thus, if the receiver is on the ground far beyond the horizon, the field will be exponentially small (see Formula (39.8)). As we rise above the surface, the field at first attenuates somewhat (see (39.10)), in the measure of boundary condition (34.6), and then begins a rapid exponential rise (see the expansion of (39.11)). As we approach the plane of the horizon $\left(\sin\psi \leqslant \frac{1}{\sqrt[3]{ka}}\right)$, then (this will be demonstrated below, (39.32)) the field will increase just as on emergence from the shadow at the edge of the flat screen (Fresnel diffraction), and, in the immediate vicinity of the horizon plane – below it, above it, and on the plane itself, as long as $\left|\sqrt[3]{ka}\sin\psi\right| \ll 1$, the field will be described by Formulas (37.11) (see also (39.15)). Finally, as we ascend further above the plane of the horizon, we enter the illuminated region, where the interference (reflection) formulas apply (they take account of the sphericity of the earth in the divergence coefficient (35.7)).

For a rigorous derivation of the reflection formulas from the general solution (39.2), the reader is referred to Fok's work [4].

Up to this point, we have assumed that the source is situated on the ground. However, Solution (39.1) can be generalized without difficulty to the case of arbitrary source position.

First of all, applying the reciprocity theorem, we may rewrite its solution for the field at the earth's surface created by a dipole elevated to a height h_0:

$$\left.\begin{aligned} u(\vartheta, 0; h_0) &= \frac{e^{ika\vartheta}}{a\vartheta} V(x, 0; y_0), \\ V(x, 0; y_0) &= 2\sqrt{\pi x} \sum_s \frac{e^{ixt_s}}{t_s - q^2} f_s(y_0), \\ f_s(y_0) &= \frac{w(t_s - y_0)}{w(t_s)}, \quad y_0 = \sqrt[3]{2(ka)^2}\,\frac{h_0}{a}. \end{aligned}\right\} \quad (39.19)$$

But we have seen that the solution in space at a height h_A is obtained from the series that gives the solution for the surface by multiplication of each of its terms by a height factor. Consequently,

$$\left.\begin{aligned} u(\vartheta, h_A; h_0) &= \frac{e^{ika\vartheta}}{a\vartheta} V(x, y_A; y_0), \\ V(x, y_A; y_0) &= 2\sqrt{\pi i x} \sum_s \frac{e^{ixt_s}}{t_s - q^2} f_s(y_0) f_s(y_A), \\ y_A &= \sqrt[3]{2(ka)^2}\,\frac{h_A}{a}, \quad f_s(y_A) = \frac{w(t_s - y_A)}{w(t_s)}. \end{aligned}\right\} \tag{39.20}$$

Indeed, we have seen that each term of the series, considered as a function of $\underline{x}$ and y_A, satisfies the differential equation and boundary conditions. As concerns the behavior of the solution at the position of the source, we have satisfied ourselves that it is determined by the terms of the series for which $\underline{t}$ is very large. Formulas (38.35) are valid in this region. They indicate that the introduction of a new height factor $f_s(y_0)$ for large $\underline{t}$ is equivalent to substituting $y_A + y_0$ for y_A. Therefore, passing to the integral form of the solution, we may easily satisfy ourselves that the entire proof described in §38 can be repeated, since the change is manifest only in the substitution of $y_0 + + y_A$ for $\underline{y}$ in the asymptotic expression of the integrand (38.35a). This demonstrates the correctness of Solution (39.20).

In the case $y_0 = 0$, the solution can be presented not only in series form, but also in the form of an integral the sum of whose residues is this series (see (39.2)). In much the same way for $y_0 \neq 0$, Series (39.20) is equivalent, as Fok showed [4], to the integral (compare also Furutsu [9]):

$$V(x, y_A; y_0) = V(x, y_0; y_A) = e^{-i\frac{\pi}{4}}\sqrt{\frac{x}{\pi}} \int_C e^{ixt} F(t, y_0, y_A)\,dt. \tag{39.21}$$

$$F(t, y_0, y_A) = \frac{i}{2} w_1(t - y_0) w_1(t - y_A) \left\{ \frac{w_2(t - y_0)}{w_1(t - y_0)} - \frac{w_2'(t) - q w_2(t)}{w_1'(t) - q w_1(t)} \right\}. \tag{39.21a}$$

Here w_1 and w_2 are the same complex Airy functions, i.e., solutions of Eq. (38.27) that show different asymptotic behavior for large negative $\underline{t}$:

$$w_1(t) = e^{i\frac{\pi}{4}}(-t)^{-\frac{1}{4}} e^{i\frac{2}{3}(-t)^{3/2}} = \frac{1}{\sqrt{2}} e^{i\frac{2\pi}{3}} \sqrt{\frac{\pi}{3}}\,(-t)^{\frac{1}{2}} H^{(1)}_{1/3}\left(\frac{2}{3}(-t)^{3/2}\right);$$

$$w_2(t) = e^{-i\frac{\pi}{4}}(-t)^{-\frac{1}{4}} e^{-i\frac{2}{3}(-t)^{3/2}} = \frac{1}{\sqrt{2}} e^{-\frac{2\pi}{3}} \sqrt{\frac{\pi}{3}}\,(-t)^{\frac{1}{2}} H^{(2)}_{1/3}\left(\frac{2}{3}(-t)^{3/2}\right). \tag{39.22}$$

The contour C may be selected in the form of a broken line running from infinity in the second quadrant, $\infty \cdot \exp\left(i\frac{2}{3}\pi\right)$, to 0 and from 0 to $+\infty$. It is just this representation in integral form that enables us to investigate the solution in the region in which the series converges poorly.

3. Let us consider in greater detail the field as a function of depth of immersion in the shadow. We observed above that for an observation point on the ground far beyond the horizon, we may limit ourselves to a single term of the series, so that the field diminishes exponentially as we move away from the source, as $D = a\vartheta$ increases. On the other hand, near the plane of the horizon, the decrease with distance must be much slower (according to Formula (37.13), the attenuation function w does not depend on distance, and so forth). And indeed, in the shadow near the horizon, we can obtain a simple and important result: the attenuation function is the same as that obtained for Fresnel diffraction on a certain equivalent flat screen. This conclusion is of great practical importance, in particular for diffraction of the field of a remote extraterrestrial source or on diffraction on hills and other convex obstacles. We present the analytical results of Fok [7] and Furutsu [9].

Persistent attempts have been made to apply the ordinary theory of Fresnel diffraction at a rectilinear edge of an opaque, infinitesimally thin screen (§11, Fig. 11.3) to description of diffraction at a curved edge and even for diffraction around the earth. This application was unjustified and, as will be clear from the above, leads to fundamentally incorrect results on the surface of the earth. As we noted in §34, diffraction at a straight edge of a thin screen gives a comparatively slow decrease in intensity as we move deeper into the umbra, $E \sim \frac{1}{\sqrt{d}}$, if d is the distance from the geometrical boundary of the shadow.

We have considered diffraction around the earth on the assumption that the essential zone on the surface of the sphere covers a region whose dimensions are small as compared with the earth's radius a. Thus we were able to replace the equation of the surface by an approximate parabolic formula (34.1), $z = -\frac{x^2+y^2}{2a}$, where x, y and z are rectangular coordinates with the plane z = 0, which is tangent to the sphere at a certain point, taken as the coordinate origin. Description with this formula is admissible even when the shape of the surface deviates from the parabolic or from the spherical outside the essential region. Hence the results can also be applied to diffraction on any convex body for which we may limit ourselves to second-order terms in expanding the equation of the surface within the essential zone.

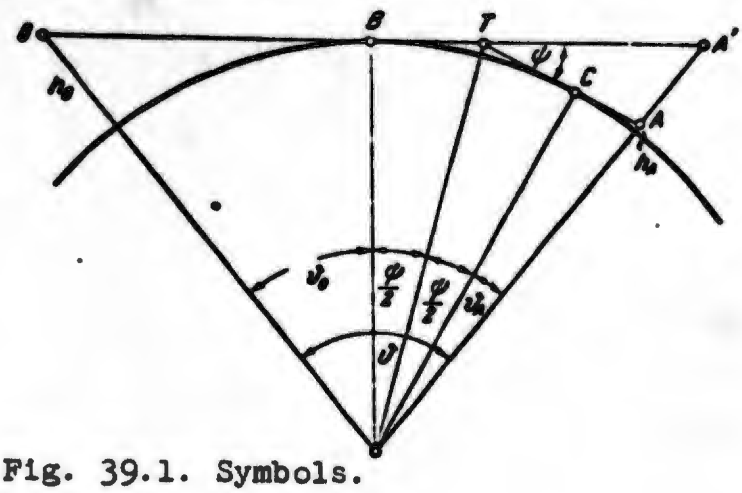

Fig. 39.1. Symbols.

It will be recalled that formulas were derived in §37 for the field at the surface of a sphere (near the geometrical boundary of the shadow) when the incident radiation is of the nature of a plane wave, i.e., the source is very far away and elevated above the ground, and also for the converse case – source on the surface of the sphere, observation point remote near the plane of the horizon.

It was clear from the derivation of these formulas that they are also valid when we are speaking, respectively, of the field of a glancing wave incident on a convex elevation if the observation is made near its peak, and also, conversely, of the field of a source situated, for example, at the top of a hill and observed at a distance from it. However, if extend these conclusions to the case of a hill, for example, on a flat earth, it is necessary to remember that not only the plane wave from the source but also its reflection from the ground surface in front of the hill is incident at the top of the hill. We shall consider this important circumstance later (§53). For the time being, we shall assume every that we are dealing with an isolated convex body in a vacuum.

We shall use the nomenclature of Fig. 39.1, where we have marked a point T to indicate the line of intersection of the horizon planes for the source O and the observation point A. The angle ψ between these planes is the diffraction angle of the incident wave, and the angular distance of point T from the points of tangency B and C of the horizon planes with the sphere is, as is easily seen, equal to $\psi/2$. We are obviously interested in the region of values of $\psi \ll 1$. Considerable importance will subsequently attach to the distances $D_O \approx a\vartheta_O$ and $D_A \approx a\vartheta_A$ from the source and from the observation point to points B and C, respectively.

Let us consider the dimensionless quantity

$$\xi = x - \sqrt{y_0} - \sqrt{y_A} = \sqrt[3]{\frac{k}{2a^2}}\left(D - \sqrt{2ah_0} - \sqrt{2ah_A}\right) \quad (39.23)$$

(for $h_0 = 0$, the quantity ξ is identical with z (39.16)). Since

$\sqrt{2\lambda h_0} \approx OB = r_1 = a\vartheta_0$ and $\sqrt{2\lambda h_A} \approx AC = r_2 = a\vartheta_A$, and D is approximately equal to the distance from O to A, ξ is a measure of the arc BC. If we imagine an elastic filament to be stretched between points O and A, then ξ measures the length of the segment on which this filament is applied to the surface of the sphere. Clearly, when point A approaches the horizon plane of the source, OA', point C will approach point B and ξ will vanish. Thus, ξ is at the same time a measure of immersion of the observation point into the umbra. To within higher-order terms in the small angles ψ and ϑ,

$$\xi = \sqrt[3]{\frac{k}{2a^2}}\, a\psi = \sqrt[3]{\frac{ka}{2}} \cdot \psi, \qquad (39.23a)$$

and we have $d = r_2\psi$, where r_2 is the distance AT from the observation point to point T, for the depth d of immersion of the observation point beneath the source horizon plane. As we shall see, it differs little from $a\vartheta_A$. The approximate expression for the exact formula (39.20)-(39.21) of which we shall be speaking pertains to the case in which ξ is not very large, while the distance x and the heights h_0 and h_A are quite large, i.e., for the condition

$$\mu \gg 1, \qquad (39.24)$$

where

$$\mu^2 = \frac{\sqrt{y_0 y_A}}{\sqrt{y_0} + \sqrt{y_A}} = \sqrt[6]{\frac{2k^4}{a}} \frac{\sqrt{h_0 h_A}}{\sqrt{h_0} + \sqrt{h_A}}. \qquad (39.25)$$

Thus, neither of the corresponding points need be situated on the earth. This condition may also be expressed in another way. According to Formulas (39.23) and (39.24),

$$\mu^2 = \frac{\sqrt{y_0 y_A}}{x - \xi} = \frac{\sqrt{h_0 h_A} \cdot \sqrt[3]{\frac{4k}{a^2}}}{\vartheta - \psi} = \frac{\vartheta_0 \vartheta_A}{\vartheta - \psi} \sqrt[3]{\frac{ka}{2}} = \frac{\vartheta_0 \vartheta_A}{\vartheta_0 + \vartheta_A} \sqrt[3]{\frac{ka}{2}}. \qquad (39.26)$$

If one of the angular distances ϑ_0 and ϑ_A is much larger than the other, for example, for incidence of a plane wave, $\vartheta_O \gg \vartheta_A$, then

$$\mu^2 \approx a\vartheta_A \cdot \sqrt[3]{\frac{k}{2a^2}} = \sqrt[3]{4p_2}, \quad p_2 = \frac{k(a\vartheta_A)^3}{8a^2}. \tag{39.27}$$

where p_2 is the parameter (34.20) for the distance of point A from point C. The requirement that $\mu \gg 1$ signifies that the parameter p_2 must be very large for the distances of the two corresponding points from their horizons (if one of the points, for example, point O, is lowered to the earth's surface, then $\vartheta_O = 0$ for it and Condition (39.24) will be violated).

The attenuation multiplier of (39.21) was transformed in [7] for conditions under which x, y_O, y_A and μ are large while ξ is small or finite. Here we present only the final result:

$$V = \frac{\sqrt{x}}{\sqrt[4]{y_O y_A}} e^{i(\omega_0 - \omega(x))} \left\{ \mu g_1(\mu\xi) - g_2(\xi) + \frac{i}{4\mu^2} \ddot{g}_2(\xi) \right\}, \quad \xi > 0 \text{ (umbra)}; \tag{39.28a}$$

$$V = 1 - \frac{\sqrt{x}}{\sqrt[4]{y_O y_A}} e^{i(\omega_0 - \omega(x))} \left\{ \mu g_1(-\mu\xi) + g_2(\xi) - \frac{i}{4\mu^2} \ddot{g}_2(\xi) \right\}, \quad \xi < 0 \tag{39.28b}$$

(in illuminated region)

where $\omega_0 - \omega(x) = \frac{2}{3}(y_O^{3/2} + y_A^{3/2}) - k(R - D)$ (R is the straight-line distance from the source). Two functions appear in these formulas: g_1 and g_2. As we shall see, one of them coincides with the expression that would apply for Fresnel diffraction at a straight-line screen edge:

$$g_1(\mu\xi) = e^{-i\mu^2\xi^2 - i\frac{\pi}{4}} \frac{1}{\sqrt{\pi}} \int_{\mu\xi}^{\infty} e^{i\alpha^2} d\alpha. \tag{39.29}$$

The other has a complex form and behaves asymptotically as follows:

$$g_2(\xi) = \frac{e^{i\frac{\pi}{4}}}{2\sqrt{\pi}} \cdot \frac{1}{\xi} \text{ for } \xi \gg 1, \ \xi > 0 \text{ (umbra)}; \tag{39.30a}$$

$$g_2(\xi) = \frac{e^{i\frac{\pi}{4}}}{2\sqrt{\pi}} \frac{1}{\xi} + \frac{\sqrt{-\xi}}{2} \frac{q + i\frac{\xi}{2}}{q - i\frac{\xi}{2}} e^{-\frac{i}{12}\xi^3}, \quad |\xi| \gg 1, \ \xi < 0 \tag{39.30b}$$

(in illuminated region).

For finite and small ξ, this function and its derivatives are of the order of unity, and, generally speaking, they depend on the parameter q, which characterizes the electrical properties of the sphere. On the other hand, $g_1(\mu\xi)$ does not depend on q and, as long as we consider $\mu \gg 1$, gives the principal term in the umbra. Let us show that it does actually describe Fresnel diffraction from a rectilinear screen edge. According to Formulas (39.26) and (39.23a), the argument of g_1 is

$$\mu\xi = \sqrt{\frac{ka\vartheta_0\vartheta_A}{2(\vartheta_0+\vartheta_A)}}\psi = \sqrt{\frac{kD_0D_A}{2(D_0+D_A)}}\psi, \qquad (39.31)$$

where $D_0 = a\vartheta_0$ and $D_A = a\vartheta_A$ are the distances of the corresponding points O and A from their apparent horizons, i.e., to B and C, respectively, taken along the arc of the great circle. We are interested in the case in which the quantity ξ is small or finite, and is not very large in any event. According to Formula (39.23a), this means that $\psi < (ka)^{-\frac{1}{3}}$. At the same time, according to Formulas (39.27) and (39.24), the ϑ_0 and ϑ_A of interest to us are such that $\vartheta \gg (ka)^{-\frac{1}{3}}$. Consequently, $\psi \ll \vartheta_0, \vartheta_A$. Under these conditions, $D_0 \approx r_1 = OT$ and $D_A \approx r_2 = AT$. Hence the quantity $\mu\xi \approx \sqrt{\frac{kr_1r_2}{2(r_1+r_2)}}\psi$ is the argument of the function (39.29). If, on the other hand, we set $\alpha = \sqrt{\frac{\pi}{2}}u$, then

$$g_1(\mu\xi) = e^{-i\frac{\pi}{2}u_1^3-i\frac{\pi}{4}}\cdot\frac{1}{\sqrt{2}}\int_{-u_1}^{\infty} e^{i\frac{\pi}{2}u^2}du; \quad -u_1 = \sqrt{\frac{kr_1r_2}{\pi(r_1+r_2)}}\psi. \qquad (39.29a)$$

On the other hand, if we replace the earth by a flat screen with a straight edge passing through point T on Fig. 39.1, then, according to Formula (11.8b), we should obtain an attenuation function

$$w_F = e^{-i\frac{\pi}{4}}\frac{1}{\sqrt{2}}\int_{-u_1}^{\infty} e^{i\frac{\pi}{2}u^2}du \qquad (39.32)$$

for Fresnel diffraction, with the same value of u_1 (see (11.8c)). Hence

$$g_1(\mu\xi) = e^{-i\mu^3\xi^3}w_F = e^{-i\frac{\pi}{2}u_1^3}w_F. \qquad (39.33)$$

Thus, to within the phase multiplier $\exp(-i\mu^2\xi^2)$, which, together with the other factors in Formulas (39.28a), (39.28b), provides the necessary phase lead, the field behind the convex body is identical under these conditions ($\mu^2 \gg 1$, $\xi \leqslant 1$) to the Fresnel diffraction field behind a flat screen. This result was first obtained semiqualitatively by Bremmer ([1, 11], page 77). We note that for greater accuracy, as will be seen from the above, the distance should be reckoned from points O and A not to T, but to points B and C, i.e., Formula (39.31) rather than the expression for u_1 (39.29a) should be used for the argument.

This conclusion assists us in understanding many of the results already known. Thus, it becomes clear why the diffraction of light from a distant star at the limb of the lunar disc produces the pattern of ordinary Fresnel diffraction at the edge of a screen when viewed from the earth, and so forth.

The physical origin of this striking result can be pointed up even by reference to the example of short waves, for which the parameter δ is small.

According to the Kirchhoff theorem, the field at point A can be obtained (compare Fig. 34.2 and the remarks pertaining to it), for example, by integration over a closed surface containing, firstly, the plane MQ passing through the free part of the imaginary screen and partly through the screen itself to the surface of the sphere (see Fig. 39.2) and, secondly, the surface of the sphere on the side toward A, beginning from Q. However, the first integral may be replaced by an integral over only the free part TM of the screen, which will give $g_1(\mu\xi)$ exactly. Actually, the segment $TQ = \frac{a\psi^2}{8}$, since $\psi \leqslant (ku)^{-\frac{1}{3}}$ is much smaller than the width of the first Fresnel zone constructed in the vertical plane TM for an observation point A; its width is equal to

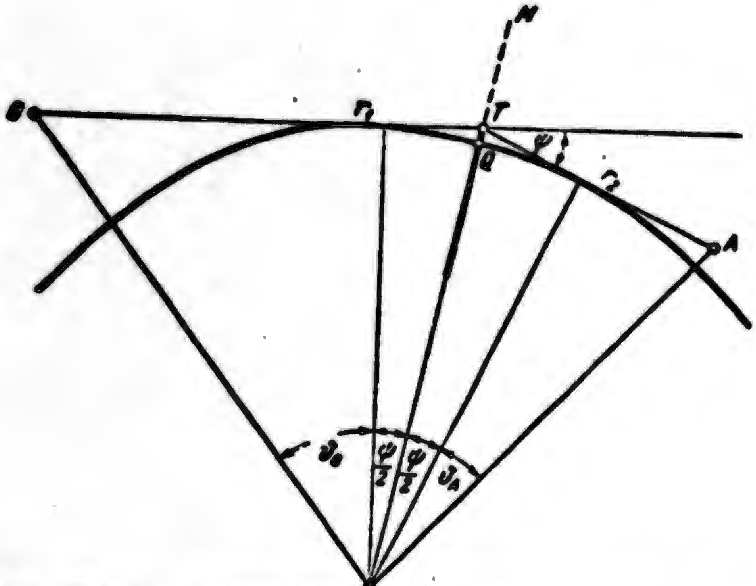

Fig. 39.2. Transition from diffraction on a sphere to diffraction at a straight edge of a flat screen.

$\sqrt{\frac{a\vartheta_A}{k}}$ and the ratio of the two quantities satisfies the inequality

$$\sqrt{\frac{a\vartheta_A}{k}} : \frac{a\psi^2}{8} \geqslant 8\sqrt{\vartheta_A \sqrt[3]{ka}} \gg 8$$

(see considerations leading up to Formula (39.29a)). Consequently, integration over QT introduces nothing that is not negligible (see §11). As concerns the field of virtual sources distributed over the surface of the sphere near the horizon, which is superimposed on $g_1(\mu\xi)$, it is described by Formulas (37.8c) and (37.11). For small δ, it is correspondingly small and can be omitted.

However, the value of the result set forth is compromised by the fact that it is valid only in a very limited range of heights h_A. The requirement $\mu^2 \gg 1$ is very rigid. Thus, according to Formula (39.25), we should have for the earth $(a \approx 6.4\cdot 10^6$ m) for heights $h_0 \sim h_A \sim h$, expressed in meters

$$h_m^{1/3} \gg 10\cdot\lambda_m^{1/3}. \tag{39.24a}$$

i.e., even for ultrashort waves, $\lambda \sim 1\ \mu$, $h_{\mu}^{1/8} \gg 10$. For λ = 10 cm and h = = 100 m, the parameter μ^2 is equal only to about 2. It diminishes when h diminishes and as λ increases. On the other hand, this conclusion is applicable only to a limited depth in the shadow, as long as ξ is not very large. Actually, for $\xi \gg 1$, it is definite that $\mu\xi \gg 1$, so that according to Formula (10.13),

$$g_1(\mu\xi) \approx \frac{e^{i\frac{\pi}{4}}}{2\sqrt{\pi}\,\mu\xi}. \qquad (39.34)$$

Applying Formula (39.30a), we see that in (39.28a) the first terms in the braces cancel exactly. A more detailed examination shows that subsequent terms also cancel, as they indeed should, when V diminishes exponentially with increasing immersion in the shadow. Thus, we may replace diffraction from the sphere by Fresnel diffraction from a straight edge of a thin screen only as long as $\xi \leqslant 1$, i.e., according to Formula (39.23a), as long as

$$\psi < \sqrt[3]{\frac{2}{ka}}, \quad d \leqslant a\theta_A \sqrt[3]{\frac{2}{ka}} = d_0. \qquad (39.35)$$

If we compare d with the distance from the plane of the shadow boundary to the surface of the sphere $d_{max} \approx \frac{(a\vartheta_A)^2}{2a}$, then this becomes a very small quantity (compare (39.26)):

$$\frac{d_0}{d_{maks}} \sim \sqrt[3]{\frac{2}{ka}}\,\frac{2a}{a\vartheta_A} = 2\,\frac{\vartheta_0}{\vartheta_0 + \theta_A}\,\frac{1}{\mu^2} \ll 1. \qquad (39.35a)$$

Thus, the region of Fresnel diffraction covers only the region of comparatively small depths in the shadow $d \ll d_{maks}$.

If we remain near the boundary of the geometrical shadow, within the limits indicated by Formulas (39.35) and (39.35a), we may expect the picture of diffraction from the screen to be applicable to a rather broad range of cases to the extent that two important generalizations are possible. It has been considered thus far that we are dealing with

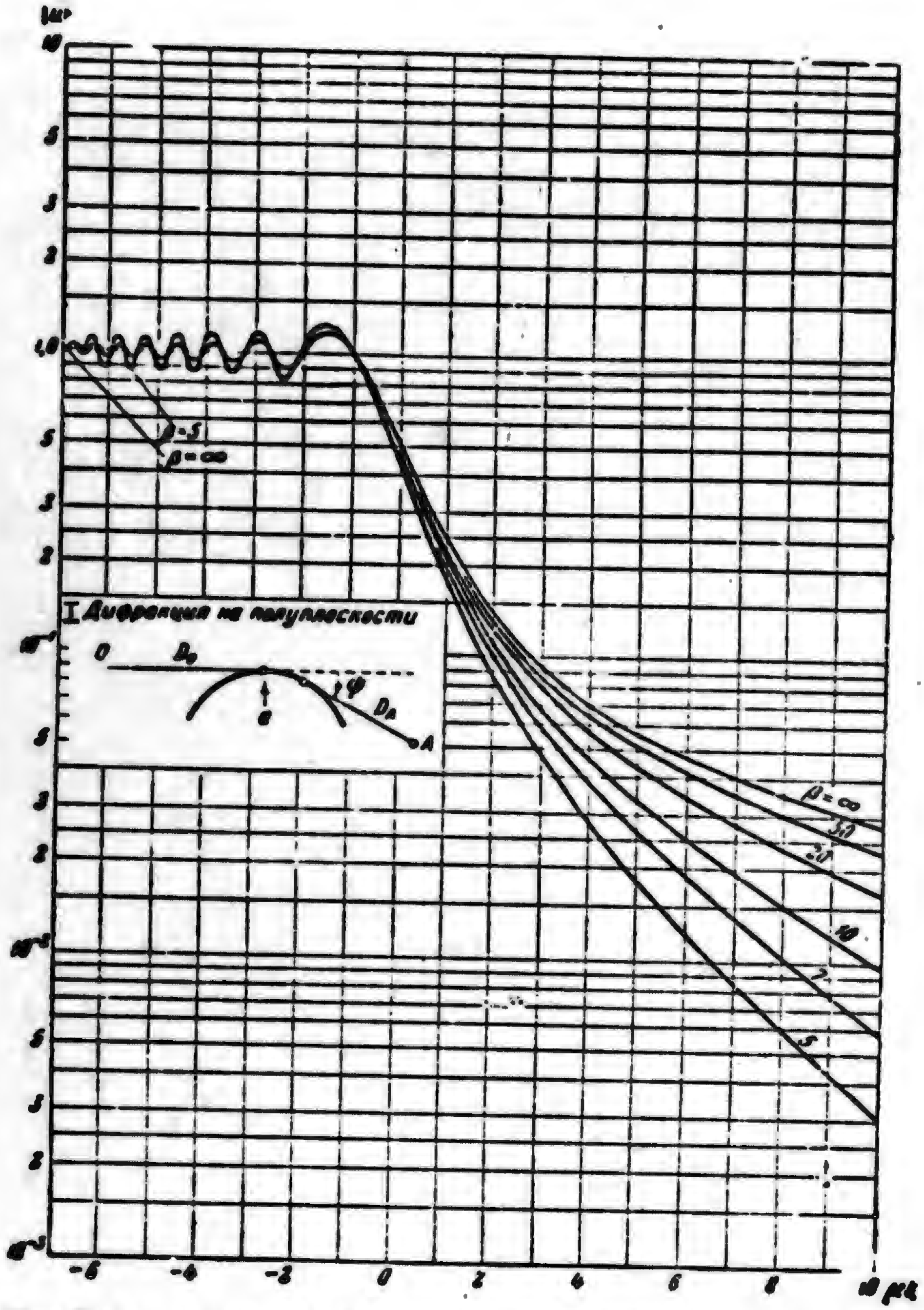

Fig. 39.3. Absolute value of attenuation function $\underline{w}$ for diffraction on a sphere for a source at high elevation and an observation point at $q = \infty$. The curves are drawn for various values of $\beta \approx 0.909\mu$ (from [VI, 9]). $\mu\xi$ is plotted against the axis of abscissas;

$$\mu\xi = \sqrt{\frac{kD_0 D_A}{2(D_0 + D_A)}}\,\psi,$$

where ψ is the angle of diffraction. I) Diffraction on half-plane.

a spherical surface, with the source and observation point both removed from point T by a distance that is small by comparison with the radius of curvature a, so that $\theta_0, \theta_A \ll 1$. Firstly, the result obtained remains valid no matter how remote point O, so that the incident wave is flat [4] (see comparison with the result of §37 presented above). Secondly, we may expect it to be valid even when the observation point A is arbitrarily remote, provided that the immersion d into the shadow remains correspondingly small. It is this last assumption that serves as a basis for application of the theory of diffraction from the edge of a thin screen to diffraction on hills and mountain ranges.

A notion as to the applicability of the approximation expressed by diffraction from a flat screen is given by the curves of Figs. 39.3 and 39.4, which have been borrowed from Furutsu's work [9]. The first diagram gives the attenuation function for $q = \infty$ as a function of $\mu\xi$ for various μ (the corresponding values of the parameter $\beta = \sqrt[3]{\frac{\mu}{4}}$ are indicated on the curves). It is seen that even for $\mu \sim 20$, the difference from diffraction at the screen is quite significant, and that it increases as we move away from the screen. The second diagram gives the attenuation function as it depends on $\rho = \left(\frac{3}{4}\right)^{2/3}\xi$ (i.e., essentially on ψ, see (39.23a)) for various q ($\alpha = -i\sqrt[3]{\frac{4}{3}}q$ are marked on the curves) for $\mu\xi \gg 7$. In either case, it is assumed that $y_0, y_A \gg 1$ and that $\operatorname{Im}\varepsilon \ll \operatorname{Re}\varepsilon$. In the former case, the function $w = W/2$ is plotted against the axis of ordinates, while in the latter it is the function $\beta w = \beta W/2$.

A comparison of the corollaries of the rigorous theory with the theory of diffraction from a screen and with experiment is carried out in great detail in [10], which also contains a rather comprehensive bibliography.

4. Let us consider briefly the questions of calculating field strength and the attenuation functions for sources other than the vertical dipole that we have been considering up to this point.

As we have already said, the field in the case of the <u>vertical magnetic dipole</u> is described by the Debye function $\underline{v}$ (and not by $\underline{u}$), which differs from (39.1) in having q_m in place of $\underline{q}$ (39.2b). In calculating the intensity components of the electric and magnetic fields, it is necessary to use Formulas (3.32) for the electric dipole and Formulas (3.33) for the magnetic dipole. Thus, if the field strengths E^v, H^v of the electric dipole have already been obtained from $\underline{u}$, we shall obtain the field strengths E_m^v, H_m^v of the vertical magnetic dipole from them by replacing $\underline{q}$ by q_m and taking $-H^v(q_m)$ as E_m^v and $E^v(q_m)$ as H_m^v. Just as in the case of the flat eartn (§§ 27, 28, 29), we may differentiate only the multiplier exp (ikR) in differentiating with respect to r, ϑ, φ unless we are interested in terms of the order of $1/R^2$ and higher.

First suppose that both the transmitter and the receiver are on the ground. Since $R^2 = a^2 + (a+h)^2 - 2a(a+h)\cos\vartheta$, $R \approx a\vartheta$,

$$\frac{\partial R}{\partial \vartheta} = \frac{1}{2R} 2a\,(a+h)\sin\vartheta \approx \frac{ar\vartheta}{R} \approx \frac{a^2\vartheta}{R} \approx a.$$

Thus, differentiation of $\underline{u}$ with respect to ϑ may be replaced by multiplication by $ik\frac{\partial R}{\partial \vartheta} \approx ika$. Further, with interest in the field at the surface of the earth, we may, by virtue of Condition (34.6), replace differentiation with respect to $\underline{r}$ by multiplication by $-ik\eta$. Consequently, provided that $ka\vartheta \approx kR \gg 1$, we shall have for the vertical electric dipole

$$\begin{aligned} E_r^v &= k^2 au, \\ E_\vartheta^v &= \frac{k^2 a}{\sqrt{\varepsilon'}} u \text{ for } r = a, \\ H_\varphi^v &= -k^2 au. \end{aligned} \qquad (39.36)$$

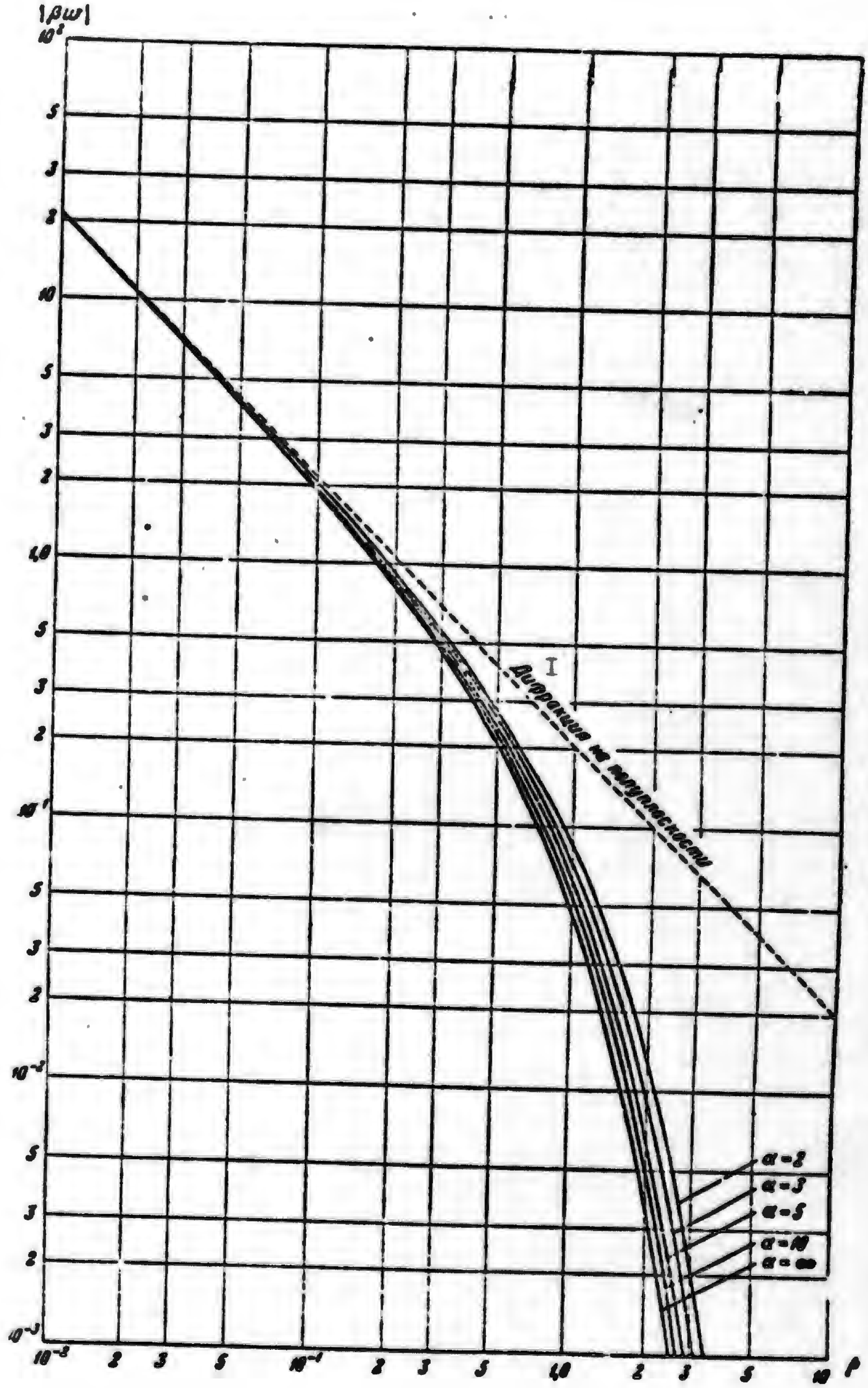

Fig. 39.4. Absolute value of attenuation function for diffraction at a sphere for a source at high elevation and an observation point with $\mu\xi \geq 7$ and various q as a function of diffraction angle ψ on an arbitrary scale

$$\rho = \left(\frac{3}{4}\right)^{2/3} \sqrt[3]{\frac{ka}{2}}\,\psi;$$

$$z = -i\sqrt[3]{\frac{4}{3}}\,q.$$

The absolute value of the attenuation function is plotted against the axis of ordinates after multiplication by $\beta \approx 0.090\mu$ (from [VI, 9]). I) Diffraction on half-plane.

The remaining components are equal to zero. It is clear from this that having selected $u = \frac{e^{ikR}}{R}$ for a dipole in free space, we have thereby assumed that the dipole moment is equal not to unity, but to a (actually, $\Pi = ru \approx au$).

The field $E^{h\vartheta}$ of a horizontal electric dipole oriented along the line of propagation, i.e., along the ϑ-axis, is obtained from this on the basis of the reciprocity theorem ($E_r^{h\vartheta} = E_\vartheta^v$) and from the usual boundary condition $E_x = \frac{1}{\sqrt{\varepsilon'}} E_z$, i.e., $E_\vartheta = \frac{1}{\sqrt{\varepsilon'}} E_r$:

$$\left.\begin{aligned} E_r^{h\vartheta} &= \frac{k^2 a}{\sqrt{\varepsilon'}} u = -H_\varphi^{h\vartheta}, \\ E_\vartheta^{h\vartheta} &= \frac{k^2 a}{\varepsilon'} u, \end{aligned}\right\} \text{for } r = a. \qquad (39.37)$$

According to the rule set forth above, the field of a vertical magnetic dipole, is

$$\begin{aligned} E_{m\varphi}^v &= k^2 av, \\ H_{mr}^v &= k^2 av, \\ H_{m\vartheta}^v &= k^2 a \sqrt{\varepsilon - \cos^2\psi}\; v \text{ for } r = a, \end{aligned} \qquad (39.38)$$

where v differs from u by the substitution of q_m for q, $v \equiv u(q_m)$. Consequently, as in the case of the flat earth (see, for example, Formula (28.7)), this field is small, e.g., proportional to $1/R^2$. As in §29, this magnetic dipole may be replaced by a square frame with two sides parallel to the direction of propagation and hence producing mutually extinguishing fields, and two sides perpendicular to the line of propagation. The field E^{hy} of one such horizontal electric dipole perpendicular to the line of propagation differs from $E_{m\varphi}^v$ by division by $-ikl \cos\psi$. Consequently, if we select a moment for the electric dipole that corresponds to the frame moment, we obtain by analogy to Formula (29.8)

$$E_\varphi^{hy} = H_r^{hy} = k^2 av. \qquad (39.39)$$

Lowered to the ground, therefore, the dipole produces at the ground surface a field of excessively high order in R. Moreover, the functions

$\underline{u}$ and $\underline{v}$ decrease exponentially beyond the horizon.

Under real conditions, therefore, the case important for the horizontal electric dipole is that in which it is elevated above the surface of the ground and a wave reflected in accordance with the interference formulas (which take the divergence coefficient of (35.7) into account) makes it appearance.

In the case of a horizontal magnetic dipole situated on the ground, the dipole may, as in the case of the plane wave, be replaced by a frame in which only its vertical sides produce a field that does not contain an additional small factor $1/\sqrt{\varepsilon}$ or 1/kR. Hence it differs by a factor $l \sin\varphi \frac{\partial}{\partial r} = ikl \sin\varphi$ from the field of the vertical electric dipole of moment m/ikl. As a result, we arrive again at the formulas for E^V with an additional sin φ, where φ is the angle formed by the direction of propagation with the axis of the magnetic dipole.

Thus, only the field of the horizontal magnetic dipole is as significant as the field of the vertical electric dipole if both corresponding points are on the ground. As concerns other cases, they are of practical interest only in the illuminated region, where the interference formulas are applicable.

In conclusion, we present a summary of the final formulas for the fields of the sources considered above. These formulas [4b] are valid throughout the space above the ground for the condition $|\sqrt{\varepsilon}| \gg 1$. (Terms of the order of 1/M and $1/\sqrt{\varepsilon}$ have been dropped as small compared to the others.)

a) For the vertical electric dipole:

$$E_r^e = -H_\varphi^e = E_0 V(x, y_0, y_A, q).$$

$$E_\theta = \frac{i}{M} E_0 \frac{\partial V(x, y_0, y_A, q)}{\partial y_0}. \qquad (39.40)$$

$$E_\varphi^e = H_r^e = H_\theta^e = 0.$$

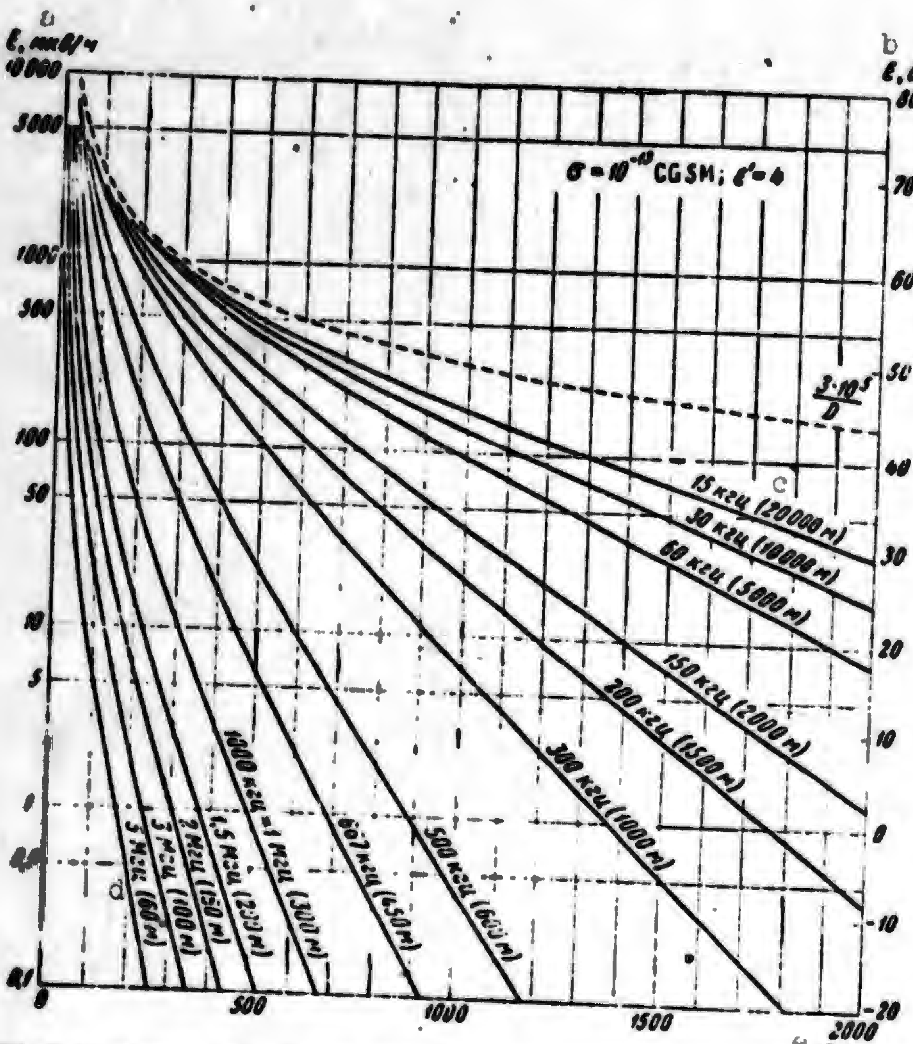

Fig. 39.5a. Field strength over ground of average moisture content as a function of distance D from transmitter for various wavelengths and a radiated power of 1 kw. Both corresponding points on the ground (Bremmer, [I, 11]). a) E, μb/m; b) E, db; c) kHz; d) MHz; e) D, km.

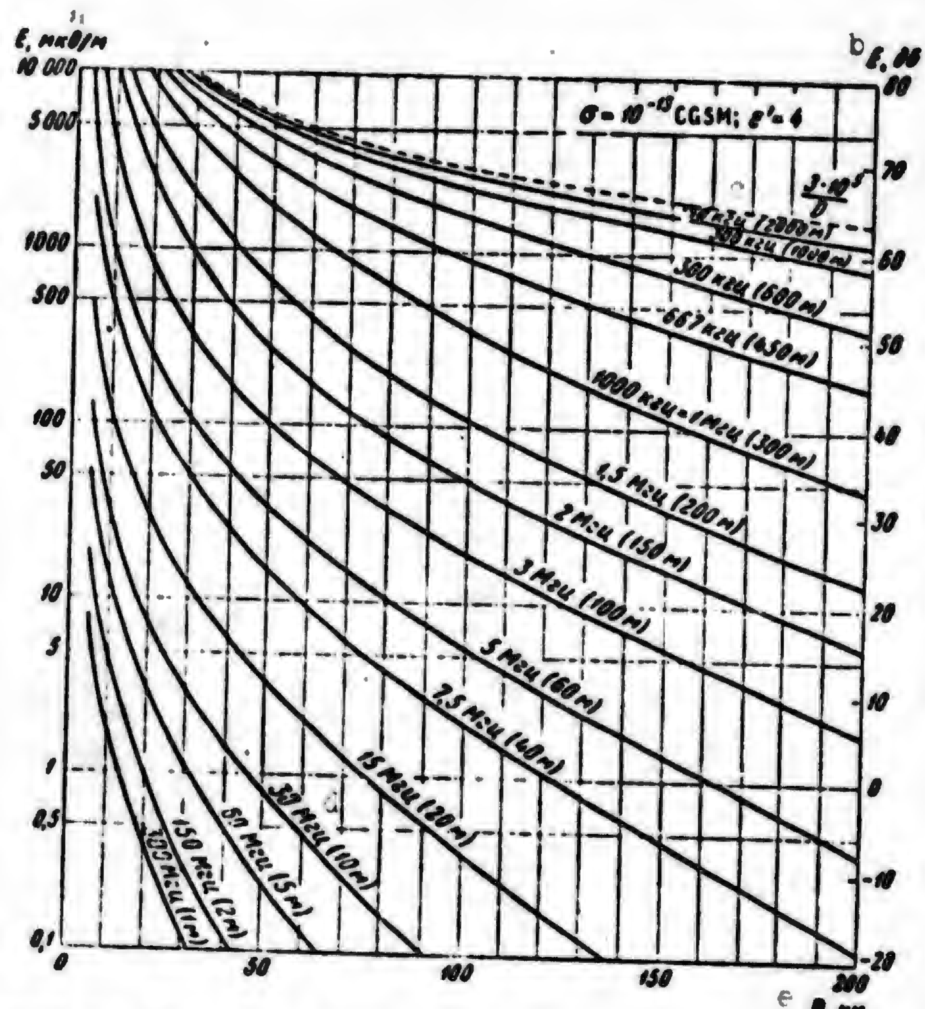

Fig. 39.5b. Same as Fig. 39.5a; shorter waves, shorter distances. a) E, μb/m; b) E, db; c) kHz; d) MHz; e) D, km.

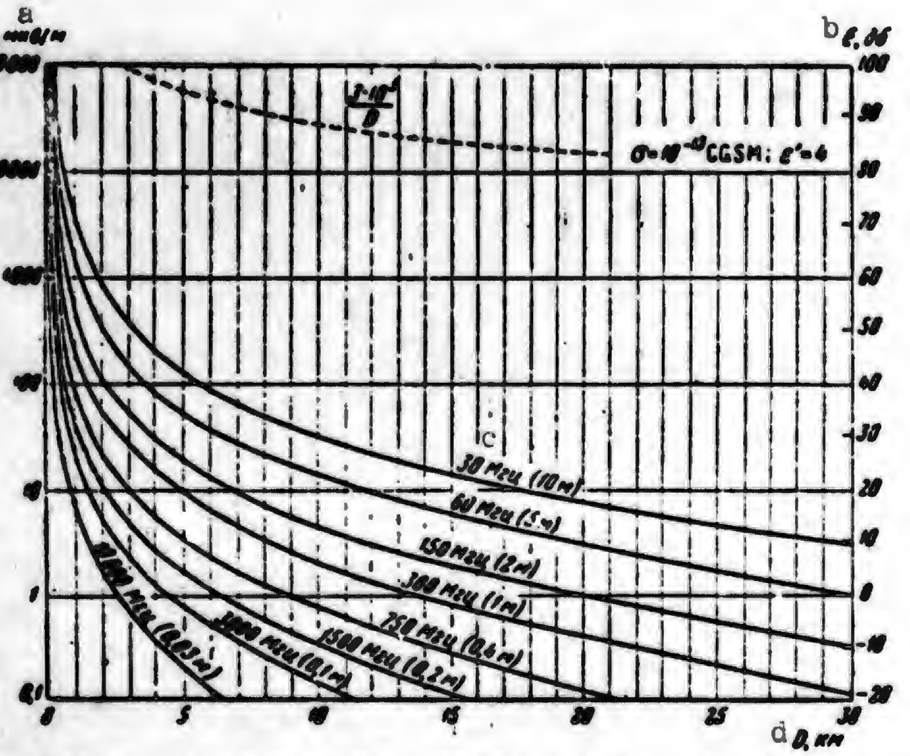

Fig. 39.5c. Same as Fig. 39.5a; meter, decimeter and centimeter waves. a) E, μv/m; b) E, db; c) MHz; d) D, km.

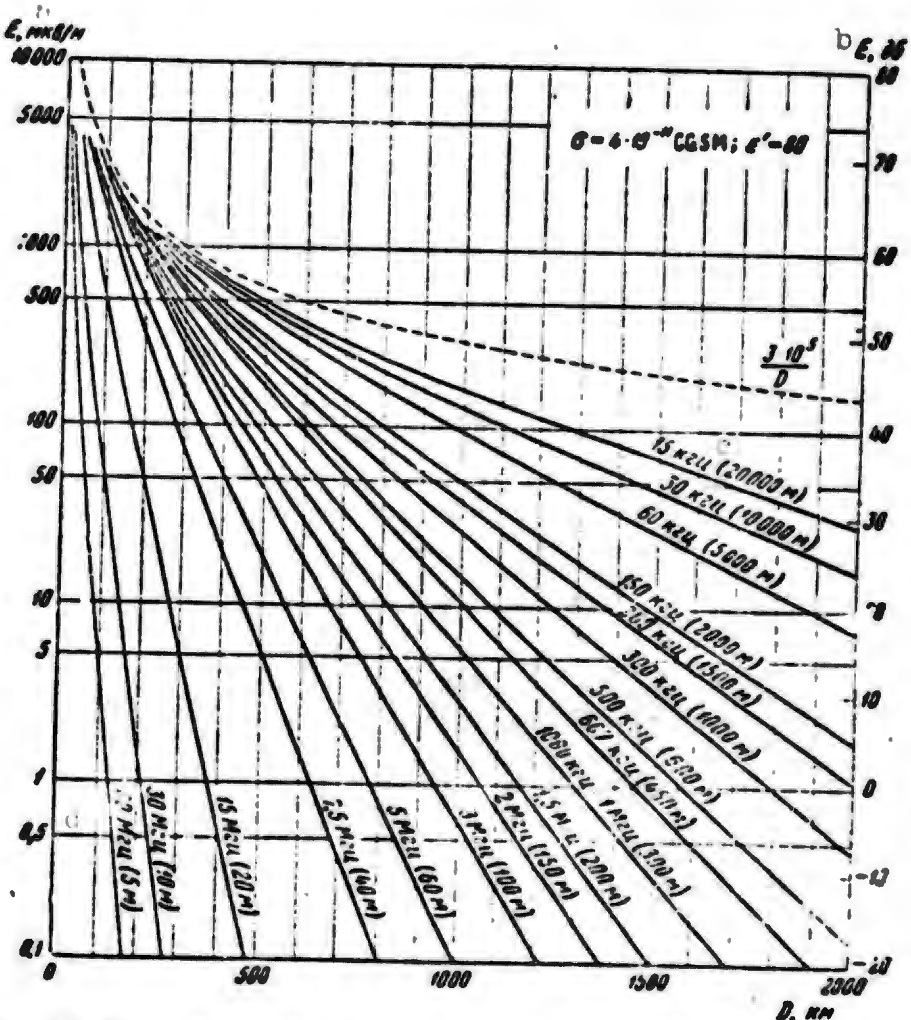

Fig. 39.6a. Same as Fig. 39.5a, but over the ocean. a) E, μb/m; b) E, db; c) kHz; d) MHz; e) D, km.

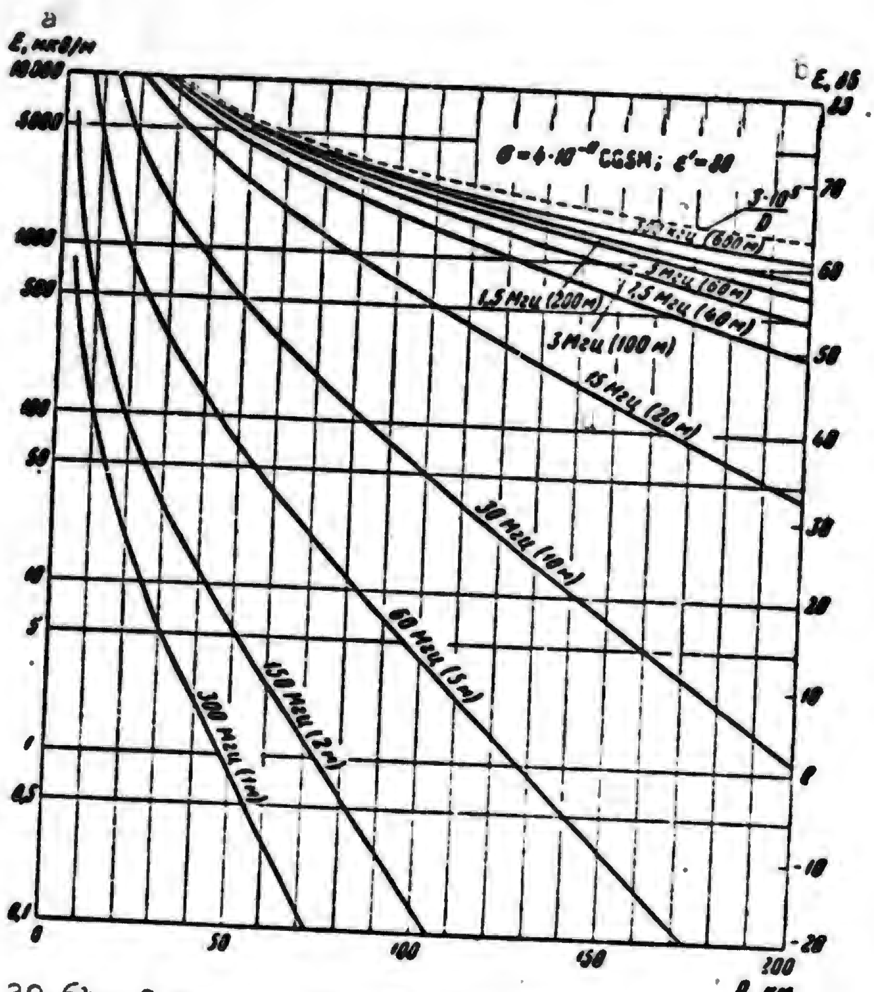

Fig. 39.6b. Same as Fig. 39.5b, but over the ocean. a) E, μv/m; b) E, db; c) kHz; d) MHz; e) D, km.

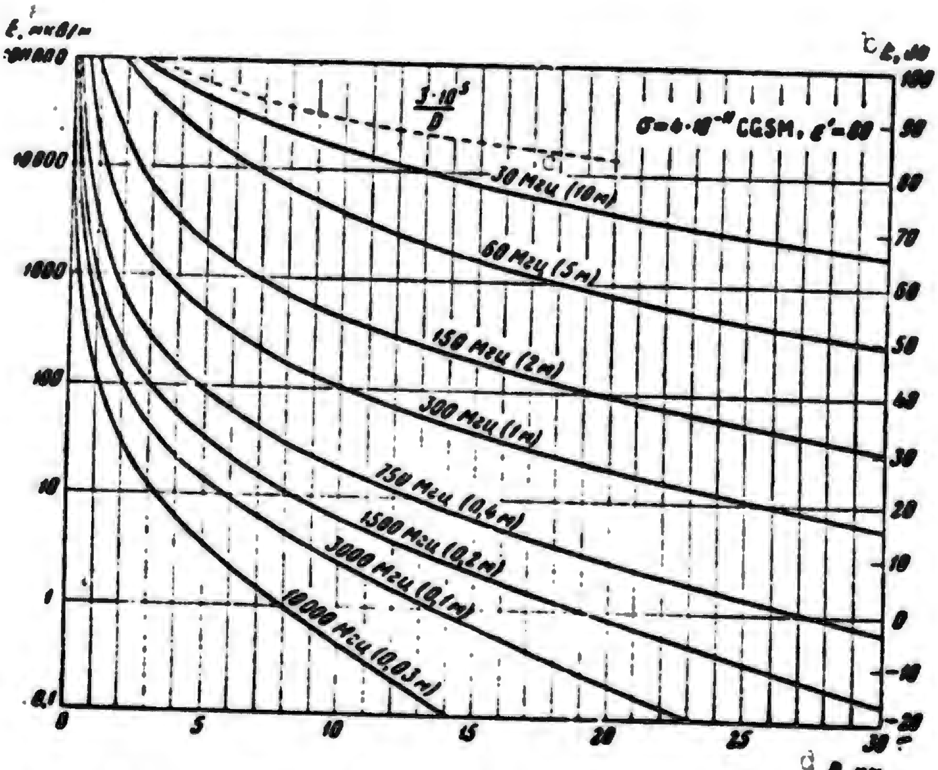

Fig. 39.6c. Same as Fig. 39.5c, but over the ocean. a) E, μv/m; b) E, db; c) MHz; d) D, km.

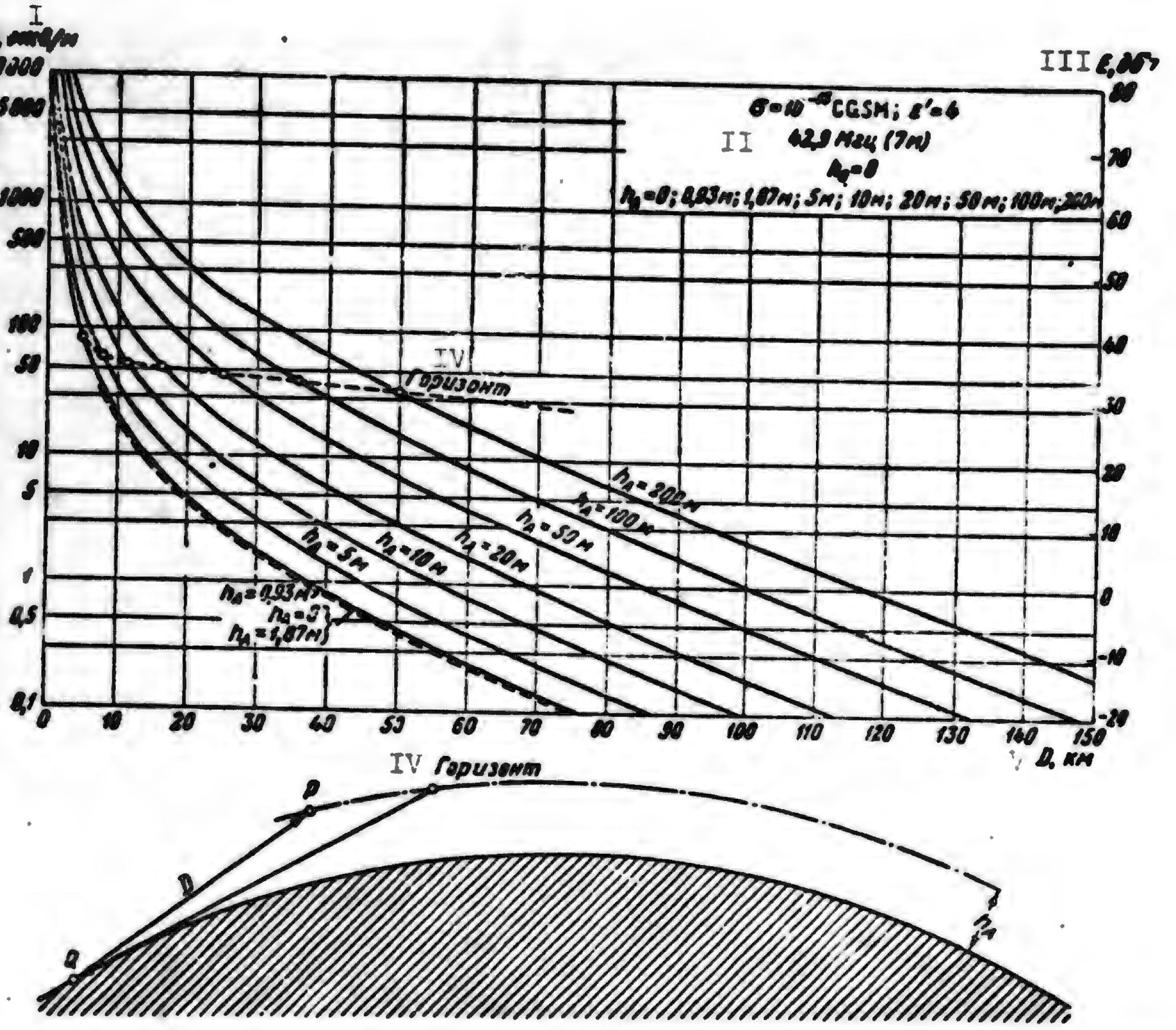

Fig. 39.7a. Field strength as a function of distance from transmitter (power 1 kw) on the ground ($h_0 = 0$), for various receiver heights h_A for a wavelength $\lambda = 7$ m, above ground of average moisture content. I) E, μv/m; II) 42.9 MHz (7 meters); III) E, db; IV) horizon; V) D, km.

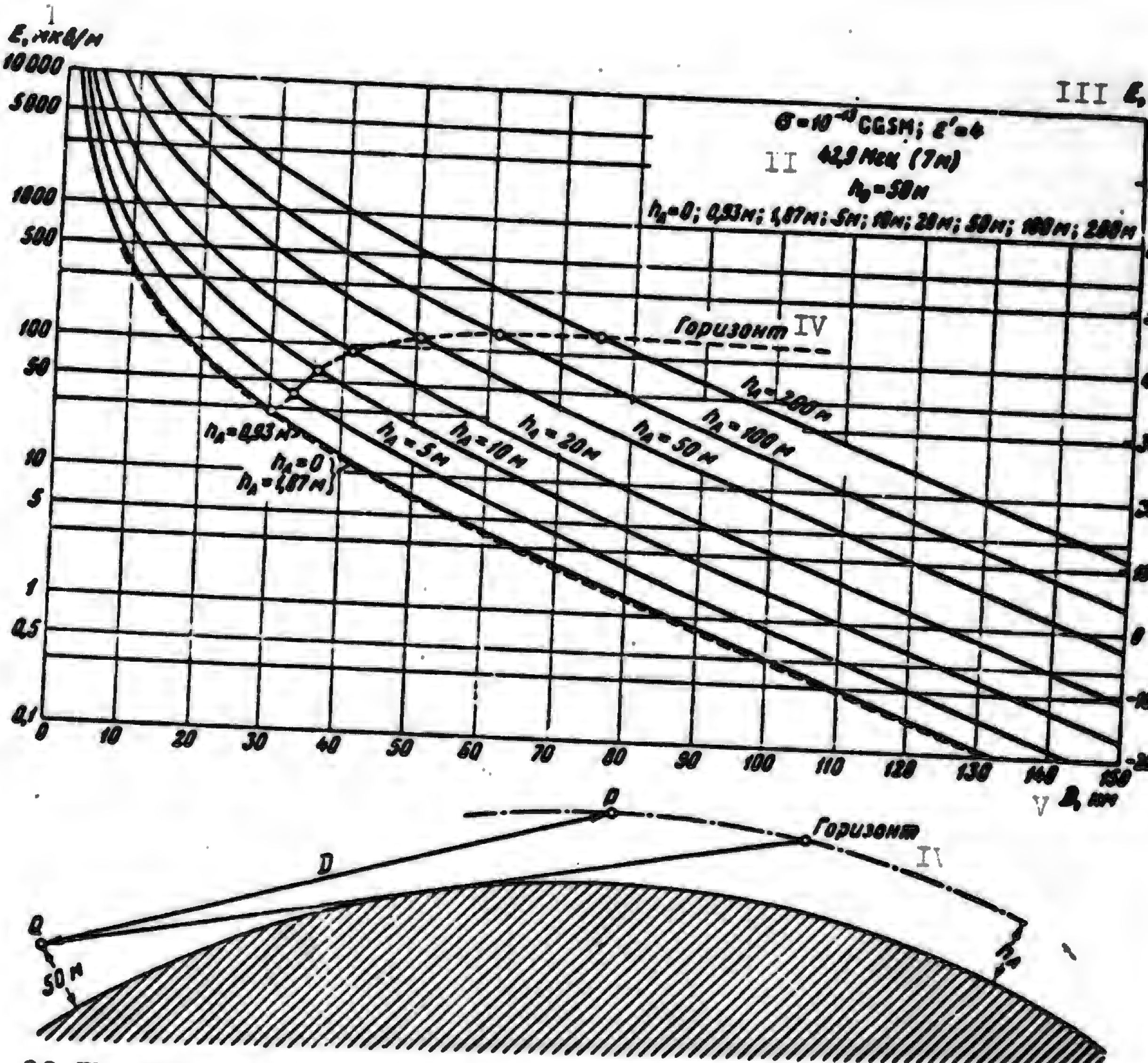

Fig. 39.7b. Same as Fig. 39.7a, but for a transmitter raised to a height h_0 = 50 m. I) E, μv/m; II) 42.9 MHz (7 meters); III) E, db; IV) horizon; V) D, km.

b) For the vertical magnetic dipole:

$$H^{v}_{mr} = H^{v}_{m\varphi} = E_0 V(x, y_O, y_A, q_m).$$

$$H^{v}_{m\theta} = \frac{i}{M} E_0 \frac{\partial V(x, y_O, y_A, q_m)}{\partial y_O}. \tag{39.41}$$

$$E^{v}_{mr} = E^{v}_{m\theta} = H^{v}_{m\varphi} = 0.$$

c) For a horizontal electric dipole situated in the plane $\varphi = 0$:

$$E^{h}_{\varphi} = H^{h}_{r} = -E_0 V(x, y_O, y_A, q_m) \sin\varphi,$$

$$E^{h}_{r} = -H^{h}_{\varphi} = \frac{i}{M} E_0 \frac{\partial V(x, y_O, y_A, q)}{\partial y_A} \cos\varphi,$$

$$H^{h}_{\theta} = \frac{i}{M} E_0 \frac{\partial V(x, y_O, y_A, q_m)}{\partial y_O} \sin\varphi. \tag{39.42}$$

$$E^{h}_{\theta} = \frac{1}{\sqrt{\varepsilon}} E^{h}_{r}.$$

d) For a horizontal magnetic dipole (vertical frame) located in the plane $\varphi = 0$:

$$H^{h}_{m\varphi} = -E^{h}_{mr} = -E_0 V(x, y_O, y_A, q) \sin\varphi,$$

$$H^{h}_{mr} = E^{h}_{m\varphi} = -\frac{i}{M} E_0 \frac{\partial V(x, y_O, y_A, q_m)}{\partial y_A} \cos\varphi, \tag{39.43}$$

$$E^{h}_{m\theta} = \frac{i}{M} E_0 \frac{\partial V(x, y_O, y_A, q)}{\partial y_O} \sin\varphi,$$

$$H^{h}_{m\theta} = 0.$$

For the electric dipole in all of these cases

$$E_0 = p \frac{k^2 e^{ika\theta}}{a\sqrt{\theta \sin\theta}}, \tag{39.44}$$

while for the magnetic dipole p (the moment of the electric dipole, $p = iIh_{eff}/\omega$ (6.4b)) should be replaced by m (the frame moment, $m = IS/c$, where I is the current in the source; h_{eff} is the effective height of the electric dipole; S is the area of the frame).

The rest of the notation is as usual: M is a large parameter (38.7); for q and q_m see Formulas (39.2a), (39.2b); y_O and y_A are the reduced heights of the source and observation point, y kh/M (see Formula (39.2a)); $\underline{x}$ is the reduced angular distance between them, x $M\theta$. Finally, V is the function of (39.21), in which the dependence on $\underline{q}$ or q_m, as the case may be, is written out in explicit form.

The attenuation functions for a sphere have now been tabulated and presented in diagram form for a large number of particular cases (see, for example, [I, 11], [5]). The problem has been elaborated in particular detail in the book by Azrilyant and Belkina [5], in which a comprehensive atlas of curves has been compiled for two extreme types of soil, q = 0 and q = ∞, including V and its derivatives, so that it is possible to carry out highly accurate calculations for the phase and amplitude of the attenuation function. Among other things, it is possi-

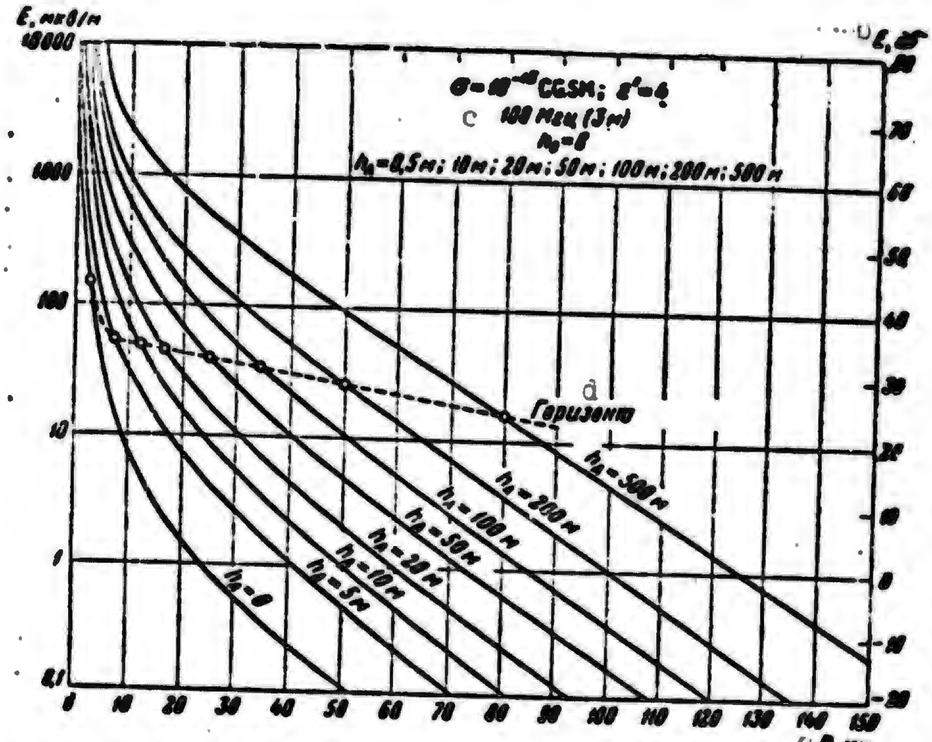

Fig. 39.8a. Same as Fig. 39.7a, but λ = 3 m. a) E, μv/m; b) E, db; c) 100 MHz (3 meters); d) horizon; e) D, km.

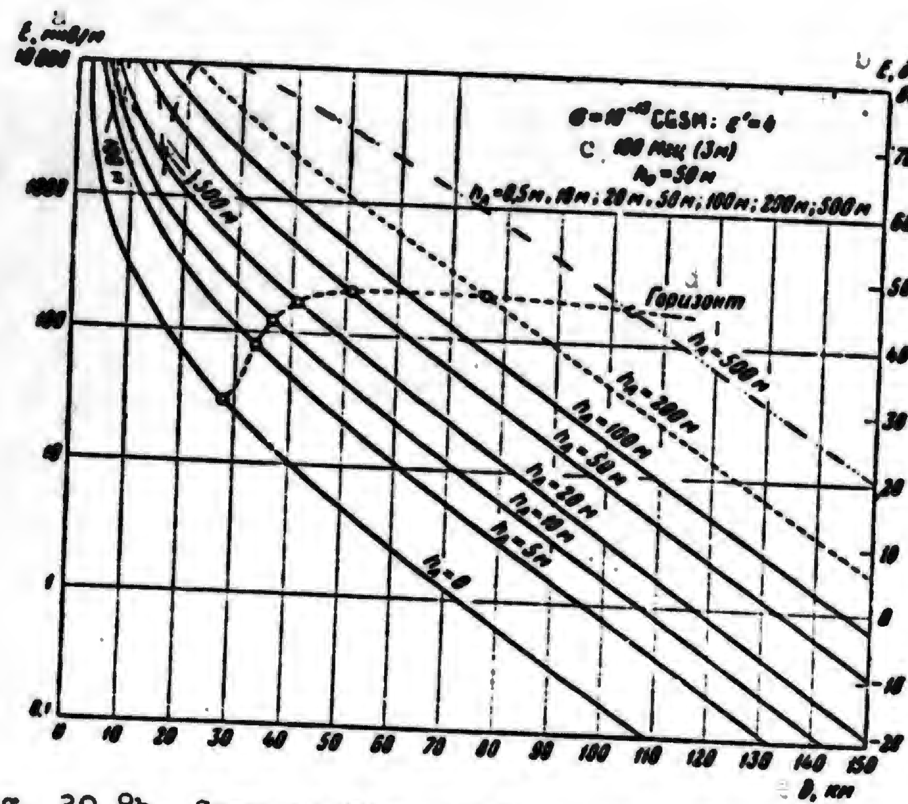

Fig. 39.8b. Same as Fig. 39.8a, but h_0 = 50 m. a) E, μv/m; b) E, db; c) 100 MHz (3 meters); d) horizon; e) D, km.

ble to devaluate rigorously the applicability of the reflection (interference) formulas. For the purpose of illustration, we present here certain less detailed but informative diagrams (Figs. 39.5-39.9) taken from Bremmer's book [I, 11]. They give a conception of the influence of soil properties, wavelength and the heights of the corresponding points on the attenuation of the radial field component.

However, experimental data indicate that the theoretical curves for a sphere surrounded by a homogeneous atmosphere describe the true field attenuation satisfactorily in far from all cases. The disagreement is particularly manifest for ultrashort waves and even shorter waves (decimeter, centimeter) beyond the limit of line-of-sight visibility, where the actually observed field is stronger than that predicted by Formula (39.20). This divergence stems from the inhomogeneity

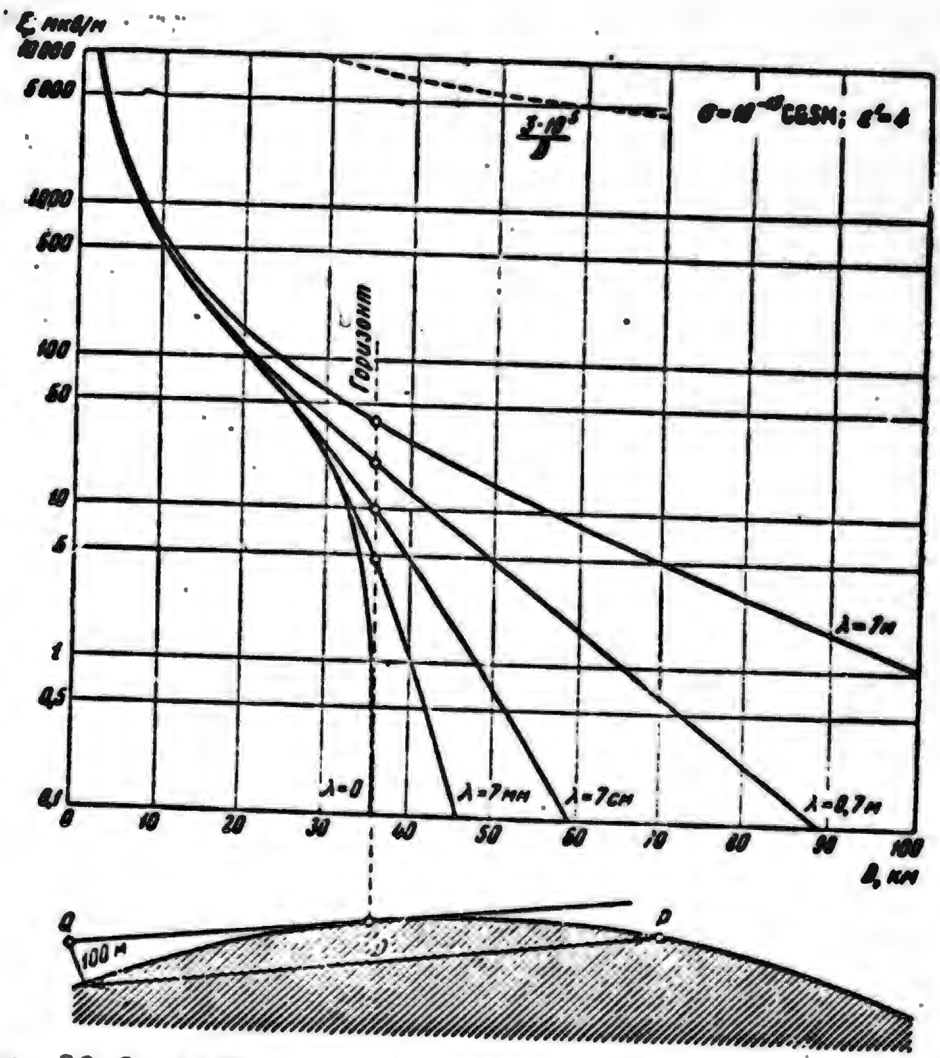

Fig. 39.9. Influence of passage across horizon line; soil of medium moisture content, h_0 = 100 m, h_A = 0 (Bremmer [I, 11]). a) E, μv/m; b) horizon.

of the atmosphere. Deviations in the range of longer-than-metric waves are accounted for by the normal decrease in the density of the air with increasing altitude (see §15), and, as we find, are described in good approximation by the same formula (39.20), in which it is necessary only to replace the earth's true radius a by the effective radius a_e, which is considerably larger than a (see §58).

In the region of shorter wavelengths, however, the situation is found to be rather complex. The formulas of the present section, even on substitution of a_e for a, at best describe diffraction only for certain average conditions. For the most part, however, they are quite inapplicable for such short wavelengths, both due to the formation of waveguide channels by inversions in the altitude variation of the tropospheric refractive index and because of diffusive scattering of the radio waves on nonregular fluctuations of air density. These problems will be examined in Chapters 9 and 10.

Manu-script Page No.	[Footnotes]
353	Not to be confused with the attenuation function of the preceding sections.

[Transliterated Symbols]

375	макс = maks = maksimal'nyy = maximum
390	эфф = eff = effektivnyy = effective

Chapter 7

FIELD ABOVE AN ELECTRICALLY INHOMOGENEOUS GROUND SURFACE

In the preceding discussion, we idealized the problem of radio-wave propagation along the ground in a variety of respects. Firstly, the interface between the ground and the atmosphere was regarded either as ideally flat or as strictly spherical. Secondly, the atmosphere was regarded as perfectly uniform. Thirdly and lastly, the soil was also considered to be fully uniform as regards its electrical properties.

The geometrical idealization is very often found to be inadmissible, primarily when the irregularity of the terrain is not negligible as compared with wavelength. We shall defer consideration of this fact to the next chapter. The inhomogeneity of the troposphere, which is particularly substantial for very short waves, will be examined in Chapters 9 and 10. Here, however, we shall dispense with the idea of homogeneous soil, which, of course, does not by any means reflect the state of affairs in the case of land masses. The small depth of penetration of radio waves into soil (Fig. 17.1) has as a consequence that no substantial "averaging over depth" takes place, while local variations of the electrical parameters along the transmission path, which are due primarily to fluctuations in the moisture contents of the soils, are quite significant. Moreover, even such practically important cases as radio communication between ship and shore indicate that it is necessary to investigation propagation of radio waves over an inhomogeneous earth.

We shall see below that the propagation of radio waves above an inhomogeneous surface exhibits highly unique properties. A number of interesting properties arise for short and longer waves. In other ranges, however, cases arise in which, for example, the field does not diminish with increasing distance, but instead increases under special but frequently encountered conditions, etc. Study of these properties is essential, on the one hand, for understanding of the radio-wave propagation mechanism and, on the other, it assists in selecting optimum conditions for radio transmission and taking account of the distortions engendered by the inhomogeneity of the soil.

In the course of time, the theory of the problem was developed for a flat earth [1, 2, 3, 4] (and generalized subsequently for a spherical earth [5, 6]) on the basis of the same method with approximate boundary conditions of the type of (21.14) and the integral equation for attenuation functions of the type of Eq. (24.12). Here, however, they were reviewed and generalized, first and foremost so that they would be applicable for variable ε and, secondly, so that they might be solved effectively under the much more complex conditions that arose.

On the other hand, other methods [7, 8, 9] also came into use later. They confirmed the conclusions of the reasoning followed above. A theory was also developed for the first time for a spherical surface [9] (the results of which were also obtained on the basis of a consistent integral equation, see [6] and §46 below). These studies culminated, for one thing, in the compilation of tables and diagrams for the final formulas [8b, 10]. We shall present some of these below. The question of disturbances to the phase and direction of propagation of disturbances to the phase and direction of propagation of radio waves on crossing the boundaries between different types of soils and the problems of the influence exerted by the numerous random inhomogeneities have been taken out of this chapter and will be considered together with other similar terrain effects in Chapter 8.

§40. APPROXIMATE BOUNDARY CONDITIONS

First of all, let us generalize the approximate boundary conditions (2.19) for the case of variable ε. We shall conduct the discussion for a flat earth, but the local relationships obtained will, of course, actually be valid also for a nonflat surface (for example, for the sphere), provided that its radius of curvature is large enough (see Formula (34.2)). We shall, moreover, assume that the source is emitting vertically polarized radiation.

Let us introduce the concept of the length over which the properties of the soil vary noticeably:

$$b = \frac{|\varepsilon|}{|\operatorname{grad} \varepsilon|}, \tag{40.1}$$

and consider first a soil that may be regarded as homogeneous within distances of the order of the wavelength in air,

$$b \gg \lambda, \quad kb \gg 1. \tag{40.2}$$

In this case, the considerations presented in connection with Formula (21.19) remain in force. Indeed, in §21 we required only that the field $\vec{E}$ in air satisfy the condition $\left(\frac{\partial E}{\partial r}\right) \approx ik_z E,$ which is the case if the attenuation function varies little over the wavelength. If the inhomogeneity of the soil has not yet made itself felt at this distance, the behavior of the attenuation function may not deteriorate. Further, as we know, the field dies out rapidly as we move deeper into the earth, so that it vanishes on a segment considerably exceeding [(17.7)]

$$l = \frac{1}{\operatorname{Im}\{k\sqrt{\varepsilon}\}} = \frac{\lambda}{\sqrt{|\varepsilon|}\sin\left(\frac{1}{2}\operatorname{arctg}\frac{4\pi\sigma}{\varepsilon'\omega}\right)},$$
$$l \geqslant \frac{1}{k|\sqrt{\varepsilon}|}. \tag{40.3}$$

Hence if Condition (40.2) is satisfied, then $b \gg l$ and within the depth of penetration of the field, the variability of soil properties in the vertical direction will not be able to manifest. Consequently, simply by regarding the dielectric constant

$$\varepsilon = \varepsilon(x, y)$$

as a slowly varying (by comparison with $\mathbf{E}$) quantity, we may recast Conditions (21.14), (21.19), (21.26) and (21.27) in the form

$$\left.\begin{aligned} \frac{\partial E_z}{\partial z} &= -\frac{ik}{\sqrt{\varepsilon^0(x, y)}} E_{z}, && (40.4)\\ E_x &= \frac{1}{\cos\psi} \frac{\cos\varphi}{\sqrt{\varepsilon^0(x, y)}} E_{z}, && (40.5a)\\ E_y &= \frac{1}{\cos\psi} \frac{\sin\varphi}{\sqrt{\varepsilon^0(x, y)}} E_{z}, && (40.5b) \end{aligned}\right\} \text{for } z = 0$$

where

$$\varepsilon^0(x, y) = \frac{\varepsilon^2(x, y)}{\varepsilon(x, y) - \cos^2\psi} \approx \varepsilon(x, y) + \cos^2\psi \approx \varepsilon(x, y). \quad (40.6)$$

Here φ is the angle formed by the line of propagation with the x-axis; ψ is the glancing angle of the wave at the plane $z = 0$,

$$k_x = k\cos\psi\cos\varphi, \quad k_y = k\cos\psi\sin\varphi, \quad k_z = k\sin\psi.$$

As usual, Condition (40.4) is valid not only for E_z, but also for the Hertzian vector (Π or Π_m) of the vertical (electric or magnetic) dipole or, in the case of a spherical surface, for the Debye functions u or v (in the magnetic case, of course, $[\varepsilon^0(x, y)]^{-1/2}$ is replaced by $[\varepsilon(x, y) - \cos^2\psi]^{1/2}$, compare Formulas (34.5)-(34.6)).

Thus, the problem is simple to solve if Condition (40.2) is observed. The situation becomes more complicated for sharply inhomogeneous soils, when $b \lesssim \lambda$. In the general case here it appears that there are no local conditions of the type (40.4)-(40.5b) at all. However, it will be appropriate to examine the important particular case in which it is possible to advance far enough into the region of small b. Boundary conditions of this type may be written, if the complex permittivity is very large,

$$|\varepsilon| \gg 1, \qquad (40.7)$$

$$b \gg \frac{\lambda}{\sqrt{|\varepsilon|}}, \quad kb|\sqrt{\varepsilon}| \gg 1, \qquad (40.8)$$

i.e., if the characteristic length of the inhomogeneity exceeds the "soil wavelength," even if it is small as compared with the wavelength in air. This region of b-values occurs when Condition (40.7) holds. Hence we may turn to the original Leontovich conditions (see §21, Subsection 4). They were based on the fact that for very large $|\varepsilon|$, the value of $k\sqrt{\varepsilon}$ is very large, so that the ray approximation of geometrical optics can be used in the soil. Consequently, as with the plane wave, Formulas (21.29) must be valid.

If the soil is inhomogeneous, but its properties do not have time to change within the limits of a segment $1/k\sqrt{\varepsilon}$, i.e., if Inequality (40.8) is satisfied, the ray approximation must be valid as before. Thus, we may rewrite Relationships (21.30) for variable ε:

$$E_x = -\frac{1}{\sqrt{\varepsilon(x, y)}} H_y, \quad E_y = \frac{1}{\sqrt{\varepsilon(x, y)}} H_x \text{ for } z = 0. \quad (40.9)$$

In exactly the same way as on conversion from (21.30) to (21.31), we shall use the field equations. However, in differentiating Eqs. (40.9), it must be remembered that ε is variable. It follows from div $\overline{E} = 0$ that

$$\frac{\partial E_z}{\partial z} = -\left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y}\right) = \frac{1}{\sqrt{\varepsilon}}\left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right) - \frac{1}{2}\frac{1}{\varepsilon^{3/2}}\left(\frac{\partial \varepsilon}{\partial x} H_y - \frac{\partial \varepsilon}{\partial y} H_x\right).$$

Since rot $\overline{H} = -ik\overline{E}$, we have

$$\frac{\partial E_z}{\partial z} = -\frac{ik}{\sqrt{\varepsilon}} E_z + \frac{1}{2\varepsilon}\left(\frac{\partial \varepsilon}{\partial x} E_x + \frac{\partial \varepsilon}{\partial y} E_y\right) \text{ for } z = 0. \qquad (40.10)$$

This formula generalizes Condition (40.4) for the case of a rapidly varying soil with large $|\varepsilon|$. It is easily seen that since $E_{x,y} \sim \frac{1}{\sqrt{\varepsilon}} E_z$, the ratio of the term added on the right to the first term is of the order of 1/kb. Within the framework of Condition (40.8), this quantity may be very large. Essentially $\partial E_z/\partial_z$ is an anomalously small quantity for a soil that conducts well (as compared with the "natural" scale of variation, kE_z). Hence it may be necessary to take account of the inhomogeneity of even a soil that conducts well.

Further, within the limits of a segment of the order of several λ, the variation of ε (which remains large everywhere) may influence the principal field components E_z, H_x and H_y only in terms of the order of $1/\sqrt{\varepsilon}$ (this will be demonstrated by direct calculations in §49 on the basis of Condition (40.10)). Hence accurate to terms of higher order in this small quantity, the relation of E_z with H_x and H_y is the same as for constant ε. As a consequence, according to (21.26) and (21.30), it follows from Equalities (40.9) that

$$\left.\begin{aligned} E_x &= \frac{\cos\varphi}{\cos\psi\sqrt{\varepsilon(x,y)}}E_z, && (40.11a)\\ E_y &= \frac{\sin\varphi}{\cos\psi\sqrt{\varepsilon(x,y)}}E_z && (40.11b) \end{aligned}\right\}\ \text{for}\ z=0$$

(for another derivation of (40.10), see [I, 1], §40). Formulas (40.11) can be obtained in a different way if we remember that Formula (40.10) can be rewritten in the form

$$\frac{\partial E_z}{\partial z} = -\frac{ik}{\sqrt{\varepsilon}}E_z + \frac{1}{\sqrt{\varepsilon}}\left(\frac{\partial}{\partial x}(\sqrt{\varepsilon}E_x) + \frac{\partial}{\partial y}(\sqrt{\varepsilon}E_y)\right) - \left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y}\right),$$

or, remembering that div $\mathbf{E} = 0$,

$$\frac{\partial}{\partial x}(\sqrt{\varepsilon}E_x) + \frac{\partial}{\partial y}(\sqrt{\varepsilon}E_y) = ikE_z.$$

Since we may consider $E_z \sim \exp(i(k_x x + k_y y))$ with an accuracy to terms of the order of $1/\sqrt{\varepsilon}$, Formulas (40.11) are the solution to this equation, which vanishes together with E_z.

Now Formula (40.10) can also be rewritten:

$$\frac{\partial E_z}{\partial z} = -\frac{ik}{\sqrt{\varepsilon}}E_z + \frac{1}{2k\varepsilon\cos^2\psi}\left(k_x\frac{\partial\varepsilon}{\partial x} + k_y\frac{\partial\varepsilon}{\partial y}\right)E_z\ \text{for}\ z=0. \quad (40.11c)$$

If, however, it is remembered that $\frac{\partial}{\partial x}E_z \approx ik_xE_z$, then

$$\frac{\partial E_z}{\partial z} = -\frac{1}{k\cos^2\psi}\left(\bar{k}, \operatorname{grad}\frac{E_z}{\sqrt{\varepsilon(x,y)}}\right) \text{ for } z=0,$$

where $\bar{k}$ is a two-dimensional vector $\bar{k}(k_x, k_y)$.

Formulas (40.11a) and (40.11b) indicate that for rapidly varying ε, $kb < 1$, marked variation of E_x and E_y, as distinct from E_z, takes place on segments much shorter than the wavelength in air: $\frac{\partial}{\partial x}E_x \sim \frac{1}{b}E_x$, etc.

A more detailed evaluation of the error of the above formulas is obtained if we apply the approximation of geometrical optics successively to the Maxwell equations with variable ε and do not stop at the first nonvanishing approximation in $1/\sqrt{\varepsilon}$, but also calculate the next one. This gives [IV, 3] for a nonuniform and uneven surface

$$E_x = \sqrt{\frac{\mu}{\varepsilon}}\,H_y\left\{1+\frac{1}{ik\sqrt{\varepsilon\mu}}\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}+\frac{\partial\ln\sqrt{\varepsilon}}{\partial z}\right)\right\}. \quad (40.13)$$

where μ is the permeability, which is assumed constant (we assume $\mu = 1$ everywhere); ρ_1 and ρ_2 are the principal radii of curvature of the surface; $\underline{z}$ is the normal to the surface. Thus, the boundary conditions (40.9) are particularly sensitive to variability of soil properties along the vertical $\underline{z}$ (the derivatives with respect to $\underline{x}$ and $\underline{y}$ appear only in the next order in $1/\sqrt{\varepsilon}$). The range of applicability of the approximate conditions (40.9) is, as we see, as follows: the surface radii of curvature must be large as compared with the depth of radio wave penetration:

$$\rho_{1,2} \gg \frac{1}{k|\sqrt{\varepsilon}|}.$$

The properties of the soil may vary only slightly on this same interval.

Thus, in much the same way as in the case of a homogeneous soil, we have obtained an impedance-type boundary condition for the function E_z, i.e., a condition that contains neither the field in the earth nor any other unknown functions. Using this condition, E_z can be found in the upper half-space from the wave equation. Then we can determine E_x and E_y on the surface $z = 0$ from Formulas (40.11a and b) and, finally, solve the equations for E_x and E_y for $z > 0$. On the other hand, staying within the framework of the geometrical-optics approximation, all field components may, for small negative z, be regarded as diminishing in accordance with the law

$$E_1(x, y, z) = E_1(x, y, 0)\, e^{-ik\sqrt{\varepsilon}\, z}. \tag{40.14}$$

Consequently, the field in the ground is known if we know the field $E(x, y, 0)$, which, with the exact boundary conditions, gives $E_1(x, y, 0)$ uniquely.

Usually the soils are sharply inhomogeneous in nature $kb < 1$, for long waves, when the displacement currents may be disregarded. Hence the applicability condition (40.8) for the formulas obtained may be written in the form

$$b \gg b^{(0)} \approx \sqrt{\frac{c\lambda}{8\pi^2\sigma}} \approx 2\cdot 10^3 \sqrt{\frac{\lambda_m}{\sigma}}, \tag{40.15}$$

where the last expression gives the smallest admissible b in meters if the wavelength in air is expressed in meters and σ is expressed in CGSE units. Thus, for typical soils, $\sigma \sim 10^8$, and for $\lambda_m \sim 100$ m, the characteristic length of the inhomogeneities must exceed 2 m substantially. In particular, in considering the propagation of radio waves near the shoreline, the transitional zone between sea and land may always be regarded as wide by comparison with this quantity $b^{(0)}$ and then the theory can be used.

§41. THE GENERALIZED INTEGRAL EQUATION

The theory for a nonuniform surface is simplified substantially if we base it on the integral equation that generalizes Eq. (24.6) for this case and on the generalized boundary conditions (40.4) and (40.11).

In the case of an inhomogeneous soil, we shall not be able to state in advance the field of even a vertical electric dipole is described by the Hertzian vector with a single nonzero component. Hence we shall consider the field $\mathbf{E}$ at once. The Green's theorem is, of course, also valid for it, so that according to Formula (8.5) we have for E_x and E_y on introduction of the Green's function v_-, if $\mathbf{E}^{(0)}$ is the field of the same source above an ideal-conductor plane,

$$E_{x,y}(x, y, z) = E^0_{x,y} + \frac{1}{2\pi}\int E_{x,y}(x', y', 0)\left(\frac{\partial}{\partial z'}\frac{e^{ik\rho}}{\rho}\right)dx'\,dy',$$
$$\rho = \sqrt{(x-x')^2+(y-y')^2+(z-z')^2}. \qquad (41.1a)$$

On the other hand, we have for E_z according to Formula (8.7) (Green's function v_+),

$$E_z(x, y, z) = E^0_z - \frac{1}{2\pi}\int\frac{\partial E_z(x', y', 0)}{\partial z'}\frac{e^{ik\rho}}{\rho}dx'\,dy'. \qquad (41.1b)$$

Indeed, for $\varepsilon = \infty$, both E_x and E_y and $\partial E_z/\partial z$ vanish for $z' = 0$. But $\frac{\partial\rho}{\partial z'} = -\frac{\partial\rho}{\partial z}$ in Formula (41.1a), so that differentiation may be taken out of the integrand. On the other hand, substituting Expression (40.12) in Formula (41.1b), we can perform integration with respect to x' and y' by parts. Hence the fields vanish at infinity and the integrated expressions drop out. Then, substituting $\frac{\partial\rho}{\partial x'} = -\frac{\partial\rho}{\partial x}$, $\frac{\partial\rho}{\partial y'} = -\frac{\partial\rho}{\partial y}$, we see that all three expressions of (41.1) may be written in the following symmetric form (assuming $\frac{\partial E_z}{\partial x} = ik_x E_z$, and so forth):

$$E_x = E^0_x - \frac{\partial A_x}{\partial z}, \quad E_y = E^0_y - \frac{\partial A_y}{\partial z}. \qquad (41.2a)$$

$$E_z = E^0_z + \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y}. \qquad (41.2b)$$

$$A_{x,y} = \frac{1}{2\pi}\int\frac{k_{x,y}}{k\cos^2\psi}\frac{1}{\sqrt{\varepsilon'(x', y')}}E_z(x', y', 0)\frac{e^{ik\rho}}{\rho}dx'\,dy', \qquad (41.2c)$$

where the expression for A_y differs from the expression for A_x by having $k_x = k\cos\psi\cos\varphi$ in the integrand replaced by $k_y = k\cos\psi\sin\varphi$. If the source is a vertical electric dipole at point $(0, 0, z_0)$, then, on suitable selection of the dipole moment, we have in the plane $z = 0$

$$E_z^0 = \frac{e^{ikD}}{D}\cos^2\psi, \quad D = \sqrt{x^2 + y^2 + z_0^2}. \qquad (41.3a)$$

Converting to the attenuation function w

$$E_z = E_z^0 w, \qquad (41.3b)$$

we obtain the following equation for it:

$$w = 1 + \frac{ikD}{2\pi}\int \frac{w(x', y')}{\sqrt{\varepsilon^0(x', y')}}\left\{1 + \frac{i}{k^2\cos^2\psi}\left(k\,\mathrm{grad}\ln\sqrt{\varepsilon^0}\right)\right\}\frac{e^{ik(r+\rho)}}{r\rho}\,dS',$$
$$r = \sqrt{x'^2 + y'^2 + z_0^2}, \quad \rho = \sqrt{(x - x')^2 + (y - y')^2 + z^2}. \qquad (41.4)$$

Here the multiplier $\frac{\cos^2\psi(x', y')}{\cos^2\psi(x, y)}$ also appears in the integrand, but we have dropped it on the assumption that z_0 and z are quite small.

The situation is particularly simple if the departure from the field above an infinitely conductive surface is found to be small. Then, in virtue of the small factor $1/\sqrt{\varepsilon^0}$, the integral may be regarded as a minor disturbance and an undisturbed field put for E_z on the right. Examples of this will be given further on. However, a peculiarity of the propagation process over the earth consists in the fact that even if the absolute value $|\varepsilon^0|$ is large, the field may differ substantially from the field above an infinitely conductive surface, i.e., the entire integral may not be small. This is clear even from the fact that ε^0 = const, when the normal attenuation function is the solution, the departure of the field from the field over an ideal conductor becomes very large at large numerical distances. Hence in the general case the solution by successive approximations i inapplicable and some other methods will be necessary.

If the term containing grad ε^0 can be disregarded (for example, if $kb \gg 1$, but not only in this case under certain special circumstances; see below), then the braces in Eq. (41.4) can be dropped. This equation may be solved directly with a special form of the function

$\varepsilon^0(x, y)$, even in the case of arbitrarily large soil-homogeneity distortions. It has been solved [1, 2, 3] for the case $z = 0$ and

$$\left.\begin{array}{ll}\varepsilon^0 = \infty, & x < 0,\\ \varepsilon^0 = \varepsilon_0^0 = \text{const}, & x > 0,\end{array}\right\} \tag{41.5}$$

which describes the propagation of radio waves above a surface consisting of ocean ($\varepsilon \approx \infty$) and dry land ($\varepsilon \neq \infty$). Needless to say, it is still necessary to prove here that the region of the ε discontinuity, where grad $\varepsilon = \infty$, is nonessential; this is actually the case if the corresponding points are situated on opposite sides of the boundary. However, if both of these points are on the same side of the boundary ("coastline reflection"), then even though the disturbance is small, it will depend essentially on the transitional zone and here the term with grad ε is very important (see §50).

This special case, which has come to be known as coastline refraction, since the entire question grew out of study of radio direction-finding errors in the vicinity of coastlines, was the first inhomogeneous-surface problem to be solved. It brought out a number of peculiar properties of the process and therefore acquired fundamental importance.

However, even in slightly more complex cases, even, for example, for two different soils of finite conductivity,

$$\left.\begin{array}{ll}\varepsilon^0 = \varepsilon_1^0 = \text{const} \neq \infty, & x < 0,\\ \varepsilon^0 = \varepsilon_2^0 = \text{const} \neq \infty, & x > 0,\end{array}\right\} \tag{41.6}$$

it was possible to obtain a solution only by a different method that permits simplifying the solution also for the case of (41.5). The method is based on substitution of another more general equation for the integral equation (41.4) [3]. We shall now pass to derivation of this equation.

Our point of departure in earlier chapters was the integral equation for field strength or for the Hertzian vector, which is based on

Formula (5.8) or (5.14). It expresses the field at a given point in terms of the sum of the volume integral over assigned sources and a surface integral containing values of the field itself and its derivative on a certain surface. In deriving these formulas, which give the solution to the wave equation (5.1),

$$\nabla^2 u + k^2 u = U, \tag{41.7}$$

essential use is made of an auxiliary function – the Green's function $v_0 = \frac{1}{r} e^{ikr}$, where $r = |R - R'|$ is the distance between the observation point and the point in question. The entire method and the integral relation based on it are fundamental to the theory of propagation of wave processes in general.

However, it was emphasized back in §8 that such selection of the Green's function is not the only one possible. Even in the case of an ideally conductive surface (§20), we used as the Green's function the combination of the function v_0 and a function describing reflection at a plane. In particular, we saw in §8 that it is required of the Green's function only that it 1) be a solution of the wave equation (41.7) without sources ($U = 0$); 2) that it increase without limit, as $1/r$, as $r \to 0$, and 3) that it diminish more rapidly at infinity than $1/r$ or, on the other hand, that it diminish as $1/r$ and satisfy the radiation condition. The Green's function v_0 that we have used everywhere may be interpreted as the field of a point dipole in a vacuum. It is obvious from physical considerations that all requirements made of the Green's function are also satisfied by the field of a dipole situated above a flat ground surface of finite conductivity. In other words, we may use not only v_0, but also the function

$$v = v_0 w_0 (R, R') = \frac{e^{ik|R-R'|}}{|R-R'|} w_0 (R, R'), \tag{41.8}$$

as the Green's function; here w_0 is the attenuation function for the placement of the corresponding points above a surface characterized by a certain $\varepsilon^0 = \varepsilon_0^0 = \text{const}$, as indicated in the argument. In particular, if w_0 denotes the attenuation function that we found in Chapter 5 for the case of a vertical dipole situated at a point $\vec{R}'$ and observed at a point $\vec{R}$, i.e., the normal attenuation function given by Formula (25.30), then the function (41.8) will be a suitable Green's function, since (as will be clear from the physical interpretation of the function y), it satisfies all three of the requirements enumerated above and, in particular, is an approximate solution to the wave equation (41.7). All results found using it will also be valid with the accuracy with which it is a solution of the wave equation.

In the nomenclature used in Formulas (26.21)-(26.22a), this function may be written as $y_0(|r-r'|;\ z',\ z)$, where $\vec{r}$ and $\vec{r}'$ are the projections of $\vec{R}$ and $\vec{R}'$ onto the xOy-plane; z' is the current point in the integral; z is the ordinate of the observation point; or, since z' and z appear only in the form of their sum, as $y(|r-r'|;\ z+z')$. For z + z' = = 0 this function satisfies the condition

$$\frac{\partial y_0(|r-r'|;\,0)}{\partial z} = -\frac{ik}{\sqrt{\varepsilon_0^0}}\,y_0(|r-r'|;\,0). \qquad (41.9)$$

Here

$$y_0(\rho;\ z+z') = 1 + 2\sqrt{s_0\rho}\int\limits_{\sqrt{s_0\rho}\left(1+\sqrt{\varepsilon_0^0}\sin\psi\right)}^{i\infty} e^{v^2 - s_0\rho\left(1+\sqrt{\varepsilon_0^0}\sin\psi\right)^2}\,dv, \qquad (41.10)$$

where

$$\sin\psi \approx \frac{z+z'}{D}, \quad s_0 = \frac{ik}{2\varepsilon_0^0}.$$

We stress that here the parameter ε_0^0 is arbitrary. In other words, we select as the Green's function the field of a dipole above a certain fictitious auxiliary homogeneous surface. Having this parameter, we obtain an opportunity to simplify the solution of various problems.

According to (5.14), we may now write for the field at any point in space (x, y, z)

$$E_z = -\frac{1}{4\pi}\int U(R')\frac{e^{ik\rho}}{\rho}y_0\,dV' + \frac{1}{4\pi}\int_S\left\{\frac{\partial E_z}{\partial n}\frac{e^{ik\rho}}{\rho}y_0 - \frac{\partial}{\partial n}\left(\frac{e^{ik\rho}}{\rho}y_0\right)E_z\right\}dS'. \quad (41.11)$$

Wishing to select the interface plane between the earth and the atmosphere $\left(\frac{\partial}{\partial n} = -\frac{\partial}{\partial z'}\right)$ as our surface S, we must differentiate the function e^{ikr}/r with respect to z with a certain caution, since here r may vanish. Hence it is necessary to proceed as outlined in §36 (see Formulas (36.6) and (36.7)). [We might also take another approach, selecting, as in §20, not Function (41.8), but the function

$$v_0 = \left\{\frac{e^{ik|R-R'|}}{|R-R'|} + \frac{e^{ik|R_1-R'|}}{|R_1-R'|}\right\}w_0(R-R') \quad (41.12)$$

as the Green's function.] Differentiating the product with respect to z and applying Formula (41.9), we obtain

$$\int\frac{\partial}{\partial n}\left(\frac{e^{ik\rho}}{\rho}y_0\right)E_z\,dS' = \int y_0E_z\left(ik - \frac{1}{\rho}\right)\frac{e^{ik\rho}}{\rho}\left(\frac{\partial\rho}{\partial n}\right)_0 dS' - 2\pi E_z y_0 - \quad (41.12a)$$

$$-ik\int\frac{1}{\sqrt{\varepsilon_0^0}}y_0E_z\frac{e^{ik\rho}}{\rho}dS'.$$

Since for the plane $\left(\frac{\partial\rho}{\partial n}\right)_0 = 0$, the following general formula is valid in particular:

$$\lim_{z\to 0}\int f(x', y')\frac{\partial}{\partial z'}\frac{e^{ik\rho}}{\rho}\bigg|_{z'=0}dx'\,dy' = -2\pi f(x, y). \quad (41.12b)$$

Thus, we obtain

$$E_z = -\frac{1}{2\pi}\int_V U(R')\frac{e^{ik\rho}}{\rho}y_0\,dV - \frac{1}{2\pi}\int_S\left(\frac{\partial E_z}{\partial z'} + \frac{ikE_z}{\sqrt{\varepsilon_0^0}}\right)\frac{e^{ik\rho}}{\rho}y_0\,dx'\,dy'. \quad (41.13)$$

Here $\rho = \sqrt{(x-x')^2 + (y-y')^2}$; in the integral with respect to S', $y_0 \equiv y_0(\rho; 0)$ is the attenuation function for a dipole situated at a point (x', y', 0), the field of which is considered at the point (x, y, 0); y_0 is calculated for a certain auxiliary homogeneous earth with $\varepsilon^0 = \varepsilon_0^0$; U is the source density of the field of (3.9b), a more detailed expression for which will not be necessary. Since $\partial E_z/\partial z$ is defined in terms of Ez for z = 0, we also obtain an integral equation for Ez. Substituting the value of (40.12) and taking y_0 out from under the volume-integral sign by virtue of the point character of U, we obtain

$$E_z = E_z^0 y_0(D; z_0) + \frac{ik}{2\pi}\int\left\{\frac{1}{\sqrt{\varepsilon^0(x, y)}}\left[1+\frac{i}{k^2\cos^2\psi}(k \operatorname{grad}\ln\sqrt{\varepsilon^0})\right]-\right.$$

$$\left.-\frac{1}{\sqrt{\varepsilon_0^0}}\right\} E_z y_0(\rho; 0)\frac{e^{ik\rho}}{\rho}dx'\,dy'. \qquad (41.14)$$

where

$$E_z^0 = -\frac{1}{2\pi}\int_V U(R')\frac{e^{ik\rho}}{\rho}dV' \qquad (41.15)$$

is the field that the given sources U would create at the observation point if the earth were ideally conductive; $D = \sqrt{x^2+y^2}$ is the projection of the distance from the source to the observation point onto the plane z = 0. Thus, the first term in the right member gives the field that the given source would create at the given point if the earth were homogeneous and had the same ε^0 as the selected auxiliary soil ε_0^0.

At the surface of the earth, we have for a source of the nature of a vertical dipole, to within the constant multiplier

$$\left.\begin{aligned} E_z^0 &= \frac{e^{ikR}}{R}\frac{D^2}{R^2} \approx \frac{e^{ikR}}{R}, \\ D &= \sqrt{x^2+y^2},\ R = \sqrt{D^2+z_0^2}. \end{aligned}\right\} \qquad (41.16)$$

Below we shall devote detailed analysis to the fundamental case $z_0 = 0$, i.e., the case in which not only the receiver but also the transmitter are on the plane z = 0. For it, the normal attenuation function is a function only of numerical distance. Hence we shall write y_0 in the form $y(s_0\rho)$, etc.

The principal interest attaches to the case of a function ε^0 not varying very rapidly, when Condition (40.2) is satisfied and the square brackets in (41.14) can be omitted. Then, proceeding from the field E_z to the attenuation function $\underline{w}$ by Formula (41.3b), we obtain instead of Eq. (41.4) the <u>generalized integral equation</u> [2, 3]

$$w(D; z_0) = y_0(D; z_0) + \frac{ikD}{2\pi}\int\left(\frac{1}{\sqrt{\varepsilon^0(x', y')}} - \frac{1}{\sqrt{\varepsilon_0^0}}\right) e^{ik(r'+\rho-D)} y(s_0\rho) dS', \quad (41.17)$$

which is equivalent to Eq. (41.4) with $\varepsilon_0^0 = \infty$, $y(s_0\rho) = y_0(D; z_0) = 1$. As usual, the exponential factor isolates the first Fresnel zone as the most essential region. If ε^0 is constant in the transverse direction within the zone, then, expanding $\underline{r}$ and ρ in series in y'^2 and integrating with respect to y', we obtain finally

$$w(D; z_0) = y_0(D; z_0) + i\sqrt{\frac{D}{\pi}}\int_0^x\left(\sqrt{s(x', y')} - \sqrt{s_0}\right)\frac{w(x'; z_0)\, y(s_0\rho)}{\sqrt{x'(D-x')}} dx', \quad (41.18)$$

$$s(x', y') = \frac{ik}{2\varepsilon^0(x', y')}; \quad s_0 = \frac{ik}{2\varepsilon_0^0} = \text{const.}$$

This equation, which is generalized by comparison with Eq. (41.4) (the square brackets from Eq. (41.4) may, of course, be carried over) by the introduction of the arbitrary constant parameter s_0, is very useful.

All of the considerations presented above may also be generalized for the case of a spherical earth [5, 6].

The Green's theorem for the radial field component at a point A above an inhomogeneous sphere, $E_r(A)$, gives

$$E_r(A) = E_r^0(A) + \frac{1}{4\pi}\int\left\{\frac{\partial v}{\partial n}E_r - \frac{\partial E_r}{\partial n}v\right\} dS, \quad (41.19)$$

where E_r^0, as we shall see below, is the field above a certain homogeneous sphere. Here the observation point is not on the sphere itself. To pass to the case r = a, we must apply the formulas of the analogous discussion in §36. As we did there (see Fig. 36.1), let us denote the distance from the observation point A in space to the current integration surface by r', the central angle between point A and the current point by θ, and the height of point A above the ground by h_A. By way of the Green's function v, we select the field of a point dipole above a homogeneous sphere characterized by a certain diameter ε_0^0 (and by a corresponding value of $q_0 = q(\varepsilon_0^0)$ (39.2a)). According to Formulas (39.1)-(39.2),

$$v = \frac{e^{ika\theta}}{r'} V(x', h_A; q_0), \tag{41.20}$$

where x' corresponds to the distance $a\theta$ according to Formula (39.2). Since for r = a

$$\frac{\partial V}{\partial n} = -\frac{\partial V}{\partial r} = \frac{ik}{\sqrt{\varepsilon_0^0}} V, \quad \frac{\partial E_r}{\partial n} = \frac{ik}{\sqrt{\varepsilon_0}} E_r, \tag{41.21}$$

and, consequently,

$$\frac{\partial v}{\partial n} = \frac{e^{ika\theta}}{r'} \left(-\frac{1}{r'} \frac{\partial r'}{\partial n} + \frac{ik}{\sqrt{\varepsilon_0^0}} \right) V, \tag{41.22}$$

then, substituting Expression (36.2) into (41.19), we get

$$E_r(A) = E_r^0(A) + \frac{ik}{4\pi} \int \left(\frac{1}{\sqrt{\varepsilon^0}} - \frac{1}{\sqrt{\varepsilon_0^0}} - \frac{ia\theta^2}{2kr'^2} - \frac{h}{r'^2} \right) \frac{e^{ika\theta}}{r'} V E_r \, dS. \tag{41.23}$$

The next-to-last term in the parentheses is very small. Indeed, as we approach h ($r' = |R - R'|$ must be distinguished from the distance r to the center of the earth),

$$\frac{ia\theta^2}{2kr'^2} \rightarrow \frac{i}{2ka}. \tag{41.24}$$

The integral that arises from this term is of the order of $E_r^0 \cdot \frac{1}{2ka}$ and can therefore be dropped (this is essentially the same term that we dropped in (36.6) when it was taken into consideration that kr' >> 1). The very last term in the parentheses in Formula (41.23) gives, as in §36 $(dS' \approx 2\pi r_0' dr_0' = 2\pi r' dr';\ r'^2 = r_0'^2 + h^2)$.

$$h\int e^{ika\theta} V E_r \frac{dS}{r^2} = 2\pi h \int\limits_h^{\infty} e^{ika\theta(r')} V(r') E_r(r') \frac{dr'}{r'^2} =$$

$$= 2\pi h \left\{ -\frac{e^{ika\theta(r')}}{r'} V(r') E_r(r') \Big|_{r'=h}^{\infty} + \int \frac{\partial}{\partial r'} \left(e^{ika\theta(r')} V(r') E_r(r') \right) \frac{dr'}{r'} \right\}. \quad (41.25)$$

In the limit, $h \to 0$, the derivatives in the integrand are in any event finite and, since the entire integral is multiplied by h, it drops out of the result. The very first term gives $2\pi E_r(A)$. Transposing this term to the left member in Formula (41.23) and multiplying the equation by 2, we obtain for observation points on the sphere itself

$$E_r(A) = E_r^0(A) + \frac{ik}{2\pi} \int \left(\frac{1}{\sqrt{\varepsilon^0}} - \frac{1}{\sqrt{\varepsilon_0^0}} \right) \frac{e^{ika(\vartheta_A - \vartheta')}}{a(\vartheta_A - \vartheta')} V(\vartheta_A - \vartheta'; q_0) E_r(\vartheta') dS, \quad (41.26)$$

where ϑ' is the central angle of the current point on the surface, reckoned from an axis passing through the source; $V(\vartheta_A - \vartheta'; q_0)$ is the attenuation function for the angular distance $\theta = \vartheta_A - \vartheta'$ at $h_A = 0$ (39.2). In the variables $D = a\vartheta_A$, $D' = a\vartheta'$, introducing instead of ε^0 and ε_0^0 the parameters q(S) and q_0 corresponding to tnem, together with the large parameter M, this equation may be rewritten in the following obvious notation:

$$E_r(D; q) = E_r^0(D; q_0) + \frac{k}{2\pi M} \int \frac{q - q_0}{D - D'} E_r(D'; q) e^{ik(D-D')} V(\vartheta - \vartheta'; q_0) dS'. \quad (41.26a)$$

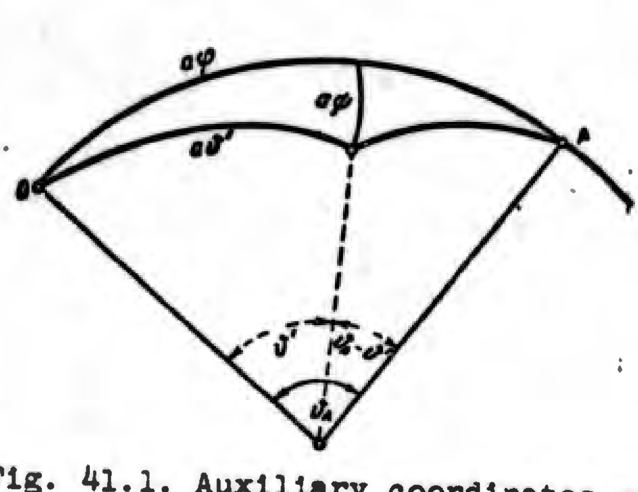

Fig. 41.1. Auxiliary coordinates on the sphere.

As in the case of the plane, it may be reduced to an equation with an integral along a line - in this case, along the great circle joining the source and the observation point. For it, we introduce coordinates on the sphere as follows (Fig. 41.1). Let the great circle passing through the source O and the observation point A be the equator. The longitude of the current point $\varphi(\varphi_0 = 0,\ \varphi_A = \vartheta_A)$ is reckoned along it. The latitude ψ of the current point is always small within the limits of the essential region (if ϑ_A is not very large, $\vartheta_A < 1$). Hence it may be assumed in approximation that

$$\vartheta' \approx \varphi + \frac{\psi^2}{2\varphi},\quad \vartheta - \vartheta' \approx \varphi_A - \varphi + \frac{\psi^2}{2(\varphi_A - \varphi)}. \tag{41.27}$$

We seek the solution in the form

$$E_r(\vartheta; q) = \frac{e^{ika\vartheta}}{a\vartheta} W(\vartheta; q), \tag{41.28}$$

where W is a certain slowly varying attenuation function (like E_r, it is a functional with respect to $q = q(\varphi, \psi)$. In all slowly varying functions W, V, $1/\vartheta$, etc., we may set $\psi = 0$ and extend integration from $-\infty$ to $+\infty$. After all of these substitutions and canceling $D^{-1}\exp(ikD)$, Eq. (41.26a) assumes the form

$$W(\vartheta_A; q) = V(\vartheta_A; q_0) + \frac{ka\vartheta_A}{4\pi i}\int_0^{\vartheta_A} W(\varphi; q)\, V(\vartheta_A - \varphi; q_0)\, d\varphi \int_{-\infty}^{+\infty} \frac{q(\varphi, \psi) - q_0}{(\vartheta_A - \varphi)\varphi}\, e^{i\frac{ka}{2}\psi^2\left(\frac{1}{\varphi} + \frac{1}{\vartheta_A - \varphi}\right)} d\psi. \tag{41.29}$$

Let us consider the case $q = q(\varphi)$, i.e., let us assume that the properties of the soil vary only along the arc of the great circle connecting the transmitter and receiver and that the boundaries separating different soils are perpendicular to this arc. Subsequently, we shall also consider an inclined boundary and satisfy ourselves that the correction for such a boundary is very small. In this case $q(\varphi) - q_0$ is taken out from under the integral with respect to ψ, which gives $(ka\vartheta_A)^{-\frac{1}{2}}(2\pi i\varphi(\vartheta_A - \varphi))^{\frac{1}{2}}$. As a result, the following equation is obtained for the attenuation function W; it is valid for q, which depends on the point (here we substitute ϑ for φ, since they are the same at the equator):

$$W(\vartheta_A; q) = V(\vartheta_A; q_0) + \sqrt{\frac{iM\vartheta_A}{\pi}} \int_0^{\vartheta_A} (q(\vartheta) - q_0) \frac{W(\vartheta; q) V(\vartheta_A - \vartheta; q_0)}{\sqrt{\vartheta(\vartheta_A - \vartheta)}} d\vartheta. \quad (41.30)$$

This equation has exactly the same form as the equation for a flat earth (41.18). Indeed, using the values of M and q (38.7), (38.12a),

$$M = \sqrt[3]{\frac{ka}{2}}, \quad q = i\sqrt[3]{\frac{ka}{2}} \frac{1}{\sqrt{\varepsilon'}} = \frac{iM}{\sqrt{\varepsilon'}}, \quad (41.30a)$$

we see that

$$\sqrt{\frac{iM\vartheta_A}{\pi}}(q - q_0) = \sqrt{\frac{i^3M^3\vartheta_A}{\pi}}\left(\frac{1}{\sqrt{\varepsilon'(\vartheta)}} - \frac{1}{\sqrt{\varepsilon_0'}}\right) =$$

$$= i\sqrt{\frac{D}{\pi}}(\sqrt{s} - \sqrt{s_0}). \quad (D = a\vartheta_A). \quad (41.30b)$$

The only difference from Eq. (41.18) consists in the substitution of W for the unknown attenuation function w and replacement of the auxiliary function for a homogeneous flat earth, y_0, by the corresponding function V for a spherical earth (if we set $D' = a\vartheta'$ and $D = a\vartheta_A$, there will also be a difference in the substitution of D' and D for x' and x). This equation is obtained for soil properties that do not vary very rapidly, i.e., properties satisfying Condition (40.2). It would not be difficult, of course, to obtain the equation for the case $kb < 1$ also, using a boundary condition analogous to Condition (40.12).

§42. PIECEWISE-HOMOGENEOUS PATH. GENERAL FORMULAS

Consider a flat surface. Suppose that the path of the radio wave (the first Fresnel zone, which takes the form of a highly eccentric ellipse with corresponding points as foci in a case $z_0 = z_Z = 0$) can be broken up into a certain number N of segments, each of which is homogeneous and characterized by a constant parameter $\varepsilon^0 = \varepsilon_j^0$, $j = 1, 2, \ldots, N$. We shall assume that the transitional regions are small as compared with the dimensions of the segments themselves (although they are large as compared with the wavelength; all regions of the first zone send a disturbance with practically the same phase to the observation point, so that only the relative magnitude of the area of a given region is a factor). Here, however, we shall assume for simplicity that, as is frequently the case, particularly for short and medium waves, the transitional regions are large as compared with the "wavelength" λ, $kb \gg 1$. In this case, Eq. (41.18) is valid with $z_0 = 0$, $y_0 = y(s_0\rho)$.

If we break up the path into homogeneous segments, this equation may be rewritten as follows:

$$w(D) = y(s_0 D) + i\sqrt{\frac{D}{\pi}} \sum_{j=1}^{N} \int_{x_{j-1}}^{x_j} y(s_0(D - x')) \frac{(\sqrt{s_j} - \sqrt{s_0})\, w(x')\, dx'}{\sqrt{x'(D - x')}}, \tag{42.1}$$

$$s_j = \frac{ik}{2\varepsilon_j^0}.$$

Here x_{j-1} and x_j are the boundaries of the jth segment, $x_0 = 0$, $x_N \equiv D$.

This equation [2] will be basic in our examination of the radio-wave propagation process above a flat piecewise-homogeneous surface. We note that on passing, in accordance with this equation, to a homogeneous surface with $\varepsilon^0 = \varepsilon_1^0$, i.e., setting $N = 1$, we must obtain $w(D) = y(s_1 D)$, where $y(s_1 D)$ is the same normal attenuation function as $y(s_0 D)$, but calculated for $\varepsilon^0 = \varepsilon_1^0$. This means that any two normal attenuation functions $y(s_0 D)$ and $y(s_1 D)$ satisfy the relation

$$y(s_1 D) - y(s_0 D) = \frac{i}{\sqrt{\pi}} (\sqrt{s_1 D} - \sqrt{s_0 D}) \int_0^D \frac{y_1(s_1 x')\, y(s_0(D - x'))}{\sqrt{x'(D - x')}} dx'. \tag{42.2}$$

Equation (24.12) is a particular case of this equation for $s_0 = 0$, $y(s_0 D) = 1$.

In the case of a piecewise-homogeneous spherical earth, Eq. (41.30) gives, with the same assumptions, instead of (42.1)

$$W(\theta_A; q) = V_0(\theta_A; q_0) + \sqrt{\frac{ikl\theta_A}{\pi}} \sum_{j=1}^{N} (q_j - q_0) \int_{\theta_{j-1}}^{\theta_j} \frac{W(\theta; q)\, V(\theta_A - \theta; q_0)}{\sqrt{\theta(\theta_A - \theta)}} d\theta, \quad (42.1a)$$

where q_j is a parameter characterizing the properties of the soil on the jth segment.

The generalized equation (42.1) (and also (42.1a)) enables us to indicate a regular method that will, in principle, make it possible to use successive calculation of quadratures to find the attenuation function for a piecewise-homogeneous surface. Analytical calculation of these quadratures is impossible in the most general case, but a considerable number of interesting particular formulations of the problem can be investigated to completion. In other cases, numerical integration is necessary.

The attenuation function w is a complex function of distance from the source. On passage across each boundary between two segments, the very nature of its behavior changes. Consequently, fixing the position of the source, it is expedient to denote it by a special symbol on each segment (Fig. 42.1). We shall denote it by $w_1(D)$ on the interval of D-values between 0 and x_1. When D is larger than x_1 but smaller than x_2, we shall denote w by w_2, and so forth, or, in general

$$w_j(D) \equiv w(D) \text{ for } x_{j-1} < D < x_j \quad (42.3)$$

(the case in which w increases with distance is represented in Fig. 42.1; we shall see later that this behavior of w is actually possible). Here, for the sake of concreteness, we shall speak of a flat earth, although all considerations will, of course, retain their force in the spherical case as well.

To determine the field at a given point, according to Formula

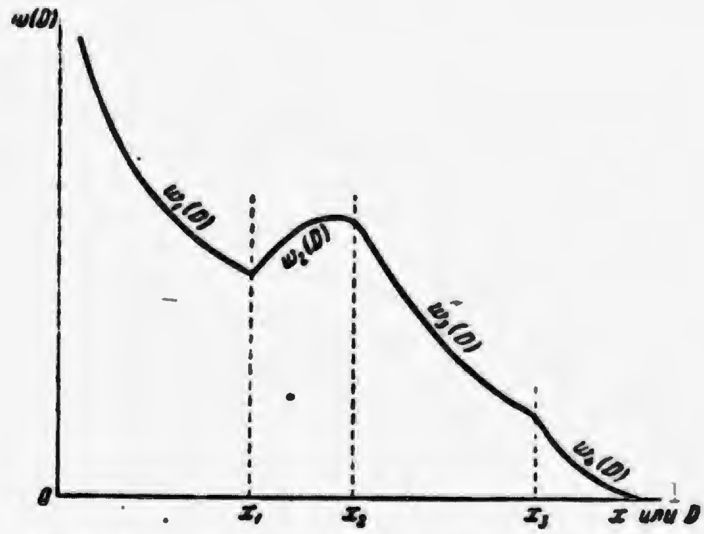

Fig. 42.1. Schematic representation of possible attenuation-function variation. 1) Or.

(42.1), it is necessary to integrate only over the region of x'-values lying between the source and the point of interest to us. But for the first (homogeneous) segment, the attenuation function is known; this is the normal function for the corresponding ε^0:

$$w_1(D) = y(s_1 D). \qquad (42.4)$$

Strictly speaking, of course, w_1 depends not only on D, but also on the distance to the more distant point D on the boundary of the next homogeneous segment. <u>In this chapter</u>, however, we shall disregard the corrections stemming from the effect of field "reflection" from subsequent segments. The reader is referred to Formula (50.27) for the corresponding evaluation. Similar effects are of independent importance in a number of practical problems (we shall examine them in §50).

Let us find $w_2(D)$ – the attenuation function for points on the second segment. The sum in Formula (42.1) contains only two terms for this case. We set the arbitrary parameter s_0 equal to s_2. Then one of the two terms of the sum – the integral over the second segment – will drop out. However, in the remaining integral over the first segment, the attenuation function is already known - it is $y(s_1D)$ (42.4). Consequently, replacing all quantities with the subscript 0 by the corresponding quantities with the subscript 2, we obtain ($N = 2$, $w \to w_2$)

$$w_2(D) = y(s_2D) + i\sqrt{\frac{D}{\pi}}(\sqrt{s_1} - \sqrt{s_2})\int_0^{x_1} \frac{y(s_1x')\,y(s_2(D-x'))}{\sqrt{x'(D-x')}}dx'. \quad (42.5)$$

Thus, to determine w on the second segment, it is necessary to calculate the integral of the unknown functions. When the function $w_2(D)$ is found, we may seek $w_3(D)$, i.e., the attenuation function for the third segment. For this purpose, we take $s_0 = s_3$ in Eq. (42.1), where the sum now contains three terms, $N = 3$. The integral containing the unknown function w_3 drops out, leaving

$$w_3(D) = y(s_3D) + i\sqrt{\frac{D}{\pi}}\left\{(\sqrt{s_1} - \sqrt{s_3})\int_0^{x_1} \frac{y(s_1x')\,y(s_3(D-x'))}{\sqrt{x'(D-x')}}dx' + \right.$$
$$\left. + (\sqrt{s_2} - \sqrt{s_3})\int_{x_1}^{x_2} \frac{w_2(x')\,y(s_3(D-x'))}{\sqrt{x'(D-x')}}dx'\right\}, \quad (42.6)$$

the function w_2 being assumed here to have been determined already from (42.5).

In general, it is obvious that we may write for the kth segment, setting $\varepsilon_0^0 = \varepsilon_k^0$

$$w_k(D) = y(s_kD) + i\sqrt{\frac{D}{\pi}}\sum_{j=1}^{j=k-1}(\sqrt{s_j} - \sqrt{s_k})\int_{x_{j-1}}^{x_j} \frac{w_j(x')\,y(s_k(D-x'))}{\sqrt{x'(D-x')}}dx' \quad (42.7)$$

(where $x_0 = 0$).

Similarly, we find for a spherical earth

$$W_k(\vartheta_A; q) = V(\vartheta_A; q_k) + \sqrt{\frac{iM\vartheta_A}{\pi}}\sum_{j=1}^{k-1}(q_j - q_k)\int_{\vartheta_{j-1}}^{\vartheta_j} \frac{W_j(\vartheta; q)\,V(\vartheta_A - \vartheta; q_k)}{\sqrt{\vartheta(\vartheta_A - \vartheta)}}d\vartheta, \quad (42.7a)$$

where integrals are again taken only of the attenuation function on the preceding segments.

Needless to say, actual application of this recurrent method may encounter difficulties of a purely computational nature. Below we shall illustrate its application to a series of elementary examples that yield practically interesting results.

§43. PATH COMPOSED OF TWO HOMOGENEOUS SEGMENTS

In this section and those to follow (to §45 inclusive), we shall consider a flat inhomogeneous earth and then convert to the spherical earth.

Suppose (Fig. 43.1) we have $s = s_1$ on the segment $0 < x < x_1$, with $x_1 < x$, $s = s_2$. The attenuation function on the first segment is, of course, known (42.4).

Equation (42.5) is valid for w_2 (i.e., for $x_1 < D$). To calculate w_2, it is necessary to substitute $\underline{y}$ from (41.10) for $z = z' = 0$ in the integral. Then there appear, in particular, multiple integrals of the error integrals Φ for the complex argument. They cannot always be reduced again to the same Φ of other arguments. Terms of the type

$$\int_0^a du \int_0^b dv e^{u^2+v^2}. \qquad (43.1)$$

also remain. They must be tabulated specially, or else it is necessary to consider their asymptotic behavior for large and small values of the corresponding arguments. It is not difficult to write all expressions

Fig. 43.1. Essential region for "land-sea" path.

for w_2 containing these integrals. However, it is an extremely cumbersome task. This result was represented in graphical form for certain cases in Furutsu's paper [8], where it was arrived at by another route.

This diagram will be presented later. In the meantime, we shall examine elementary limiting cases. Instead of writing out this complex complete expression and using it to find the results for these limiting cases, we shall use asymptotic representations for $\Phi(x)$ and integrals of the type of (43.1) to simplify Formula (42.5) at once, in accordance with the particular case being considered. It is also convenient to compute tables of exact values of $w_2(D)$, substituting the real and imaginary parts of $\underline{y}$ into Eq. (42.5) and performing numerical integration.

Let us consider first the <u>case</u> $x_1 \ll D$ (one of the corresponding points situated relatively close to a boundary between segments). Then we can set in the integrand

$$\frac{1}{\sqrt{D-x'}}\, y(s_2(D-x')) \approx \frac{y(s_2 D)}{\sqrt{D}} \qquad (43.2)$$

(since x' varies from 0 to x_1 and, consequently D − x' varies from D to $D - x_1 \approx D$)* and take this multiplier out from under the integral sign. The remaining integral can be calculated, using Formula (37.7),

$$\int_0^{x_1} \frac{y(s_1 x')\, dx'}{\sqrt{x'}} = i\sqrt{\frac{\pi}{s_1}}\left\{1 - \frac{i}{\sqrt{\pi s_1 x_1}}(1 - y(s_1 x_1))\right\}, \qquad (43.3)$$

so that

$$w_2(D) = y(s_2 D)\left\{\sqrt{\frac{s_2}{s_1}} + \frac{\sqrt{s_1} - \sqrt{s_2}}{\sqrt{s_1}}\, \frac{i}{\sqrt{\pi s_1 x_1}}(1 - y(s_1 x_1))\right\}. \quad (43.4)$$

In particular, if a longer segment exhibits infinite conductivity, $s_2 = 0$, then according to Formula (41.10) for $z = z' = 0$,

$$w_2(D) = \frac{i}{\sqrt{\pi}}\, \frac{1 - y(s_1 x_1)}{\sqrt{s_1 x_1}} = e^{-s_1 x_1}\left\{1 + \frac{2i}{\sqrt{\pi}} \int_0^{\sqrt{s_1 x_1}} e^{u^2}\, du\right\} \qquad (43.5)$$

(x_1 is the length of a relatively short nonideally conducting segment).

If it is admissible to regard the ocean as possessing ideal conductivity ($|s_2 D| \ll 1$), this case may be related to attenuation of the radio-wave field, for example, in radio communication between a ship and a shore radio station when the distance x_1 from the radio station to the shoreline is much shorter than the distance to the ship. The attenuation function (43.5) [2, 11]*

$$w_1(D) = z(s_1x_1) = \frac{i}{\sqrt{\pi s_1 x_1}}(1 - y(s_1x_1)) \qquad (43.6)$$

solves the elementary problem of propagation over the inhomogeneous surface. It forms a basis for the theory of coastline radio-wave re fraction. Its asymptotic behavior was found earlier [1]. We obtain it by substituting the expansions of y for small and large numerical distances (these will be the numerical distances to the shore for the shorter of the segments – that passing over dry land). According to Formula (25.6),

$$z(s_1x_1) \approx 1 + \frac{2i}{\sqrt{\pi}}\sqrt{s_1x_1} \text{ for} |s_1x_1| \ll 1, \qquad (43.6a)$$

and, according to Formula (25.20),

$$z(s_1x_1) \approx \frac{i}{\sqrt{\pi s_1 x_1}} \text{ for } |s_1x_1| \gg 1. \qquad (43.6b)$$

This case is characterized firstly by a comparatively slow decrease in the attenuation function with distance for large $|s_1x_1|$ (in inverse proportion to the square root of the numerical distance to the shore, while over a homogeneous surface it is inversely proportional to the first power of the numerical distance between corresponding points). This circumstance is accounted for by the fact that land (the nonideally conducting segment) occupies only a small fraction of the area of the essential region (the first zone; see Fig. 43.1) and then has a shape such that the increase in the length of this fraction with increasing x_1 causes a proportionally small change in the part of the ellipse covered by dry land.

Another important peculiarity of this formula (at long distances) is the fact that the velocity of radio-wave propagation, as over a homogeneous surface, is equal to the velocity c in air. Actually, at long distances from the shore, the distance-dependent attenuation multiplier gives an additional phase that does not depend on distance:

$$z = \left|\frac{1}{\sqrt{\pi s_1 x_1}}\right| e^{i\left(\frac{\pi}{2}-\frac{\chi}{2}\right)}, \quad \chi = \arg s_1. \tag{43.7}$$

The constant phase shift $\frac{\pi}{2} - \frac{1}{2}\arg s_1$ is established in the transitional region between ocean and land, which we shall not consider here (see §50), and then remains constant.

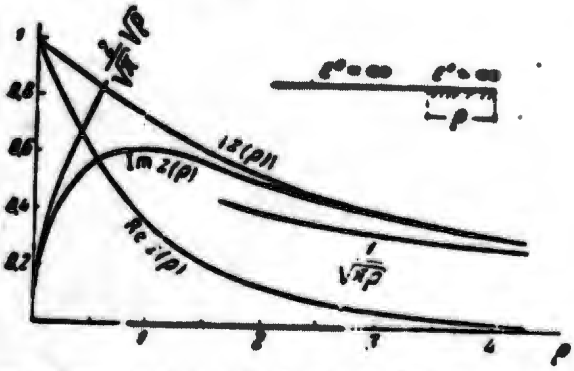

Fig. 43.2. Attenuation function $z(\rho)$ for "sea-land" path as a function of the real numerical distance $\rho = s_1 x_1$ corresponding to the dry-land segment, which is regarded as relatively short. The curves of $\frac{2}{\sqrt{\pi}}\sqrt{\rho}$ and $\frac{1}{\sqrt{\pi\rho}}$ are asymptotic for IM $z(\rho)$.

Explanation of these peculiarities of the asymptotic behavior of z [1] is of fundamental importance for understanding of the radio-wave propagation mechanism (see §45).

The function z is shown on the diagram of Fig. 43.2 for real s_1.

Let us now consider Formula (43.4) for the case in which neither s_1 nor s_2 is zero (two different soils of finite conductivity). Further, let both numerical distances $s_1 x_1$ and $s_2(D - x_1)$ be large, so that both attenuation functions can be replaced by their asymptotic expressions

$$y(s_1 x_1) = -\frac{1}{2 s_1 x_1}, \quad y(s_2(D - x_1)) = -\frac{1}{2 s_2 (D - x_1)}.$$

We obtain from (43.4) [2, 4]

$$\left.\begin{array}{l} w_2(D) = -\frac{1}{2\sqrt{s_1 s_2 D}}, \\ |s_1 x_1| \gg 1, \quad |s_2(D-x_1)| \gg 1. \end{array}\right\} \qquad (43.8)$$

Thus, it may be stated briefly that for large numerical distances, the attenuation function of a composite transmission path is the geometric mean of the attenuation functions that each of the soils would give for the distance in question (i.e., of $y(s_1D) = -(2s_1D)^{-1}$ and $y(s_2D) = -(2s_2D)^{-1}$).

This simple relationship was obtained above for the case of unequal segments, $x_1 \ll D - x_1$. However, Expression (43.8) was obtained fully symmetrical with respect to the two segments, with their respective lengths not appearing separately in the result at all. In particular (applying the reciprocity theorem), this means that Formula (43.8) holds for both $x_1 \ll D - x_1$ and $x_1 \gg D - x_1$, as long as both of the numerical distances are large.

Suppose that we are moving away from a source situated on a segment $s = s_1$. As the distance increases, the attenuation function will vary as $y(s_1D)$ and, at a large enough distance D, will go to $-1/2s_1D$. When we cross the boundary between the first and second segments at $D = x_1$, then at distances $D - x_1$ from the boundary that are small as compared with x_1, Formula (43.4) will become valid (with x_1 replaced by $D - x_1$ and s_1 by s_2 and vice versa)

$$w_2(D) = y(s_1D)\left\{\sqrt{\frac{s_1}{s_2}} + i\frac{\sqrt{s_2}-\sqrt{s_1}}{\sqrt{s_2}}\frac{1-y(s_2(D-x_1))}{\sqrt{\pi s_2(D-x_1)}}\right\}, \quad (43.4a)$$

which, as we move away from the boundary to a distance such that $|s_2(D-x_1)| \gg 1$, becomes the formula $w_2(D) \approx -\frac{1}{2\sqrt{s_1 s_2 D}}$, which is valid as long as $D - x_1 \ll x_1$. We have not calculated $\underline{w}$ for longer distances, when $D - x_1$ becomes of the same order as x_1. However, if we omit this region and go beyond it to the contrary case $D - x_1 \gg x_1$, we shall again have, according to Formula (43.8), $w_2 = -\frac{1}{2\sqrt{s_1 s_2 D}}$, since one of the segments is again much smaller than the other. From this we conclude that this formula must also be valid for the intermediate case $D-x_1 \sim x_1$. Thus, for Formula (43.8) to be valid, it is actually necessary only to observe the conditions $|s_1 x_1| \gg 1$ and $|s_2(D-x_1)| \gg 1$.

The relationships under consideration are illustrated in Fig. 43.3 for the case $s_1 = 2.0$ km^{-1}, $s_2 = \frac{1}{4}s_1 = 0.5$ km^{-1}. Distance from the transmitter in kilometers is plotted against the axis of abscissas, while the ordinates are field amplitudes in microvolts per meter for a transmitter with a power W = 1 kw. It is calculated using Formula (18.9):

$$|E| = \frac{3\cdot 10^5 \sqrt{W_{kw}}}{D_{km}} |w(D)|. \qquad (43.9)$$

Here it is assumed that the length of the segment with $s = s_1$ is 20 km. After passing across the boundary between different soils, the solid curve, which was constructed by Formula (43.4), merges with the curve of (43.8) even at very short distances from the boundary, of the order of 1.5-2 km, when $s_2(D-x_1) \approx 0.75 \div 1.0$. Thus, the applicability conditions of the simplified asymptotic formula (43.8) are not actually very rigid: the sign in these conditions may be replaced by the sign $\gtrsim$. We shall find a confirmation of this later (see Fig. 44.1). Remembering that the normal attenuation function $y(\rho)$ differs markedly from its asymptotic expression $-1/2\rho$ even for $\rho \approx 5$, we see that in the case of composite paths, the applicability conditions for the asymptotic expressions may be relaxed in a certain sense. This can be accounted for by the fact that the simplifications consist in substituting $y(\rho) \approx (-1/2\rho)$ in the integrands, in which it is necessary to integrate over the entire region, so that while this substitution is not very accurate in the initial range of integration, the situation improves at more remote points and the error incurred does not have a very strong effect over the entire integral.

$$\left.\begin{array}{l} w_2(D) = -\frac{1}{2\sqrt{s_1 s_2 D}}, \\ |s_1 x_1| \gg 1, \quad |s_2(D-x_1)| \gg 1. \end{array}\right\} \qquad (43.8)$$

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Suppose that we are moving away from a source situated on a segment $s = s_1$. As the distance increases, the attenuation function will vary as $y(s_1D)$ and, at a large enough distance D, will go to $-1/2s_1D$. When we cross the boundary between the first and second segments at $D = x_1$, then at distances $D - x_1$ from the boundary that are small as compared with x_1, Formula (43.4) will become valid (with x_1 replaced by $D - x_1$ and s_1 by s_2 and vice versa)

$$w_2(D) = y(s_1D)\left\{\sqrt{\frac{s_1}{s_2}} + i\frac{\sqrt{s_2}-\sqrt{s_1}}{\sqrt{s_2}}\,\frac{1-y(s_2(D-x_1))}{\sqrt{\pi s_2(D-x_1)}}\right\}, \quad (43.4a)$$

which, as we move away from the boundary to a distance such that $|s_2(D-x_1)| \gg 1$, becomes the formula $w_2(D) \approx -\frac{1}{2\sqrt{s_1 s_2 D}}$, which is valid as long as $D-x_1 \ll x_1$. We have not calculated $\underline{w}$ for longer distances, when $D - x_1$ becomes of the same order as x_1. However, if we omit this region and go beyond it to the contrary case $D-x_1 \gg x_1$, we shall again have, according to Formula (43.8), $w_2 = -\frac{1}{2\sqrt{s_1 s_2 D}}$, since one of the segments is again much smaller than the other. From this we conclude that this formula must also be valid for the intermediate case $D-x_1 \sim x_1$. Thus, for Formula (43.8) to be valid, it is actually necessary only to observe the conditions $|s_1x_1| \gg 1$ and $|s_2(D-x_1)| \gg 1$.

The relationships under consideration are illustrated in Fig. 43.3 for the case $s_1 = 2.0$ km^{-1}, $s_2 = \frac{1}{4}s_1 = 0{,}5$ km^{-1}. Distance from the transmitter in kilometers is plotted against the axis of abscissas, while the ordinates are field amplitudes in microvolts per meter for a transmitter with a power W = 1 kw. It is calculated using Formula (18.9):

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Here it is assumed that the length of the segment with $s = s_1$ is 20 km. After passing across the boundary between different soils, the solid curve, which was constructed by Formula (43.4), merges with the curve of (43.8) even at very short distances from the boundary, of the order of 1.5-2 km, when $s_2(D-x_1) \approx 0{,}75 \div 1{,}0$. Thus, the applicability conditions of the simplified asymptotic formula (43.8) are not actually very rig id: the sign in these conditions may be replaced by the sign $>$. We shall find a confirmation of this later (see Fig. 44.1). Remembering that the normal attenuation function $y(\rho)$ differs markedly from its asymptotic expression $-1/2\rho$ even for $\rho \approx 5$, we see that in the case of composite paths, the applicability conditions for the asymptotic expressions may be relaxed in a certain sense. This can be accounted for by the fact that the simplifications consist in substituting $y(\rho) \approx$ $\approx (-1/2\rho)$ in the integrands, in which it is necessary to integrate over the entire region, so that while this substitution is not very accurate in the initial range of integration, the situation improves at more remote points and the error incurred does not have a very strong effect over the entire integral.

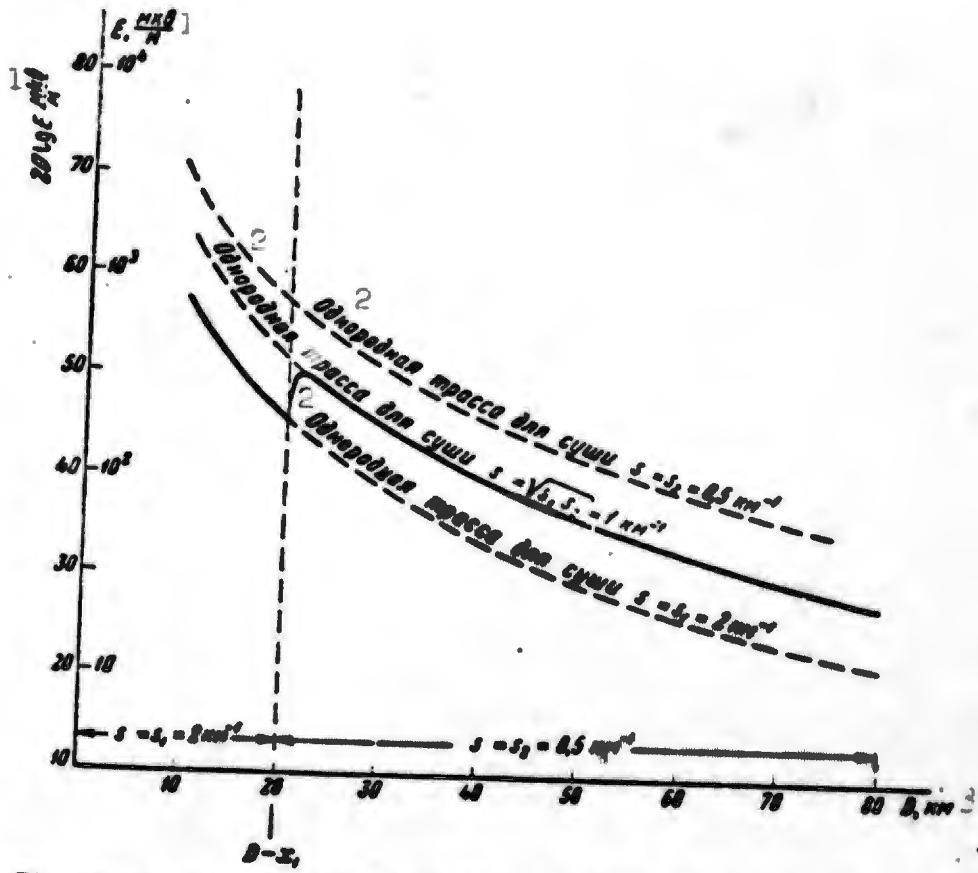

Fig. 43.3. Strength of electric field in decibels (to 1-μv/m) level for a power of 1 kw as a function of distance for a transmission path composed of two segments with different electrical properties. s = 0.5 km^{-1} ("good soil") and s = 2 km^{-1} ("poor soil"). 1) μv/m; 2) homogeneous path for dry land; 3) kilometers.

The rise in the curve as it passes onto better soil is interesting and of practical importance. It indicates that intelligibility may sometimes be improved by moving the receiver somewhat farther away from the transmitter if in this process it is placed on a soil with better conductivity [2b], and, conversely, that even a relatively short increase in distance may result in an extremely sharp attenuation of the field if the segment of the path immediately before the receiver now has a poorer conductivity. This effect is quite distinct at relatively short wavelengths. At longer wavelengths, it would become noticeable only at distances so great that the curvature of the earth could no longer be disregarded.

The specific influence of the properties of the initial and final segments of the path, which leads to phenomena like that noted above, is particularly distinct when we consider the extreme case of inhomogeneity, i.e., that which occurs when one of the segments is infinitely conductive, s = 0 ("sea"). We shall now examine it in greater detail. The corresponding formula (43.5) is valid only for $x_1 \ll D - x_1$. Consequently, it gives only the field around the boundary between segments and can, for example, be applied to study phase relationships in the field (see §50; at longer distances from the boundary, as we noted above, the phase is shifted by a constant amount that does not depend on distance, while near the boundary we observe the peculiar distortions of the wave front one of whose consequences is coastline refraction). For $x_1 \lessdot D-x_1$, the corresponding formula must be derived anew. The case $D - x_1 \ll x_1$ (major part of the path over dry land) can be obtained from the general formula (43.4), letting s_1 in this formula approach zero, which gives for $|s_2 D| \gg 1$

$$w_2(D) = -\frac{1}{2s_2 D}\left\{1 - \frac{2i}{\sqrt{\pi}}\sqrt{s_2 x_1}\right\}. \qquad (43.10)$$

Here x_1 is the length of the (shorter) infinitely conductive segment. But for $x_1 \approx D-x_1$, Formula (43.10) is invalid.

We return to the general formula (42.5) and examine it for an arbitrary relation between x_1 and $D - x_1$, assuming, however, that the numerical distance traversed by the radio waves above dry land is large. Let segment 1 of length x_1 be infinitely conductive, $s_1 = 0$. Consequently, we require that $|s_2(D-x_1)| \gg 1$.

Setting in Formula (42.5)

$$y(s_1 x') = 1, \quad y(s_2(D-x')) = -\frac{1}{2s_2(D-x')},$$

we obtain

$$w_2(D) = -\frac{1}{2s_2D} + i\sqrt{\frac{Ds_2}{\pi}}\frac{1}{2s_2}\int_0^{x_1}\frac{dx'}{\sqrt{x'(D-x')^3}},$$

from which

$$w_2(D) = -\frac{1}{2s_2D}\left(1 - \frac{2i}{\sqrt{\pi}}\sqrt{s_2x_1}\sqrt{\frac{D}{D-x_1}}\right) =$$

$$= -\frac{1}{2s_2D} + i\frac{1}{\sqrt{\pi}}\sqrt{\frac{x_1}{D}}\frac{1}{\sqrt{s_2D\left(1-\frac{x_1}{D}\right)}} \quad \text{for } |s_2(D-x_1)| \gg 1. \tag{43.11}$$

Here x_1 is the length of the ideally conductive segment; $D - x_1$ is the length of the segment characterized by the parameter s_2.

(It is obvious that in the limiting case $x_1 \ll D$, this formula becomes Formula (43.10), while for $D - s_1 \ll D$ and with the appropriate changes in nomenclature, it becomes Formula (43.6b), as indeed it should.)

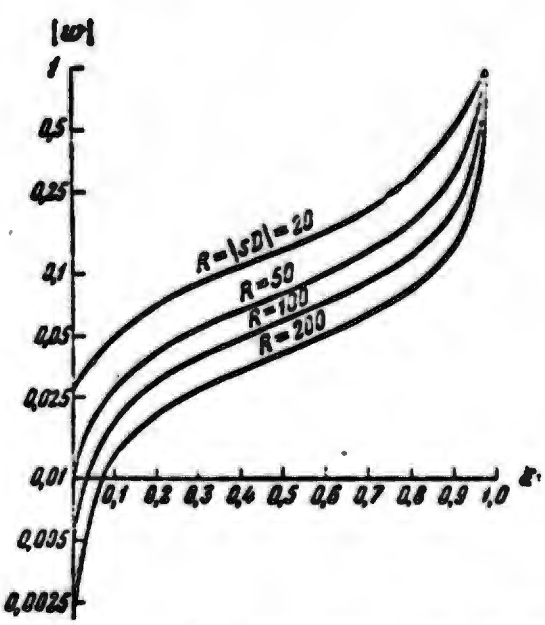

Fig. 43.4. Dependence of attenuation function $\bar{w}$ on the filling of a path of length D by an infinitely conductive segment of length x_1. The filling fraction is plotted against the axis of abscissas. The curves pertain to different soil properties, which corresponds to different numerical distances R for a given D.

The absolute value of the attenuation function for the real numerical distance is

$$|w_2(D)| = \sqrt{\frac{1}{4(s_2D)^2} + \frac{1}{\pi}\frac{x_1}{D}\frac{1}{1-\frac{x_1}{D}}\frac{1}{s_2D}}. \qquad (43.11a)$$

Combining Formula (43.11) and (for $|s_2(D-x_1)| < 1$)) Formula (43.6), in which it is necessary to replace x_1 by $D - x_1$ for the given case, we may construct curves of the attenuation function. Figure 43.4 presents curves giving the absolute value of the attenuation function for various soils and distances between corresponding points, for various $R = |s_2D|$, as it depends on the filling of the path by an infinitely conductive surface (the fraction ξ of the of the path occupied by it is plotted against the axis of abscissas). Further, Fig. 43.5 gives curves indicating the field amplitude for several types of soils and positions of the boundary, reduced to an emitter power W = 1 kw. All of these curves indicate the effect of field amplification with increasing distance when the terminal segment becomes a good conductor.

§44. PATH COMPOSED OF THREE HOMOGENEOUS SEGMENTS

In examining a path composed of three homogeneous segments, we shall again limit ourselves to derivation of the attenuation functions for specific length ratios and electrical properties of the segments.

As was seen from §43, it is convenient to examine the field for a path in which no segment is an ideal conductor separately from the case in which at least one segment has s = 0.

First suppose that none of the segments has s = 0. We shall further limit ourselves to the case in which the numerical distances corresponding to each of the segments are quite large. That is to say, we require that

$$|s_1x_1| \gg 1, \quad |s_2(x_2-x_1)| \gg 1, \quad |s_3(D-x_2)| \gg 1. \qquad (44.1)$$

As will be seen below from comparison with experimental data, these conditions are not actually very rigid, so that the sign $\gg$ may be read simply as the sign $>$, much as we said in connection with Fig. 43.3. The field on the first and second segments is given by the formu-

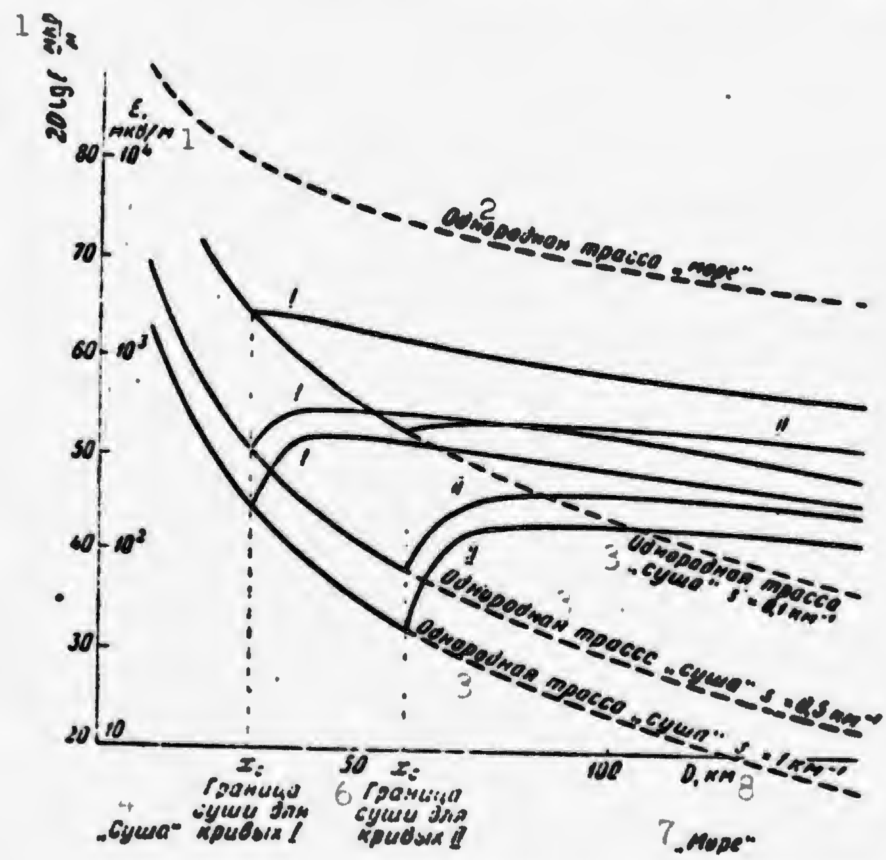

Fig. 43.5. Same as Fig. 43.3, for the case in which the radio waves pass from "land" to "sea," with two different "land" distances and for three values of the parameter s. 1) μv/m; 2) homogeneous "sea" path; 3) homogeneous "land" path; 4) land; 5) boundary of land for curves I; 6) boundary of land for curves II; 7) sea; 8) kilometers.

las of the preceding section: $w_1(x) = y(s_1 x)$, while Formula (43.8) gives $w_2(x)$ as we move away from x_1 to a distance such that $|s_2(x-x_1)| \approx 1$. Since by definition $|s_2(x_2-x_1)| \gg 1$, the region of inapplicability of Formula (43.8) covers a very small fraction of the second segment and we may consider $w_2(D) = -\frac{1}{2\sqrt{s_1 s_2 D}}$. Finally, by virtue of the third condition of (44.1), we may make the substitution $y(s_3(D-x')) \approx -\frac{1}{2s_3(D-x')}$ in the integrands in Formula (42.6). Hence Formula (42.6) is reduced to

$$w_3(D) = -\frac{1}{2s_3 D} - i\sqrt{\frac{D}{\pi}}\left\{(\sqrt{s_1}-\sqrt{s_2})\int_0^{x_1}\frac{g(s_1,x')}{\sqrt{x'}}\frac{dx'}{2s_3(D-x')^{3/2}} - \right. \tag{44.2}$$

$$\left. -(\sqrt{s_2}-\sqrt{s_3})\int_{x_1}^{x_2}\frac{1}{2\sqrt{s_1 s_2}\,x'^{3/2}}\frac{dx'}{2s_3(D-x')^{3/2}}\right\}.$$

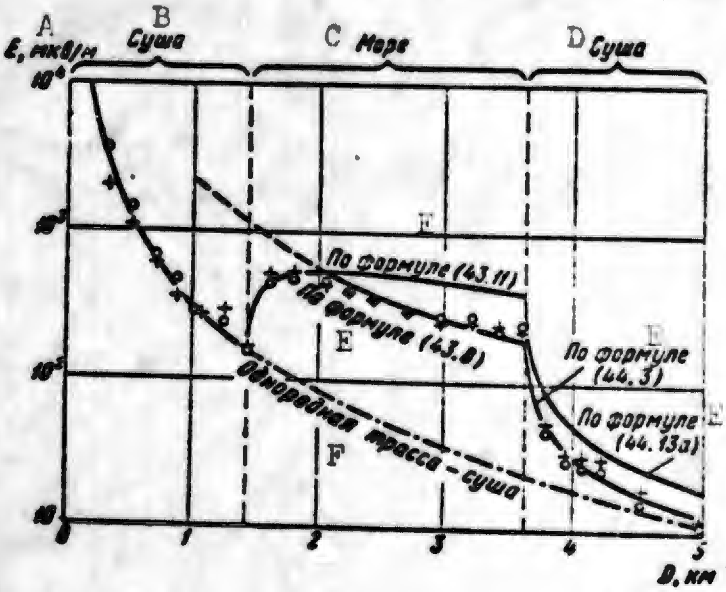

Fig. 44.1. Theoretical curves and experimental data for the case of three segments with different electrical properties (the intermediate segment has high conductivity). Wavelength 3.9 m, power 11 w, crosses and circles indicating data from two series of measurements. A) μv/m; B) land; C) sea; D) land; E) according to formula; F) homogeneous land path; G) kilometers.

The final calculation is easy in the next two cases. First, let the properties of the terminal segments be the same: $s_1 = s_3$. Then the first integral drops out and we obtain

$$w_3 = -\frac{1}{2s_3 D}\left\{1 - \frac{i}{\sqrt{\pi}}\frac{\sqrt{s_1}-\sqrt{s_2}}{\sqrt{s_1 s_2 D}}\left[\frac{D-2x_1}{\sqrt{x_1(D-x_1)}} + \frac{D-2(D-x_2)}{\sqrt{x_2(D-x_2)}}\right]\right\}. \tag{44.3}$$

Figure 44.1 presents the theoretical curves and experimental data [12] corresponding to the case under consideration. Although the intermediate segment was filled by ocean in this case, it may not be regarded as ideally conducting because the frequency was very high (λ = 3.9 m), i.e., $s_2 = 0.86\ \text{km}^{-1}$. This case is also interesting in that s_1 and s_3 are not real: $\varepsilon_3 \approx 15+2{,}3i$, $s_1 = s_3 \approx (7+i\cdot 50)\ \text{km}^{-1}$. Developed theory requires only that there be time for attenuation of waves passing through the soil both from the source itself and from the boundary between soils, and hence that there be no region – or only a very small one – in which the attenuation function varies markedly at the wavelength in air. In principle, such a region can be encountered near a boundary between soils if in one of them $|\varepsilon| \sim 1$, while in the other $|\varepsilon| \gg 1$. But here even this is impossible, since the absolute value $|\varepsilon|$ is large everywhere. Hence although s is almost purely imaginary here, we are justified in applying the theory even to this case of ultrashort waves. The theoretical curves are drawn in accordance with Formulas (43.8) (on the second segment) and (44.3) (on the third segment). Also plotted are theoretical curves obtained on the assumption of infinite conductivity in the second segment, i.e., by Formulas (43.11) (for the second segment) and (44.13a) (see below, for the third segment). It is understood that a more exact curve might be constructed by Formula (43.4) on the second segment around $D = x_1$. However, it is evident that even the use of the elementary asymptotic formulas yields a satisfactory result. In particular, although $s_2(x_2 - x_1) \approx 2$ in this case, Conditions (44.1) are observed for all practical purposes.

The other case in which (44.2) is calculated in an elementary fashion is the case of a small initial segment $x_1 \ll D$, when the substitution $D - x' \approx D$ may be made in the first integral. Application of Formula (43.3) indicates that for $s_1 \neq s_3$, but small x_1, the difference from Formula (44.3) reduces to addition of a term $\sqrt{\frac{s_3}{s_1}} - 1$ in the braces, so that

$$w_3(D) \approx -\frac{1}{2\sqrt{s_1 s_3 D}}\left\{1 - \frac{i}{\sqrt{\pi}}\frac{\sqrt{s_3} - \sqrt{s_2}}{\sqrt{s_2 s_3 D}}\left[\frac{D - 2x_1}{\sqrt{x_1(D - x_1)}} + \frac{D - 2(D - x_2)}{\sqrt{x_2(D - x_2)}}\right]\right\}. \quad (44.4)$$

(The integral in Eq. (44.2) can also be calculated for x_1 not small. However, the result is cumbersome.)

In the case of soils that are not very good conductors, the second term in the braces can be disregarded when all numerical distances are large, and we arrive at the formula

$$w_3 \approx -\frac{1}{2\sqrt{s_1 s_3 D}}. \qquad (44.5)$$

In this expression, which is analogous in a certain sense to Formula (43.8), the constants characterizing the intermediate segment are lacking altogether. The attenuation function is determined solely by the properties of the terminal segments, i.e., the "takeoff" and "landing" areas (terms introduced by L.I. Mandel'shtam). However, it is valid only if certain conditions reflecting the properties of the intermediate segment are observed.

Since we should have $w \approx -(1/2s_2D)$ for a homogeneous path with the properties of the intermediate segment, Formula (44.5) signifies that "good" or "poor" terminal segments even of relatively short length may result in substantially better or worse reception, respectively. As noted above, this effect is particularly marked at short wavelengths ($\lambda < 100$ m). At longer wavelengths, it is significant only at absolute segment lengths such that it is no longer possible to disregard the curvature of the earth. Its influence is particularly substantial when one or two segments exhibit infinite conductivity ("sea"). Here Formulas (44.1)-(44.5) are inapplicable, and we shall examine this case separately. Let us investigate two particular cases, which might be referred to as "sea-land-sea" and "land-sea-land."

In the "sea-land-sea" case, we obtain, setting $s_1 = s_3 = 0$, in Formula (42.6),

$$w_3(D) = 1 + i\sqrt{\frac{s_2 D}{\pi}} \int_{x_1}^{x_2} \frac{w_2(x')}{\sqrt{x'(D-x')}}\, dx'. \qquad (44.6)$$

However, if $w_2(x')$ in the above is replaced by its approximate expression (43.10), the result obtained will not be useful for $x_1 \to 0$, since a nonintegrable singularity forms in the integral at this limit. For this not to happen, it would be necessary to replace $w_2(x')$ by the exact function (which, unlike Function (43.11) is also valid for $s_2D \to 0$), which we have not derived. We shall therefore use a slightly different approach.

Together with the case under consideration (Fig. 44.2a), we shall consider another case, that represented in Fig. 44.2b; let the third segment have properties identical with those of the second, $s_2 = s_3$. Equation (42.2) may be used for the attenuation function, not taking s_0 equal to s_2, as would follow from the general method (which would give (42.5)), but instead taking $s_0 = s_1 = 0$.

Then we obtain

$$w_2(D) = 1 + \frac{i}{\sqrt{\pi}} \sqrt{s_2 D} \int\limits_{x_2}^{D} \frac{w_2(x')\,dx'}{\sqrt{x'(D-x')}}. \qquad (44.7)$$

The solution of this equation was found earlier in the form of (42.5). According to Formula (43.11), the function $w_2(x')$ under the integral sign is approximately equal to

$$w_2(x') \approx -\frac{1}{2s_2x'} + \frac{i\sqrt{x_1}}{\sqrt{\pi s_2 x'(x'-x_1)}}. \qquad (44.8)$$

This solution is valid only for points far enough removed from x_1, but it is in any event valid for the entire interval $x_2 < x' < D$.

Subtracting Eqs. (44.7) and (44.6), we obtain (it is noted that neither of these equations embodies any special assumptions regarding the dimensions of the segments, etc.)

$$w_3(D) = w_2(D) - \frac{i}{\sqrt{\pi}} \sqrt{s_2 D} \int\limits_{x_2}^{D} \frac{w_2(x')\,dx'}{\sqrt{x'(D-x')}}. \qquad (44.9)$$

Now we have eliminated the circumstance that prevented us from substituting the simple expression for $w_2(x')$, (43.10), in the integrand. Introducing it, we may perform elementary integration to obtain the result

$$w_3(D) = -\frac{1}{2s_2D} + \frac{i}{\sqrt{\pi s_2 D}}\left(\sqrt{\frac{x_1}{D-x_1}} + \sqrt{\frac{D-x_2}{x_2}}\right) + \\ + \frac{1}{2}\left\{1 - \frac{2}{\pi}\arcsin\frac{(x_2-x_1)D - x_1(D-x_2)}{x_2(D-x_1)}\right\}. \qquad (44.10)$$

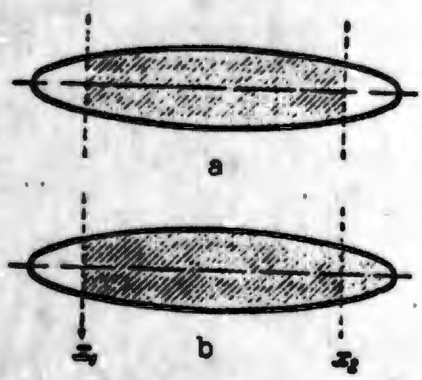

Fig. 44.2. Illustrating calculation of attenuation function for "sea-land-sea" pass.

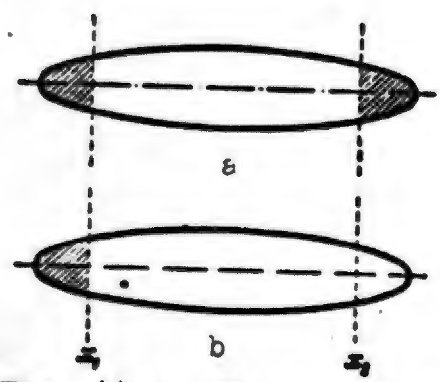

Fig. 44.3. Illustrating calculation of attenuation function for "land-sea-land" pass.

If "land" occupies the entire path, then $x_1 = 0$, $x_2 = D$ and, as indeed it must, $w_3 = -(1/2s_2D)$. Otherwise, $x_1 = x_2$, arc sin gives $-\pi/2$, and, dropping terms of the order of $1/\sqrt{s_2D}$ and $1/s_2D$ as small by comparison with the principal terms, we obtain $w_3 = 1$, which is as it should be. In general, when the terminal segments are short, $x_1 \ll D$, $D - x_2 \ll \ll D$, $\arcsin \approx \frac{\pi}{2} - 2\sqrt{\frac{x_1(D-x_2)}{D^2}}$, so that

$$w_3(D) = -\frac{1}{2s_2D} + \frac{i}{\sqrt{\pi s_2 D}}\left(\sqrt{\frac{x_1}{D}} + \sqrt{\frac{D-x_2}{D}}\right) + \frac{2}{\pi}\sqrt{\frac{x_1(D-x_2)}{D^2}}. \qquad (44.10a)$$

If the path is very long, $|s_2D| \gg 1$, then relatively small terminal segments (last term) give a strong increase in the field.

We pass now to the "land-sea-land case (Fig. 44.3a). Here Eq. (42.6) for $s_1 = s_3$ gives for $s_2 = 0$

$$w_3(D) = y(s_1D) - i\sqrt{\frac{s_1D}{\pi}}\int_{x_1}^{x_2}\frac{w_2(x')\,y(s_1(D-x'))}{\sqrt{x'(D-x')}}\,dx'. \qquad (44.11)$$

Substituting the following approximate expression for $w_2(x')$ according to Formula (43.11) (s_2 is replaced by s_1 and x_1 by $x' - x_1$):

$$w_2(x') = -\frac{1}{2s_1x'} + \frac{i}{\sqrt{\pi}}\sqrt{\frac{x'-x_1}{x'}}\sqrt{\frac{1}{s_1x_1}} \qquad (44.12)$$

and substituting $y(s_1(D-x')) = -\frac{1}{2s_1(D-x')}$ (accordingly, the result obtained will be valid only for $|s_1(D-x_2)| \gg 1$), we can perform integration to obtain

$$w_3(D) = \frac{-i}{\sqrt{\pi}}\frac{1}{2(s_1D)^{3/2}}\left[\frac{D-2x_1}{\sqrt{x_1(D-x_1)}} - \frac{D-2x_2}{\sqrt{x_2(D-x_2)}}\right] - \\ -\frac{1}{\pi s_1\sqrt{x_1D}}\sqrt{\frac{x_2-x_1}{D-x_2}} - \frac{1}{\pi s_1D}\arcsin\sqrt{\frac{x_1(D-x_2)}{x_2(D-x_1)}}. \qquad (44.13)$$

Since we are regarding all numerical distances as large, the first term, whose denominator contains the additional factors $\sqrt{s_1x_1}$, $\sqrt{s_1(D-x_2)}$, etc., can be dropped. This leaves

$$w_3(D) = -\frac{1}{\pi s_1D}\left\{\sqrt{\frac{D(x_2-x_1)}{x_1(D-x_2)}} + \arcsin\sqrt{\frac{x_1(D-x_2)}{x_2(D-x_1)}}\right\}. \qquad (44.13a)$$

If the intermediate segment is vanishingly small, $x_1 = x_2$, then w_3 becomes $-1/2s_1D$, as indeed it should. The converse limiting case $x_1 \to 0$, $x_2 \to D$, is not covered by this formula, since in the derivation we have assumed the numerical distances to be large for each of the segments. On the other hand, however, here we may use the condition $x_1 \ll D$, $D - x_2 \ll D$ at an earlier stage. That is to say, we proceed as follows. Together with the case of Fig. 44.3a that is under consideration, we consider the case of Fig. 44.3b, i.e., we assume that the segment $x_2 < x' < D$ is also ideally conductive. Setting $s_2 = 0$ in Eq. (42.1) (for this case, the sum over j contains two terms) and taking the arbitrary parameter s_0 equal to s_1, we get for the attenuation function

$$w_2(D) = y(s_1D) - i\sqrt{\frac{s_1D}{\pi}}\int_{x_1}^{D}\frac{w_2(x')\,y(s_1(D-x'))\,dx'}{\sqrt{x'(D-x')}}. \qquad (44.14)$$

Subtracting this equation from Eq. (44.11), we obtain

$$w_3(D) = w_2(D) + i\sqrt{\frac{s_1 D}{\pi}}\int_{x_2}^{D}\frac{w_2(x')\,y(s_1(D-x'))\,dx'}{\sqrt{x'(D-x')}}. \qquad (44.15)$$

If we consider $x_1 \ll D$ and $D - x_2 \ll D$, the attenuation function w_2 for a case of this type ("land-sea") in which the high-conductivity segment is considerably longer than the low-conductivity segment appears under the integral sign. Hence we may substitute the following function (43.6) for $w_2(x')$:

$$w_2(x') \approx \mathrm{const} \approx z(s_1 x_1) = \frac{i}{\sqrt{\pi s_1 x_1}}\left(1 - y(s_1 x_1)\right). \qquad (44.16)$$

Further, taking $\frac{1}{\sqrt{x'}} \approx \frac{1}{\sqrt{D}}$, out from under the integral sign, we get

$$w_3(D) = z(s_1 x_1)\left(1 - i\sqrt{\frac{s_1}{\pi}}\int_{x_2}^{D}\frac{y(s_1(D-x'))\,dx'}{\sqrt{D-x'}}\right). \qquad (44.17)$$

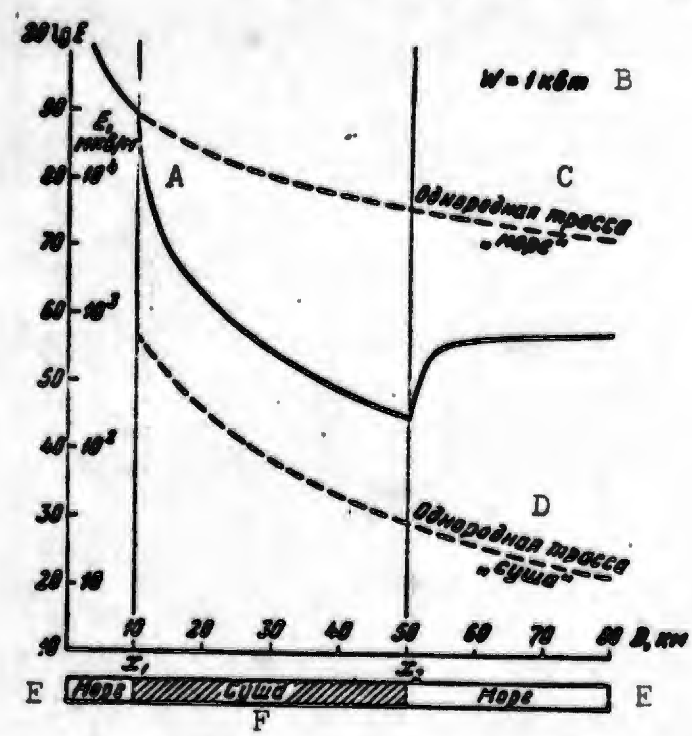

Fig. 44.4. Electric field strength as a function of distance on a "sea-land-sea" path for land with s = 2 km^{-1}. A) μv/m; B) W = 1 kw; C) homogeneous "sea" path; D) homogeneous "land" path; E) sea; F) land.

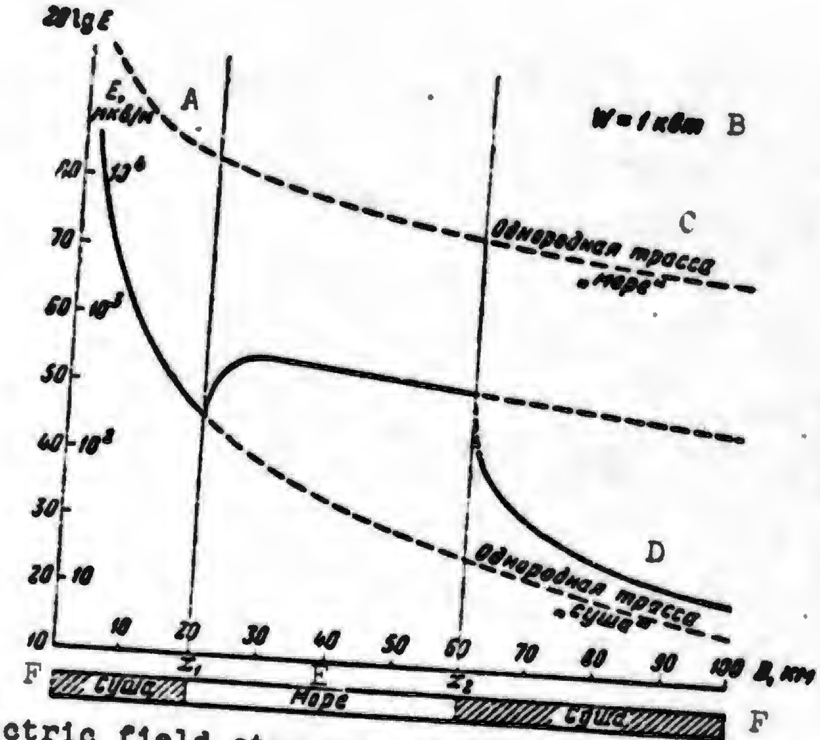

Fig. 44.5. Electric field strength as a function of distance on a "land-sea-land" path for land with s = 2 km^{-1}. A) μv/m; B) W = 1 kw; C) homogeneous "sea" path; D) homogeneous "land" path; E) sea; F) land.

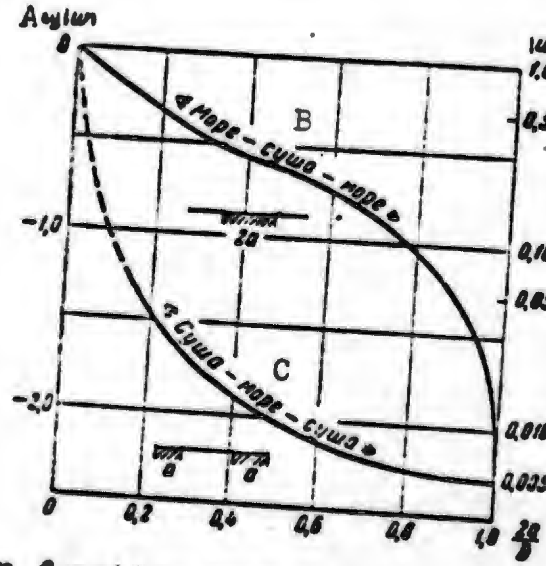

Fig. 44.6. Attenuation function versus filling of path by "land" for filling beginning at the ends (lower curve) and from the middle (upper curve). The "land" is such that the numerical distance is 100 when the path is completely filled by the "land." A) log |w|; B) sea-land-sea; C) land-sea-land.

We have already encountered the integral that appears here (43.3):

$$\int_{x_2}^{D} \frac{y(s_1(D-x'))\,dx'}{\sqrt{D-x'}} = \int_{0}^{D-x_2} \frac{y(s_1\xi)\,d\xi}{\sqrt{\xi}} = i\sqrt{\frac{\pi}{s_1}} + \frac{1-y(s_1(D-x_2))}{\sqrt{s_1}\sqrt{s_1(D-x_2)}},$$

and, consequently,

$$w_3(D) = z(s_1x_1)\,z(s_1(D-x_2));\ (x_1 < D,\ D-x_2 < D). \qquad (44.18)$$

Thus, in the case of two identical not infinitely conductive segments placed at the ends of a relatively much longer ideally conductive segment, the attenuation function decomposes into the product of two attenuation functions each of which would obtain if one such poorly conductive segment were present in each case. In particular, when their numerical distances are large,

$$w_3 = -\frac{1}{\pi s_1\sqrt{x_1(D-x_2)}}. \qquad (44.18a)$$

Figures 44.4 and 44.5 show field-amplitude curves for the "sea-land-sea" and "land-sea-land" cases when the "land" is characterized by the constant $|s_2| = 2\ \mathrm{km}^{-1}$ and the segment lengths are selected in a certain definite fashion. The field strength is shown in μv/m and in <u>decibels</u> referred to the threshold E = 1 μv/m, with the assumption that W = = 1 kw.

Figure 44.6 illustrates the part taken by the filling of the path by segments with good conductivity: for a homogeneous, at first infinitely conductive path of fixed length, it shows the variation of <u>w</u> as the terminal segments (lower curve) or the center (upper curve) is filled with land." The properties of the "land" are so selected that |sD| = 100 when the path is entirely filled with "land."

These curves show that terminal segments with good conductivity sharply increase the field level even when they are relatively short (right-hand part of upper curve), while relatively short terminal segments with poor conductivity reduce the field level sharply (left-hand part of lower curve). Conversely, introduction of a segment with good conductivity into the middle of the path (right-hand part of lower curve) or introduction of a segment with poor conductivity (left-hand part of upper curve) have little effect on field strength.

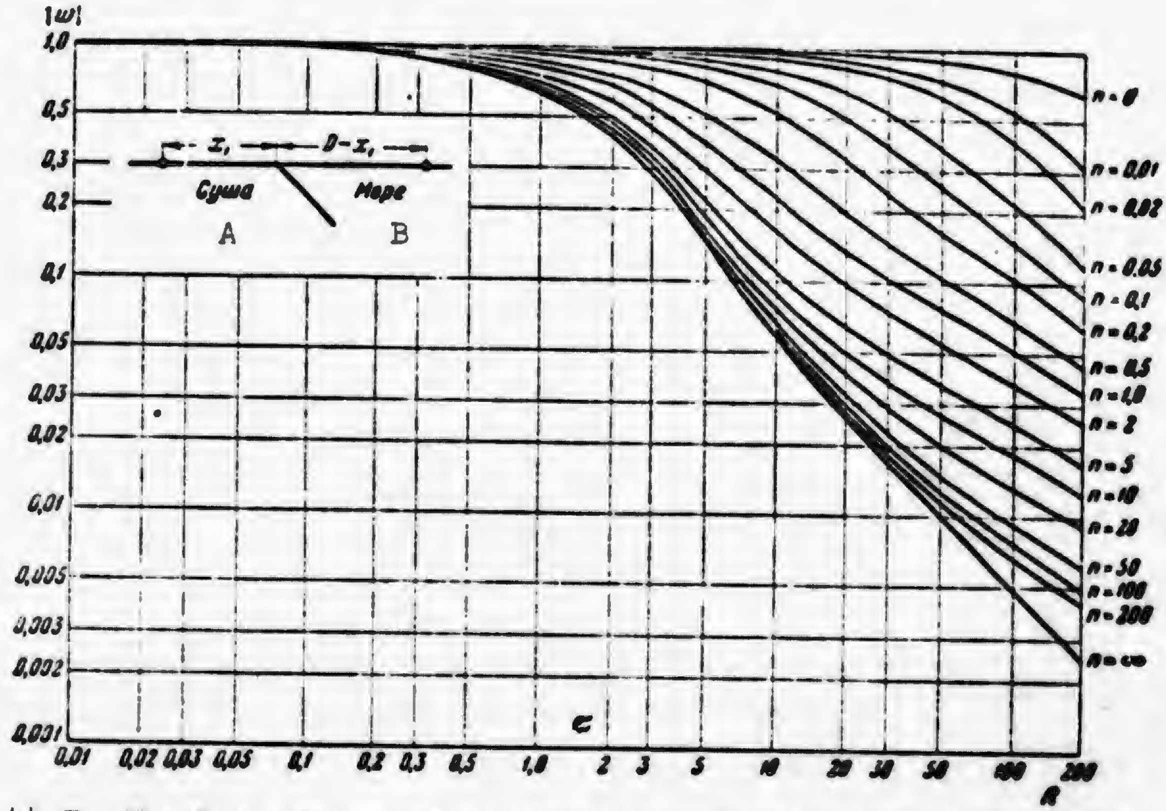

Fig. 44.7. Absolute value of attenuation function for "land-sea" path for various relationships n between lengths of individual path segments, $n = x_1/(D - x_1)$ (the ratio of conductivities is assumed equal to $\sigma_m/\sigma_s = 200$) [8]. The abscissa is numerical distance when the entire path is filled with land $(n = \infty)$, $R = s_c D \approx -\frac{ik}{2\varepsilon_c} D$. A) Land; B) sea.

These relationships again stress the importance of the "takeoff" and "landing" areas for propagation of radio waves above a flat earth's surface. The theory of the field over a piecewise-homogeneous path consisting of two and three segments has been elaborated in great detail by other methods in the work of Clemmow [7] and Furutsu [8]. The physical assumptions on which these methods were based are the same as those in the method of the generalized integral equation, which was set forth

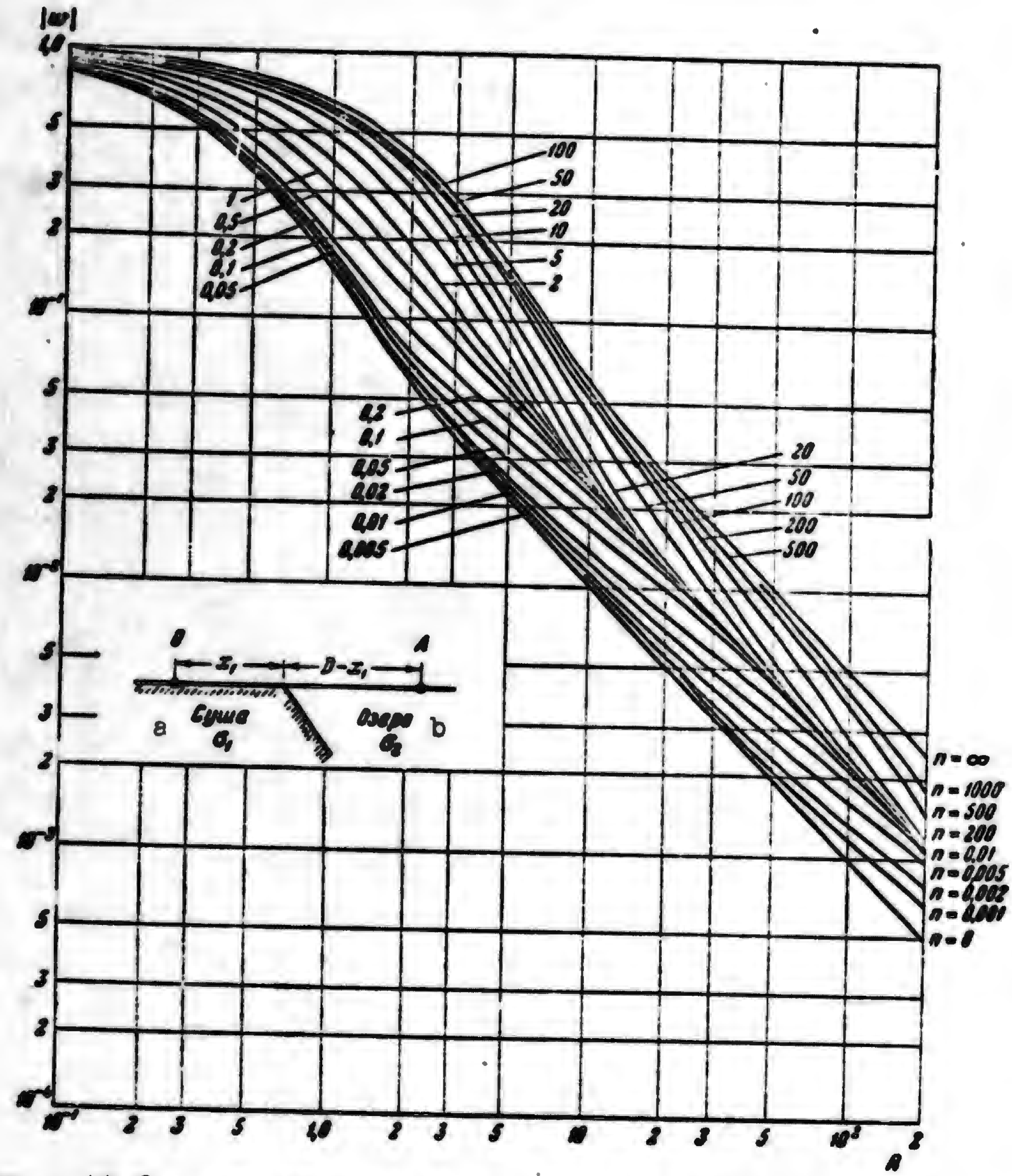

Fig. 44.8. Same as Fig. 44.7, but with conductivity ratio $\sigma_1/\sigma_2 = 5$ ("land-lake" case) [8]. a) Land; b) lake.

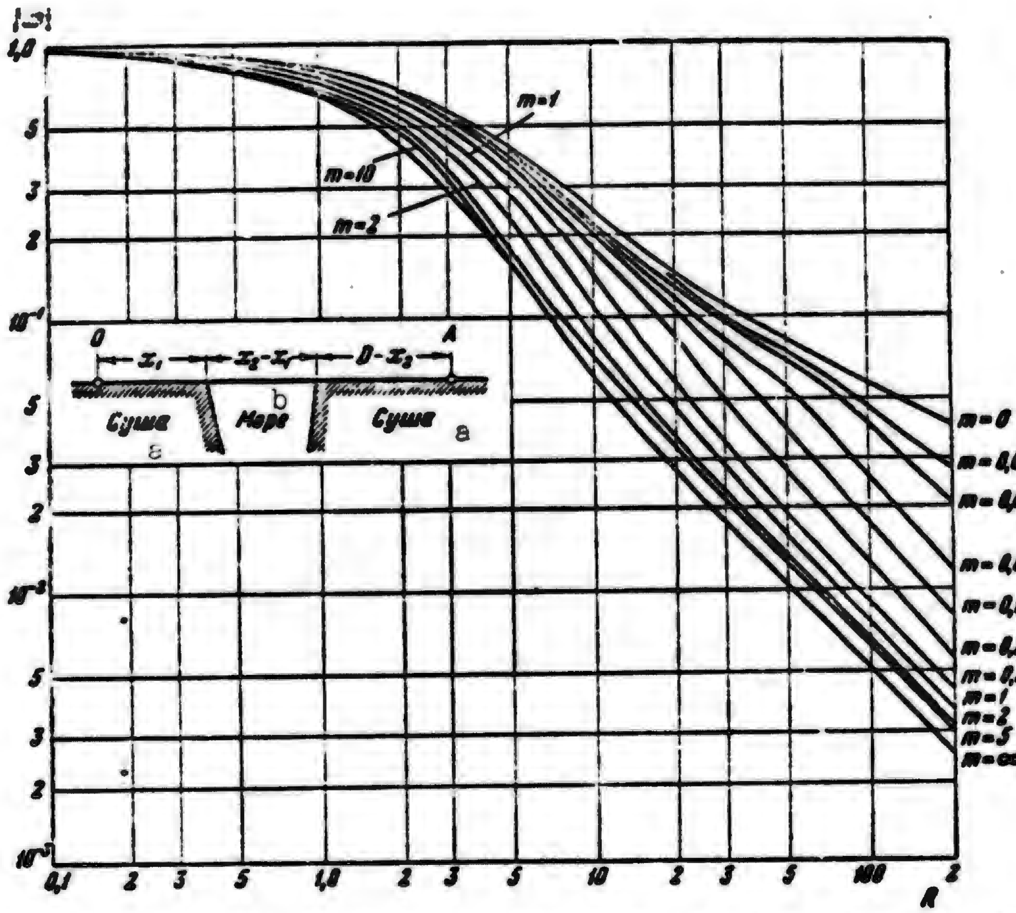

Fig. 44.9. Absolute value of attenuation function for "land-sea-land" path when the lengths of the first two segments are equal, $x_2 - x_1 = x_1$, while the third is m times longer than each of the first two, $D - x_2 = m(x_2 - x_1) = mx_1$, for various m; $\sigma_m = \infty$. The numerical distance for complete filling of the entire path by land, $R = s_c D = \frac{\pi}{\lambda x_c} D$ [8] is plotted against the axis of ordinates. a) Land; b) sea.

above. Among other things, these authors did a great deal of work on the compilation of diagrams and tables for attenuation functions that are not expressed in terms of elementary functions (see, for example, (43.1)). In those limiting cases in which expressions are obtained in elementary functions, these results are, of course, in agreement with those presented above. Some of the most important of the diagrams appear in Figs. 44.7-44.9.

The formulas derived above are somewhat complicated for application in engineering practice. As a consequence, a semiempirical method that is rather easy to handle has made its appearance (Millington [13]). As is indicated by comparison with the theoretical formulas ([14]; see also [17]), it gives an excellent approximation to the rigorous theory in a number of cases and a quite satisfactory one in others. Thus, for the "land-sea" and "land-sea-land" and similar cases, the error in the amplitude does not exceed 2-4 db, while the phase error is 21.5°.

§45. GENERAL PICTURE OF RADIO-WAVE PROPAGATION ABOVE A FLAT EARTH

Before going over to a spherical earth, it will be expedient to summarize the results obtained in the present chapter for a flat earth. Together with the derivations for the field in space above a homogeneous plane (§25, 26, 27), they lead us to a definite physical picture of the radio-wave propagation process above a flat earth. The general features of this picture were outlined in his time, even before the theory had been developed for an inhomogeneous earth, by L.I. Mandel'shtam.

The field at each given point represents a superposition of the "direct-arrival" field of the source and the field of the secondary sources that it induces in the soil. If the soil is infinitely conductive, the field of the secondary sources duplicates the "direct-arrival" field. The departure from infinite conductivity is equivalent to the appearance of additional secondary sources that disturb the field.

Great importance attaches to the distribution of these disturbing sources along the path or, in general, over the zone that is essential for formation of reflections. It can be stated that the entire essential zone radiates together with the source. The "takeoff" and "landing" areas play a special role, as we have stressed more than once in the course of the discussion. The chief conclusion at which we arrive on the basis of the inhomogeneous-surface cases considered is that the <u>influence of the individual path segments is not additive</u>. In the literature, and particularly in that dealing with the genesis of the theory set forth above, we repeatedly encountered attempts to construct the attenuation function for an inhomogeneous path on the basis of various naive considerations and arbitrary assumptions. They have always been based on the concept of a wave glancing along the surface of the earth and suffering disturbance in accordance with the properties of the surface segments over which it passes. Thus, there was an attempt to use the normal attenuation function with <u>s</u> in this function set equal to its average value along the path [15].* This, of course, cannot be correct, since the center part of the path in this calculation is just as important as the terminal segments. These methods are definitely incapable of explaining the increase in the field with distance (Fig. 44.1) and similar phenomena. At the same time, such attempts were still being made up to the middle of the 'Fifties.

In actuality, the propagation of radio waves is to a considerable degree a three-dimensional process. Each point of the field in space serves as a source of spherical waves. In the absence of the earth, all of these sources would produce waves that, interfering with one another, would form a single wave with a spherical front. In the presence of an infinitely conductive earth, the fields of the secondary radiators, which are distributed over the surface, not only compensate for the absence of a field in the lower half-space, but also intensify the field in the upper half-space, and do so in such a way that the front again becomes spherical, but practically flat to small heights above the earth.

The deviation of the earth's conductivity from infinity results in absorption of energy in the soil and bending of the wave front. Near the ground, the equiphase surfaces are bent back and, accordingly, the normal to the front and with it the energy flux vector have a component directed toward the ground.* Hence the field is continuously being redistributed in space as we move away from the source. Energy flows toward the surface from space. At long enough distances from the source, maintenance of the field at the ground surface is provided by those regions of the field in space that are so high that they have not been subject to the attenuating effect of the ground surface (it was shown at the end of §26 that for large enough angles of elevation, $|\sqrt{\varepsilon'}\sin\psi| \gg 1$, the field is created by the source and an essential zone of the ground surface in the immediate vicinity of the source; here this zone is so small that the earth within it may be regarded as infinitely conductive; see Formula (26.30) and Fig. 26.2). It is precisely for this reason that the attenuation function at great distances does not take the form $e^{-\rho}$, as it would for progressive absorption of a glancing wave (§23), but diminishes much more slowly, in proportion to $1/\rho$.

At each given observation point, of course, the fields of the secondary radiators of the immediately adjacent segment of the surface (with dimensions of the order of $1/s$, since these fields are propagated along the ground and attenuate according to the law $y(sr)$), which are excited by the same wave arriving from space, will also make themselves felt. This is why <u>both</u> terminal segments are so important.

One of the practical conclusions to which this general conception and the concrete formulas of the present chapter lead consists in the possibility of amplifying the received field by fortunate location of the receiver and transmitter. Even the curves of Fig. 43.3 indicate that by moving farther from the source, we may obtain an <u>amplification</u> of the field if this results in one end of the path being covered by a soil with better conductivity. The same thing is seen from the formulas and curves pertaining to the "sea-land" case (Figs. 44.1 and 44.5).

Let us compare, for example, the fields of the following two cases. Suppose first that the path is homogeneous, the distance $D = 50$ km, $\lambda = 70$ m and $\sigma = 10^8$ CGSE. According to Table 2 (page 239), this means that $s \approx 1.0\ \text{km}^{-1}$ and $sD \approx 50$. The attenuation function gives a factor $w \approx -\frac{1}{2sD} = -0.01$. Now let the distance between the corresponding points be increased to 70 km by addition of a 10-km ideally conductive (sea) segment at either end. This case is appropriately handled by Formula (44.10) or directly by (44.10a). Here $\frac{x_1}{D} = \frac{D-x_2}{D} = \frac{10}{70}$, $s_2D = 70$. Substituting numerical values, we get

$$\frac{1}{\sqrt{\pi s_2 D}}\left(\sqrt{\frac{x_1}{D}} + \sqrt{\frac{D-x_2}{D}}\right) = \frac{2}{\sqrt{7}} \cdot \frac{1}{\sqrt{70\pi}} \approx \frac{1}{20}.$$

It is easily seen that this term is the principal one. As a result, it is found that

$$|w_2| \approx 0.05.$$

Thus, increasing the distance by a factor of almost one and one-half should result in a five-fold increase in the field. This case should occur, for example, in radio communication between two ships at opposite ends of an island as both of them move away from the shoreline.

Experimental data (see, for example, Fig. 44.1) have confirmed the correctness of these theoretical predictions.

The qualitative considerations set forth above also enable us to offer certain remarks concerning the behavior of a field in space as the observation point is raised higher above an inhomogeneous surface.

Suppose, for example, that the soil on the segment $0 < x < x_1$ is characterized by a constant s_1, while the constant is s_2 for $x > x_1$. Assuming that the lengths of all segments are long enough, we may state that for an observation point lowered to the ground, $z = 0$, Formula (43.8) applies:

$$w(D, 0) = -\frac{1}{2\sqrt{s_1 s_2}\, D}.$$

As the height of the observation point increases, the essential zone of the surface will be shortened, as described in §26 (Fig. 26.2), becoming compressed toward a transmitter situated on the ground. In the final analysis, we shall reach a height z such that the entire essential zone has been displaced into the region of the first segment with $s = s_1$. Beginning at this value of $\underline{z}$, the attenuation function will be identical to the attenuation function in space for homogeneous soil, i.e., since for $|s_1 D| \gg 1$ we definitely have $|s_1 x_1| \gg 1$ (see Formula (26.26)),

$$y(D, z) \approx \frac{\sqrt{\varepsilon_1' \sin\psi}}{1 + \sqrt{\varepsilon_1' \sin\psi}} - \frac{1}{2\pi r (1 + \sqrt{\varepsilon_1' \sin\psi})^2}, \quad \sin\psi = \frac{z}{D}.$$

If the first segment is "better" than the second, this will give rise to an improvement in reception more significant than at the ordinary elevation, or conversely.

More complete formulas for the field in space above a piecewise-homogeneous path can be obtained at once in the form of quadratures, since the field right at the ground is already known and, according to Green's theorem, this is sufficient to find the field in space (see [VII, 11]). However, the integrals that appear here are not reduced to any form of simple analytical functions (see [7, 8], where they were derived by another method and tabulated).

However, the above picture of radio-wave propagation is found to be too simple when we pass from a flat earth to a spherical one, or, more precisely, to the field far beyond the horizon. Here, as we find, the concept of a wave skipping along the surface or "sticking" to it is more acceptable. We shall investigate this case in the next section.

§46. PIECEWISE-HOMOGENEOUS PATH ON A SPHERICAL EARTH'S SURFACE

Let us consider certain special cases of an inhomogeneous spherical surface on the basis of Eq. (42.7a) [6].

1. First we obtain the attenuation function $W_2(\Delta\vartheta_1, \Delta\vartheta_2; q_1, q_2)$ for a path consisting of two homogeneous (N = 2) having angular lengths $\Delta\vartheta_1 = \vartheta_1$ and $\Delta\vartheta_2 = \vartheta_A - \vartheta_1$ and characterized by the parameters $q = q_1$ and $q = q_2$. The attenuation function W_1 is known on the first segment. It is $V(\vartheta; q_1)$. In Formula (42.7a), the sum contains only one term, j = = 1, with

$$W_j(\vartheta; q) = W_1(\vartheta; q_1) = V(\vartheta; q_1);$$

$$W_2(\Delta\vartheta_1, \Delta\vartheta_2; q_1, q_2) = V(\vartheta_A; q_2) + \sqrt{\frac{iM\vartheta_A}{\pi}}(q_1 - q_2)\int_0^{\vartheta_1} \frac{V(\vartheta; q_1)\,V(\vartheta_A - \vartheta_1; q_2)}{\sqrt{\vartheta(\vartheta_A - \vartheta)}}\,d\vartheta \qquad (46.1)$$

From Formulas (39.1)-(39.2) for y = 0, we substitute the functions V in series form and integrate over ϑ. This will give

$$W_2(\Delta\vartheta_1, \Delta\vartheta_2; q_1, q_2) = V(\vartheta_A; q_2) + \sqrt{\pi i M\vartheta_A}(q_1 - q_2) \times$$

$$\times \sum_{k,\, l=1}^{\infty} \frac{e^{iM[\Delta\vartheta_1 \cdot t_k(q_1) + \Delta\vartheta_2 \cdot t_l(q_2)]}}{(t_k(q_1) - q_1^2)(t_l(q_2) - q_2^2)(t_k(q_1) - t_l(q_2))} - \qquad (46.2)$$

$$- \sqrt{\pi i M\vartheta_A}(q_1 - q_2) \sum_{k,\, l=1}^{\infty} \frac{e^{iM\vartheta_A t_l(q_2)}}{(t_k(q_1) - q_1^2)(t_l(q_2) - q_2^2)(t_k(q_1) - t_l(q_2))}.$$

The sum over <u>k</u> that appears in the last term on the right is easily calculated using the identity (which is essentially an expansion in elementary fractions)

$$\sum_k \frac{1}{t_k(q_1)-q_1^2} \cdot \frac{1}{t-t_k(q_1)} = \frac{w(t)}{w'(t)-q_1 w(t)} \qquad (46.3)$$

(if $t_k(q_1)$ are the roots of the equation $w'(t_k) = q_1 w(t_k)$).
The validity of this relationship is demonstrated by the fact that both sides of Equality (46.3), after multiplication by an arbitrary function $f(t)$ and integration over the contour C, which encloses all poles $t = t_k$, give the same result. Actually, if we expand $f(t)$ into a Fourier integral in e^{ixt} functions, each term of the expansion will give the required equality (see Formula (39.2) for $y = 0$), which means that the whole function $f(t)$ will also give it.

Now, taking into account that $w'(t_l) = q_2 w(t_l)$, we obtain

$$\sum_k \frac{1}{t_k(q_1)-q_1^2} \frac{1}{t_k(q_1)-t_l(q_2)} = -\frac{w(t_l)}{w'(t_l)-q_1 w(t_l)} = \frac{1}{q_1-q_2}. \qquad (46.4)$$

We substitute this result in the last term in the right member of Relationship (46.2). We satisfy ourselves that it cancels with $V(\vartheta_A; q_2)$ (see Formula (39.2)), and finally obtain for a path consisting of two segments with the angular dimensions $\Delta\vartheta_1$ and $\Delta\vartheta_2$ and characterized by the parameters q_1 and q_2.

$$W_2(\Delta\vartheta_1, \Delta\vartheta_2; q_1, q_2) = \sqrt{i\pi M\vartheta_A}\,(q_1-q_2) \sum_{k,\, l=1}^{\infty} \frac{e^{iM(t_k(q_1)\Delta\vartheta_1 + t_l(q_2)\Delta\vartheta_2)}}{(t_k(q_1)-q_1^2)(t_l(q_2)-q_2^2)(t_k(q_1)-t_l(q_2))}. \qquad (46.5)$$

This formula was first derived by Furutsu [8] by a different method.

In the limiting case of distances so great that only the first term may be left in each sum (i.e., if each segment of the path corresponds to passage over the horizon, $M\Delta\vartheta_1 \gg 1$, $M\Delta\vartheta_2 \gg 1$), we obtain

$$W_2 = \sqrt{\pi i M\vartheta_A}\,\frac{q_1-q_2}{t_1(q_1)-t_1(q_2)} \cdot \frac{e^{i[t_1(q_1)M\Delta\vartheta_1 + t_1(q_2)M\Delta\vartheta_2]}}{(t_1(q_1)-q_1^2)(t_1(q_2)-q_2^2)}. \qquad (46.6)$$

Introducing the attenuation functions $V(\Delta\vartheta_1; q_1)$ and $V(\Delta\vartheta_2; q_2)$ for each of the path segments, we may rewrite Formula (46.6) as follows:

$$W_2(\Delta\vartheta_1, \Delta\vartheta_2; q_1, q_2) = \sqrt{\frac{\vartheta_A}{M\,\Delta\vartheta_1\,\Delta\vartheta_2}}\,\frac{q_1-q_2}{t_1(q_1)-t_1(q_2)}\,V(\Delta\vartheta_1; q_1)\,V(\Delta\vartheta_2; q_2). \quad (46.7)$$

In many cases, the first two algebraic factors are of secondary importance. If this is the case, then, as we see, the field attenuation is of the nature of the progressive attenuation of a wave that diminishes exponentially first over the first segment and then over the second. This buildup of attenuation provides a sharp distinction between the spherical-earth and flat-earth cases.

2. A formula for the attenuation function in the three-segment case can also be obtained in a similar fashion.

Setting k = 3 in Formula (42.7a), we obtain

$$\begin{aligned} W_3(\Delta\vartheta_1, \Delta\vartheta_2, \Delta\vartheta_3; q_1, q_2, q_3) &= V(\vartheta_A; q_3) + \\ &+ \sqrt{\frac{iM\vartheta_A}{\pi}}(q_1-q_2)\int_0^{\vartheta_1}\frac{V(\vartheta; q_1)V(\vartheta_A-\vartheta; q_3)}{\sqrt{\vartheta(\vartheta_A-\vartheta)}}d\vartheta + \sqrt{\frac{iM\vartheta_A}{\pi}}(q_2-q_3)\times \\ &\times\int_{\vartheta_1}^{\vartheta_2}\frac{W_2(\Delta\vartheta_1, \vartheta-\vartheta_1; q_1, q_2)V(\vartheta_A-\vartheta; q_3)}{\sqrt{\vartheta(\vartheta_A-\vartheta)}}d\vartheta. \end{aligned} \quad (46.8)$$

Substituting V here from Formula (39.2) and W_2 from Formula (46.5) and performing elementary integration with respect to ϑ, we obtain three terms, two of which contain triple series. One of these triple series will contain the following sum (see (46.4)):

$$\begin{aligned} &\sum_l \frac{1}{(t_l(q_2)-q_2^2)(t_h(q_1)-t_l(q_2))(t_l(q_2)-t_m(q_3))} = \frac{1}{t_h(q_1)-t_m(q_3)}\times \\ &\times\left\{\sum_l \frac{1}{(t_l(q_2)-q_2^2)(t_l(q_2)-t_m(q_3))} - \sum_l \frac{1}{(t_l(q_2)-q_2^2)(t_l(q_2)-t_h(q_1))}\right\} = \\ &= \frac{1}{t_h(q_1)-t_m(q_3)}\left\{\frac{1}{q_3-q_2}-\frac{1}{q_1-q_2}\right\} = \frac{q_1-q_3}{(q_1-q_2)(q_3-q_2)(t_h(q_1)-t_m(q_3))}, \end{aligned}$$

where $t_m(q_3)$ is a root of the equation $w'(t) = q_3 w(t)$. Then two of the three terms cancel, leaving, if $\Delta\vartheta_1 = \vartheta_1$, $\Delta\vartheta_2 = \vartheta_2 - \vartheta_1$, $\Delta\vartheta_3 = \vartheta_A - \vartheta_2$ are the angular lengths of the three segments of the path,

$$W_3(\Delta\vartheta_1, \Delta\vartheta_2, \Delta\vartheta_3; q_1, q_2, q_3) = \sqrt{i\pi M\vartheta_A}\,(q_1 - q_2)(q_2 - q_3) \times$$

$$\times \sum_{k, l, m=1}^{\infty} \frac{e^{iM[\Delta\vartheta_1 t_k(q_1) + \Delta\vartheta_2 t_l(q_2) + \Delta\vartheta_3 t_m(q_3)]}}{(t_k(q_1) - q_1^2)(t_l(q_2) - q_2^2)(t_m(q_3) - q_3^2)(t_k(q_1) - t_l(q_2))(t_l(q_2) - t_m(q_3))}. \tag{46.9}$$

The manner in which this formula is extended to the case N > 3 is quite obvious.

Let us analyze Formula (46.9) for the case in which the terminal segments have identical properties $q_1 = q_3$. We shall consider the influence of an intermediate strip with the properties $q = q_2$. If observation is conducted far beyond the horizon and the width of the intermediate strip of soil $\Delta\vartheta_2$ and its position are such that, in addition, the conditions $M\Delta\vartheta_1 \gg 1$ and $M\Delta\vartheta_3 \gg 1$ are satisfied, we may restrict ourselves to the first term in summing over k and m, and Formula (46.9) assumes the form

$$W_3(\Delta\vartheta_1, \Delta\vartheta_2, \Delta\vartheta_3; q_1, q_2, q_1) \approx$$

$$\approx V(\vartheta_A; q_1) \sum_{l=1}^{\infty} \frac{e^{iM\Delta\vartheta_2 (t_l(q_2) - t_1(q_1))}}{(t_l(q_2) - q_2^2)(t_l(q_2) - t_1(q_1))^2} \frac{(q_1 - q_2)^2}{(t_1(q_1) - q_1^2)^2}. \tag{46.10}$$

Thus, a strip with different soil properties situated in the middle of an otherwise homogeneous long path influences the field attenuation (for a homogeneous path we should have $V(\vartheta_A; q_1)$). However, this influence is determined solely by the width and electrical properties of the strip, and is in no way dependent on the position of the strip on the path. The disturbance of the amplitude and phase caused by this strip are transferred to the observation point without changes, no matter where this strip is. This circumstance provides a radical distinction between the cases of the spherical and flat earths. Actually, one of the most characteristic properties of ground-wave propagation above a flat surface is the weak dependence on soil properties on intermediate segments if the over-all path is sufficiently long (§44, and Fig. 44.6 in particular). In the limiting case of very long paths, the "geometric mean" formula (44.5) is valid; according to it, the attenuation function is the geometric mean of the attenuation-function values calculated for the entire path length with respect to the properties of the terminal segments. Here the properties of the soil on intermediate segments drop out altogether. Propagation as a whole is determined by the "takeoff" and "landing" areas. On the other hand, in the case of observation beyond the horizon, Formula (46.10) indicates that the situation is fundamentally different. This does not, of course, mean that the electrical properties of the path's terminal segments are of no particular importance at all. Let us consider, for example, a case in which not only $M\Delta\theta_1 \gg 1, M\Delta\theta_3 \gg 1$, but also $M\Delta\theta_2 \gg 1$, i.e., all segments are sufficiently long. Then only the first term may be left in the sum in Formula (46.10) and

$$W_3(\Delta\theta_1, \Delta\theta_2, \Delta\theta_3; q_1, q_2, q_1) = \\ = \sqrt{i\pi M\theta_A} \frac{e^{iM[(\Delta\theta_1+\Delta\theta_3)t_1(q_1)+\Delta\theta_2 t_2(q_2)]}}{(t_1(q_1)-q_1^2)^2 (t_2(q_2)-q_2^2)(t_2(q_2)-t_1(q_1))^2}(q_1-q_2)^2. \tag{46.11}$$

We consider further the two corresponding cases distinguished by the fact that the soil that was at the ends of the path in one case is in the middle in the other and vice versa. Thus, for example, we might, for the sake of clarity, envisage "land-sea-land" (SMS, case a) and "sea-land-sea" (MSM, case b) cases in much the same way as we did for the flat earth (§44). We shall compare these cases for identical filling of the path with the soil in question, the path having a given total length. That is to say, suppose that in case a, when the segment lengths are $\Delta\theta_1^{(a)}, \Delta\theta_2^{(a)}, \Delta\theta_3^{(a)}$, we have $q_1^{(a)} = q_3^{(a)} = q_C$, while $q_2^{(a)} = q_M$; in case b, suppose that $\Delta\theta_1^{(b)} + \Delta\theta_3^{(b)} = \Delta\theta_2^{(a)}$ and $\Delta\theta_2^{(b)} = \Delta\theta_1^{(a)} + \Delta\theta_3^{(a)}$, with $q_1^{(b)} = q_3^{(b)} = q_2^{(a)} = q_M$, $q_2^{(b)} = q_1^{(a)} = q_3^{(a)} = q_C$. It follows from Formula (46.11) that the ratio of the attenuation functions in this case is

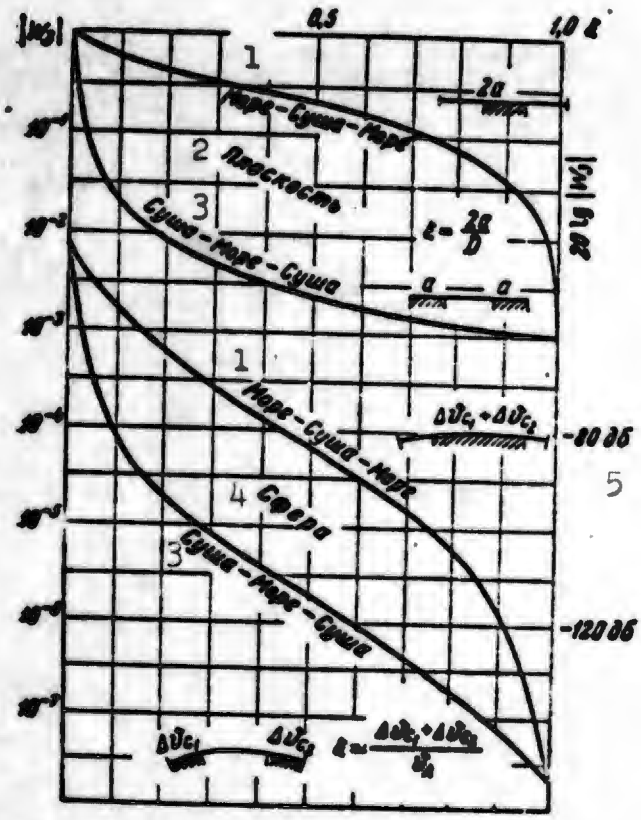

Fig. 46.1. Dependence of absolute value of attenuation function on filling of the path with land when the filling begins at the ends (lower curves) or from the middle (upper curves) of the path for plane and spherical surfaces, $\lambda = 100$ m. 1) Sea-land-sea; 2) plane; 3) land-sea-land; 4) sphere; 5) db.

$$\frac{W_3^{(a)}}{W_3^{(b)}} = \frac{t_1(q_M) - q_M^2}{t_1(q_C) - q_C^2}, \qquad (46.12)$$

and, consequently, in contrast to the case of the flat earth, this ratio is not only independent of the position of the strip $\Delta\vartheta_2$, but is even totally independent of the relative portion of the path filled by the soil in question. However, the field depends quite strongly on the kind of soil at the end of the path. Thus, for example, in the case of land with $\sigma_S = 9\cdot 10^7$ CGSE, sea with $\sigma_M = 4\cdot 10^{10}$ CGSE, and a wavelength $\lambda = 300$ m, we have

$$q_C \approx 3{,}02 e^{i\frac{\pi}{4}}, \quad q_M \approx 0{,}14 e^{i\frac{\pi}{4}}, \quad M \approx 40{,}6.$$

The corresponding values of t_1 may be found approximately, for example,

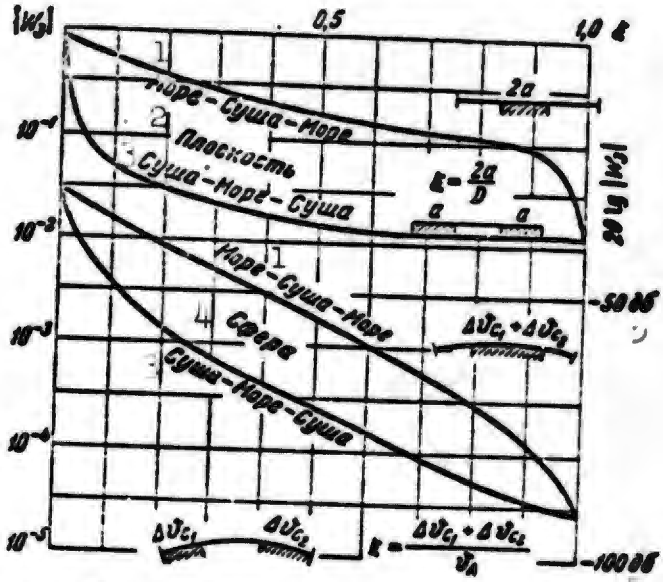

Fig. 46.2. Same as Fig. 46.1, but λ = = 300 m. 1) Sea-land-sea; 2) plane; 3) land-sea-land; 4) sphere; 5) db.

from the asymptotic expansions with Formulas (39.5) and (39.6):

$$t_1(q_M) = t_1(0) + \frac{q_M}{t_1(0)} + \ldots \approx 0,55 + 0,87i,$$

$$t_1(q_C) = t_1(\infty) + \frac{1}{q_C} + \ldots \approx 1,4 + 1,8i.$$

From which

$$\frac{W_3^{CMC}}{W_3^{MCM}} \approx \frac{1,04}{7,4} \approx 0,14. \qquad (46.12a)$$

Figures 46.1 and 46.2 show W_3^{SMS} and W_3^{MSM} as functions of the land filling of the path, i.e., the quantity $\xi = \frac{\Delta\vartheta_1^{(a)} + \Delta\vartheta_2^{(a)}}{\vartheta_A} = \frac{\Delta\vartheta_2^{(b)}}{\vartheta_A}$, in logarithmic scale for λ = 100 m and λ = 300 m. It is assumed that the entire path has a length $\vartheta_A = \frac{8}{M} \approx 0,14$, which corresponds to $D = a\vartheta_A \approx 900$ км. Since Formula (46.11) may be used as long as $M \cdot \Delta\vartheta_i \gg 1$, the curves are calculated only in the region $\frac{1}{4} < \xi < \frac{3}{4}$, and smoothly joined on the other segments with the known points for ξ = 0 and ξ = 1. The same diagrams show similar curves calculated from the formulas for a flat inhomogeneous path (which are, of course, inapplicable here for D = 900 km and

λ = 100 and 300 m). It is evident from this (see also Fig. 44.6) that, although the properties of the soil are in themselves a significant factor in the case of a spherical earth (for example, for λ = 300 m, the field drops by 92 db as ξ varies from 0 to 1), the distribution of this soil over the path is far from being as essential as in the case of the flat earth.

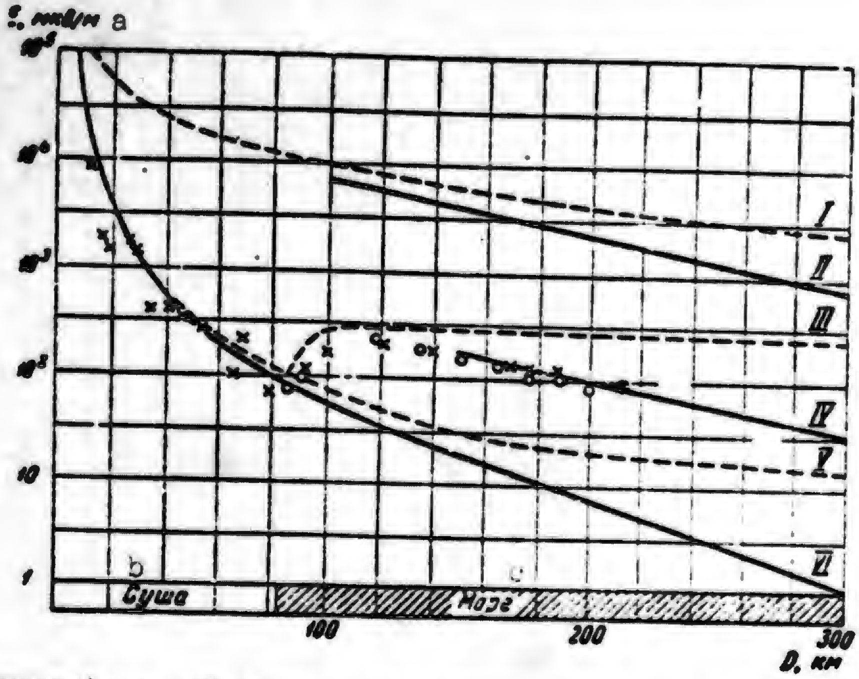

Fig. 46.3. Comparison of theory with experiment (crosses and circles). The broken lines represent the theory for a flat earth and the solid curves that for a spherical earth. I, II) for a homogeneous sea path; V, VI) for homogeneous land; III, IV) for a "land-sea" path. $\sigma_S = 9 \cdot 10^7$ CGSE, transmitter power 10 kw, λ = 96 m. a) μv/m; b) land; c) sea.

Actually, if a segment with good conductivity is situated in the middle of the path, then, as indicated, for example, by the lower of the top pair of curves on Fig. 46.2, this has no effect on the field in the case of a flat earth even if it occupies half of the entire path (for ξ = 0.5 and ξ = 1, we have almost identical values, $20\lg|w| \approx -38$ дб). For a spherical earth, on the other hand, we obtain in the analogous case with ξ = 0.5 a field 20 db higher than for a path completely filled by land (for $\xi = 1$ we obtain $20 \lg|w| = -92$ db, while we have -72 db for $\xi = 0.5$).

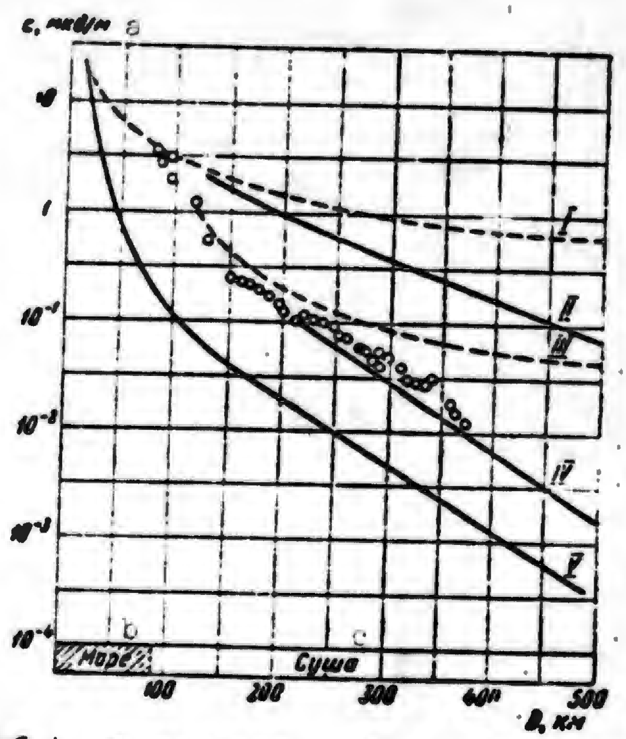

Fig. 46.4. Same as Fig. 46.3, for $\sigma_S = 10^8$ CGSE, $\lambda = 250$ m. a) μv/m; b) land; c) sea.

Thus, in the case of a spherical ground surface – if we are speaking of observation far beyond the horizon –, the "takeoff" and "landing" areas for the radio waves have a distinct influence (the distance between the two curves of the lower pair amounts to 14-18 db), but in contrast to the case of the flat earth, the concept of a ground wave skipping over the surface and accumulating attenuation progressively and additively is in general much more appropriate here. In this connection, it is interesting to recall that, as was noted earlier (see [I, 1], §122), the additional phase of the wave due to attenuation, which does not increase in the case of a homogeneous flat earth, increases linearly with D for a sphere.

Figure 46.3 presents a comparison of theory with experiment for

λ = 96 m [12b], while Fig. 46.4 shows it for λ = 250 m [16]. As will be seen from these figures, the deviation from the curves for an inhomogeneous flat earth is quite clearly manifest here and the theoretical formulas of this section are in good agreement with the experimental data.

Formulas (46.5) and (46.9) are easily generalized to the case of a receiver elevated to a height z_A above the ground surface and a transmitter raised to a height z_0. For this purpose, it is necessary to introduce height factors under the summation sign (see Formula (39.20) and the arguments justifying it). In Series (46.5), we introduce $w(t_k(q_1) - bz_0)/w(t_k(q_1))$ and $w(t_l(q_2) - bz_A)/w(t_l(q_2))$, while in Series (46.9) we introduce $w(t_k(q_1) - bz_0)/w(t_k(q_1))$ and $w(t_m(q_3) - bz_A)/w(t_m(q_3))$, where $b = (2M^2/a)$. These series were derived in [9] in precisely this generalized form.

Extremely detailed tables and diagrams that facilitate calculation of the attenuation function for arbitrary positions of the corresponding points and an arbitrary number of homogeneous path segments on a spherical earth's surface will be found in [10].

Manu-script Page No.	[Footnotes]
420	Here and in similar cases, we shall henceforth use the "smoothness" of the function y and its derivatives, i.e., the circumstance that a relatively small change in the argument causes a relatively small change in the function and its derivatives as well.
421	For it, we introduce the special symbol z, because it plays a substantial role in other problems of the inhomogeneous path as well. It is this function that is tabulated in the tables of [II, 8].
443	Here the apparent agreement with experiment was based on the fact that the experiment was limited to short numerical distances, $\lvert \rho D \rvert \sim 0.5 \div 1.0$. When the numerical distances are so small, all methods lead to closely similar results. As we have already noted, the specific nature of the phenomenon, to

the extent that we disregard the curvature of the earth and, consequently, restrict ourselves to short distances, comes clearly into evidence only for $\lambda \leqslant 100$ meters.

444 As we saw in §22, the energy flux vector inclination angle is

$$\alpha \approx \frac{1}{|\sqrt{\varepsilon \div 1}|} \cos \frac{\chi}{2}, \quad \chi = \operatorname{arctg} \frac{4\pi\sigma}{(\varepsilon' + 1)\omega}$$

(see Formula (22.9)).

[Transliterated Symbols]

Manu-script Page No.	
439	с = s = susha = land
439	м = m = more = sea

Chapter 8

PROPAGATION OF RADIO WAVES OVER AN UNEVEN SURFACE

§47. INTRODUCTION

Having studied the fundamental laws of radio wave propagation above ideal flat and spherical surfaces, it is natural to pass on to investigation of the role of unevenness of the surface. It is hardly possible to indicate practical conditions under which such irregularities are absent. However, as long as long and medium waves were used in most cases, it was very often possible to assume that roughnesses small by comparison with the wavelength would have no substantial influence. We shall see below that even this is not always correct (influence on accuracy of radio direction finding, §50; variation of effective soil conductivity, §51). In general, however, this was the case.

Only with the trend to shorter wavelengths did the situation begin to deteriorate steadily. In certain cases, surface roughness changes the propagation picture radically. It is sufficient to point out that radar scanning of the ground surface from aircraft would be impossible (more precisely, points on a strictly horizontal surface could be observed provided that they were directly below the aircraft) if it were not for the roughness of this surface, which provides scattering of the radio waves at angles other than the angle of incidence onto a certain average ground surface. In this connection, more and more work is being devoted to the theory of radio wave propagation above an uneven surface (see, for example, [1]).

The difficulty of the problem stems, for one thing, from the fact that the conditions subject to investigation are extremely varied. For this reason alone, the theory of the problem cannot be unified for all cases.

Usually, and very crudely, the problems that arise are classified on the basis of the criterion ratio of characteristic height of roughness ζ to radiation wavelength λ. In actuality, however, it is not the "wavelength" that is a factor, but its reciprocal, the wave number k, in itself together with the projection of the corresponding vector onto the vertical. Actually, we shall consider incidence of a plane wave at a glancing angle ψ onto a flat surface with a depression of depth ζ (Fig. 47.1).

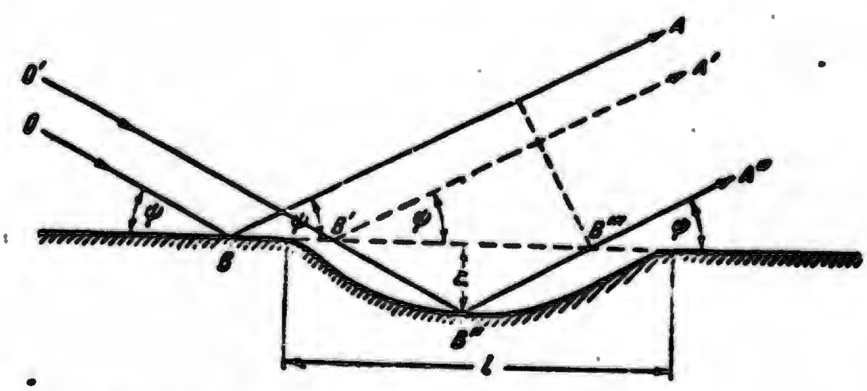

Fig. 47.1. Phase shift due to surface irregularity.

Even if the ray O'B" incident in the depression finds a horizontal surface and is reflected in the direction A" at the angle of incidence, it will acquire a phase difference $\Delta\varphi$ from the reflection O'B'A' from the undisturbed plane

$$\Delta\varphi = k\cdot 2\zeta \sin\psi. \quad (47.1)$$

This phase difference can be disregarded if

$$2\zeta \sin\psi \ll \lambda. \quad (47.2)$$

In this case, coherence of the rays is not disturbed. As concerns the meaning of the "much less than" sign, it is not possible to make any definite statement. It is usually considered that $\Delta\varphi < \frac{\pi}{2}$ and hence $8\zeta\sin\psi < \lambda$ are necessary. However, it goes without saying that this criterion depends on the purpose of the investigation: even small $\Delta\varphi$ are essential for the accuracy of radio direction finding.

Condition (47.2) is, of course, essential if the irregularities cover the entire surface, since the disturbance affects a considerable number of the rays. If, however, the irregularity is isolated or if there are only a few of them, they will, even when Condition (47.2) is violated, disturb the field noticeably only when they cover a large part of the region essential for reflection. Suppose that the nearest of the corresponding points is situated at a distance r, and that the <u>horizontal</u> dimensions of the irregularity (or of all irregularities taken together) are of the order of l. Then, according to Formula (12.7), provided that Condition (47.2) is not satisfied, the disturbance will still be small when

$$l \ll \frac{1}{\sin\psi}\sqrt{\pi\lambda r}. \tag{47.3}$$

These two conditions indicate that the smaller the angle ψ, the weaker will be the disturbing effect of irregularities. As we know, this is why even a rough sheet of paper will reflect brilliantly if it is seen at a very small glancing angle.

Thus, in addition to the classification of the problems that arise by the <u>short wave-long wave</u> criterion (comparison with ζ), there is still another important division: <u>glancing waves</u> (small angle ψ) and <u>steeply incident waves</u> (large angle ψ). In a considerable number of cases, these two breakdowns coincide, because in practical short-wave transmission, the source and observation point are elevated high above the ground surface. Hence long waves are usually glancing waves, while short waves are steeply incident. Nevertheless, this is by no means a general rule.

In themselves, however, the height and length of the irregularity are not quite decisive factors. Consider, for example, a plane wave glancing along the $z = 0$ plane. If we raise the entire plane to a height $z = \zeta$, the field in the space above it will remain as before. A disturbance can arise only from the margins of the displaced surface segment. But this boundary effect may always be regarded as small at great distances from the edges provided that the horizontal dimensions l of the segment are large. The field will be changed sharply only in the region $0 < z < \zeta$ from which it is "displaced." In the rest of the space, however, the disturbance will be small, no matter how large both ζ and l become. Thus, the most important geometrical parameter is the <u>surface inclination angle</u> γ. For gently sloping irregularities, $|\gamma| << 1$, the disturbance created by this segment will be smaller than for steep irregularities at the same ζ. It is found that a quite general method can be developed for gently sloping irregularities (§49). Only when $\gamma \sim 1$ do Conditions (47.2) and (47.3) begin to play a decisive role. Consequently, we must distinguish the case $\gamma << 1$ from cases with $\gamma \gtrsim 1$. This is a third criterion that must be kept in mind.

Finally, the problems are also classified on the basis of the type of answer required. In some cases, we are concerned with the disturbance of the field by a given specific irregularity. Such, for example, are the problems of distortion of radio wave propagation direction by a coastal inlet (radio direction-finding error) or the extremely important problem of field amplification behind a hill ("obstacle gain"). But these are only selected cases. More frequently, it is necessary to know the <u>influence of a large number of similar irreg-</u>

ularities: sea waves, vegetation cover, hilliness. A closely related problem is that of scattering from the lower boundary of a turbulent layer in the troposphere, which, in the opinion of certain investigators, should explain the anomalous range of very short radio waves beyond the horizon. In all of these cases, the statistical approach is necessary: knowing the statistical characteristics of the surface (number of irregularities per unit area, average height, average horizontal extent and the elevation angle of the individual irregularities, the distribution of these quantities, and their mutual correlation), we must find the average values, fluctuations and correlation characteristics of the field amplitude and phase.

Certain general considerations may be advanced for all such problems.

We have already seen in the case of a flat homogeneous (or even inhomogeneous) surface that its influence can be reduced to the appearance of secondary "virtual" radiators excited by the incident "primary" wave. Since ε is usually not essential, their fields arise with a phase shift, for example, in Formula (24.5), the factor

$$\frac{ik}{\sqrt{\varepsilon^0}} = \frac{k}{|\sqrt{\varepsilon^0}|} e^{i\left(\frac{\pi}{2} - \frac{1}{2}\arg\varepsilon\right)}.$$

However, if the soil is homogeneous, this shift is constant. The field of the secondary radiators at the observation point has a phase $\varphi = k(r + \rho) + \text{const}$, determined primarily by the path length r from the source to the secondary radiator in question and then the distance ρ to the observation point. It is precisely the region of the minimum of this phase that determines the essential region on the plane – the first Fresnel zones.

When the surface is disturbed – either electrically or geometrically – then each secondary radiator will have a different amplitude and will give rise to a different phase shift other than k(r+). However, if we have a large number of more or less equivalent secondary radiators, the phase k(r+) will again determine the region of the surface whose radiators emit oscillations that are, on the average, in phase at the observation point. The fields of these secondary radiators will add coherently. It is clear that in general, this coherent scattering will be significant in the same direction as that in which the wave mirror-reflected from a plane taking the role of a certair average plane would be propagated.

Needless to say, surface irregularities are also responsible for scattering in directions other than that of mirror reflection from an averaging plane. For this reason alone, the scattering fields sent out by different surface segments will be basically incoherent in this case.

Considerable progress has been made in study of the influence of periodic irregularities: strictly sinusoidal surface, regularly spaced rectangular-section grooves, an endless array of uniformly spaced half-cylinders placed parallel to one another. However, these cases, which play an important role in the theory of diffraction gratings, corrugated waveguides, and the like, are of minor importance for the problem of radio wave propagation along the earth's surface. Here we are never presented with strictly periodic surfaces. Moreover, in the most difficult case, in which rigorous solutions would be necessary for steeply sloping surfaces and short wavelengths – the effect of the irregularities is not additive (see §52). Deviations from the shape of the periodic surface for which a given theory has been constructed may change the result completely. For this reason, we shall not set forth the work that has been done in this connection (see, for example, the survey [1]), but concentrate our attention on statistical problems.

In a number of cases, we shall combine study of electrical-property nonuniformities with our examination of the influence of irregularities. This will apply to problems in which joint investigation of both types of deviation from the ideal is suggested by practical conditions and by the uniformity of the mathematical methods (distortion of field near a coastal inlet, statistical influence of random irregularities and inhomogeneities).

The present chapter will set forth the theory of problems for which a solution has been obtained in one or another degree. The fundamental ways to solution of the problem of irregularities – not necessarily gently sloping – that are small by comparison with wavelength have been known even since Rayleigh's day [2]. Here the simple method of disturbances developed in [3, 4, 5] is possible. However, it is applicable only to rather steep (with respect to the averaging plane) incidence of the radio waves. For glancing waves (source and observation point near the ground surface), a different approach is generally required [VII, 3]. It is found to be successful only for gently sloping surfaces, but here it is quite general and applicable, if the surface is sloped gently enough, for irregularity heights that are not even small by comparison with λ.

The main reason why these methods are not applicable to steep and large irregularities consists in the difficulty of accounting for the mutual influence of the irregularities on one another (only in the theory just mentioned is it found possible to take this fact into account). Mutual shadowing is a particular case of this effect. However, theoretical analysis is successful in a certain converse case: if incidence is steep enough, shadowing may be disregarded, and if the wavelength is short enough by comparison with ζ, the reradiation is insignificant [6].

In §48, the field of both glancing and steeply incident waves is considered in the presence of projections on a plane with good conductivity that are small by comparison with wavelength. If the projections are spaced widely enough, statistical consideration is possible and leads to conclusions of general importance.

In §49, we derive general starting formulas for a theory that will be valid for a surface with gently sloping irregularities (even in the presence of inhomogeneities); these formulas will apply for any wave glancing angle if the irregularity heights are small, $\zeta \ll \lambda$, and even for rather large ζ for a glancing wave ($\psi \to 0$) and a flat enough surface.

In §50, this theory is used for analysis of individual irregularities and inhomogeneities – coastal declivities and coastline variation of ε ("coastline refraction").

In §51, the statistical characteristics of the field are examined on the same basis: these include average field-strength values and the root-mean-square values that characterize scattering of the radiation by irregularities and inhomogeneities.

In §52, the same statistical characteristics of the scattered radiation are considered for gentle irregularities whose height is large as compared with wavelength, but in the case of rather large angles of incidence on the basis of what might be called a Kirchhoff approximation.

Finally, §53 considers diffraction on an isolated obstacle of the hill type.

§48. SMALL PROJECTIONS ON AN IDEALLY CONDUCTING PLANE

1. It was shown in §20 that if a projection with arbitrary electrical properties is placed on an ideally conducting plane, the field of any source can be obtained by superposing the fields of this source

and its image in the plane (constructed by the conventional rules), taken in vacuo in the presence of the body formed by the projection in question and its reflection in the same plane. In the present section, we shall apply this result to the propagation of radio waves near an ideally conducting plane on which there are projections that are small by comparison with wavelength in air but with arbitrary elevation angles γ.

The problem of interest to us will be reduced to consideration of scattering - in vacuo - of waves on small particles. The general method of solving such problems, which was indicated by Rayleigh [7], consists in examination first of all of the quasistationary field in the immediate vicinity of the object, where the linear parameter determining the rapidity of field variation in space is not wavelength, but the dimensions of the object. The external incident field is constant throughout this region in space and varies simultaneously in time at all points. We substitute in the equation for any of the field components

$$\nabla^2\Pi + k^2\Pi = 0 \qquad (48.1)$$

the function

$$\Pi = \Pi^0(x, z)\,w(x, y, z);\ \Pi^0 = Ae^{i(k_x x + k_z z - \omega t)} \qquad (48.2)$$

where Π^0 is the incident field, which is propagating from the source toward the object along a line in the xz-plane. For the time being, let $k_z = 0$.

The distortion function w near the object varies on a segment of the order of its dimensions, i.e., since these dimensions are small by comparison with wavelength,

$$\frac{\partial w}{\partial x},\ \frac{\partial w}{\partial y},\ \frac{\partial w}{\partial z} \sim \frac{w}{d} \gg kw, \qquad (48.3)$$

where d is a linear parameter characterizing the dimensions of the object, kd << 1.

Substituting Π (48.2) into Equation (48.1), we obtain, with application of (48.3)

$$\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}+ik\frac{\partial w}{\partial x}=0. \qquad (48.3a)$$

We set $w = w_0 + w_1 + \ldots$, where the ratio of successive terms is of the order of kd. Introducing this expansion into Equation (48.3a) and equating terms of the same order, we obtain

$$\frac{\partial^2 w_0}{\partial x^2}+\frac{\partial^2 w_0}{\partial y^2}+\frac{\partial^2 w_0}{\partial z^2}=0, \qquad (48.3b)$$

$$\frac{\partial^2 w_1}{\partial x^2}+\frac{\partial^2 w_1}{\partial y^2}+\frac{\partial^2 w_1}{\partial z^2}=-ik\frac{\partial w_0}{\partial x}. \qquad (48.3c)$$

$$\cdots\cdots\quad\cdots\cdots\cdot$$

In the zeroth approximation, therefore, it is necessary to solve the static problem $\nabla^2 w_0 = 0$, which satisfies certain boundary conditions for the field at the surface of the object. Corrections of this order may be found from the inhomogeneous equations for subsequent approximations.

After the field near the object has been found, the function w for the remaining space is determined by sewing to the solution of the wave equation.

Thus, the problem is reduced to study of the field in a body situated in a (spatially) homogeneous and (temporarily) variable incident-wave field that is, however, quasistationary. If the value of ε is such that the wavelength is large by comparison with the dimensions of the body not only outside the body but inside it as well, the field of the wave will give rise to an electrical polarization that varies in phase at all points of the body. No closed currents arise inside the body and the polarization may be regarded as homogeneous. If, however, the wavelength inside the body is short, the field does not penetrate deep into it and, on the other hand, it has different phases at different points. In this case, closed circular currents arise along the surface, so that the body as a whole acquires not only an electrical moment but also a magnetic moment. It is necessary to consider the radiation of both the electrical dipole moment induced by the electric field of the incident wave and the magnetic dipole moment, which, naturally, is proportional to its magnetic field.

The complete scattering theory exists for the sphere and the cylinder (see, for example, [8; II, 9]), as well as for the ellipsoid [9]. In application to particles of small radius, it gives the values of the indicated electrical and magnetic moments.

2. Let us consider a homogeneous hemisphere of radius a lying flat-side-down on a plane. We take the center of its base B as the coordinate origin and assume that radiation is incident at an angle ψ to the plane from a dipole situated at point O (Fig. 48.1). If the polarization is horizontal, there will be no resultant (tangential to the plane $z = 0$) incident field at the position of the projection B. In the case of vertical polarization, however, the incident electric field at point B has a z-component

$$2E_z^{(0)} = 2|E^{(0)}|\cos\psi = 2k^2 p\cos^2\psi\frac{e^{ikR}}{R}, \qquad (48.4)$$

where $\vec{E}^{(0)}$ is the field of the source in question in a vacuum.

Let us consider first the case of not very large ε, $k|\sqrt{\varepsilon}|a \ll 1$. The field inside the body will be homogeneous and in a sphere of radius a it will create a dipole moment equal, as we know from electrostatics (see, for example, [10]), to

$$p' = p_z' = a^3\frac{\varepsilon - 1}{\varepsilon + 2}2E_z^{(0)}. \qquad (48.5)$$

From this, using the formulas of §18, we obtain the field $\vec{E}^{(e)}$, scat-

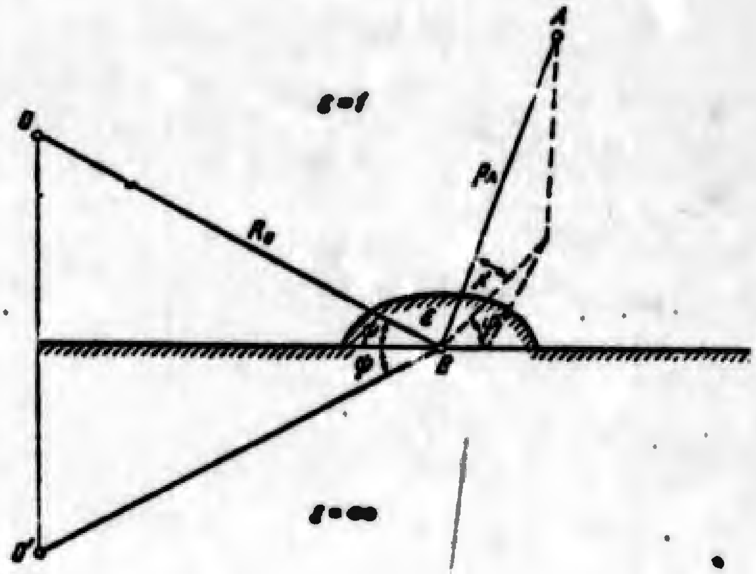

Fig. 48.1. Scattering by an ellipsoidal projection.

tered at an angle χ (to the moment $\vec{p}'$),

$$E_\chi^{(s)'} = k^2 p' \cos\chi \, \frac{e^{ik\rho_A}}{\rho_A}, \tag{48.6}$$

where $\hat{E}_\chi$ points in the direction of increasing angle χ and ρ_A is the distance from point B. Dividing the energy flux $(\Sigma \cdot d\Omega\rho_A^2)$ in the solid angle $d\Omega$ by the flux of primary radiation arriving on a unit surface perpendicular to the incident ray, Σ_0, we obtain the effective differential scattering cross section of the hemisphere (it is hoped that th the use of the same symbol σ for different quantities – conductivity and effective cross section – will not give rise to confusion):

$$d\sigma = \frac{\Sigma\rho_A^2\, d\Omega}{\Sigma_0} = \frac{\frac{c}{8\pi}\,|E_\chi^{(s)'}|^2 \rho_A^2\, d\Omega}{\frac{c}{8\pi}\,|E^{(0)}|^2}. \tag{48.7}$$

$$d\sigma = 4a^6k^4 \left|\frac{\varepsilon-1}{\varepsilon+2}\right|^2 \cos^2\chi \cos^2\psi \, d\Omega. \tag{48.8}$$

The photoscattering cross section (the integral over all angles) is

$$\sigma = \frac{32\pi}{3} k^4 a^6 \left|\frac{\varepsilon - 1}{\varepsilon + 2}\right|^2 \cos^2 \psi. \tag{48.9}$$

This quantity has the dimensions of area.

If the absolute value of ε is large, Formula (48.5) may be used as before, but, in addition, it is necessary to take into account the radiation of the magnetic moment m'. If we were speaking of a sphere with a permeability μ, it would acquire a magnetic moment

$$m' = m'_y = a^3 \frac{\mu - 1}{\mu + 2} 2H_y^{(0)}. \tag{48.10}$$

in a static field $2\bar{H}^{(0)}$. From Formulas (48.5) and (48.10), we can find the radiation of an infinitely conductive sphere by the following method – which was used by Rayleigh. We consider the fact that the field does not penetrate into the interior of such a sphere. Thus the continuous (on passage across the interface) normal component of magnetic induction and the continuous tangential component of the electric field must vanish outside the sphere, at its surface. At the same time, since it is quasistationary, this field may be described by the formulas of electro- and magnetostatics. They will yield the required external field (which does not contain the components mentioned above) if we ascribe the values $\varepsilon = \infty$ and $\mu = 0$ to the material of the sphere. According to Formulas (48.5) and (48.10), this means that we are assigning to the sphere an electrical dipole moment $p' = p'_z = a^3 \cdot 2E_z^{(0)}$ and a magnetic dipole moment $m' = m'_y = -\frac{1}{2} a^3 \cdot 2H_y^{(0)}$. Consequently, the field of the radiation will be the sum of the fields of the electric dipole (48.6) for this p'_z and the field of the magnetic dipole

$$\left.\begin{aligned} E_x^{(m)'} &= k^2 \sin\chi \frac{e^{ik\rho}}{\rho} m', \\ E_y^{(m)'} &= 0, \\ E_z^{(m)'} &= k^2 \cos\chi \cos\varphi \frac{e^{ik\rho}}{\rho} m' \end{aligned}\right\} \tag{48.11}$$

at the indicated m'.

Remembering that for any vector $\vec{A}$ directed along the χ axis, $A_x =$ $= -A \sin \chi \cos \varphi$, $A_y = -A \sin \chi \sin \varphi$, $A_z = A \cos \chi$, and substituting the values of p' and m', we obtain

$$E' = 2E_z^0 \cdot k^2 \frac{e^{ik\rho}}{\rho} F(\chi, \varphi), \qquad (48.12)$$

$$\left.\begin{aligned} F_x &= a^3\left(-\cos\chi\cos\varphi + \frac{1}{2}\right)\sin\chi, \\ F_y &= -a^3\sin\chi\cos\chi\sin\varphi, \\ F_z &= a^3\left(\cos\chi - \frac{1}{2}\cos\varphi\right)\cos\chi. \end{aligned}\right\} \qquad (48.12a)$$

We note that Formula (48.6) can also be written in the form of (48.12). The corresponding values of the function $\vec{F}$, which may be called the scattering function (it is analogous to the so-called scattering amplitude), are given by the first terms of the right members in Formulas (48.12a), which need only be multiplied by $(\varepsilon - 1)/(\varepsilon + 2)$.

Here it is necessary to take note of the polarization properties of the scattered radiation. As we have already noticed, horizontally polarized radiation is not at all disturbed by the projection. On the other hand, the radiation of a vertical dipole is, according to Formulas (48.12a), strongly depolarized by a conducting projection. Indeed, on scattering at an azimuth $\varphi = \pi/2$, the field has a horizontal component $F_y = -a^3 \sin \chi \cos \chi$. At the same time, a poorly conducting projection does not produce such an effect. And this is understandable: only a vertically polarized electric dipole, which radiates the field (48.6), is induced in a weakly conducting projection. At high conductivity, on the other hand, the field of a magnetic dipole oriented along the y-axis is superimposed upon it. Its electric lines of force form a circle about this axis. As a result of addition of the two fields, we might, for example, have lines on which the vertical component of the field is zero: according to Formulas (48.12a), this occurs at $\cos\varphi = 2\cos\chi$. Here $F_y = -a^3\sqrt{1-4\cos^2\chi}\,\sin\chi\cos\chi$. Thus, the field has horizontal polarization on this line and vanishes entirely at the point $\chi = 0$, $\varphi = \pi/2$.

Let us calculate the scattered-radiation energy flux in the solid-angle element $d\Omega$ separately for the vertical component E'_z and for the horizontal component, as would correspond to reception on a vertical antenna in one case and on a horizontal frame in the other. These fluxes are respectively proportional to $F_z^2 = a^6\cos^2\chi\left(\cos\chi - \frac{1}{2}\cos\varphi\right)^2$ and $F_x^2 + F_y^2 = a^6\sin^2\chi\left\{\cos^2\chi - \cos\varphi\cos\chi + \frac{1}{4}\right\}$. Thus, for the vertically polarized component of the received radiation

$$d\sigma^{(v)} = 4k^4a^6\cos^2\psi\cos^2\chi\left(\cos\chi - \frac{1}{2}\cos\varphi\right)^2 d\Omega; \qquad (48.13a)$$

and for the horizontal component

$$d\sigma^{(h)} = 4k^4a^6\cos^2\psi\sin^2\chi\left(\cos^2\chi - \cos\varphi\cos\chi + \frac{1}{4}\right)d\Omega. \quad (48.13b)$$

The resultant cross section of the projection is

$$d\sigma = d\sigma^{(v)} + d\sigma^{(h)} = 4k^4a^6\cos^2\psi\left(\cos^2\chi\left(1 + \frac{1}{4}\cos^2\varphi\right) + \right.$$

$$\left. + \frac{1}{4}\sin^2\chi - \cos\chi\cos\varphi\right)d\Omega, \qquad (48.13c)$$

and the total cross section

$$\sigma = \frac{40\pi}{3}k^4a^6\cos^2\psi. \qquad (48.14)$$

More detailed calculations for the case of large but finite conductivity give (see [10], page 383)

$$\sigma = \frac{32\pi a^6k^4}{3}(1 + |\gamma|^2)\cos^2\psi, \qquad (48.15)$$

where

$$\gamma = \frac{1}{2}\left(1 + \frac{3}{ka\sqrt{\varepsilon}}\operatorname{ctg} ka\sqrt{\varepsilon} - \frac{3}{(ka\sqrt{\varepsilon})^2}\right) \qquad (48.15a)$$

and as $\varepsilon \to \infty$, the parameter $\gamma \to 1/2$.

The same procedure for determining the magnetic polarizability μ of a highly conductive body has been applied to the case of an ellipsoid [7]. Suppose that half of an ellipsoid with the semiaxes a, b, and c directed along the x, y, and z-axes, respectively, lies with its flat side on the plane $z = 0$. In this case, the scattering will correspond to that on the whole ellipsoid in vacuo. The field $2E_z^{(0)}$, which is polarized along the z-axis, creates a moment

$$p' = p_z^{\circ} = \frac{\varepsilon - 1}{4\pi + (\varepsilon - 1) N} T \cdot 2E_z^{(0)} \tag{48.16a}$$

in a dielectric and nonconductive ellipsoid and, in a nonconducting magnetic with the permeability μ, a magnetic moment

$$m' = m_y^{\circ} = \frac{\mu - 1}{4\pi + (\mu - 1) M} T \cdot 2H_y^{(0)}, \tag{48.16b}$$

where

$$T = \frac{4}{3}\pi abc,$$

$$\left.\begin{aligned} M &= 2\pi abc \int_0^{\infty} \frac{d\lambda}{(a^2+\lambda)^{1/2}(b^2+\lambda)^{3/2}(c^2+\lambda)^{1/2}}, \\ N &= 2\pi abc \int_0^{\infty} \frac{d\lambda}{(a^2+\lambda)^{1/2}(b^2+\lambda)^{1/2}(c^2+\lambda)^{3/2}}. \end{aligned}\right\} \tag{48.16c}$$

From this we obtain the fields of these dipoles, and then, letting ε increase without limit and μ vanish, the resultant field radiated by the scattering ellipsoid; this field, of course, has the form (48.12), but with a different scattering function $\vec{F}$:

$$\left.\begin{aligned} F_x &= -\left(\frac{T}{N}\cos\chi\cos\varphi - \frac{T}{4\pi - M}\right)\sin\chi, \\ F_y &= -\frac{T}{N}\sin\chi\cos\chi\sin\varphi, \\ F_z &= \left(\frac{T}{N}\cos\chi - \frac{T}{4\pi - M}\cos\varphi\right)\cos\chi. \end{aligned}\right\} \tag{48.17}$$

In the case of a sphere, $a = b = c$, $T = \frac{4}{3}\pi a^3$, $N = M = 4\pi/3$ and we again obtain Formula (48.12a). Here and below, the differential section $d\sigma^{(v)}$ for scattered vertically polarized radiation is proportional to $|F_z|^2$, while that for horizontally polarized radiation is proportional to $|F_x|^2 + |F_y|^2$. The resultant differential and total cross sections for the ellipsoid are

$$d\sigma = 4k^4 \cos^2\psi \left\{ \frac{T^2}{N^2} \cos^2\chi + \left(\frac{T}{4\pi - M} \right)^2 (1 - \cos^2\chi \sin^2\varphi) - \right.$$

$$\left. - \frac{2T^2}{N(4\pi - M)} \cos\chi \cos\varphi \right\} d\Omega, \tag{48.18}$$

$$\sigma = \frac{32\pi}{3} k^4 T^2 \left(\frac{1}{N^2} + \frac{1}{(4\pi - M)^2} \right) \cos^2\psi. \tag{48.19}$$

Thus, we have depolarization of the same type as for the hemisphere.

3. Let us now consider the field above an ideally conducting plane over which many projections of the type considered above [VII, 3d] are scattered at random. A simple result can be obtained only when the projections are quite widely spaced, so that each of them may be regarded as situated in a field $\bar{E} = 2E^{(0)} + \bar{E}'$, composed of the undisturbed field $2\vec{E}^{(0)}$ that would prevail in the absence of the projections and an average ("locally averaged" or average-over-ensemble; for greater detail on the averaging concept, see §51, Subsection 1) scattering field of all other projections. We may thus disregard local variations of the scattered field only when the distances between projections are considerably larger than their dimensions. Actually, the field of each projection is small, since it is proportional to $(ka)^6$. Only the combined field of many remote projections can produce a noticeable effect. Let ν be the number of projections per unit of area. $\nu(\vec{r}\,')dS'$ projections on an area element at point $\vec{r}\,'$ possess induced electric and magnetic moments in the over-all averaged field $\bar{\vec{E}}$ such that at a certain point $\vec{r}$ in space they create a scattered average

field

$$\overline{E_z(r)} = \overline{E_z(r')}\, k^2 \frac{e^{ik|r'-r|}}{|r'-r|} F_z(\chi, \varphi)\, \nu\, dS', \qquad (48.20)$$

where χ and φ are the angles of the vector $\vec{r}' - \vec{r} = \vec{\rho}$. Hence the total field at an observation point A is equal to (we take, for example, the z-component)

$$\overline{E_z(A)} = 2E_z^{(0)}(A) + \int \overline{E_z(r)}\, \nu k^2 F_z(\chi, \varphi) \frac{e^{ik\rho}}{\rho} dS'. \qquad (48.21)$$

Selecting a point A in the plane $z = 0$ ($\chi = 0$), we obtain an integral equation for the average field $\overline{E_z(r)}$:

$$\overline{E_z(r)} = 2E_z^{(0)}(r) + \int \nu k^2 F_z(0, \varphi)\, \overline{E_z(r)} \frac{e^{ik\rho}}{\rho} dS', \qquad (48.22)$$

$$\rho = \sqrt{(x-x')^2 + (y-y')^2}.$$

It has exactly the same form as the ordinary equation for a homogeneous flat surface, which we have used several times, as, for example, Equation (24.6).

Suppose at first that the source is situated in the plane $z = 0$. Assuming

$$\overline{E_z(x, y)} = 2E_z^{(0)}(x, y)\, w(x, y), \quad 2E_z^{(0)}(x, y) = \frac{e^{ikr}}{r}, \quad r = \sqrt{x^2 + y^2} \qquad (48.23)$$

(field of vertical dipole on $z = 0$ plane), we can determine the attenuation function w for the average field above a surface covered with randomly placed projections. Even here, of course, the essential zone is a narrow ellipse – the first Fresnel zone. Hence for a slowly varying function F_z, we may take its value on the ellipse axis $F_z(0,0)$. The identity of the equations for these two cases is not surprising. It was stressed back in §§6 and 8 that the integral equation obtained by application of Green's theorem admits of interpretation in terms of the fields of secondary dipoles (and sometimes even quadrupoles) distributed on a plane. In the case of an uneven surface, the secondary dipoles acquire tangible shape.

Thus, the average field above an ideally conductive surface with projections is subject to the same laws as the field above a flat surface with a certain effective finite $\varepsilon^0 = \varepsilon_g^0$. This value ε_g^0 can be found by comparing coefficients in Equations (24.6) and (48.22):

$$\frac{1}{\sqrt{\varepsilon_g^0}} = \frac{2\pi}{ik} \nu k^3 F_z(0, 0). \tag{48.23}$$

For poorly conducting hemispheres, (48.5), this gives

$$\frac{1}{\sqrt{\varepsilon_g^0}} = 2\pi e^{-i\frac{\pi}{2}} \frac{\varepsilon - 1}{\varepsilon + 2} k \nu a^3; \tag{48.24a}$$

for hemispheres with good conductivity, (48.12a), –

$$\frac{1}{\sqrt{\varepsilon_g^0}} = \pi e^{-i\frac{\pi}{2}} k \nu a^3; \tag{48.24b}$$

and for hemiellipsoids, (48.17), –

$$\frac{1}{\sqrt{\varepsilon_g^0}} = 2\pi e^{-i\frac{\pi}{2}} k \nu T \left(\frac{1}{N} - \frac{1}{4\pi - M}\right). \tag{48.24c}$$

The solution to Equation (48.22) will again be the normal attenuation function (25.18)

$$w(r) = y(s_g r), \tag{48.25}$$

$$s_g = \frac{ik}{2\varepsilon_g^0} = e^{-i\frac{\pi}{2}} \frac{\pi^2 k^3 \nu^2}{2} \cdot \begin{cases} 4a^6 \left(\frac{\varepsilon - 1}{\varepsilon + 2}\right)^2, \\ a^6, \\ 4T^2 \left(\frac{1}{N} - \frac{1}{4\pi - M}\right)^2 \end{cases} \tag{48.25a}$$

respectively, for the three cases under analysis (48.24a)-(48.24c). Indeed, it was shown in §25 that y(sr) is a solution of this equation for any s if the phase of the integration constant C is appropriately selected. That is to say, $\frac{\pi}{4} - \frac{1}{2}\arg s < \arg \sqrt{C} < \frac{3\pi}{4} - \frac{1}{2}\arg s$. In this case, arg $s = -\pi/2$ and hence $\frac{\pi}{2} < \arg \sqrt{C} < \pi$. Consequently, we may again consider $\sqrt{C} = i\infty$.

The behavior of the attenuation function for small $|s_g r|$ is again given by Expression (25.6) for $s = s_g$ (this is clear from the fact that Equation (48.22) can also be solved by the iteration method for small $|s_g r|$). At great distances, the expansion of (25.19a) will again be valid. For large enough ε, we shall have in the case of hemispheres, according to Formulas (48.25a),

$$s_g = e^{-i\frac{\pi}{2}} \frac{\pi^2 k^3 a^6 \nu^2}{2}. \tag{48.26}$$

Reducing everything to a certain effective conductivity σ_g, we may consider $\varepsilon_g^0 = 4\pi i \sigma_g/\omega$, with

$$\sigma_g = ic\frac{1}{4\pi^3 k a^6 \nu^2}. \tag{48.27}$$

The area occupied by a single projection is $S = \pi a^2$. The average height $h = a/2$. Hence the expression for σ_g can also be rewritten in the form

$$\sigma_g = \frac{ic}{16\pi k h^2 \Psi^2}, \quad \Psi = S\nu, \tag{48.27a}$$

where Ψ is the total area occupied by all projections situated on the unit area. It may be called the fill factor. By hypothesis, $\Psi \ll 1$. Thus, even projections with very good conductivity will give rise to scattering of radio waves and hence cause attenuation of the average field in the plane $z = 0$ equivalent to the appearance of nonideal soil conductivity.

For a built-up area, therefore, if we substitute scattering from an ideally conducting hemisphere for scattering from buildings and assume that the fill factor Ψ is small, for example $\Psi \simeq 0.1$, setting $h \simeq 10$ m $= 10^3$ cm, we obtain for propagation of broadcast-band waves, $\lambda \simeq 300$ m $= 3 \cdot 10^4$ cm,

$$|\sigma_g| \approx \frac{3 \cdot 10^{10} \cdot 3 \cdot 10^4}{32 \cdot \pi^2 \cdot 10^6 \cdot 10^{-2}} \approx 3 \cdot 10^8 \text{ CGSE}. \tag{48.28}$$

This value is close to normal soil conductivity, so that the secondary attenuation may be found substantial even for such Ψ, h and λ. For cities with larger ν, where the formulas are no longer strictly accurate, we may expect very strong effects. This is consistent with semi-quantitative experimental results. Thus, measurements over Amsterdam at 299 meters give $\sigma_{eff} \cong \sigma_g \cdot 10^5$ CGSE. For details see [11].

4. Up to this point, we have been considering the average field of a source in the plane $z = 0$. It was found that in the plane itself, it decays in proportion to the attenuation function $y(s_g r)$. Since the field in space is determined by the field on the plane $z = 0$, it follows from this that even for $z_A \neq 0$, at elevated observation points, the average field is described by the function y for an elevated observation point at the same effective soil constants. From this it follows on the basis of the reciprocity theorem that the same attenuation function with the same effective constants will also describe the average field of an elevated source. In particular, if the glancing angles ψ are large enough, $|\sqrt{\varepsilon_g^0}\sin\psi| > 1$, the interference (reflection) formulas are, as we know, valid. The reflection coefficient $f_{\parallel}$ for an average field polarized in the plane of incidence will be

$$f_{\parallel} = \frac{\sqrt{\varepsilon_g^0}\sin\psi - 1}{\sqrt{\varepsilon_g^0}\sin\psi + 1}. \tag{48.29}$$

Thus this average field arises from the corresponding segment of the $z = 0$ plane disturbed by irregularities in the direction of mirror reflection. In other directions, it is absent, provided that this segment is not too large.

However, the field attenuation described by the function $y(s_g r)$ is of quite a different nature than in the case of a flat surface with finite conductivity. It is governed not by drainage of energy into the soil and joulean heat absorption, but by scattering into space. In particular, even scattering by an isolated projection produces a field with an electric-field component directed along the y-axis (see, for example, Formula (48.12a)), and even the energy flux along the y-axis is different from zero (see Formulas (48.8) and (48.13)), which is impossible for a flat absorbing surface. The average field due to many random projections does not produce such a flux. However, this is not to say that there is no scattering in these directions. To find it, we must average the <u>energy flux</u>, i.e., the root-mean-square values. For all three of the cases under consideration (48.8), (48.13) and (48.18), the effective cross section of the surface area S can, after multiplication by νS, be obtained in the form ($d\sigma^{(v)}$ and $d\sigma^{(h)}$ are obtained in the same way for scattered radiation of a given polarization – vertical or horizontal):

$$d\sigma = 4k^4\nu S\cos^2\psi\, G(\chi, \varphi)\, d\Omega, \qquad (48.30)$$

$$G(\chi, \varphi) = \begin{cases} a^6\left|\dfrac{\varepsilon-1}{\varepsilon+2}\right|^2\cos^2\chi, & (48.30a) \\ a^6\left\{\cos^2\chi\left(1+\dfrac{1}{4}\cos^2\varphi\right)+\dfrac{1}{4}\sin^2\chi-\cos\chi\cos\varphi\right\}, & (48.30b) \\ \dfrac{T^2}{N^4}\cos^2\chi+\left(\dfrac{T}{4\pi-M}\right)^2(1-\cos^2\chi\sin^2\varphi)-\dfrac{2T^2}{N(4\pi-M)}\cos\chi\cos\varphi, & (48.30c) \end{cases}$$

respectively, for poorly conducting hemispheres ($k|\sqrt{\varepsilon}|a \ll 1$), for highly conductive hemispheres and for highly conductive hemiellipsoids.

The common factor $4\cos^2\psi$ is present because $d\sigma$ is calculated by division by the energy flux from an elevated source in empty space. The moment induced in the projection is proportional to the vertical component of the incident field, which is equal to $2\cos\psi|E^{(0)}|$, where $\vec{E}^{(0)}$ is the field of the isolated dipole. Wishing to obtain the absolute value of the energy scattered by the surface segment S in the direction (χ, φ), we must multiply $d\sigma$ by $\frac{c}{8\pi}|E^{(0)}|^2$, if the source is up high enough so that its field on area S may be regarded as still unattenuated due to scattering on projections. In the general case, however, it is necessary to multiply $d\sigma$ further by $|y(r; z_0; 0)|^2$, where the attenuation function of the incident field on area S is taken for a source at height z_0 and for an observation point on the plane $z = 0$.

By way of example, let us write the values of $G(\chi, \varphi)$ for two practically imporantt positions of the observation point - in the direction of regular reflection from a plane ($\chi = \psi$, $\varphi = 0$) and for back scattering, at the source ($\chi = \psi$, $\varphi = \pi$; the "radar case"), in either case for hemispherical projections - a) poorly conducting and b) highly conductive (see Formulas (48.30a) and (48.30b)):

$$\begin{array}{l|c|l} & \text{a} & \text{b} \\ \chi = \psi,\ \varphi = 0 & \multirow{2}{*}{$\Bigg\}\, a^6 \left|\frac{\varepsilon - 1}{\varepsilon + 2}\right|^2 \cos^2\psi$} & \frac{1}{4} + \cos^2\psi - \cos\psi \\ \chi = \psi,\ \varphi = \pi & & \frac{1}{4} + \cos^2\psi + \cos\psi \end{array}$$

§49. GENTLY SLOPING IRREGULARITIES

1. A highly general method can be developed for uneven surfaces that are in a certain sense slightly inclined from a certain plane, provided that these irregularities are gently sloped, i.e., the gradients are small everywhere.

We shall at first assume that the surface is ideally conducting, $\varepsilon = \infty$. Let it be described by a certain function (Fig. 49.1)

$$z = \zeta(x, y), \tag{49.1}$$

with introduction of a special symbol for the derivatives of this function

$$\gamma_x = \frac{\partial\zeta}{\partial x}, \quad \gamma_y = \frac{\partial\zeta}{\partial y} \tag{49.1a}$$

and let us consider them to be small:

$$\left|\frac{\partial\zeta}{\partial x}\right| \ll 1, \quad \left|\frac{\partial\zeta}{\partial y}\right| \ll 1. \tag{49.2}$$

Later we shall present a criterion for smallness of the inclination of ζ itself from zero, which is placed on a certain average plane. We state in advance that, as will become evident, the heights ζ may exceed the wavelengths λ of the radiation for a skipping incident field and small enough γ.

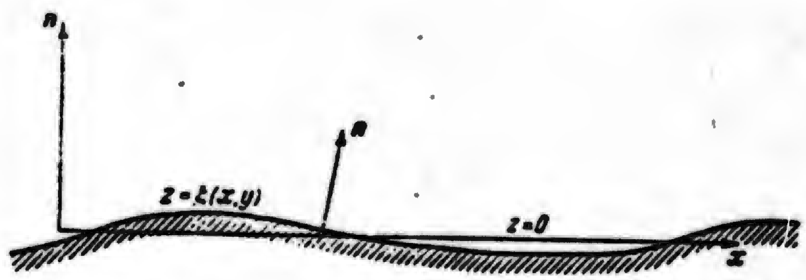

Fig. 49.1. Gently sloping irregularities.

The boundary condition for the electric field on this surface requires, according to Formula (21.26), that the tangential components of $\vec{E}$ vanish:

$$[En] = 0 \text{ for } z = \zeta, \qquad (49.3)$$

where $\vec{n}$ is the unit vector of the normal to the surface $z = \zeta$ at the point in question;

$$n_x = \frac{-\gamma_x}{\sqrt{1+\gamma_x^2+\gamma_y^2}}, \quad n_y = \frac{-\gamma_y}{\sqrt{1+\gamma_x^2+\gamma_y^2}}, \quad n_z = \frac{1}{\sqrt{1+\gamma_x^2+\gamma_y^2}}. \qquad (49.2a)$$

Disregarding quantities of the third order with respect to γ, we have, according to Formula (49.3),

$$E_y + \gamma_y E_z = 0, \quad E_x + \gamma_x E_z = 0 \text{ for } z = \zeta. \qquad (49.3a)$$

The third of the relationships given in (49.3) is a corollary of those written. All of them obtain on the surface $z = \zeta(x,y)$.

Let us pass the plane $z = 0$ and transfer these conditions to it, expanding $\vec{E}$ in series in the inclination ζ:

$$E(x, y, \zeta) = E(x, y, 0) + \left(\frac{\partial E}{\partial z}\right)_{z=0} \zeta + \dots. \qquad (49.4)$$

Keeping terms with ζ in powers no higher than the first and disregarding the products of ζ by γ, we obtain

$$\left.\begin{aligned}E_x(x, y, 0) &= -E_z(x, y, 0)\gamma_x - \frac{\partial E_x(x, y, 0)}{\partial z}\zeta,\\ E_y(x, y, 0) &= -E_z(x, y, 0)\gamma_y - \frac{\partial E_y(x, y, 0)}{\partial z}\zeta\end{aligned}\right\}\text{for } z=0. \quad (49.5)$$

Since E_x and E_y themselves and, as will become evident below, also $\partial E_s/\partial z$ are already quantities of the order of $\gamma_{x,y}$, we are actually taking terms of the <u>second</u> order in ζ into account here.

Thus, if the form of the surface (49.1) and the positions of the sources are given, the problem reduces to determination of the field in the space $z > 0$ when the boundary conditions (49.5) are valid on the plane $z = 0$.

Let us now generalize these conditions for the case of a soil with finite but <u>large</u> $|\varepsilon|$. This is particularly simple if the field is polarized in the plane of incidence, for example, for vertical polarization. If we assume that $|\varepsilon^0|$ is large, we may drop terms containing the products of ζ and γ by the small quantity $1/\sqrt{\varepsilon^0}$.

In the case of a truly flat surface with finite ε and vertical polarization, (21.26) are valid and the horizontal components E_x and E_y are small by comparison with the vertical component E_z. In the case of an ideally conductive but uneven (with small ζ, γ) surface, the components E_x and E_y on the plane $z = 0$ are again small according to Formula (49.5). Disregarding the crossover effect, we may assume in the general case of large but finite $|\varepsilon|$ and a gently irregular surface that small tangential components of the field due to two different causes will make themselves felt at the plane $z = 0$:

$$\left.\begin{aligned}E_x &= \left(\frac{k_x}{k\cos^2\psi}\,\frac{1}{\sqrt{\varepsilon^0}} - \gamma_x\right)E_z - \frac{\partial E_x}{\partial z}\zeta, \quad (49.6a)\\ E_y &= \left(\frac{k_y}{k\cos^2\psi}\,\frac{1}{\sqrt{\varepsilon^0}} - \gamma_y\right)E_z - \frac{\partial E_y}{\partial z}\zeta \quad (49.6b)\end{aligned}\right\}\text{for } z=0.$$

The last terms on the right are of the second order with respect to ζ (more precisely, of the order of $\gamma\cdot\zeta$) and are necessary only for certain special purposes (see §51).

Differentiating these expressions with respect to x and y and using the equation div $\vec{E} = 0$, we obtain a condition that generalizes Condition (40.12)

$$\frac{\partial E_z}{\partial z} = -\frac{ik}{\sqrt{\varepsilon^0}}\left\{1 + \frac{i}{k^2}(k \operatorname{grad} \ln \sqrt{\varepsilon^0})\right\} E_z + \frac{\partial}{\partial x}\left(\gamma_x E_z + \frac{\partial E_x}{\partial z}\zeta\right) +$$
$$+ \frac{\partial}{\partial y}\left(\gamma_y E_z + \frac{\partial E_y}{\partial z}\zeta\right) = -\frac{1}{k\cos^2\psi}(k \operatorname{grad})\frac{E_z}{\sqrt{\varepsilon^0}} + (\gamma \operatorname{grad} E_z) +$$
$$+ E_z\left(\frac{\partial \gamma_x}{\partial x} + \frac{\partial \gamma_y}{\partial y}\right) + \frac{\partial}{\partial x}\left(\zeta\frac{\partial E_x}{\partial z}\right) + \frac{\partial}{\partial y}\left(\zeta\frac{\partial E_y}{\partial z}\right) \text{ for } z = 0. \qquad (49.6c)$$

If we disregard terms of the second order, Formulas (49.6) can be interpreted as follows. For a radio wave skipping along the plane ($\psi = 0$) on the x-axis ($k_y = 0$), if ζ varies only in this direction, the appearance of irregularities is equivalent to substituting ε^0_{eff} for ε^0 or γ_{eff} for γ.

$$\frac{1}{\sqrt{\varepsilon^0}} \to \frac{1}{\sqrt{\varepsilon^0_{\text{эфф}}}} = \frac{1}{\sqrt{\varepsilon^0}} - \gamma = -\gamma_{\text{эфф}}, \qquad (49.7)$$

where, if

$$\frac{1}{\sqrt{\varepsilon^0}} = \frac{1}{\sqrt{|\varepsilon^0|}}\left(\cos\frac{\chi}{2} - i\sin\frac{\chi}{2}\right), \quad \frac{1}{\sqrt{\varepsilon^0_{\text{эфф}}}} = \frac{1}{\sqrt{|\varepsilon^0_{\text{эфф}}|}}\left(\cos\frac{\chi_{\text{эфф}}}{2} - i\sin\frac{\chi_{\text{эфф}}}{2}\right);$$

then

$$\frac{1}{\sqrt{|\varepsilon^0_{\text{эфф}}|}} = \sqrt{\left(\frac{\cos\frac{\chi}{2}}{\sqrt{|\varepsilon^0|}} - \gamma\right)^2 + \frac{\sin^2\frac{\chi}{2}}{\sqrt{|\varepsilon^0|}}}, \qquad (49.7a)$$

$$\operatorname{tg}\frac{\chi_{\text{эфф}}}{2} = \frac{\sin\frac{\chi}{2}}{\cos\frac{\chi}{2} - \sqrt{|\varepsilon^0|}\,\gamma}. \qquad (49.7b)$$

The entire method is applicable only for $\gamma \ll 1$. Even then, however, if the absolute value $|\varepsilon^0|$ is large enough, as in the case of infinite conductivity, when $\chi \approx \pi/2$, it may be found that $\tan\frac{\chi_{eff}}{2} < 0$, i.e., $\frac{1}{2}\chi_{eff} > \pi/2$.

Conversely, taking finite conductivity into account is equivalent to replacing the inclination angles γ_x and γ_y by

$$\gamma_{x\,\text{эфф}} = \gamma_x - \frac{h_x}{k\cos^2\psi}\frac{1}{\sqrt{\varepsilon^0}}, \quad \gamma_{y\,\text{эфф}} = \gamma_y - \frac{h_y}{k\cos^2\psi}\frac{1}{\sqrt{\varepsilon^0}}, \tag{49.7c}$$

(with ζ left as before in terms of the second order).

The boundary condition (49.6c), which generalizes Condition (40.12) for the case of a gently irregular surface, also enables us to rewrite the generalized integral equation (41.13) and (41.14) for this case:

$$E_z(x, y, 0) = E_z^0 y_0(D, z_0) - \frac{1}{2\pi}\int\left(\frac{\partial E_z}{\partial z'} + \frac{ikE_z}{\sqrt{\varepsilon_0^0}}\right)\frac{e^{ik\rho}}{\rho}\, y_0(\rho, 0)\,dx'dy', \tag{49.8}$$

where $\partial E_z/\partial z'$ should be substituted from Formula (49.6c). Here ε_0^0 is a constant auxiliary parameter (ε^0 for a certain fictive soil); it is assumed that the radiator is raised to a height z_0; D is the distance from the radiator along the horizontal to an observation point situated in the plane $z = 0$; y_0 is the normal attenuation function for the given positioning of the sources above a soil with $\varepsilon^0 = \varepsilon_0^0 = \text{const}$. As always, integration extends over the plane $z = 0$.

Similarly, for the other components of $\vec{E}$ we select the Green's function in the form

$$v_- = \left(\frac{e^{ik\rho}}{\rho} - \frac{e^{ik\rho_1}}{\rho_1}\right) y_0(\rho_0;\, z - z'), \tag{49.8a}$$

$$\rho = \sqrt{\rho_0^2 + (z - z')^2}, \quad \rho_1 = \sqrt{\rho_0^2 + (z + z')^2}, \quad \rho_0 = \sqrt{(x - x')^2 + (y - y')^2}$$

and, using Green's theorem (5.8) (it must be remembered that as $z' \to 0$, we have $v_- = 0$ and $\frac{\partial v}{\partial n} = -\frac{\partial v}{\partial z'} = +2\frac{\partial}{\partial z}\left(\frac{e^{ik\rho}}{\rho}\right) y_0(\rho_0;\, z)$), we obtain, for example,

$$E_x(x, y, z; z_0) = E_{0x}^{A}(x, y, z; z_0) - \frac{1}{2\pi}\int E_x(x', y', 0; z_0)\frac{\partial}{\partial z}\left(\frac{e^{ik\rho}}{\rho}\right)\cdot y_0(\rho_0;\, z)\,dS' =$$

$$= E_{0x}^{A}(x, y, z; z_0) - \frac{1}{2\pi}\left(\frac{\partial}{\partial z} + \frac{ik}{\sqrt{\varepsilon_0^0}}\right)\int E_x(x', y', 0; z_0)\,v\,dS'. \tag{49.8b}$$

Here it is taken into account that $\frac{\partial u_0}{\partial z} = -\frac{ik}{\sqrt{\varepsilon_0^0}} y_0$; E^A_{0x} denotes the corresponding volume integral over the sources. In the case $\varepsilon_0^0 = 0$ – this is simply the field of the same sources for ideal conductivity and a flat earth (for z = 0 we have $E^A_{0x} = E^A_{0y} = 0$) – and v should be understood as

$$v = \frac{e^{ik\rho}}{\rho} y_0(\rho;\ z). \tag{49.8c}$$

Since $\partial E_z/\partial z'$ in Formula (49.8) may be substituted by $-\left(\frac{\partial E_x}{\partial x'} + \frac{\partial E_y}{\partial y'}\right)$ and integration by parts performed with respect to x' and y', respectively, with $\frac{\partial v}{\partial x', y'} = -\frac{\partial v}{\partial x, y}$. Formulas (48.8)-(49.8c) can also be written as follows:

$$E_x = E^A_{0x} + \left(\frac{\partial}{\partial z} + \frac{ik}{\sqrt{\varepsilon_0^0}}\right) A_x,\ E_y = E^A_{0y} + \left(\frac{\partial}{\partial z} + \frac{ik}{\sqrt{\varepsilon_0^0}}\right) A_y, \tag{49.9a}$$

$$E_z = E^A_{0z} - \frac{ik}{2\pi\sqrt{\varepsilon_0^0}} \int E_z v dS' - \frac{\partial A_x}{\partial x} - \frac{\partial A_y}{\partial y}, \tag{49.9b}$$

$$A_x = -\frac{1}{2\pi} \int E_x v dS' = -\frac{1}{2\pi} \int \left(\eta_x E_z - \frac{\partial E_x}{\partial z};\right) v dS',$$

$$A_y = -\frac{1}{2\pi} \int E_y v dS' = -\frac{1}{2\pi} \int \left(\eta_y E_z - \frac{\partial E_y}{\partial z};\right) v dS', \tag{49.9c}$$

where, in the case of vertical polarization of the field $\vec{E}^A$,

$$\eta_{x,y} = \frac{k_{x,y}}{k \cos^2 \psi} \frac{1}{\sqrt{\varepsilon^0}} - \gamma_{x,y} \tag{49.9d}$$

(in the case of horizontal polarization we consider only an irregular ideally conductive surface, $\varepsilon^0 = \infty$). The quantity E^A_{0z} near the plane z = 0 or for a lowered source ($z_0 = 0$) is simply

$$E^A_{0z} = E^0 y_0(\rho_0;\ z + z_0). \tag{49.9e}$$

This system of equations may serve as a basis for solution of a number of problems (see, for example, §§50, 51).

2. Let us consider the influence of irregularities and departures of ε^0 from infinity as a small disturbance. The position of the source

and observation point will be arbitrary. We shall use boundary conditions (49.6). We shall first drop terms with ε^0 and write all formulas for an ideally conductive surface. The transition to finite conductivity is accomplished simply by substituting according to Formula (49.7c).

We set

$$E = E^{(0)} + E^{(1)} + E^{(2)} + \dots, \tag{49.10}$$

where $E^{(0)}$ is the field that would obtain for $\zeta \equiv 0$. It is known from the rigorous reflection theorem (§20). Consequently, $E^{(1)}$ is defined by Formulas (49.9a)-(49.9d) (setting $\varepsilon_0^0 = \infty$, $y_0 = 1$) in these formulas), if we use boundary conditions (49.6a)-(49.6b), which, according to Formula (49.10), assume the form

$$E_{x,y}^{(1)} = -\frac{\partial E_{x,y}^{(0)}}{\partial z}\zeta - \gamma_{x,y}E_z^{(0)} \text{ for } z = 0. \tag{49.10a}$$

Let us consider three particular cases.

a) Suppose that a vertical electric dipole with moment p is situated at point O (Fig. 49.2). Its hertzian vector in free space is reduced to a single component directed along the moment and equal to $\Pi = p\frac{e^{ikR}}{R}$. Thus, its moment is twice that which we selected in Chapter 5. Its reflection at O_1 must blso be directed vertically. Let us introduce the spherical coordinate systems (R, ϑ, φ) and $(R_1, \vartheta_1, \varphi_1 = \varphi)$ for each dipole. According to Formula (20.4a), we obtain in the plane $z = 0$ ($R = R_1$, $\vartheta = \pi - \vartheta_1$), dropping the time multiplier,

$$E_x^{(0)} = E_y^{(0)} \doteq 0,\ E_z^{(0)} = 2k^2 p \sin^2\vartheta\frac{e^{ikR}}{R}. \tag{49.11}$$

Further, since for $kR \gg 1$,

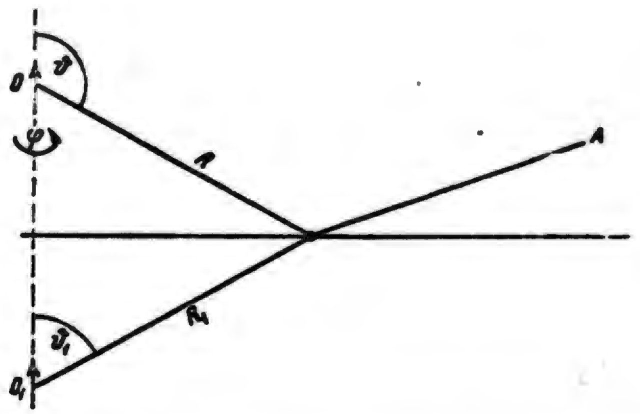

Fig. 49.2. Nomenclature in derivation of expression for scattered field.

$$\left.\begin{aligned}
&\frac{\partial}{\partial z}=\frac{\sin\vartheta}{R}\frac{\partial}{\partial\vartheta}+\cos\vartheta\frac{\partial}{\partial R}\approx\cos\vartheta\frac{\partial}{\partial R},\\
&\frac{\partial}{\partial z_1}=\frac{\sin\vartheta_1}{R}\frac{\partial}{\partial\vartheta_1}+\cos\vartheta_1\frac{\partial}{\partial R_1}\approx\cos\vartheta_1\frac{\partial}{\partial R_1},\\
&\frac{\partial}{\partial x}\approx\sin\vartheta\cos\varphi\frac{\partial}{\partial R},\quad \frac{\partial}{\partial y}\approx\sin\vartheta\sin\varphi\frac{\partial}{\partial R},\\
&\left(\frac{\partial}{\partial z}\right)_{z_1=0}=-\left(\frac{\partial}{\partial z_1}\right)_{z_1=0},
\end{aligned}\right\} \tag{49.12}$$

we may assume that

$$\left(\frac{\partial E_x^{(0)}}{\partial z}\right)_{z=0}=-2ik^3p\sin\vartheta\cos^2\vartheta\frac{e^{ikR}}{R}\cos\varphi, \tag{49.13a}$$

$$\left(\frac{\partial E_y^{(0)}}{\partial z}\right)_{z=0}=-2ik^3p\sin\vartheta\cos^2\vartheta\frac{e^{ikR}}{R}\sin\varphi. \tag{49.13b}$$

For z = 0, therefore, according to Formula (49.10a),

$$E_x^{(1)}=2k^2p\frac{e^{ikR}}{R}\{ik\zeta\cos^2\vartheta\cos\varphi-\gamma_x\sin\vartheta\}\sin\vartheta, \tag{49.14a}$$

$$E_y^{(1)}=2k^2p\frac{e^{ikR}}{R}\{ik\zeta\cos^2\vartheta\sin\varphi-\gamma_y\sin\vartheta\}\sin\vartheta; \tag{49.14b}$$

here ϑ is the angle formed by the ray with the dipole axis.

b) Let a horizontal dipole directed along the y-axis be situated at point O. Its image at O_1 will be oriented in the opposite direction. At the surface z = 0, according to Formula (20.4c), we shall have

$$E_x^{(0)} = -2k^2p\,\frac{e^{ikR}}{R}\sin\varphi\sin\vartheta\cos\vartheta. \qquad (49.15)$$

If, as before, we assume

$$\begin{aligned} \frac{\partial f(R)}{\partial z} &\approx \cos\vartheta\,\frac{\partial f}{\partial R}, \\ \frac{\partial f(R_1)}{\partial z} &\approx \cos\vartheta_1\,\frac{\partial f}{\partial R_1}, \end{aligned} \qquad (49.16)$$

where ϑ is the angle formed by the ray drawn from the dipole to the observation point with the z-axis, $\vartheta_1 = \pi - \vartheta$, then, according to Formula (20.4c),

$$\left.\begin{aligned} \left(\frac{\partial E_x^{(0)}}{\partial z}\right)_{z=0} &= \\ \left(\frac{\partial E_y^{(0)}}{\partial z}\right)_{z=0} &= \end{aligned}\right\} -2ik^3p\,\frac{e^{ikR}}{R}\cos\vartheta \begin{cases} \sin^2\vartheta\cos\varphi\sin\varphi; \\ \sin^2\vartheta\sin^2\varphi - 1. \end{cases} \qquad (49.17)$$

Therefore

$$\left.\begin{aligned} E_x^{(1)} &= \\ E_y^{(1)} &= \end{aligned}\right\} 2k^2p\,\frac{e^{ikR}}{R}\cos\vartheta \begin{cases} (ik_0^2\sin\vartheta\cos\varphi + \gamma_x)\sin\vartheta\sin\varphi, \\ ik_0^2(\sin^2\vartheta\sin^2\varphi - 1) + \gamma_y\sin\vartheta\sin\varphi, \text{ при } z = 0. \end{cases} \qquad (49.18)$$

c) For a <u>horizontal dipole oriented along the x-axis</u>,

$$\left.\begin{aligned} E_x^{(1)} &= \\ E_y^{(1)} &= \end{aligned}\right\} 2k^2p\,\frac{e^{ikR}}{R}\cos\vartheta \begin{cases} ik_0^2(\sin^2\vartheta\cos^2\varphi - 1) + \sin\vartheta\cos\varphi\cdot\gamma_x; \\ (ik_0^2\sin\vartheta\sin\varphi + \gamma_y)\sin\vartheta\cos\varphi. \end{cases} \qquad (49.19)$$

Thus, we know the values of $E_x^{(1)}$ and $E_y^{(1)}$ on the plane $z = 0$ [(49.14) for the vertical dipole and (49.18) and (49.19) for the horizontal dipole], so that we can without difficulty determine the fields $E_x^{(1)}$ and $E_y^{(1)}$ at any point in the space $z > 0$ (needless to say, physical significance must be attributed to the resulting solution only at points $z > \zeta$).

On the other hand, in order to determine $\vec{E}_z^{(1)}$, it is necessary to use the formula div $E^{(1)} = 0$. Since $\partial E_x^{(1)}/\partial x$ and $\partial E_y^{(1)}/\partial y$ can be found from Formulas (49.14) and, respectively, (49.18)-(49.19), we also know $\partial E_z^{(1)}/\partial z$ (the field $\vec{E}^{(1)}$ has no sources other than virtual sources dis-

tributed over the plane z = 0). Now we can use Formulas (49.9a)-(49.9c) or Formula (8.5), which gives the same result at once:

$$E_x^{(1)}(x, y, z) = \frac{\partial A_x}{\partial z}, \tag{49.20a}$$

$$E_y^{(1)}(x, y, z) = \frac{\partial A_y}{\partial z}, \tag{49.20b}$$

where

$$A_x = -\frac{1}{2\pi}\int E_x^{(1)}(x', y', 0)\,\frac{e^{ik\rho}}{\rho}\,dx'dy',\quad A_y = -\frac{1}{2\pi}\int E_y^{(1)}(x', y', 0)\,\frac{e^{ik\rho}}{\rho}\,dx'dy'. \tag{49.21}$$

Further, following Relationship (8.7) or according to Formula (49.9b), we find

$$E_z^{(1)} = -\frac{\partial A_x}{\partial x} - \frac{\partial A_y}{\partial y}. \tag{49.22}$$

Using Formulas (49.14), (49.18) and (49.19), we obtain (the upper expression in each formula is for the x-component and the lower expression is for the y-component):

a) for a vertical electric dipole

$$A_{x,y} = -\frac{k^2 p}{\pi}\int\left\{ik\zeta \begin{matrix}\cos\varphi\\ \sin\varphi\end{matrix} \cos^2\vartheta - \sin\vartheta\cdot\gamma_{x,y}\right\}\sin\vartheta\,\frac{e^{ik(R+\rho)}}{R\rho}\,dx'dy'; \tag{49.23}$$

b) for a horizontal electric dipole directed along the x-axis,

$$A_{x,y} = -\frac{k^2 p}{\pi}\int\left\{\begin{matrix} ik\zeta(\sin^2\vartheta\cos^2\varphi - 1) + \sin\vartheta\cos\varphi\gamma_x \\ (ik\zeta\sin\vartheta\sin\varphi + \gamma_y)\sin\vartheta\cos\varphi\end{matrix}\right\}\cos\vartheta\,\frac{e^{ik(R+\rho)}}{R\rho}\,dx'dy'; \tag{49.24}$$

c) for a horizontal electric dipole directed along the y-axis,

$$A_{x,y} = -\frac{k^2 p}{\pi}\int\left\{\begin{matrix} (ik\zeta\sin\vartheta\cos\varphi - \gamma_x)\sin\vartheta\sin\varphi \\ ik\zeta(\sin^2\vartheta\sin^2\varphi - 1) + \gamma_y\sin\vartheta\sin\varphi\end{matrix}\right\}\cos\vartheta\,\frac{e^{ik(R+\rho)}}{R\rho}\,dx'dy'. \tag{49.25}$$

Together with Formulas (49.23)-(49.25), Formulas (49.20) and (49.22) give the field at any point if the surface shape ζ is known. In many cases, it is convenient to introduce differentiation under the integral sign. Thus, for example, for E_z (here we take into account that only ρ depends on x and y under the integral sign, with $\frac{\partial\rho}{\partial x, y} = -\frac{\partial\rho}{\partial x', y'}$, and again perform integration by parts with respect to x' and y'), we have:

a) for a vertical dipole

$$E_z^{(1)}(x, y, z) = \frac{k^2 p}{\pi} \int \left\{ ik(\cos^2\vartheta - \sin^2\vartheta)(\gamma_x \cos\varphi + \gamma_y \sin\varphi) - \left(\frac{\partial}{\partial x'}\gamma_x + \frac{\partial}{\partial y'}\gamma_y\right) \sin\vartheta - k^2\zeta \sin\vartheta \cos^2\vartheta \right\} \sin\vartheta \frac{e^{ik(R+\rho)}}{R\rho} dx'dy'. \quad (49.26)$$

b) for a horizontal dipole directed along the y-axis,

$$E_z^{(1)}(x, y, z) = \frac{k^2 p}{\pi} \int \cos\vartheta \left\{ \sin\vartheta \sin\varphi \left[k^2\zeta \cos^2\vartheta + 2ik \sin\vartheta (\gamma_x \cos\varphi + \gamma_y \sin\varphi) + \frac{\partial}{\partial x'}\gamma_x + \frac{\partial}{\partial y'}\gamma_y \right] - ik\gamma_y \right\} \frac{e^{ik(R+\rho)}}{R\rho} dx'dy'. \quad (49.27)$$

The formulas simplify substantially in the two limiting cases. Firstly, if the source is situated in the plane z = 0, when $\vartheta = \pi/2$. Secondly, if the source is very far from the zone of the plane z = 0 that is essential for reflection, so that the incident wave may be regarded as plane. In the former case (dipole in plane z = 0, $\vartheta = \pi/2$, R = r), we have: for a vertical dipole, according to Formula (49.23),

$$A_{x,y} = \frac{k^2 p}{\pi} \int \gamma_{x,y} \frac{e^{ik(r+\rho)}}{r\rho} dx'dy'; \quad (49.28)$$

for either of the horizontal dipoles, according to Formulas (49.24) and (49.25),

$$A_{x,y} = 0. \quad (49.29)$$

In this approximation, therefore (i.e., disregarding terms of the order of 1/kr and higher-order terms in ζ), the field of the low-lying horizontal dipole is not at all disturbed by surface irregularities. For the vertical dipole, on the other hand, Formula (49.26) gives

$$E_z^{(1)}(x, y, z) = -\frac{k^2 p}{\pi} \int \left\{ ik(\gamma_x \cos\varphi + \gamma_y \sin\varphi) + \frac{\partial}{\partial x'}\gamma_x + \frac{\partial}{\partial y'}\gamma_y \right\} \frac{e^{ik(r+\rho)}}{r\rho} dx'dy'. \quad (49.30)$$

In particular, this formula illustrates the remark made above

(§47) to the effect that for the field to be disturbed, it is not the height ζ in itself that is essential, but the angles of inclination γ of the surface (in Formulas (49.26), (49.27) there is also a term containing ζ; it transfers the influence of the displacement of the reflected source as the horizontal interface is raised). Further,

$$E^{(1)}_{x,y}(x,\ y,\ z) = \frac{k^2 p}{\pi}\frac{\partial}{\partial z}\int \gamma_{x,y}\frac{e^{ik(r+\rho)}}{r\rho}\,dx'dy'. \qquad (49.30a)$$

Let us pass to another particular case – <u>incidence of a plane wave</u>. Taking the plane y = 0 as the plane of incidence and accordingly setting φ = 0 in the integrals,

$$p\frac{e^{ikR}}{R}\approx q e^{ikx'\sin\vartheta}, \quad q = p\frac{e^{ikR_0}}{R_0}, \qquad (49.31)$$

where q and R_0 are constants, we may write:

a) for a vertical dipole

$$A_{x,y} = \frac{-k^2 q}{\pi}\int\left\{ik\zeta\begin{Bmatrix}1\\0\end{Bmatrix}\cos^2\vartheta - \sin\vartheta\,\gamma_{x,y}\right\}\sin\vartheta\,\frac{e^{ik(x'\sin\vartheta+\rho)}}{\rho}\,dx'dy', \qquad (49.32)$$

from which

$$E^{(1)}_x = \frac{k^2 q}{\pi}\int(-ik\zeta\cos^2\vartheta + \sin\vartheta\,\gamma_x)\sin\vartheta\,\frac{\partial}{\partial z}\frac{e^{ik(x'\sin\vartheta+\rho)}}{\rho}\,dx'dy',$$

$$E^{(1)}_y = \frac{k^2 q}{\pi}\int\left\{\sin^2\vartheta\,\gamma_y\frac{\partial}{\partial z}\frac{e^{ik(x'\sin\vartheta+\rho)}}{\rho}\,dx'dy',\right.$$

$$E^{(1)}_z = \frac{k^2 q}{\pi}\int\left\{ik(\cos^2\vartheta - \sin^2\vartheta)\,\gamma_x - \right.$$

$$\left. - \left(\frac{\partial}{\partial x'}\gamma_x + \frac{\partial}{\partial y'}\gamma_y\right)\sin\vartheta - k^2\zeta\cos^2\vartheta\sin\vartheta\right\}\frac{e^{ik(x'\sin\vartheta+\rho)}}{\rho}\,dx'dy'; \qquad (49.33)$$

b) for a horizontal dipole oriented along the x-axis,

$$A_{x,y} = -\frac{k^2 q}{\pi}\int\left.\begin{matrix}-ik\zeta\cos^2\vartheta + \sin\vartheta\,\gamma_x\\ \sin\vartheta\,\gamma_y\end{matrix}\right\}\cos\vartheta\,\frac{e^{ik(x'\sin\vartheta+\rho)}}{\rho}\,dx'dy', \qquad (49.34)$$

c) for a horizontal dipole oriented along the y-axis,

$$A_{x,y} = -\frac{k^2 q}{\pi}\int\begin{Bmatrix}0\\ -ik\zeta\end{Bmatrix}\cos\vartheta\,\frac{e^{ik(x'\sin\vartheta+\rho)}}{\rho}\,dx'dy'. \qquad (49.35)$$

$$E_x^{(1)} = 0,$$
$$E_y^{(1)} \neq 0,$$
$$E_z^{(1)} \neq 0, \quad E_z^{(1)} = -\frac{k^2 q}{\pi}\int ik\gamma_y \cos\theta \, \frac{e^{ik(x'\sin\theta+\rho)}}{\rho} \, dx'dy'. \qquad (49.36)$$

These formulas again indicate that the field of a horizontal dipole is subject to considerably less disturbance on the part of soil irregularities than the field of a vertical radiator. This fact is one of the reasons for the preference given horizontal radiators in radar, where the wavelengths are short and the disturbances may, generally speaking, be significant as a result.

It must be noted that for observation points in the plane z = 0, the formula for $E_x^{(1)}$ and $E_y^{(1)}$, which contains the differentiation of the expression $\frac{1}{\rho}e^{ik\rho}$ with respect to z, would appear to give a divergence in the integral with respect to (x', y'). However, according to Formula (41.12b),

$$\lim_{z\to 0}\int f(x', y')\frac{\partial}{\partial z}\frac{e^{ik\rho}}{\rho}\,dx'dy' = -2\pi f(x, y),$$

so that we may write for observation in the plane z = 0 if a plane wave is incident:

a) for a vertical dipole, from Formulas (49.33),

$$E_x^{(1)} = -2k^2 q(-ik''_0\cos^2\theta - \sin\theta\gamma_x)e^{ikx\sin\theta},$$
$$E_y^{(1)} = -2k^2 q\sin^2\theta\gamma_y e^{ikx\sin\theta},$$
$E_z^{(1)}$ retains the same form; (49.33a)

b) for a horizontal dipole oriented along the x-axis, from Formula (49.34),

$$E_x^{(1)} = 2k^2 q(-ik''_0\cos^2\theta + \sin\theta\gamma_x)\cos\theta e^{ikx\sin\theta},$$
$$E_y^{(1)} = 2k^2 q\sin\theta\cos\theta\gamma_y e^{ikx\sin\theta}, \qquad (49.34a)$$

c) for a horizontal dipole oriented along the y-axis, from Formula (49.35) ($E_z^{(1)}$ remains as before):

$$E_x^{(1)} = 0,$$

$$E_y^{(1)} = -2ik^2 q\zeta \cos\theta\, e^{ikx\sin\theta}. \qquad (49.35a)$$

Those of the formulas from (49.9) to (49.35) that pertain to the field $\vec{E}^{(0)}$ polarized in the plane incidence also cover, as we have already noted, the case of an irregular surface with finite ε if by γ_x and γ_y we understand $\gamma_{x_{eff}}$ and $\gamma_{y_{eff}}$ (49.7c). In the case of a flat surface with finite ε (even if it varies within the framework of Condition (40.8)), these formulas describe a weak (by comparison with the case $\varepsilon = \infty$) disturbance of the field if we set

$$\gamma_x = -\frac{k_x}{k\cos^2\psi}\frac{1}{\sqrt{\bar{\varepsilon}}},\quad \gamma_y = -\frac{k_y}{k\cos^2\psi}\frac{1}{\sqrt{\bar{\varepsilon}}},\quad \zeta = 0. \qquad (49.37)$$

3. The limits of applicability of the gentle-irregularity method proceed primarily from the expansion (49.4)-(49.5), in which only explicitly written terms are kept. Moreover, the method of perturbations has subsequently been applied: in calculating $\vec{E}^{(1)}$, the undisturbed field $\vec{E}^{(0)}$ was substituted in the integrals. Thus, we have assumed – in terms of absolute values –

$$\left(\frac{\partial^2 E_{x,y}}{\partial z^2}\right)_{z=0}\zeta \ll \left(\frac{\partial E_{x,y}}{\partial z}\right)_{z=0};\quad \zeta\left(\frac{\partial E_z}{\partial z}\right)_{z=0} \ll (E_z)_{z=0}. \qquad (49.38)$$

Within the framework of the perturbation method, the values of these derivatives are determined by the behavior of the unperturbed field. For it, however, differentiation with respect to z for reflection of a wave incident at a glancing angle ψ is equivalent to multiplication by $ik_z = ik\sin\psi$. Consequently, one of the conditions for validity of the method set forth is

$$k\zeta\sin\psi \ll 1. \qquad (49.38a)$$

Thus, for a glancing ray, $\psi \ll 1$, we may consider even heights ζ that exceed the radiation wavelength. However, it must be remembered that at very small ψ, the effective scattering region of the surface may become very large. The right members in Formula (49.26) and in other similar formulas increase accordingly, so that $\bar{E}^{(1)}$ may prove not small by comparison with $\bar{E}^{(0)}$, and the expansion of (49.10) and the method of perturbations may be inapplicable. Accordingly, the criterion for Relationships (49.38) may take a form other than Condition (49.38a). This case is examined in §51.

4. Let us consider the field distortion caused by local deviations of the soil properties from the average or of the surface ζ from the zero plane in the case in which the dimensions of such an isolated inhomogeneity spot b are small by comparison with the wavelength in air.

It might appear that its influence should be correspondingly small and, moreover, should vanish as wavelength increases. We shall see that this is not always the case. For this purpose, let us examine Eqs. (49.6c)-(49.8) with $\varepsilon_0^0 = \infty$. We drop the term of the second negative order $\zeta(\partial E_{x,y}/\partial z)$. Further, instead of considering both the irregularity and the inhomogeneity of electrical properties, let us set $\gamma = 0$, knowing that it will be possible as a result to substitute $\varepsilon^0_{\text{eff}}$ for ε^0 according to Formula (49.7).

Suppose that the inhomogeneity region has dimensions of the order of b and that within the limits of this region, the maximum deviation of the quantity $1/\sqrt{\varepsilon^0}$ from its value outside the spot $1/\sqrt{\varepsilon^0_\infty}$ is η, so that

$$\frac{1}{\sqrt{\varepsilon^0}} = \frac{1}{\sqrt{\varepsilon^0_\infty}} + \eta f\left(\frac{x-x_0}{b}, \frac{y-y_0}{b}\right), \qquad (49.39)$$

where f is a dimensionless function of the order of unity with derivatives also of the order of unity; it decreases to zero when the distance $x - x_0$, $y - y_0$ from the conditional spot center (x_0, y_0) is considerably larger than b.

Since we have assumed that kb << 1, and since $\frac{\partial}{\partial x}\frac{1}{\sqrt{\varepsilon^0}} \sim \frac{1}{b\sqrt{\varepsilon^0}}$, the principal term in the general expression (49.6c) is the term with $\operatorname{grad}\ln\sqrt{\varepsilon^0}$. Here the condition

$$\frac{1}{|\sqrt{\varepsilon^0}|} \ll kb \ll 1 \qquad (49.40)$$

may still be observed for sufficiently long wavelengths.

Let the incident field take the form of a plane wave propagating along the x-axis,

$$E_z^0 = e^{ikx}.$$

We set $E_z = E_z^0$ under the integral sign. Retaining principal terms, we obtain

$$E_z = e^{ikx}\left\{1 + \frac{1}{2\pi b\sqrt{\varepsilon^0}}\iint f'\left(\frac{x'-x_0}{b}, \frac{y'-y_0}{b}\right)\frac{e^{ik(\rho+x'-x)}}{\rho}\,dx'dy'\right\}, \qquad (49.41)$$

where $\rho = \sqrt{(x-x')^2 + (y-y')^2 + z^2}$ and the prime on the f denote differentiation with respect to the first argument.

Two limiting cases are possible.

a) The observation point is far outside the spot $\rho \gg b$. Placing the coordinate origin somewhere inside the spot, we can disregard the variation of the denominator ρ and, replacing it by the distance from the observation point to the coordinate origin ρ_0, take it out from under the integral sign. Further, in view of the smallness of $k(\rho-\rho_0) \sim kb \ll 1$, we may expand the exponential expression in series:

$$E_z = e^{ikx}\left\{1 + \frac{e^{ik(\rho_0-x)}}{2\pi b\rho_0\sqrt{\varepsilon^0}}\iint f'\left(\frac{x'-x_0}{b}, \frac{y'-y_0}{b}\right)\times\right.$$
$$\left.\times[1 + ik(\rho-\rho_0+x') + \ldots]\,dx'dy'\right\}.$$

After integration, the first term in the expansion will give zero (since $f(\pm\infty) = 0$), while we obtain the following evaluation from the second term, considering that in order of magnitude $\rho-\rho_0+x' \sim b$:

$$E_z \approx e^{ikx}\left\{1 + \frac{e^{ik(\rho_0-x)}}{\rho_0}\frac{kb^2}{\sqrt{\varepsilon^0}}c\right\}, \qquad (49.41a)$$

where we have set

$$i\iint f\left(\frac{x'-x_0}{b},\ \frac{y'-y_0}{b}\right)\frac{\rho-\rho_0+x'}{b}\frac{dx'dy'}{b^2}=2\pi c, \quad |2\pi c|\sim 1.$$

The perturbation is proportional to the product of three small quantities b/ρ_0, kb and $1/\sqrt{\varepsilon^0}$, so that it is vanishingly small.

b) We consider the other limiting case: the observation point is within the boundaries of the inhomogeneity spot, near its center, $\rho_0 \ll b$. In this case, the exponential factor in the integrand may be assumed equal to unity. Introducing the polar coordinates $dx'dy' = \rho_0 d\rho_0 d\varphi_0$, we obtain

$$E_z \approx e^{ikz}\left(1+\frac{1}{2\pi kb\sqrt{\varepsilon^0}}\iint f\left(\frac{x'-x_0}{b},\ \frac{y'-y_0}{b}\right)d\rho_0 d\varphi_0\right). \quad (49.41b)$$

and the correction term is found to be of the order of

$$\frac{1}{b\sqrt{\varepsilon^0}}b=\frac{1}{\sqrt{\varepsilon^0}}.$$

Consequently,

$$E_z = e^{ikz}\left[1+0\left(\frac{1}{\sqrt{\varepsilon^0}}\right)\right]. \quad (49.42)$$

Thus, as long as the observation point is within the boundaries of the spot, the field perturbation will be determined solely by the maximum deviation of the soil electric properties from the "unperturbed" values. It is totally independent of the ratio of wavelength to spot dimensions and diminishes with increasing wavelength only because ε^0 increases. This results in an even more significant distortion of the propagation direction and the associated radio direction finding error (see §50).

§50. PERTURBATION OF FIELD NEAR THE BOUNDARY OF AN INHOMOGENEITY

1. Special interest attaches to the phase disturbance that arises when a gradient or a boundary between two soils occurs on the path of the radio waves – for example, when the radio waves cross from the space above the ocean to the space above dry land. The formulas of the

preceding sections, and those of §§43 and 49 in particular, answer the questions that arise here. In practice, two problems are of particular importance: the distortion of the propagation direction (in marine radio navigation practice, for example, in radio direction finding between a shore station and a ship, this problem has been given a special name – "shoreline refraction") and variation of the propagation phase velocity (radiogeodesy, phase methods of radio direction finding).

As long as we are in the vicinity of such a boundary of an electrically or geometrically uniform region, we can use the theory of perturbations (§49), which was applied to an incident plane wave. Generally speaking, this theory is stated in Formulas (49.9a)-(49.9d) et seq., and specifically for the plane wave in Formulas (49.33) (vertical dipole) and (49.34)-(49.36) (horizontal dipole), with the derived formulas to take account of electrical inhomogeneity in the case of vertical polarization,

$$\zeta = 0,\ \gamma_{x,y} \to -\frac{k_{x,y}}{k\cos^2\psi}\frac{1}{\sqrt{\varepsilon'}}. \tag{50.1}$$

In view of the smallness of the distortion, we may assume

$$E_i = E_i^{(0)} + E_i^{(1)} = E_i^{(0)}(1+f_i) \approx E_i^{(0)} e^{f_i}, \tag{50.2}$$

where

$$f_i = \frac{E_i^{(1)}}{E_i^{(0)}} = f_{i\text{п}} + f_{i\text{р}}, \tag{50.3}$$

where (we shall drop the subscript i) f_P is the correction for soil inhomogeneity and f_R is that for the terrain.

The phase distortion $\varphi_1 = -\mathrm{Im}\, f$ is determined by the imaginary part of f.

Accordingly, the distortion α of the propagation direction,

$$\alpha = \alpha_{\text{п}} + \alpha_{\text{р}}, \tag{50.4}$$

is determined as the angle through which the normal to the wave front is rotated with respect to the normal for the undisturbed wave. α may be obtained from f as follows (we shall be referring to a glancing wave, $k = k_x$).

Draw the coordinate system (x, y) and then (ξ, η) in the plane $z = 0$, with the ξ-axis normal to the boundary between soils (coastline) or to the base of a gradient (Fig. 50.1), while the x-axis is placed along the propagation direction of the undisturbed wave with its origin at the position of the source. The equation of the equiphase line is

$$\left(\frac{dy}{dx}\right)_{\varphi=\mathrm{const}} = -\operatorname{ctg}\alpha \approx -\frac{1}{\alpha}. \qquad (50.5)$$

But $\varphi = \varphi_0 + \varphi_1$, where the unperturbed phase $\varphi_0 = kD = kx$ (D is the distance from the source). Therefore

$$-\frac{\frac{\partial\varphi}{\partial x}}{\frac{\partial\varphi}{\partial y}} = -\frac{\frac{\partial\varphi_0}{\partial x}+\frac{\partial\varphi_1}{\partial x}}{\frac{\partial\varphi_0}{\partial y}+\frac{\partial\varphi_1}{\partial y}}.$$

Neglecting $\partial\varphi_1/\partial x$ as small by comparison with $\partial\varphi_0/\partial x = k$ and considering that $\partial\varphi_0/\partial y = 0$, we have

$$\alpha = \frac{1}{k}\frac{\partial\varphi_1}{\partial y}. \qquad (50.6)$$

Below, φ_1 will frequently be expressed in terms of x_A and D, where x_A is the distance from the observation point A along the line of wave propagation to its intersection with the coast. But

$$\left(\frac{\partial x_A}{\partial y}\right)_{D=\mathrm{const}} = -\frac{D-x_A}{D}\operatorname{tg}\theta, \qquad (50.7)$$

where θ is the "wave incidence angle on the shoreline." Therefore

$$\alpha = -\frac{\operatorname{tg}\theta}{k}\frac{D-x_A}{D}\operatorname{Im}\frac{\partial f(x_A,D)}{\partial x_A}. \qquad (50.8)$$

For a plane wave $(D \to \infty)$, we have simply

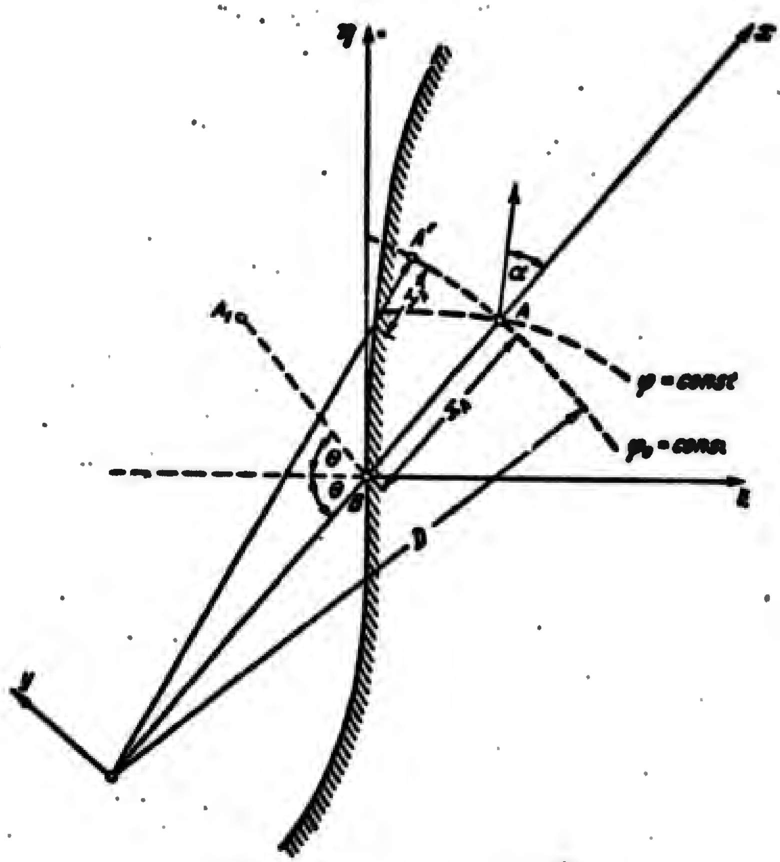

Fig. 50.1. Coastline refraction.

$$\alpha = -\frac{\operatorname{tg}\theta}{k}\frac{\partial \operatorname{Im} f}{\partial x_A} = -\frac{\sin\theta}{k}\operatorname{Im}\frac{\partial f}{\partial \xi_A}. \tag{50.9}$$

By definition, the phase velocity v(x) at a given point is given by the relation

$$\varphi = \varphi_0 + \varphi_1 = \omega\int\frac{dx}{v(x)}, \tag{50.10}$$

and since $\varphi_0 = kD \equiv kx$, we differentiate with respect to x and remember that $\varphi_1 = \operatorname{Im} f$, to obtain

$$\frac{c}{v(x)} = 1 + \frac{c}{\omega}\frac{\partial \operatorname{Im} f}{\partial x} = 1 - \frac{1}{\operatorname{tg}\theta}\alpha. \tag{50.11}$$

The concept of the average velocity $\bar{v}$ on the path from the source to the observation point is used often. It is defined as the velocity at which a given total phase advance $\varphi_0 + \varphi_1$ would be $(\omega/\bar{v})D$ (in a vacuum, it is $(\omega/c)D$). Therefore,

$$\frac{c}{v} = \frac{\kappa_0 + \kappa_1}{\omega D} c = 1 + \frac{c\,|m|}{\omega D}. \tag{50.12}$$

We shall limit ourselves to a study of the phase disturbance to the vertical field component of a vertical dipole situated on the ground ($z_0 = 0$).

2. At first we shall assume that an incident wave is incident, i.e., the source is very far away:

$$E^{(0)} = e^{ikx} = e^{ik(\xi\cos\theta + \eta\sin\theta)}. \tag{50.13}$$

According to Formulas (49.22) and (49.32), for $\vartheta = \pi/2$ and $q = 1$,

$$\begin{gathered} E_z^{(1)} = -\frac{\partial A_x}{\partial x} - \frac{\partial A_y}{\partial y}, \\ A_{x,y} = \frac{k^2}{\pi}\int \gamma_{x,y} e^{ikx' + ik\rho} \frac{dx'\,dy'}{\rho}. \end{gathered} \tag{50.14}$$

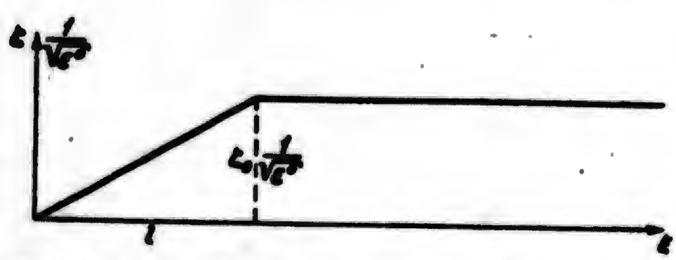

Fig. 50.2. Gentle gradient. Transition strip, sea to land.

Let us consider the disturbance due to a gentle gradient (Fig. 50.2):

$$\zeta = \begin{cases} 0 & \text{for } -\infty < \xi < 0, \\ \zeta_0 \cdot \dfrac{\xi}{l} = \gamma_0 \xi & \text{for } 0 < \xi < l, \\ \zeta_0 & \text{for } l < \xi < \infty. \end{cases} \tag{50.15}$$

By the applicability condition of the method, it is necessary that $\gamma_0 \ll 1$. At the break points of the surface, the method is not, strictly speaking, correct, since the higher derivatives of the field with respect to the coordinates are large here. A special investigation [12a] in which the formulas given below are generalized for the case $z \neq 0$ indicates how smooth these transitions must be.

Substituting Expressions (50.13) and (50.15) into Formulas (50.14), we can perform integration in the variables ξ, η, with the integration with respect to η resulting in Hankel functions $H_0^{(1)}$,

$$\int_{-\infty}^{+\infty} e^{ik(\xi+\eta\sin\theta)} \frac{d\eta}{\rho} = \pi H_0^{(1)}(k|\xi_A-\xi|\cos\theta) e^{ik\eta_A\sin\theta}, \quad (50.16)$$

while in integrating with respect to ξ it is necessary to remember that

$$\int e^{\pm it} H_0^{(1)}(t)\,dt = te^{\pm it}[H_0^{(1)}(t) \mp iH_1^{(1)}(t)] = F_{\pm}(t),$$

$$\int e^{-it} H_0^{(1)}(t)\,t\,dt = \frac{1}{3} te^{-it}[tH_0^{(1)}(t) + (it+1)H_1^{(1)}(t)] = G(t). \quad (50.17)$$

This will enable us to obtain the disturbance field $E^{(1)}$ at points a) in front of the slope; b) on the slope ($0 < \xi < l$); c) behind the slope ($l < \xi$) by appropriate selection of the limits of integration with respect to ξ. The field is expressed in terms of functions $F_{\pm}$ of the arguments

$$R_1 = k\xi_A\cos\theta,\ R_2 = k(\xi_A - l)\cos\theta = R_1 - R_0,\ R_0 = kl\cos\theta, \quad (50.18)$$
$$\xi_A = x_A\cos\theta.$$

If we restrict ourselves to sufficiently long distances $|R_1|$, $|R_2| \gg 1$, the formulas are quite simple [VII, 3d]:

a) $$f = -\sqrt{\frac{l}{8\pi}}\gamma_0\left(\frac{1}{\sqrt{|R_1|}}e^{2iR_1} - \frac{1}{\sqrt{|R_2|}}e^{2iR_2}\right),$$

b) $$f = \sqrt{\frac{2}{\pi i}}\gamma_0\sqrt{R_1} + \sqrt{\frac{l}{8\pi}}\gamma_0\frac{1}{\sqrt{|R_2|}}e^{2iR_2},$$

c) $$f = -\sqrt{\frac{2}{\pi i}}\gamma_0\left(\sqrt{R_1} - \sqrt{R_2}\right). \quad (50.19)$$

In front of the slope, therefore, the disturbance depends in an oscillatory manner on the distance to the slope, on k and on $\cos\theta$, while behind the slope it is monotonic. This is a feature common to all such disturbances. At great distances behind the slope $\sqrt{R_1} - \sqrt{R_2} \approx \frac{l\sqrt{k}}{2\sqrt{\xi_A}}$

so that the disturbance gradually dies out.

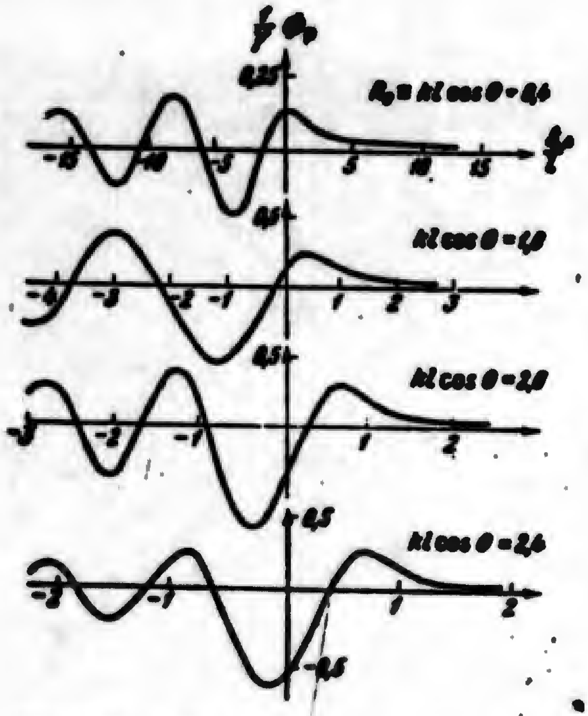

Fig. 50.3. Curves of function determining variation of propagation direction and phase velocity near a gentle slope.

In order to obtain α_R and v, it is necessary to carry out the differentiations indicated by Formulas (50.9) and (50.11). Calculations yield [VII, 3d]

$$\alpha_p = -\frac{1}{2}\sin 2\theta\cdot\Phi_p;\ \frac{c}{v}-1=\cos^2\theta\cdot\Phi_p;$$

$$\Phi_p=\frac{\gamma}{2}\{[J_1(R_1)\cos R_1+N_1(R_1)\sin R_1]-[J_1(R_2)\cos R_2+N_1(R_2)\sin R_2]\}, \tag{50.20}$$

where it is assumed that we are formally regarding not only the Bessel function J_1, but also the Neumann function N_1 as odd functions of the argument, $N_1(-x) = -N_1(x)$.

Figure 50.3 presents diagrams of the function $(1/\gamma)\Phi$ for certain values of $kl\cos\theta$; the diagrams were borrowed from [12].

The disturbance due to inhomogeneity of the soil (coastline) is calculated in a perfectly analogous manner. We set (see Fig. 50.2)

$$\frac{1}{\sqrt{\varepsilon^0(\xi)}}=\begin{cases}0 & \text{for} -\infty<\xi<0,\\ \frac{1}{\sqrt{\varepsilon^0}}\frac{\xi}{l} & \text{for } 0<\xi<l,\\ \frac{1}{\sqrt{\varepsilon^0}} & \text{for } l<\xi<\infty.\end{cases} \tag{50.21}$$

On the right in the above, ε^0 is a constant characterizing the land. The results differ only when the imaginary part of f is separated, since ζ is real and ε^0 is complex. We obtain

$$\alpha_n=-\frac{1}{2}\sin 2\theta\cdot\Phi_n;\quad \frac{c}{v}-1=\cos^2\theta\cdot\Phi_n=-\frac{1}{\operatorname{tg}\theta}\alpha_n;$$

$$\Phi_n=\frac{1}{2\sqrt{|\varepsilon^0|}}\{\Phi(R_1)-\Phi(R_2)\}\cdot\frac{1}{kl},$$

$$\Phi(R)=J_0(R)\cos\left(R+\frac{\chi}{2}\right)+N_0(R)\sin\left(R+\frac{\chi}{2}\right)+$$

$$+\frac{R}{\cos^2\theta}\left\{[J_0(R)-N_1(R)]\sin\left(R+\frac{\chi}{2}\right)-[N_0(R)+J_1(R)]\cos\left(R+\frac{\chi}{2}\right)\right\} \tag{50.22}$$

(where it is again necessary to take $N_1(-x)\to -N_1(x)$, $N_0(-x)\to N_0(x)$ for the negative argument).

Using the asymptotic expressions for the cylindrical functions, we may pass to simpler expressions in certain limiting cases.

a) Observer at sea, $R_1<0$, $|R_1|\gg 1$, $|R_2|\gg R_0$:

$$\alpha_n=\frac{\operatorname{tg}\theta\cos 2\theta}{\sqrt{2\pi kr_A|\varepsilon^0|}}\cos\left[\frac{\pi}{4}-\frac{\chi}{2}+2k\left(|\xi_A|+\frac{l}{2}\right)\cos\theta\right]\frac{\sin(kl\cos\theta)}{kl\cos\theta}. \tag{50.23}$$

b) Observer on land, $R_1>0$, $R_1\gg 1$, $R_1\gg R_0$:

$$\alpha_n=-\frac{\operatorname{tg}\theta}{\sqrt{2\pi kr_A|\varepsilon^0|}}\sin\left(\frac{\pi}{4}+\frac{\chi}{2}\right). \tag{50.24}$$

We see that in the latter case, the influence of the transition zone has dropped out of the picture entirely. The other difference between cases a) and b) is the oscillatory behavior of α in front of the shoreline and the monotonic dependence on ξ_A and k on land — in full analogy to the case of a geometric inhomogeneity. On going over to numerical results, it is helpful to remember that if we disregard the displacement currents $\left(\chi = \frac{\pi}{2},\ \varepsilon^{\circ} \approx 4\pi i \frac{\sigma}{\omega}\right)$ and express ξ_A in kilometers and σ in units of 10^7CGSE, $\sigma_0 = 10^{-7}\sigma$CGSE, then

$$\frac{1}{\sqrt{2\pi k \xi_A |\varepsilon^{\circ}|}} \approx 1{,}1 \frac{1}{\sqrt{\sigma_0 \xi_{A\,km}}} \text{ deg.} \qquad (50.25)$$

We note that this common factor does not depend explicitly on wavelength. As we move away from the shoreline, the disturbance vanishes. Diagrams for certain cases – borrowed from [12] – appear in Figs. 50.4 and 50.5. The existence of the phase surges (peaks on curves at boundaries of transition region, where ϕ_P goes to infinity in the present idealized case with sharp boundaries) was recently detected experimentally (for greater detail, refer to [11]).

3. Let us consider the special question of "radio-wave reflection from the shore," i.e., the case in which both the transmitter and the receiver are on the ocean surface at the respective points O and A' (Fig. 50.1). If the transmitter is very far away, we may speak of incidence and disturbance of a plane wave. We shall, however, consider a more general case. In the formulas §49, we must now substitute as the undisturbed field not the plane wave, but the function $(1/r)e^{ikr}$. The function $\exp(ik(r + \rho))$ will figure in the interval over the region occupied by land ($0 < \xi < \infty_2 - \infty < \eta < +\infty$). In its exponent, as usual, we expand in the vicinity of a stationary point. This point is point B on the boundary between land and sea, at which reflection should take place with a "reflection angle" equal to the "angle of incidence" θ. Then integration is carried out in the usual manner and we obtain

$$E = \frac{e^{ikD}}{D}\left(1 + \frac{D}{\sqrt{kD' \xi_0 \xi_{A'}}} \sqrt{\frac{i}{2\pi\varepsilon^{\circ}}} \frac{\cos 2\theta}{2\cos\theta} e^{ik(D'-D)}\right). \qquad (50.26)$$

Here D is the straight-line distance from the source O to the observa-

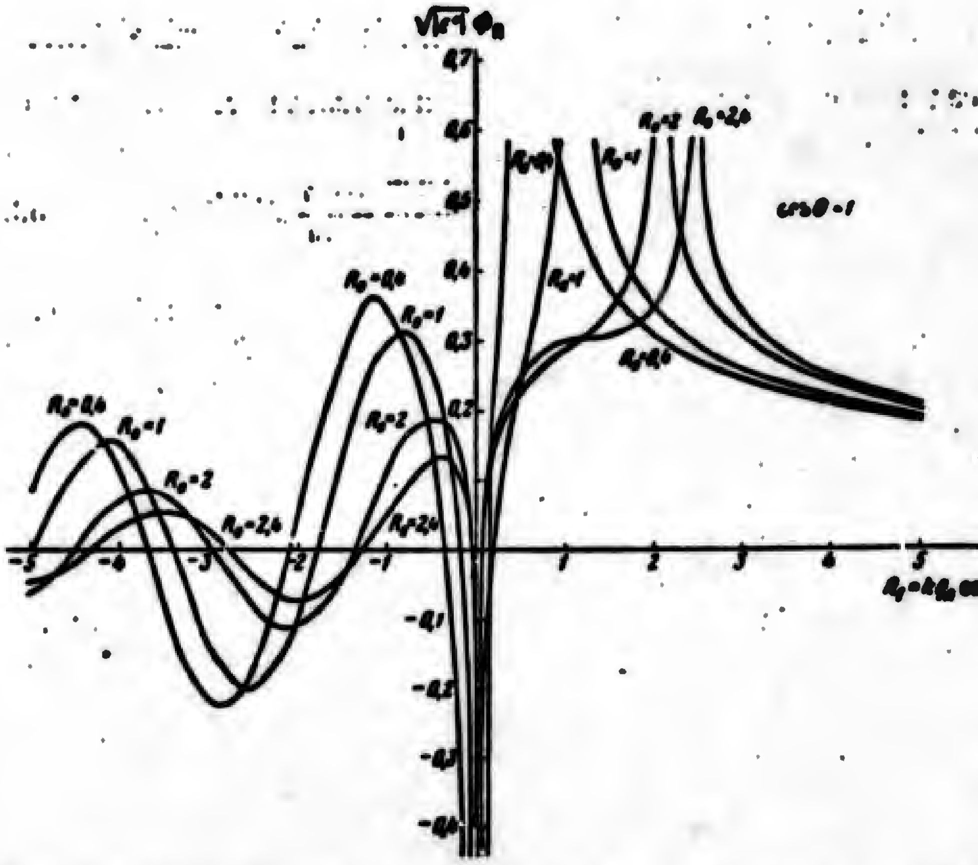

Fig. 50.4. Auxiliary curves for determining distortions of propagation direction and phase velocity due to inhomogeneity of soil electrical properties near the shore for normal incidence of radio waves ($\theta = 0$). The different curves correspond to different transition-zone dimensions.

tion point A', $D' = x_0 + x_{A'}$ is the path traveled by the ray OBA' reflected from the shore. Thus, $k(D' - D)$ is an additional phase lead; $\xi_0 = x_0 \cos\theta$ and $\xi_{A'} = x_{A'} \cos\theta$ are the distances to the shore (more precisely, to the tangent to it at the point of regular reflection) from the source and from the observation point.

This result is essentially determined by the fact that we have taken the transition zone into account. Attention should be drawn to the fact that the field perturbation diminishes as we move away from the shore. This circumstance justifies the approximate method of calculating the field along a piecewise-homogeneous path, as set forth in

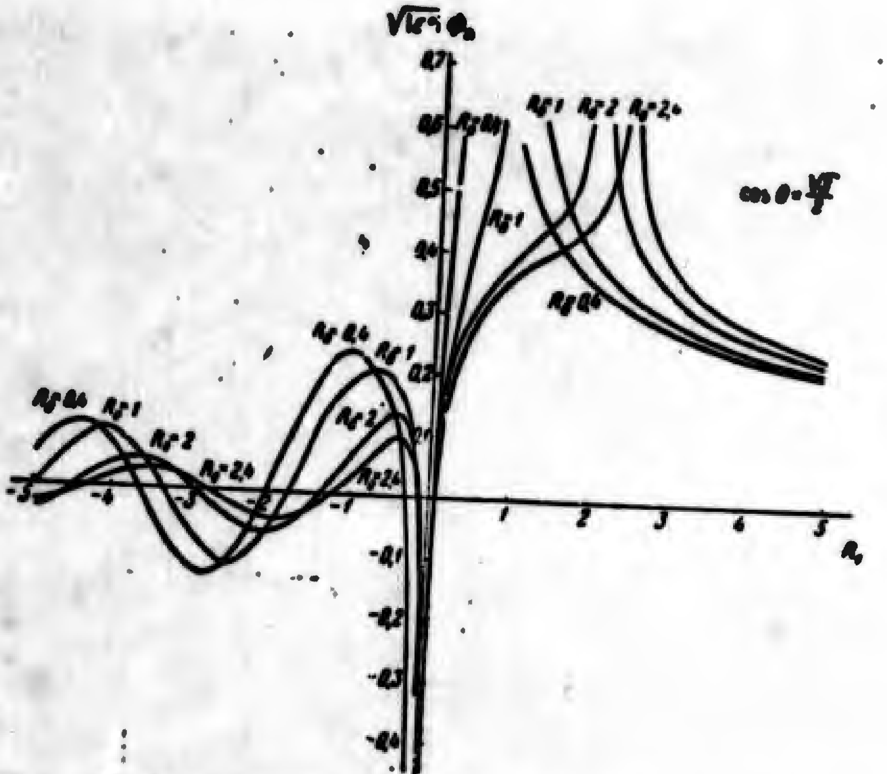

Fig. 50.5. Same as Fig. 50.4 but for incidence of radio waves at a 30° angle.

Chapter 7. There we assumed consistently that the properties of the soil behind the observation point (outside the line connecting the source and the observation point) do not influence the observed field. Accordingly, reducing the integral over the plane to an integral along a line, we restricted the integration to the range between points O and A. According to Formula (50.26), this is admissible if

$$\frac{\cos 2\theta}{2\cos\theta}\,\frac{D}{\sqrt{kD'\cos\delta_{A'}}}\left|\frac{1}{\sqrt{2\pi\varepsilon^0}}\right| \ll 1. \qquad (50.26a)$$

This means that if we drop the trigonometric factors and set $D - D' - x_0$, then we may disregard the reflected wave even at the minimum $|\varepsilon^0| - 4$ (see (21.23)) if the distance to the boundary behind the observation point, $x_{A'}$, satisfies the condition

$$\sqrt{kx_{A'}} \gg \frac{1}{4\sqrt{2\pi}}. \qquad (50.27)$$

Even at a distance of one wavelength, this inequality is observed to within about 5%. In general, however, neglecting the reflected wave (a characteristic, for example, for all of Chapter 7) introduces an error of the order of the left member of Formula (50.26a).

4. Now suppose that the observation point and the source are situated at a finite distance from the boundary on either side of it. We saw that the influence of the transition zone is not manifest in this case far from the shore, so that it is possible to set, for example, $l = 0$. This means that we are assuming all distances $- x_0$, x_A, $x_0 \cos\theta = \xi$ and $x_A \cos\theta = \xi_A$ – to be large by comparison with l. Since the disturbance is small, we may substitute the undisturbed field in the integral expression here as well:

$$E = E^{(0)}(r) + \frac{ik}{2\pi\sqrt{\varepsilon^0}} \iint_s E^{(0)}(r') \frac{e^{ik\rho}}{\rho} dr', \quad \rho = |r - r'|. \qquad (50.28)$$

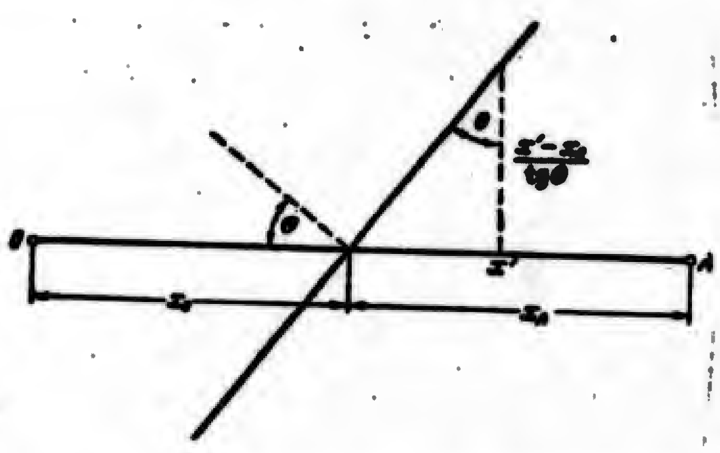

Fig. 50.6. Illustrating calculation of coastline refraction.

The integral is extended over the region occupied by "land," where $\varepsilon^0 \neq \infty$ and $E^{(0)}(\vec{r}) = e^{ikr}/r$. On the assumption that $kD \gg 1$, it could be calculated in elliptical coordinates, using the method of steepest descents (see [VII, 3d] and [VII, 4a]). However, this is not necessary. We can expand the exponent in powers of the deviation from kD in the usual manner:

$$r' + \rho \approx D + \frac{y'^2}{2}\left(\frac{1}{x'} + \frac{1}{D - x'}\right),$$

and integrate over y' from $-\infty$ to $(x' - x_0)/\tan\theta$ and over x' from x_0 to D (Fig. 50.6). Dividing the integral over y' into two parts, from $-\infty$ to 0 and from 0 to $(x' - x_0)/\tan\theta$, we see after converting to the variable $t = y'\sqrt{\frac{k}{2}\left(\frac{1}{x'} - \frac{1}{D - x'}\right)}$ that the limit of the second integral is large if cot θ is not very small. That is to say, using Formula (10.13), we obtain

$$E = \frac{e^{ikD}}{D}\left\{1 + i\sqrt{\frac{ikD}{2\pi x^0}}\int_{x_0}^{D}\frac{dx'}{\sqrt{x'(D - x')}}\left(1 - \frac{e^{i\left(\frac{x' - x_0}{\mathrm{tg}\,\theta}\right)^2 \frac{kD}{2x'(D - x')}}}{2\frac{x' - x_0}{\mathrm{tg}\,\theta}\sqrt{\frac{i\pi kD}{x'(D - x')}}}\right)\right\}. \tag{50.29}$$

In integrating over x', all points of the range of integration are in general of equal importance. If we assume $x_0 \sim x_A \sim D$, then this means that $x' \sim D - x' \sim D$ also. Consequently, we may drop the second term in the parentheses under the integral sign if $kD \gg \tan^2\theta$. If, on the other hand, $x_0 \gg x_A$, then $x' - x_0 \sim D - x' \sim x_A$, $x' \sim D$ and the neglect is admissible if $kx_A \gg \tan^2\theta$. In general, it can be stated that necessarily

$$kx \gg \mathrm{tg}^2\,\theta, \tag{50.30}$$

where x is the smaller of the two distances to the shore: that from the receiver and that from the transmitter.

This condition has a simple significance. The width of the first Fresnel zone, which determines the region essential for formation of the disturbance, is $\sqrt{\frac{x}{k}}$ at a distance x from the nearest focus.

For incidence of radio waves at an angle θ on a straight shoreline, the deviation of the shore from the normal to the line of propagation will not exceed $\tan\theta\sqrt{\frac{x}{k}}$ even at the boundary of this zone. In general, all points of the zone make contributions of the same order to the integral. The deflection of the shoreline from the normal will have little effect on the result if it is small as compared to the length x of the region in the zone over which integration is extended. Consequently, we must have $\tan\theta \ll \sqrt{kx}$, which leads us to (50.30). From this we may conclude that the formulas derived are also valid for the case in which the shore is not a straight line, provided that the shoreline does not run too close to the axis of the essential zone.

Since we are assuming $kx \geq 1$, Condition (50.30) means in practice that this neglect is not admissible for radio waves skipping along the shore when θ is near $\pi/2$. Setting in this case $\sin\theta \sim 1$, Condition (50.30) may be written as follows:

$$kx\cos^2\theta \gg 1. \tag{50.30a}$$

When these limitations are observed, Expression (50.29) is simplified and we obtain on integration

$$E = \frac{e^{ikD}}{D}\left(1 + \frac{i}{2}\sqrt{\pi sD}\left(1 + \frac{2}{\pi}\arcsin\frac{x_A - x_0}{x_A + x_0}\right)\right). \tag{50.31}$$

$(x_A + x_0 = D)$. If $x_0 < x_A$, we obtain, assuming that $\arcsin\frac{x_A - x_0}{D} = \frac{\pi}{2} - \chi$, $\cos\chi = \frac{x_A - x_0}{D}$, $\sin\chi = \sqrt{\frac{4x_Ax_0}{D^2}}$,

$$E = \frac{1}{D}e^{ikD}\left\{1 + i\sqrt{\pi sD}\left(1 - \frac{1}{\pi}\arcsin\sqrt{\frac{4x_Ax_0}{D^2}}\right)\right\}. \tag{50.32}$$

If, on the other hand, $x_0 > x_A$, then $\arcsin\frac{x_A - x_0}{D} = -\left(\frac{\pi}{2} - \chi\right)$ and

$$E = \frac{1}{D}e^{ikD}\left\{1 + \frac{i}{\sqrt{\pi}}\sqrt{sD}\arcsin\sqrt{\frac{4x_Ax_0}{D^2}}\right\}. \tag{50.33}$$

Accordingly, for incidence of a plane wave on land from the direction of the sea, $x_A \ll x_0 \sim D$, it follows from Formula (50.33) that

$$E = \frac{1}{D} e^{ikD} \left\{1 + \frac{2i}{\sqrt{\pi}} \sqrt{sx_A}\right\}. \tag{50.34}$$

If, however, $x_0 \ll x_A \sim D$ (by the reciprocity theorem, this corresponds to incidence of a plane wave from the land side), we find from Formula (50.32)

$$E = \frac{1}{D} e^{ikD} \left\{1 + i\sqrt{\pi sD} - \frac{i}{\sqrt{\pi}} \sqrt{sx_0}\right\}. \tag{50.35}$$

If $x_0 = 0$, then, as we should expect, we obtain the first terms of the expansion in sD for the attenuation function (see Formula (25.1)).

All of these expressions yield corrections to the undisturbed field that increase with increasing dimensions of the disturbing region (x_A or x_0, respectively). This reflects the inapplicability of the perturbations method when the numerical distance traversed by the wave above the disturbing surface is long. The physical cause of this is to be sought in our disregard of the fact that the field of the secondary radiators, which is superimposed on the primary field, itself dies out as propagation advances and is in general subject to the influence of the tertiary, etc., radiators that it excites. This is not taken into consideration in the method of perturbations. Other methods are required to take full account of "reradiation." They have been set forth in Chapter 7. They are based on taking not the function $v_0 = e^{ik\rho}/\rho$, which, physically, gives the dipole field above an ideally conductive surface, but the function $v = v_0 y(s\rho)$ (where $y(s\rho)$ is the normal attenuation function), which describes the field of the dipole above an absorbing surface, as the Green's function. As was shown in 43, the phase shift actually reaches a constant magnitude at large numerical distances. For the problems analyzed in the present section, it is not of interest: for Im f = const, we have $\alpha = 0$ and $v = c$.

Formulas (50.32) and (50.33) enable us to calculate α_p and v by the same rules as for a plane wave (see Formulas (50.9) and (50.11)). A peculiarity of the problem consists in the fact that it is necessary to exercise caution here in applying the reciprocity theorem. Although E_z and, consequently, f do not vary for a vertical dipole when points O and A are transposed, the propagation-direction distortions α are found to differ. In one case, differentiation of f in Formula (50.9) is carried out with respect to x_A, while in the other it is over x_0. As we see from Formula (50.31), these quantities appear asymmetrically. If the transmitter is at sea and the receiver is on land, we obtain instead of Formula (50.24)

$$\alpha_{п} = -\frac{\operatorname{tg}\theta}{\sqrt{2\pi k x_A |\varepsilon^0|}} \sqrt{1 - \frac{x_A}{D}} \sin\left(\frac{\pi}{4} + \frac{\chi}{2}\right). \quad (50.36)$$

If, however, the transmitter is on land and the receiver at sea, it is necessary to change the sign of the entire expression and understand x_A as the over-water segment of the path. When this segment is considerably longer than the land segment, the df error α_p diminishes sharply in absolute value when the receiver and transmitter change places. The values of α_p obtained from the formula are given for certain cases in Fig. 50.7.

The formulas derived here for α_p give $v(x_A)$ according to (50.11) and $\bar{v}$ according to (50.12).

5. Up to this point, we have been considering the boundary between regions with $\varepsilon^0 = \infty$ and finite ε^0. In the general case, we are concerned with the boundary between regions with two different finite ε^0. Let us denote them by ε_1^0 and ε_2^0. Here we shall apply Eq. (41.17), which was derived with the aid of the Green's function $e^{ik\rho}y(s_0\rho)$. We take the arbitrary parameter ε_0^0 equal to ε_1^0, where the subscript 1 denotes soil, for example, on the left of the boundary. Then the integrand is zero to the left of the boundary, where $\varepsilon^0(x, y) = \varepsilon_1^0 = \text{const}$.

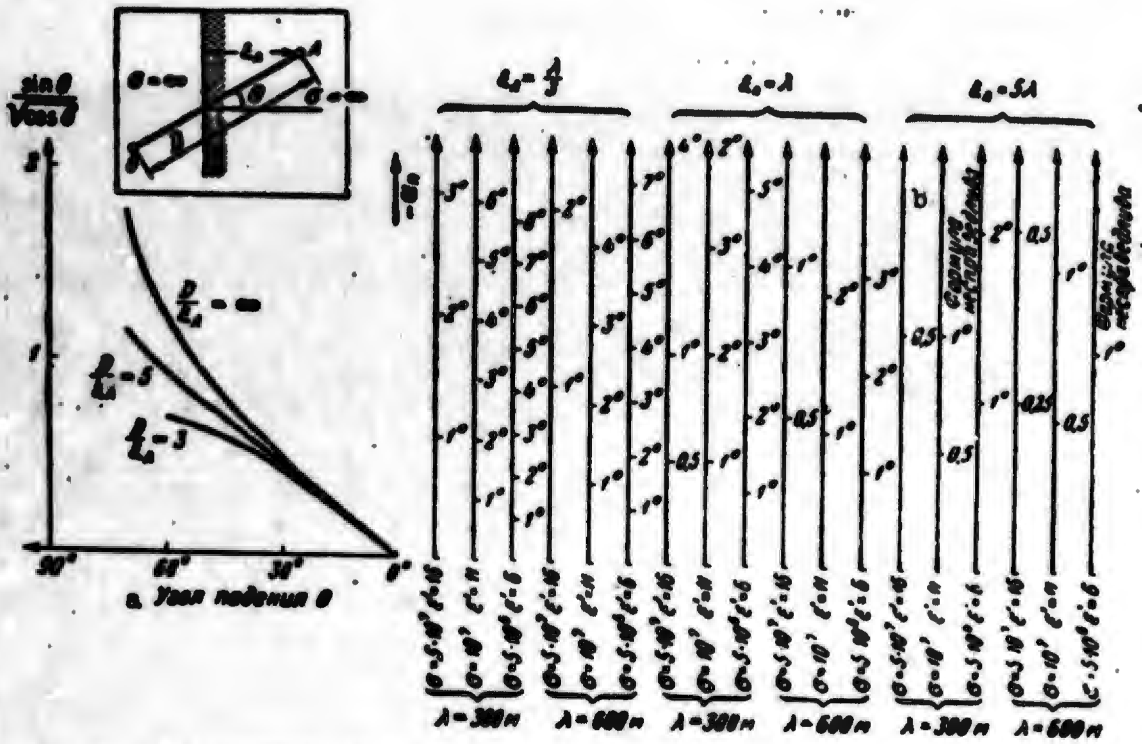

Fig. 50.7. Alignment chart for determining angle of shoreline refraction ("soil error") α_p. The ordinate scales appear on the right; they indicate $-\alpha_p$ in degrees for various λ, ξ_A, σ and ε'. a) Incidence angle θ; b) formula invalid.

The integral is left extended only over the region to the right of the boundary, where $\varepsilon^0(x, y) = \varepsilon_2^0 = \text{const}$. We can take the constant multiplier $\frac{1}{\sqrt{\varepsilon_1^0}} - \frac{1}{\sqrt{\varepsilon_2^0}}$ out of it. Further, restricting ourselves to distances from the boundary for which all numerical distances are s still small, we may consider $y_0 = 1$. Finally, since we are limiting ourselves to the method of perturbations, we may set $w = 1$ in the integrand (or, if the source is very far away and its numerical distance from the shore is not small, w can be replaced by the value of y for the soil to the left of the boundary taken at the shore, and moved out from under the integral sign). Then the function w is expressed in exactly the same way as for the sea-land case analyzed earlier – for which we obtained Formula (50.31), the only difference being that

$1/\sqrt{\varepsilon_0}$ is replaced by the difference

$$\frac{1}{\sqrt{\varepsilon^0_{eff}}}=\frac{1}{\sqrt{\varepsilon^0_1}}-\frac{1}{\sqrt{\varepsilon^0_2}}. \qquad (50.37)$$

Accordingly, Formula (50.36) remains in force for a_p, but the absolute value and phase of the quantity ε^0_{eff} from Formula (50.37) must be substituted for $|\varepsilon^0|$ and χ.

In particular, if both soils are good conductors and displacement currents can be disregarded, $\chi = \pi/2$ and

$$\frac{1}{\sqrt{|\varepsilon^0_{eff}|}}=\frac{1}{\sqrt{|\varepsilon^0_1|}}\left(1-\sqrt{\frac{\sigma_1}{\sigma_2}}\right). \qquad (50.37a)$$

where σ_1 and σ_2 are the conductivities of these soils. For a more detailed exposition of these questions, refer to [VII, 3, c and d], [11] and [12].

It is considerably more difficult to investigate the case in which Condition (50.30) is violated. In the limit $\theta = \pi/2$, we are concerned with the field of a source situated on the dividing line between sea and land, observed in the vicinity of this line. At points remote from the shoreline, it goes over on one side into the field above the over-water pass, and on the other into the field above the dry-land pass. The corrections due to the presence of inhomogeneity behind the source can be found by the method of perturbations in each case. Formulas indicating how these fields merge with one another when the observation point is moved across the soil's dividing line have not yet been derived in general form. Certain results pertaining to this will be found in [11].

§51. FIELD ABOVE A GENTLY SLOPING, RANDOMLY IRREGULAR AND INHOMOGENEOUS SURFACE (GLANCING WAVES OR LOW IRREGULARITIES)

1. In this section and those that follow, we extend the statistical analysis, begun in §48, of the field above a surface with random irregularities, but for other relationships between the characteris-

tics of the field and the surface. Further, in the present section we shall assume the existence of random irregularities in the electrical properties of the soil. Here we shall assume the irregularities to be gentle, i.e., the angles γ_x and γ_y formed by the surface with a certain averaging plane to be small,

$$|\gamma_x| \ll 1, \quad |\gamma_y| \ll 1, \tag{51.1}$$

and the parameter ε to be large and to vary quite smoothly:

$$\left|\frac{1}{\sqrt{\varepsilon}}\right| \ll 1, \quad |\operatorname{grad} \ln \sqrt{\varepsilon}\,| \ll \sqrt{\varepsilon}\,k. \tag{51.2}$$

The analyses in §§51 and 52 are distinguished by the fact that in the former, we assume the height ζ_0 of the irregularities to be either small by comparison with λ (although their horizontal dimensions are arbitrary) or, if the glancing angle ψ of the incident wave is small enough, even to exceed the wavelength, although it still satisfies the criterion (see below, Formula (51.30a))

$$2\pi \sqrt{\gamma\zeta_0\psi} \ll \sqrt{\lambda}. \tag{51.3}$$

Thus, we shall be able to employ the approximation developed in §49. Here the reciprocal influence of the irregularities will be taken into account fully. In the following section, we shall consider a case that is in a certain sense the converse: irregularities whose height may be <u>larger</u> than the wavelength provided that the glancing angles ψ of the incident wave are correspondingly <u>large</u> enough.

2. The example of §48 has already shown that the combined action of many irregularities, each of which distorts the field only very weakly outside its immediate neighborhood, is capable of influencing the <u>average</u> field substantially if the irregularities cover a large enough area. In the case of gentle irregularities and high conductivity, the vertical component is the principal component of the field. As was shown in Subsection 4 of §49, it experiences only minor dis-

turbance due to an individual irregularity or inhomogeneity and this disturbance diminishes as we move away to a distance of the order of the horizontal dimensions l of the irregularity or inhomogeneity. Even here, therefore, we shall be interested in the superimposed effect of many disturbances, the effective number of which increases with increasing distance from the transmitter to the receiver. We shall follow [VII, 3, b and d].

Thus, suppose that the deviation of the surface from a certain averaging plane $z = \gamma = 0$ and also that the small gradient angles γ_x and γ_y average zero. The variations of $\eta = 1/\sqrt{\varepsilon}$ — we shall denote them by ξ — about its average value $\eta_0 = \overline{\frac{1}{\sqrt{\varepsilon}}}$

$$\frac{1}{\sqrt{\varepsilon}} = \eta = \eta_0 + \xi, \quad \bar{\xi} = 0, \quad \eta_0 = \overline{\frac{1}{\sqrt{\varepsilon}}} \tag{51.4}$$

must also be smooth enough (small at distances of the order of the wavelength in the soil).

Thus we have four independent small parameters in addition to $\overline{\frac{1}{\sqrt{\varepsilon}}} = \eta_0$: the root-mean-square values ζ_0 and ξ_0 of the quantities ζ and ξ and the amplitudes of their gradients, γ, and, for example, β. Instead of this, we might characterize the surface by the amplitudes ζ_0 and ξ_0 and by the average <u>horizontal</u> dimensions l_g and l_e of geometric and electrical inhomogeneities, since

$$\gamma \sim |\nabla\zeta| \sim \frac{\zeta_0}{l_g}, \quad \beta \sim |\nabla\xi| \sim \frac{\xi_0}{l_e}. \tag{51.5}$$

Moreover, the parameters l_g and l_e may be different along different horizontal axes, so that it is necessary to introduce l_{gx} and l_{gy} instead of l_g and l_{ex} and l_{ey} instead of l_e. Only if the surface is <u>statistically isotropic</u> do they reduce to the two parameters l_e and l_g. We shall describe the statistical properties of the surface (this is sufficient in the present section) by the correlation functions at two points, which are given by the two-dimensional radius-vectors $\vec{r}(x, y)$ and $\vec{r}'(x', y')$. In the general case

$$\overline{\zeta(r)\zeta(r')} = \zeta_0^2 F_g\left(\frac{x'-x}{l_{gx}}, \frac{y'-y}{l_{gy}}\right), \qquad (51.6)$$

$$\overline{\zeta(r)\zeta^*(r')} = \zeta_0^2 F_e\left(\frac{x'-x}{l_{ex}}, \frac{y'-y}{l_{ey}}\right). \qquad (51.7)$$

Here it is assumed that the axes are placed along the principal directions of variation of F (strictly speaking, they might not coincide for F_e and F_g). The correlation functions F are equal to unity when the argument is equal to zero (consequently, for example, ζ_0^2 is the mean-square height ($\zeta_0^2 = \overline{\zeta^2}$)) and diminish to zero when the distance is $|x' - x|$ and $|y' - y|$ exceed, respectively, the characteristic lengths l_x and l_y, which are known as the correlation lengths along the x- and y-axes or the radii of correlation. Thus, for ocean covered by regular waves (for example, after a storm at a certain distance from it), the correlation of heights along the wave is retained at distances considerably longer than across the waves. This means that if we direct the x-axis across the waves, $l_{gx} \ll l_{gy}$ (if the axes are arbitrarily directed, the argument of the function F is more complex to write). The fact that the functions F are assumed to depend only on the differences of the coordinates expresses the statistical homogeneity of the surface. If their statistical properties change over considerable distances, we may regard all parameters to be slowly varying functions of the coordinates.

Subsequently, we shall assume that the surface is also statistically isotropic from a certain point on, i.e., we shall set $l_x = l_y = l$. Although this assumption is not fundamental, it does simplify some of the bookkeeping.

A remark on the concept of the mean is in order. Here and below, we refer to averaging over the ensemble. This means that, leaving the positions (relative to the plane z = 0) of the source and observation point (or points $\vec{r}$ and $\vec{r}'$ in (51.6) and (51.7)) unchanged, we "insert" different copies of the surface possessing identical statistical characteristics, i.e., surfaces having identical statistical parameters and, in general, identical correlation functions F. Instead of this we might say that, fixing the <u>relative positions</u> of the source and observation point, we place them in different positions with respect to the given surface. In practice, the "average field" is frequently understood as the result of averaging quantities obtained with a fixed source, when measurements are taken about a certain average point at various observation stations within distances that are small by comparison with the distance to the transmitter. These two definitions are not, generally speaking, consistent.

We shall regard the electrical and geometric parameters as independent. In actuality, this is not always the case: zones of terrain at higher elevations are usually drier and therefore poorer conductors and vice versa. It would not be difficult to take this into account. However, we shall limit ourselves to the simpler case indicated.

Often, instead of the actual functions ζ, ε, F_g and F_e, it is convenient to use their Fourier transforms:

$$\zeta(r) = \zeta_0 \int g(q) e^{iqr} dq; \tag{51.8a}$$

$$\varepsilon(r) = \varepsilon_0 \int e(q) e^{iqr} dq; \tag{51.8b}$$

$$F_g(\rho) = \int G_g(q) e^{iq\rho} \frac{dq}{q_g^2}; \quad G_g(q) = \int F_g(\rho) e^{-iq\rho} \frac{d\rho}{l_g^2};$$

$$F_e(\rho) = \int G_e(q) e^{iq\rho} \frac{dq}{q_e^2}; \quad G_e(q) = \int F_e(\rho) e^{-iq\rho} \frac{d\rho}{l_e^2}; \tag{51.9}$$

$$q_g = \frac{2\pi}{l_g}; \quad q_e = \frac{2\pi}{l_e}; \quad l_g^2 = \frac{1}{2}(l_{gx}^2 + l_{gy}^2); \quad l_e^2 = \frac{1}{2}(l_{ex}^2 + l_{ey}^2); \quad \rho = r' - r. \tag{51.10}$$

Here all vectors are two-dimensional.

Finally, attention should be drawn to the rules for conversion from the average gradient products to correlation functions. Since the operations of averaging and differentiation are interchangeable,

$$\left.\begin{aligned}
&\bar{\gamma}_x = \bar{\gamma}_y = \frac{\partial\bar{\zeta}}{\partial x} = \frac{\partial\bar{\zeta}}{\partial y} = 0,\\
&\overline{\gamma_x(r)\zeta(r')} = \frac{\partial}{\partial x}\overline{\zeta(r)\zeta(r')} = \frac{\partial}{\partial x}\zeta_0^2 F_\zeta = -\zeta_0^2\frac{\partial}{\partial\rho_x}F_\zeta,\\
&\overline{\zeta(r)\gamma_x(r')} = -\overline{\gamma_x(r)\zeta(r')} = -\zeta_0^2\frac{\partial}{\partial x}F_\zeta = +\zeta_0^2\frac{\partial}{\partial\rho_x}F_\zeta,\\
&\overline{\gamma_x(r)\gamma_y(r')} = -\overline{\gamma_y(r)\gamma_x(r')} = \frac{\partial}{\partial x}\frac{\partial}{\partial y}\overline{\zeta(r)\zeta(r')} = -\zeta_0^2\frac{\partial^2}{\partial\rho_x\,\partial\rho_y}F_\zeta,
\end{aligned}\right\}\qquad(51.11)$$

and so forth. Here $\rho_x = x' - x$, $\rho_y = y' - y$.

Obviously,

$$\frac{\partial}{\partial\rho_x}F_\zeta(0) = \frac{\partial}{\partial\rho_y}F_\zeta(0) = 0. \qquad (51.11a)$$

Indeed, at a given point any values of the inclinations are possible, so that $\overline{\gamma_x(r)\zeta(r)} = \overline{\gamma_y(r)\zeta(r)} = 0$.

3. Let us consider the average field of a vertical dipole situated at the point $(0, 0, z_0)$ and observed on the plane $z = 0$ ($z_A = 0$). We shall assume that (see below) the field at each point can be written in the form

$$E_i(r) = \overline{E_i(r)}\left(1 + \sum_k \alpha_k f_k^{(i)}(r)\right), \qquad (51.12)$$

where α_k are the small parameters of our problem: the root-mean-square values of ζ, γ, ξ and $1/\sqrt{\varepsilon^0}$; $\bar{E}_i$ is the average value of the <u>ith</u> field component; $f_k^{(i)}(r)$ are factors describing the fluctuations of these components. By the definition of $\bar{E}_i$, their average values are equal to zero.

We may further write

$$\overline{E_z(r)} = E_z^0(r)\,w(r), \qquad (51.13)$$

where $w(\vec{r})$ is a slowly varying attenuation function and E_z^0 is the field that would be present in the case of an ideally conductive plane.

Obviously, the usual relationships

$$\frac{\partial}{\partial x}\overline{E_z(r)} \approx ik_x\overline{E_z(r)}, \tag{51.14}$$

are also valid here, with

$$k_x^2 + k_y^2 = k^2\cos^2\psi = \varkappa^2, \tag{51.15}$$

where ψ is the wave glancing angle and $\vec{\varkappa}$ denotes a two-dimensional vector, $\varkappa_x = k_x$, $\varkappa_y = k_y$.

Let us first derive the equation for E_z. Averaging Eq. (49.9), we have (we take the value $\varepsilon_0^0 = \infty$ for the arbitrary parameter)

$$\overline{E}_z = \overline{E}_z^0 - \frac{\partial\overline{A}_x}{\partial x} - \frac{\partial\overline{A}_y}{\partial y}, \tag{51.16}$$

$$\overline{A}_{x,y} = -\frac{1}{2\pi}\int\left\{\frac{k_{x,y}}{k\cos^2\psi}\eta_0\overline{E_z(r')} + \overline{\left(\frac{k_{x,y}}{k\cos^2\psi}\xi(r') - \gamma_{x,y}(r')\right)E_z(r')} - \overline{\frac{\partial\varepsilon_{x,y}(r')}{\partial z}\zeta(r')}\right\}\cdot v(r'-r)\,dr'. \tag{51.17}$$

On the other hand, multiplying $E_z(\vec{r}\,')$ by the corresponding multipliers and averaging only after this is done, we can, applying the iteration method (i.e., substituting the expression for E_z in terms of $A_{x,y}$ and, consequently, in terms of E_z again in the right member), obtain the necessary integrands. Thus, we have

$$\begin{aligned}\overline{\left(\frac{k_x}{k\cos^2\psi}\xi(r') - \gamma_x(r')\right)E_z(r')} = \frac{1}{2\pi}\int\Big\{\Big(\frac{k_x^2}{k^2\cos^4\psi}\overline{\xi(r')\xi(r'')} +{}\\ + \overline{\gamma_x(r')\gamma_x(r'')}\Big)\frac{\partial v(r''-r')}{\partial x'} + \Big(\frac{k_x k_y}{k^2\cos^4\psi}\overline{\xi(r')\xi(r'')} +{}\\ + \overline{\gamma_x(r')\gamma_y(r'')}\Big)\frac{\partial v(r''-r')}{\partial y'}\Big\}\overline{E}_z(r'')\,dr'',\end{aligned} \tag{51.18}$$

$$\overline{\frac{\partial\varepsilon_x(r')}{\partial z}\zeta(r')} = \frac{\partial^2}{\partial z^2}\cdot\frac{1}{2\pi}\int\overline{\zeta(r')\gamma_x(r')}\,\overline{E_x(r')}\,v(r''-r')\,dr'' \tag{51.19}$$

and two additional expressions: one that differs from (51.18) in the substitution of k_y and $\gamma_y(\vec{r}\,')$ for k_x and $\gamma_x(\vec{r}\,')$ and one that differs from (51.19) in having E_x replaced by E_y and γ_x by γ_y. In deriving them, we took into account that, according to Formula (51.12), for example, $\overline{\gamma_x\gamma_y E_z} = \overline{\gamma_x\gamma_y}\cdot\overline{E_z}$, $\overline{\xi E_z} = \overline{\xi}\cdot\overline{E_z}$, etc., to within higher-order terms in ζ and that $\frac{\partial E_{x,y}}{\partial z}\zeta$ for vertically polarized radiation is now a quantity of the second negative order, so that its product by ζ or γ can be dropped, and so forth. The admissibility of these neglects determines the limits of applicability of the method (see below).

All of these expressions can be simplified substantially if we recall that average products of the type $\overline{\xi(r)\xi(r')}$, $\overline{\gamma_x\gamma_y}$ and the like are expressed in terms of the correlation functions and are therefore nonzero only as long as the coordinate difference $\rho = \vec{r}'' - \vec{r}'$ is of the order of l, i.e., small. Consequently, the effective range of integration over $\vec{r}''$ is small, the attenuation functions do not vary within it, and we may assume that

$$E_z(r'') \approx E_z(r')\, e^{ik(r''-r')}. \qquad (51.20)$$

Here $E_z(\vec{r}')$ can be taken out from under the sign of the integral over $\vec{r}''$. In (51.19), of course, it is necessary to assume $\rho = \sqrt{(x''-x')^2+(y''-y')^2+z'^2}$ and set $z' = 0$ only after the calculations. Here it will be convenient to take advantage of the fact that the function v satisfies the wave equation

$$\frac{\partial^2}{\partial z'^2}v = -\left(k^2+\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)v. \qquad (51.20a)$$

After these simplifications, we can substitute Expressions (51.18) and (51.19) into Formula (51.17) and then substitute the resulting expressions into Formula (51.16). Since all field components save E_z have now been excluded from $\bar{A}_x$ and $\bar{A}_y$, we obtain an integral equation for E_z, which was our objective:

$$\overline{E_z(r)} = E_z^0 + \frac{ik}{2\pi}\int\left(\frac{1}{\sqrt{\varepsilon_0}}+\frac{1}{\sqrt{\varepsilon_g}}+\eta_0\right)\overline{E_z(r')}\,v(r'-r)\,dr', \qquad (51.21)$$

where

$$\frac{ik}{\sqrt{\varepsilon_e}} = \frac{1}{2\pi k^2\cos^4\psi}\int e^{i\varkappa(r''-r')}\Big\{\Big(k_x^2\frac{\partial}{\partial x'} + k_xk_y\frac{\partial}{\partial y'}\Big)v(r''-r')\frac{\partial}{\partial x} +$$

$$+\Big(k_xk_y\frac{\partial}{\partial x'} + k_y^2\frac{\partial}{\partial y'}\Big)v(r''-r')\frac{\partial}{\partial y}\Big\}dr''; \qquad (51.22)$$

$$\frac{ik}{\sqrt{\varepsilon_g}} = \frac{1}{2\pi}\int e^{i\varkappa(r''-r')}\Big\{\Big(\overline{\gamma_x(r')\gamma_x(r'')}\frac{\partial}{\partial x'} + \overline{\gamma_x(r')\gamma_y(r'')}\frac{\partial}{\partial y'} -$$

$$-\overline{\zeta(r')\gamma_x(r'')}\frac{\partial^2}{\partial z'^2}\Big)v(r''-r')\frac{\partial}{\partial x} + \Big(\overline{\gamma_y(r')\gamma_x(r'')}\frac{\partial}{\partial x'} +$$

$$+\overline{\gamma_y(r')\gamma_y(r'')}\frac{\partial}{\partial y'} - \overline{\zeta(r')\gamma_y(r'')}\frac{\partial^2}{\partial z'^2}\Big)v(r''-r')\frac{\partial}{\partial y}\Big\}dr'' \qquad (51.23)$$

(it must be remembered that the function $v(\vec{r}'' - \vec{r}')$ is taken from the argument $\rho = \sqrt{(x''-x')^2+(y''-y')^2+z'^2}$, and that only after differentiation with respect to z' do we set z' = 0). The equation obtained for $\bar{E}_z$ is the same as that which would be obtained with these sources in the case of a flat earth characterized by the electrical parameter

$$\frac{1}{\sqrt{\varepsilon_{tot}}} = \overline{\frac{1}{\sqrt{\varepsilon}}} + \frac{1}{\sqrt{\varepsilon_g}} + \frac{1}{\sqrt{\varepsilon_e}}. \qquad (51.24)$$

Here this parameter still has the form of an operator (since it contains ∂/∂x and ∂/∂y). But if we take into account that the average products (51.6), (51.7) depend only on the coordinate difference $\vec{\rho} = \vec{r}'' - \vec{r}'$, we may pass from integrals over $\vec{r}''$ to integrals over $\vec{\rho}$. Then it will be convenient to substitute the operators ∂/∂x and ∂/∂y, which operate on $v(\vec{r}' - \vec{r})$ by −∂/∂x' and −∂/∂y' and conduct integration by parts in (51.21). In accordance with Formula (51.14), all of this simply reduces to substituting ik_x for ∂/∂x and ik_y for ∂/∂y. Finally, therefore, converting to correlation functions with (51.11), etc., and substituting the Fourier expansions if convenient, we obtain one of the following three forms:

$$\frac{1}{\sqrt{\varepsilon_e}} = \frac{-k_0^2}{2\pi k\cos^4\psi}\int e^{i\varkappa\rho}F_e(\rho)(\varkappa\nabla)v(\rho)d\rho =$$

$$= \frac{ik_0^2}{2\pi k\cos^4\psi}\int e^{i\varkappa\rho}v(\rho)(\varkappa^2 - i(\varkappa\nabla))F_e(\rho)d\rho =$$

$$= \frac{\xi_0^2}{k\cos^2\psi}\int\frac{\varkappa^2+(\varkappa q)}{\sqrt{q^2+2(\varkappa q)-k^2\sin^2\psi}}G_z(q)\frac{dq}{q_0^2}; \qquad (51.25)$$

$$\frac{1}{\sqrt{\varepsilon_g}} = -\frac{\xi_0^2}{2\pi k}\int e^{i\varkappa\rho}F_g(\rho)\{k^2\varkappa^2 - i(k^2+\varkappa^2)(\varkappa\nabla)-(\varkappa\nabla)(\varkappa\nabla)\}v(\rho)\,d\rho =$$

$$= \frac{\xi_0^2}{2\pi k}\int e^{i\varkappa\rho}v(\rho)\{k^2\sin^2\psi\cdot(\varkappa\nabla)+i(\varkappa\nabla)(\varkappa\nabla)\}F_g(\rho)\,d\rho =$$

$$= -\frac{\xi_0^2}{k}\int\frac{(\varkappa q)^2-k^2\sin^2\psi\cdot(\varkappa q)}{\sqrt{q^2+2\varkappa q-k^2\sin^2\psi}}G_g(q)\frac{dq}{q_g^2}. \qquad (51.26)$$

All of these vectors are two-dimensional. On converting to the last expression, we used the integral [14]

$$\int_0^\infty e^{ik\rho}J_0(p\rho)\,d\rho = \lim_{a\to 0}\int_0^\infty e^{ik\rho-a\rho}J_0(p\rho)\,d\rho = \lim_{a\to 0}\frac{1}{\sqrt{p^2+(a-ik)^2}}, \qquad (51.27)$$

where the sign of the radical is to be so selected that

$$|a-ik+\sqrt{p^2+(a-ik)^2}\,|>|p|. \qquad (51.27a)$$

In practice, for $a=0,\ p^2=\varkappa^2=k^2\cos^2\psi$: this means that

$$\sqrt{\varkappa^2-k^2} = -ik\sin\psi.$$

With the usual substitution (51.13), Eq. (51.21) used here reduces to an equation for the attenuation function w(r) that is in agreement with the usual equation (24.6) for a given source above a flat and homogeneous surface with an effective permittivity ε_{tot}. As we know, for a vertical dipole the solution of this equation is the normal attenuation function or Sommerfeld function. The numerical distance in it must be calculated using the parameter $s_{tot} = ik/2\varepsilon_{tot}$ (for greater detail see below). We may proceed even further. If the statistical characteristics of the surface are different in different regions of this surface, the average field over it must be regarded as the field above a plane with an effective permittivity ε_{tot} that has corresponding values in these regions. It must be sought, for example, by the methods of Chapter 7 or §50.

We may, of course, dispense with the limitation expressed by Formulas (51.6) and (51.7) and consisting in the fact that we assumed the axes of statistical anisotropy to be coincident for the geometrical and electrical inhomogeneities. If the principal axes for F_g coincide with x and y and those for F_e coincide with certain axes x_1 and y_1 that are rotated relative to x and y, this will not affect the result: the statistical irregularity and statistical inhomogeneity introduce independent contributions into $1/\sqrt{\epsilon_{tot}}$ and may be calculated independently by Formulas (51.25) and (51.26).

4. Before evaluating ϵ_{tot} by the derived formulas in the various cases, let us consider the limits of applicability of the entire method and those of these formulas. They are determined:

a) by the assumption that the deviations of $\vec{E}^{(1)}$ of the field $\vec{E}$ from the average $\bar{\vec{E}}$ are relatively small, $E^{(1)} \ll \bar{E}$. On the strength of this assumption, we replaced $\overline{\gamma_z \gamma_z E}$ by $\overline{\gamma_z \gamma_z} \cdot \bar{E}$, etc., i.e., we assumed, let us say, that

$$\overline{\gamma_z(r)\gamma_z(r')E_z^{(1)}} \ll \overline{\gamma_z(r)\gamma_z(r')} \cdot \overline{E_z(r')} \tag{51.28a}$$

$$\overline{\xi(r)\xi(r')E_z^{(1)}} \ll \overline{\xi(r)\xi(r')} \cdot \overline{E_z(r)} \tag{51.28b}$$

and so forth;

b) by dropping $\overline{\gamma\xi} \cdot \frac{\partial \mathfrak{H}_z}{\omega s}$, i.e., by the assumption, for example, that

$$\overline{\gamma_z(r)\xi(r')} \cdot \frac{\partial \mathfrak{H}_z(r')}{\omega s} \ll \overline{\gamma_z(r)\gamma_z(r')} \cdot \overline{E_z(r')}. \tag{51.28c}$$

We note that this condition is not mandatory. In [13], the problem is considered without dropping these terms on the same basis as in the present work. However, we shall see that they do not influence the most essential results;

c) finally, by the conditions embodied in the basic expansion of the method of gentle irregularities (49.4):

$$\zeta\left(\frac{\partial^2 E}{\partial z^2}\right)_{z=0} \ll \left(\frac{\partial E}{\partial z}\right)_{z=0} \tag{51.28d}$$

and in the method of approximate boundary conditions (40.8)

$$kl_0\sqrt{\varepsilon} \gg 1 \tag{51.28e}$$

(in all inequalities we are, of course, speaking of absolute values).

Let us examine these limitations.

According to Formulas (49.9), (51.16) and (51.17), considering only the influence of irregularities, we have

$$E_z^{(1)} = -\frac{1}{2\pi}\frac{\partial}{\partial x}\int(\gamma_x E_z - \overline{\gamma_x E_z^{(1)}})\,v\,dS - \frac{1}{2\pi}\frac{\partial}{\partial y}\int(\gamma_y E_z -$$

$$-\overline{\gamma_y E_z})\,v\,dS + O(\gamma E_z^{(1)}) + O\left(\zeta\frac{\partial E_{v,o}}{\partial z}\right).$$

Consequently, leaving, let us say, only x terms

$$\overline{\gamma_x(r)\gamma_x(r')E_z^{(1)}(r')} = -\frac{1}{2\pi}\frac{\partial}{\partial x'}\int\overline{\gamma_x(r)\gamma_x(r')\gamma_x(r'')}\cdot$$

$$\cdot\overline{E_z(r'')}\,v(r'-r'')\,dS'' + O(\gamma^4) + O\left(\gamma^2\zeta\frac{\partial E_{v,o}}{\partial z}\right).$$

The average of the triple product is small provided that at least one of the three distances, $\vec{r} - \vec{r}'$, $\vec{r} - \vec{r}''$, $\vec{r}' - \vec{r}''$ is greater than l. Hence we may assume for estimates that

$$\overline{\gamma_x(r)\gamma_x(r')\gamma_x(r'')} = \gamma_0^3 F\left(\frac{r-r'}{l_g}\right)F_1\left(\frac{r'-r''}{l_g};\ r'\right),$$

where F_1 depends parametrically on $\vec{r}'$ and has properties similar to those of F. Then, substituting Formula (51.20) as usual and $\vec{\rho} = \vec{r}' - \vec{r}''$, $d\vec{\rho} = -d\vec{r}''$, we have

$$\overline{\gamma_x(r)\gamma_x(r')E_z(r')} \sim \gamma_0^3\frac{\partial}{\partial x'}\left\{\overline{E}_z(r')F\left(\frac{r-r'}{l_g}\right)\cdot J\right\},$$

$$J = \frac{1}{2\pi}\int F_1\left(\frac{\rho}{l_g};\ r_1\right)e^{ik\rho(1-\cos\varphi)}\,d\rho \approx$$

$$\approx l_g\begin{cases}\int F_2(x)\,dx & \text{for } kl_g \ll 1,\\ \frac{1}{\sqrt{kl_g}}\int\frac{1}{\sqrt{x}}F_2\,dx & \text{for } kl_g \gg 1,\end{cases}$$

where $F_2(x)$ is a function with the properties of a correlation function; for more detail, see below. In other words,

$$\overline{\gamma_x(r)\gamma_x(r')E_z(r')} \sim \gamma^2 E_z \begin{cases} F & \text{for } kl_g \ll 1, \\ \sqrt{kl_g}\,F & \text{for } kl_g \gg 1. \end{cases}$$

We see that the condition $|\overline{\gamma_x\gamma_x E_z^{(1)}}| < |\overline{\gamma_x\gamma_x}\cdot E_z|$ is always satisfied for $kl_g \ll 1$, but for $kl_g \gg 1$ only provided that

$$\gamma_0\sqrt{kl_g} \sim \frac{\zeta_0}{\sqrt{l_g\lambda}} \ll 1. \qquad (51.28f)$$

This important condition indicates that ζ_0 need not be small by comparison with λ if the gradient angles γ of the surface are small enough.

The admissibility of replacing $\overline{\xi\cdot\xi\cdot E_z}$ by $\overline{\xi\xi}\cdot E_z$ is evaluated in exactly the same way. Since we know that passage from irregularities to inhomogeneities is equivalent to substituting $\frac{1}{\sqrt{\varepsilon}} \sim \xi$ for γ and vice versa (see Formula (49.7)), we may state at once that for $kl_e \ll 1$, Condition (51.28b) is always satisfied, but for $kl_e \gg 1$ only provided that

$$\xi\sqrt{kl_e} \ll 1.$$

Conditions (51.28c) and (51.28d) contain derivatives of $E_{x,y}$ with respect to the vertical. Here it is necessary to distinguish two possibilities. If the glancing angles ψ are very large, the disturbance of the incident field is small, as was noted in §49, Subsection 3. Accordingly, the derivatives of the field component with respect to the vertical are the same as for an undisturbed field, $\frac{\partial E}{\partial z} = ik_z E = ik\sin\psi E$. In particular, if a horizontally polarized wave is incident, then E_z is absent in the space and $\frac{\partial E_x}{\partial z} = ik\sin\psi E_x$ is not expressed in terms of E_z. If, however, the influence of the nonideal soil is strong (glancing incidence), then E_x and E_z are connected by a relationship that depends on ε_{tot}:

$$\bar{E}_z \sim \frac{1}{\sqrt{\varepsilon_{tot}}} \bar{E}_{z0}, \quad \frac{\partial \bar{E}_x}{\partial z} \sim \frac{1}{\sqrt{\varepsilon_{tot}}} \frac{\partial \bar{E}_z}{\partial z} = \frac{-ik}{\sqrt{\varepsilon_{tot}}} \bar{E}_z.$$

In particular, the influence of the original polarization is not felt here.

Noting these remarks, let us consider Condition (51.28c). Replacing $\overline{\gamma_x \gamma_y}$ by $\frac{1}{l_g}\overline{\gamma_x \zeta}$ in this condition, we arrive at the following criteria.

For large ψ (the sense of this concept will be made clear below), $E_x = \sin \psi \, E_z$ for vertical polarization and we have

$$kl_g \sin^2 \psi \ll 1.$$

For horizontal polarization, E_z is absent altogether and the above neglect is inadmissible. The corresponding case is considered in [13].

At small ψ, when the role of surface nonideality is a major one, we should have

$$kl_g \ll |\varepsilon_{tot}|.$$

Finally, when irregularities have little influence, Conditions (51.28d) and (51.28e) have the same sense as before, (49.38a), $k\zeta \sin \psi \ll 1$. In the case of a strong influence, however, a more detailed examination is necessary, and we shall provide it. For this purpose, let us consider the role of the terrain. According to (49.9), we may substitute γ_x for n_x and drop the y-terms in view of the orientational nature of the discussion to obtain

$$E_z = \frac{1}{2\pi} \frac{\partial}{\partial z} \int \gamma_x E_z v \, dS.$$

Substituting E_z from Formula (51.12), we drop higher-order terms; applying (51.20), we take the slowly varying factors out from under the integral sign and substitute the Fourier expansion (51.8a) for ζ. This will give

$$E_z = -\frac{\zeta_0}{2\pi} \bar{E}_z(x, y) \int g(q) \, dq \, iq_x \frac{\partial}{\partial z} \int \frac{dS'}{\rho} e^{i(q-\varkappa, \rho_0) + ik\rho},$$

$$(\rho^2 = \rho_0^2 + z^2).$$

Instead of comparing successive terms of the expansion $\frac{\partial^{k+1}E_z}{\partial z^{k+1}}\zeta$ and $\frac{\partial^k E_z}{\partial z^k}$, it is convenient to compare the terms $\frac{\partial^{k+2}E_z}{\partial z^{k+2}}\zeta^2$ and $\frac{\partial^k E_z}{\partial z^k}$. Addition of another double differentiation with respect to z may, as in (51.20a), be reduced to differentiation with respect to x and y and, after integration by parts, to differentiation with respect to x' and y':

$$\frac{\partial^2}{\partial z^2}\int\frac{dS'}{\rho}e^{i(q-\varkappa,\,\rho_0)+ik\rho} = \int\left(-\frac{\partial^2}{\partial x'^2}-\frac{\partial^2}{\partial y'^2}-\right.$$

$$\left.-k^2\right)e^{i(q-\varkappa,\,\rho_0)}\cdot\frac{e^{ik\rho}}{\rho}dS' = ((q-\varkappa)^2-k^2)\int\frac{dS'}{\rho}e^{i(q-\varkappa,\,\rho_0)+ik\rho}.$$

Thus, the terms compared are distinguished by the addition of an extra multiplier $\zeta_0^2(q^2 - 2k\vec{q})$ in the integral over $\vec{q}$. For order-of-magnitude evaluations, a certain average value of this multiplier may be taken out from the integrand. The last term of the expansion in ζ will therefore be small by comparison with that preceding it if

$$\left|\zeta_0\sqrt{q_g^2 - 2kq_g}\right| \ll 1.$$

To evaluate $\partial^2E_z/\partial z^2$, we apply the same method to the expression for E_z. This will lead to the same result.

In the long-wave case, $k \ll q_g$, the condition obtained yields nothing new, since $\zeta_0 q_g \sim \gamma_0$ and we again have the requirement that the gradient angles be small, $\gamma_0 \ll 1$. For short waves, on the other hand, $k \gg q_g = 2\pi/l_g$, we have

$$2\pi\sqrt{\frac{\zeta}{l_g\lambda}} \ll 1,$$

which agrees with Formula (51.28f).

Thus, the results obtained above are valid for the following conditions:

$$\gamma_0 = \frac{\zeta_0}{l_g} \ll 1; \qquad (51.29a)$$

$$\frac{1}{\sqrt{\varepsilon}} \ll 1; \qquad (51.29b)$$

$$k l_e \sqrt{\varepsilon} \gg 1; \qquad (51.30a)$$

$$\left.\begin{aligned} &\sqrt{k l_g}\,\gamma_0 \ll 1 \left(\text{i.e. } 2\pi \frac{\zeta_0}{\sqrt{l_g \lambda}} \ll 1\right); \\ &\sqrt{k l_e}\,\frac{1}{\sqrt{\varepsilon}} \ll 1; \end{aligned}\right\} \text{if } \begin{aligned} &k l_g \gg 1, \qquad (51.30b) \\ &k l_e \gg 1. \qquad (51.30c) \end{aligned}$$

Moreover, in the event that the influence of irregularities and inhomogeneities is small (large ψ), the conditions

$$k l_g \sin^2 \psi \ll 1; \qquad (51.31a)$$

$$k \zeta \sin \psi \ll 1; \qquad (51.31b)$$

apply for a vertically polarized wave, while if their influence is strong we have the comparatively weak condition

$$k l_g \ll |\varepsilon_{tot}|. \qquad (51.31c)$$

For horizontally polarized incident radiation in the region where departures from ideality have weak influence (large ψ), Inequality (51.28c) may not be satisfied and it is necessary to refine the analysis (see [13]). However, the theory of perturbations may be used here precisely because this effect is small (see below, Subsection 5).

5. Let us examine the effective surface constant more closely. We shall restrict ourselves (introducing this limitation for the first time) to the case of a statistically isotropic surface, $l_{ex} = l_{ey} = l_e$, $l_{gx} = l_{gy} = l_g$. Accordingly, the functions G will depend only on the absolute value of $\vec{q}$, and integration can still be carried out over the angles in Formulss (51.25) and (51.26). Since $(\varkappa\nabla) F(\rho) = \frac{\varkappa\rho dF}{\rho d\rho}$ etc., and, moreover, $(\varkappa\rho \equiv \varkappa\rho \cos\varphi,\ d\rho = \rho d\rho d\varphi)$,

$$\int_0^{2\pi} e^{i\varkappa\rho} d\varphi = 2\pi J_0(\varkappa\rho).$$

$$\int \varkappa\rho e^{i\varkappa\rho}d\varphi = -2\pi i\varkappa\rho J_0'(\varkappa\rho),$$

$$\int_0^{2\pi} (\varkappa\rho)^2 e^{i\varkappa\rho}d\varphi = -2\pi\varkappa^2 J_0''(\varkappa\rho) = 2\pi\varkappa^2\left(J_0(\varkappa\rho) + \frac{1}{\varkappa\rho}J_0'(\varkappa\rho)\right),$$

we may, applying the first of the forms (51.25) and the second form of (51.26), write (in the term containing $d^2F/d\rho^2$, we integrate by parts and apply the general properties of correlation functions (51.11a), $F'(0) = F'(\infty) = 0$)

$$\frac{1}{\sqrt{\varepsilon_e}} = \frac{i\xi_0^2}{\cos\psi}\int_0^\infty e^{ik\rho}J_1(\varkappa\rho)(1 - ik\rho)F_e(\rho)\frac{d\rho}{\rho}; \qquad (51.25a)$$

$$\frac{1}{\sqrt{\varepsilon_g}} = \frac{i\varkappa\xi_0^2}{k}\int_0^\infty e^{ik\rho}\left\{(1 - ik\rho)J_0(\varkappa\rho) + \left(\frac{k^2\rho}{\varkappa} + \frac{ik}{\varkappa} - \frac{1}{\varkappa\rho}\right)J_1(\varkappa\rho)\right\}\frac{dF_g}{d\rho}\frac{d\rho}{\rho}. \qquad (51.26a)$$

These expressions may be regarded as definitive for an isotropic surface. Let us consider the limiting cases.

The electrical inhomogeneity makes a statistical contribution to the effective soil constant $1/\sqrt{\varepsilon_{tot}}$ that is, according to Formula (51.25a), of the second order in ξ_0. If we set

$$\frac{1}{\sqrt{\varepsilon(x, y)}} = \eta' - i\eta'', \quad \eta' = \eta_0' + \xi', \; \eta'' = \eta_0'' + \xi'',$$

then since η' and η'' are positive and ξ' and ξ'' fluctuate uniformly in both directions, it is obvious that $|\xi'| < \eta_0'$, $|\xi''| < \eta_0''$. Therefore

$$\xi_0^2 = |\xi'^2 + \xi''^2| < |\eta_0|^2 = \left|\frac{1}{\sqrt{\varepsilon}}\right|^2. \qquad (51.25b)$$

Consequently, $1/\sqrt{\varepsilon_e}$ is a quantity of at least the second order of smallness with respect to $1/\sqrt{\varepsilon(r)}$.

In the case of small inhomogeneities, $kl_e \ll 1$, we may at once set $k\rho \ll 1$, $J_0(\varkappa\rho) = 1$, $J_1(\varkappa\rho) = \frac{1}{2}\varkappa\rho$, $\exp(ik\rho) = 1$ in Formula (51.25a). Indeed, by virtue of the factor $F_e(\rho)$, the range of integration is effectively limited to the interval $0 < \rho \leq l_e$. Introducing the dimensionless variable $x = \rho/l_e$, on which, accordingly, F_e also depends, we obtain

$$\frac{1}{\sqrt{\varepsilon_e}} = -\frac{i\varkappa_0^2}{\cos\psi}\int_0^\infty F_e(x)\frac{dx}{x}, \quad (kl_e \ll 1). \tag{51.32a}$$

In the converse case of large inhomogeneities, $kl_e \gg 1$, we consider first the region of angles so small that

$$kl_e(1-\cos\psi) = 2kl_e\sin^2\frac{\psi}{2} \ll 1. \tag{51.32b}$$

Under these conditions, the product $\exp(ik\rho)J_n(\varkappa\rho)$ contains a slowly varying part and very large distances, $k\rho \sim kl_e \gg 1$, become important. Hence, replacing the Bessel functions by their asymptotic expressions,

$$J_0(\varkappa\rho) \sim \sqrt{\frac{2}{\pi\varkappa\rho}}\left[\cos\left(\varkappa\rho - \frac{\pi}{4}\right) - \frac{1}{8\varkappa\rho}\sin\left(\varkappa\rho - \frac{\pi}{4}\right)\right],$$
$$J_1(\varkappa\rho) \sim \sqrt{\frac{2}{\pi\varkappa\rho}}\left[\cos\left(\varkappa\rho - \frac{3\pi}{4}\right) - \frac{3}{8\varkappa\rho}\sin\left(\varkappa\rho - \frac{3\pi}{4}\right)\right], \tag{51.32c}$$

converting from trigonometric to exponential functions, dropping the $\exp(i\varkappa\rho)$ terms, which, on multiplication by $\exp(ik\rho)$, give a rapidly oscillating integrand, and substituting $\exp(i(k-\varkappa)\rho) = \exp(ik(1-\cos\psi))$, and 1 for $\cos\psi$, we obtain (again assuming $\vec{\rho} = xl_e$)

$$\frac{1}{\sqrt{\varepsilon_e}} = e^{i\frac{3\pi}{4}}\frac{\varkappa_0^2\sqrt{kl_e}}{\sqrt{2\pi}}\int_0^\infty \frac{dx}{\sqrt{x}}F_e(x), \quad kl_e \gg 1, \quad kl_e(1-\cos\psi) \ll 1. \tag{51.32d}$$

It is easily seen, however, that both limit formulas (51.32a) and (51.32d) make a negligibly small contribution to ε_{tot} – one that lies beyond the limits of accuracy of the discussion. Indeed, if we apply (51.25b), we see that, according to Formula (51.32a), $\frac{1}{\sqrt{\varepsilon_e}} \leq \left(\frac{1}{\sqrt{\bar{\varepsilon}}}\right)^3 \ll \frac{1}{\sqrt{\bar{\varepsilon}}}$, while according to (51.30c), the same also applies for Formula (51.32d), i.e., for $kl_e \gg 1$. This relieves us of the necessity of considering the region of even larger angles (when ε_e can only be less important) and, in Formula (51.24), enables us to disregard the term $1/\sqrt{\varepsilon_e}$, which is small by comparison with $\overline{1/\sqrt{\varepsilon^0}}$.

Let us now consider the influence of roughness.

For $kl_g \ll 1$ – small irregularities, long waves – we may again assume $k\rho \ll 1$ and $J_0(\varkappa\rho) \approx \exp(ik\rho) \approx 1$, so that Formula (51.26a) gives

$$\frac{1}{\sqrt{\varepsilon_g}} = \frac{ik\zeta_0^2}{2l_g}\cos^2\psi\int_0^\infty \frac{1}{x}\frac{dF_g}{dx}dx, \quad kl_g \ll 1. \qquad (51.33a)$$

On the other hand, for $kl_g \gg 1$, in the range of small angles, (51.32b), large ρ_0 again assume importance. Substituting Formulas (51.32c) and dropping terms containing rapidly oscillating multipliers, we obtain

$$\frac{1}{\sqrt{\varepsilon_g}} = e^{i\frac{3\pi}{4}}\frac{\zeta_0^2\sqrt{kl_g}}{2\sqrt{2\pi}\,l_g^2}\int_0^\infty \frac{dx}{\sqrt{x^1}}\frac{dF_g(x)}{dx}, \quad 1 \ll kl_g. \qquad (51.33b)$$

These quantities are, of course, small by comparison with unity: (51.33a) by virtue of Formula (51.29a) and (51.33b) by virtue of Formulas (51.29a) and (51.30b). However, they may not be small by comparison with $1/\sqrt{\varepsilon}$, and for this reason they must, generally speaking, be taken into account. In particular, for skipping propagation over an ideally conductive rough surface (in general, if $\frac{1}{\sqrt{\varepsilon_g}} > \frac{1}{\sqrt{\varepsilon}}$), even a small $1/\sqrt{\varepsilon_g}$ may give rise to any amount of average-field attenuation at long enough ranges.

The region of larger angles, $kl_g(1 - \cos\psi) \geq 1$, with $kl_g \gg 1$, is of no essential interest. As we know, if $|\sqrt{\varepsilon_g}\,\sin\psi| > 1$, the distortion of the field due to irregularities must be vanishingly small (see Formula (26.29)). But even at the boundary of this range of angles, $\sin\psi = 2\frac{1}{\sqrt{2kl_g}}$, Formula (51.33b) gives

$$\sqrt{\varepsilon_g}\sin\psi \sim \frac{l_g^2}{\zeta_0^2\sqrt{kl_g}}\frac{1}{\sqrt{kl_g}} = \left(\frac{1}{\gamma_0\sqrt{kl_g}}\right)^2.$$

According to Formula (51.30b), this is a very large quantity within the range of applicability of the method. Consequently, even here the average field differs little from the field above an ideal surface.

Hence it can be considered here by the methods of the perturbation theory.

In summary, we may state that

$$\frac{1}{\sqrt{\varepsilon_{eff}}}=\overline{\frac{1}{\sqrt{\varepsilon}}}+\frac{1}{\sqrt{\varepsilon_g}}. \qquad (51.34)$$

where ε_g is given by Formulas (51.33a) and (51.33b) and Formula (51.33b) can be used without constraints on the values of ψ, since the influence of ε_g vanishes for $2kl_g \sin^2\frac{\psi}{2} \gg 1$.

We note that in averaging the electrical properties, it is not ε, but $1/\sqrt{\varepsilon}$ that is the subject of this averaging. This means, for example, that if $4\pi\sigma \gg \varepsilon'\omega$, we are averaging $1/\sqrt{\sigma}$. For uniform fluctuations of σ through a factor of m between σ_1 and $\sigma_2 = m\sigma_1$, the effective value is

$$\sigma_{eff}=\frac{4\sigma_1}{\left(1+\frac{1}{\sqrt{m}}\right)^2}. \qquad (51.34a)$$

Consequently, if the conductivity differences are very great ($m \gg 1$), σ_{eff} is approximately four times larger than for the segments with poor conductivity.

The effective constants that we have found are somewhat unusual in nature.

For $kl_g \ll 1$, we have $\arg 1/\sqrt{\varepsilon_g} = \pi/2$ (51.33a), and, for $kl_g \gg 1$, according to Formula (51.33b), $\arg 1/\sqrt{\varepsilon_g} = 3\pi/4$, while this quantity is negative for the soil. We note that in the case of small steep irregularities (48.23), we had $\arg\sqrt{\varepsilon_g^0} = -\frac{\pi}{2}$. But the effect found for steep irregularities need not, after all, coincide with the effect for gentle irregularities. In order to pass from the results of §48 to Formula (51.33a), we should consider very flat ellipsoids, $a \sim \zeta_0 \gg b \sim c$. But then as $a \to 0$, according to Formula (48.24c), we would obtain $1/\sqrt{\varepsilon_g^0} \sim T \sim a \to 0$. Indeed, in the case of shallow irregularities, the effect remained only in terms of the second order in ζ_0. As was shown in §25, Subsection 1, Formula (25.18a) is for any ε a solution of the equation of $w(r)$ that satisfies Condition (25.1) as $r = D \to 0$ (this condition is a corollary of the fact that for small r, Eq. (25.1) can be solved by iteration). However, in determining the constant C in Formula (25.16), we now take into account the fact that $\arg \sqrt{s} = \pi/2 + \arg 1/\sqrt{\varepsilon}$, to obtain the value of $\arg \sqrt{s_g} = 3\pi/4$ for $kl_g \ll 1$ and $\arg \sqrt{s_g} = \pi$ for $kl_g \gg 1$. Hence in the case of ideally conductive soil, when $s_{tot} = s_g$, by Formula (25.16c) and the fact that $\arg \sqrt{C}$ may differ from the value indicated in it by $\pm\pi/4$, we obtain

$$
\begin{aligned}
&\text{for } kl_g \ll 1: && -\frac{\pi}{2} < \arg \sqrt{C} < 0, \\
&\text{for } kl_g \gg 1: && -\frac{3\pi}{4} < \arg \sqrt{C} < -\frac{\pi}{4}.
\end{aligned} \tag{51.35}
$$

Hence we may set $\sqrt{C} = -i\infty$. For the rest, however, Solution (25.15) is retained.

It follows from this, for one thing, that the Leontovich condition

$$\frac{\partial \bar{E}_z}{\partial z} = -\frac{ik}{\sqrt{\varepsilon_{tot}}} \bar{E}_z \text{ for } z = 0 \tag{51.36}$$

applies for the attenuation function of the average field and hence also for the average field E_z itself on the plane $z = 0$.

The effective parameters of an uneven and inhomogeneous surface can also be obtained directly by averaging the approximate boundary condition (40.9) [13].

As we have already noted, the formulas given above for ε_{tot} are valid if we disregard the quantity $\partial E_x/\partial z$ (see Formula (51.28c)). This neglect is not necessary, and if we use the boundary conditions (21.28a) and (21.28c) in the process of deriving Eq. (51.21), we may not only improve the expressions for ε_g so that they become applicable for larger ψ (where, admittedly, ε_g is now very large, so that the importance of irregularities is minor), but also – and this is more important – obtain ε_g for a horizontally polarized source field [13]. That is to say, an additional factor $(1 - \tan^2\psi)$ appears in Formula (51.33a), and if we examine the region $\sqrt{kl_g}\sin\psi \ll 1$ specifically for $kl_g \gg 1$, then $1/\sqrt{\varepsilon_g} = k^2\zeta_0^2\sin^3\psi$ (we have not discussed this region previously, since $1/\sqrt{\varepsilon_g}$ is very small here). It is found for horizontal polarization that

$$\frac{1}{\sqrt{\varepsilon_g}} = -\frac{ik\zeta_0^2}{2l_g}\int_0^\infty \frac{1}{x}\frac{dF_\rho(x)}{dx}dx, \quad kl_g \ll 1; \qquad (51.37a)$$

$$\frac{1}{\sqrt{\varepsilon_g}} = -e^{i\frac{\pi}{4}}\frac{2k\zeta_0^2}{\sqrt{2\pi kl_g}}\int_0^\infty \frac{1}{\sqrt{x}}\frac{dF_\rho(x)}{dx}dx, \quad kl_g \gg 1, \quad \sqrt{kl_g}\sin\psi \ll 1; \qquad (51.37b)$$

$$\frac{1}{\sqrt{\varepsilon_g}} = k^2\zeta_0^2\sin\psi, \quad kl_g \gg 1, \quad \sqrt{kl_g}\sin\psi \gg 1. \qquad (51.37c)$$

6. Up to this point, we have been interested only in the average field. We satisfied ourselves in particular that it is described in the space above the surface by the same formulas as apply above a flat and homogeneous surface with certain effective characteristics (that depend on the polarization of the field and the glancing angle). Thus, for sufficient heights of the corresponding points, i.e., if

$$k(z_A + z_0) \gg |\sqrt{\varepsilon_{tot}}|,$$

the average field is described by the interference (reflection) formulas and is therefore nonzero only in the direction of mirror reflection from the plane z = 0. However, we know that irregular scattering must also take place in other directions (precisely because of this scattering, the average field dies out with propagation over an ideally conductive irregular surface). Hence the average values of the root-mean-square field quantities, for example, the energy flux vector and the correlation functions of field strength, do not, generally speaking, vanish in any other direction either. In view of this, we pass to consideration of the root-mean-square quantities. Here we shall limit ourselves to the following formulation of the problem: an ideally conductive irregular scattering surface has the dimension $S \sim b^2$, where b is small by comparison with the distances R_0 and ρ_A from the source and the observation point to a certain central point of the surface, which is taken as the coordinate origin $x = 0$, $y = 0$ in the averaging plane $z = 0$ (although, of course, it is large by comparison with both λ and l_g). The source is situated in the plane $y = 0$ and the glancing angle of the incident ray ψ and the related incidence angle $\vartheta = \pi/2 - \psi$ may be regarded as approximately constant for the entire segment S, since $b \ll R_0$. Similarly, the line from the center point of the scattering segment to the observation point A forms a glancing angle of χ with the $z = 0$ plane, and the azimuth with respect to the x-axis is φ (see Fig. 51.1).

We introduce two unit vectors for the propagation directions of the incident and scattered waves:

$$\left.\begin{aligned} \boldsymbol{\alpha}(\alpha_x, \alpha_y, \alpha_z) &= \boldsymbol{\alpha}(\cos\psi, 0, -\sin\psi) = \boldsymbol{\alpha}\left(\frac{\partial R}{\partial x}, 0, \frac{\partial R}{\partial z}\right); \\ \boldsymbol{\beta}(\beta_x, \beta_y, \beta_z) &= \boldsymbol{\beta}(\cos\chi\cos\varphi, \cos\chi\sin\varphi, \sin\chi) = \boldsymbol{\beta}\left(\frac{\partial\rho}{\partial x}, \frac{\partial\rho}{\partial y}, \frac{\partial\rho}{\partial z}\right). \end{aligned}\right\} \quad (51.38)$$

We shall also use the two-dimensional vectors

$$\boldsymbol{\alpha}_0(\alpha_x, \alpha_y) = \boldsymbol{\alpha}_0\left(\frac{\partial R}{\partial x}, 0\right); \quad \boldsymbol{\beta}_0(\beta_x, \beta_y) = \boldsymbol{\beta}_0\left(\frac{\partial\rho}{\partial x}, \frac{\partial\rho}{\partial y}\right). \qquad (51.38a)$$

In the plane $z = 0$, within the limits of the segment S, each field component may be presented in the form (51.12)

$$E_i = \overline{E}_i\left(1 + \sum_k a_{ik} f_k^{(i)}\right).$$

Here, as we know, we may write

$$\overline{E}_i = E_i^{(0)} \cdot f(\psi),$$

where $E_1^{(0)}$ is the undisturbed incident field; f is the corresponding plane-wave reflection coefficient (for an effective permittivity ε_{tot}).

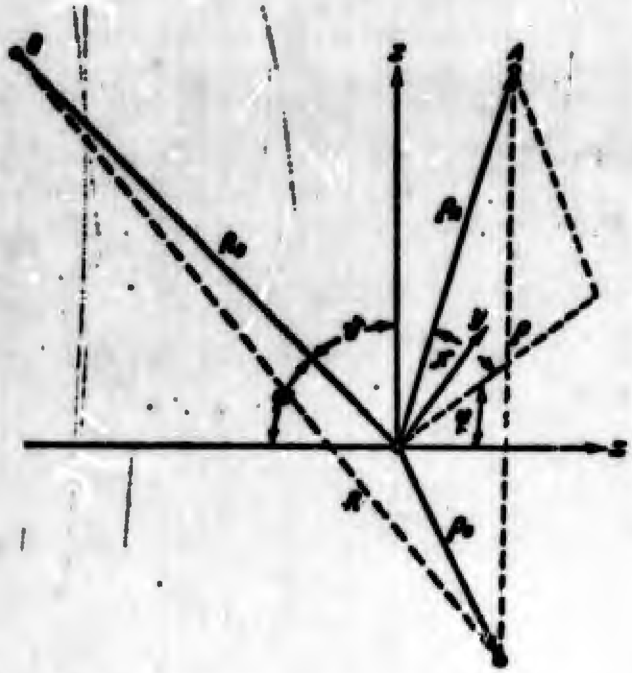

Fig. 51.1. Nomenclature for calculating scattering by a gently sloped rough surface by the perturbation method.

Assuming that the angles ψ and χ are large enough, we may disregard the difference between f and unity (this is a nonessential neglect: f appears as a common factor in the final result). Hence we are speaking essentially of <u>calculation of the field by the perturbation theory</u>, as developed in §49. This enables us to obtain the result easily. In the subsequent exposition, we shall follow [15]. We can use the corresponding formulas for the field $E_1^{(1)}$, which coincides with $E_i \cdot \sum_h a_h f_h^{(1)}$, (49.31)-(49.36). We also take into account the fact that for points of the integration within S

$$\frac{e^{ik\rho}}{\rho} \approx \frac{e^{ik\rho_A}}{\rho_A} e^{-ik\beta_0 r}, \quad r = (x, y),$$

$$\frac{e^{ikR}}{R} \approx \frac{e^{ikR_0}}{R_0} e^{ik\alpha_0 r}. \qquad (51.39)$$

We consider further that

$$\frac{\partial}{\partial x}\frac{e^{ik\rho}}{\rho}\approx ik\frac{\partial\rho}{\partial x}\frac{e^{ik\rho}}{\rho}=-ik\frac{e^{ik\rho}}{\rho}\beta_x.$$

For the fields of a vertical dipole at O, therefore, we have, according to Formulas (49.33), after taking the weakly varying factors out from under the integral sign:

$$E_x^{(1)}=\frac{p}{\pi}k^2\frac{e^{ik(R_0+\rho_A)}}{R_0\rho_A}\alpha_z ik\beta_x\int\left(-ik\zeta(1-\alpha_z^2)+\alpha_x\gamma_x\right)e^{ik(\alpha_0-\beta_0,\,r)}dr, \quad (51.40a)$$

$$E_y^{(1)}=\frac{p}{\pi}k^2\frac{e^{ik(R_0+\rho_A)}}{R_0\rho_A}\alpha_z^2 ik\beta_y\int\gamma_y e^{ik(\alpha_0-\beta_0,\,r)}dr, \quad (51.40b)$$

$$E_z^{(1)}=\frac{p}{\pi}k^2\frac{e^{ik(R_0+\rho_A)}}{R_0\rho_A}\int\left\{ik(1-2\alpha_z^2)\gamma_x-\left(\frac{\partial\gamma_x}{\partial x}+\frac{\partial\gamma_y}{\partial y}\right)\alpha_z-k^2\zeta\alpha_z(1-\alpha_z^2)\right\}\times$$
$$\times e^{ik(\alpha_0-\beta_0,\,r)}dr.$$

First we calculate the correlation functions, for example,

$$S_{yy}=\overline{E_y^{(1)}E_y^{(1)*}}=\frac{p^2k^6}{\pi^2R_0^2\rho_A^2}\alpha_z^4\beta_y^2 I,$$

where, according to Formulas (51.8a) and (51.9),

$$I=\int\overline{\gamma_y(r)\gamma_y(r')}\,e^{ik(\alpha_0-\beta_0,\,r-r')}dr\,dr'=$$
$$=\int G_g(q)\frac{dq}{q_g^2}\zeta_0^2q_y^2\int e^{i(k(\alpha_0-\beta_0)-q,\,r'-r)}dr'\,dr.$$

We substitute $r'-r=\rho,\ dr'dr=d\rho dr$ and remember that the area of integration is S. We obtain (using the δ-function, $2\pi\delta(x)=\int_{-\infty}^{+\infty}e^{i\xi x}d\xi$)

$$I=\frac{\zeta_0^2S}{q_g^2}\int G_g(q)\,q_y^2\,dq\,(2\pi)^2\,\delta(k\alpha_x-k\beta_x-q_x)\,\delta(k\alpha_y-k\beta_y-q_y)=$$
$$=\frac{4\pi^2\zeta_0^2S}{q_g^2}k^2(\alpha_y-\beta_y)^2G_g(k(\alpha_0-\beta_0)). \quad 51.41)$$

Here $G_g(k(\vec{\alpha}_0-\vec{\beta}_0))$ is the Fourier component of the correlation function for the elevations ζ, taken for $\vec{q}=k(\vec{\alpha}_0-\vec{\beta}_0)$.

Thus,

$$S_{yy}=Q\beta_x^2\beta_y^2\alpha_z^4G_g(k(\alpha_0-\beta_0)).$$

$$Q = \frac{4p^2k^6\zeta_0^2 S}{R_O^2\rho_A^2 q_z^2} = \frac{4p^2k^6 S}{R_O^2\rho_A^2}\left(\frac{l_g\zeta_0}{2\pi}\right)^2. \qquad (51.42)$$

If we introduce the expression for the flux of incident energy Σ_0 onto the surface S for a <u>vertical dipole</u>, which is equal to

$$\Sigma_0 = \Sigma_{0\parallel} = \frac{cp^2k^4}{8\pi R_0^2}\cos^2\psi,$$

we may assume accordingly

$$Q = Q_\parallel = \frac{8}{\pi\cos^2\psi}\frac{S\Sigma_{0\parallel}}{c\rho_A^2}\left(\frac{l_g\zeta_0}{\lambda^2}\right)^2. \qquad (51.42a)$$

For a <u>horizontal dipole</u> pointed along the y-axis

$$\Sigma_0 = \Sigma_{0\perp} = \frac{cp^2k^4}{8\pi R_0^2},\ Q = Q_\perp = \frac{8}{\pi}\frac{S\Sigma_{0\perp}}{c\rho_A^2}\left(\frac{l_g\zeta_0}{\lambda^2}\right)^2. \qquad (51.42b)$$

The integration of (51.41) was particularly characteristic in this calculation. It indicates that components with $\vec{q} = k(\vec{\alpha}_0 - \vec{\beta}_0)$, where $\vec{\alpha}_0$ and $\vec{\beta}_0$ characterize the directions of incidence and scattering, have been picked out of the Fourier expansion of the surface-irregularity correlation function. The same integral is encountered with other degrees of q_x and q_y in calculating the other field correlation functions. Hence even before the calculations we may set in Formulas (49.33)-(49.36)

$$\gamma_y(r') \to iq_y\zeta \to ik(\alpha_y - \beta_y)\zeta,$$

$$\frac{\partial\gamma_x(r')}{\partial x'} \to (iq_x)^2\zeta \to -k^2(\alpha_x - \beta_x)^2\zeta$$

and so forth, i.e., we may write for a vertical dipole

$$E_x^{(1)} \approx Aik\beta_z\alpha_z(-ik(1-\alpha_z^2) + ik\alpha_x(\alpha_x - \beta_x))J,$$

$$E_y^{(1)} \approx Aik\beta_z\alpha_z^2 ik(\alpha_y - \beta_y)J,$$

$$E_z^{(1)} \approx A(ik(1-2\alpha_z^2)ik(\alpha_x - \beta_x) + \alpha_x\cdot ik^2[(\alpha_x - \beta_x)^2 + (\alpha_y - \beta_y)^2] -$$

$$-k^2\alpha_z(1-\alpha_z^2))J; \qquad (51.43)$$

$$J = \int\zeta e^{ik(\alpha_0 - \beta_0,\, r)}dr,\ A = \frac{p}{\pi}k^2\frac{e^{ik(R_O+\rho_A)}}{R_O\rho_A}. \qquad (51.43a)$$

We obtain as a result of these calculations [15]

$$\mathcal{E}_{ik} = \overline{E_i^{(1)}E_k^{(1)*}} = QG_\varepsilon(k(\alpha_0 - \beta))\,\Phi_{ik}, \tag{51.44}$$

where:

a) for a <u>vertical</u> dipole

$$\left.\begin{aligned}
\Phi_{xx} &= \beta_z^2(1-\alpha_x\beta_x)^2\alpha_z^2,\\
\Phi_{yy} &= \beta_z^2\beta_y^2\alpha_z^4,\\
\Phi_{zz} &= \alpha_z^2[\beta_z - \alpha_z(\beta_x^2+\beta_y^2)]^2,\\
\Phi_{xy} &= -\beta_z^2(1-\alpha_x\beta_x)\alpha_z^2\beta_y,\\
\Phi_{yz} &= \beta_y[\beta_z - \alpha_z(\beta_x^2+\beta_y^2)]\alpha_z^2\beta_y,\\
\Phi_{zx} &= -\beta_z(1-\alpha_x\beta_x)\alpha_z[\beta_z - \alpha_z(\beta_x^2+\beta_y^2)];
\end{aligned}\right\} \tag{51.45}$$

b) for a <u>horizontal dipole</u> oriented along the y-axis,

$$\left.\begin{aligned}
\Phi_{xx} &= \Phi_{xy} = \Phi_{xz} = 0,\\
\Phi_{yy} &= \beta_z^2(1-\alpha_y^2),\\
\Phi_{zz} &= \beta_y^2(1-\alpha_y^2),\\
\Phi_{yz} &= -\beta_y\beta_z(1-\alpha_y^2).
\end{aligned}\right\} \tag{51.46}$$

A similar calculation gives the scattered-energy flux vector $\vec{W}$, whose components are equal to (after averaging over time, which gives a factor 1/2)

$$W_i = \frac{c}{8\pi}\sum_k[\beta_i\mathcal{E}_{kk} - \beta_k\mathcal{E}_{ik}]. \tag{51.47}$$

Hence

$$W = \frac{c}{8\pi}QG_\varepsilon(k(\alpha_0-\beta))\,B\beta, \tag{51.48}$$

where

$$B = \begin{cases} (\beta_y^2+\beta_z^2)(1-\alpha_y^2) \\ \text{for a horizontal dipole oriented on the y-axis;} \\ (\beta_z^2+\beta_y^2)(1-\alpha_x\beta_x)^2\alpha_z^2 - 2\beta_z\beta_y^2\alpha_z^3(1-\alpha_x\beta_x) + (\beta_y^2+\beta_z^2)\beta_y^2\alpha_z^4 \\ \qquad \text{for a vertical dipole.} \end{cases} \tag{51.48a}$$

It is convenient to express the scattering in the form of an effective differential section $d\sigma$, which has already been used in §48. The scattered-energy flux in a solid angle $d\Omega$ is $W\rho_A^2 d\Omega$. Dividing it by Σ_0, we obtain, according to Formula (51.42), for the radiation of a vertical dipole

$$d\sigma^{(v)} = \frac{1}{\cos^2\psi} B(\psi;\ \chi,\ \varphi)\, d\sigma_0, \qquad (51.48b)$$

where

$$d\sigma_0 = \frac{1}{\pi^2} G_\varepsilon(k(\alpha_0 - \beta_0))\left(\frac{l_\varepsilon \xi_0}{\lambda^2}\right)^2 S\, d\Omega. \qquad (51.48c)$$

The other sections will also be expressed in terms of this quantity. For the radiation of a horizontal dipole, the factor $\cos^2\psi$ is absent from the denominator.

This quantity combines energy fluxes with different polarizations. We may (and this is a matter of particular interest) write the scattering sections separately for different polarizations. Thus, for example, the χ-component of the scattered field is expressed in terms of the cartesian components as follows ($\vec{\chi}_I$ is the unit vector in the direction of increasing χ):

$$E_\chi^{(1)} = E_x^{(1)}\chi_{1x} + E_y^{(1)}\chi_{1y} + E_z^{(1)}\chi_{1z},\quad \chi_1 = \chi_1(-\sin\chi\cos\varphi,\ -\sin\chi\sin\varphi,\ \cos\chi),$$

and the radiation flux polarized parallel to the plane of observation (vertical polarization) is, according to Formula (51.44), equal to

$$W_\parallel = \frac{c}{8\pi}\overline{E_\chi^{(1)}E_\chi^{(1)*}} = \frac{c}{8\pi} Q G_\varepsilon(k(\alpha_0 - \beta_0)) \times$$
$$\times (\chi_{1x}^2\Phi_{xx} + \chi_{1y}^2\Phi_{yy} + \chi_{1z}^2\Phi_{zz} + 2\chi_{1x}\chi_{1y}\Phi_{xy} + 2\chi_{1x}\chi_{1z}\Phi_{xz} + 2\chi_{1y}\chi_{1z}\Phi_{yz}).$$

Here we have for the scattered field of a horizontal dipole oriented along the y-axis, according to Formulas (51.46), $(\,) = \sin^2\varphi(1 - \alpha_x^2)$. Multiplying by $\rho_A^2 d\Omega$ and dividing by Σ_{01} (51.42b), we obtain the section for scattering into the component polarized in the plane of observation:

$$d\sigma_\parallel^{(h)} = \sin^2\varphi(1 - \alpha_x^2)\, d\sigma_0. \qquad (51.49)$$

If we substract this quantity from the total section (51.48a) for $B = (\beta_y^2 + \beta_z^2)(1 - \alpha_x^2)$ (and drop $\cos^2\psi$ in the denominator), we obtain the section for scattering into the component polarized perpendicular to the observation plane:

$$d\sigma_{\perp}^{(h)} = \sin^2\psi \sin^2\chi \cos^2\varphi\, d\sigma_{\varphi}. \qquad (51.49a)$$

Formula (51.49) gives a measure of the depolarization of the scattered radiation. In practice, however, it is often necessary to know the scattering cross section corresponding to reception at a vertical antenna, i.e., to know the energy flux transferred by the z-component (and not by the x-component) of the scattered field $E_g^{(1)}$. This flux is $W_z = \frac{c}{8\pi} E_z^{(1)} E_z^{(1)*}$, which, according to Formulas (51.46), gives

$$d\sigma_z^{(h)} = \sin^2\psi \cos^2\chi \sin^2\varphi\, d\sigma_{\varphi}. \qquad (51.49b)$$

According to the reciprocity theorem, we can obtain the scattering cross section from the above formula for the radiation of a <u>vertical dipole</u> if the <u>horizontally polarized</u> (perpendicular to the observation plane) component is observed. It is only necessary to substitute χ for ψ (and vice versa), i.e., to replace $\beta_y(1-\alpha_z^2) = \cos^2\chi \sin^2\psi \sin^2\varphi$ by $\sin^2\chi \cos^2\psi \sin^2\varphi$ and remember that the section is obtained by division not by $\Sigma_{0\perp}$, but by $\Sigma_{0\|}$, which contains $\cos^2\psi$. Consequently,

$$d\sigma_{\perp}^{(v)} = \sin^2\chi \sin^2\varphi\, d\sigma_{\varphi}. \qquad (51.49c)$$

For the z-component of the field of a vertical dipole (both reception and transmission on a vertical antenna), we obtain according to Formulas (51.42a) and (51.45)

$$d\sigma_z^{(v)} = \cos^2\chi\, [\cos\varphi - \cos\psi \cos\chi]^2\, d\sigma_{\varphi}. \qquad (51.49d)$$

Let us consider the values of G_g in the limiting cases. a) For <u>short waves</u>, $|k(\alpha_0-\beta_0)|l_g \gg 1$, i.e., if $\psi \sim 1$, $kl_g \gg 1$, the correlation function is a relatively slowly varying one. Hence in Expression (51.9) for G_g taken for $\vec{q} = k(\vec{\alpha}_0 - \vec{\beta}_0)$, we may substitute F_g by its value at $\vec{\rho} = 0$ and take it out from under the integral sign. In the integral that remains, we extend the limits to infinity and, using the property of the δ-function, $\delta(ax) = \frac{1}{a}\delta(x)$, we obtain

$$G_\zeta(k(\alpha_0-\beta_0)) = q_\zeta^2\delta(k(\alpha_x-\beta_x))\,\delta(k(\alpha_y-\beta_y)) = \frac{q_\zeta^2}{k^2}\delta(\alpha_0-\beta_0). \quad (51.50)$$

We see that for short enough wavelengths, the scattered radiation is propagated only in the direction of regular reflection from the plane $z = 0$, $\vec{\alpha}_0 = \vec{\beta}_0$.

b) In the long-wave case, $kl_g|\vec{\alpha}_0 - \vec{\beta}_0| \ll 1$, on the other hand, we may disregard the factor $\exp(-i\vec{q}\vec{\rho})$ in Formulas (51.9). Then

$$G(k(\alpha_0-\beta_0)) = \frac{1}{q_\zeta^2}\int\frac{\overline{\zeta(\rho)\zeta(\rho+\rho')}}{\zeta_0^2}d\rho' = \text{const}. \quad (51.51)$$

Thus, the factor G does not depend on the angles at all; angular directivity is determined solely by the smoothly varying quantities Φ_{ik} and B.

7. By way of example, let us consider back scattering – the field observed at the point at which the transmitter is located ("radar observation" or "reverberation"). In this case, $\chi = \psi$, $\varphi = \pi$, so that

$$\alpha = \alpha(\cos\psi, 0, -\sin\psi),\ \beta = \beta(-\cos\psi, 0, \sin\psi). \quad (51.52)$$

i.e.,

$$\alpha = -\beta.$$

Consequently, according to Formulas (51.45) and (51.47):

a) for a vertical dipole

$$\left.\begin{aligned}
\Phi_{xx} &= \sin^2\psi\cos^2\psi(1+\cos^2\psi)^2 = \frac{1}{4}\sin^2 2\psi(1+\cos^2\psi)^2,\\
\Phi_{yy} &= \Phi_{xy} = \Phi_{yz} = 0,\\
\Phi_{zz} &= \cos^4\psi(1+\cos^2\psi)^2,\\
\Phi_{xz} &= \sin\psi\cos^3\psi(1+\cos^2\psi),\\
B &= (1+\cos^2\psi)^2\cos^2\psi;
\end{aligned}\right\} \quad (51.53)$$

b) for a horizontal dipole directed along the y-axis, according to Formulas (51.46) and (51.47),

$$\begin{aligned}
\Phi_{xx} &= \Phi_{xy} = \Phi_{zz} = \Phi_{xz} = \Phi_{yz} = 0,\\
\Phi_{yy} &= \sin^4\psi,\\
B &= \sin^4\psi.
\end{aligned} \quad (51.54)$$

Further, in the case of short waves ($kl_g \gg 1$), the scattered field at the observation point in this approximation vanishes, according to Formula (51.50), for all angles ψ except for the case of vertical incidence $\psi = \pi/2$ (since $\delta(\alpha_x - \beta_x) \to \delta(2\cos\psi) = 0$, if $\cos\psi \neq 0$). But this corresponds to the direction of mirror reflection from a plane and is not of interest to us at the moment. In the case of <u>long waves</u>, on the other hand, according to Formula (51.51), the quantity G_g is a constant of the order of unity, which we denote by G_g^0. Formulas (51.49)-(51.49c) give for this case:

a) the radiation of a <u>horizontal dipole</u> directed along the y-axis. Reception at a horizontal antenna (51.49a)

$$d\sigma_{\perp}^{(h)} = \frac{1}{\pi^4} \sin^4\psi \, G_g^0 \left(\frac{l_g \zeta_0}{\lambda^2}\right)^2 S \, d\Omega. \tag{51.55}$$

Reception at a vertical antenna (51.49b)

$$d\sigma_{g}^{(h)} = 0; \tag{51.55a}$$

b) the radiation of a <u>vertical dipole</u>. Reception at a horizontal (perpendicular to the propagation plane) antenna (51.49c)

$$d\sigma_{\perp}^{(v)} = 0. \tag{51.56}$$

Rece, tion at a vertical antenna (51.49d)

$$d\sigma_{g}^{(v)} = \frac{1}{\pi^3} \cos^2\psi \, (1 + \cos^2\psi)^2 \, G_g^0 \left(\frac{l_g \zeta_0}{\lambda^2}\right)^2 S \, d\Omega. \tag{51.56a}$$

In this approximation, therefore, there is no depolarization of radar-received scattered radiation. With gaussian correlation $\left(F_g = \exp\left(-\frac{\rho^2}{l_g^2}\right)\right)$ $G_g^0 = \pi$. Since we are concerned with long waves, $(l_g \zeta_0 \lambda^{-2})^2 = (kl_g)^4 \gamma_0^2 \ll 1$. As concerns short waves, on the other hand, it must be remembered that with $\psi \sim 1$ we are concerned with the case $k\zeta_0 \ll 1$. This is the only reason why the scattering does not disappear in the mirror direction. We shall see in §52 that for $k\zeta_0 \geq 1$, considerable short-wave scattering appears in all directions. It is evident from Formulas (51.55) and (51.56) why it is advisable to use a horizontally polarized radiator for radar purposes if we wish to suppress "glare" from rough surfaces (needless to say, if the problem is radiation from a scattering surface, this result has the opposite significance).

Since the scattered field is expressed directly in terms of the Fourier components of the correlation function for the heights of points on the surface, study of scattering may serve as a method for determining the statistical characteristics of the surface.

The correlation functions for fields observed at two different points, $E^{(1)}(\vec{R})E_k^{(1)}(\vec{R}')$, can be found by a similar method. In this connection, it may be found helpful to have certain general relationships between the correlation functions in an electromagnetic field, as obtained in [27].

The irregularities have been assumed to be stationary in the entire analysis given above. At the same time, for example, scattering of radio waves on sea swells gives rise to interesting phenomena as a result of the motion of the irregularities – first and foremost, variation of the frequency of the scattered waves. In view of the slowness of the surface movement as compared with the velocity of the radio waves, and considering the shift and inclination angles γ in boundary condition (49.5) as slowly varying functions of time, it will not be difficult to calculate the modulation of the scattered radiation, which resembles combination scattering of light.

The shift $\Delta\omega$ of the frequency ω is small, $\Delta\omega \sim \omega\frac{v}{c}$, where v is the velocity of irregularity movement. In principle, however, it enables us, for example, to use radio methods to measure the velocity of sea waves. This question is investigated in the paper by F.G. Bass [28] for the case of gently sloping irregularities in an approximation corresponding to that adopted under the present heading.

§52. FIELD ABOVE A GENTLY SLOPING RANDOMLY IRREGULAR SURFACE (STEEP INCIDENCE, HIGH IRREGULARITIES)

1. In §51 we considered a field scattered by a surface with gently sloping irregularities. The results obtained are applicable to the case of irregularities that are high by comparison with wavelength only provided that the surface is very gently sloped, (51.30):

$$2\pi\zeta_0 \ll \sqrt{l_g\lambda}, \tag{52.1}$$

where l_g is the characteristic length of the surface roughness; ζ_0 is their height. On the other hand, this theory is valid in the case of glancing waves, when $\psi \to 0$; here the theory is rigorous in the respect that reradiated waves (field scattered by nearby irregularities) is taken into account rigorously at each point of the surface.

In the case of arbitrarily high irregularities and arbitrary angles of incidence, we encounter the difficult problem of accounting for a generally strong reradiated field at each scattering point. In particular, it is essential that certain points of the surface are shaded from the primary ray, and even from the reradiated field. All of this represents an obstacle to the creation of a unified and complete theory. One of the cases that admit of special simplification is the problem considered in §51. Another is scattering of waves incident at a sufficiently large angle ψ (reckoned from a certain averaging surface) onto a surface whose irregularities are of arbitrary height, but whose gradient angles are small enough so that we may disregard: a) the shadowing effect on individual segments; b) the squares of the terrain gradient angles (in the factors before exponential functions; consequently, this requirement is rather weak); c) the reradiated field. That is to say, it is necessary that it be possible at each point on the surface to regard the field as the sum of the incident (undisturbed) field and the reflected field in accordance with the boundary condition at the point in question.

This last requirement is characteristic for the Kirchhoff approximation in diffraction theory. Actually (see §11), it is assumed in diffraction at a hole in a screen according to this approximation that the field is equal to the undisturbed incident-wave field in the plane of the hole. Accordingly, the approximation of which we are ppeaking under the present heading may also be called a Kirchhoff approximation.

It is obvious that the third requirement can be met only for short waves

$$\lambda \ll l_\zeta. \qquad (52.2)$$

In this case, each element of the surface may be replaced by the plane tangent to it and we may assume that the following relation holds for the ith component of E:

$$E_i = E_i^0(1 + f_i) \text{ for } z = \zeta(x, y). \qquad (52.3)$$

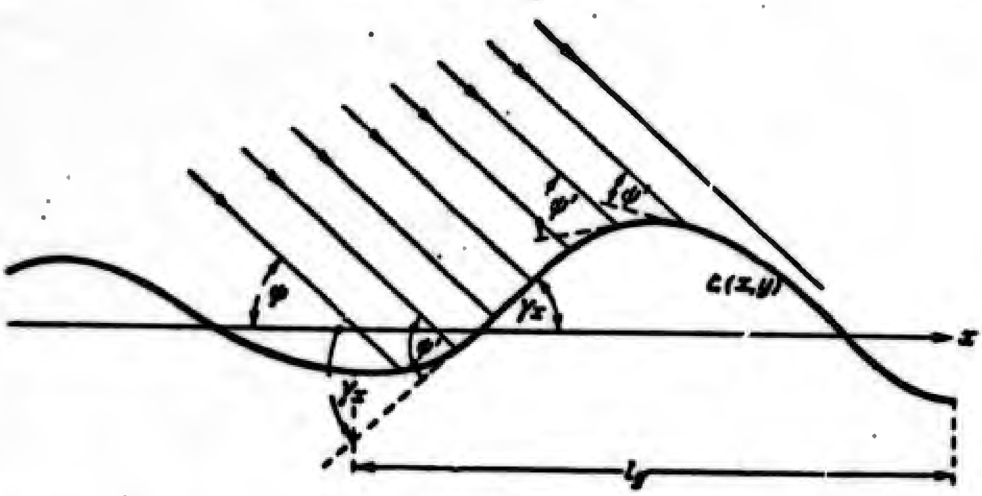

Fig. 52.1. Symbols in the case of scattering by an irregular surface with rather steep wave incidence.

where E_i^0 is the undisturbed field at the given point; f_i is the coefficient of reflection of the ith component for a surface with the given electrical properties at a given angle of incidence of the undisturbed field onto the tangential plane. We have already judged the admissibility of using the reflection coefficient in the case of a spherical surface. This evaluation can also be applied here. Obviously, the reflection will be the same as from the tangential plane if, according to Formula (34.17),

$$\sin\psi' \gg \frac{1}{\sqrt[3]{ka}}, \tag{52.4}$$

where ψ' is the glancing angle of the incident wave at the tangential plane; a is the radius of curvature at the point in question, i.e., generally speaking, a quantity of the order of l_g^2/ζ_0. All three conditions will be satisfied only provided that the glancing angle ψ of the primary wave at the plane $z = 0$ is, as a rule, different from the gradient angles of the surface, which are of the order of $\gamma \sim \zeta_0/l_g$ (see Fig. 52.1, where $\psi' = \psi \pm \gamma_x$). Under these conditions, there will be no shaded areas and rays reflected from one segment of the surface will not be incident on another, while for large enough k it is also possible to satisfy Condition (52.4). Thus, the method under consideration, which was developed in [6, 16, 17, 18, 19], is applicable to gently sloping surfaces,

$$\gamma \ll 1, \tag{52.5}$$

for large angles of incidence and for short waves:

$$|\psi-\gamma| \gg \frac{1}{\sqrt[3]{ka}}, \quad a \sim \frac{l_g^2}{\zeta_0} \sim \frac{l_g}{\gamma}. \tag{52.6}$$

In any event, therefore, we must have $\sqrt[3]{ka} \gg 1$, which can be rewritten

$$\zeta_0 \ll \frac{l_g}{\lambda}\sqrt{l_g\lambda}. \tag{52.7}$$

In comparing with Formula (52.1), we see that in this method, to the extent that we are speaking of short waves, (52.2), we may consider heights much greater than in the method of §51.

We note that, as in the case of the Kirchhoff diffraction theory, this method is satisfactory only for scattering directions close to those that follow from geometrical optics – to directions corresponding to regular reflection from each element, or (since $\gamma \ll 1$) from an averaging horizontal plane (compare remark after Formula (52.42)).

2. Below we shall follow the paper by M.A. Isakovich [6]. We shall start from the vector form of the Kirchhoff formula (5.14b) and limit ourselves to an ideally conductive surface, $[\vec{n}\vec{E}] = 0$. Then we have for the scattered field at the observation point A

$$E(A) = -\frac{1}{4\pi}\int\left\{ik[nH]v + (nE)\,\mathrm{grad}\,v\right\}dS, \qquad (52.8)$$

where $v = \frac{1}{\rho}\exp(ik\rho)$. For longer distances ρ from the scattering surface to the observation point, we may assume

$$v \approx \frac{1}{\rho_A}e^{ik\rho_A - ik\beta r}, \quad \mathrm{grad}\,v \approx -\frac{ik}{\rho_A}\beta v, \qquad (52.9)$$

where, as in §51, ρ_A is the distance from observation point A to a certain point C on the scattering surface, which is taken as the origin of the vectors $\vec{r}$, which denote the current point of the surface, and $\vec{\beta}$ is the vector of (51.38) (see also Fig. 51.1). Similarly, we may write for the incident field of a source O remote from the same point of the surface S by a distance R_0

$$E^0 = E^0(C)e^{ik\alpha r}, \; E^0(C) = P\frac{e^{ikR_0}}{R_0}, \qquad (52.10)$$

where $\vec{\alpha}$ is defined by Formula (51.38) and $\vec{P}$ is the constant amplitude of the incident wave. We can denote the scattering vector by

$$q = k(\alpha - \beta) \qquad (52.11)$$

In the case of an ideal conductor, the reflection coefficients for the vertical component of the electric field and the horizontal component of the magnetic field are unity, so that in Formula (52.8)

$$[nH] = 2[nH^0], \; (nE) = 2(nE^0), \; E = E^0 + E^1.$$

In the incident wave, $E^0 = -\frac{1}{k}[kH^0] = -[\alpha H^0]$ and $H^0 = [\alpha E^0] = \frac{e^{ikR}}{R}[\alpha P]$. Therefore, taking the slowly varying factors $R \approx R_0$ and $\rho \approx \rho_A$ out from under the integral sign, we have

$$E^1 = -\frac{ik}{2\pi}\frac{e^{ik(R_0+\rho_A)}}{R_0\rho_A}\int ([n[\alpha P]] - \beta(nP))\, e^{ik(\alpha-\beta,\, r)} dS. \quad (52.12)$$

In the factor in front of the exponential, only $\vec{n}$ depends on the variable of integration. If, therefore, we denote

$$I = \int n e^{iqr} dS \quad (52.13)$$

and introduce a vector in the incident wave to indicate the vector amplitude of the magnetic field,

$$Q = [\alpha P], \quad P = -[\alpha Q], \quad (\alpha P) = (\alpha Q) = 0; \quad (52.14)$$

then the integral in (52.12) may be written in a form that admits of convenient manipulation:

$$\int ([n[\alpha P]] - \beta(nP))\, e^{iqr} dS = [IQ] + \beta(I[\alpha Q]). \quad (52.15)$$

In Formula (52.13), we substitute $\vec{n}$ by its expression according to Formulas (49.2a) and, by virtue of the smallness of the $\gamma_{x,y}$ (52.5), we disregard their squares in the denominators. Further, we take into account that we have $\vec{q}\vec{r} = q_x x + q_y y + q_z \zeta(x, y)$ on the surface S. Therefore, for example,

$$\frac{\partial}{\partial x} e^{iqr} = i\left(q_x + q_z \frac{\partial \zeta}{\partial x}\right) e^{iqr} \approx i(q_x + q_z\gamma_x)\, e^{iqr}$$

(and similarly for $\partial/\partial y$). This means that

$$-\gamma_x e^{iqr} = \frac{1}{q_z}\left(q_x + i\frac{\partial}{\partial x}\right) e^{iqr}.$$

Substituting these expressions for γ_x and γ_y in $\vec{I}$ we may drop integrals containing derivatives of the function to be integrated (they vanish after integration over the entire surface, since the field vanishes at a distance and they produce a negligibly small boundary effect in integration over a limited surface). Consequently,

$$\vec{I} = \int \vec{n} e^{i\vec{q}\vec{r}} dS = \frac{\vec{q}}{q_z} \int e^{i\vec{q}\vec{r}} dS, \tag{52.16}$$

and the direction of the vector $\vec{I}$ coincides with the direction of $\vec{q}$. Consequently, remembering that $(\vec{\alpha}\vec{Q}) = 0$, we may write for the integral of (52.15) (we take advantage of the fact that $[qQ] \equiv \beta^2 [qQ] - (\beta\beta)[qQ] = [\beta[[qQ], \beta]] + \beta(\beta[qQ])$:

$$[qQ] + \beta(q[\alpha Q]) = -[\beta[\beta[qQ]]] + \beta(\beta[qQ]) - \beta(\alpha[qQ]) = -[\beta[\beta[qQ]]] - \beta(q[qQ]).$$

But the last term is zero. The scattered field is therefore finally

$$E^1 = \frac{ik}{2\pi} \frac{e^{ik(R_0 + r_A)}}{R_0 r_A q_z} [\beta[\beta[qQ]]] I^{(0)}, \tag{52.17}$$

where all static [sic] properties of the surface are covered in the multiplier

$$I^{(0)} = \int e^{i\vec{q}\vec{r}} dS. \tag{52.17a}$$

If the surface were flat, we should have $\zeta(x, y) = 0$ and the integral

$$I^{(0)} = \int e^{i(q_x x + q_y y)} dx\, dy = (2\pi)^2 \delta(q_x) \delta(q_y) \tag{52.18}$$

would indicate that the field is nonzero only in the direction of mirror reflection: $\alpha_x = \beta_x$, $\alpha_y = \beta_y$. In view of the smallness of the surface gradient angles, we may again set $dS = dxdy$ now. In this case, the only difference between the fields $\vec{E}$ and the field above a flat surface is due to the presence of the factor $\exp(iq_z\zeta(x,y))$ in the integrand.

3. Let us consider the statistical properties of the integral $I^{(0)}$. They are determined as a function of the height distribution of the points on the surface, $W(\zeta)$. The average field $\bar{\vec{E}}$ will contain

$$\overline{I^{(0)}} = \int dx\, dy\, e^{i(q_x x + q_y y)} \overline{e^{iq_z\zeta}}, \tag{52.19}$$

$$\overline{e^{iq_z\zeta}} = \int W(\zeta) e^{iq_z\zeta} d\zeta = f(q_z). \tag{52.19a}$$

It differs from the field above a plane only in the numerical factor

$f(q_z)$, which has the sense of a reflection coefficient. As before, this field will differ from zero only in the direction of mirror reflection, but it will have a different intensity. Since $\int W(\zeta)d\zeta = 1$, it is obvious that $f < 1$, and the reflection will be attenuated due to scattering in other directions. We obtained a similar result in §51 for arbitrary angles of incidence and low surface roughness ($k\zeta_0 \ll 1$), as well as for higher irregularities with glancing incidence.

Thus, for example, if the height distribution is the normal gaussian distribution,

$$W(\zeta) = \frac{1}{\zeta_0\sqrt{2\pi}} e^{-\frac{\zeta^2}{2\zeta_0^2}}, \tag{52.20}$$

then (we take into account the fact that for the direction of mirror reflection $\psi = \chi$, $\varphi = 0$, $q_z = -2k \sin \psi$)

$$f(q_z) = e^{-\frac{1}{2}q_z^2\zeta_0^2} = e^{-2k^2\zeta_0^2 \sin^2\psi}. \tag{52.21}$$

Thus, regular reflection is disturbed when the projection of the average irregularity height onto the direction of the incident ray $\zeta_0 \sin\psi$ exceeds the wavelength divided by 2π.

4. The scattered radiation, which will also be present in directions other than that of mirror reflection from a plane, is of considerably greater interest that the average field. The average (over copies of the surface) field will be zero here, but the mean-square field, which determines the field-strength fluctuations, and the average energy flux are, generally speaking, nonzero. The (time-) average energy flux in the direction of the vector $\vec{\beta}$, which is determined by the vector

$$\Sigma = \frac{c}{8\pi}[EH^*], \quad H = [\beta, E],$$

is, according to Formula (52.17),

$$\Sigma = \frac{ck^4}{32\pi^3}\frac{1}{R_0^2\rho_A^2}\frac{1}{q_z^2}\left|[\beta[\beta[\alpha-\beta, Q]]]\right|^2 \overline{|I^{(\omega)}|^2}. \qquad (52.22)$$

This expression must still be averaged over copies of the surface.

We introduce the binary distribution function, the probability $W_2(\zeta_1, \zeta_2)$ that the surface heights will be equal to ζ_1 and ζ_2 at two points defined by the two-dimensional radius vectors $\vec{r}_1(x_1, y_1)$ and $\vec{r}_2(x_2, y_2)$. In the case of an isotropic surface, this function will depend only on the distance $\rho = |\vec{r}_1 - \vec{r}_2|$. For normal distribution

$$W_2(\zeta_1, \zeta_2, \rho) = \frac{1}{2\pi\zeta_0^2\sqrt{1-F_g^2(\rho)}} e^{-\frac{1}{2\zeta_0^2(1-F_g^2)}(\zeta_1^2-2F_g\zeta_1\zeta_2+\zeta_2^2)}, \qquad (52.23)$$

where ζ_0^2 and $F_g(\rho)$ are determined by Formula (51.6). F_g is the height correlation coefficient at the two points on the surface. The average of interest to us takes the form (we replace $\vec{r}_2$ by $\vec{r}_1 + \vec{\rho}$, and one integration – over $\vec{r}_1$ – gives the area of the scattering surface S):

$$\overline{|I^{(\omega)}|^2} = \int e^{i(q(r_1-r_2))+iq_z(\zeta_1(r_1)-\zeta_2(r_2))} W_2(\zeta_1, \zeta_2; \rho)\, dr_1\, dr_2\, d\zeta_1\, d\zeta_2 =$$

$$= S\int e^{iq\rho}\, d\rho \int e^{iq_z(\zeta_1-\zeta_2)} W_2(\zeta_1, \zeta_2; \rho)\, d\zeta_1\, d\zeta_2. \qquad (52.24)$$

If all heights are small, then, expanding $\exp(iq_z(\zeta_1 - \zeta_2))$ in series and limiting ourselves to the first nonvanishing powers, we shall arrive at the case analyzed in §51 (since the distribution W_2 is normalized to unity, the integral over ζ_1 and ζ_2 will be equal to $1 - q_z^2\overline{\zeta^2} + q_z^2\overline{\zeta_1\zeta_2} + \ldots$). Here, however, we are concerned with significant heights, and assume that

$$\overline{|I^{(\omega)}|^2} = 2\pi S\int_0^\infty J_0(|q|\rho)\rho\, d\rho\, f\left(q_z, -q_z; \frac{\rho}{l_g}\right) =$$

$$= 2\pi S l_g^2 \int_0^\infty J_0(|q|\, l_g x)\, x\, dx\, f(q_z, -q_z; x). \qquad (52.24a)$$

Here we have performed integration over the angles of the two-dimensional vector $\vec{q}(q_x, q_y)$ and introduced the so-called characteristic distribution function

$$f(a,b;x)=\int e^{i(a\zeta_1+b\zeta_2)}W_2(\zeta_1,\zeta_2;x)\,d\zeta_1\,d\zeta_2, \qquad (52.25)$$

which is the Fourier-transformed function W_2. In the case of normal distribution (52.23),

$$f(q_{z1},-q_{z2};x)=e^{-q_z^2\sigma_0^2(1-F_g(x))},\quad x=\frac{\rho}{l_g}. \qquad (52.26)$$

Formulas (52.22), (52.24) and (52.25) fully determine the intensity of the scattered radiation in different directions if the function W_2 is known. The converse is also obvious: by experimental study of the distribution of scattered radiation, we can obtain information concerning the correlation characteristics of the scattering surface.

In the present approximation, when wavelength is small by comparison with the geometrical characteristics of the surface, expressions of the type $|\vec{q}|l_g$ and $q_z\sigma_0$ are large. This means that we shall be interested in the high-frequency Fourier components of the function W_2 in Formula (52.25). Accordingly, values of W_2 for small ρ/l_g appear effectively under the integral sign, where the function F_g is close to unity and W_2 has a pronounced maximum at $\zeta_1=\zeta_2$. This has a simple physical significance: in our approximation, reflection takes place from small segments on isolated rough spots turned at an angle such that regular reflection from this zone ensures arrival of the scattered wave at the observation point. Within the limits of the corresponding small region of the surface, the height ζ_1 is always close to the height ζ_2. Thus, for example, with the normal distribution according to Formula (52.26) f vanishes quickly when F_g is still little different from unity. Hence we may expand the function F_g in series about the zero of its argument and restrict ourselves to the first nonvanishing term. Since by virtue of the general properties of the correlation function

$$F_g(0) = 1, \quad F_g'(0) = 0, \quad F_g''(0) < 0, \tag{52.27}$$

we may assume $F_g(x) \approx 1 + \frac{1}{2} F_g''(0) x^2$,

$$f\left(q_x, q_y; \frac{\rho}{l_g}\right) \approx \exp\left\{-\frac{1}{2} q_z^2 \zeta_0^2 \frac{\rho^2}{l_g^2} |F_g''(0)|\right\}. \tag{52.28}$$

Substituting f in Formula (52.24a), we integrate with the formula

$$\int_0^\infty J_0(ax) e^{-bx^2} x dx = \frac{1}{2b} e^{-\frac{a^2}{4b}};$$

we obtain

$$\overline{|I(q)|^2} = \frac{2\pi S l_g^2}{q_z^2 \zeta_0^2 |F_g''(0)|} e^{-\frac{1}{2}\frac{l_g^2}{\zeta_0^2}\frac{q_x^2+q_y^2}{q_z^2|F_g''(0)|}}. \tag{52.29}$$

Thus, for example, for gaussian correlation, $F(x) = \exp(-x^2)$, we have $|F_g''(0)| = 2$ and

$$\Sigma = \frac{ck^4}{32\pi^3} \frac{1}{R_0^2 \rho_A^2} \frac{S l_g^2}{q_z^4 \zeta_0^2} [\beta[\beta[\alpha - \beta, Q]]]^2 e^{-\frac{l_g^2}{4\zeta_0^2}\frac{q_x^2+q_y^2}{q_z^2}}. \tag{52.30}$$

The energy flux in the primary ray (we take into account that $(\vec{\alpha}\vec{P}) = 0$) is equal to

$$\Sigma_0 = \frac{c}{8\pi} [H^0 [\alpha H^0]] = \frac{cQ^2}{8\pi R_0^2} \alpha. \tag{52.30a}$$

Multiplying Σ by $\rho_A^2 d\Omega$, where $d\Omega$ is a solid-angle element around the scattering direction, and dividing by Σ_0, we obtain the effective differential section of the surface for scattering in the given direction:

$$d\sigma = \frac{\Sigma \rho_A^2 d\Omega}{\Sigma_0} = \frac{k^4 l_g^2}{4\pi q_z^4 \zeta_0^2} \left[\beta\left[\beta\left[\alpha - \beta, \frac{Q}{Q}\right]\right]\right]^2 e^{-\frac{l_g^2}{4\zeta_0^2}\frac{q_x^2+q_y^2}{q_z^2}} S d\Omega. \tag{52.31}$$

In the general case, if we make no assumptions concerning the statistical properties of the surface, it follows from Formulas (52.22) and (52.30a) that for scattering of radiation incident in the direction of the vector $\vec{\alpha}$ and having polarization determined by the unit vector of the incident-wave magnetic field

$$h = \frac{Q}{Q} = \frac{H^0}{H^0},$$

in the direction of the unit vector $\vec{\beta}$, the following formula is valid for the differential cross section:

$$d\sigma(\alpha, \beta; h) = \frac{[\beta[\beta[\alpha-\beta, h]]]^2}{4\pi^2(\alpha_z-\beta_z)^2} k^2 \overline{|I^{(0)}|^2}\, d\Omega, \tag{52.32}$$

where

$$\overline{|I^{(0)}|^2} = \int e^{iq(r_1-r_2)+iq_z(\zeta_1(r_1)-\zeta_2(r_2))} W(\zeta_1(r_1), \zeta_2(r_2))\, dr_1\, dr_2\, d\zeta_1\, d\zeta_2 =$$

$$= \int e^{iq(r_1-r_2)} f(q_z, -q_z; r_1, r_2)\, dr_1\, dr_2. \tag{52.32a}$$

Thus, $d\sigma$ is expressed in terms of the Fourier-transformed characteristic function f. For a statistically homogeneous surface f depends only on $\vec{r} = \vec{r}_1 - \vec{r}_2$ (if the surface is statistically isotropic, it depends only on the modulus of $\vec{r}$). In this case we may write the inverse formula

$$f(q_z, -q_z; r) = \int \frac{(\alpha_z-\beta_z)^2}{[\beta[\beta[\alpha-\beta, h]]]^2} \frac{d\sigma}{d\Omega} \frac{dq}{S} e^{-iqr}, \tag{52.32b}$$

so that the surface characteristic function (for specific values of its arguments) may be determined if the scattering is studied in adequate detail.

In the case of vertical polarization of the incident radiation, the vector $\vec{Q}$ is parallel to the plane $z = 0$, $Q = Q_y$. In the case of horizontal polarization along the y-axis we have $P = P_y$.

Let us consider Formula (52.31), which was derived for normal correlation. The statistical factor (exponential cofactor) has a maximum in the direction of regular reflection from the plane $z = 0$, $q_x = q_y = 0$, when $q_z = -2k \sin\psi$. Since $l_g \gg \zeta_0$, this peak is sharp. If $\vec{\alpha}$ lies in the xz-plane, we may expand the vector product to find that for either of the two polarizations

$$[\beta[\beta[\alpha-\beta, Q]]]^2 = 4\sin^2\psi \cdot Q^2. \qquad (52.33)$$

Around the maximum we may set

$$\chi = \psi + \Delta\chi, \quad \Delta\varphi = \varphi \ll 1.$$

Expanding the exponent in increments $\Delta\psi$ and $\Delta\varphi$ of the angles ψ and φ, we obtain

$$\frac{q_x^2 + q_y^2}{q_z^2} = \frac{\cos^2\psi + \cos^2\chi - 2\cos\psi\cos\chi\cos\varphi}{(\sin\psi + \sin\chi)^2} \approx \frac{(\Delta\chi)^2 + \mathrm{ctg}^2\chi\,(\Delta\varphi)^2}{4}.$$

Hence near the maximum

$$d\sigma = \frac{S\,d\Omega}{\pi} \cdot \frac{l_g^2}{16\zeta_0^2}\, e^{-\frac{l_g^2}{16\zeta_0^2}\left((\Delta\chi)^2 + \mathrm{ctg}^2\chi\,(\Delta\varphi)^2\right)}, \qquad (52.34)$$

where $d\Omega \approx \cos\chi\, d(\Delta\chi)\, d(\Delta\varphi)$. From this it is evident that the peak has an angular (root-mean-square) width

$$(\Delta\chi)_{\text{ск}} \sim \sqrt{8}\,\frac{\zeta_0}{l_g}.$$

This means that for $l_g \sim 10\zeta_0$ (which, as regards order of magnitude, is the case for the sea waves), the peak is smeared over an angle of the order of $2(\Delta\chi) \sim 0.6$, i.e., of the order of several tens of degrees.

We obtain the total scattering by integration over $\Delta\chi$ and $\Delta\varphi$ within the limits $-\infty$ and $+\infty$:

$$\sigma = \frac{S}{\pi}\,\frac{\pi\cos\psi}{\mathrm{ctg}\,\psi} = S\sin\psi. \qquad (52.34a)$$

As it should, the total effective section of the area S equals its projection onto the direction perpendicular to the incident ray. All energy incident on this projection is scattered inside a cone around the direction of mirror reflection from the horizontal plane. Within these limits, the distribution is given by Formula (52.34).

The typical independence of scattering on wavelength, which is expressed by Formulas (52.31) and (52.34), is explained by the fact that for λ much smaller than the radius of curvature of the surface at the point in question, as was shown in §35, the reflection coefficient differs from that for the plane tangent to the surface only in the "divergence factor," which does not depend on wavelength (see (35.6)). Reflection takes place from many minute almost flat mirrors whose inclination corresponds to the possibility of regular reflection in the direction of interest to us. The number of these mirrors determines the intensity of scattering. This number depends only on the properties of the surface and not on the properties of the radiation. In accordance with this conception, a method has even been proposed for treating scattering (see below) without invoking an electrodynamic picture of the process, proceeding solely from the statistics of reflector placement.

For back scattering to the observation point ("radar observation"), $\alpha = -\beta,\ q = 2k\alpha,\ |\beta[\beta[\alpha-\beta, Q]]| = 2|\alpha Q|,\ q_z = -2k\sin\psi,\ q_x = 2k\cos\psi,$ we obtain (irrespective of polarization)

$$d\sigma = \frac{l_g^2}{16\pi\zeta_0^2} e^{-\frac{l_g^2}{4\zeta_0^2}\cdot\frac{\cos^2\psi}{\sin^2\psi}} \frac{S\,d\Omega}{\sin^4\psi}. \qquad (52.35)$$

Basically, the angle dependence of scattering is given in the general case by the exponential factor in Formula (52.31):

$$d\sigma \sim e^{-\frac{l_g^2}{4\zeta_0^2}\frac{\cos^2\psi+\cos^2\chi-2\cos\psi\cos\chi\cos\varphi}{(\sin\psi+\sin\chi)^2}}. \qquad (52.36)$$

A number of diagrams and evaluations may be found in [6], where the results set forth above are for the most part derived.

It must be stressed that many of the results depend exceedingly strongly on the height distribution law and other statistical characteristics of the surface. In order to satisfy ourselves of this, let us consider a surface on which the heights are distributed as in a sine wave:

$$\zeta = \zeta_0 \cos(Kx), \quad K = \frac{2\pi}{l_g}. \qquad (52.37)$$

The probability of finding a height ζ in an interval $d\zeta$ is proportional to the ratio of the time dx during which the height is in the range from ζ to $\zeta + d\zeta$ to the period l_g (the coefficient 2 takes account of the fact that there are two such points on the length l_g):

$$W_1(\zeta)d\zeta = 2\frac{dx}{l_g} = \frac{K d\zeta}{\pi\left|\frac{d\zeta}{dx}\right|} = \frac{d\zeta}{\pi\sqrt{\zeta_0^2-\zeta^2}} \text{ for } \zeta<\zeta_0 \qquad (52.37a)$$

($W_1(\zeta) = 0$ for $|\zeta| > \zeta_0$). Consequently, for example, the characteristic function, which here takes the part of the reflection coefficient for the average field (which is present only in the direction of regular reflection from the plane) is equal to

$$f(q_z) = \int_{-\zeta_0}^{\zeta_0} e^{iq_z\zeta} W_1(\zeta)d\zeta = \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{iq_z\zeta_0\sin\varphi} d\varphi = J_0(q_z\zeta_0). \qquad (52.38)$$

As we see, this result differs sharply from Expression (52.21). The function f is quite large; for large $q_{z\,0}$, the decrease is extremely slow, and, moreover, oscillatory, and the function J_0 takes both signs $\left(J_0(q_z\zeta_0) \approx \sqrt{\frac{2}{\pi q_z\zeta_0}}\cos\left(q_z\zeta_0 - \frac{\pi}{4}\right)\right)$.

Let us further consider the root-mean-square quantities, for example, the flux of scattered energy for a surface with the same correlation of nearby points as on a sinusoidal surface (52.37). We are dealing with short waves scattered as a result of mirror reflections from various segments on the sinusoid. Hence we shall disregard the correlation of points on the surface at distances $\gtrsim l_g$. It may be assumed that the surface is composed of pieces (of length $\sim l_g$) of a sinusoid with phase shifts between them that are distributed completely at random. The boundary effect at the joints between segments of the sinusoid will be disregarded.

For this case, the function $W_2(\zeta_1,\zeta_2;\ \vec{r}_1,\ \vec{r}_2)$ will contain the complete correlation. The breakdown of this correlation near the boundaries of each segment may be left out of consideration, since we are interested only in the correlation on segments of the order of $1/q_x$, which we assume to be must smaller than l_g. The probability of finding the height ζ_2 at a point $\vec{r}_2(x_2,y_2)$ if the height is ζ_1 at the point $\vec{r}_1(x_1,\ y_1)$ is

$$W_2(\zeta_1,\zeta_2;r_1,r_2)=W_1(\zeta_1)\,\delta(\zeta_2-\zeta_1-\zeta_0\cdot(\cos(Kx_2)-\cos(Kx_1))). \quad (52.39)$$

Substituting this value into Formula (52.32a), we obtain

$$\overline{|I^{(\omega)}|^2}=\int e^{iq_x(x_1-x_2)+iq_y(y_1-y_2)+iq_z\zeta_0\cdot 2\sin\frac{K(x_1+x_2)}{2}\sin\frac{K(x_1-x_2)}{2}}\,W_1(\zeta_1)\,d\zeta_1\,dr_1\,dr_2. \quad (52.39a)$$

The integrals over y_1 and y_2 give $2\pi L\delta(q_y)$, where $L=\sqrt{S}$ is the linear dimension of the scattering surface. We substitute $x_1+x_2=2X$, $x_1-x_2=x$, $dx_1dx_2=dxdX$, and break the interval L of integration over X into $KL/2\pi$ segments, each of length $2\pi/K$. As a result, setting $KX=\varphi$, we have

$$\int_0^L e^{iq_z\zeta_0\cdot 2\sin KX\sin\frac{Kx}{2}}\,dX=\frac{KL}{2\pi}\frac{1}{K}\int e^{iq_z\zeta_0\cdot 2\sin\frac{Kx}{2}\sin\varphi}\,d\varphi. \quad (52.39b)$$

By virtue of the normalization of $W_1(\zeta)$, integration over ζ_1 gives unity. Further, the integral over x is taken effectively within the limits of a small segment (since the heights on different segments of length l_g are considered to be uncorrelated). Hence we may set $\sin\frac{Kx}{2}\approx\frac{Kx}{2}$ and move the limits of the integral to infinity:

$$\overline{|I^{(\omega)}|^2}=L^2\delta(q_y)\cdot 2\pi\int_0^{2\pi}\delta(q_x+Kq_z\zeta_0\sin\varphi)\,d\varphi=$$

$$= 2\pi\delta(q_y)S\int_{-Kq_z\zeta_0}^{+Kq_z\zeta_0}\delta(q_x-t)\frac{dt}{Kq_z\zeta_0\cos\varphi},$$

where the variable has been substituted in the last integral: $t = Kq_z\zeta_0\sin\varphi$. The remaining integral is equal to unity if $|q_x| < Kq_z\zeta_0$, and zero if $|q_x| > Kq_z\zeta_0$. Thus, finally, according to Formula (52.32),

$$d\sigma = \frac{[\mathbf{s}[\mathbf{s}[\alpha-\beta, \mathbf{h}]]]^2}{4\pi^2(\alpha_z-\beta_z)^2}\cdot\frac{2\pi S\delta(q_y)k^2}{\sqrt{K^2q_z^2\zeta_0^2-q_x^2}}, \qquad (52.40)$$

if $q_x^2 < K^2q_z^2\zeta_0^2$; $d\sigma = 0$ for $q_x^2 > K^2q_z^2\zeta_0^2$. If we remember that $\delta(q_y) = \frac{1}{K}\delta(\alpha_y - \beta_y)$, then the last cofactor in Formula (52.40) may be written in the form

$$\frac{2\pi S\delta(\alpha_y-\beta_y)}{\sqrt{K^2\zeta_0^2(\alpha_z-\beta_z)^2-(\alpha_x-\beta_x)^2}}. \qquad (52.40a)$$

The expression obtained for $d\sigma$ is substantially different from Expression (52.31). As q_x^2 increases, the scattered flux does not diminish, as indicated by Formula (52.31), but instead increases. In the direction of regular reflection, at $q_x = 0$, strictly speaking, we may not replace $\sin K\frac{x}{2}$ in the integral over x by its argument; it is necessary to extend integration over x from 0 to l with the formula $\int_0^\pi J_0(2z\sin\varphi)d\varphi = \pi J_0^2(z)$, and then take into account that $J_0^2(q_z\zeta_0) \approx \frac{2}{\pi q_z\zeta_0}\cos^2\left(q_z\zeta_0-\frac{\pi}{4}\right)$, where the square of the cosine may be replaced by its average value, which is equal to 1/2. As a result, as from Formula (52.40), we get

$$d\sigma = \frac{S\delta(\alpha_y-\beta_y)}{4\pi K\zeta_0}\sin\psi\, d\Omega \text{ for } q_x = 0. \qquad (52.41)$$

This by no means resembles Expression (52.34). The scattering diminishes with diminishing ψ (Formula (52.41) is, of course, valid only as long as $\psi \gtrsim K\zeta_0 \sim \gamma$, see Inequality (52.6)). For back scattering to the source, $(\alpha_x-\beta_x)^2 = 4\cos^2\psi$, $(\alpha_z-\beta_z)^2 = 4\sin^2\psi$,

$$d\sigma = \frac{S\delta(\alpha_\nu - \beta_\mu)\,d\Omega}{16\pi\sin^4\psi\sqrt{K^2\overline{\zeta_0^{-2}}\sin^2\psi - \cos^2\psi}} \quad \text{for } K^2\overline{\zeta_0^{-2}}\,\mathrm{tg}^2\psi > 1,$$

$$d\sigma = 0 \qquad \text{for } K^2\overline{\zeta_0^{-2}}\,\mathrm{tg}^2\psi < 1 \tag{52.42}$$

(needless to say, zero is obtained only within the framework of the Kirchhoff approximation being used; in actuality, some diffracted field is always present). Indeed, if the angle ψ is large enough, the surface will have no segment perpendicular to the direction of incidence. This formula should be compared with Formula (52.35).

Comparison of concrete results obtained for two statistically different surfaces indicates an exceptionally strong dependence of the conclusions on the statistical characteristics. Hence the final formulas must be used only with great caution in practical conditions.

5. An important property of statistically scattered radiation is the nonadditive nature of the effect. Suppose, for example, that the function describing the surface $\zeta(x, y)$ can be represented as the sum of two independent random functions, $\zeta(x, y) = \xi(x, y) + \eta(x, y)$ having different natures (for example, small ripples on heavy sea swells). Then, say, if we are seeking the average field, the expression

$$\overline{e^{iq_z(\xi+\eta)}} = \int e^{iq_z(\xi+\eta)}\, W_1(\xi)\, W_1'(\eta)\, d\xi\, d\eta, \tag{52.43}$$

where W_1 and W_1' are the respective distribution functions, will figure instead of Expression (52.19a). Integration over ξ and η can be performed separately and E will be proportional not to $f(q_z)$, but to the product of the characteristic functions $f(q_z)f'(q_z)$, where f' corresponds to the distribution W_1'. The same applies for two-dimensional distribution functions W_2 and the functions $f(q_z, -q_z; x)$ corresponding to them. The product of two such functions f appears in Formula (52.24). If both quantities are normally distributed, the situation is simplified, since, according to (52.28), the product of the two functions f is identical with one of them taken with another argument.

For example, with gaussian correlation $F_g''(0) = -2$ we obtain

$$f\left(q_{z_0} - q_z; \frac{\rho}{l_g^{(1)}}\right) \cdot f\left(q_{z_0} - q_z; \frac{\rho}{l_g^{(2)}}\right) = e^{-q_z^2 \rho^2 \left(\frac{\zeta_0^{(1)2}}{l_g^{(1)2}} + \frac{\zeta_0^{(2)2}}{l_g^{(2)2}}\right)}, \quad (52.44)$$

where $\zeta_0^{(1)}$, $\zeta_0^{(2)}$, $l_g^{(1)}$ and $l_g^{(2)}$ are the parameters of the two distributions. Thus we have simple addition of the squares of the average surface inclination angles in these distributions, $\gamma_0^2 = \zeta_0^2/l_g^2$. The attenuation of the scattered radiation will be determined by irregularities with larger gradient angles.

Interest attaches to the case in which ζ is formed by superposition of a regular part ξ (for example, the regular wave) and a smaller-scale random part η. Then

$$f(q_{z_0} - q_z) = \int e^{iq_z(\xi(r_1) + \eta(r_1) - \xi(r_2) - \eta(r_2))} W_2(\eta_1, \eta_2)\, d\eta_1 d\eta_2. \quad (52.45)$$

We see that the function f contains an additional regular factor $\exp\{iq_z(\xi(r_1) - \xi(r_2))\}$, which appears under the integral sign in Formula (52.24). Suppose, for example, that ξ is sinusoidal waves with an amplitude ξ and a wave vector $\vec{K}$,

$$\xi = \xi_0 \cos(Kr),\; K l_g^{(\eta)} \ll 1, \quad (52.46)$$

where $l_g^{(n)}$ is the horizontal scale of the random irregularities. Then, denoting the new integral $I^{(0)}$ by I_ξ, we have

$$\overline{|I_\xi|^2} = \int e^{iq(r_1 - r_2) + iq_z \xi_0 \cdot 2 \sin\left[\frac{K}{2}(r_1 + r_2)\right] \sin\left[\frac{K}{2}(r_1 - r_2)\right]} f\left(q_{z_0} - q_z; \frac{\rho}{l_g^{(\eta)}}\right) dr_1 dr_2.$$

(52.47)

In the coordinates $R = \frac{1}{2}(r_1 + r_2)$, $\rho = r_1 - r_2$, $dr_1 dr_2 = dR d\rho$ we take first the integral over $\vec{R}$, directing the x-axis along $\vec{K}$. Proceeding in the same way as on conversion from Expression (53.39a) to Expression (52.39b), we obtain

$$\int e^{2iq_z \xi_0 \sin\left(\frac{1}{2} K\rho\right) \sin KX} dX dy = \frac{L^2}{2\pi} \int_0^{2\pi} e^{2iq_z \xi_0 \sin\left(\frac{1}{2} K\rho\right) \sin\varphi} d\varphi.$$

Substituting this expression in the integral over $\vec{\rho}$, we remember that $K \ll q$ and that, according to Condition (52.46), $Kl_g^{(n)} \ll 1$. Therefore $\sin\left(\frac{1}{2}K\rho_0\right)$ is a comparatively slow function and can be replaced by the sine of its argument:

$$\overline{|I_\xi|^2} = \frac{S}{2\pi}\int e^{i(q+q_z\xi_0\sin\varphi K,\rho)} f\left(q_z, -q_z; \frac{\rho}{l_g^{(\eta)}}\right) d\rho d\varphi =$$

$$= S\int_0^\infty\int_0^{2\pi} J_0(|q+q_z\xi_0\sin\varphi K|\rho) f\left(q_z, -q_z; \frac{\rho}{l_g^{(\eta)}}\right) d\rho d\varphi. \quad (52.48)$$

Comparing this result with Formula (52.24a), we see that since $K\xi_0 < 1$ (the surface gradients must be small), a small vector is added to $\vec{q}$ as a result of superposition of the regular part ξ. If we again assume normal distribution and gaussian correlation (52.26) for η, with the root-mean-square value η_0 of the quantity η, we obtain instead of (52.29)

$$\overline{|I_\xi|^2} = \frac{S l_g^2}{8\pi^2 q_z^2\eta_0^2}\int_0^{2\pi} e^{-\frac{l_g^{(\eta)2}}{8\eta_0^2}\cdot\frac{|q+q_z\xi_0\sin\varphi K|^2}{q_z^2}} d\varphi,$$

and, assuming $|q+q_z\xi_0\sin\varphi K|^2 \approx q^2 + 2q_z\xi_0(Kq)\sin\varphi$, we find

$$\overline{|I_\xi|^2} = \overline{|I^{(0)}|^2}\cdot I_0\left(\frac{l_g^{(\eta)2}}{2\eta_0^2}\cdot\frac{\xi_0}{q_z}(Kq)\right), \quad (52.49)$$

where $I^{(0)}$ is the former value of the integral and I_0 is a Bessel function of an imaginary argument. As $\xi_0 \to 0$, it gives unity (random irregularities on the plane, i.e., the same case as before). If, however, the modulus of its argument is large, $I_0(z) \approx \frac{1}{\sqrt{2\pi|z|}}e^{|z|}$. This additional multiplier may be found to be quite large (although the relative variation of the exponent in (52.29) remains small; the exponent is multiplied by a factor $1-2q^{-2}\xi_0 q_z(Kq)$, which differs little from unity. It is interesting to note that in this additional factor, the argument is a complex combination of the characteristics of both types of irregularity, one that brings out their nonadditive character.

We again see in this result how the conclusions depend on the distribution statistics of surface-point heights. Clearly, if the heights of the large-scale irregularities were normally distributed, we should again necessarily obtain Formula (52.44), even with regularity of their positions. In this case, the addition of large-scale irregularities would reduce scattering. Indeed, the chief role in integration in Formula (52.43) is taken by very short distances that are determined by small-scale irregularities. The presence of long-range order in irregularities with a correlation length $\sim 1/K \gg l_g^{(n)}$ cannot influence the result. With sinusoidal height distribution (52.46), however, the result is found to be fundamentally different – scattering increases.

At the same time, if we compare this result with the formula for $d\sigma$ in the case of a sine-wave-like surface (52.40), it becomes obvious that the superposition of small-scale irregularities η with a normal distribution law on such a surface lowers scattering sharply in the main range of angles, but that scattering then appears where $d\sigma$ was 0 in the absence of the small-scale roughness. We again see that the chief influence on scattering comes from irregularities with the largest gradient angles.

The great sensitivity of scattering to the statistical properties of the surface prevents the derivation of final scattering formulas that are applicable for all surfaces. On the other hand, it follows from this that radiation scattering can be investigated by determining the statistical properties of the surfaces. For example, this investigation might be carried out not only with the aid of electromagnetic waves, but also by acoustic methods, for which the formulas have the same structure [6].

6. In the case of high irregularities under consideration, polarization relationships arise – in particular, depolarization phenomena similar to those considered for small irregularities in §§48 and 51. In this case, they are all embodied in the factor $|\beta|\beta|qQ|||$. in Formula (52.17) or in the square of the analogous expression in Formula (52.22). On subsequent averaging, these factors do not change, so that the polarization relationships are independent of the statistical properties of the surface. It is found [6] that for an incident wave that is <u>horizontally</u> (i.e., perpendicular to the plane of incidence) polarized, $P = P_y$, while for a <u>vertically</u> (i.e., parallel to the plane of incidence) polarized incident wave, $Q = Q_y$, the components of the corresponding electric-field strengths of the scattered wave, $E^{1(h)}$ and $E^{1(v)}$ have the following angle dependences:

$$E_{\varphi}^{1(v)} = E_{\chi}^{1(h)} = A \sin\varphi, \tag{52.50a}$$

$$E_{\varphi}^{1(h)} = E_{\chi}^{1(v)} = A \frac{\cos\varphi\cos\chi - \cos\varphi(1+\sin\psi\sin\chi)}{\sin\psi + \sin\chi}, \tag{52.50b}$$

where A denotes the aggregate of all remaining factors in Formula (52.17) other than $|\beta|\beta|qQ|||$. Here E_φ gives the horizontal component of the received radiation directly, while E_χ differs from the (vertical) z-component by the factor $\cos\chi$. The squares of the angle factors give the corresponding partial sections. Generally speaking, the polarization relationships are found to differ from those prevailing for small irregularities. In certain cases, however, the difference is not great. Thus, the vertical component of the field of a horizontal dipole contains, according to Formula (51.49b), the factor $\sin^2\psi\cdot\cos^2\chi\cdot\sin^2\varphi$, while, according to Formula (52.50a), we obtain $\cos^2\chi\cdot\sin^2\varphi$. In either case, the maximum occurs at $\varphi = \pi/2$, in the line forming a right angle with the plane of incidence.

7. Experimental study of the problems set forth in the present

chapter is still in a rather unsatisfactory state. Without going into all of the available studies, we note only that it was found in [20] for the case of reflection of radio waves with $\lambda = 3$ cm from the ocean (wave height about 30 cm) that, as the angle ψ increased from zero to 80° in the direction of regular reflection from the averaging plane, the average reflection coefficient varied, diminishing from approximately one (in the range $\psi < 10°$) to values of the order of (−25)-(−30) db, which was reached at $\psi \sim 20$-$25°$, and then remained practically constant (decrease by a few decibels). We note that even at small angles, the theory developed in §51 is inapplicable. Actually, it may be assumed for sea waves that l_g is ten times larger than the amplitude ζ_0, which is equal to half the height of the waves, i.e., $\zeta_0 \sim 15$ cm, $l_g \sim 150$ cm. Consequently, Condition (51.30b), $2\pi\zeta_0 \ll \sqrt{l_g\lambda}$, is not satisfied. On the other hand, the approximation of the present Section is applicable, according to Inequality (52.6), only when the angles ψ exceed the surface inclination angle γ. Then for f we may rely upon the applicability of Formula (52.19a) and its particular cases for various surface statistical properties, (52.21) and (52.38). Since for sea waves, $\gamma_0 \sim 1/4 \sim 15°$, we are concerned precisely with that region in which the f found experimentally reaches a constant value. However, according to Formula (52.21), we should expect rather strong variation of f for $k\zeta_0 = \frac{2\pi}{3} 15 \approx 30$, when $\sin\psi$ varies from $\sin 30° = 1/2$ to $\sin 80° \approx 1$ (by $20 \log 4 \approx 12$ db). According to [20], this does not actually occur. On the other hand, Formula (52.38) gives $f \approx J_0(30 \sin \psi)$. Replacing the Bessel function by its asymptotic value and $\cos(30 \sin \psi - \pi/4)$ by its root-mean-square value $1/\sqrt{2}$, we find that the reflection coefficient f must vary from $\sqrt{\frac{2}{\pi k\zeta_0 \sin 30°}} \cdot \frac{1}{\sqrt{2}} \approx 7$ to $\sim 1/10$, or, in decibels, from $-20 \log 7 \approx$

≈ -17 db to ~ -20 db. This corresponds rather closely to the experimental data cited above (particularly if the error in the assignment of ζ_0 is taken into account.

Thus, the sinusoidal type of distribution (52.37a) of the heights ζ would appear more plausible than the normal distribution (52.20). On the other hand, there are rather weighty theoretical considerations, based on the mechanism of sea-swell formation, that favor the normal distribution. The situation may by no means be regarded as clarified, and the experimental analysis given above is primarily illustrative in nature.

8. Both in the present section and in §§48 and 51, we have been considering average values of the fields and their quadratic formations. For one thing, this gives average values of the reflection coefficient. Very frequently, however, these quantities are inadequate for practical application, and it is necessary to have full knowledge of the statistical field-amplitude or reflection-coefficient distribution. Sometimes these characteristics are reduced to the probability of the field amplitude (or reflection coefficient) exceeding a given threshold value. At the same time, it is only in the case of gaussian normal distribution of the given quantity that the entire distribution is fully determined solely by two parameters – the average and root-mean-square values of this quantity. We have seen that the statistical properties of the surface have a substantial influence on the values of these parameters. Obviously, they will also have a strong influence on the distribution law of the scattered-field amplitudes as a whole.

We present the theoretical results for one special case, as obtained by Beckmann [21]. This study considered scattering from a surface with a broken profile in the direction of regular reflection from the horizontal plane (Fig. 52.2), this scattering being regarded as

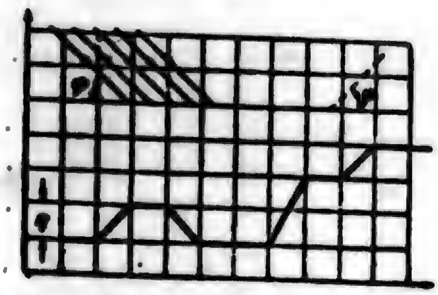

Fig. 52.2. Illustrating calculation of mirror reflection from a surface with a broken-line profile.

superposition of waves reflected regularly from randomly distributed horizontal zones of the surface. The author disregards reradiation and mutual shadowing (with the consequence that the incidence angle ψ must be rather large and the wavelength λ rather small). Certain assumptions are made concerning the distribution law of

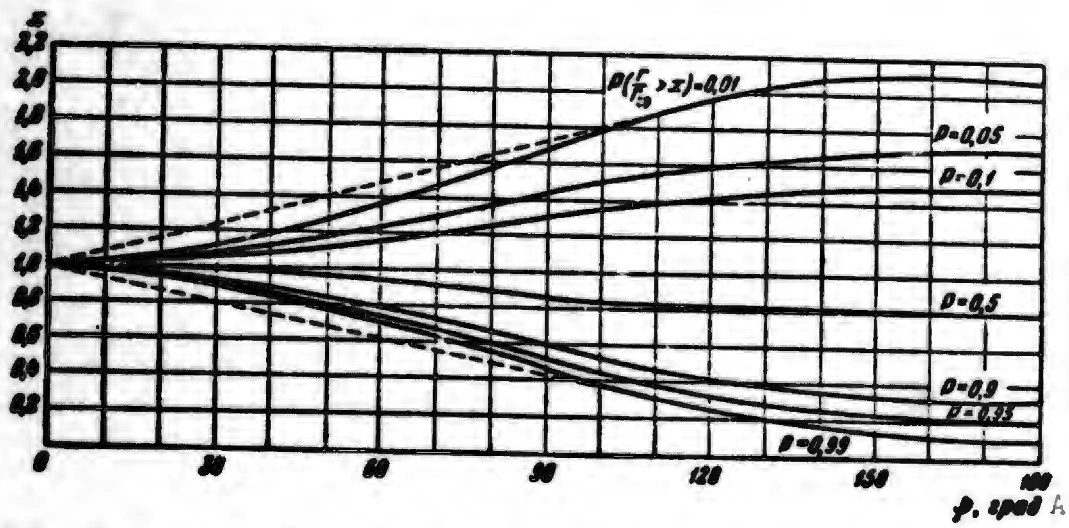

Fig. 52.3. Probability P(x) of the ratio of amplitude to the root-mean-square value of r/r_{sr} exceeding a given value x for a given glancing angle ψ. P is constant along each curve. A) degrees.

the heights y_i of the horizontal segments and the total number of these segments: there is a maximum difference $((\Delta y)_{maks} = s\eta)$ among the values of y_i. Accordingly, there exists a maximum possible difference among the phases of the reflected waves – the maximum phase shift is $\varphi_s = 4\pi(s-1)\eta \sin\psi$, where s is the maximum number of elementary plateaus. Within the limits of $(\Delta y)_{maks}$, the heights are assumed to be uniformly distributed. The phase shifts φ in the range between $-a$ and $+a$ are accordingly also uniformly distributed; here

$$a = \frac{\varphi_s}{2} = 2\pi(s-1)\eta \sin\psi.$$

This quantity a essentially determines the distribution of the resultant scattered-wave amplitude r. At $a = \pi$, the problem is identical to the classical Rayleigh problem [22], and, consequently, r has a Rayleigh distribution: $p(r) = \text{const}\cdot re^{-r^2/r_0^2}$. The same holds for $a \gg \pi$. The formulas for the more general case $a \neq \pi$ were derived in [21]. We shall present only the resultant diagram. Figure 52.3 gives curves of constant values of constant values of the probability P that the resultant amplitude r, expressed in fractions of its root-mean-square value, $r/\sqrt{\overline{r^2}}$, will exceed a certain value x for a given a. The problem was not studied for other statistical properties of the surface.

§53. OBSTACLE GAIN

Soil irregularities may give rise to an interesting effect that enables radio waves to propagate far beyond the horizon in some cases. It is detected experimentally ("obstacle amplification" or "obstacle gain") [23, 24] and has practical applications.

As follows from the theory of diffraction on a convex body (see §39, Subsection 3), the field of a plane wave incident upon this body is described near the plane of the horizon, which is the approximate boundary between the illuminated and shaded regions, by the formulas of Fresnel diffraction on a certain equivalent straight-edged flat screen (see Fig. 39.2). "Near the plane" – here this implies a small dimensionless coordinate (39.23), (39.23a)

$$\xi = \sqrt[3]{\frac{ka}{2}}\,\psi \lesssim 1.$$

Here a is the radius of curvature of the obstacle and ψ is the diffraction angle. Here it is assumed that $\sqrt[3]{ka} \gg 1$. In the case of a hill with $a \sim 100$-1000 m and waves with $\lambda \sim 10$-100 cm, the parameter

$\sqrt[3]{mu} \sim 10-50$ and the range of angles covered by this approximation is very large. Physically, substitution of a flat screen is admissible because the radiation of the virtual dipoles induced on the shadow side of the convex body is small around the plane of the horizon (§§37, 39).

But there is still another limitation that must be kept in mind. Let us consider an equivalent screen (Fig. 53.1, scales greatly distorted). We place the origin of the rectangular coordinate system x, y, z at point T, with x directed along the projection of the propagation line onto the plane of the horizon and z directed along the vertical. It will in any event be inadmissible to use the formulas for diffraction from a straight edge if the dimensions of the obstacle <u>along the y-axis</u> are small by comparison with the dimensions of the Fresnel zone, i.e., by comparison with $\sqrt{\lambda r}$. In this case, it would be necessary to consider instead the diffraction at a rounded peak. This is an essential limitation. Thus, in the experiments of [25], $\lambda = 3$ m, $r = 80$ km and 256 km, and the heights H of the hills in these two cases were 465 m and 1700 m. As a result, $H/\sqrt{\lambda r}$ assumed the respective values 1.15 and 2.5. We might expect the effective radius of curvature at the peak to be substantially smaller than H, so that the theoretical conclusions for diffraction around a sphere are not, strictly speaking, applicable to this case. If, however, we are concerned not with an isolated hill, but with a ridge or range of hills, the problem can be reduced systematically to diffraction from a cylinder. This again leads us to the formulas for diffraction from the edge of a screen in much the same way as was explained above for a sphere.

The phenomenon of "obstacle gain" consists in the possibility that the field at point A behind the obstacle may prove to be many times <u>stronger</u> than when the source, observation point and round surface have the same relative positions in the absence of this obstacle.

To understand the essential nature of this phenomenon, let us first consider the case of a flat earth (Fig. 53.2). We substitute a flat screen for the elevation.

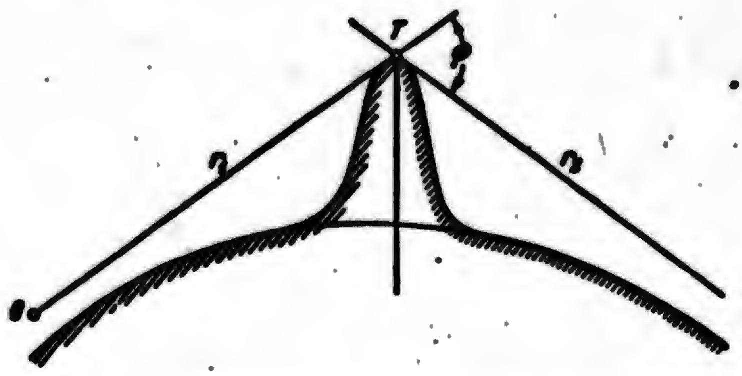

Fig. 53.1. Substitution of flat screen for hill.

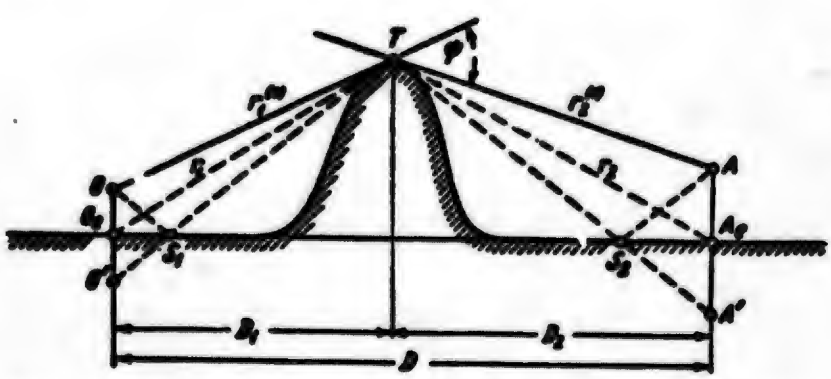

Fig. 53.2. Illustrating calculation of obstacle gain for the case of a flat earth.

Let the polarization of the waves be horizontal, and let the absolute value of ε be so large that the reflection coefficient from the ground is −1 for a glancing wave. We assume further that the distance D between the corresponding points O and A (with heights h_0 and h_A) is large enough so that in calculating the field in the absence of the obstacle we may use simple formulas of the type (19.42), (19.43). The absolute value of the attenuation function will be

$$|\varpi_A| = 2\sin\frac{kh_0h_A}{D} \approx \frac{2kh_0h_A}{D}. \qquad (53.1)$$

Now let an obstacle of height H, which we may replace with a thin straight-edged screen, be placed at a distance D_1 from O. The field at A can be obtained by the imaging method; the case of interest to us was examined in §20. That is to say, it is necessary to construct the image O' of the source and the image T' of the screen T and analyze the diffraction of radiation from both sources on a double screen in empty space (Fig. 20.3) or, on the other hand, to add the fields at the true observation point A and at its image A'. Here it is necessary to introduce the phase π, which takes account of the reflection coefficient under these conditions, for the rays OS_1TA and OTS_2A (or the equivalent rays O'TA and OTA'). For each of the fields we may apply Formula (11.8) or (39.2), although the values of u_1 for them will differ.

If we take u_1 in the form (11.8d), it is necessary to construct for each ray a screen perpendicular to the line joining the two corresponding points in question: O and A, O and A', O' and A, O' and A':

$$u_1^{(i)} = -H^{(i)}\sqrt{\frac{k}{\pi}\frac{r_1^{(i)}+r_2^{(i)}}{r_1^{(i)}r_2^{(i)}}}, \quad i = 1, 2, 3, 4, \qquad (53.2)$$

where $r_1^{(i)}$ and $r_2^{(i)}$ are the distances from the source (O or O') and from the observation point (A or A') to the perpendicular dropped from T to the line joining the source in question to the observation point in question. We may assume with sufficient accuracy in Formula (53.2) that $r_1^{(i)} = D_1$, $r_2^{(i)} = D_2$, i.e.,

$$u_1^{(i)} = -H^{(i)}\sqrt{\frac{k}{\pi}\frac{D}{D_1D_2}}. \qquad (53.2a)$$

and if $D_1 >> D_2$, then $u_1^{(i)} \approx -H^{(i)}\sqrt{\frac{k}{\pi D_2}}$.

Thus, the field at point A will take the form

$$E(A)\sim e^{ik\left(r_1^{(1)}+r_2^{(1)}\right)}w_F\left(u_1^{(1)}\right)-e^{ik\left(r_1^{(2)}+r_2^{(2)}\right)}w_F\left(u_1^{(2)}\right)-$$

$$-e^{ik\left(r_1^{(3)}+r_2^{(3)}\right)}w_F\left(u_1^{(3)}\right)+e^{ik\left(r_1^{(4)}+r_2^{(4)}\right)}w_F\left(u_1^{(4)}\right). \qquad (53.3)$$

Considering that w_F varies little over wavelength, we may introduce average distances to T from the midpoint O_e between the sources O and O', $r_1 \approx \frac{1}{2}(OT + O'T)$, and from the midpoint A_e between the observation points, $r_2 \approx \frac{1}{2}(AT + A'T)$. Further, denoting by ψ the angle between r_1 and r_2 (diffraction angle), we may assume that a resultant wave $E_0 \sim 2 \sin (kHh_0/r_1)$, which is diffracted through an angle ψ, falls on the exposed part of the screen plane around T. The fact that two waves – the directly diffracted wave and the wave reflected after diffraction at point S_2 – arrive at the real observation point can be taken into account by introducing a factor $2 \sin (kHh_A/r_2)$.

Instead of Formula (53.3), therefore, we may express the field at A by the attenuation function (see Formulas (11.8c) and (11.8d)):

$$w_A' \sim 4\sin\frac{kHh_0}{r_1}\sin\frac{kHh_A}{r_2}w_F(u_1),$$

$$u_1=-\sqrt{\frac{k}{\pi}\frac{r_1r_2}{r_1+r_2}}\sin\psi=-H\sqrt{\frac{k}{\pi}\frac{r_1+r_2}{r_1r_2}}. \qquad (53.4)$$

At the same time, we had Expression (53.1) in the absence of the obstacle. The ratio of these quantities may be either smaller or larger than unity. In the latter case, it is known as the obstacle gain

$$G=\frac{|w_A'|}{|w_A|}. \qquad (5\ .5)$$

Actually, H may be large enough to place the point T, for example, in the maximum of the directional-pattern lobe, and then w_F will still not be very small. Thus, if $|u_1| \gg 1$, we obtain in the general case, using Formula (10.13b) for w_F,

$$G = \frac{4\sin\frac{kHh_O}{r_1}\sin\frac{kHh_A}{r}}{2\sin\frac{kh_Oh_A}{r_1+r_2}}\cdot\frac{1}{\pi\sqrt{2}\,|u_1|}. \qquad (53.6)$$

If

$$\frac{h_O}{r_1}\sim\frac{h_A}{r_2}\sim\frac{\pi}{2}\cdot\frac{1}{kH}, \qquad \frac{h_O}{H},\ \frac{h_A}{H}\ll 1, \qquad (53.6a)$$

then

$$G\approx 2\frac{r_1+r_2}{kh_Oh_A}\,\frac{1}{\pi\sqrt{2}H}\sqrt{\frac{\pi r_1r_2}{k(r_1+r_2)}}=\sqrt{\frac{2}{\pi}}\sqrt{\frac{r_1+r_2}{kr_1r_2}}\,\frac{r_1}{h_O}\,\frac{r_2}{h_A}\cdot\frac{1}{kH},$$

i.e.,

$$G\approx\left(\frac{2}{\pi}\right)^{\frac{5}{2}}kH\sqrt{\frac{r_1+r_2}{kr_1r_2}}=\sqrt{2}\cdot\frac{4}{\pi^2}|u_1|. \qquad (53.7)$$

This maximum value of G may be very large (Formula (53.7) has, after all, been derived on the assumption that $|u_1| >> 1$), if, as is evident from (53.6a), h_O/r_1 and h_O/r_2 are small. Thus, with $\lambda = 1$ m, $h_O = h_A = 10$ m, and $r_1 = r_2 = \frac{1}{4}\cdot 10^5$ meters, Inequality (53.6a) is satisfied if $kH = \frac{\pi}{4}\cdot 10^4$, i.e., if $H = \frac{1}{8}\cdot 10^4 = 1250$ meters. Here $|u_1| \approx 24$ and $G \approx 13.5$. Thus, if a hill is on the path, the field is amplified by almost 43 db.

In characterizing the physical content of this interesting effect, it is frequently said that the edge of the screen acts as a passive rebroadcaster. This, of course, is not accurate. If the same obstacle peak were suspended at the same point in the absence of the entire lower part of the hill, its effect would be nullified. It is important that the entire obstacle blocks access to the observation point A for rays from O passing along the ground and experiencing strong attenuation. The <u>phase</u> interference relationships for rays arriving at A on different paths vary widely. They are such that low-lying rays (which are more strongly attenuated by the presence of the ground), taken in the aggregate, still cancel the effect of less attenuated rays passing through the upper layers of the space. As before, the zone essential for passage of rays in the absence of the obstacle covers ellipsoids with relatively small ordinal numbers (compare §11), although the field is weaker here. The obstacle screens out and eliminates the "harmful" rays and leaves access open for the unattenuated space waves.

A field diffracting through a hole and having originally, according to the Huygens principle, the form of a superposition of the fields from virtual sources distributed <u>over the plane</u> of the hole, can always be reduced to an expression having the form of the field of radiators distributed <u>around the perimeter</u> of the hole (see remark at the end of §11). Here again, therefore, we have the distance from the observation point to the crest of the hill, i.e., to the edge of the screen. The fact that the properties of the screen material do not figure in the field expression once again indicates that we may speak of "hilltop rebroadcasting" only in a conventional sense and then with caution.

The analysis given above rests essentially on a theory whose applicability in the case of real obstacles – hills, mountain ranges, etc. – is obvious in far from all cases. Even if we agree on the conclusions of the theory set forth in §39, Subsection 3, at great distances from the sphere, $D_2 \gg a$, and if we describe the hilltop schematically by an equation of the second order with a consistent radius of curvature a, then, according to Formula (39.35), this theory can be used only for rather small angles ψ. In the example given above, $\psi \approx \approx 2H/r \approx 1/40$, and for the theory to be valid it is necessary that

$$\psi \sim \frac{1}{4} < \sqrt[3]{\frac{2}{ka}},$$

i.e.,

$$ka < 3 \cdot 10^4.$$

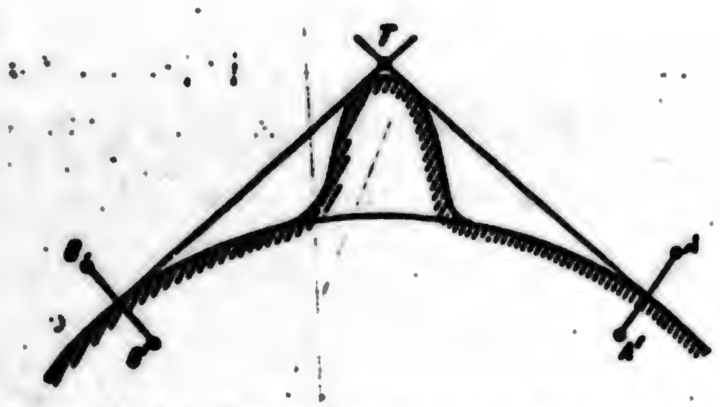

Fig. 53.3. Obstacle gain in the case of a spherical earth.

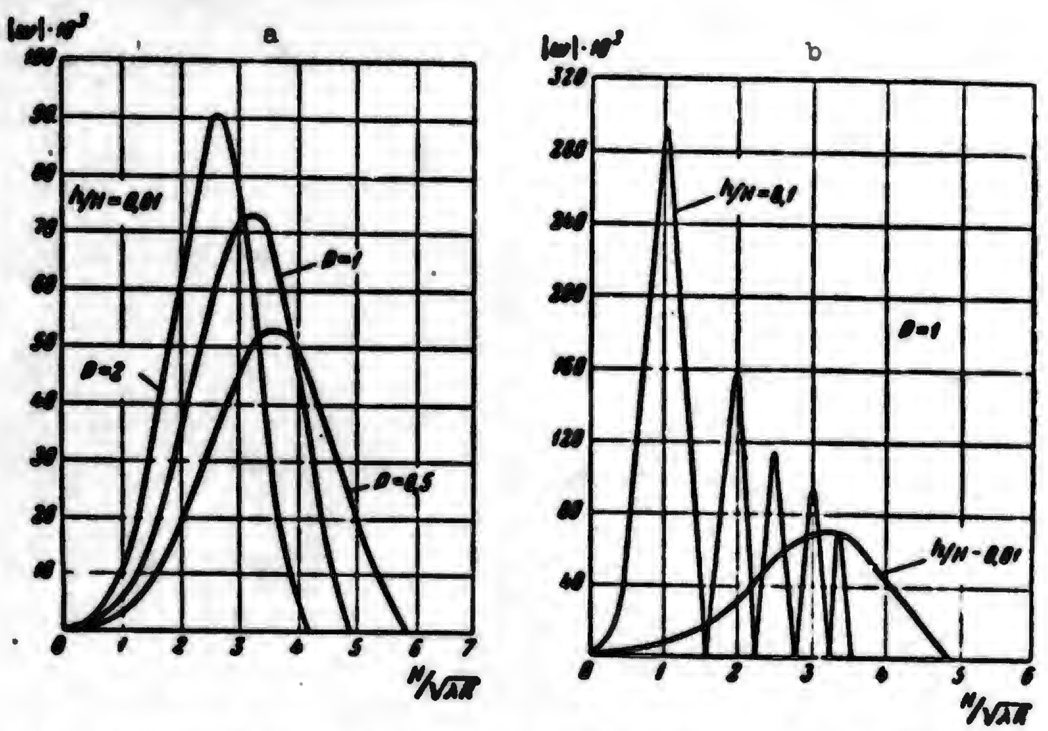

Fig. 53.4. Attenuation multiplier (with respect to free space) in the presence of obstacle gain: a) for a given ratio of source and screen-edge heights and various observation-point heights h_A ($p = h_A/h_0$); b) for equal source and observation-point heights ($p = 1$) and various ratios of source height *h* to screen height.

Thus with $\lambda = 1$ m, we must have $a \lesssim 5 \cdot 10^3$ meters. This is undoubtedly the case for real hills (we leave aside the problem of geometrical regularity of the hill's shape), and the theory may be regarded as applicable.

The physical interpretation of the effect suggests – and this is found to be true – that obstacle gain must be even more significant when the earth's curvature plays a substantial role in the absence of an obstacle, provided that A is beyond the horizon of O. It is necessary only that the obstacle height H be sufficiently large, i.e., that the edge of the equivalent screen T be in the illuminated region with respect to both O and A (Fig. 53.3). In this case, the field incident at T can again be obtained with the reflection formulas, by which it is also possible to take into account reflection of the diffracted wave from the ground. (Needless to say, the curvature of the earth's surface can be taken into account in these formulas.) Roughly speaking, the field at the observation point will then be almost the same as though the earth were flat, and the obstacle height H were reckoned from the line OA. But the field in the absence of the obstacle (w_A), with respect to which G is reckoned, will be much smaller. Hence the obstacle gain increases. In [26], from which we have borrowed the curves of Fig. 53.4, *a* and *b*, S.Ya. Braude gave a similar calculation for this case. Here the values of the attenuation multiplier (with respect to free space) are given as functions of the quantity $H/\sqrt{\lambda(r_1 + r_2)}$ for (optimum) conditions such that Relationships (53.6a) are observed and $h_O/H = 0.01$ (Fig. 53.4a; here different curves correspond to different values of $p = h_A/h_O$), and also such that $h_O/H = h_A/H = 0.01$ and 0.001 (Fig. 53.4b).

Manuscript Page No.	[Transliterated Symbols]
478	эфф = eff = effektivnyy = effective
497	П = P = pochva = soil
497	Р = R = rel'yef = terrain (relief)
568	ср = sr = sredniy = average
568	макс = maks = maksimal'nyy = maximum

Chapter 9

PROPAGATION OF RADIO WAVES IN A LAYERWISE-INHOMOGENEOUS MEDIUM

§54. INTRODUCTION

It is hardly likely that the main problem of radio wave propagation theory is anything other than over-the-horizon radio transmission. The classical theory of diffraction around a homogeneous spherical earth with a fully homogeneous atmosphere (Chapter 6) explained the rapid attenuation of long and medium waves with immersion in the geometrical shadow, but it was found incapable of predicting or even explaining the unexpectedly long penetration of meter-, centimeter- and even shorter waves beyond the horizon, which sometimes on the state of the atmosphere and sometimes does not (as before, of course, we are speaking of cases in which the ionosphere does not take an essential role). In explaining this effect, we rely on three physical effects, which sometimes can be separated clearly, so that the part played by each of them stands out distinctly. In certain cases, however, competing theories arise to interpret the same fact, and occasionally these effects act jointly. They are: a) the amplifying action of isolated irregularities (§53); b) random scattering on turbulent inhomogeneities of the atmosphere (Chapter 10) and c) the influence of layerwise inhomogeneity of the atmosphere, which is always present, if due only to the normal decrease in air density with increasing altitude. This last phenomenon, which is also sometimes known as refraction in the troposphere, is the principal subject of the present chap-

A layerwise-inhomogeneous medium is a space in which the electrical characteristics depend only on one of the coordinates of an orthogonal system, for example, on z in the rectangular and cylindrical systems or on r in the spherical system. Strictly speaking, the simplest problems of this type are cases that we have already analyzed – propagation of radio waves in a homogeneous atmosphere separated from a likewise homogeneous earth by a plane (Chapter 5) or spherical (Chapter 6) surface. In the present chapter, our interest will focus on more complex phenomena that arise basically in idealized formulations in the following three practically important problems, which accordingly determine the specific formulations of the problem and the approximations employed.

1) A layerwise-inhomogeneous soil encountered in geophysical survey work ("radio geology"). Here we may be speaking of sharply demarcated homogeneous horizontal layers with essentially different electrical characteristics. Since only long waves penetrate to appreciable depth, we may refer first of all to large λ, for which the absolute value of ε is large. The wavelength in air is usually long as compared with the thickness of the layers. For theory, however, the wavelength in the layer itself is essential; this may be small by comparison with the layer thickness. In practice, we have usually satisfied ourselves with wavelengths so long that we may limit ourselves to the quasistationary treatment. We shall not examine these problems.

2) The problem of ultralong-distance propagation of radio waves in a space bounded by the spherical surface of the earth and the reflecting surface of the corresponding ionospheric layer, which is likewise spherical (concentric with the surface of the earth). Here it is sometimes possible to regard each of the three layers (earth, atmosphere up to reflecting layer, atmosphere above it) as homogeneous.

In actuality, however, ε varies continuously with altitude, so that even the effective height of the reflecting surface in the ionosphere depends on the wavelength of the radiation and the reflection coefficient depends on the entire curve of ε. The reflection of waves shorter than, for example, 10-100 m is the object of the theory of radio-wave propagation in the ionosphere and hence will not be considered here.

3) Refraction in the troposphere and, in particular, the formation of tropospheric waveguide channels ("super refraction," "ultrarefraction"). This theoretically most difficult problem reduces to study of radio propagation in a medium in which the permittivity ε is very close to unity, with $\varepsilon - 1$ depending only on one vertical coordinate, and marked variation of this quantity occurring on intervals that are usually very large as compared with wavelength. The thickness of the waveguide channel is correspondingly large by comparison with λ. In many cases, the presence of the ground surface also plays a substantial role. As a rule, ε may be regarded as real. It is this third problem that will be the principal object of our analysis.

In many cases, therefore, the vertical scales of the medium are large as compared with λ. Under these conditions, it is natural to take recourse to the method of geometrical optics. Moreover, an attempt may be made to examine passage of radio waves as a consequence of successive reflections from the boundaries of homogeneous layers. However, this approach is inconvenient when a small number of reflections is involved. However, in cases in which we are concerned with distances along the layer that are large by comparison with its thickness (and these are the most essential cases), it is sometimes necessary to take very many reflections into account.

The problems enumerated above have been investigated in a very large number of papers, and the circumstantial monograph [I, 9], to which we shall refer the reader in many cases, has been devoted to the problem as a whole. Here we shall set forth only certain basic aspects.

§55. THREE-LAYERED SPACE

1. The physical content of many aspects of the process can be understood if we start from the following elementary problem.

Fig. 55.1. Three homogeneous plane layers.

Suppose that a flat homogeneous layer with $\varepsilon = \varepsilon_2$, which has infinite extent in the horizontal directions x and y, borders at $z = 0$ on a lower half-space in which $\varepsilon = \varepsilon_3$, while at $z = H$ it borders on an upper half-space, $\varepsilon = \varepsilon_1$ ($H < z < \infty$; Fig. 55.1). Let the source be either an electrical or a magnetic vertical dipole. By virtue of the cylindrical symmetry of the problem, we can describe the field by the single-component hertzian vector, $\Pi \equiv \Pi_z$ in the electrical case or $\Pi_m \equiv \Pi_{mz}$ in the magnetic case.

In each homogeneous region, and in particular inside the layer $\varepsilon = \varepsilon_2$, we have ($k = \omega/c$)

$$(\nabla^2 + \varepsilon_2 k^2)u = 0, \quad u = \Pi \text{ or } u = \Pi_m. \tag{55.1}$$

In the simplest case, if $\varepsilon_1 = \varepsilon_3 = \infty$, i.e., the boundaries of the layer are absolutely reflecting (and, consequently, the source is inside the layer), the conditions $E_x = E_y = 0$ obtain at these boundaries, which means, according to Formulas (4.8) and (4.11), that for both $z = 0$ and $z = H$

for $u = \Pi$

$$\frac{\partial u}{\partial z} = 0; \tag{55.2a}$$

for $u = \Pi_m$

$$u = 0. \tag{55.2b}$$

There are two methods for analyzing this problem (they are extremely similar and each can be reduced to the other at once). The first – the "normal-wave method" – consists in separating variables and finding the entire complete system of partial solutions of Eq. (55.1), which describe an orthogonal system of functions, the "natural functions" or "normal waves." Then a linear combination of these is selected that satisfies the radiation condition and the condition at the source. For the two-layer problem – flat homogeneous earth and homogeneous atmosphere – this method reduces to the Sommerfeld method, §31, when the natural functions (31.4a) and (31.4b) form a continuous sequence – they exist for all positive values of the parameter ν.

The second method – the "method of plane waves" – consists in representing the source field near the source as a superposition of plane waves (generally speaking, with complex direction cosines) and considering the passage of each plane wave separately. In the case of a two-layered space, this method reduces to the one set forth in §32. It was shown there that application of the plane-wave method leads to the results given by the first method. These methods were developed by Eckersley [1] in application to a more complex layerwise-inhomogeneous medium and subsequently elaborated successfully by a number of authors (in particular, see [2; I, 9; I, 11]. Eckersley based his work principally on the geometrical-optical approximation (see below, §57 and §61).

Let us first turn to the normal-wave method. Introducing the cylindrical coordinates (r, φ, z) with the z-axis passing lengthwise through the dipole, we may, by virtue of the problem's symmetry, limit ourselves to solutions that depend only on r and z. We obtain a particular solution (index ν) by separating the variables, i.e., substituting u into Formula (55.1) in the form of the product $u = ZR$ of a function $Z(z)$ that depends only on z and is subject to the equation

$$\frac{d^2Z}{dz^2} + (k^2\varepsilon_2 - \nu^2)Z = 0, \qquad (55.3)$$

by a function $R(r)$ that depends only on r and is subject to the equation

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + \nu^2 R = 0, \qquad (55.4)$$

where ν is the separation parameter. The first equation is solved in sines and cosines, while the second is solved in zero<u>th</u> order cylindrical functions. To satisfy the conditions at the boundaries of the layer, it is expedient to take the cosine of the argument $\sqrt{k^2\varepsilon_2 - \nu^2}\,z$ for Π and the sine for Π_m. Conditions (55.2a) and (55.2b) then give in both cases $\sqrt{k^2\varepsilon_2 - \nu^2}\,H = l\pi,\ l = \pm 1, \ldots,$ i.e.,

$$\nu^2 = \nu_l^2 = k^2\varepsilon_2 - \frac{l^2\pi^2}{H^2}. \qquad (55.5)$$

Consequently,

$$\Pi = \Pi_0 \cos\frac{l\pi z}{H}, \qquad (55.6a)$$

$$\Pi_m = \Pi_{m0} \sin\frac{l\pi z}{H}. \qquad (55.6b)$$

However, Eq. (55.4) is solved in zero<u>th</u>-order cylindrical functions of the argument νr. The radiation condition (receding wave at large r) may be satisfied by taking a Hankel function of the first kind for values of the parameter $\nu = \nu_l$ defined by Condition (55.5),

$$R = R_l \sim H_0^{(1)}(\nu_l r) = H_0^{(1)}\left(k\sqrt{\varepsilon_2 - \left(\frac{l\pi}{kH}\right)^2}\, r\right). \qquad (55.7)$$

At infinity, $H_0^{(1)} \sim \exp i\nu_l r$. Consequently, by selecting the sign for the radical that gives a nondamping wave in free homogeneous space ($\varepsilon_2 = 1$ and $H = \infty$),

$$\operatorname{Re}\sqrt{\varepsilon_2 - \left(\frac{l\pi}{kH}\right)^2} > 0. \qquad (55.7a)$$

we see that even for $\varepsilon_2 = 1$, waves with large enough l,

$$l > l_c = \frac{H}{\pi\lambda}, \qquad (55.8)$$

will damp exponentially at infinity, and the layer will be opaque to them. There is a minimum layer thickness that still admits of a solution in the form of a nondamping traveling wave (for real ε_2). Indeed, if

$$H < H_{min} = \frac{\pi}{k\sqrt{\varepsilon_2}} = \frac{\lambda}{2\sqrt{\varepsilon_2}}, \qquad (55.8a)$$

then even at $l = 1$, the parameter ν will be complex and $H_0^{(1)}$ will damp exponentially at infinity. At small distances, even these waves must, generally speaking, be taken into account in the solution. In the general case, the field is a superposition of the particular solutions cited above, which are known as "<u>normal oscillations</u>," "<u>normal solutions</u>," or "<u>modes</u>" (English) with certain coefficients B_l:

$$\left.\begin{matrix}\Pi\\ \Pi_m\end{matrix}\right\} = \sum B_l \left.\begin{matrix}\cos\\ \sin\end{matrix}\right\} \frac{l\pi z}{H} \cdot H_0^{(1)}\left(k\sqrt{\varepsilon_2 - \left(\frac{l\pi}{kH}\right)^2}\, r\right). \qquad (55.9)$$

The conditions at the source $\left(\lim_{r\to r_0}(ru) = \text{const}\right)$ must be satisfied by selection of the coefficients. The number of modes that do not damp at infinity, i.e., the number of terms of the series that make contributions to the field at great distances, is, roughly speaking, of the order of H/λ at $\varepsilon \sim 1$, as will be seen from Formula (55.5).

All of the above is, of course, the ordinary theory of a flat waveguide with ideally conductive walls. We note that the permitted values of $\nu = \nu_l$ form a discontinuous sequence, and the waveguide has a discontinuous spectrum. If, for example, we replace the cosine by the half-sum of the exponents, then each partial solution assumes the nature of a plane-wave superposition at long enough distances. That is to say, since for $|\nu r| \gg 1$

$$H_0^{(1)}(\nu r) \approx \sqrt{\frac{2}{\pi\nu r}} \exp\left(i\left(\nu r - \frac{\pi}{4}\right)\right).$$

we have

$$\cos\frac{l\pi z}{H} H_0^{(1)}(\nu_l r) \approx \sqrt{\frac{1}{2\pi\nu_l r}}\, e^{-i\frac{\pi}{4}} \left\{ e^{i\left(\nu_l r + \frac{\pi l}{H} z\right)} + e^{i\left(\nu_l r - \frac{\pi l}{H} z\right)} \right\}. \qquad (55.9a)$$

Thus, each normal solution may be regarded as a superposition of plane waves starting at angles θ_l to the z-axis:

$$\operatorname{ctg}\theta_l = \pm\frac{\pi l}{H\nu_l}. \qquad (55.9b)$$

It may be said that the lth wave is, generally speaking, propagated at a complex angle α_l to the z-axis, with

$$k\sin\alpha_l = \nu_l, \quad k\cos\alpha_l = \frac{\pi l}{H}. \qquad (55.9c)$$

The phase velocity of each of these waves, for example, for $l < l_c$ and $\varepsilon_2 = \mathrm{Re}\,\varepsilon_2$, is

$$v_l = \frac{\omega}{\sqrt{\nu_l^2 + \left(\frac{\pi l}{H}\right)^2}} = \frac{\omega}{k\sqrt{\varepsilon_2}} < c. \qquad (55.9d)$$

If, however, we seek the propagation phase velocity along the r-axis directly from Formula (55.9), we obtain

$$v_l^{(r)} = \frac{\omega}{\nu_l} = \frac{\omega}{k\sqrt{\varepsilon_2 - \left(\frac{l\pi}{kH}\right)^2}} > \frac{\omega}{k\sqrt{\varepsilon_2}}, \qquad (55.9e)$$

i.e., for $\varepsilon_2 = 1$, we have $v_l^{(r)} > c$. Each of these plane waves is reflected in turn from the boundaries of the layer. The superposition of all waves and their reflections forms the field at infinity.

This result also arises when we construct the same solution on the basis of the plane-wave method, using the image theory. The source O (Fig. 55.2) emits a field that has the same form as in homogeneous space and goes directly to the observation point A (ray OA). However, this field does not meet the boundary conditions at z = 0 and z = H. In order to satisfy them, it is necessary (see §20) to add the field of the images O_1' and O_2' of the source in the planes z = 0 and z = H, respectively. This gives rise to once-reflected waves arriving at A.

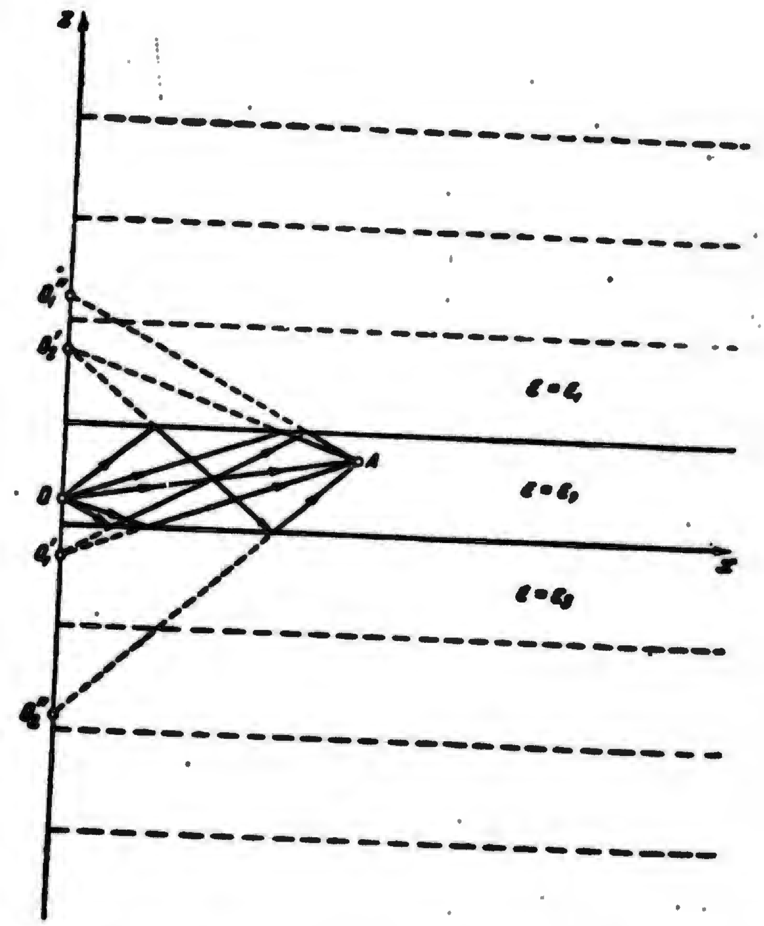

Fig. 55.2. Calculation of field in a flat waveguide with absolutely reflective walls by the method of reflected sources.

However, these added fields assist in satisfying the boundary conditions at one of the pair of planes, but they do not satisfy the conditions at the second of these planes: the field of source O_1' does not satisfy the condition on plane $z = H$, so that it is necessary to add its image O_1'' in this plane, and since the field of source O_2' does not satisfy the condition at $z = 0$, it is necessary to add O_2''. Accordingly, twice-reflected rays are added and also arrive at point A. Continuing further in this manner, we obtain an infinite chain of sources on the z-axis and a correspondingly infinite set of waves arriving at point A by different paths. This also corresponds to the superposition of waves (55.9a) arriving at points A at different angles θ_l (55.9b) or

α_l (55.9c).

2. The first generalization of the resulting picture consists in the fact that we assume noninfinite values for ϵ_1 and ϵ_3. Accordingly, the reflection from the planes $z = 0$ and $z = H$ is no longer total. Energy will leak from the layer. On the other hand, radiation will also pass to A by other paths, through the regions outside. This will change the character of the solution substantially: a continuous spectrum is added, as we see, to the discontinuous spectrum of values of $\nu = \nu_l$. Here we shall follow Brekhovskikh [3; I, 9]. Let the source, a vertical dipole, be situated inside the layer, on the z-axis at a height z_0.

We introduce the relative coefficients of refraction

$$\left.\begin{aligned} n = n_1(z) = \sqrt{\frac{\epsilon_1(z)}{\epsilon_2}}, \quad z > H, \\ n = n_3(z) = \sqrt{\frac{\epsilon_3(z)}{\epsilon_2}}, \quad z < 0. \end{aligned}\right\} \qquad (55.10)$$

They may also be complex if ϵ_1, ϵ_2 and/or ϵ_3 are complex.

At a distance R from it, the field radiated by a point dipole may be decomposed into plane waves (some with complex directions of propagation). This decomposition is given by Formula (32.2b). On striking the boundary of the layer, each of the plane waves represented will be reflected from it, then from the opposite boundary, then again from the first boundary, and so forth. At each reflection from the lower boundary $z = 0$, it will be multiplied by the corresponding reflection $f_3(\alpha)$, and at each reflection from the upper boundary by $f_1(\alpha)$. Moreover, on the way from one boundary to the other, it will increase its phase by $2ikH \cos \alpha = 2ik_z(\alpha)H$. Numbering the reflections by the subscript l and adding waves that have arrived after all possible reflections, we obtain, after writing out conscientiously the terms corresponding to various l and adding the resulting geometrical progression, the following expression for the total field (for greater detail see [I, 9], §27), for example, for $z > z_0$,

$$\Pi = \frac{ik}{2}\int_{\Gamma} \sin\alpha \, d\alpha \, \Phi_{>}(\alpha) J_0(kr \sin\alpha), \tag{55.11}$$

where we have used Formula (32.9) and the notation

$$\Phi_{>}(\alpha) = \frac{(e^{-ik_z z_0} + f_3 e^{ik_z z_0})(e^{-i(H-z)k_z} + f_1 e^{i(H-z)k_z})}{e^{-iHk_z}(1 - f_1 f_3 e^{2iHk_z})}, \quad z > z_0. \tag{55.12a}$$

The contour of integration Γ must be so selected that as $r \to 0$, $z \to z_0$, the expression for Π will become the expression valid for an isolated dipole. Consequently, the contour Γ must coincide with that indicated in §32.

The same expression (55.11) is valid for $z < z_0$ with $\Phi_{>}$ replaced by $\Phi_{<}$, which is obtained by transposing z and z_0 in $\Phi_{>}$:

$$\Phi_{<} = \frac{(e^{-izk_z} + f_3 e^{izk_z})(e^{-i(H-z_0)k_z} + f_1 e^{i(H-z_0)k_z})}{e^{-iHk_z}(1 - f_1 f_3 e^{2iHk_z})}, \quad z < z_0. \tag{55.12b}$$

Let us consider certain limiting cases.

a) If $\varepsilon_1 = \varepsilon_2 = 1$, we return to the case of a dipole in the atmosphere at height z_0 above a flat homogeneous earth. There is no reflection from the level H, $f_1 = 0$, and, for example, for an electrical dipole $f_3 = f_{\parallel}$ (32.5). This leads to the same results as before, for example at $z_0 = 0$, if we use the nomenclature of (55.12a), to Formula (32.10). The spectrum is continuous with respect to the parameter $\nu = k \sin\alpha$.

b) If both surfaces are absolutely reflective, $\varepsilon_1 = \varepsilon_3 = \infty$, $f_1 = f_3 = 1$, we should obtain the series (55.9) instead of the integral. This is actually the case, as we shall see presently. However, it is better to consider first the more complex case of finite ε_1 and ε_3.

We substitute J_0 by the formula $J_0 = \frac{1}{2}(H_0^{(1)} + H_0^{(2)})$ and take advantage of the fact that $H_0^{(1)}(\nu r) = -H_0^{(2)}(-\nu r)$. Complementing the contour of integration over α by its reflection at the point $\alpha = 0$, so that (see Fig. 55.3) the new contour Γ_1 proceeds from $\alpha = -\frac{\pi}{2} + i\infty$ to $\alpha = -\pi/2$, then along the real axis to $\alpha = +\pi/2$ and then to $\alpha = \frac{\pi}{2} - i\infty$ (this is equivalent to changing in integration over ν from 0 to ∞ to integration from $-\infty$ to $+\infty$), we obtain, for example, for $z > z_0$

$$\Pi = \frac{ik}{4}\int_{\Gamma_1} \sin\alpha\, d\alpha \cdot \Phi_>(\alpha) H_0^{(1)}(kr\sin\alpha). \qquad (55.13)$$

Here it is taken into account that, according to Formula (32.5), f_1 and f_3 satisfy the relationships

$$f(-\alpha) = \frac{1}{f(\alpha)}, \quad \Phi(\alpha) = -\Phi(-\alpha). \qquad (55.14)$$

The integral of (55.13) can be transformed by replacing integration over the contour Γ_1 by integration over other contours, mainly (see Fig. 55.4): a) along a straight line passing from $\alpha = \frac{\pi}{2} + i\infty$ to $\alpha = \frac{\pi}{2} - i\infty$; b) along an infinitely remote circular arc in the upper half-plane, on which $H_0^{(1)}$ is exponentially small (as $\alpha \to i\infty$, we have $\sin\alpha \to -\frac{1}{2i}e^{+\infty} \sim i\infty$, so that $H_0^{(1)}(\nu r) \sim \exp(-\infty)$); c) along circles about the poles P_1, P_2, ... of the expression under the integral sign, which gives a sum of residues, and d) alongside segments (broken line on Fig. 55.4) drawn from the branch points A_k to $+\infty$. But the first of these integrals vanishes, since each of $H_0^{(1)}(kr \sin\alpha)$, $\sin\alpha$ and, according to (55.14), $\Phi(\alpha)$ is an odd function of α, and hence so is their product. The second integral is also zero. The poles P_1, P_2, ... are determined by vanishing of the denominator, which gives the equation $1 - f_1 f_3 e^{2ik_z H} = 0$, $k_z = k\cos\alpha$, or

$$\ln f_1(\alpha_l) + \ln f_3(\alpha_l) + 2ikH\cos\alpha_l = 2\pi i l, \qquad (55.15)$$

where l is an arbitrary integer numbering the pole.

The branch points A_1 and A_2 appear due to the presence of the square roots in f_1 and f_3. That is to say, substituting n_1^2 and n_3^2 for

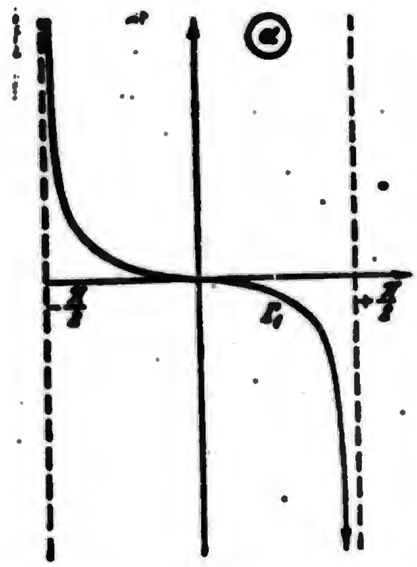

Fig. 55.3. Integration contour in Formula (55.13).

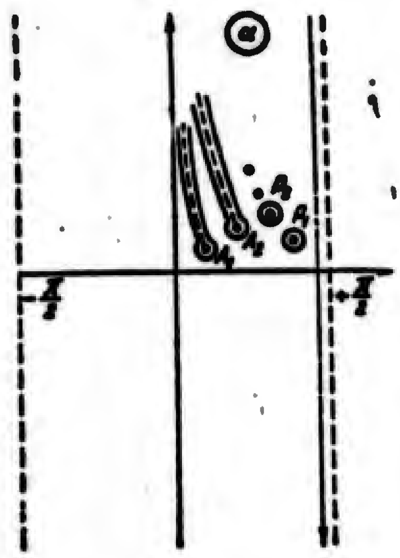

Fig. 55.4. Replacement of integration contour Γ_1 (Fig. 55.3) on conversion from Formula (55.13) to Formula (55.17).

ε in Formula (32.5) for the upper and lower boundaries, respectively, we see that we are speaking of points α_{A_1} and α_{A_2} for which

$$\sqrt{n_1^2-\sin^2\alpha_{A_1}}=0,$$

$$\sqrt{n_2^2-\sin^2\alpha_{A_2}}=0. \qquad (55.16)$$

Finally we obtain

$$\Pi = \frac{ik}{4}\left\{2\pi i \sum_{l=0}^{\infty} \operatorname{Res}\Phi(\alpha_l) H_0^{(1)}(kr\sin\alpha_l)\sin\alpha_l + \right.$$

$$\left. + \sum_{k=1}^{3} \int^{(A_k)} \Phi(\alpha) H_0^{(1)}(kr\sin\alpha)\sin\alpha\, d\alpha \right\}. \tag{55.17}$$

If n_1 and n_3 are very large, then, according to Equalities (55.16), the α_{A_k} have large positive imaginary parts. For this reason, $H_0^{(1)}$ is exponentially small along the path of integration alongside the segments, and these integrals can be dropped. Thus, Π reduces to a sum of residues, a sum of discrete terms. Here $f_1 \approx f_3 \approx 1$, and it follows from Condition (55.15), if we denote $k \sin \alpha_l = \nu_l$, that $H\sqrt{k^2-\nu_l^2} = \pi l$, i.e., ν_l is the same as in Formula (55.5). Further, for $l \neq 0$,

$$\operatorname{Res}\Phi_>(\alpha_l) = [1 - f_1 f_3 \exp(2iHk_z)]\,\Phi_>(\alpha_l) =$$
$$= 4\cos(z_0 k_z)\cos((H-z)k_z).$$

Thus, a solution of the form (55.9), (55.9c) appears. (We shall not dwell on the detailed determination of the coefficients B_l for the vertical dipole, which must result in identity of the two formulas.)

The elementary example analyzed here indicates that waveguide propagation is expressed the discontinuous part of the ν (or α) spectrum, while the continuous part of the spectrum (the integrals in Formula (55.17)) describes propagation through the external layers. In the limiting case of an infinitesimally thin waveguide, $H \to 0$ (or, which is of course equivalent, for identical properties of the waveguide and one of the external regions, for example, $\varepsilon_1 \to \varepsilon_2$), as we have seen, only the continuous part of the spectrum (32.10) remains, while in the other limiting case – when there is no leakage into the surrounding regions – only the discontinuous part is left. In the more general case, the coefficients f differ from both zero and unity. If they are known, then we may determine the poles α_l from Formula (55.15) and the branch points α_{A_k} from Equalities (55.16) to arrive at the sums and quadratures together with the components of Π.

The fully homogeneous layers considered above, for f_1 and f_3 reduce to Fresnel coefficients, represent the simplest case. It may be applied, for example, in problems of radio geology (but it can also be used to study propagation of radio waves near a spherical earth's surface). Application of this theory to (vertically) inhomogeneous layers is more complex (see §§56, 57).

§56. REFRACTION IN THE TROPOSPHERE. ELEMENTARY ANALYSIS

1. First of all, the inhomogeneity of the troposphere gives rise to two important effects in radio-wave propagation. Firstly, the monotonic decrease of ε with altitude gives rise to refraction that strengthens the penetration of waves of any wavelength beyond the horizon if the gradient of ε satisfies certain conditions. Secondly, regions of a waveguide type may arise on nonmonotonic variation of ε, channeling rather short waves far beyond the horizon. Accordingly, we are obliged to turn from consideration of flat layers to analysis of a spherically layered atmosphere, i.e., we are obliged to consider the inhomogeneity of the medium and diffraction around the earth simultaneously. It is found that this more complex problem reduces in many respects either to a plane-layered atmosphere or to diffraction around an earth surrounded by a homogeneous atmosphere.

Thus, we consider an inhomogeneity such that ε depends only on height above the earth:

$$\varepsilon = \varepsilon(h), \quad h = r - a, \tag{56.1}$$

where r is the radial distance from the center of the earth (and not one of the coordinates of the cylindrical system, as it was, for example, in §55).

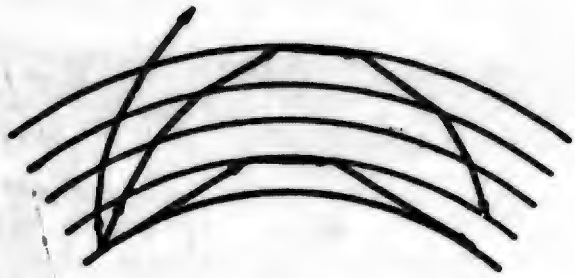

Fig. 56.1. Possible ray trajectories in a layerwise-inhomogeneous atmosphere with a monotonic variation of refractive index.

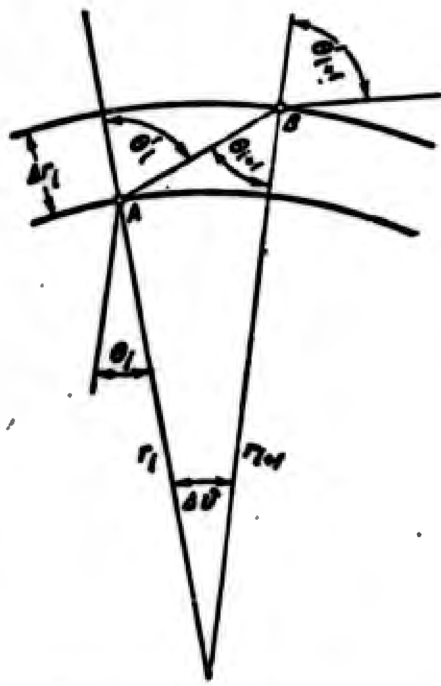

Fig. 56.2. Refraction on passage of a ray through a thin homogeneous spherical layer with sharp boundaries.

The refractive index $n = \sqrt{\varepsilon}$ is often used instead of the permittivity, and by virtue of the smallness of the difference between n and unity, $\varepsilon - 1 \approx 2(n - 1)$.

To obtain a rough orientation, let us assume an inhomogeneity of the nature of homogeneous concentric layers of progressively decreasing density. We represent the source as being at the surface of the earth and break up its radiation into a set of rays leaving the ground at various angles to the horizon. On passing from one layer to another higher layer, which, consequently, has a lower density, each ray will experience a deflection from the normal, and its glancing angle ψ will progressive diminish (Fig. 56.1). A ray that left the ground at a relatively small angle $\psi = \psi_0$ may have acquired horizontal direction at a certain height and may then begin to deflect downward, passing through the same layers in the reverse order with a progressively increasing ψ, until it returns to the ground. Obviously, this return to the ground is possible for a given ψ_0 only for certain laws of decrease of ε. Conversely, it may happen that rays that emerged at too large an angle ψ_0 experience a certain deviation from their original direction on reaching a region where ε reaches its limiting value $\varepsilon = 1$, after which there is no further deflection, the ray leaves the troposphere and is able to return only after reflection from the ionosphere (if these circumstances do not cause it to leave the atmosphere altogether in this case).

Indeed, suppose that the thin spherical layers have a thickness Δr (see Fig. 56.2). A ray incident upon the ith layer at point A at an angle θ_i to the radius r_i is refracted and propagates in the layer at another angle θ_i' (to the same radius) and, in accordance with Snell's law,

$$n_{i-1} \sin \theta_i = n_i \sin \theta_i'. \tag{56.2}$$

However, it is incident upon the next, (i + 1)th layer in a direction that forms an angle θ_{i+1} different from θ_i' with the radius r_{i+1}, since the point of incidence B is displaced away from A and the direction of the radius r_{i+1} itself no longer coincides with the direction r_i here. According to the law of sines, which we apply to triangle OAB, we have $\sin(\pi - \theta_i'):\sin \theta_{i+1} = r_{i+1}:r_i$. Using this relation to express sin θ_i' and substituting it into Formula (56.2), where we may set $(n_{i-1}/n_i) \approx (n_i/n_{i+1})$. with an accuracy to higher-order infinitesimals, we obtain an important result: the product $n_i r_i \sin \theta_i$ remains constant on refraction,

$$n(r)\, r \sin\theta(r) = \text{const} = n_0 r_0 \sin\theta_0. \qquad (56.3)$$

Here the subscript "0" marks the values of these quantities at some point, for example, at the point of emergence of the ray. It follows from this that the angle $\theta(r)$ is known at each point if the refractive index $n(r)$ and the parameters at the beginning of the trajectory are known. In particular, it is thus possible to find conditions under which $\theta = \pi/2$, i.e., the ray acquires horizontal direction (after which, by virtue of the full symmetry of the problem, the ray will be bent in the direction toward the earth, passing through lower and lower layers at the same angles as on the ascending trajectory, provided that ε depends only on h).

The ray radius of curvature ρ at a given point is equal to the ratio of the path increment Δs = AB to the increment in the angle $\Delta\varphi$ of the tangent, i.e., on a layer thickness Δr (see Fig. 56.2), $\rho = \frac{\Delta s}{\Delta\varphi} = \frac{\Delta r}{(\theta_i' - \theta_i)\cos\theta}$. Since $\Delta\theta$ is determined as $\theta_{i+1} - \theta_i$, we have

$$\Delta\varphi = \theta_i' - \theta_i = \theta_i' - \theta_{i+1} + \Delta\theta = \Delta\theta + \Delta\theta = \frac{\Delta r \cdot \operatorname{tg}\theta_{i+1}}{r_i} + \Delta\theta.$$

Further, on differentiating Formula (56.3), we find that $nr \cdot \cos\theta \cdot \Delta\theta = -\sin\theta \cdot \Delta(nr)$. From this we obtain

$$\rho = -\frac{n}{\frac{dn}{dr}\sin\theta}. \qquad (56.4)$$

Obviously, the ray reaches its greatest distance from the surface of the earth when $\rho(r) = r$, $\sin\theta = 1$. This means that $-n = \frac{dn}{dr} \cdot r$, i.e.,

$$\frac{d(nr)}{dr} \equiv n + r\frac{dn}{dr} = 0. \qquad (56.5)$$

Thus, the "reflection point" of the ray is determined by the condition $\frac{dn}{dr} = -\frac{n}{r}$. However, it is precisely in the reflection region that ray-path analysis (geometrical optics) is not accurate (see §57).

An extremely helpful analytical method proceeds directly from the law of refraction for spherical layers (56.3) [VIII, 23]; it consists in introduction of an <u>effective earth's radius</u> and, more broadly, enables us to use the results of the theory developed for the homogeneous atmosphere for propagation of radio waves in an inhomogeneous atmosphere, and vice versa.

If we limit ourselves to small altitude intervals, n may be expanded in series in r − a and limited to the first terms, so that $n \approx n_0 + \left(\frac{dn}{dr}\right)_0 (r-a)$. Setting r − a = h, r = a(1 + h/a), we have instead of Formula (56.3)

$$n_0 a \left(1 + h\left(\frac{1}{a} + \frac{1}{n_0}\left(\frac{dn}{dh}\right)_0\right)\right) \sin\theta = \mathrm{const}. \qquad (56.6)$$

Thus, the ray equation contains additively terms that reflect the influence of the earth's sphericity, h/a, and the influence of the atmospheric inhomogeneity, $(dn/dh)_0$. Hence if the expansion that we use for n is valid, i.e., if the gradient of n may be assumed constant, it will be convenient to set

$$\frac{1}{a_e} = \frac{1}{a} + \frac{1}{n_0}\left(\frac{dn}{dh}\right)_0, \qquad (56.7)$$

where a_e is a new constant, the "effective earth radius." Thus the problem of propagation of the rays in an inhomogeneous atmosphere reduces to the problem of propagation in a homogeneous atmosphere above a spherical earth with a different, <u>larger</u> (since $(dn/an)_0 < 0$) radius. Further, considering a <u>plane</u> inhomogeneous atmosphere ($a = \infty$, see §57), we may substitute for it a homogeneous atmosphere above a spherical surface with $a_e = n_0/(dn/dh)_0$. On the other hand, the converse applies: propagation of radio waves in a homogeneous atmosphere above a spherical earth may be regarded as propagation above a flat earth in an atmosphere with a changed, modified, refractive index. Indeed, setting $r = a(1 + h/a)$ in Formula (56.3) and not even assuming constancy of the gradient of n, we arrive at the conclusion that a picture of the radio wave propagation can be obtained by considering a plane layered atmosphere with $n = n_{mod}$, where, in view of the smallness of the quantities h/a and $n - 1$,

$$n_{\text{мод}} = n(h)\left(1 + \frac{h}{a}\right) \approx n(h) + \frac{h}{a}. \qquad (56.8)$$

Since this quantity is usually of the order of 10^{-4}, it is customary to use the refractive index M,

$$M = (n_{\text{мод}} - 1)\cdot 10^6 = (n - 1)\cdot 10^6 + 0{,}157\cdot h_{м}, \qquad (56.9)$$

where h is expressed in meters. Similarly, the modified permittivity ε_{mod} may also be introduced:

$$\varepsilon_{\text{мод}} = n^2_{\text{мод}} \approx 1 + 2(n_{\text{мод}} - 1). \qquad (56.10)$$

Thus, we shall henceforth be able to extend all results derived for a plane-layered atmosphere (§57) to the spherically layered atmosphere. This last approach – elimination of the curvature of the layers – is the most important in practice.

We note that for $\sin\theta$ near unity (usually the most essential case), we may write according to Formulas (56.4) and (56.7)

$$\frac{1}{a_e} = \frac{1}{a} - \frac{1}{\rho}, \qquad (56.11)$$

i.e., the effective radius a_e is a measure of the "relative curvature" of the earth's surface and the ray trajectory. Deferring the rigorous foundation of the effective-radius method to §58, let us now examine certain corollaries.

2. The monotonic decrease of $\varepsilon(h)$ with a constant gradient is an idealization that is rather seldom justified. However, it will be convenient to use precisely this case as a point of departure.

a) The presence of the negative gradient $\varepsilon (d\varepsilon/dh < 0)$ increases the effective earth radius. Accordingly, the effective distance to the horizon increases: for a source elevated to a height h_0, it will not be $\sqrt{2ah}$, but instead $\sqrt{2a_e h}$. The physical implication of this effect is bending of the rays in an inhomogeneous atmosphere, so that they are able to penetrate beyond the visible horizon (lowermost ray on Fig. 56.1). Formula (15.3) applies for dn/dh (see also the remarks that follow). From Formula (15.4), we obtain for the so-called "standard atmosphere" ("normal refraction")

$$a_e \approx \frac{a}{1+a\frac{dn}{dh}} \approx \frac{4}{3} a \approx 8500 \text{ km}. \qquad (56.12)$$

In general, however, a_e may reach even larger values. Thus, for a vapor-saturated atmosphere, we may obtain $a_e \approx 10{,}000$ km from Formulas (15.3) and (56.7).

b) It may be found that the gradient of ε fully compensates the influence of the curvature, the right member of Formula (56.7) vanishes, and $a_e \to \infty$ ("critical refraction"). This means that as a result of refraction, the radio waves are propagated indefinitely, as above a flat earth (here we dispense everywhere with absorption in the soil). This will occur for $-\frac{dn}{dh} \to 15{,}7 \cdot 10^{-8}$ m^{-1}.

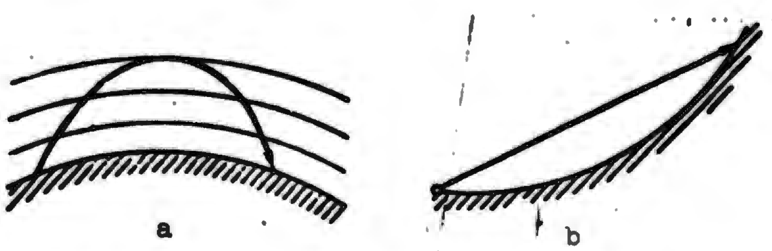

Fig. 56.3. Superrefraction (a) and representation of it for transition to the equivalent radius of an earth with negative curvature (b).

c) If $a_e < 0$, the rays will be bent toward the ground so sharply,

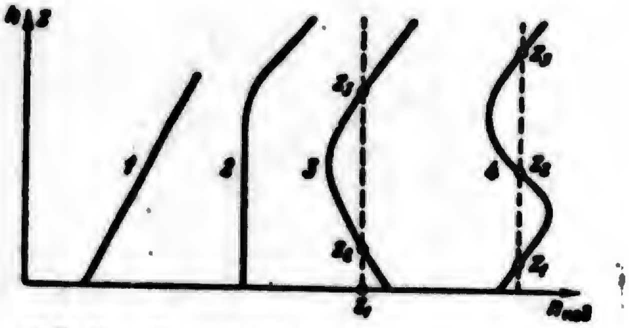

Fig. 56.4. Certain important forms of profiles n_{mod} (h).

experiencing a similar deflection even after reflection from the earth's surface, that they will remain in the layer next to the ground ("superrefraction"). This corresponds to waveguide propagation and, since we are essentially speaking of the case $\rho < r$ here, we have in accordance with Formula (56.4) $-\frac{dn}{dr} > \frac{n}{r} \approx \frac{1}{a}$, and the effective radius is negative.

Figure 56.3 shows how the ray trajectories appear in tropospheric refraction that causes them to penetrate beyond the horizon: in reality on the one hand (Fig. 56.3a) and, on the other, after we pass to a homogeneous troposphere with a modified earth's radius for the case in which $-dn/dh$ is extremely large, all rays are capable of returning to the ground surface, and a_e is negative (Fig. 56.3b). The effective change in the sign of the earth's curvature permits the now rectilinear ("homogeneous atmosphere") rays to reach the ground surface directly.

Figure 56.4 represents certain possible forms of n_{mod} as a function of h; as is customary, the inverse relationship $h(n_{mod})$ is plotted. If the slope is a ($dn_{mod}/dh = 1/a$, $dh/dn_{mod} = a$, compare (56.8)), line 1 describes a homogeneous atmosphere above a spherical earth. If this line is steeper, this corresponds to a uniform (dn/dh = const) decrease of n with altitude, i.e., to the standard atmosphere, for example. Special interest attaches to "anomalous" cases like those represented by curves 2-4.

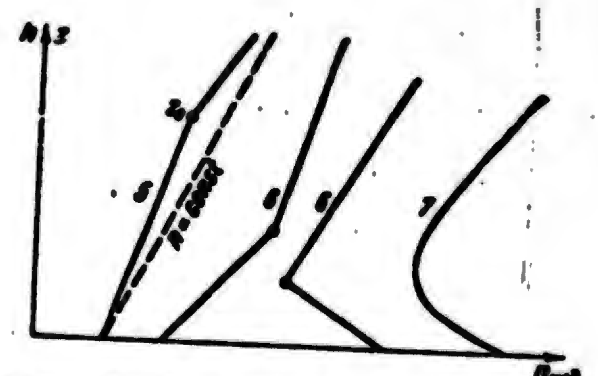

Fig. 56.5. Idealized $n_{mod}(h)$ profiles used in theoretical calculations. A) n_{mod}.

Curve 2 corresponds to the case in which at first, near the ground, $dh/dn_{mod} >> a$, i.e., $dn/dh \approx -1/a$ – "critical refraction," which then goes over into a rapid decrease of n with altitude. Curves 3 and 4 indicate a nonmonotonic curve of $n(h)$, which causes inconstancy of the signs of dn_{mod}/dh. Among other things, we have here regions in which n_{mod} and, consequently, $k^2\epsilon_{mod}$ are so large that, as we shall see later (§57), the theoretically important coefficient $k^2\epsilon_{mod} - \nu^2$ in Eq. (57.3) may become positive for certain real values of admissible . Accordingly, particular solutions arise in which the multipliers $H_0^{(1)}(\nu r)$ will not decay with increasing r for a certain set of discontinuous ν.

Just as for a homogeneous three-layered space (§55), this will correspond to waveguide propagation (for details see §57). Such a situation may arise in the region $z_1 < z < z_2$ in the case of curve 3 ("ground waveguide") and in the case of curve 4 ("elevated waveguide"). Indeed, here we may draw lines (dashed verticals) that have segments to the left of the curves. Needless to say, the presence of such segments on the curves of $h(n_{mod})$ is a necessary but not sufficient condition. Thus, for example, it is also necessary to satisfy Condition (57.25), $\lambda \lesssim 2H_{eff}\sqrt{\delta\varepsilon}$. Otherwise, no discontinuous spectrum of positive v^2 will arise.

Curves 5, 6 and 7 on Fig. 56.5 indicate three idealized types of M-profiles that have been used extensively in theoretical calculations. They are:

a) profile 5, which, in the view of certain authors [5], gives a good representation of mean conditions in the troposphere: a decrease in n with altitude up to a certain z_0, after which n remains constant at unity. Actually, the decrease in n cannot continue indefinitely, since the diminishing density of the air cannot make ε smaller than unity (to the extent that we dispense with the influence of the much higher ionosphere);

b) the "bilinear" or "broken-line linear" profile (curve 6) [I, 13], Chapter 2; [I, 9], §40);

c) the "power-law" profile (curve 7) [6]

$$n_{mod} = 1 + q\left[h - \left(\frac{h}{\Delta}\right)^p \frac{\Delta}{p}\right], \tag{56.13}$$

where q, Δ and p are constants, with p acquiring various values between 0 and 1, q taken from the standard atmosphere, $q = \frac{1}{a} + \frac{1}{n}\left(\frac{dn}{dh}\right)_{stand}$, and Δ a length parameter that has the sense of an effective waveguide thickness.

Using these – or other – relationships for $\varepsilon_{mod}(h)$ in the theory of the plane-inhomogeneous layered medium, for example, in the approximation of geometrical optics (§57), or in the exact solution of Eqs. (57.2)-(57.3) (§58), we can, in principle, obtain the unknown field.

Obviously, curves 6 and 7 must schematize the conditions that prevail in the case of curves 3 and 4. Here, however, as for the scheme expressed by curve 5, the question as to the justifiability of substituting the broken line for the smooth curve remains open.

3. To bring the ray-path treatment to its culmination, let us set forth a more rigorous method that does not assume the medium to be broken down into elementary homogeneous layers with sharp discontinuities of n at their boundaries.

Let us consider the passage of a ray on the basis of the Fermat principle [7]. According to this principle (see (61.45) below and, in general §61, Subsection 3), the path of a ray between two given points 1 and 2 corresponds to an extreme value of the integral giving its propagation time. In the coordinates (r, ϑ), if $n = n(r)$, we are speaking of the two-dimensional problem. The ray path element may be assumed equal to

$$dl = \sqrt{(dr)^2 + r^2 (d\vartheta)^2} = d\vartheta \sqrt{r^2 + \left(\frac{dr}{d\vartheta}\right)^2}$$

and it is necessary to find $r(\vartheta)$ such that

$$J = \int_{\vartheta_1}^{\vartheta_2} F(r, r') d\vartheta = \text{extr}, \qquad (56.14)$$

$$F(r, r') = n(r) \sqrt{r^2 + (r')^2}, \quad r' = \frac{dr}{d\vartheta}. \qquad (56.14a)$$

This means that the variation of J vanishes if we vary $r(\vartheta)$ with fixed values $r(\vartheta_1) = r_1$, $r(\vartheta_2) = r_2$. A peculiarity of a layerwise-inhomogeneous medium is the independence of n and hence of F on ϑ. In the general case, the requirement $\delta J = 0$ leads to the Euler equations, second-order differential equations for $r(\vartheta)$. However, when F does not contain the independent variable in explicit form, a single integration is performed at once and we arrive at the first-order equation

$$F - r' \frac{\partial F}{\partial r'} = C, \qquad (56.15)$$

where C is the constant of integration.

We introduce the angle α between the ray and the horizon at the given point:

$$r' = r\,\mathrm{tg}\,\alpha. \qquad (56.16)$$

At the beginning of the path, the angle α is the emergence angle α_1.

Substituting Expressions (56.14a) and (56.16) into Eq. (56.15), we find

$$C = r_1 n(r_1)\cos\alpha_1 = rn(r)\cos\alpha = \frac{rn(r)}{\sqrt{1+\left(\frac{r'}{r}\right)^2}}. \qquad (56.17)$$

This equation is the same as (56.3). Solving it for r', we obtain

$$\frac{dr}{d\vartheta} = r\sqrt{\frac{r^2n^2(r)}{r_1^2n^2(r_1)}\frac{1}{\cos^2\alpha_1} - 1}, \qquad (56.18)$$

from which the trajectory of the ray, $r(\vartheta)$, is determined for the specific n(r). This method was applied in [7] for the case

$$\varepsilon(r) = n^2(r) = \delta + \frac{\gamma}{r^2},$$

where δ and γ are constants. This dependence corresponds to an actually possible $\varepsilon(r)$. Indeed, expanding in series in h = r − a, we find

$$\varepsilon(h) \approx \delta + \frac{\gamma}{a^2}\left(1 - 2\frac{h}{a}\right) = n^2(a)\left(1 - \frac{2\gamma}{a^2\delta+\gamma}\frac{h}{a}\right).$$

For $\gamma = 0.13a^2$, $\delta = 0.87$, we obtain the standard atmosphere (the case $\gamma = 0.2a^2$, $\delta = 0.8$ was analyzed in detail [7]). We shall consider a more general form with linear variation of n:

$$n = n(a)\left(1 - \gamma\frac{h}{a}\right) \equiv n(r_1)\left(1 - \gamma\frac{r - r_1}{a}\right). \qquad (56.19)$$

First of all, let us determine the conditions under which a ray, having reached its highest point, turns back toward the ground. At this point $r_0 = a + h_0$, we have r' = 0, and, consequently, Formulas (56.16) and (56.19), if it is taken into account that

$$r^2n^2(r) \approx r_1^2n^2(r_1)\left(1 + 2(1-\gamma)\frac{r - r_1}{a}\right).$$

give

$$h_0 - h_1 = -\left(h_1 + \frac{a}{2(1-\gamma)}\right)\sin^2\alpha_1.$$

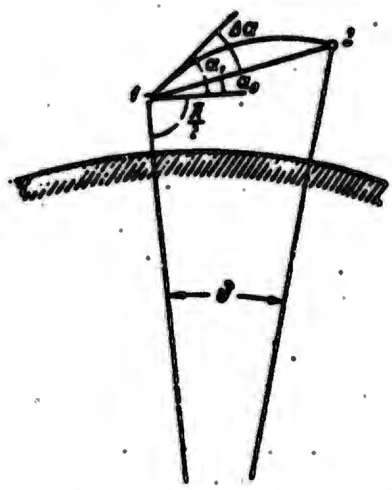

Fig. 56.6. Difference between emergence angle of ray and angle of direction to arrival point that arises as a result of refraction.

Thus, a turning point $h_0 > h_1$ exists only if $1 - \gamma < 0$, as we found earlier, i.e., in the superrefraction region. In the case of normal refraction, an upward-directed ray ($\alpha_1 > 0$) will be deflected progressively downward from the straight line, but the angle that it forms with the horizon will not vanish. The ray will pass to point 2 (Fig. 56.6), which is seen at an angle α_0 with the horizon from the starting point 1, despite the fact that the initial angle α_1 was larger than α_0. Their difference, the refraction angle $\Delta\alpha = \alpha_1 - \alpha_0$, can be found by integrating the ray equation (56.18). For linear n (56.19), disregarding the squares of $(r - r_1)/a$, we can bring Eq. (56.18) to the form

$$\frac{d(r-r_1)}{d\vartheta} = \frac{r_1}{\cos\alpha_1}\sqrt{\sin^2\alpha_1 + 2(1-\gamma+\sin^2\alpha_1)\frac{r-r_1}{a}}. \quad (56.18a)$$

Integrating over r from r_1 to r_2 and over ϑ from ϑ_1 to ϑ_2, we obtain (dropping the common factor $(r_1/a) \approx 1$)

$$\vartheta = \frac{\cos\alpha_1}{1-\gamma+\sin^2\alpha_1}\left\{\sqrt{\sin^2\alpha_1 + 2(1-\gamma+\sin^2\alpha_1)\frac{\Delta h}{a}} - \sin\alpha_1\right\} \quad (56.20)$$

($\Delta h = h_2 - h_1 = r_2 - r_1$). Let us consider the practically important case $\alpha_1 \ll 1$. Since, for example, in the standard atmosphere, $\gamma \approx 1/4$, we drop α_1^2 as small by comparison with unity and with $1 - \gamma$ to obtain

$$(1-\gamma)\vartheta + \alpha_1 = \sqrt{\alpha_1^2 + 2(1-\gamma)\frac{\Delta h}{a}},$$

or, after squaring,

$$\alpha_1 = \frac{\Delta h}{a\vartheta} - \frac{1-\gamma}{2}\vartheta, \quad (\alpha_1 \ll |1-\gamma|). \qquad (56.21)$$

We note that in the other limiting case, $\sin\alpha_1 \sim 1$, since $\Delta h \ll a$, we can expand the radical in powers of Δh, which, in the present approximation, yields

$$\operatorname{tg}\alpha_1 \approx \frac{\Delta h}{a\vartheta}, \quad (\sin^2\alpha_1 \sim 1). \qquad (56.21a)$$

Let us now calculate α_0. Using the sine theorem, we may write $r_2 \sin\left(\pi - \frac{\pi}{2} - \alpha_0 - \vartheta\right) = r_1 \sin\left(\frac{\pi}{2} + \alpha_0\right)$, so that

$$\alpha_0 = \operatorname{arctg}\frac{\cos\vartheta - \frac{r_1}{r_2}}{\sin\vartheta} \approx \operatorname{arctg}\frac{r_2 - r_1 - \frac{a\vartheta^2}{2}}{a\vartheta} \approx \frac{\Delta h}{a\vartheta} - \frac{\vartheta}{2}. \quad (56.22)$$

Here it is taken into account that $\vartheta = \vartheta_2 - \vartheta_1$ is small (or, in other words, $\Delta h/a$ is not small), and that α_0 is also small. Comparing with Formula (56.21), we obtain

$$\Delta\alpha = \alpha_1 - \alpha_0 = \frac{\gamma\vartheta}{2}. \qquad (56.23)$$

In this approximation, therefore, $(\alpha_1^2 \ll |1-\gamma|,\ \Delta h \ll a,\ \vartheta \ll 1)$ the refraction angle does not depend at all on the angle of emergence α_1. It is determined entirely by the angular distance to the target and by the refractive-index gradient $\gamma = -\frac{a}{n(r_1)}\frac{dn(r)}{dr}$.

A more developed and practically highly useful application of the ray treatment in the case of a nonmonotonic variation of $n(r)$ was given in [21].

§57. STRATIFIED MEDIUM WITH ARBITRARILY VARYING PERMITTIVITY. THE APPROXIMATION OF GEOMETRICAL OPTICS

1. In the foregoing section, we showed experimentally that a spherically layered medium can be replaced by a plane-layered medium if the true permittivity $\varepsilon(h)$, which depends on altitude h above the earth's surface, is replaced by the modified permittivity ε_{mod} or by the modified index of refraction n_{mod}:

$$\varepsilon_{mod} = n^2_{mod}, \tag{57.1}$$

$$n_{mod} = n(h) + \frac{h}{a}, \tag{57.1a}$$

where a is the radius of the earth (the rigorous justification and limitations of this rule will be examined in §58). Hence our problem is now reduced to study of a plane-layered medium with $\varepsilon = \varepsilon_{mod}(h)$ above a flat earth's surface with $\varepsilon = \varepsilon_3$ = const. In the present section, the wave equation for the field will be studied as it applies to this problem on the basis of the geometrical-optical approximation, which is frequently quite adequate, in a cartesian coordinate system in which z is reckoned along the vertical and has the sense of height above the earth's surface. Throughout this section, $\varepsilon(z)$ should be understood as $\varepsilon_{mod}(h)$, although the subscript "mod" will not always be given.

As was noted in §15, the normal variation of air density with altitude is enough to create an inhomogeneous medium. Usually, however, the shape of the function $\varepsilon(h)$ is quite complex. The relation of the refractive index $n = \sqrt{\varepsilon}$ to the density, temperature and moisture content of the air is given by Formula (15.1), which is valid even for centimeter waves.

At high altitudes, beyond a few tens of kilometers, we begin to find regions with marked ionization of the air – the ionosphere, where $\varepsilon(z) = n^2(z)$ varies with altitude in a complex manner (and differently at different times of day and in different seasons). Here we may assume in most cases that

$$\varepsilon = 1 - \frac{4\pi N e^2}{m(\omega^2 + \nu^2_{eff})}\left(1 - i\frac{\nu_{eff}}{\omega}\right),$$

where N is the number of electrons in a unit volume, m is the mass of the electron and e is its charge, and ν_{eff} is the effective number of electron collisions per second, with N = N(h). The propagation of radio waves in the ionosphere is the object of a special theory (see, for example, the monographs [I, 2], [I, 3]). However, we shall be interested in the troposphere, where Formula (15.1) applies and ε may be regarded as real with good approximation.

As was indicated in §3, Subsection 4, the field of a vertical electric or magnetic dipole may be described for a layerwise-inhomogeneous medium, as in the case of a homogeneous medium, by the single-component hertzian vectors – $\Pi = \Pi_z$ and $\Pi_m = \Pi_{mz}$, respectively; these are subject to Eq. (3.37a) or (3.37b). That is to say, if we are speaking of a magnetic dipole, the equation takes the form

$$\nabla^2 u + k^2\varepsilon(z)u = 0, \tag{57.2}$$

$u = \Pi_m$, $k = \omega/c$. For the electric dipole, $u = \Pi$, on the other hand, it is, strictly speaking, necessary to take $\varepsilon(z)$ as implying $\varepsilon(z) - \sqrt{\varepsilon}\frac{d^2}{dz^2}\frac{1}{\sqrt{\varepsilon}}$. Even here, however, it is almost always possible to disregard the departure from $\varepsilon(z)$.

Let us apply the normal-wave method. Separating the variables in the cylindrical coordinates r, z as in §55, Subsection 1, i.e., $u = Z(z)R(r)$, we see that the entire difference from the case of homogeneous layers reduces to substitution of the following equation system for (55.3) and (55.4):

$$\frac{d^2Z}{dz^2} + (k^2\varepsilon(z) - \nu^2)Z = 0, \tag{57.3}$$

$$\frac{d^2R}{dr^2}+\frac{1}{r}\frac{dR}{dr}+\nu^2R=0. \qquad (57.3a)$$

The latter equation is solved in zeroth-order cylindrical functions of νr, so that the complete solution takes a form similar to (31.9a),

$$u=2\int_C B(\nu)J_0(\nu r)Z(z;\nu)d\nu, \qquad (57.3b)$$

or, if, as usual, the Bessel function is replaced by a Hankel function (compare (55.11) and (55.13)),

$$u=\int_{C_0} B(\nu)H_0^{(1)}(\nu r)Z(z;\nu)d\nu. \qquad (57.3c)$$

The coefficients $B(\nu)$ of the expansion and the contour of integration are so selected as to satisfy the conditions at the source and on the plane $z = 0$. If it is found that these conditions can be satisfied for real ν, then, much as was shown in §55 and in §31, the relation to r is not of the nature of an exponential decrease. However, it is possible to find $Z(z;\nu)$, to solve Eq. (57.3) analytically, only for certain special forms of the function $\varepsilon(z)$. We shall enumerate the most important cases here, assuming $\nu = 0$ for the sake of simplicity. If $\nu \neq 0$, it will be necessary to substitute $\varepsilon(z) - \nu^2/k^2$ for $\varepsilon(z)$.

a) "Linear layer" [8]:

$$\varepsilon=c+bz, \qquad (57.4)$$

where c and b are constants; $\varepsilon(z)$ vanishes at the point $z_0 = -c/b$. Introducing the new independent variable

$$\zeta=\left(\frac{k}{|b|}\right)^{2/3}\varepsilon(z)=\left(\frac{k}{|b|}\right)^{2/3}c\left(1-\frac{z}{z_0}\right), \qquad (57.5)$$

we reduce Eq. (57.3) to the form $\frac{d^2Z}{d\zeta^2}+\zeta Z=0$. The solution is expressed in terms of cylindrical functions of order 1/3 and exhibits different kinds of asymptotic behavior on either side of the point z_0 at great distances from it. The two independent solutions of the equation can be combined to form a single solution that behaves in a definite fashion at infinity, for example, diminishing as $z \to +\infty$, i.e., if σ and z_0 are positive, as $\zeta \to -\infty$,

$$Z = A\zeta^{1/2}\left\{J_{1/3}\left(\frac{2}{3}\zeta^{3/2}\right) + J_{-1/3}\left(\frac{2}{3}\zeta^{3/2}\right)\right\}, \quad \zeta > 0; \tag{57.6}$$

$$Z = A(-\zeta)^{1/2}\left\{-I_{1/3}\left[\frac{2}{3}(-\zeta)^{3/2}\right] + I_{-1/3}\left[\frac{2}{3}(-\zeta)^{3/2}\right]\right\}, \quad \zeta < 0.$$

These two formulas describe a function that is continuous together with its derivatives at $\zeta = 0$.

For $\zeta \gg 1$ (remote from the plane on which $\varepsilon = 0$), we have, using the asymptotic expressions for the Bessel functions, $J_p(x) \sim \sqrt{\frac{2}{\pi x}} \times \cos\left(x - \frac{(2p+1)}{4}\pi\right)$,

$$Z = \frac{2A}{\sqrt{\pi}\,\zeta^{1/4}} \cos\left(\frac{2}{3}\zeta^{3/2} - \frac{\pi}{4}\right), \tag{57.7}$$

and the solutions of the standing-wave type. For $\zeta < 0$, $-\zeta \gg 1$, the field decays exponentially. Figure 57.1 shows schematically the nature of the solutions in the two regions. A decrease in ε with increasing z corresponds to this figure, i.e., $b < 0$. The standing wave (57.7) can be interpreted as a superposition of waves traveling in opposite directions: one incident from the negative-z side and one reflected back from the region around the point $z = z_0$, beyond which the field decreases exponentially.

This type of $\varepsilon(z)$ function is particularly important for two reasons. Firstly, it is used to determine the constant of integration in a highly general approximate method (see below, Subsection 2). Secondly, with ε_{mod} substituted for ε, it corresponds to the problem of a homogeneous atmosphere and a spherical earth, which was considered in Chapter 6. Indeed, the solution of this problem can be obtained by the method set forth here if we assume, according to Equality (57.1),

$$\varepsilon_{mod}(h) = \left(1 + \frac{h}{a}\right)^2 \approx 1 + 2\frac{h}{a}. \tag{57.4a}$$

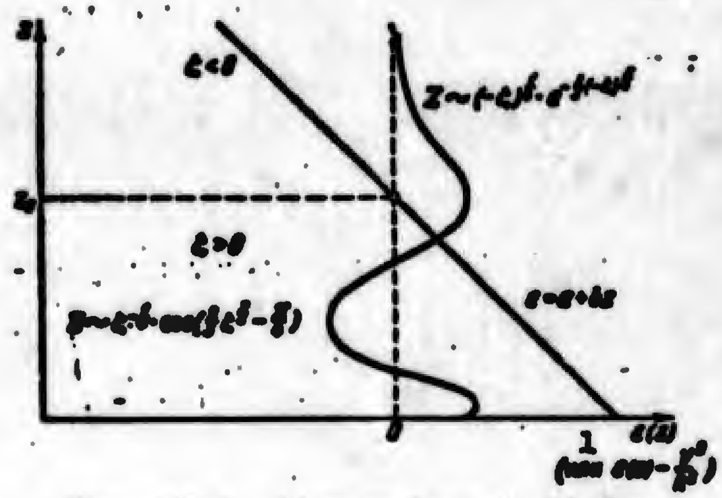

Fig. 57.1. Linear layer. 1) Or.

where a is the earth's radius, so that it is necessary to consider $c = 1$, $b = 2/a$ in Equality (57.4) and subsequent formulas. Accordingly, the solution of the diffraction problem in Chapter 6 also contained Airy functions as altitude factors, i.e., Hankel functions of order 1/3 (38.28). In the case of an atmosphere that is inhomogeneous but has ε linearly dependent on h, it is only necessary to assume that $c = \varepsilon(0)$, $b = 2/a_e$.

b) The "parabolic layer" when in a certain interval of z-values from $-z_m$ to $+z_m$ [9] (c and b are constants),

$$\varepsilon = c + bz^2, \; b > 0. \qquad (57.8)$$

The results vary essentially depending on the sign of c. For $c > 0$, the function $\varepsilon(z)$ does not vanish anywhere. Equation (57.3) is solved in Weber functions – in functions of a parabolic cylinder [10]. That is to say, if for $\nu = 0$ we set

$$u = \sqrt[4]{4k^2 b}\, e^{-i\frac{\pi}{8}} z, \quad p = -\frac{1}{2} + i\frac{kc}{2\sqrt{b}}, \qquad (57.8a)$$

then Eq. (57.3) is brought to the form

$$\frac{d^2 Z}{du^2} + \left(p + \frac{1}{2} - \frac{1}{4}u^2\right) Z = 0 \qquad (57.8b)$$

and has independent solutions [II, 11]

$$Z_1 = D_p(u),\quad Z_2 = D_{-p-1}(iu). \tag{57.8c}$$

From these we can form the solutions that are twice shown schematically in Fig. 57.2.

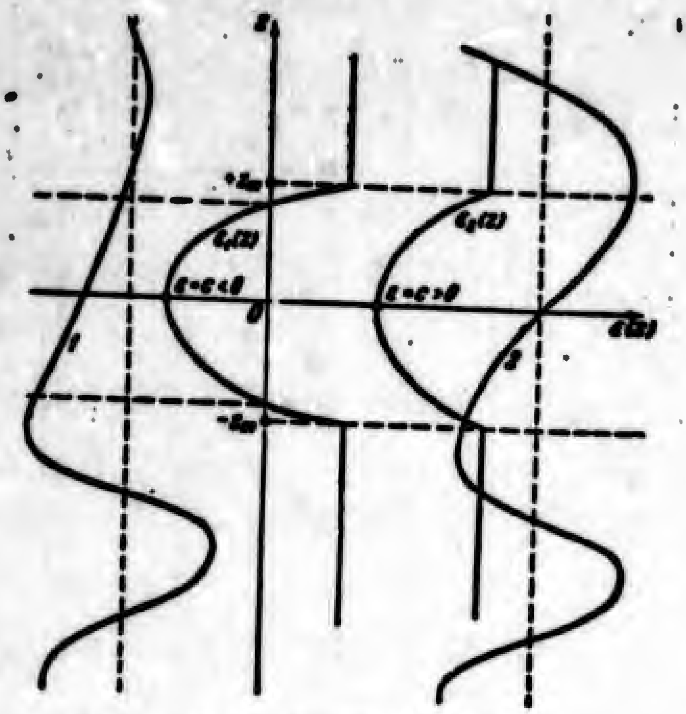

Fig. 57.2. Parabolic layer.

In the case $c > 0$, the solution can be given the form of a standing wave (with varying distances between the nodes and a varying amplitude). But there also exist solutions of form such that in one half-space, say as $z \to \infty$, the solution takes the form of a receding wave, while for $z \to -\infty$ it takes the form of a superposition of the incident and reflected waves. Another independent solution in this case will take the form of a receding wave for $z \to -\infty$ and a superposition of the wave incident from $z = +\infty$ and the reflected wave for $z \to +\infty$. The reflected-wave amplitude will be the larger the smaller $\varepsilon(0) = c$.

On the other hand, for $c < 0$, an oscillating solution (standing wave or superposition of incident and reflected waves) arises only in regions where $\varepsilon > 0$. The solution, which contains a receding wave for $z \to \infty$, contains incident and reflected waves for $z \to -\infty$, and an exponentially diminishing field in the region where $\varepsilon < 0$. Here, obviously, part of the wave is transmitted and part reflected.

c) Layer [11]

$$\varepsilon = 1 - P\frac{e^{mz'}}{1+e^{mz'}} - M\frac{4e^{mz'}}{(1+e^{mz'})^2}, \tag{57.9}$$

which leads to a solution in the form of hypergeometric functions. For P = 0 and M real and positive, this is the so-called "symmetrical layer," while for M = 0 and P real and positive, it is the "transition layer" (Fig. 57.3). Here z' is reckoned from the middle of the layer, $z' = z - z_0$.

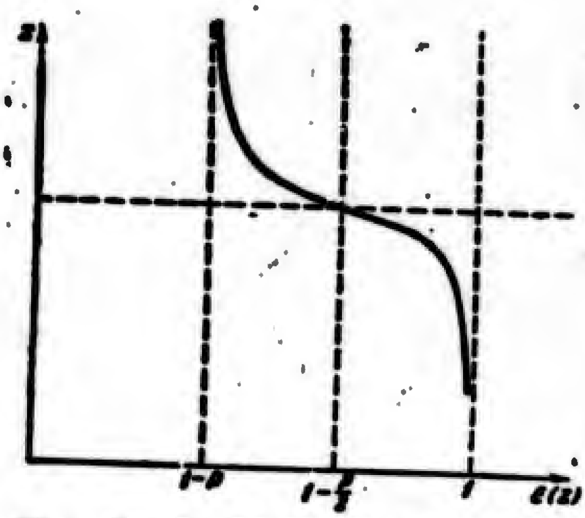

Fig. 57.3. "Transition layer."

d) Layer with

$$\varepsilon = \frac{1}{d+z}; \tag{57.10}$$

after variable substitution $d + z = \zeta$, this leads to an equation that can be solved in degenerate hypergeometric functions.

c) Layer with

$$\varepsilon = \frac{c}{b^2 + z^2}; \qquad (57.11)$$

after the same substitution, the equation is reduced to an equation for Bessel functions.

However, it is often possible to get by without exact solution in problems of atmospheric radio-wave propagation. A specific aspect of the problem consists in the fact that the vertical parameters of the layers that are of interest to us – the thicknesses on which ε changes appreciably and accordingly the propagation conditions also change appreciably – are usually very large as compared with wavelength. This is not the case, for example, in the propagation of superlong waves ($f \sim 100$-1000 Hz, i.e., $\lambda \sim 3000$-300 km), which are formed by lightning discharges and propagate in the layer between the ground and the ionosphere. In this last case, the wavelength is longer than the waveguide height H or, in any event, comparable to it. Further, the relation $\lambda \sim H$ also obtains for tropospheric waveguides on inversion of the altitude curve of ε (see §58 below). Accordingly, the number of modes that must be taken into account for long distances is small. It will be helpful, however, to consider first the inverse case, which not only permits simplified treatment in many respects, but also facilitates orientation to the problem.

2. We shall first proceed on the assumption that ν is real. The smallness of the variation of ε over wavelength is a circumstance highly favorable to the use of the so-called geometrical-optical approximation. Indeed, we introduce the characteristic length l_0:

$$\left|\frac{1}{\varepsilon(z)}\frac{d\varepsilon}{dz}\right| \sim \frac{1}{l_0}.$$

If

$$kl_0 \gg 1, \qquad (57.12)$$

then the coefficient of Z varies slowly in Formula (57.3) and the solution must be either "almost sinusoidal," if $k^2\varepsilon(z) - \nu^2 > 0$, or "almost exponential," if $k^2\varepsilon(z) - \nu^2 < 0$. Indeed, the second-order ordinary differential equation (57.3) has two independent solutions. If at long distances, for example, as $z \to -\infty$ and $z \to +\infty$, the function $\varepsilon(z)$ goes to certain constant values ε_1 and ε_2, then for correspondingly large $|z|$, we may consider

$$Z_1^{\pm} \sim e^{\pm i\sqrt{k^2\varepsilon_1 - \nu^2}\,z} \text{ for } z \to -\infty;$$

$$Z_2^{\pm} \sim e^{\pm i\sqrt{k^2\varepsilon_2 - \nu^2}\,z} \text{ for } z \to +\infty \qquad (57.13)$$

as solutions to the equation. The first solution in each pair yields a wave traveling along the z-axis, while the second gives a wave traveling in the opposite direction.

However, this is true only provided that the radical $\sqrt{k^2\varepsilon_{1,2} - \nu^2}$ is real. When it is imaginary, the solutions show an exponential increase or decrease. Instead of these solutions, we might use as the basic solutions their linearly independent combinations

$$Z_1^{s,c} \sim Z_1^{+} \mp Z_1^{-} \sim \frac{\sin}{\cos}\left\{\sqrt{k^2\varepsilon_1 - \nu^2}\,z,\right.$$

$$Z_2^{s,c} \sim Z_2^{+} \mp Z_2^{-} \sim \frac{\sin}{\cos}\left\{\sqrt{k^2\varepsilon_2 - \nu^2}\,z.\right. \qquad (57.13a)$$

The usual form of this geometrical-optical approximation (for refinements, see below in this subsection and §61) consists in seeking the solution of our equation

$$\frac{d^2Z}{dz^2} + p^2(z)Z = 0, \qquad (57.14)$$

$$p^2(z) = k^2\varepsilon(z) - \nu^2 \qquad (57.14a)$$

in a form resembling Solution (57.13), i.e.,

$$Z = A(z)\,e^{ik\varphi(z)}, \qquad (57.15)$$

and considering in succession the smallness of $1/kl_0$ or simply $1/k$. Substituting Z into Eq. (57.3), we obtain (the prime denotes differentiation with respect to z)

$$A'' + i(2A'\varphi' + A\varphi'')k - A\varphi'^2k^2 + p^2A = 0. \tag{57.15a}$$

If in first approximation we retain only terms of the highest order in k (and in $p \sim k$), we obtain an equation for φ that has two solutions (they differ in sign):

$$k^2\varphi'^2 = p^2(z),\ k\varphi = \pm \int^z p(z)\,dz. \tag{57.16}$$

Equating the terms of the next order in k to zero in Eq. (57.15a), we obtain an equation for determination of A:

$$2A'\varphi' + A\varphi'' = 0,$$

i.e.,

$$\ln A = -\ln \sqrt{\varphi'} + \text{const},$$

$$A \sim \frac{1}{\sqrt{p(z)}}. \tag{57.16a}$$

Thus, we obtain two independent solutions, which can be written as follows:

$$Z^{\pm} = \frac{B}{\sqrt{p(z)}} e^{\pm i\left\{\int_{z_0}^{z} p(z)\,dz + \psi\right\}}, \tag{57.17}$$

where B, ψ and z_0 are constants of integration (essentially, one independent complex or two real: ψ and z_0 are interrelated).

It is obvious that Eq. (57.15a) can be solved by the method of successive approximations only provided that not only k^2, but also the values of p(z) are large enough. This second condition is violated at the point $z = z_0$ (if such a point exists), where $p^2(z) = 0$. But this point is generally of particular importance. On one side of it, the function $p^2(z)$ is positive, the phase Z of the wave is real and the solution is oscillatory. On the other side, the solution diminishes or increases exponentially. Thus, in the neighborhood of z_0, the solution found by the geometrical-optical method is invalid. However, in this small region we can approximate the curve of $\varepsilon(z)$ by a linear function, setting, for example, $(\nu^2/k^2) = c + bz$. Then it is possible to obtain an exact solution for this small region, in the form of (57.6) or (57.7) (where $\varepsilon - \nu^2/k^2$ must be substituted for ε), and to sew the two solutions together somewhere on the boundary of the region. In particular, we obtain from Formula (57.7), taking the half-sum of the particular solutions in the region where $k^2\varepsilon - \nu^2 > 0$,

$$\mathcal{E} = \frac{B}{\sqrt[4]{k^2\varepsilon(z) - \nu^2}} \cos\left\{\int_{z_0}^{z} \sqrt{k^2\varepsilon(z) - \nu^2}\,dz + \psi\right. \qquad (57.17a)$$

This solution must be sewed to the exact solution in the region $z \approx z_0$. But Expression (57.17a) has exactly the same structure as (57.7). Equating the multipliers for the cosines, for example, at $\nu = 0$, we determine the nonessential constant factor, while comparison of the cosines themselves gives (we use Formula (57.4) in performing the integration):

$$\cos\left(\frac{2}{3}\frac{k}{|b|}\varepsilon^{3/2}(z) - \frac{\pi}{4}\right) = \cos\left(k\int_{z_0}^{z} \sqrt{c + bz}\,dz + \psi\right) =$$
$$= \cos\left(\frac{2}{3}\frac{k}{b}\varepsilon^{3/2}(z) + \psi\right). \qquad (57.17b)$$

If, to cite a concrete case, $b < 0$ (see Fig. 57.1), then $\psi = \pi/4$. Thus, for a function $\varepsilon(z)$ that diminishes with z, the geometrical-optical approximation, correctly sewn to the exact solution in the region $z = z_0$ (where the approximation is unsuitable), gives the two independent solutions (57.17) with $\psi = \pi/4$. Where $p^2(z) > 0$, the solution with the plus sign in the exponent describes a wave propagating in the positive direction of the z-axis, while a solution with a minus sign describes the reflected wave. The reflection coefficient f for observation at point z is therefore

$$f=\frac{E^-}{E^+}=e^{-i\left\{\int_{z_0}^{z}\sqrt{k^2\varepsilon(z)-\nu^2}\,dz+\frac{\pi}{4}\right\}},\quad z<z_0. \qquad (57.18a)$$

If, however, $\varepsilon(z)$ increases with altitude, $b > 0$, we may speak only of upward reflection of a wave coming down from $z = +\infty$. Then, according to Formula (57.17b), we obtain $\psi = -\pi/4$. Hence with such a variation of $\varepsilon(z)$ at a point z above point z_0, at which the radical vanishes, we shall have

$$f=\frac{E^+}{E^-}=e^{i\left\{\int_{z_0}^{z}\sqrt{k^2\varepsilon(z)-\nu^2}\,dz-\frac{\pi}{4}\right\}},\quad z>z_0. \qquad (57.18b)$$

The integral can be evaluated within the limits of the linear layer. It is equal (for $\nu = 0$) to $\frac{2}{3}\frac{k}{b}\varepsilon^{3/2}(z)$ (see Formula (57.17b)). However, as is clear from the derivation, the formulas for f are valid for an arbitrary function $\varepsilon(z)$, provided that the approximation of geometrical optics is valid: we must introduce a linear $\varepsilon(z)$ only in a small region near the "turning points" z_0, where the geometrical-optical approximation is invalid, and then only to determine the constant phase shift ψ, since it cannot be determined from the approximate solution.

However, it is clear that the expressions found cannot be valid if the point z_0 is near an extreme of the $\varepsilon(z)$-curve, where $d\varepsilon/dz$ vanishes and the linear approximation is not suitable for $\varepsilon(z)$. In such a case, however, the parabolic approximation using the parabolic-cylinder functions (57.8c) instead of (57.6) as the exact solution in a small region becomes possible. However, we shall set forth this case below (see Subsection 6).

3. On the basis of Solution (57.17), it can be stated that for given k and (real) ν, the space breaks down along the z-axis into a region in which the difference $k^2\varepsilon(z) - \nu^2$ is positive and, accord-

ingly, the solution is oscillatory, and a region in which it is negative and the independent solutions are of the nature of an exponential decrease or increase. In any event, it is obvious that an oscillating solution is possible only for sufficiently small ν^2.

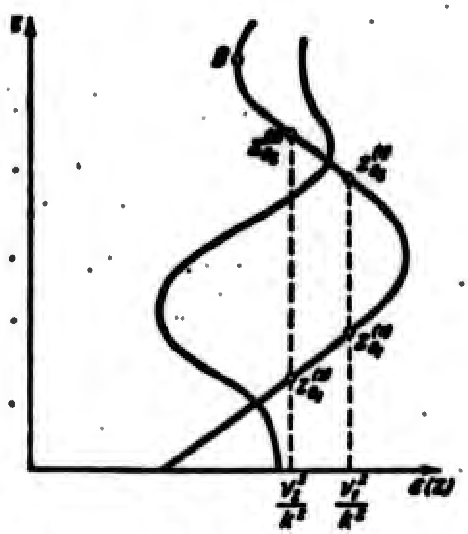

Fig. 57.4. Elevated waveguide.

Let us now show that if such oscillatory solutions occur only in a limited region of z-values, the possible values of ν will form a discontinuous sequence – they will have, as in the case of a homogeneous layer between reflective boundaries (55.5), a discontinuous spectrum.

Suppose that there are values of the number ν such that $k^2\epsilon(z) - \nu^2$ vanishes at points z_{01} and $z_{02} > z_{01}$, between which this expression is positive (Fig. 57.4). For $z_{01} < z < z_{02}$, therefore, the solution is oscillatory, and outside this segment it diminishes exponentially (this curve of the solution is shown schematically on the same figure; see left-hand curve). Above we considered only one turning point, z_0. It was possible to satisfy the condition of sewing to the exact solution at this point by proper selection of the integration constant ψ. Now we are also obliged to satisfy the sewing condition at the second turning point. This can be done if we have the sole remaining parameter ν. Thus, the solution, which is finite throughout the space (and accordingly decreases exponentially at the points z_{01} and z_{02}), can exist for selected values of ν, which are known as natural values. Then the solutions themselves will be normal solutions or normal waves for the medium in question. They are determined from the following considerations.

Considering the oscillatory solution below the return point with an $\varepsilon(z)$ that decreases with altitude, i.e., in this case in the region $z < z_0 = z_{02}$, we found it in the form (57.17a) with $\psi = \pi/4$:

$$Z^c = \frac{B}{\sqrt[4]{k^2\varepsilon - \nu^2}} \cos\left\{\int_{z_{02}}^{z} \sqrt{k^2\varepsilon(z) - \nu^2}\, dz + \frac{\pi}{4}\right\}. \qquad (57.19)$$

When we instead assumed $\varepsilon(z)$ to be an <u>increasing</u> function of z, i.e., when we set $b > 0$ in Formula (57.4), the same reasoning led us to the conclusion that $\psi = -\pi/4$. Here the oscillatory solution occurs for $z > z_0$. Setting $z_0 = z_{01}$, we have

$$Z^c = \frac{B}{\sqrt[4]{k^2\varepsilon - \nu^2}} \cos\left\{\int_{z_{01}}^{z} \sqrt{k^2\varepsilon - \nu^2}\, dz - \frac{\pi}{4}\right\}. \qquad (57.20)$$

However, these two solutions must agree for $z_{01} < z < z_{02}$, which occurs only provided that the arguments of the cosines differ by $l\pi$, where $l = 0, 1, 2, \ldots$, is a whole number:

$$\int_{z_{02}}^{z} \sqrt{k^2\varepsilon - \nu^2}\, dz + \frac{\pi}{4} = \int_{z_{01}}^{z} \sqrt{k^2\varepsilon - \nu^2}\, dz - \frac{\pi}{4} - l\pi,$$

or

$$\int_{z_{01}}^{z_{02}} \sqrt{k^2\varepsilon(z) - \nu^2}\, dz = \left(l + \frac{1}{2}\right)\pi. \qquad (57.21)$$

Going through all possible values of l, we obtain the corresponding ν_l and, consequently, the normal oscillations of Z_l^c generated by either of Formulas (57.19)-(57.20), which are in agreement after substituting ν_l. We note that from a purely formal point of view, this process of selecting normal solutions corresponds fully to the atomic-orbit quantizing procedure in quasiclassical approximation; Formula (57.21) corresponds to the quantizing rule, while integrals of the type (57.21) correspond to the phase integrals in the Bohr-Sommerfeld theory; hence the original name given this method in the theory of radio-wave propa-

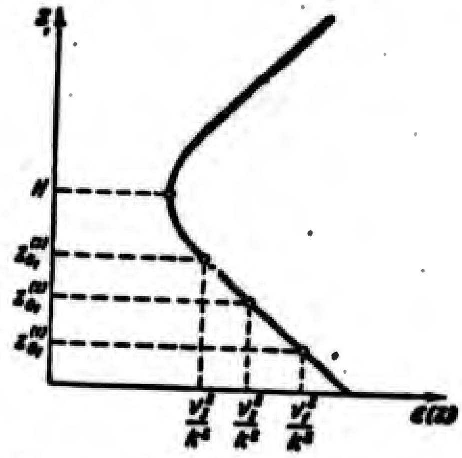

Fig. 57.5. Ground waveguide.

gation – the "phase-integral method" [1].

A similar selection of possible ν_l arises if $\varepsilon(z)$ decreases monotonically with altitude in a certain region, but the region of values where the difference $k^2\varepsilon(z) - \nu^2$ can be positive is bounded below by the earth's surface $z = 0$ (Fig. 57.5). In the region $0 < z < z_{01}$, where z_{01} is still to be determined, we may again have a solution of the form (57.17a), which must satisfy the boundary condition for a vertically polarized wave at $z = 0$:

$$\frac{dZ^e}{dz} = -\frac{ik}{\sqrt{\varepsilon_3}} Z^e \text{ for } z = 0, \tag{57.22a}$$

or – for a horizontally polarized wave –

$$\frac{dZ^e}{dz} = -ik\sqrt{\varepsilon_3 - 1}\, Z^e \text{ for } z = 0 \tag{57.22b}$$

(ε_z is the permittivity of the soil). If we assume $|\varepsilon_z| \gg 1$, these conditions signify for practical purposes that

$$\text{vertical polarization: } \left(\frac{dZ^e}{dz}\right)_{z=0} = 0; \tag{57.22c}$$

$$\text{horizontal polarization: } (Z^e)_{z=0} = 0. \tag{57.22d}$$

In the former case, substituting Solution (57.17a) with $\phi = \pi/4$, $z_0 = z_{01}$, we arrive at the condition

$$\sin\left\{\int_0^{z_{01}} \sqrt{k^2\varepsilon(z) - \nu^2}\, dz - \frac{\pi}{4}\right\} = 0, \tag{57.23a}$$

i.e., the argument of the sine is $\pm l\pi$, where l is a whole number. Since the root under the integral sign is positive, we have finally for vertical polarization

$$\int_0^{z_{01}} \sqrt{k^2\varepsilon(z)-\nu^2}\,dz = \left(l+\frac{1}{4}\right)\pi,\quad l=0,1,2,\ldots \tag{57.23b}$$

For horizontal polarization we obtain in a similar manner

$$\int_0^{z_{01}} \sqrt{k^2\varepsilon(z)-\nu^2}\,dz = \left(l-\frac{1}{4}\right)\pi,\quad l=1,2,\ldots \tag{57.23c}$$

Since the point z_{01} is determined by the condition $k^2\varepsilon(z_{01}) - \nu^2 = 0$, these relationships may be written somewhat differently:

$$\int_0^{z_{01}} \sqrt{\varepsilon(z)-\varepsilon(z_{01})}\,dz = \frac{\lambda}{2}\begin{cases} l+\frac{1}{4};\ l=0,1,\ldots\ \text{(vert. pol.)} \\ l-\frac{1}{4};\ l=1,2,\ldots\ \text{(horiz. pol.)} \end{cases} \tag{57.23d}$$

Assigning values to l, we find the turning points $z_{01}^{(1)}$, $z_{01}^{(2)}$ (Fig. 57.5), which determine the height of the region above the earth's surface within which the field oscillates along the z-axis and, at the same time, real values of the parameter $\nu_l = k\sqrt{\varepsilon(z_{01}^{(l)})}$ exist.

4. We return to the complete expression for $u_\nu = Z_\nu(z)R_\nu(r)$ and examine it for the function $\varepsilon(z)$ shown in Fig. 57.4 or 57.5. The function $R_\nu(r)$ will, as already noted, be a solution of an equation identical to the equation for a homogeneous layer (55.4), i.e., it will reduce to Hankel functions $H_0^{(1)}(\nu_l r)$ for values of ν_l taken according to Formula (57.21) or (57.23).

Thus, in the interval between minimal ε_{min} and maximal ε_{maks} values of ε, for $k^2\varepsilon_{min} < \nu^2 < k^2\varepsilon_{maks}$, the ν-spectrum is discontinuous. All ν with $\nu^2 > k^2\varepsilon_{maks}$ are forbidden (the solutions rise or decay exponentially for all z), but for $\nu^2 < k^2\varepsilon_{min}$, no limitations at all are imposed and the spectrum is continuous. The corresponding Z_ν are partial waves propagating freely from the source.

We shall subsequently see that this rough approximation is inadequate for many problems of tropospheric propagation, and we shall improve it in two respects. First, near the extreme $\varepsilon(z) = \varepsilon(Z_B)$, the linear approximation is not admissible and it is necessary to sew the solution of type (57.17), (57.17a) to the exact solution not for a linear ε (57.4), but, for example, for a parabolic one (57.8). Secondly, even if $p^2(z) > 0$ for all z, the passage of waves along z is not entirely free. However, the discontinuous segment of the spectrum in the region of real ν (provided that the value of ν_l determined from Formula (57.21) or (57.23d) is not near the extreme of $\varepsilon(z)$) is determined quite satisfactorily by the procedure presented above. The selected solutions, if there are any (Fig. 57.4 shows the case in which the first two values, and Fig. 57.5 that in which the first three values are such solutions: $\nu_1^2 = k^2\varepsilon(z_{01}^{(1)})$, $\nu_2^2 = k^2\varepsilon(z_{01}^{(2)})$ and $\nu_3^2 = k^2\varepsilon(z_{01}^{(3)})$), indicate that the region $0 < z < Z_B$ (Fig. 57.4) or $0 < z < H$ (Fig. 57.5) possesses waveguide properties. It is customary to say that these normal waves are "trapped" by the waveguide. The correspondence with the case of three homogeneous layers (§55) is obvious (although it must be noted that the correspondence is not literal: in the case of §55, there is no field outside the waveguide because the existence of conductivity is assumed here. Otherwise, with $\varepsilon_1 = \varepsilon_3 = \infty$, wave propagation would be possible here for any ν). Figure 57.5 shows the "ground waveguide," and Fig. 57.4 the "elevated waveguide." Above and below the waveguide, the solutions corresponding to the natural values of ν_l^2 cannot propagate along z – they decay exponentially.

If the radiator is placed in the layer $z_{01} < z < z_{02}$ its field will contain a superposition of solutions $u_\nu = R_\nu(r)Z_\nu(z)$ that has the necessary singularity at the position of the radiator. Generally speaking, all ν will be represented here. But those of them for which $\nu^2 < k^2\varepsilon_{min}$ – in the approximation being considered – diverge freely in space. Only the partial waves trapped by the waveguide remain inside it and are propagated indefinitely along r.

Suppose that the radiator is located outside the waveguide. The discontinuous-spectrum waves cannot propagate outside the waveguide and decay exponentially along z in either direction from the source. However, if the distance to the true waveguide region is not very long, the exponentially damping solution may still be rather large at its boundary. It must be sewn to the oscillating solution at this boundary, and it may undergo waveguide propagation along r. This means that the field is partly drawn into the waveguide and will then propagate in it without attenuation. Similarly, if the observation point is not in the waveguide itself, but near it, the field received will be exponentially small as compared with the field at the nearest point in the waveguide.

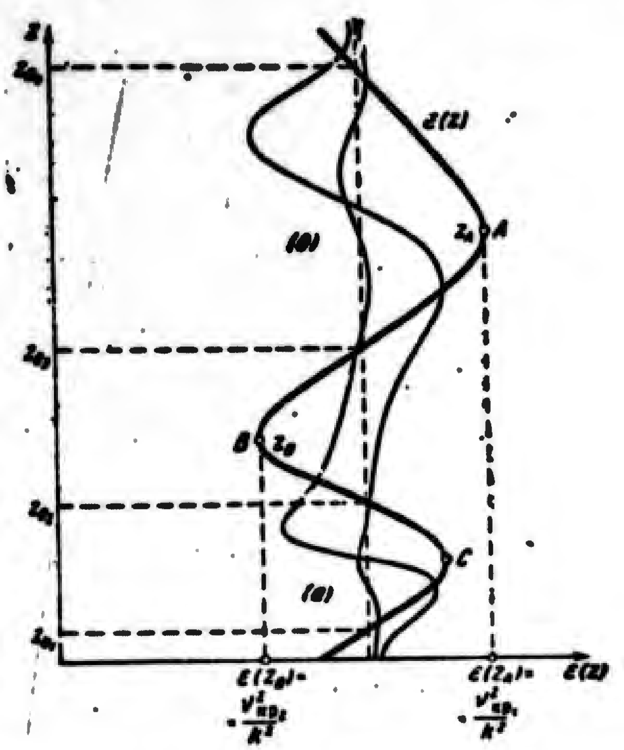

Fig. 57.6. Two connected waveguides.

Let us now consider the case of an $\varepsilon(z)$ such that the difference $k^2\varepsilon(z) - \nu_l^2$ can be positive not in one but in several z-regions (Fig.

57.6). Then it is possible for the solution to be nonzero and oscillating in two regions: in region a, $z_{01} < z < z_{02}$, and in region b, $z_{03} < z < z_{04}$. It is obvious that the trapped waves from a source inside the waveguide region a will not only propagate along the layer, but will also be drawn in in weakened form into the other region b, in which they will again propagate without attenuation if the waveguide parameters are appropriate. The second waveguide will draw in energy from the first until, in the region of large enough r, back suction into the first waveguide becomes substantial. In the final analysis, a kind of equilibrium will be reached and the fields in the two waveguide channels will propagate matched to one another.

Let us now consider the question as to the minimum width of the region necessary for the appearance of waveguide propagation. Suppose that as we move away from point z_0, at which $k^2\varepsilon - \nu^2 = 0$, $\varepsilon(z)$ increases linearly, i.e., $\varepsilon(z) = \varepsilon(z_0) + \beta z$ until we have reached the middle of the channel (point $z = z_0 + H_{eff}/2$) and then decreases, again linearly. Then Formula (57.21) gives the following condition even for $l = 0$:

$$2\int_{z_0}^{z_0+\frac{H_{\text{эфф}}}{2}} \sqrt{k^2(\varepsilon(z_0)+\beta z)-\nu^2}\,dz = 2\frac{kH_{\text{эфф}}\sqrt{\delta\varepsilon}}{3\sqrt{2}} > \frac{\pi}{2},$$

where $\delta\varepsilon = \varepsilon\left(z_0 + \frac{1}{2}H_{\text{эфф}}\right) - \varepsilon(z_0)$ is the maximum deviation of the function ε from its value at z_0. Consequently, we must have

$$\lambda < \frac{4\sqrt{2}}{3} H_{\text{эфф}} \sqrt{\delta\varepsilon}. \qquad (57.24)$$

Since this formula was obtained for a specific profile $\varepsilon(z)$, it will be better to speak of orders of magnitude. Thus, the condition takes the form

$$\lambda \lesssim 2H_{\text{эфф}} \sqrt{\delta\varepsilon}. \qquad (57.25)$$

Consequently, waveguide channels may be expected to form only for short waves. With $\delta\varepsilon \sim \varepsilon - 1 \sim 10^{-4}$ and $H_{eff} \sim$ 10-100 m

$$\lambda \lesssim 10 \div 100 \text{ cm}. \tag{57.26}$$

5. Separation of variables using cylindrical coordinates was necessary because we used a vertical dipole as our source. If instead we consider a plane wave with a propagation vector lying, for example, in the xz-plane, incident at an angle α_0 from the homogeneous half-space $z < 0$ onto a layerwise-inhomogeneous half-space $z > 0$,

$$\Pi = \Pi_0 e^{i(k_x x + k_z z)}, \quad z < 0, \quad k_x = k \sin \alpha_0, \quad k_z = k \cos \alpha_0, \tag{57.27}$$

then for $z > 0$ the solution is reasonably sought in the form

$$\Pi = \Pi_0 e^{ik_x x} Z(z),$$

where substitution of

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + \varepsilon(z) k^2\right) \Pi = 0 \tag{57.28}$$

in the waveguide equation gives an equation for Z:

$$\frac{d^2 Z}{dz^2} + k^2 (\varepsilon(z) - \sin^2 \alpha_0) Z = 0. \tag{57.29}$$

If we use the approximation of geometrical optics, we obtain, by analogy to Formula (57.17),

$$Z^{\pm} = \frac{B}{\sqrt[4]{k^2(\varepsilon(z) - \sin^2 \alpha_0)}} e^{\pm i \left\{ k \int\limits_{z_0}^{z} \sqrt{\varepsilon(z) - \sin^2 \alpha_0}\, dz + \psi \right\}}, \tag{57.30}$$

where $\psi = \pi/4$ again if we are speaking of the region $z < z_0$ (z_0 is the point at which the radicand vanishes) and $\varepsilon(z)$ diminishes with altitude. Otherwise $\psi = -\pi/4$.

Thus, a wave propagated upward toward point z_0 will be described by the function

$$\Pi = \frac{B}{\sqrt[4]{k(\varepsilon(z) - \sin^2 \alpha_0)}} e^{i \left\{ k \int\limits_{z_0}^{z} \sqrt{\varepsilon(z) - \sin^2 \alpha_0}\, dz + k \sin \alpha_0 \cdot x + \frac{\pi}{4} \right\}}. \tag{57.31}$$

which can be interpreted as a plane wave with a continuously varying angle of inclination of the normal to the z-axis. By superposition of two such solutions, $\Pi(\alpha_0)$ and $\Pi^*(-\alpha_0)$, we may form the field

$$\Pi^c = \frac{B}{\sqrt[4]{k^2(\varepsilon(z)-\sin^2\alpha_0)}} e^{ikx\sin\alpha_0}\cos\left\{k\int_{z_0}^{z}\sqrt{\varepsilon(z)-\sin^2\alpha_0}\,dz+\frac{\pi}{4}\right\}. \quad (57.32)$$

It describes a wave propagating along the x-axis (attenuating if α_0 is complex) and representing a standing wave along the z-axis. With a profile $\varepsilon(z)$ that gives a waveguide (Fig. 57.4 or (57.5), Condition (57.21) (or (57.23d)) will separate out the "captured" normal oscillations for a ground waveguide.

6. Let us now pass to more delicate problems, one of which – refinement of the solution for the region $p^2 > 0$ – is particularly important for tropospheric radio propagation.

The geometrical-optical approximation gives a solution that has an important singularity: each of the waves $Z^{\pm}$ (57.17) is (in this approximation) a solution of the waveguide equation and exists independently of the other if there are no regions in which this solution is invalid, i.e., if there are no turning points $p(z) = 0$. Consequently, a wave propagating, for example, upward, can be reflected, i.e., generate a wave that propagates downward, only provided that the approximation of geometrical optics becomes invalid at a certain point and the condition for sewing to the exact solution in this region obliges us to add the second wave of the pair. The subsequent approximations within the framework of the same method, obtained taking higher terms of the expansion in $1/k$ into account in Eq. (57.15a), can give only corrections of an algebraic nature to the zeroth approximation, i.e., independent corrections to each of the separately existing waves (57.17). Thus, for example, for the $\varepsilon(z)$ profile represented in Fig.

57.6, if ν^2 exceeds $k^2\varepsilon(z_A)$, where A is the extreme point of the $\varepsilon(z)$-curve on the right, we shall have $p^2(z) < 0$ at all z. Consequently, the two independent solutions here will diminish exponentially, one upward and the other downward. If, on the other hand, ν^2 is smaller than $k^2\varepsilon(z_B)$, where B is the extreme point of the $\varepsilon(z)$-curve on the left, then $p^2(z) > 0$ everywhere and the two independent solutions will be waves that pass upward and downward freely without any reflection.

At the same time, it may be expected on the basis of physical considerations that partial reflection must occur even with inhomogeneities and incidence angles α_0 such that $p^2(z)$ is everywhere positive. Consequently, we may conclude that the reflection will be exponentially small in this case. Hence arises the important problem of a generalization of the geometrical-optical approximation that will enable us to take account of the above exponentially small distortion, and reflection in particular, while observing Condition (57.12).

It is also necessary to improve the solution in another respect. Above, under heading 2, we limited ourselves to the case in which the turning points z_{01} and z_{02} were far enough away from one another to permit linear approximation for $p^2(z)$, e.g., to make the dashed straight line on Fig. 57.6 pass at a sufficient distance to the right of point B. At the same time, it is necessary to have a solution for all possible ν^2, i.e., for all positions of the dashed ordinate, even including the case in which it passes through point B.

For this purpose, it will be necessary to turn again to the exact equation (57.14). The problem of wave propagation through a region of variable $\varepsilon(z)$ is formally identical to the problem of passage of a particle through a region of variable potential in quantum mechanics. In the former case, the approximation of geometrical optics corresponds to the quasiclassical approximation (or to the Wentzel-Kramers-

Brillouin method) in the second case [sic]. In this juxtaposition, it is necessary to set $k^2(\varepsilon(z) - 1) = -U(z)$, where $U(z)$ is the potential energy of the particle and $k^2 - \nu^2 = k^2 \cos^2 \alpha_0 = E$, where E is the total energy of the particle; $p^2(z) = E - U$ is the kinetic energy "at a given point." If ν^2/k^2 is taken to the left of point B on Fig. 57.6, $\nu^2 < k^2\varepsilon(z_B)$, this corresponds to the values of $E - U > 0$, i.e., to "passage of the particle over a barrier," and in the approximation of geometrical optics and the quasiclassical approximation of quantum mechanics, the wave, like the particle in classical mechanics, passes unimpeded in any direction. If ν^2/k^2 is taken to the right of point B but to the left of C, then $p^2 = E - U < 0$. In classical physics, the passage of a particle, for example, from region *a* to region *b* is altogether impossible. But "pumping under the barrier" takes place in the approximation of geometrical optics and in the quasiclassical approximation of quantum mechanics. Finally, if $k^2\varepsilon(z) < \nu^2 < k^2\varepsilon(z_A)$, then captured normal waves may arise in region *b*. Their counterparts in quantum mechanics are stationary bound states of a particle in a potential trough. In rigorous quantum-mechanical treatment, as in wave optics, partial reflection $p^2 > 0$ will still take place. For a given barrier, it will be smaller the larger the "particle kinetic energy" at its minimum (above the crest of the barrier). Here it must be stressed that the case of large energies in our problem corresponds not only to large *k*, but also to large imaginary ν. The entire problem has been analyzed in application to quantum mechanics in a number of papers (see, for example, [12, 13]), and the most complete solution was given in [14], where a formula valid for any E and, consequently, any ν^2 was derived.

We first present a not fully rigorous but concise examination. It departs from the fact that both the equation for Z and its solutions can be regarded not only for real z and ν, but also for complex ones. In particular, if we stated that $p^2(z)$ is greater than zero everywhere, we had real z in mind. Let us limit ourselves at first to real ν. When ν^2 is varied in such a way that the dashed straight line on Fig. 57.6 is shifted from right to left in the direction of point B, the real roots of the function $p^2(z)$, for example, z_{02} and z_{03}, move closer together and ultimately merge with the double root z_B. With further variation of ν^2, the roots become complex and, since two roots remain, complex-conjugate: $z_{02} = z_{03}^*$. If we move out far beyond point B, they will be so far from one another that $p^2(z)$ may be replaced near each of them by the linear function, $p^2(z) \sim (z - z_{01})$, like Function (57.4). In the small region in which this expansion is valid, cylindrical functions of order 1/3 of the (now complex) argument ζ will again form the exact solution. In the intermediate region, however, that which contains point B, a different, parabolic approximation is required. In either case, we have the exact solution in a small region around the roots. Far from them, where $p^2(z)$ is large, we may, as before, write the solution in the form (57.17), but now it is not absolutely necessary to perform integration in phase over the real z-axis. Around the roots, we can shift the contour of integration into the complex plane, extending it to the region around one of the roots, in which the linear (or, if necessary, parabolic) approximation is valid and the exact solution is known, and sew Solution (57.17) to this exact solution. It is obvious that in the linear approximation, this is exactly the same procedure as for real roots, and we shall obtain the same formulas as under heading 2, the only difference being that z_{01} is to be understood as the complex root of the equation $p^2(z) = 0$ and integration in the exponential must be performed over a contour in the complex z-plane between the real observation point z

and the root z_{0i}. Special analysis should be given the question as to which of the two roots z_{0i} is to be taken. It is resolved by the procedure of sewing solutions, where it may be found that two waves – incident from $-\infty$ and reflected back – sew to the wave going out in the direction toward $+\infty$ if the root lying in the <u>upper</u> half-plane is taken. Conversely, if we are speaking of a wave arriving from $+\infty$ and partly reflected back, the root lying in the <u>lower</u> half-plane must be taken.

For a wave incident from $z = -\infty$ and propagating in the positive direction of the z-axis, the reflection coefficient naturally has the same form (57.18a) as for the real root:

$$f = \frac{Z^-}{Z^+} = e^{-2i\left\{\int_{z_0}^{z} p(z)dz + \frac{\pi}{4}\right\}}, \quad z \to -\infty, \tag{57.33}$$

where $\operatorname{Im} z_0 > 0$ and the sign for extraction of the root of $p^2(z)$ is determined if only by the fact that $|f| < 1$ is necessary. For the wave incident from $z = +\infty$, the reflection coefficient, in analogy to Formula (57.18b), is

$$f = \frac{Z^+}{Z^-} = e^{2i\left\{\int_{z_0}^{z} p(z)dz - \frac{\pi}{4}\right\}}, \quad z \to +\infty, \tag{57.33a}$$

where $\operatorname{Im} z_0 < 0$.

Up to this point we have been speaking of real ν. In the expansion of the general solution (57.3b), (57.3c), which satisfies the conditions given at the source, it is also possible that we may encounter complex ν (as was the case, for example, in §32 for $\sin^2 \alpha \equiv \nu^2/k^2$). However, in generalizing the expressions for the coefficients of the expansion for the case of complex turning points z_{0i}, we spoke of roots of the quantity $p^2(z)$ in which ν^2 appears on an equal footing with $\varepsilon(z)$. In actuality, therefore, Formulas (57.33) and (57.33a) are also valid for <u>complex</u> ν.

We note that these results can also be obtained by a different, more rigorous method, and without separation consideration of regions near the turning point (real or complex) and far from it. For this purpose, it is necessary [15] to replace Eq. (57.14) by the modified equation

$$\frac{d^2Z}{dz^2}+(p^2(z)-\theta(z))Z=0, \qquad (57.34)$$

where the additional term $\theta(z)$ is so selected (actually, it is not necessary to express it in explicit form) that it will vanish at infinity, while ensuring validity around point z_0 of a solution form

$$Z(z)=A\xi^{\frac{1}{2}}p^{-\frac{1}{2}}J_{\pm m}(\xi), \qquad (57.34a)$$

where (essentially, $\xi = 2\zeta/3$ in Formula (57.5))

$$\xi=\int_{z_0}^{z}p(z)dz, \quad m=\frac{1}{n+2} \qquad (57.35)$$

and n is the exponent in the formula expressing the behavior of $p(z)$ near point z_0, $p^2(z) \sim (z - z_0)^n$. For the cases examined above, $n = 1$ (simple root). Such a solution (for greater detail, see the monograph [16]) merges asymptotically with Solution (57.17), but, beyond that, automatically yields the transition between the regions $z < z_0$ and $z > z_0$ in the case of real roots or between the regions to the left of the $p^2(z)$-extreme region and to the right of it for complex roots, with the correct phase ψ. Here, as before, it is necessary to require that the second root of the function $p^2(z)$ be far enough away so that it will be possible to take account only of the nearest root. In particular, if we are concerned with the case represented in Fig. 57.6, then ν^2 need not be close to ν^2_{krit}, at which the ordinate passes through the extreme point of the curve. It can be stated that the incidence angle of the wave α_0, $\nu^2=k^2\sin^2\alpha_0$, must be not very close to the critical angle (in the quantum-mechanical case, this corresponds to the requirement that the kinetic energy of the particle not be small even near the crest of the barrier).

However, it is necessary to state that even in the case the problem cannot be regarded a solved in general form. On transfer of the differential equation from the real axis into the complex plane, the solution is found to be highly sensitive to details of the analytical behavior of the function $\varepsilon(z)$. First of all, the presence of <u>poles</u> on $p^2(z)$ is important. Thus, for example, two bell functions, $\varepsilon(z) - 1 \sim \sim \exp\left(-\frac{(z-z_B)^2}{a^2}\right)$ and $\varepsilon(z) - 1 \sim \left(a^2 + (z-z_B)^2\right)^{-1}$, may lead to sharply different reflection coefficients. In the case of passage over a barrier, we are interested in a very fine effect – an exponentially small reflection. It depends essentially on the distance between the roots z_0 and the poles z_ε. This question is examined, for example, in [13]. It was found that in the presence of poles at a finite distance, the reflection coefficient f (57.33) must be multiplied by a certain function of $k\mu$, where

$$\mu = z_0 - z_\varepsilon \tag{57.36}$$

is the distance between the root and the pole, which is different for poles of the first and second orders. In either case, this function tends to 1 as $k\mu \to \infty$, but has a substantial influence on the result for $k\mu \lesssim 1$.

Let us pass to the case of merging roots, i.e., to ν^2 or a_0 which may be close, for example, in Fig. 57.6, to the ordinate passing through point A or B. As we have already noted, the general method of analysis here consists in sewing the asymptotic solution to the exact solution not for a linear layer, but for a parabolic layer (57.8). Hence for ν around ν_{krit}, the reflection coefficient reduces asymptotically to the coefficient of reflection from a parabolic layer [9]. The

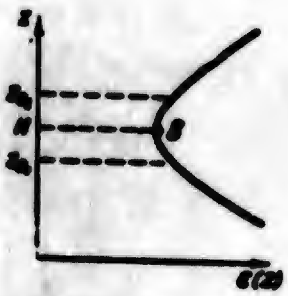

Fig. 57.7. Turning points near the minimum of $\varepsilon(z)$.

most general analysis within the framework of the geometrical-optical approximation was carried through in [14] and [17]. In the first of these papers, the method of replacing the equation by a similar equation (57.34) was generalized substantially and, in particular, was applied to the case of complex and merging roots. The investigation of radio wave propagation in a stratified atmosphere that was developed in [17] by another method led to similar results. It is this last form of the solution that we shall set forth here.

Together with the variable exponents

$$\xi(z) = \int_0^z p(z)\,dz \qquad (57.37)$$

we introduce the quantity (see Fig. 57.7)

$$i\pi\rho = \int_{z_{10}}^{z_{20}} p(z)\,dz, \qquad (57.37a)$$

$$\xi_0 = \frac{1}{2}\int_0^{z_{10}} p(z)\,dz + \frac{1}{2}\int_0^{z_{20}} p(z)\,dz. \qquad (57.37b)$$

The parameter ρ is a function of ν^2 and replaces ν^2. If $\nu^2 < k^2\varepsilon(z_B)$, we may state: if we are to the left of point B, we have $p^2(z) > 0$ for all z; the roots z_{10} and z_{20} are complex and the integrand in Formula (57.37a) is real. In the converse case, $\nu^2 > k^2\varepsilon(z_B)$, z_{10} and z_{20} are real "to the right of point B" and near it, since $p^2(z) < 0$ in the region between z_{10} and z_{20} and $p^2(z) > 0$ for $z < z_{10}$ and $z > z_{20}$. Thus, the integrand is imaginary here. We take the sign before the root such that p(z) will be positive to the left of B, $p(z) = +\sqrt{|p^2(z)|}$, and expand p(z) in series in the integrand around z_B. Keeping quadratic terms and remembering that $\varepsilon'(z_B) = 0$, $\varepsilon''(z_B) > 0$,

we obtain

$$\rho = \frac{k^2\varepsilon(z_B) - \nu^2}{\sqrt{2k^2\varepsilon''(z_B)}} \equiv \frac{p^2(z_B)}{\sqrt{2p'(z_B)}}. \qquad (57.37c)$$

"Left of z_B" (i.e., for $\nu^2 < k^2\varepsilon(z_B)$), therefore, ρ is real and positive. Continuing this expression analytically into the region $\nu^2 > k^2\varepsilon(z_B)$, we find that for "passage under the barrier," "right of z_B," ρ is real and negative. Proceeding from physical considerations, we may conclude that as $\rho \to +\infty$, the reflection coefficient must exponentially to zero, or to unity as $\rho \to -\infty$.

Let us consider the functions Z^+ and Z^-, which merge with the functions of (57.17) <u>high above the layer</u>, $z \gg z_B$, i.e., into the departing and arriving waves, respectively. Sewing them to the functions of a parabolic cylinder, which give the solution for a parabolic layer (57.8), we obtain an expression for Z^+ in the entire range of variation of z [17]. We present the final result. Expressing the constant B in Formula (57.17) in such a way that high above the layer

$$Z^+ = \frac{1}{\sqrt{p(z)}} e^{i\left(\xi - 2\xi_B + \frac{\pi}{4}\right)} = \frac{1}{\sqrt{p(z)}} e^{i\left(\int_{z_B}^{z} p(z)dz - \int_{0}^{z_B} p(z)dz + \frac{\pi}{4}\right)}, \qquad (57.38a)$$

we have far <u>below</u> the layer ($z \ll z_B$)

$$Z^+ = \chi_1(\rho) \frac{1}{\sqrt{p(z)}} e^{i\left(\xi - 2\xi_B + \frac{\pi}{4}\right)} + e^{-\rho\pi} \frac{e^{-i\left(\xi + \frac{\pi}{4}\right)}}{\sqrt{p(z)}}, \qquad (57.38b)$$

where

$$\chi_1(\rho) = \frac{\sqrt{2\pi}}{\Gamma\left(\frac{1}{2} - i\rho\right)} e^{-\frac{\rho\pi}{2}} e^{i(\rho - \rho\ln\rho)}. \qquad (57.38c)$$

Thus, we have a reflected wave below the layer in addition to the wave moving upward (compare (57.38a); first term in Formula (57.38b)). The reflection coefficient is given by the ratio of the second term in this formula to the first:

$$f = \frac{e^{-\rho\pi}}{\chi_1(\rho)} e^{-i\left(\zeta-\zeta_0+\frac{\pi}{4}\right)} = \frac{e^{-2\pi\rho}}{\chi_1(\rho)} e^{-i\left(\int\limits_{z_{10}}^{z} p(z)dz+\frac{\pi}{4}\right)}. \qquad (57.39)$$

Thus, the departure from the f (57.18a) obtained far from the extreme point B (even if the pole of the function $p(z)$ is quite far from z_{10}) is expressed by an additional factor $\exp(-2\pi\rho)\chi_1^{-1}(\rho)$. On the other hand, the ratio of the wave amplitudes Z^+ above the layer and below the layer gives the "permeability of the barrier" (in amplitude!) g:

$$g = \frac{1}{\chi_1(\rho)}. \qquad (57.40)$$

We note, first of all, that for $\rho = 0$, i.e., for a wave "touching the barrier," $\nu^2 = k^2\varepsilon(z_B0$, we find in view of the fact that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$,

$$g(0) = |f(0)| = \frac{1}{\sqrt{2}} \qquad (57.40a)$$

here $z_{10} \equiv z_B$ and in Formula (57.39) the integral in the exponent is real, giving only a phase shift). Thus, in this case half of the energy passes through and half is reflected. This is, of course, also approximately true for small positive ρ. At the same time, elementary geometrical optics would give $g = 1$, $f = 0$ for such an over-barrier approximation. In the general case, the transparency with respect to energy is ([14], Formula 37), where in our notation $E = 2\rho$)

$$|g(\rho)|^2 = \frac{1}{|\chi_1(\rho)|^2} = \frac{1}{1+e^{-2\pi\rho}} \qquad (57.40b)$$

(here we have used the property of Γ-functions $\left|\Gamma\left(\frac{1}{2}+i\rho\right)\right|^2 = \frac{\pi}{\operatorname{ch}\pi\rho}$ [11, 11]).

Thus, as it should be, we have full transparency for $\rho \to +\infty$, in passage asymptotically high "above the barrier." For a wave "much lower than the barrier," $\rho \to -\infty$, transparency disappears.

Thus, in over-barrier passage not far from the crest of the barrier ($p(z)$ does not vanish anywhere, but in a certain region it is small by comparison with k or comparable to it), significant reflection may occur. With a waveguide profile $\varepsilon(z)$, the waveguide will not hold such waves completely, but the range of their propagation along r will still be increased substantially. Conversely, part of the wave will be pulled outside in passage "slightly below the barrier." Concrete numerical examples examined in accordance with the more complete theory [17], which is to be set forth below (§58), show [20] how leakage increases as we pass from the under-barrier to the over-barrier region.

We note that even for completely entrapped waves, the linear approximation of the function (z) and, accordingly, calculations based on examination of the turning points and trapping conditions (57.21), (57.23d) are comparatively less exact for the first captured waves, for which only a few wavelengths fit into the waveguide (along the z-axis). It is obvious that the basic condition of the geometrical-optical approximation (57.12), is not observed very well here (accordingly, the quasiclassical approximation is not exact for the first steady-state orbits in the atom).

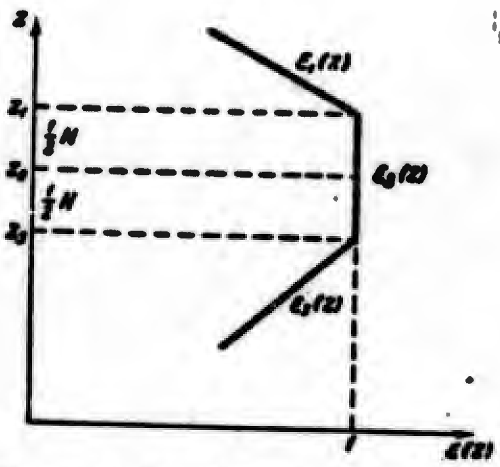

Fig. 57.8. Illustrating calculation in geometrical-optical approximation for the case of the profile of (57.41).

7. By way of illustration, let us apply the geometrical-optical approximation to two problems. The first of them is the elementary case of superrefraction. The second problem, which was considered rigorously in Chapter 6, is that of diffraction in a homogeneous atmosphere above a homogeneous (for the sake of simplicity) infinitely conductive spherical earth. Let the source be at a height z_0 in the middle of a homogeneous layer of height H (which we shall subsequently let tend to zero), and let ε vary linearly in either direction from this layer (see Fig. 57.8, which represents the case $\beta > 0$):

$$\varepsilon = \varepsilon_1(z) = 1 - \beta(z - z_1) \text{ for } z > z_1 \equiv z_0 + \frac{H}{2},$$
$$\varepsilon = \varepsilon_3(z) = 1 + \beta(z - z_3) \text{ for } z < z_3 \equiv z_0 - \frac{H}{2}. \qquad (57.41)$$

We obtain a three-layered space and the field in the homogeneous layer is expressed by Formula (55.17). A series of residues is taken at the points defined by Formula (55.15), and it is necessary to find the coefficients f_1 and f_3 of reflection from the upper and lower boundaries of the layer. In the approximation of geometrical optics, they are expressed by Formulas (57.18a) and (57.18b), with $\varepsilon(z)$ coinciding with $\varepsilon_1(z)$ in the former case, while in the latter $\varepsilon(z) = \varepsilon_3(z)$. Within the limits of the integral, z must be replaced by the ordinate of the point for which we are calculating the amplitude ratio of the incident and returning waves, i.e., we must set $z = z_1$ in Formula (57.18a) for the upper boundary of the homogeneous layer and $z = z_3$ in Formula (57.18b) for the lower boundary. Then in either case z_0 is the point at which the corresponding radicand vanishes.

We note that on introducing into the integral the new variable ζ, $\zeta^2 = \varepsilon_1(z) - \sin^2\alpha$, $dz = \frac{2\zeta d\zeta}{\frac{d\varepsilon_1}{dz_j}}$, the reflection coefficient can be rewritten in the form

$$f_1(\alpha)=\exp\left\{-4ik\int_0^{\cos\alpha}\zeta^2\frac{d\zeta}{\frac{d\varepsilon_1}{dz}}-i\frac{\pi}{2}\right\}, \tag{57.42}$$

where z in $d\varepsilon_1/dz$ must be expressed in terms of ζ. According to (57.41),

$$\frac{d\varepsilon_1}{dz}=-\beta,\quad f_1(\alpha)=\exp\left\{\frac{4}{3\beta}ik\cos^3\alpha-i\frac{\pi}{2}\right\}. \tag{57.42a}$$

For a parabolic layer $\varepsilon_1(z)=1-\frac{g}{a^2}(z-z_1)^2$, where g and a are certain constants,

$$f_1(\alpha)=\exp\left\{\frac{ika}{\sqrt{g}}\frac{\pi}{4}\cos^2\alpha-i\frac{\pi}{2}\right\} \tag{57.42b}$$

According to Formula (57.18b), the coefficient of reflection from the lower boundary, $z = z_3$, will differ in the sign before the integral. As a result, the equation for determining the poles (55.15) will be brought to the form

$$\ln f_1(\alpha_l)+\ln f_3(\alpha_l)+2ikH\cos\alpha_l=2\pi il,$$

$$2ik\int_{z_{00}}^{z_3}\sqrt{\varepsilon_3-\sin^2\alpha_l}\,dz+2ik\int_{z_1}^{z_{10}}\sqrt{\varepsilon_1-\sin^2\alpha_l}\,dz-i\pi+2ikH\cos\alpha_l=2\pi il, \tag{57.43}$$

where l is a whole number. Since $H\cos\alpha_l\equiv\int_{z_3}^{z_1}\sqrt{\varepsilon_2-\sin^2\alpha_l}\,dz$, where $\varepsilon_2\equiv 1$, we may combine all three integrals and rewrite the equation in the form

$$k\int_{z_{00}}^{z_{10}}\sqrt{\varepsilon(z)-\sin^2\alpha_l}\,dz=\pi\left(l+\frac{1}{2}\right). \tag{57.43a}$$

We have, of course, come back to Condition (57.21). The integral is taken from the "lower" to the "upper" turning point.

For $\varepsilon(z)$ (57.41), this gives

$$\frac{1}{3}\cos^3\alpha_l+\beta H\cos\alpha_l=\frac{1}{2}\left(l+\frac{1}{2}\right)\beta\lambda. \tag{57.43b}$$

According to the applicability condition for geometrical optics, the variation of ε over wavelength must be small. Consequently, $|\beta\lambda| \ll 1$ and for small l, α_l is close to $\pi/2$. For $\beta > 0$ and $H \to 0$,

$$\cos\alpha_l = \left[\frac{3}{8}\left(l+\frac{1}{2}\right)\beta\lambda\right]^{2/3}. \qquad (57.44)$$

For

$$l < l_{\text{max}} = \frac{8}{3\beta\lambda} - \frac{1}{2} \qquad (57.45)$$

there are "permitted" (waveguide-trapped) values α_l among the real values of α. The corresponding normal oscillations do not damp in propagation along r.

Let us now pass to the other problem – a homogeneous atmosphere with $\varepsilon = 1$ and a spherical earth, for simplicity with $\varepsilon_3 \to \infty$. According to Formula (57.1a), we must write simply for all positive z

$$\varepsilon(z) = 1 + \frac{2z}{a}. \qquad (57.46)$$

Further, the boundary condition corresponding to the polarization of the radiation is imposed at $z = 0$. For a wave propagating upward with $\varepsilon(z)$ (57.46), a real root $p^2(z)$ exists for arbitrary real ν:

$$z_0 = \frac{a}{2}\left(\frac{\nu^2}{k^2} - 1\right).$$

In the case of complex ν, the turning point z_0 will also be complex. However, the necessity of satisfying the boundary condition at $z = 0$ yields yet another condition for the field Z_ν, and the requirement of uniqueness of the solution in the interval $0 < z < z_0$ imposes on the value of ν (the only remaining free parameter) the limitation that we have already obtained for real z_0. Since complexity of z_0 and ν does not influence the form of f, we again have Formulas (57.23b) and (57.23c) for selection of the permitted ν. The necessary integral has just been evaluated: again assuming $\nu^2 = k^2 \sin^2 \alpha$, we need only set $\beta = -2/a$ in Formula (57.42a), as is seen from comparison of Formulas (57.41) and (57.46); this yields

$$\int_0^{z_0} \sqrt{k^2\varepsilon(z)-\nu^2}\,dz = -\frac{1}{3}ka\cos^3\alpha.$$

The permitted ν or α are determined by Formula (57.23d). Substituting the value found for the integral in this formula, we see that they are essentially complex:

$$\cos^3\alpha_l = -\frac{3\lambda}{2a}\begin{cases} l+\frac{1}{4} \\ l-\frac{1}{4} \end{cases} = -\frac{3\pi}{ka}\begin{cases} l+\frac{1}{4}(l=0,1,\ldots,-\text{ vert. pol.}), \\ l-\frac{1}{4}(l=1,2,\ldots,-\text{horiz. pol.}). \end{cases} \quad (57.47)$$

Since $ka \gg 1$, the solution α_l of this equation is close to $\pi/2$ for not too large l. Setting $-1 = e^{\pm i\pi}$, and introducing the quantity

$$\beta_l = \frac{\pi}{2} - \alpha_l, \quad (57.48)$$

we have, after extraction of the cube root, one of the two possibilities

$$\sin\beta_l \approx \beta_l \approx \left[\frac{3\pi}{ka}\left(l\pm\frac{1}{4}\right)\right]^{1/3} e^{\pm i\frac{\pi}{3}} \approx \cos\alpha_l. \quad (57.49)$$

According to Formula (57.42a), this means that the reflection coefficient is exponentially small, as it should be, since all turning points z_0 are complex for complex ν. In the complete field Expression (55.17), the Hankel function $H_0^{(1)}(\nu_l r)$ may be replaced at large distances from the source by its asymptotic expression*

$$H_0^{(1)}(\nu_l r) \equiv H_0^{(1)}(kr\cos\beta_l) \approx \sqrt{\frac{2}{\pi kr\cos\beta_l}}\, e^{ikr\cos\beta_l} \approx$$

$$\approx \sqrt{\frac{2}{\pi kr}}\, e^{ikr - i\frac{kr}{2}\beta_l^2}. \quad (57.50)$$

The presence of an imaginary part of β_l results in a physically intelligent nonincreasing solution if the minus sign is taken before i in Formula (57.49). (Here $z_0 = \frac{a}{2}(\sin^2\alpha_l - 1) \sim \frac{a}{2}\left(-\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$, i.e., z_0 lies in the upper half-plane in agreement with the rule for selection of the root in Formula (57.33)). Consequently, it is necessary to assume that $l + \frac{1}{4}$ corresponds to vertical polarization and $l - \frac{1}{4}$ corresponds to horizontal polarization; see Formula (54.47).

$$\operatorname{Im}\beta = \left[\frac{3\pi}{ka}\left(l \pm \frac{1}{4}\right)\right]^{2/3} \operatorname{Im} e^{-\frac{2\pi}{3}i}, \tag{57.51}$$

the attenuation of the field with distance is described by the multiplier

$$\frac{1}{\sqrt{kr}} e^{-\frac{\sqrt{3}}{4}\left[\frac{3\pi}{ka}\left(l \pm \frac{1}{4}\right)\right]^{2/3} kr},$$

or, considering that $r = a\vartheta$,

$$\frac{1}{\sqrt{ka\vartheta}} \exp\left\{-\frac{\sqrt{3}}{4}\left[3\pi\left(l \pm \frac{1}{4}\right)\right]^{2/3} (ka)^{1/3} \vartheta\right\}. \tag{57.52}$$

At the same time, we have seen (see Formula (39.7)) that in the more rigorous solution the field of a vertical electric dipole at the surface of a spherical earth is described by exactly the same series, in which the exponents giving the attenuation are equal in successive terms to

$$-x \operatorname{Im} t_s = -\sqrt[3]{\frac{ka}{2}}\, \vartheta \operatorname{Im} t_s,$$

where the value of t_s' from §39 must be taken for t_s in the case of an infinitely conductive earth.

Thus, instead of Im t_s' we obtained $\left[\frac{3\pi}{2}\left(l + \frac{1}{4}\right)\right]^{2/3} \operatorname{Im} e^{-\frac{2\pi i}{3}}$, where it is necessary to set $l + 1 = s = 1, 2, 3$ ... For horizontal polarization, the boundary condition at the earth's surface (57.22d) corresponds to the "short-wave" case in §39, i.e., $q = \infty$. Hence the values of $\left[\frac{3\pi}{2}\left(l - \frac{1}{4}\right)\right]^{2/3} \operatorname{Im} e^{-\frac{2\pi i}{3}}$ obtained in the approximation of geometrical optics should be compared with the Im t_s^0 from §39, setting $s = l =$ $= 1, 2, 3$ ... We obtain Table 3 for $t_s / \left|\operatorname{Im} \exp\left(-\frac{2\pi i}{3}\right)\right|$.

This example is a good illustration of the accuracy of the geometrical-optical approximation: it can be inexact only for the lowest value of l, where the error in the exponent reaches 10%.

TABLE 3

l	A Округленные точные значения		B Приближение геометрической оптики	
	C вертикальная поляризация	D горизонтальная поляризация	C вертикальная поляризация	D горизонтальная поляризация
1	1,02	2,34	1,12	2,58
2	3,24	4,09	3,26	4,08
3	4,82	5,52	4,83	5,53
4	6,163	6,77	6,170	6,79

A) Rounded exact values; B) geometrical-optics approximation; C) vertical polarization; D) horizontal polarization.

We shall not dwell on the question as to the extent of agreement between the preexponential constant factors in the individual terms of the series, i.e., the extent of agreement between the solutions as a whole. What is essential is that it is the smallest l that is important at great distances. For it, therefore, we must look for $f(\alpha)$ by more rigorous methods. But it is precisely in the case of a spherical earth and homogeneous atmosphere that this is not difficult: for a linear $\epsilon_{mod}(z)$, the equation for $Z(z)$ may be solved exactly; by the substitution of variables (57.5), it is brought to an equation that can be solved in Airy functions or cylindrical functions of order 1/3. We note that here we can no longer be bound by the requirement that each term of the series decrease with increasing z. To the contrary, we know from §39 that the field increases with increasing elevation above the ground and that the individual terms of the series increase exponentially. Accordingly, the cylindrical functions of order 1/3 that are selected will be Hankel and not Bessel functions. However, by virtue of the imaginariness of the argument, these functions coincide for large values of the argument, as indeed they should, since the method of geometrical optics, which yields "standing" waves (trigonometric or Bessel functions) in the region $z < z_0$, is valid only for large phase values.

In addition to the residue series, the complete solution (55.17) contains integrals taken along the sections drawn to infinity from the branching points of the two reflection coefficients. In this case, however, the coefficient of reflection from the upper boundary f_1 (57.42b) has no branch points and the coefficient of reflection from the lower boundary has, since $|\varepsilon_3| \to \infty$, a branch point at infinity; the integral vanishes, as noted in connection with Formula (55.15). At large but finite $|\varepsilon_3|$, when it is exponentially small, this integral describes a "side wave," a wave that passes to the observation point through the earth.

Up to this point, we have assumed that both the observation point and the source are on the ground, $z_0 = z = 0$. Now suppose $z \neq 0$. As before, the solution is expressed by a residue series (55.15), but the "altitude factors" $\Phi(\alpha_l)$ depend on z (and terms of the series with $l > 0$ become increasingly significant, so that the inaccuracy of the geometrical-optical approximation is reduced). Since they can differ from the solution $Z(z)$ of Eq. (57.3) only by a constant factor, their form in the approximation of geometrical optics is known to us as functions (57.19) with (z) (57.46) and with z_{01} determined by the relationship $\varepsilon(z_{0l}) = 1 + \frac{2z_{0l}}{a} = \sin^2\alpha_l$, where α_l is given by Formula (57.47). From this, we find by evaluating the integral in much the same way as in the derivation of Formula (57.42a),

$$Z_l = \frac{C}{\sqrt[4]{2\frac{z}{a} + \cos^2\alpha_l}} \cos\left\{\frac{1}{3}\left[2(ka)^{2/3}\frac{z}{a} + (ka)^{2/3}\cos^2\alpha_l\right]^{3/2} + \frac{\pi}{4}\right\}. \quad (57.53)$$

But it is seen at once that $(ka)^{2/3}\frac{z}{a} = \frac{1}{2^{1/3}} y$, where y is the dimensionless height introduced in §39.

Since the geometrical-optical approximation is valid only for large values of the argument, we must assume either $y \gg 1$ or $l \gg 1$. For this case, as was noted earlier, the cosine may be replaced by the exponent alone (or, what is the same thing, the Bessel function may be replaced by the Hankel function) by virtue of the imaginariness of the argument α. Thus, for example, setting for $y \gg l$

$$\frac{2}{3}\left\{y+\left[-\frac{3\pi\left(l+\frac{1}{4}\right)}{2}\right]^{2/3}\right\}^{3/2}+\frac{\pi}{4}\approx\frac{2}{3}y^{3/2}+$$
$$+y^{1/2}\left[\frac{3\pi\left(l+\frac{1}{4}\right)}{2}\right]e^{-\frac{2\pi}{3}i}+\dots+\frac{\pi}{4},$$

we obtain, selecting the constant factor C such that as $y \to 0$ the altitude Z_l becomes unity,

$$Z_l\approx\frac{\exp\left\{\frac{2}{3}iy^{3/2}+y^{1/2}\frac{\sqrt{3}}{2}\left(\frac{3\pi\left(l+\frac{1}{4}\right)}{2}\right)^{2/3}\right\}}{\sqrt[4]{1+\left[-\frac{2}{3\pi\left(l+\frac{1}{4}\right)}\right]^{2/3}y}}. \qquad (57.54)$$

This altitude factor is identical to the factor given by the rigorous theory (§39) for $y \gg 1$, where it is accordingly necessary to take the asymptotic expression for $w(t-y)$, i.e., for $H^{(1)}_{1/3}$. We note that it is Formula (57.53) that is the asymptotic expression for $I_{1/3}$.

We have analyzed application of the normal-wave-in-waveguide method to cases in which f can be determined approximately by analytical solution of Eq. (57.43) (or (55.15)). In more realistic cases, it is necessary to resort to a rather complex numerical procedure. By way of example, we refer to the solution [4] of the propagation problem for superlong waves (thousands and tens of thousands of cycles) generated, for example, by lightning discharges (atmospherics) in the layer between the surface of the earth (which may be regarded here as an ideal conductor) and the ionosphere. In this study, the altitude curve $\varepsilon(z)$ in the ionosphere was characterized by the formula for the "transition layer" (57.9) with M = 0 (Fig. 57.3), with the values of the constants P and m selected separately for day and night from ionospheric radiosonde data. The exact solution for such a layer [11] gives the value of the reflection coefficient f_1 (while we consider f_3 equal to unity). Then the ν_l were found by numerical solution of Eq. (55.15) and the fields of the corresponding modes were summed (sometimes taking about ten first harmonics into account) to plot the attenuation function of the field as a function of distance r along the horizontal. The resulting dependences on r and frequency for the daytime and nighttime fields were found to be rather fantastic. They agreed extremely well with those found experimentally (for detail see the monographs [I, 2] and [4]).

§58. SPHERICALLY STRATIFIED ATMOSPHERE. GENERAL THEORY

1. We shall consider systematically the solution of the field equations for a spherically stratified atmosphere. For an arbitrary radial distribution of $\varepsilon(h)$ above a spherical earth, we could use the method of the modified refractive index, which was introduced, admittedly not at all rigorously, in §56, and then the solution for a three-layered space (55.17). Here it would be necessary to assume that a source at height z_0 is situated in a homogeneous layer of infinitesimally small thickness, bounded above and below by inhomogeneous layers. In this case, we should be obliged to calculate the coefficients of reflection from the boundaries of these layers, taking this reflection coefficient into account for the lower layer in addition to the inhomogeneous tropospheric layer $0 < z < z_0$, as well as the influence of the earth. Special methods (see [I, 9]) have been developed for calculation of these reflection coefficients (and, from them, of the functions

$\Phi_{\gtrless}(\alpha)$). However, the methods of analysis in common use (it is not difficult to prove the identity of the two approaches) proceed directly from integration of the field equations in an inhomogeneous medium above the earth (see [I, 13], Chapter 2; [18]). We shall now set them forth in a manner that also justifies the modified-refractive-index method.

As was shown in §3, Subsection 4, the solution of the Maxwell equations may in this case be placed by the solution of a scalar equation for the Debye functions u(r) (in the case of the field of a vertical electric dipole) or v(r) (in the case of the field of a vertical magnetic dipole), which satisfy Eqs. (3.39a)-(3.39c), which we shall write in the spherical coordinate system (r, ϑ, φ) with its center at the center of the earth and with the polar axis passing through the dipole. By virtue of symmetry, the functions u and v need not depend on φ. Hence we have

$$\left.\begin{aligned} &\frac{1}{r}\frac{\partial^2}{\partial r^2}(ru) + \frac{1}{r^2\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\,\frac{\partial u}{\partial\vartheta}\right) + k^{*2}u = 0,\\ &\frac{1}{r}\frac{\partial^2}{\partial r^2}(rv) + \frac{1}{r^2\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\,\frac{\partial v}{\partial\vartheta}\right) + k^2v = 0;\end{aligned}\right\} \qquad (58.1)$$

$$k^2 = \varepsilon(r)k_0^2, \quad k_0 = \frac{\omega}{c}, \quad k^{*2} = k^2 - k\frac{d^2}{dr^2}\frac{1}{k}. \qquad (58.1a)$$

In many cases we may set $k^* = k$. This is valid primarily when ε experiences only monotonic variation due to the normal decrease in density in the troposphere, i.e., when it varies by an amount of the order of 10^{-3}-10^{-4} on an interval of the order of 10 km = 10^6 cm. Actually, since ε is close to unity, we may write

$$k(r) \equiv \sqrt{1 + (\varepsilon(h) - 1)}\,k_0 \approx \left(1 + \frac{\varepsilon(h)-1}{2}\right)k_0$$

and hence (r = a + h)

$$k\frac{d^2}{dr^2}\frac{1}{k} \approx \sqrt{\varepsilon(h)}\frac{d^2}{dr^2}\frac{1}{1+\frac{\varepsilon-1}{2}} \approx \frac{d^2}{dr^2}\left(1 - \frac{\varepsilon-1}{2}\right) \doteq -\frac{1}{2}\frac{d^2\varepsilon}{dr^2}. \qquad (58.2)$$

On the other hand, the principal term in the left member is determined by the length scale for the vertical variations of the field, i.e., according to Formula (38.9),

$$\frac{d^2u}{dr^2} \sim \left(\frac{k^2}{a}\right)^{2/3} \cdot u.$$

Hence the difference between k^* and k can be disregarded if

$$\frac{d^2\varepsilon}{dr^2} < \left(\frac{k^2}{a}\right)^{2/3}. \qquad (58.3)$$

It is easy to see that Inequality (58.3) is satisfied with a large margin even for the longest waves, since $|d^2\varepsilon/dr^2| \sim 10^{-15}\ cm^2$. If, however, we are speaking of sharp variations of $\varepsilon(r)$, variations of the inversion type, when the question as to the formation of waveguides may arise, such variations of $\varepsilon(r)$ will interest us on intervals of the order of H_{eff}, where H_{eff} is the effective thickness of the inversion layer:

$$\frac{d^2\varepsilon}{dr^2} \sim \frac{\delta\varepsilon}{H^2_{eff}}.$$

On the other hand, for example, for the waveguide-trapped field $|\partial^2u/\partial r^2| \sim |u|/H^2_{eff}$ in this case, and the difference between k^* and k can again be disregarded, since $|\delta\varepsilon| \ll 1$. Hence we may examine a single equation, (51.1a), for both polarizations of the field, if necessary understanding k^* for k.

But Eq. (58.1a) differs from the equation for a homogeneous atmosphere only in the presence of $k^2 = \varepsilon k_0^2$ instead of $k_0^2 = \omega^2/c^2$. It will subsequently be convenient to introduce a (nonunity) value of the permittivity of air at ground level, $\varepsilon_0 = \varepsilon(a)$, and understand by k_0^2 the quantity $\varepsilon_0 \frac{\omega^2}{c^2}$. After this it will be convenient to make the same transformations as in §38, to wit: firstly, from the vertical variable r and the horizontal ϑ to pass to the variables y and x (38.8) and (38.9):

$$x=\sqrt[3]{\frac{k_0a}{2}}\,\theta=\sqrt[3]{\frac{k_0}{2a^2}}\,a\theta=M\theta=\frac{k_0a\theta}{2M^2}=\frac{a\theta}{D_0(\lambda)}, \tag{58.4}$$

$$y=(r-a)\cdot\sqrt[3]{2k_0^2a^{-1}}=\frac{h}{h_0(\lambda)}=\frac{k_0h}{M}, \tag{58.5}$$

$$M=\sqrt[3]{\frac{k_0a}{2}},\quad D_0(\lambda)=\frac{2M^2}{k_0},\quad h_0(\lambda)=\frac{M}{k_0}, \tag{58.6}$$

where $M \gg 1$ is the large parameter of the problem; D_0 and h_0 are the horizontal and vertical scales; secondly, we convert from the field function u (which we shall also understand as the Debye function v of a vertical magnetic dipole) to the attenuation function U by the formula

$$u=\frac{e^{ik_0a\theta}}{R}\sqrt{x}\,U \tag{58.7}$$

(see Formulas (38.19) and (38.20); U replaces W_1). As a result, expanding the small quantity M^{-1} in series and retaining the lowest-order terms, we obtain for U, in analogy to (38.21), an equation that differs in the presence of the quantity $\varepsilon(r) - \varepsilon_0$ [18],

$$\frac{\partial^2U}{\partial y^2}+i\frac{\partial U}{\partial x}+\left[y+M^2\frac{\varepsilon(r)-\varepsilon_0}{\varepsilon_0}\right]U=0, \tag{58.8}$$

with the boundary condition (compare (38.22))

$$\frac{\partial U}{\partial y}=-qU \text{ for } y=0. \tag{58.9}$$

Here for an electric dipole (vertical polarization)

$$q=\frac{iM}{\sqrt{\varepsilon_3^c}}=\frac{iM}{\varepsilon_3}\sqrt{\varepsilon_3-\sin^2\alpha}, \tag{58.10}$$

and for a magnetic dipole (horizontal polarization)

$$q=q_m=iM\sqrt{\varepsilon_3-\sin^2\alpha}, \tag{58.11}$$

where ε_3 is the permittivity of the earth (more precisely, its ratio to the permittivity of the air at the surface of the earth, ε_0); α is the angle of incidence (which replaces the glancing angle $\psi = \frac{\pi}{2} - \alpha$) of the wave at the point in question. Hence the term $-\sin^2\alpha$ under the radicaal has explicit significance for a plane wave. But since the dipole field can be decomposed into plane waves with complex sin α, Conditions (58.9)-(58.11) will also obtain for each such partial wave. In particular, it is necessary in the plane-wave method, as used in §32 and §55, to assume $\sin^2\alpha = \nu^2/k_0^2$, where ν^2 is the separation parameter of the cylindrical variables in the Sommerfeld method, §31 (compare also §55 and §57).

Thus, the problem is reduced, as for the homogeneous atmosphere, to a parabolic equation. The transition to this from the wave equation (58.1a) was based on the statement that the attenuation function varies much more rapidly along the vertical than along the horizontal. Accordingly, the vertical scale $h_0(\lambda)$ is much smaller than the horizontal $D_0(\lambda)$ (58.6), $h_0(\lambda) = \frac{M^2}{ka} D_0(\lambda) = \frac{1}{2M} D_0(\lambda)$. In the case of a stratified inhomogeneity, this difference of scales can only increase if, for example, the field follows the variations of $\varepsilon(r)$. Hence the transition to the parabolic equation is no less admissible here than in §38.

Boundary conditions (58.9)-(58.11) typify the fact that the parameter $q = q_m$ is always very large for horizontal polarization. On the other hand, for vertical polarization, even with a large ε_3, the parameter q may also be large if we are speaking of short enough waves – if $M \gg \sqrt{\varepsilon_3}$. The physical sense of this parameter relationship was analyzed in §34 (see (34.22)), where it was shown that if $|q| \gg 1$, absorption of the waves in the soil will make itself felt before geometrical distortion of the surface in attenuation of the field of a source on the ground.

In the case of very short waves, therefore, for which tropospheric refraction is basically of interest, even with a soil that conducts well the principal case is that in which $|q| \gg 1$ and the boundary condition (58.9) takes the following form for either polarization:

$$U=0 \text{ for } y=0. \tag{58.12}$$

According to Formulas (34.22), (34.24), this will be the case for vertical polarization when, disregarding the displacement currents,

$$10^7\sigma_{CGSM}^{1/3}\lambda_{km}^{2/3} \ll 1, \tag{58.13}$$

i.e., for an average soil, $\sigma_{CGSM} \sim 10^{-13}$, $\lambda_{km} \ll 0.3$ km must be true; for the ocean, $\sigma_{CGSM} \sim 0.4 \cdot 10^{-10}$, $\lambda \ll 25$ meters. If, on the other hand, the displacement currents are small, $\varepsilon_3 \approx \mathrm{Re}\,\varepsilon_3$, as occurs for very short waves, the absolute value $|q| = M/\sqrt{\mathrm{Re}\,\varepsilon_3}$ is that much larger.

Now the formulation of the problem has been simplified considerably. An important result follows even from this formulation: if in Formula (58.8) the multiplier M^2 is taken out from the square brackets, we may, applying Formulas (58.5) and (58.6), recast the equation in the following form:

$$\frac{\partial^2 U}{\partial y^2}+i\frac{\partial U}{\partial x}+M^2\left[\frac{2h}{a}+\frac{\varepsilon(h)-\varepsilon_0}{\varepsilon_0}\right]U=0. \tag{58.14}$$

It may be understood as the equation for a <u>flat</u> earth's surface and plane stratification in the coordinates x, y for a function $\varepsilon(h)$ that varies with altitude according to

$$\varepsilon_{mod}(h)=\varepsilon(h)+\frac{2h}{a}\varepsilon_0. \tag{58.15}$$

This leads us to the method of the modified refractive index, §56, Subsection 1 (see also Formula (57.1)): the earth's curvature can be replaced by an additional linear increase in ε.

On the other hand, if the true permittivity $\varepsilon(h)$ varies linearly (or almost linearly), we may assume $\varepsilon(h)-\varepsilon_0=\varepsilon_0+h\left(\frac{d\varepsilon}{dh}\right)_0$ and introduce the effective radius a_e by the formula

$$\frac{1}{a_e}=\frac{1}{a}+\frac{1}{2}\left(\frac{d\varepsilon}{dh}\right)_0. \tag{58.16}$$

If we then redetermine x, y and M in Formulas (58.4)-(58.6), setting

$$x'=\frac{k_0 a\theta}{2M'^3}=x\frac{M^3}{M'^3}, \quad y'=\frac{k_0 h}{M'}=y\frac{M}{M'}. \tag{58.17}$$

$$M^* = \sqrt[3]{\frac{k_0 a_e}{2}}, \qquad (58.18)$$

then we get

$$\frac{\partial^2 U}{\partial y'^2} + i\frac{\partial U}{\partial x'} + y'U = 0, \qquad (58.19)$$

i.e., in the variables x', y' the same equation as for the homogeneous atmosphere above a spherical earth, for example, Eq. (38.21). It follows from this that for linear variation of $\varepsilon(h)$, the field takes exactly the same form as in the case of a homogeneous atmosphere, the only difference being that the vertical and horizontal length scales are changed by substituting the effective radius a_e for a (58.16):

$$D_0(\lambda) \to D_0^*(\lambda) = \frac{2M^{*2}}{k_0}, \quad h_0(\lambda) \to h_0^*(\lambda) = \frac{M^*}{k_0}, \qquad (58.20)$$

i.e., $D_0(\lambda)$ is increased by a factor of $\sqrt[3]{\frac{a_e^2}{a^2}}$ and $h_0(\lambda)$ by a factor of $\sqrt[3]{\frac{a_e}{a}}$.

In this justification for the methods of the modified refractive index and the effective earth radius, they occur in the same approximation, in which the wave equation may be replaced by a parabolic equation. However, we may also proceed directly from the wave equation (58.1) and, having separated the variables r and ϑ, systematically expand $\sin\vartheta$ in powers of ϑ, i.e., in powers of the ratio $a\vartheta/a$, in the equation for the function of ϑ. An estimate of the error in the value of the height factor of the field $Z(h)$ based on this indicates [19] that the relative error is of the order of $kh^{5/2}/a^{3/2}$. For centimeter waves, for example, $\lambda = 10$ cm, the error rises to 20% at an altitude of the order of 2 km. Thus, the ε_{mod} method can be used for such short waves only for limited altitude ranges, for example, in analysis of tropospheric waveguides. However, even for $\lambda \sim 10$ m, the error is negligibly small throughout the troposphere, so that the wave equation

$$\nabla^2 u + k^2 \varepsilon_{\text{возд}}(z) u = 0, \quad k = \frac{\omega}{c}, \tag{58.21}$$

may be considered, where z ≡ h is the height above a (flat) ground surface, i.e., it has the same sense as in §§56 and 57.

2. Like the waveguide equation (58.21) taken in cylindrical coordinates r, z, the solution of parabolic equation (58.14) is sought by the method of separation of variables. In the latter case, we set in Formula (58.14)

$$U = \frac{1}{2\pi i} \int_C e^{ixt} F(y; t)\, dt, \tag{58.22}$$

and, consequently, F must be a solution of the equation

$$\frac{d^2 F}{dy^2} + (p^2(y) - t) F = 0, \quad p^2(y) = \frac{M^2}{\varepsilon_0}(\varepsilon_{\text{возд}} - 1), \tag{58.23}$$

and C is a specially selected contour.

In the other case, according to Formula (57.3b), we may set (in contrast to Eqs. (58.1)-(58.3), r is a <u>cylindrical</u> coordinate here, r = aϑ):

$$u = 2 \int B(\nu) J_0(\nu r) Z(z; \nu)\, d\nu, \tag{58.24}$$

where Z must be a solution of the equation that we examined on several occasions in §§55 and 57:

$$\frac{d^2 Z}{dz^2} + (k^2 \varepsilon_{\text{возд}}(z) - \nu^2) Z = 0. \tag{58.25}$$

Since U is the attenuation function and u is a field function containing the factor r^{-1}, the conditions at the source that these functions must satisfy will be different. The contour C and the properties F in Formula (58.22), as well as the coefficient B(ν) and the function Z in Formula (58.24) must be selected to conform to these conditions, the boundary condition at z = 0 and the radiation condition. We see that the equations for the "height factors" F and Z differ only in the scale of the independent variable, $y = \frac{k_0}{M} z$, $k_0^2 = \frac{\omega^2}{c^2}\varepsilon_0$, and in the notation of the separation parameter:

$$\nu^2 = -\frac{k_0^2}{M} t + k^2. \qquad (58.26)$$

We shall use Eq. (58.25) as our point of departure.

Let us consider the problem of determining the height factors. Equation (58.25) has two independent solutions, and since as $z \to \infty$, definitely $\varepsilon(z) \to 1$ and $\varepsilon_{mod}(z)$ becomes $1 + (2z/a)\varepsilon_0$, i.e., a linear function of z, these solutions must go over into solutions of the equation for a linear layer (57.4) and admit of expression in terms of cylindrical functions of order 1/3 or Airy functions. It is clear both from this and from the solution in the geometrical-optical approximation that these will be solutions Z^+ and Z^- that behave as departing and arriving waves. The radiation condition requires that only the solution Z^+ be retained; this gives a receding wave asymptotically as $z \to \infty$. If the source is at a height z_0, the field below it may contain both fundamental solutions:

$$Z = AZ^+ \text{ for } z > z_0,$$
$$Z = C_1 Z^+ + C_2 Z^- \text{ for } z_0 > z > 0, \qquad (58.27)$$

where A, C_1 and C_2 are constants determined by sewing these functions together at $z = z_0$ and from the boundary condition at $z = 0$. We shall take it in the form (34.6), although it is now necessary to remember that the glancing angles ψ, $\cos^2\psi = \sin^2\alpha = \nu^2/k^2$ may be arbitrary, i.e., it is necessary to set

$$\frac{\partial Z}{\partial z} = -ik\eta Z \text{ for } z = 0, \qquad (58.28)$$

$\eta = \dfrac{\sqrt{\varepsilon_g - \frac{\nu^2}{k^2}}}{\varepsilon_g}$ for vertical polarization; $\eta = \sqrt{\varepsilon_g - \frac{\nu^2}{k^2}}$ for horizontal polarization. From this we obtain two equations:

$$AZ^+(z_0) = C_1 Z^+(z_0) + C_2 Z^-(z_0),$$

$$C_1 \frac{\partial Z^+(0)}{\partial z} + C_2 \frac{\partial Z^-(0)}{\partial z} = -ik\eta\,(C_1 Z^+(0) + C_2 Z^-(0)),$$

which define two of the three constants.

The common constant factor depends on how $Z^{\pm}$ is normalized. As $z \to +\infty$, their behavior is given by Formula (57.17), which, if convenient, may be regarded as an asymptotic expression of the Airy function (57.6) (in this case, $B^{\pm} = \exp\left(\pm i \frac{\pi}{4}\right)$):

$$Z^{\pm} \sim \frac{B^{\pm}}{\sqrt[4]{k^2 \varepsilon_{\text{mod}}(z) - \nu^2}}\, e^{\pm i \left\{ \int\limits_{z_0}^{z} \sqrt{k^2 \varepsilon_{\text{mod}}(z) - \nu^2}\, dz + \varphi \right\}}. \qquad (58.29)$$

Here the $B^{\pm}$ are arbitrary normalizing factors. In this case, as will be seen at once, we must set

$$C_2 = \frac{\nu}{2B^+B^-}\, i Z^+(z_0). \qquad (58.29a)$$

As a result, having determined A and C_1, we find (the primes denote derivatives with respect to z):

$$u = -\frac{i}{B^+B^-} \int\limits_0^{\infty} \left\{ \frac{Z^{-\prime}(0;\nu) + ik\eta Z^-(0;\nu)}{Z^{+\prime}(0;\nu) + ik\eta Z^+(0;\nu)} - \frac{Z^-(z_0;\nu)}{Z^+(z_0;\nu)} \right\} Z^+(z_0;\nu)\, Z^+(z;\nu)\, J_0(\nu r)\, \nu\, d\nu$$

for $z > z_0$;

$$u = -\frac{i}{B^+B^-} \int\limits_0^{\infty} \left\{ \frac{Z^{-\prime}(0;\nu) + ik\eta Z^-(0;\nu)}{Z^{+\prime}(0;\nu) + ik\eta Z^+(0;\nu)} - \frac{Z^-(z;\nu)}{Z^+(z;\nu)} \right\} Z^+(z_0;\nu)\, Z^+(z;\nu)\, J_0(\nu r)\, \nu\, d\nu$$

for $0 < z < z_0$. (58.30)

Actually, the indicated selection of C_2 and the limits are valid if u now satisfies one last requirement that we have not yet taken into consideration: as $z \to z_0$, $r \to 0$ the function u must become the Debye function for an isolated dipole (16.12):

$$u = \frac{e^{ikR}}{R} = \int\limits_0^{\infty} \frac{\nu\, d\nu}{\sqrt{\nu^2 - k^2}}\, J_0(\nu r)\, e^{\mp (z - z_0)\sqrt{\nu^2 - k^2}} \text{ for } z \gtrless z_0. \qquad (58.31)$$

But as $R \to 0$ in this integral (as in (58.30)), infinitely large ν play an important role. Hence it is sufficient to substitute the values

$Z^{\pm}$ as $\nu \to \infty$ in the integrals of (58.30).

We may use the asymptotic form of $Z^{\pm}$ (58.29) with $\nu^2 \gg k^2\varepsilon(z)$:

$$Z^{\pm} \sim \frac{B^{\pm}}{\sqrt{\nu}} e^{\mp(z-z_0)\nu \pm z_0\nu}. \qquad (58.32)$$

Substituting these expressions in (58.30) gives

$$u = \int_0^{\infty} J_0(\nu r) e^{\mp(z-z_0)\nu} d\nu \text{ for } z \gtrless z_0.$$

Thus, there actually is agreement with the integral of (58.31) (in which we must also let $\nu \to \infty$).

Thus, (58.30) is the field of a dipole in an arbitrary spherically symmetrical atmosphere if $Z^{\pm}$ implies solutions to Eq. (58.25) that have been normalized in accordance with their asymptotic behavior (58.29). We note that the factor $-1/B^+B^-$ can be expressed without reference to the asymptotic behavior of the functions. That is to say, we may set in Formulas (58.30)

$$iB^+B^- = \frac{1}{2}W^0 = \frac{1}{2}\left(Z^- \frac{\partial Z^+}{\partial z} - Z^+ \frac{\partial Z^-}{\partial z}\right). \qquad (58.33)$$

Indeed, according to Formula (58.25), $dW^0/dz = 0$, i.e., W^0 is independent of z. Hence the value of this constant can be obtained, for example, by using the values of (58.29) for $Z^{\pm}$ as $z \to \infty$ in Formula (58.33). Then we find that $W^0 = 2iB^+B^-$.

Solution (58.30) obviously generalizes Solution (31.9a), which is valid for a homogeneous atmosphere and a flat earth. Hence it is sometimes known as the generalized Sommerfeld integral. Like Formula (31.9a), it is still far from a form suitable for practical use.

First of all, remembering that, according to Formula (58.25), $Z^{\pm}$ and, according to Formula (58.28), η as well depend only on ν^2, i.e., are even functions of ν, we may, as usual, set

$$J_0(\nu r) = \frac{1}{2}(H_0^{(1)} + H_0^{(2)}) = \frac{1}{2}(H_0^{(1)}(\nu r) - H_0^{(1)}(-\nu r))$$

and extend integration over the entire real axis of ν:

$$u = \int\limits_{-\infty}^{+\infty} H_0^{(1)}(\nu r) Z^+(z_0; \nu) Z^+(z; \nu) Q(z, \nu)\, \nu d\nu, \qquad (58.34)$$

$$Q(z, \nu) = \frac{1}{W^0}\left\{\frac{Z^{-\prime}(0; \nu) + ikqZ^-(0; \nu)}{Z^{+\prime}(0; \nu) + ikqZ^+(0; \nu)} - \frac{Z^-\left(\begin{matrix} z_0 \\ z \end{matrix}; \nu\right)}{Z^+\left(\begin{matrix} z_0 \\ z \end{matrix}; \nu\right)}\right\} \text{ for } z \gtrless z_0, \qquad (58.34a)$$

where z_0 is taken for $z > z_0$ and z for $z < z_0$, i.e., smaller than the numbers z and z_0, in the arguments of the second fraction in the braces.

This form of the solution [I, 13] is now quite similar to Form (58.22), and can be reduced entirely to the latter. On the other hand, we may at once find an expression for F and the shape of the contour C on the basis of Formula (58.22). Without giving the derivation [18], which is fully analogous to the above, we present the result. It is found that the contour C must cover all poles of the integrand function. Further,

$$F = F(y, y_0, t; q) =$$

$$= \frac{1}{W^0}\left\{\frac{F^{-\prime}(0; t) + qF^-(0; t)}{F^{+\prime}(0; t) + qF^+(0; t)} - \frac{F^-\left(\begin{matrix} y_0 \\ y \end{matrix}; t\right)}{F^+\left(\begin{matrix} y_0 \\ y \end{matrix}; t\right)}\right\} F^+(y; t) F^+(y_0; t), \; y \gtrless y_0. \qquad 58.35)$$

Here the $F^{\pm}(y; t)$ are solutions of Eq. (58.23), which behave asymptotically, as $y \to \infty$, as receding and arriving waves:

$$F^{\pm}(y; t) \sim \frac{B^{\pm} e^{i\frac{\pi}{4}}}{\sqrt{P^2(y) - t}} e^{\pm i\int\limits_0^y \sqrt{P^2(y) - t}\, dy}, \qquad (58.35a)$$

$W^0 = F^+F^{-\prime} - F^-F^{+\prime}$ (the prime denotes differentiation with respect to y) and is equal to $2iB^+B^-$. For $B^+ = B^- = 1$ and $\tau = 0$, Formula (58.35a) gives an asymptotic expression of the Airy functions. Thus, $F^{\pm}$ differs from $Z^{\pm}$ only in the notation for the independent variables, while F is obviously identical to $Z^{+}(z; \nu)Z^{+}(z_0; \nu)Q(z; \nu)$. The difference between the two solution forms essentially reduces to substitution of the Hankel function $H_0^{(1)}(\nu r)$ in Formula (58.22) by its asymptotic expression for large $\nu r = \nu a$:

$$H_0^{(1)}(\nu r) \sim \sqrt{\frac{2}{\pi \nu r}}\, e^{i\left(\nu r - \frac{\pi}{4}\right)}.$$

Actually, according to Formula (58.26), provided that we may state that the values of t concerned are not very large, $t \ll M^{*2}$, then $\nu \approx k\left(1 + \frac{\delta t}{2M^{2}}\right)$ and according to Formulas (58.17), (58.34) and (58.35),

$$u \approx \sqrt{\frac{2k}{\pi r}}\, e^{i\left(kr - \frac{\pi}{4}\right)} \int\limits_{-\infty}^{+\infty} e^{ixt} F(y, y_0, t) \frac{\delta_0^2}{2M^2}\, dt. \qquad (58.35b)$$

The multiplier before the exponential corresponds to conversion from u to U, and to demonstrate the identity of Formulas (58.34) and (58.22), it would be necessary to consider, in addition, the problem of reduction of the contour C to the real axis. It proceeds from the condition at the source.

The actual calculation of u or U must consist firstly in finding the height factors Z^{*} as solutions of Eq. (58.23) or (58.25). In certain cases, it may be carried out analytically for a given function $\varepsilon(z)$, while in others it is done by numerical integration. In many cases, however, it is sufficient to take the solution in the geometrical-optical approximation (§57). Secondly, it is necessary to perform integration over t (or ν). Just as in the case of the homogeneous atmosphere (or a linear curve of $\varepsilon(z)$), the integral can frequently be reduced to a series of residues in which, if we are considering the field "beyond the horizon," we may restrict ourselves to a small number of terms. This transformation is similar to conversion from Formula (39.21) to Formula (39.20) or from (55.13) to (55.17). The agreement with the case of a three-layered space, which was considered in §55, is complete. Formula (58.34) is essentially another form of (55.13), and since $\nu d\nu = k^2 \sin\alpha\, d\sin\alpha$, the function under the integral, by which $H_0^{(1)}(\nu r)\nu d\nu$ is multiplied in Formula (58.34), may be regarded as identical with $-\frac{i}{4k\cos\alpha}\Phi_m(\alpha)$ (compare introductory remarks to the present subsection). Accordingly, the integration contour may even now be drawn away from the real axis in the direction of $+i\infty$ (the integral is exponentially small on an infinitely remote semicircle), hanging down on sections passed from the branch points of the integrand to $+i\infty$ and, moreover, leaving residues behind it at the poles of this expression (this is fully equivalent, if we substitute $\nu = k \sin \alpha$, to transition from contour Γ_1, Fig. 55.3, to the contour of Fig. 55.4). Thus, u becomes the sum of residues at the poles ν_l in the upper half-plane and the integrals alongside sections drawn from the branch points ν_k. Q exhibits these poles and branch points:

$$u = 2\pi i \sum_l H_0^{(1)}(\nu_l r) Z^+(z;\ \nu_l) Z^+(z_0;\ \nu_l) \operatorname{Res} Q(z,\ \nu_l) +$$

$$+ \sum_k \int_{\nu_k}^{i\infty} H_0^{(1)}(\nu r) Z^+(z;\ \nu) Z^+(z_0;\ \nu) Q(z,\ \nu) d\nu. \qquad (58.36)$$

The poles ν_l are determined by the equation

$$Z^{+\prime}(0;\ \nu_l) + ik\eta Z^+(0;\ \nu_l) = 0. \qquad (58.36a)$$

The branch points, on the other hand, are the zeros of the function $\eta(\nu)$ and, for $|\epsilon_3| \gg 1$, are placed very high, just as in the case of the three-layered space (§55, Formula (55.16)). As a result, the corresponding $H_0^{(1)}(\nu_k r)$ are exponentially small and the integrals alongside the sections may usually be disregarded. Assuming that the poles are simple, we may assume (the second term in the braces in (58.34a) does not make a contribution)

$$\operatorname{Res} Q(z) = \frac{v_l}{W^2} \frac{Z^{-\prime}(0;\, v_l) + ik\eta_l Z^{-}(0;\, v_l)}{\frac{d}{dv}\left[Z^{+\prime}(0;\, v) + ik\eta Z^{+}(0;\, v)\right]_{v=v_l}} \tag{58.37}$$

If we use Formula (55.22), the integral also reduces to a sum of residues at the poles of the same function taken in the variables y, t. However, to investigate the field in the "visible region," "above the horizon," or "near the horizon" (here these concepts have a somewhat different sense than in the case of the homogeneous atmosphere), it would be necessary to take too many terms of the series or it would diverge anyway. In this region, it is more convenient to use the integral representation (58.22) or (58.34).

In studying waveguide propagation of radio waves, it is most important to determine the "characteristic numbers" – the values ν_l and, in particular, those of them that have the smallest imaginary parts.

A detailed exposition of the technique for calculations by Formula (58.36) may be found, for example, in [I, 13], which also presents certain results for linear and "broken-linear" profiles, i.e., for cases in which either $\varepsilon(z)$ is a linear function of altitude at all altitudes or the slope of the line changes abruptly at a certain altitude (curves 5 and 6 on Fig. 56.5), as well as for a profile composed of two straight lines joined by a segment of a parabola and for a profile containing a linear trend and an exponential segment. The case of the broken-line profile was examined in [5]. Important results based on Fok's solution (§57, Subsection 6, Formulas (58.22) and (58.35)) were obtained in [20], where a special study was devoted to superrefraction for a profile of the type of curve 7 on Fig. 56.5 or Fig. 57.6, i.e., in the presence of a waveguide layer at the ground. This profile can be described systematically by the formula

$$\frac{M^2}{\varepsilon_0} \varepsilon_{\text{mod}}(z) \equiv p^2(y) - p^2(y_l) + \frac{(y - y_l)^2}{y + y_l}. \tag{58.38}$$

where $y_1 = k_0H_1/M^*$ is the dimensionless altitude of the middle of the layer and $y_l = p^2(y_1) - 2y_1$. The most important case, $q = \infty$, was selected for q. If the approximation of geometrical optics is used to calculate the altitude functions Z (or the function F), then Z^+ is given by the function (57.38a)-(57.38c), while Z^- is its complex conjugate. Forming from them the function $Q(z; \nu)$ (58.34a), and then converting to the dimensionless variables y, y_l, y_0 and t and applying Formula (58.35), we obtain after certain calculations [18b]

$$F(t, y_0, y) = \frac{e^{i(\xi(y_0) - \xi_0)} \sin \xi(y)}{\sqrt[4]{p^2(y_0) - t}\,\sqrt[4]{p^2(y) - t}\,[\chi_1(\rho)e^{-2i\xi_0} - ie^{-2\rho}]}. \quad (58.39)$$

Here it has been assumed that one of the corresponding points – the source – has a dimensionless height y_0 and is situated above the layer, $y_0 > y_1$, while the second is below the inversion point, inside the waveguide, $y < y_1$; the values of $\xi(y)$, ξ_0, χ and ρ are given by Formulas (57.37)-(57.37b) and (57.38c). If in the deep shadow we limit ourselves to calculation of the first few terms of the residue series (i.e., if we take into account only the first few poles covered by contour C (58.22)), we obtain propagation that is distinctly waveguide in nature. The roots of Eq. (58.6a) (at $q = \infty$ and, consequently, $\eta = \infty$, it becomes the condition $Z^+(0; \nu_l) = 0$, compare (39.3)) give the values of ν_l and, according to Formula (58.26), the values of t_l for the waves trapped by the waveguide and the first few untrapped normal waves. On the other hand, these values are also obtained as poles of the function F (58.39), i.e., as roots of the equation

$$\chi_1(\rho) = ie^{-2\rho + 2i\xi_0}, \quad (58.39a)$$

which may be compared with the same condition (55.15) in the three-layer method.

A detailed numerical analysis is given in [20] for several sets of particular values of the parameters y_1, y_0 and y_l. The waveguide nature of propagation is manifest in the fact that the imaginary parts of the roots t_l are much smaller than for a homogeneous atmosphere above a spherical earth.

In the two forms that proceed respectively from Formulas (58.22) and (58.34), and in a third form based on Formula (55.13) (all of these forms are fundamentally identical if it is considered, for example, that the asymptotic approximation is essentially taken for $H_0^{(1)}(\nu r)$ in the parabolic-equation method), the theory set forth above has also been applied for nonwaveguide profiles of $\varepsilon(h)$ in the problem of penetration beyond the horizon. Due to the difficulty of calculating the altitude functions, the first few terms of the residue series, which give the weakest attenuation with distance r, are usually studied. Naturally, they give an essentially exponential decrease of the field with distance.

Recently, however, attention was drawn anew to the entire problem. It had been found that different Z_l increased differently with altitude. Only with $z = 0$ can we speak unequivocally of predominant importance of the lower l. In [5], the equation for the altitude factors (58.25) was solved on the assumption that $\varepsilon(h)$ diminishes with altitude by the standard-atmosphere law (15.3)-(15.4) only in the interval between the surface of the earth and an altitude $h = 9.3$ km, where $\varepsilon = 1$ with such a gradient, while above this level ε does not vary. Thus, ε_{mod} has a broken-line profile with an inflection point at the altitude $h_1 = 9.3$ km. With this assumption, Eq. (58.25) was solved exactly, and not in the geometrical-optical approximation. This becomes possible because Airy functions (38.28) represent the solution with a linear variation of $\varepsilon(h)$. In the interval $0 < h < h_1$, the solution is represented by the sum of two Airy functions (whose asymptotes have the form of arriving and departing waves, respectively):

$$Z = C_1 w^- + C_2 w^+,\ 0 < h < h_1, \tag{58.40}$$

while for $h > h_1$, it is of the nature of a receding wave only,

$$Z = A w^+,\ h_1 < h. \tag{58.40a}$$

Sewing the two functions together at $h = h_1$ and satisfying the condition at the ground surface that corresponds to short waves and a large $\varepsilon_3 (q = \infty)$, we can determine not only two of the three coefficients C_1, C_2 and A, but also the natural values of ν_l. The result obtained in [5] indicates that for high enough corresponding points (which are nevertheless far beyond the horizon), the contribution of higher-number modes diminishes only for small l. For $\lambda = 6$ m and antenna heights $h_0 = h = 1000$ feet ≈ 330 m, beginning with $l \approx 90$, the contribution of higher modes begins to increase sharply, and it remains large in any event up to $l = 126$. The imaginary part of ν_l remains much smaller here than in the case of a homogeneous atmosphere above a spherical earth. It is known, for one thing, that the experimental points fall on the curve for the unbounded standard atmosphere (i.e., onto the curve corresponding to the theory of §39 with $a \to a_e = \frac{4}{3} a$) only in front of the horizon and directly behind it. Beyond, the actual field exceeds the theoretical field (for this case) by tens and even hundreds of decibels. The mechanism of radio wave scattering on tropospheric turbulence was invoked to explain this fact. At the same time, the broken-line profile calculation described above gives (if we consider 80 modes in the region of l-values in which their contribution is largest) very good agreement with experiment.

However, the broken-line profile embodies an idealization that seems very doubtful: if at a certain point the derivative of $\varepsilon(h)$ changes jumpwise, this physically unreal singularity of the curve may endow the solution with features that do not actually occur. This weak point of all calculations based on inflected profiles has not yet been subjected to exhaustive study.

We recall once again that, as was shown in [21], the simple ray treatment may be found fruitful within line-of-sight range.

Manuscript Page No.	[Footnotes]
641	It should be remembered that here r is the distance from the source along the arc of a great circle, $r = a\vartheta$, and not the distance from the center of the earth.

Manuscript Page No.	[Transliterated Symbols]
585	мин = min = minimal'nyy = minimal
598	мод = mod = modifitsirovannyy = modified
602	эфф = eff = effektivnyy = effective
602	станд = stand = standartnyy = standard
624	кр = kr = kriticheskiy = critical
633	крит = krit = kriticheskiy - critical

Chapter 10

PROPAGATION OF RADIO WAVES IN A TURBULENT TROPOSPHERE

§59. INTRODUCTION

1. We have hitherto regarded the atmosphere as either homogeneous or stratified-inhomogeneous. In actuality, however, as was noted as early as §15, the real atmosphere is always filled with random three-dimensional inhomogeneities, random fluctuations of density, temperature and humidity and, consequently, fluctuations of the permittivity ε at the refractive index n. They produce various effects that influence the propagation of radio waves, light and sound. Accordingly, they are studied experimentally by various methods that complement one another, and the conclusions are frequently common for all three forms of radiation. For the sake of definiteness, however, we shall speak of radio waves in the troposphere.

A bundle of radio waves passing through a statistically inhomogeneous medium experiences random phase shifts that are different at different points on the front, and, accordingly, refraction and scattering (we may disregard absorption if we are speaking of frequencies for which ε is real). Hence at an observation point on the path of the undisturbed pencil of rays, the signal level oscillates at the frequency of the oscillations of ε in the medium. Since these frequencies are small by comparison with the frequencies characteristic for radio transmission, what actually takes place is fading of the signal. Moreover, the direction of the normal to the front fluctuates due to refraction on inhomogeneities. On passage of starlight through the atmosphere, these two effects reduce, respectively, to twinkling and vacillation of the apparent positions of the stars as seen from the ground. Moreover, if scattering is very strong, there is a decrease in average brightness.

The effects enumerated above are detrimental to reception conditions and have become very important with the current steady increase in the sensitivity of the receivers (although useful information on the nature of turbulent motions in the troposphere can be derived from their study). On the other hand, scattering of radiation from the direct beam results in "illumination" of regions that would otherwise be in shadow. An important case here is found in penetration of radio waves beyond the horizon, where, in the absence of the effect described, the field in deep shadow is exponentially small. Thus, due to scattering on inhomogeneities, the field beyond the horizon may be substantially more stronger than that predicted by theory for a homogeneous (§39) or standard layerwise-inhomogeneous (§§56-58) atmosphere. Experimentally, as we said at the end of §58, we do, in the case of very short waves, detect an excess (of tens and hundreds of decibels) of the observed field over that calculated for a homogeneous medium. Despite the statistical nature of this field, which is subject to considerable fading as well as other types of fluctuations, it can be used (and is already in use) for over-the-horizon radio transmission. As we shall see below, the scattering effect is particularly strong for superhigh frequencies, i.e., precisely in the band in which the sphericity of the earth has a particularly harmful effect. Hence, since about the beginning of the '50's, study of the role of the atmosphere's statistical inhomogeneity has acquired particularly great importance. Since, by the very essence of the phenomenon, this study requires statistical methods, it constitutes an important and very rapidly developing branch of statistical radiophysics. An enormous - and still increasing - number of studies have been devoted to it. Detailed surveys and monographs [1, 2, 3, 4] containing also original results obtained by their authors are available even now. All of this obliges us to limit ourselves in the present book to a brief exposition of only certain elementary questions and references to the corresponding surveys, and in some cases to the original articles.

2. The effect of inhomogeneities on the field depends essentially on the relationship between the dimensions l of the individual inhomogeneity (this quantity may be determined as $l = n/|\nabla n|$), the wavelength $\lambda = 1/k$ and the characteristic dimension L of the entire scattering region (if the entire medium is filled with inhomogeneities, then $L \sim D$, where D is the distance between corresponding points). Inhomogeneities with $l >> \lambda = 1/k$ are most effective (see §60 below; compare also §48, from which it is seen that for $l << \lambda = 1/k$ the scattering is Rayleigh scattering and proportional to $(kl)^{-4}$, i.e., it is small). Hence the approximation of geometrical optics in its various modifications is the basic method. Further, the "depth of inhomogeneity" - the quantity $\delta\varepsilon$ - the deviation of ε from its mean value $\bar{\varepsilon}$ is an essential factor. In the atmosphere, it is always true that $|\delta\varepsilon| << \bar{\varepsilon}$ so that we have the second useful small parameter of the problem. If the dimensions of the single inhomogeneities and their number are not very large, the disturbance that they create is small and we may utilize the smallness of $\delta\varepsilon$ to apply an elementary method - that of small perturbations (§60), in which the field incident on each inhomogeneity is regarded as identical with the unperturbed field. This method is quite adequate and is basic in calculating scattering over "large" angles, and, in particular, in analysis of penetration of the rays beyond the horizon. However, the accumulated effect of many

inhomogeneities, even though they partially compensate one another, may ultimately render this approximation unsuitable, generally speaking, after passage over a long enough distance. Below we shall see that this is the case chiefly for observation in the line-of-sight region. It becomes necessary to take account of the "reradiated fields," and various other approximate methods to be considered in later paragraphs are possible here. Let us describe them briefly here.

If the characteristic dimensions l of the inhomogeneities are large enough and the distances L at which the observations are made are long enough, the Fresnel zone fits into a single inhomogeneity in the transverse direction. Then a theoretically important parameter, the so-called <u>wave parameter</u> P,

$$P = \frac{L}{kl^2} \ll 1, \tag{59.1}$$

is small. In this case, each inhomogeneity acts as a prism deflecting the ray and as a collecting or dispersing lens. These effects are minor. The total effect is close to that which occurs on passage through a layerwise-inhomogeneous medium with a random distribution of $\delta\varepsilon$ over one coordinate – the longitudinal one. To the extent that $kl \gg 1$ we may apply geometrical optics, as in the one-dimensional problem considered in §57, with the deviations of the refractive index from its average given statistically.

This elementary case of the geometrical-optical approximation by no means exhausts all of its possibilities. Even in the three-dimensional problem, the slowly varying phase $ik\varphi(\vec{r})$ and amplitude $A(\vec{r})$ of the wave can be found for any L by successive approximations from the solution of the wave equation for the wave function u. We can, for example, as in §57 (see Formula (57.15)), seek the function $u(\vec{r})$ in the form

$$u(r) = A(r)e^{ik\varphi(r)}. \tag{59.2}$$

by expanding A and φ in series in powers of the small parameter $1/kl$. In principle, however, it is possible to go farther by including A in the phase, as is customary, for example, in quantum mechanics, or as is always done when it is desired to extract the exact value of the small reflection coefficient from the approximation of geometrical optics (compare §57):

$$u(r) = e^{\Phi(r)}, \tag{59.3}$$

$$\Phi(r) = ik\varphi(r) + \ln A(r). \tag{59.3a}$$

In this case, we may, expanding Φ in series, also take account of relatively large amplitude distortions. However, the form (59.3) shows an advantage over (59.2) only when the first two approximations of geometrical optics, which give a solution of the form (57.17) are not adequate. At the same time, for passage of waves through a chaotically inhomogeneous atmosphere, when, in contrast to §57, we are not interested in exponentially small reflection, even the elementary solution in the form (57.17) is in many cases found to be sufficient (if necessary with the corresponding three-dimensional generalization). Moreover, it is often possible here to determine the phase Φ not only by the usual method, expanding Φ in $1/kl$ and then separating an expression of the first order with respect to δn from the principal terms, but also to expand Φ directly in δn (and then in $1/kl$). This means that we may consider in Formula (59.3)

$$\Phi = \Phi_0 + \Phi_1, \tag{59.3b}$$

where Φ_0 corresponds to the solution of the wave equation for the given sources in a homogeneous medium ($\delta n = 0$) and Φ_1 is a term proportional to δn (in the one-dimensional problem, $\Phi_0 = ikz$, for a point source $\Phi = ikr$, and so forth). As will be seen from the above, this approach, which is sometimes known as the method of smooth perturbations [1, 2] and was used in a number of papers following Rytov [5, 6], is in principle closely related to other forms of the geometrical-optical approximation and is often equivalent to them. At the same time, this question is illuminated variously in the literature. The relative merits of these methods (and the ray-diffusion method to be described below) and their ranges of applicability have been evaluated quite differently. In many cases the region of applicability was found by considering a single isolated inhomogeneity. At the same time, it is extremely important that we are interested in statistical quantities for a medium filled full with inhomogeneities (here there is extensive averaging of terms of the first order with respect to δn). On the other hand, criteria borrowed from ordinary optics and assuming implicitly that $\delta n \sim n$, which is also inadmissible, are sometimes adopted. Hence we are obliged to devote a relatively large amount of space to the question of ranges of applicability of the various methods (we note, for example, that survey papers on this problem frequently disagree with the conclusions set forth in the present chapter).

The ray-diffusion method mentioned above is extremely convenient and helpful. It is based on the concept of independent scattering of individual rays of the pencil on certain centers, with the calculation accordingly carried out with the Fokker equation. Clearly, this method also uses ideas of geometrical optics, but the relation between them is rather unique (see §62).

3. Above we spoke of a definite geometrical characteristic l of the inhomogeneities. Actually, there are direct measurements of ε that tend to the conclusion that such a geometrical parameter does appear in certain processes. In the language of statistical description, this means that, for example, the correlation function of the $\delta\varepsilon$ values at two points $\vec{r}$ and $\vec{r}'$ takes the form

$$\overline{\delta\varepsilon(r)\cdot\delta\varepsilon(r')}=\overline{(\delta\varepsilon)^2}F\left(\frac{|r-r'|}{l}\right), \qquad (59.4)$$

where $F(0) = 1$, $F(\rho) \to 0$ for $\rho \gg 1$ and the simplest case of statistically uniform and isotropic distribution of the inhomogeneities has been selected: the correlation coefficient F depends only on a single (constant) parameter l, the correlation length. In the case of a nonisotropic inhomogeneity, the correlation lengths l_x, l_y and l_z along the three axes are different (compare §51).

Instead of the correlation function, we might use the <u>structural function</u> for the fluctuations of ε (see §15):

$$D_\varepsilon(|r-r'|)=2\overline{(\delta\varepsilon)^2}\left(1-F\left(\frac{|r-r'|}{l}\right)\right), \qquad (59.5)$$

so that $D_\varepsilon(0) = 0$, $D_\varepsilon(\infty) = 2\overline{(\delta\varepsilon)^2}$. Often, instead of the ε fluctuations, it is expedient to consider fluctuations of the refractive index $n = \sqrt{\varepsilon}$. Since $\varepsilon=\varepsilon_0+\delta\varepsilon,\ n=n_0+\delta n\approx\sqrt{\varepsilon_0}+\frac{1}{2}\frac{\delta\varepsilon}{\sqrt{\varepsilon_0}}$, we have instead of Formula (59.4)

$$\overline{\delta n(r)\cdot\delta n(r')}=\overline{(\delta n)^2}F\left(\frac{|r-r'|}{l}\right). \qquad (59.6)$$

$$\overline{(\delta n)^2}=\frac{1}{4}\cdot\frac{\overline{(\delta\varepsilon)^2}}{\varepsilon_0}. \qquad (59.6a)$$

Accordingly, the structural function for n is

$$D_n(|r-r'|)=2\overline{(\delta n)^2}\left(1-F\left(\frac{|r-r'|}{l}\right)\right)=$$
$$=\frac{1}{4}D_\varepsilon(|r-r'|). \qquad (59.7)$$

From the very first studies of statistical radio-wave propagation in the troposphere, the usual analytical method has consisted in assuming some definite function for F as a hypothesis and then drawing conclusions regarding the parameters $\overline{(\delta\varepsilon)^2}$ and l from comparison of the calculated radio wave field with experiment. Thus, it was assumed in many papers that

$$F(\rho)=e^{-\rho}. \qquad (59.8)$$

For $\rho = 0$, however, it must be true that $F'(0) = 0$ (compare §51). Hence this function cannot be everywhere regular. Other assumptions ($F(\rho) = \exp(-\rho^2)$ and so forth) were also made. This phenomenological approach is at variance with the theory of [6, 7], which assumes a function F (or D) as given by the Kolmogorov-Obukhov theory of locally homogeneous turbulence (see §15; for more detail, [2]). Expressions for D_n (15.8c) (see also (15.8d)) were obtained on the basis of this theory:

$$D_n = \begin{cases} C_n^2 l_s^{2/3} \left(\frac{R}{l_s}\right)^2 & \text{for } R \ll l_s, \\ C_n^2 R^{2/3} & \text{for } l_s \ll R \ll l_0. \end{cases} \qquad (59.9)$$
$$R = |r - r'|, \quad C_n^2 = a^2 l_0^{4/3} M^2.$$

where l_0 and l_s are the dimensions of the largest and smallest vortices, respectively. Hence it must be assumed that the function F(R) vanishes (correlation disappears outside the maximum vortex) for $R \gg l_0$, which implies, according to Formula (59.5), $D_\varepsilon(l_0) \sim 2\overline{(\delta\varepsilon)^2}$, i.e., according to (59.9),

$$C_n^2 \approx \frac{2\overline{(\delta\varepsilon)^2}}{l_0^{2/3}}. \qquad (59.10)$$

Hence we may write for the correlation function

$$F = \begin{cases} 1 - \frac{R^2}{l_s^{4/3} l_0^{2/3}} & \text{for } R \ll l_s, \\ 1 - \left(\frac{R}{l_0}\right)^{2/3} & \text{for } l_s \ll R \ll l_0, \\ 0 & \text{for } l_0 \ll R. \end{cases} \qquad (59.11)$$

It is extremely important that two ength scales appear in this theory: the "internal" scale l_s and the "external" scale l_0, where $l_s \ll l_0$. Which of them is the decisive factor depends on the manifestations of turbulence that we are considering. Thus, the correlation function F is almost constant near unity over the entire interval of R from l_s to $\sim l_0$. The theory does not give the exact form of this function near $R = l_0$. For calculations, however, it is convenient to have some sort of analytical interpolation expression for it. Since l_s is negligibly small under the conditions of the troposphere as compared with l_0, we are speaking basically of the region $R \gg l_s$. We might, for example, derive a "smoothing" factor χ and assume

$$F\left(\frac{R}{l_0}\right) = 1 - \left(\frac{R}{l_0}\right)^{2/3} \chi\left(\frac{R}{l_0}\right), \quad l_s \ll R. \tag{59.12}$$

where $\chi(\rho) = 1$ for $\rho \ll 1$, $\chi(\rho) = \rho^{-\frac{2}{3}}$ for $\rho \gg 1$. Naturally, we might also simply assume

$$F\left(\frac{R}{l_0}\right) \approx e^{-\left(\frac{R}{l_0}\right)^{2/3}}, \quad l_s \ll R \tag{59.13}$$

and so forth.

If by analogy with the two-dimensional case (51.9) we introduce spectral functions (for a function F with a single length parameter l)

$$G(q) = \int F\left(\frac{R}{l}\right) e^{-iqR} \frac{dR}{l^3}, \quad F\left(\frac{R}{l}\right) = \int G(q) e^{iqR} \frac{dq}{q_0^3}. \tag{59.14}$$

$$\delta n = \nu f(r) = \frac{\nu l^3}{(2\pi)^3} \int g(\varkappa) e^{i\varkappa r} d\varkappa, \quad g(\varkappa) = \frac{1}{l^3} \int f(r) e^{-i\varkappa r} dr,$$

where $\vec{R}$, $\vec{r}$, $\vec{q}$ and $\vec{\varkappa}$ are three-dimensional vectors ($q_0 = 2\pi/l$), we have for the correlation function (59.8)

$$G(q) = \frac{8\pi}{(1 + q^2 l^2)^2}. \tag{59.15}$$

If

$$F = e^{-\rho^2} = e^{-\frac{R^2}{l^2}}, \tag{59.16}$$

then

$$G(q) = \pi^{3/2} e^{-\frac{q^2 l^2}{4}}. \tag{59.17}$$

Finally, if $F(R)$ is given by Formula (59.11), we may, taking the principal region of distances, $l_s \ll R \ll l_0$, into account and accordingly the q-values for the principal region, take the integral within the limits 0 and ∞:

$$G(q)=\int\left(1-\frac{R^{2/3}}{\xi^{2/3}}\chi\left(\frac{R}{l_0}\right)\right)e^{-iqR}\frac{dR}{\xi^3}=$$

$$=(2\pi)^3\frac{\delta(q)}{\xi^3}-\frac{1}{\xi^{11/3}}\int_0^\infty e^{-iqR}\chi\left(\frac{R}{l_0}\right)R^{2/3}dR.$$

Dropping the first term, which pertains to the lower limit of q, and substituting ξ for qR, we have

$$G(q)=\frac{-4\pi}{(q l_0)^{11/3}}\int_0^\infty\chi\left(\frac{\xi}{q l_0}\right)\sin\xi\cdot\xi^{5/3}d\xi. \qquad (59.18)$$

The arbitrariness associated with interpolation does not influence the calculated spectrum if we may consider $\chi = 1$, i.e., as seen from Formula (59.18), if

$$q l_0 \gg 1. \qquad (59.18a)$$

This determines the region of q-values in which we may use confidently the fluctuation spectrum predicted by theory. F is often interpolated with a formula proposed by Karman (see the monograph [2]):

$$F=\frac{2^{2/3}}{\Gamma\left(\frac{1}{3}\right)}\rho^{\frac{1}{3}}K_{\frac{1}{3}}(\rho),\ \rho=\frac{R}{l_0}. \qquad (59.19)$$

For this correlation function we may obtain [2]

$$G(q)=\frac{\Gamma\left(\frac{11}{6}\right)}{\Gamma\left(\frac{1}{3}\right)}\frac{8\pi^{3/2}}{(1+q^2l_0^2)^{11/6}}\approx\frac{8\Gamma\left(\frac{11}{6}\right)\pi^{3/2}}{\Gamma\left(\frac{1}{3}\right)}\frac{1}{(q l_0)^{11/3}} \qquad (59.20)$$

(this last expression is written for $ql_0 >> 1$, i.e., in the range of q-values not influenced by the type of interpolation); it is also understood that $ql_s << 1$). Hence arises the relationship usually used: $G(q) \sim q^{-11/3}$.

As was shown by V.A. Krasil'nikov and A.M. Obukhov, radio wave propagation can be analyzed on the basis of the function (59.9) [7a, 6, 7c] (even earlier, D.I. Blokhintsev had analyzed sound-wave propagation [7b] on the basis of a similar formula for the density fluctuations) in a turbulized atmosphere.

4. Certain fundamental methods of radio wave propagation theory for a statistically inhomogeneous troposphere will be examined in the paragraphs to follow. The results of application of these methods to practically important problems will be presented only on a very small scale, by way of illustration. Among these problems, we may cite determination of the root-mean-square phase fluctuation (similar to that examined in §62, Subsection 2 by the ray-diffusion method) and amplitude (such as that analyzed in §61, Subsection 4 by the "equivalent screen method") at a given point. It is not difficult to find these characteristics by other methods as well – for example, the method of smooth perturbations, averaging the squares of the imaginary and real parts of ψ accordingly (61.62a). Moreover, the problem of coherence of the radiation received at various points of the receiver raises the question of phase (and amplitude) correlation at two points situated either along or across the initial ray. Knowledge of the solution, for example, in the form (61.62) will also enable us to carry out these calculations. We note that it is a rather complex task to take consistent account of the two-parameter correlation function (59.11). Such a detailed consideration is given in the book by V.I. Tatarskiy [2]. In most cases, we shall have in mind a single parameter (usually corresponding to l_0), making appropriate reservations when necessary.

In its entirety, such analysis is adequate if, firstly, we can dispense with depolarization of the radiation on scattering, i.e., consider only the scalar wave equation (for a justification of this possibility, see §60), and, secondly, if the medium is uniformly turbulent, and, in particular, we can dispense with the altitude variation $\varepsilon(z)$, on which turbulent fluctuations are superimposed.

It is possible to account for the polarization relationships if we start from the Maxwell equations rather than from the wave equation. This procedure was developed in [8], based on an explicit separation of all field components into two parts – the statistical average and a fluctuation part.

Special attention is drawn to the simultaneous examination of random inhomogeneities and the systematic trend of ε. Two aspects must be borne in mind here.

Firstly, we are speaking of simultaneous consideration of refraction in the troposphere and scattering on inhomogeneities of turbulent origin (or on rain drops and fog). This problem has been examined in a number of papers by a variety of methods (see [2]). For example, the method of smooth perturbations (§61) is used, with the field in a quiet stratified medium, calculated in the approximation of geometrical optics [4; 16] taken as the zeroth approximation. The elementary form (59.2) of the geometrical-optical approximation is also used [15].

The second aspect consists in considering the influence of the ground surface, which, as we find, gives rise to special phenomena. They were discerned and explained in [17]. These singularities are associated with the fact that near the ground, in the range in which the quadratic formula (19.42) is valid, the resultant field is the consequence of interference between two rays – the direct ray and that reflected from the ground. Even if it perturbs each ray only slightly, the appearance of inhomogeneities may sharply disturb the conditions of interference. It is found as a result that the statistical characteristics of the field depend on distance in a manner quite different than in the absence of the earth's surface. The theory of the phenomenon was developed in [18].

We shall limit ourselves to these brief remarks, referring the reader to the surveys cited above [1, 2, 3, 4] and to the remaining quite extensive literature devoted to this specific problem.

§60. THE METHOD OF PERTURBATIONS

The propagation of radio waves in an arbitrarily inhomogeneous medium is described by the Maxwell equations (2.4a), (2.4b), which no longer give the simple wave equation for $\varepsilon = \varepsilon(\vec{r})$: applying the rot operator to Eq. (2.4b) and substituting Expression (2.4a) for rot $\vec{H}$, we obtain in the region where $\vec{j}_0 = 0$

$$\mathrm{rot}\,\mathrm{rot}\,\boldsymbol{E} = \mathrm{grad}\,\mathrm{div}\,\boldsymbol{E} - \nabla^2\boldsymbol{E} = k^2\varepsilon(r)\boldsymbol{E},$$

and applying the operator div to Eq. (2.4a), we find that $\mathrm{div}(\varepsilon\boldsymbol{E}) = \varepsilon\,\mathrm{div}\,\boldsymbol{E} + \boldsymbol{E}\,\mathrm{grad}\,\varepsilon = 0$. Consequently,

$$\nabla^2\boldsymbol{E} + k^2\varepsilon(r)\boldsymbol{E} + \mathrm{grad}(\boldsymbol{E}\,\mathrm{grad}\ln\varepsilon(r)) = 0. \qquad (60.1)$$

Since

$$\varepsilon(r) = \varepsilon_0 + \delta\varepsilon, \quad \varepsilon_0 \equiv \overline{\varepsilon(r)}, \quad |\delta\varepsilon| \ll 1, \qquad (60.2)$$

$\ln\varepsilon(r) \approx \ln\varepsilon_0 + \frac{\delta\varepsilon}{\varepsilon_0} - \frac{1}{2}\left(\frac{\delta\varepsilon}{\varepsilon_0}\right)^2$. We set

$$\boldsymbol{E}(r) = \boldsymbol{E}_0(r) + \boldsymbol{E}_1(r), \quad k^2\varepsilon_0 = k_0^2, \qquad (60.3)$$

where $\vec{E}_0(\vec{r})$ is the statistically averaged field at the point in question and $\vec{E}_1(\vec{r})$ is the fluctuation term:

$$\boldsymbol{E}_0(r) = \overline{\boldsymbol{E}(r)}, \quad \overline{\boldsymbol{E}_1(r)} = 0. \qquad (60.4)$$

Generally speaking, the average field $\vec{E}_0$ may differ sharply from its value at $\delta\varepsilon \equiv 0$. Only in the method of perturbations is it assumed that $\vec{E}_0$ coincides with its value for the homogeneous medium with $\varepsilon = \varepsilon_0 =$ const. In any event, we may expect $\vec{E}_1$ to be a quantity of the order of $\delta\varepsilon$, i.e., relatively small (with respect to the principal component of the vector $\vec{E}_0$).

Substituting Expressions (60.2) and (60.3) into Formula (60.1), we take account of terms up to the second order in $\delta\varepsilon$ to obtain

$$\nabla^2 E_0 + k_0^2 E_0 + \nabla^2 E_1 + \frac{k_0^2 \delta\varepsilon}{\varepsilon_0} E_0 + k_0^2 E_1 + \operatorname{grad}\left(E_0 \operatorname{grad}\frac{\delta\varepsilon}{\varepsilon_0}\right) +$$
$$+ k_0^2 \frac{\delta\varepsilon}{\varepsilon_0} E_1 + \operatorname{grad}\left(E_1 \operatorname{grad}\frac{\delta\varepsilon}{\varepsilon_0} - \frac{1}{2} E_0 \operatorname{grad}\left(\frac{\delta\varepsilon}{\varepsilon_0}\right)^2\right) = 0. \quad (60.5)$$

Terms of zeroth order with respect to $\delta\varepsilon$ (the first two), of the first order and, finally, of the second order of smallness appear here. These last terms cannot be dropped if we wish to take into account the attenuation (due to scattering on inhomogeneities) of the average field $\vec{E}_0$. The situation here is perfectly analogous to that considered in §51, Subsection 2 for a wave skipping over a randomly rough surface. Attenuation occurs on a length that is long by comparison with λ as well as by comparison with l. This means that the average field $\vec{E}_0$ differs from the unperturbed field by a slowly varying attenuation multiplier $w(\vec{r})$. Averaging Eq. (60.5) over the ensemble, we obtain an equation from which the first-order terms have dropped (we also take into consideration that $\operatorname{grad}\overline{(\delta\varepsilon)^2} = 0$):

$$\nabla^2 E_0 + k_0^2 E_0 + k_0^2 \overline{\frac{\delta\varepsilon}{\varepsilon_0} E_1} + \operatorname{grad}\overline{\left(E_1 \operatorname{grad}\frac{\delta\varepsilon}{\varepsilon_0}\right)} = 0. \quad (60.5a)$$

Since the gradient of w is small, the first two terms in the sum are practically equal to zero and the last – small – terms must be taken into account to determine w. In determining $\vec{E}_1$, on the other hand, they may be regarded as a correction. In Formula (60.5), therefore, we may collect terms of the first order in $\delta\varepsilon$ to get

$$\nabla^2 E_1 + k_0^2 E_1 = -k_0^2 \frac{\delta\varepsilon}{\varepsilon_0} E_0 - \operatorname{grad}\left(E_0 \operatorname{grad}\frac{\delta\varepsilon}{\varepsilon_0}\right), \quad (60.6)$$

where the right member does not contain $\vec{E}_1$. We have obtained the wave equation with sources. The field $\vec{E}_1$, which vanishes together with $\delta\varepsilon$, is obtained from this, according to Formula (5.10), as

$$E_1(r) = \frac{1}{4\pi}\int\left[k_0^2\frac{\delta\varepsilon(r')}{\varepsilon_0}E_0(r') + \mathrm{grad}_{r'}\left(E_0(r')\,\mathrm{grad}_{r'}\frac{\delta\varepsilon(r')}{\varepsilon_0}\right)\right]\times$$

$$\times\frac{e^{ik_0|r-r'|}}{|r-r'|}dr'. \qquad (60.7)$$

Under this heading, we are considering the solution by the perturbation method. The starting point is the assumption that the scattered field can be calculated by Formula (60.7) after substitution of the field $\vec{E}_0$ by its unperturbed value – the wave incident from the source in question that propagates before scattering in a homogeneous medium with $\varepsilon = \varepsilon_0 =$ const. The physical significance of this assumption is clear from the form of the right member of Eq. (60.6). The field $\vec{E}_0$ creates an excess (by comparison with that which would prevail at $\delta\varepsilon = 0$) polarization $P = \frac{\delta\varepsilon}{4\pi}E$ in the medium. Substituting this expression into Formula (3.9b) and remembering that $\mathrm{div}(\varepsilon\vec{E}) = 0$, where $\vec{E}_0$ may be substituted for $\vec{E}$ with an accuracy to higher-order terms, we can again obtain Formula (60.6). Considering $\vec{E}_0$ as identical with the unperturbed field, we thereby disregard the influence of the field scattered by one inhomogeneity on the polarization of the other inhomogeneity.

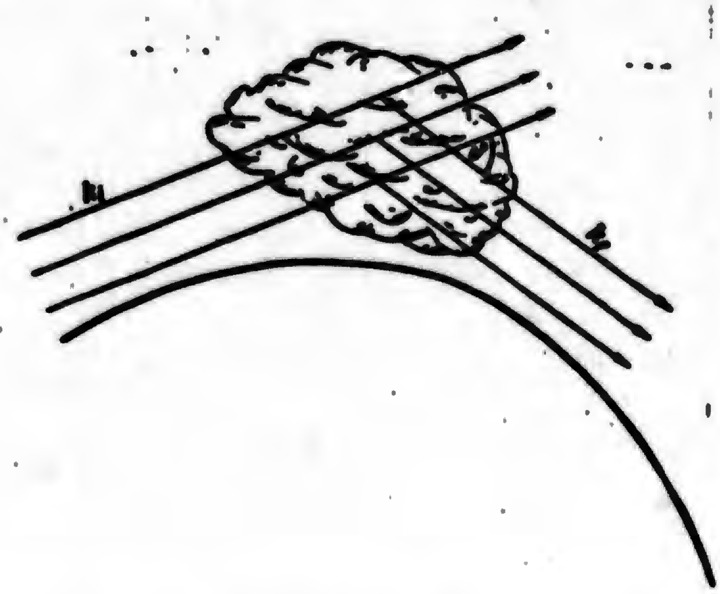

Fig. 60.1. Scattering on turbulent inhomogeneities in the troposphere.

This approach is used extensively in calculating the field that penetrates beyond the horizon as a result of scattering. It was in this form that the conception of statistical scattering on inhomogeneities was first used [9].

Thus, we shall assume (Fig. 60.1) that a plane wave is incident on the region occupied by the inhomogeneities, giving at the point $\vec{r}$

$$E_0 = A_0 e^{ik_0 r'}, \qquad (60.8)$$

while observations are conducted at a point $\vec{r}$ far outside the scattering volume, so that

$$\begin{aligned} k_0|r-r'| &= k_0\sqrt{r^2-2rr'\cos\chi+r'^2} = \\ &= k_0 r - k_0 r'\cos\chi + \ldots \approx k_0 r - k_f r', \end{aligned} \qquad (60.9)$$

where χ is the angle between $\vec{r}$ and $\vec{r}'$; $\vec{k}_f$ has the significance of the wave vector of the scattered field, $k_0 r'\cos\chi = k_f r'$, $k_f^2 = k_0^2$. Integration of the second term in the integrand of (60.7) by parts gives

$$\begin{aligned} &\int \mathrm{grad}_{r'}\left(E_0\,\mathrm{grad}\frac{\delta\varepsilon}{\varepsilon_0}\right)\frac{e^{ik_0|r-r'|}}{|r-r'|}dr' = \\ &= -\mathrm{grad}_r\int\left(E_0\,\mathrm{grad}\frac{\delta\varepsilon}{\varepsilon_0}\right)\frac{e^{ik_0|r-r'|}}{|r-r'|}dr' \to \\ &\to -ik_f\int\left(E_0\,\mathrm{grad}\frac{\delta\varepsilon}{\varepsilon_0}\right)e^{-ik_f r'}dr'\cdot\frac{e^{ik_0 r}}{r}. \end{aligned}$$

As we see, this term expresses the field pointed in the direction of propagation – the transverse electric field. Obviously, it must cancel with the appropriate first-term component. Without giving the proof, we simply subtract from the first term the component directed along $\vec{k}_f$ and drop the second term. This means that we are substituting $\vec{A}_0 - \frac{k_f(k_f A_0)}{k_0^2}$ for $\vec{A}_0$ and dropping the second term under the integral in (60.7). Thus we obtain

$$E_1(r) = \frac{e^{ik_0 r}}{4\pi r}\left(k_0^2 A_0 - k_f(k_f A_0)\right)\int\frac{\delta\varepsilon(r')}{\varepsilon_0}e^{i(k_0-k_f,\, r')}dr'. \qquad (60.10)$$

Here $k_f^2 = k_0^2$. This expression gives the scattered field far from the scattering region for a given value of the function $\delta\varepsilon(\vec{r}')$. We are interested in the scattered-energy flux in an element of the solid angle $d\Omega$, $\Sigma \cdot r^2 d\Omega = \frac{c}{8\pi}|E_1|^2 r^2 d\Omega$. It is appropriate to average it over various copies of the scattering region with given statistical properties. Further, dividing by the flux of incident energy, $\Sigma_0 = cA_0^2/8\pi$, we arrive at the differential effective cross section $d\sigma$ in exactly the same way as in §§48 and 52. It is found equal to

$$d\sigma = \frac{\Sigma r^2 d\Omega}{\Sigma_0} = \frac{k_0^4 d\Omega}{16\pi^2}\left(1 - \left(\frac{k_1 A_0}{k_0 A_0}\right)^2\right)\iint \frac{\overline{\delta\varepsilon(r')\,\delta\varepsilon(r'')}}{\varepsilon_0^2} \times$$

$$\times e^{i(k_1 - k_0,\, r'' - r')}\, dr'\, dr''. \qquad (60.11)$$

Thus, in perfect analogy to the results of §52, the effective cross section is determined by the correlation function of the inhomogeneities, in this case ε, and if only the single rankth parameter l is important,

$$\frac{\overline{\delta\varepsilon(r')\,\delta\varepsilon(r'')}}{\varepsilon_0^2} = \frac{\overline{(\delta\varepsilon)^2}}{\varepsilon_0^2} F\left(\frac{r' - r''}{l}\right). \qquad (60.12)$$

Depending on the assumptions made regarding the function F, the concrete results obtained are different. Formula (60.11) indicates that the polarization of the wave is weakly manifest if the scattering angles are small. Indeed, the entire difference from the corresponding solution for the <u>scalar</u> wave equation consists in the appearance of the factor

$$1 - \frac{(k_1 A_0)^2}{k_0^2 A_0^2} = 1 - \cos^2\theta,$$

where θ is the angle between the scattering direction and the polarization direction of the incident wave, $\vec{A}_0$. The scattering direction is characterized by the angles ϑ, φ, where the azimuth φ can be reckoned from $\vec{A}_0$. Then

$$1 - \left(\frac{k_1 A_0}{k_0 A_0}\right)^2 = 1 - \sin^2\vartheta\cos^2\varphi \qquad (60.12a)$$

and the average (over two polarizations) value of this factor is

$$1 - \frac{1}{2}\sin^2\vartheta. \tag{60.12b}$$

If, therefore, the scattering angles are small, $\vartheta \ll 1$, we may consider instead of Eq. (60.1) the wave equation

$$\nabla^2 E + k^2\varepsilon(r)E = 0, \tag{60.12c}$$

or the scalar equation for a single component of the vector $\vec{E}$.

We shall henceforth assume that $\varepsilon_0 = 1$, $k_0 = k$. Formula (60.12) assumes that the correlation length is identical in all directions. We may, of course, consider a more general case – that of anisotropic turbulence (see [19]).

We denote

$$k_f - k_i = q, \quad q^2 = 2k^2(1-\cos\vartheta) = 4k^2\sin^2\frac{\vartheta}{2}, \tag{60.13}$$

where $\vec{q}$ is the "scattering vector" and ϑ is the angle between the directions of propagation of the incident and scattered waves – the scattering angle. Using Formula (60.12), converting to the variables $\rho = \frac{r' - r''}{l}$ and $\vec{r}'$, and assuming that the effective distances $\vec{r}' - \vec{r}''$ are very small by comparison with the range of variation of $\vec{r}'$, i.e., with the dimensions of the volume V, we can integrate independently over $\vec{r}'$ (which gives the total scattering volume V) and within indefinite limits over ρ. We obtain as a result

$$d\sigma = \frac{k^4 V l^3 d\Omega}{16\pi^4}\overline{(\delta\varepsilon)^2}(1 - \sin^2\vartheta\cos^2\varphi)\int F(\rho)e^{iq\rho l}d\rho, \tag{60.14}$$

or

$$d\sigma = \frac{k^4 V l^3 d\Omega}{16\pi^4}\overline{(\delta\varepsilon)^2}(1 - \sin^2\vartheta\cos^2\varphi)G(q). \tag{60.14a}$$

Thus, the section for scattering by a given angle is determined by the component of the Fourier expansion of the correlation function for $\vec{q}$, which is equal to the scattering vector (60.13), $G(\vec{q}) = G(\vec{k}_f - \vec{k}_i)$. Even for $ql \gg 1$, when the correlation function is close to unity, scattering is determined at once by the difference between R and unity (for example, in the case of (59.11), by the small term $(R/l_0)^{2/3}$).

For long waves or extremely small angles ϑ, $ql = 2kl\sin\frac{\vartheta}{2} \ll 1$, the integral in Formula (60.14) is determined by the decrease in the function F with increasing ρ. Letting $q\rho l \to 0$, remembering that the remaining dimensionless integral is of the order of unity and disregarding the polarization factor, we obtain (here the large ρ contribute, so that the larger of the two parameters, l_0, is essential)

$$d\sigma = \frac{k^4 V l^3 d\Omega}{4\pi}\overline{(\delta\varepsilon)^2}\int_0^\infty F(\rho)\,d\rho \sim k^4 V l^3 \frac{d\Omega}{4\pi}\overline{(\delta\varepsilon)^2}. \qquad (60.14b)$$

This result is, naturally, of the nature of Rayleigh scattering on small particles, $kl \ll 1$ (§48). As always in such cases, the smallness of l is physically essential not by comparison with the wavelength λ itself, but by comparison with the reciprocal absolute value of the vector imparted to the wave, $|q|$, $ql \ll 1$. For small ϑ, even short waves, $\lambda \ll l$, will be scattered like long ones.

The scattering cross sections for the three forms of the correlation function (59.8), (59.16) and (59.19), which we shall call cases (A), (B) and (C), is expressed accordingly by Formulas (59.15), (59.17) and (59.20) (see also [1, 2 and 3]) as follows:

$$d\sigma = \frac{k^4 V l^3 d\Omega}{16\pi^2}\cdot\overline{(\delta\varepsilon)^2}\,(1-\sin^2\vartheta\cos^2\varphi)\begin{cases} \frac{8\pi}{(1+q^2l^2)^2} \text{ for } F = e^{-\rho}, & \text{(A)} \\ \pi^{3/2} e^{-\frac{q^2l^2}{4}} \text{ for } F = e^{-\rho^2}, & \text{(B)} \\ \frac{8\pi^{3/2}\Gamma\left(\frac{11}{6}\right)}{\Gamma\left(\frac{1}{3}\right)(1+q^2l^2)^{11/6}} \text{ for } F = \\ \quad = \frac{2^{2/3}}{\Gamma^{1/3}}\rho^{1/3}K_{\frac{1}{3}}(\rho). & \text{(C)} \end{cases} \qquad (60.15)$$

where the expressions appearing after the brace give the function G for cases (A), (B) and (C), respectively.

However, it is necessary to exercise caution in using these formulas in the range of large q. Actually large q correspond to small ρ in the function $F(\rho)$. At the same time, case (A) corresponds to the function (59.8), which is patently invalid in the region of small ρ, thus contradicting the condition $F'(0) = 0$. The function (59.19) (case (C)) was so selected as to give the curve of $F(\rho)$ in the region $R \gg l_s$ (see Formula (59.11)), and, consequently, is also inapplicable for $\rho \to 0$. Only function (B) is internally free of contradictions. All of these functions indicate a rapid decrease in scattering when $2\sin\frac{\vartheta}{2}$ exceeds the small quantity $1/kl$, i.e., when $ql > 1$. If, however, the scattering volume is large enough, scattering is nevertheless found to be significant and enables us to explain many cases of penetration beyond the horizon (see Fig. 60.1). We obtain for cases (A), (B) and (C), respectively, for $\vartheta \ll 1$, if $q^2l^2 \gg 1$:

$$d\sigma = \frac{V\overline{(\delta\varepsilon)^2}\,d\Omega}{16\pi^4}\cdot\left\{\begin{array}{ll} \frac{8\pi}{l\vartheta^4}, & \text{(A)} \\ \pi^{3/2}k^4l^3e^{-\frac{k^2\vartheta^2l^2}{4}}, & \text{(B)} \\ \frac{8\pi^{3/2}\Gamma\left(\frac{11}{6}\right)}{\Gamma\left(\frac{1}{3}\right)}\cdot\frac{k^{1/3}}{l^{2/3}\vartheta^{11/3}}. & \text{(C)} \end{array}\right. \qquad (60.16)$$

Experimental data on penetration beyond the horizon, which is today customarily explained by tropospheric scattering on inhomogeneities, are characterized by the fact that the field depends only very weakly on wavelength as long as the latter is small enough. This is obviously in agreement with Formula (60.16A) or, approximately, with Formula (60.16C) (admittedly, experiment [10] would predict $d\sigma \sim k^{-1/3}$ rather than $k^{1/3}$). However, as we have just stated, the functions G for (A) and (C) are inapplicable for extremely large q. Although it can still be said of (C) that this function is noncontradictory in the region $1 \ll ql_0$, $1 \gg ql_s$, where its Fourier transform does not depend on the interpolation used for the function (59.11) in the region $l_s \ll$ $\ll R \ll l_0$, we do not know at all for (A) beginning with what ρ and, consequently, up to what q it may be regarded as correct.

Knowing $d\sigma$, we can calculate the field at the observation point. It is only necessary to perform integration over the volume V in the limits given by the antenna directional pattern. Numerous calculations of this kind can be found in the literature (see, for example, the survey [3]).

It must be noted that marked scattering (which also depends weakly on wavelength) can be obtained from Formula (60.14) only if $\lambda \ll l$. Indeed, it is clear even from physical considerations that when many inhomogeneities are superimposed per wavelength, $\lambda \gg l$, their action is averaged and weakened. It also seen from Formula (60.15) that as $kl \to 0$, the scattering cross section decreases as $(l/\lambda)^4$. Hence statistical penetration over the horizon can occur only for rather short waves. If we adopt for l an estimate that appears to follow from experimental data (§15), $l \sim 10$-100 m, we arrive at the conclusion that the effect may be considerable for meter- and even shorter waves.

The scattering formula obtained (60.14) can be endowed with a somewhat different sense. Let the scattering volume in the direction of incidence of the wave have a thickness L, so that V = SL, where S is the area of the volume's cross section. If we divide the scattered energy by the primary flux incident upon the <u>entire</u> surface S, we obtain instead of $d\sigma$

$$dw = \frac{d\sigma}{S} = \frac{k^4 L l^3}{16\pi^2} d\Omega \, (\overline{\delta\varepsilon})^2 (1 - \sin^2\theta \cdot \cos^2\varphi) G(q). \qquad (60.17)$$

which indicates the directional distribution ("probability") of scattering of the waves. From this we may, in particular, find average values, for example, the mean square scattering angle $\overline{\vartheta^2}$ or, at first, the average value of $4 \sin^2 \vartheta/2 = q^2/k^2$. Since the essential ϑ are small, it agrees with $\overline{\vartheta^2}$. According to Formula (60.17), remembering that $2\pi\sin\vartheta\, d\vartheta = \frac{2\pi}{k^2} q\, dq,\ 0 < q < 2k,$ we have

$$\overline{4\sin^2\frac{\vartheta}{2}} = \frac{LP}{8\pi}\overline{(\delta\varepsilon)^2}\int_0^{2k} G(q)\, q^3\, dq. \qquad (60.18)$$

Thus, the large q make the main contribution here. If, therefore, we use the two-parameter function (59.11), l will have a value closer to l_s than to l_0 (this remark also applies to Formula (60.19), see below, but not to Formula (60.20), where, according to the remarks prefacing Formula (60.14b), large distances are a factor and $l \sim l_0$). Further, we may not simply substitute G(q) according to Formula (60.15) here, since for $kl \gg 1$, this would mean in certain cases (for example, (A) and (C)) substantial use of the range of q-values in which the function G may be incorrect (for $kl = \infty$, the integral will diverge for case (A)). For this reason, we convert from G(q) to F(ρ) by Formula (59.14) and reverse the order of integration over q and ρ. Remembering consistently that $F'(0) = 0$, introducing the variable $ql = x$ and setting $kl \to \infty$ at the upper limit, we find

$$\int_0^\infty G(x)\, x^3\, dx = \int_0^\infty x^3 dx \int_0^\infty F(\rho)\, e^{ix\rho}\, d\rho = 4\pi\int_0^\infty F(\rho)\,\rho\, d\rho\, \frac{d^2}{d\rho^2}\int_0^\infty \sin(\rho x)\, dx. \qquad (60.18a)$$

The last (internal) integral is equal to $1/\rho$ (compare Formula (10.45)). Integrating twice by parts and remembering that $F(\infty) = 0$, $F'(0) = 0$ (we assume the lower limit to be a in the integrated terms and then let a tend to zero), we obtain finally

$$\overline{4\sin^2\frac{\vartheta}{2}} \approx \overline{\vartheta^2} = -\frac{1}{2}\frac{L\overline{(\delta\varepsilon)^2}}{l}\int_0^\infty \frac{1}{\rho}\frac{d^2(F(\rho)\rho)}{d\rho^2}\, d\rho = -\frac{L\overline{(\delta\varepsilon)^2}}{l}\int_0^\infty \frac{1}{\rho}\frac{dF}{d\rho}\, d\rho. \qquad (60.19)$$

Thus, the mean square of the scattering angle in a layer of thickness L is proportional to this thickness and does not depend on wavelength.

After integration (60.17) over all q, analogous calculations give the total probability of scattering. They are distinguished by the absence of the factor q^2/k^2 in Formula (60.18) and from Formula (60.18a) by the absence of the operator $d^2/d\rho^2$. We have

$$w = \int dw = \frac{1}{2} k^2 l L \overline{(\delta\varepsilon)^2} \int_0^\infty F(\rho) d\rho. \tag{60.20}$$

The dimensionless integral is a number of the order of unity. The approximation used under this heading is valid as long as $w \ll 1$, i.e., provided that

$$k^2 l L \overline{(\delta\varepsilon)^2} \ll 1. \tag{60.21}$$

With $l \sim 100$ m, $\overline{(\delta\varepsilon)^2} \sim 10^{-20}$, we find for wavelengths $\lambda \sim 1$ m that L must be $\ll 2 \cdot 10^8$ cm.

It is also helpful to have an expression for integral scattering through an angle exceeding a certain given angle ϑ_0, i.e., for scattering with ... $> 2k \sin \frac{\vartheta_0}{2} \approx k\vartheta_0$. On integration, we encounter an integral of the type (60.18a) lacking the operator $d^2/d\rho^2$ and having as its lower limit $x = x_0 = kl\vartheta_0$

$$\lim_{\alpha \to 0} \int_{x_0}^\infty e^{-\alpha x} \sin \rho x \, dx = \frac{\cos \rho x_0}{\rho},$$

so that

$$\int_{\vartheta = \vartheta_0}^\infty dw = \frac{1}{2} k^2 l L \overline{(\delta\varepsilon)^2} \int_0^\infty F(\rho) \cos(\rho x_0) d\rho. \tag{60.22}$$

If $x_0 \ll 1$, this quantity is equal to w. But if $x_0^2 > 1$, i.e., if (taking account also of Formula (60.19), at least in the case of a single-parameter distribution, when the same l figures everywhere)

$$\vartheta_0^2 > \frac{1}{k^2 l^2} \sim \frac{\overline{\vartheta^2}}{k^2 L l \overline{(\delta\varepsilon)^2}} \sim \frac{\overline{\vartheta^2}}{w} \tag{60.23}$$

(since $w \ll 1$, this means that ϑ_0 is much larger than the root-mean-square scattering angle), then the total probability of scattering by an angle larger than ϑ_0 is small.

We note in conclusion that we have not touched at all upon the important and interesting problem of the part played by motion of the inhomogeneities, which was investigated by G.S. Gorelik [22].

§61. GEOMETRICAL OPTICS

1. Under this heading, we shall devote detailed consideration to the various forms of the geometrical-optical approximation and, in accordance with what was said in §59, Subsection 2, we shall compare their regions of applicability.

Here the conditions of radio propagation in the atmosphere introduce singularities that facilitate the solution but complicate the question (which is still disputed in the literature) as to the relationship and regions of applicability of the various forms of the method. In addition to the small parameter $1/kl$, where l is the correlation radius, i.e., essentially the average dimension of the inhomogeneities, we have the small parameter $\delta n = n - 1$, which is almost always even smaller than the first. Together with this, a certain role is taken by the conventional parameter of diffraction theory – the <u>wave parameter</u> (a term proposed by G.S. Gorelik) $P = (L\lambda/l^2)$. Thus, if we set $\delta n(r) = \nu f(r)$, where $\nu^2 \sim \overline{(\delta n)^2}$, $|f| \sim 1$, then we are concerned with three parameters:

$$\mu = \frac{1}{kl} \ll 1, \tag{61.1a}$$

$$\nu \sim \sqrt{\overline{(\delta n)^2}} \ll 1, \tag{61.1b}$$

$$P = \frac{L\lambda}{l^2} = \frac{L}{kl^2} \lesssim 1. \tag{61.1c}$$

Using the smallness of these parameters in different ways, we obtain different forms of the geometrical-optical approximation.

If $\delta n = 0$, the phase is $\varphi_0 \sim kz$ for a plane wave. After passage through an inhomogeneity, i.e., on a path $\Delta z = l$, the phase distortion is of the order $k\Delta z \cdot (n - 1) = kl(n - 1)$. Hence it is usually said that the distortion may manifest on a segment of the order of l/n and that the distortion of phase (and amplitude) is a function of $\frac{z}{l} \cdot n = \mu k z n$. In actuality, however, $n - 1$ rather than n appears here. If the parameter δn is small, the distortion will be correspondingly small, of the order of $\mu \nu k r$, after passage through a single inhomogeneity. Hence the perturbation of phase (and amplitude) is in reality a function of precisely this argument. Nevertheless, if we first take into account the smallness of only one parameter, for example, μ, then the smallness of ν, or vice versa, there will, of course, be no error at all.

2. Let us begin with the one-dimensional problem. It has already been considered in §57, where, however, we were interested in another aspect of it: a single isolated inhomogeneity was examined and, moreover, our attention was concentrated on the problem of reflection rather than transmission of the wave.

Let us once again consider the scalar equation (see remark prefacing Formula (60.12c))

$$\frac{d^2u}{dz^2} + p^2(z)u = 0, \quad p^2(z) = k^2\varepsilon(z). \tag{61.2}$$

We separate the variable p(z) from the constant part and introduce the dimensionless coordinate ξ:

$$\frac{d^2u}{d\xi^2} + n^2u = 0, \tag{61.3}$$

$$\xi = kz, \quad n = n_0 + \delta n, \quad n_0 = \text{const}. \tag{61.3a}$$

a) We shall at first the smallness of μ as a basis (61.1a). Using the same form of the geometrical-optical approximation as in §57, Subsection 2, it will be convenient to write u in dimensionless variables as follows:

$$u = A(\mu\xi)\, e^{\frac{i}{\mu}\varphi(\mu\xi)}, \quad \mu\xi = \frac{z}{l}. \tag{61.4}$$

Substitution of this expression into Eq. (61.3) gives an equation that differs from (57.15a) only in the form of notation,

$$\mu^2 A'' + i\mu(2A'\varphi' + A\varphi'') - A\varphi'^2 + n^2 A = 0. \tag{61.5}$$

Here the primes denote differentiation with respect to the argument $\mu\xi = z/l$. We expand A in powers of μ:

$$A = A_0 + \mu A_1 + \mu^2 A_2 + \dots \tag{61.6}$$

and substitute in Eq. (61.5). Equating the sums of terms of the same order with respect to the parameter μ to zero, we obtain

$$\begin{gathered} A_0(\varphi'^2 - n^2) = 0, \\ i(2A_0'\varphi' + A_0\varphi'') - A_1(\varphi'^2 - n^2) = 0, \\ A_0'' + i(2A_1'\varphi' + A_1\varphi'') - A_2(\varphi'^2 - n^2) = 0, \\ \dots\dots\dots\dots\dots \end{gathered} \tag{61.7}$$

The first two equations again yield Formulas (57.16)-(57.17), but in another form:

$$\begin{gathered} \varphi' = \frac{d\varphi}{d\mu\xi} = n, \quad \varphi = \mu\int n\,d\xi \equiv \mu k\int n\,dz; \\ \frac{A_0'}{A_0} = -\frac{\varphi''}{2\varphi'} = -\frac{n'}{2n}, \quad A_0 = \frac{1}{\sqrt{n}}. \end{gathered} \tag{61.8}$$

Consequently,

$$u = \frac{\text{const}}{\sqrt{n}}\, e^{ik\int^z n(z)\,dz}. \tag{61.9}$$

The third of Eqs. (61.7) yields

$$\begin{gathered} A_0'' = \frac{d^2}{d(\mu\xi)^2}\frac{1}{\sqrt{n}} = -i(2nA_1' + n'A_1) = -2i\sqrt{n}\,\frac{d}{d\mu\xi}(A_1\sqrt{n}), \\ A_1 = \frac{i}{2\sqrt{n}\,\mu}\int \frac{1}{\sqrt{n}}\frac{d^2}{d\xi^2}\frac{1}{\sqrt{n}}\,d\xi = \frac{i}{2\mu k\sqrt{n}}\int_0^z \frac{1}{\sqrt{n}}\frac{d^2}{dz^2}\frac{1}{\sqrt{n}}\,dz \end{gathered} \tag{61.9a}$$

(here it is assumed that the layer of inhomogeneities begins at $z = 0$), and so forth. The condition for validity of Solution (61.9) requires that $\mu|A_1| \ll |A_0|$ (and, of course, that $\mu|A_2| \ll |A_1|$ and so forth), i.e.,

$$\left|\frac{1}{2k}\int_0^z \frac{1}{\sqrt{n}}\frac{d^2}{dz^2}\frac{1}{\sqrt{n}}\,dz\right| \ll 1. \tag{61.10}$$

It is necessary to distinguish two cases. If $n - 1 \gtrsim 1$, then $\frac{dn}{dz} \sim \frac{n}{l}$, $\frac{d^2n}{dz^2} \sim \frac{n}{l^2}$. Taking certain average values out from under the integral sign, we obtain the usual condition for dense media:

$$L \ll nkl^2 = n\frac{l^2}{\bar{\lambda}}, \tag{61.11}$$

where $L \sim z$ is the effective size of the entire region of integration, i.e., of the region occupied by the inhomogeneities. When we are concerned with an isolated layer having a single maximum or minimum (which corresponds to the discussion in §57), for example, with an ionospheric layer, then $L \sim l$ and Condition (61.11) reduces to the condition

$$kln \gg 1. \tag{61.12}$$

This is the conventional condition for short wavelength in a given substance. The case $L \gg l$ corresponds to a medium filled full with inhomogeneities. Then Condition (61.11) can be recast in the form $1 \ll \frac{L}{l} \ll nkl$. It follows from this that Condition (61.12) is also a necessary condition for such a medium.

Let us now consider a medium such as the troposphere, with relatively small fluctuations of n, $|n - 1| = |\delta n| \ll 1$. Then Inequality (61.10) can be recast in the form

$$\left|\frac{1}{4k}\int_0^z\left(1-\frac{\delta n}{2}\right)\frac{d^2\delta n}{dz^2}\,dz\right| = \left|\frac{1}{4k}\frac{d\delta n}{dz} - \frac{1}{8k}\int_0^z \delta n\frac{d^2\delta n}{dz^2}\,dz\right| \ll 1. \tag{61.10a}$$

Assuming $\left|\frac{d\delta n}{dz}\right| \sim \left|\frac{\delta n}{l}\right|$ and so forth and taking average values out from under the integral sign, we obtain two conditions:

$$4kl \gg \delta n, \tag{61.13}$$

$$\frac{8kl^2}{(\delta n)^2} \gg L, \quad P \equiv \frac{L}{kl^2} \ll \frac{8}{\nu^2}. \tag{61.14}$$

The first of these is very important. It replaces Condition (61.12) and indicates, in accordance with what we said above, that for $\nu \ll 1$ (61.1b), much less stringent conditions are imposed on wavelength than in the optics of dense media. The second condition imposes a limitation on the inhomogeneity layer thickness. For an isolated layer, $L \sim l$, it is even weaker than Condition (61.13) and may be left out of consideration. In a randomly inhomogeneous medium, on the other hand, Condition (61.14) is in principle a new one and imposes limitations on the distance L up to which the approximation is valid and, in particular, it is possible to use Solution (61.9), disregarding the quantity A_1 as small by comparison with A_0. We note at once that in tropospheric propagation of radio waves, this condition is almost totally nonessential.

Actually, $\overline{(\delta n)^2} < 10^{-12}$, $l \sim 10^4$ cm here and, consequently, we must have (in cm)

$$L_{cm} \ll \frac{10^{20}}{\lambda_{cm}}, \qquad (61.15)$$

where λ is the wavelength of the radiation (in cm).

b) Let us now consider another form of the method, one which is in principle free of even the constraint (61.14).

Instead of finding the complete expression for φ from Eq. (61.5) and expanding the amplitude, we may, as is customary, for example, in quantum mechanics, introduce an amplitude factor into the phase and seek the total phase by successive approximation. That is to say, instead of Equality (61.4), we set

$$u = e^{\frac{i}{\mu}\varphi(\mu z)}, \qquad (61.16)$$

so that the equation for u reduces to an equation for φ (it is sufficient to set A = 1 in Formula (61.5)):

$$i\mu\varphi'' - \varphi'^2 + n^2 = 0, \qquad (61.17)$$

where the primes denote differentiation with respect to the argument $\mu\xi$. We seek the phase φ in the form of a series

$$\varphi = \varphi_0 + \frac{\mu}{i}\varphi_1 + \frac{\mu^2}{i^2}\varphi_2 + \dots. \qquad (61.18)$$

Equating the coefficients of identical powers of μ one by one to zero, we reduce Eq. (61.17) to a system of equations

$$\left.\begin{aligned} &\varphi_0'^2 - n^2 = 0, \\ &\varphi_0'' + 2\varphi_1'\varphi_0' = 0, \\ &\varphi_1'' + 2\varphi_2'\varphi_0' + \varphi_1'^2 = 0, \\ &\dots\dots\dots\dots\dots\dots \end{aligned}\right\} \qquad (61.19)$$

The first two of these yield exactly the result of the first approximation in the earlier analysis (61.9),

$$\varphi_0 = \mu\int^{\xi} n\,d\xi = k\mu\int^{z} n\,dz; \quad \varphi_1 = \ln\frac{1}{\sqrt{n}}; \qquad (61.20)$$

$$u = \frac{1}{\sqrt{n}}e^{ik\int^{z} n\,dz}. \qquad (61.20a)$$

Further, in a full analogy to Formula (61.9a), the third equation gives

$$\varphi_2 = -\frac{1}{2\mu k}\int^{z}\frac{1}{\sqrt{n}}\frac{d^2}{dz^2}\frac{1}{\sqrt{n}}\,dz. \qquad (61.20b)$$

If $|\mu\varphi_2| \ll 1$, we may set

$$e^{\frac{i}{\mu}\cdot\frac{\mu^2}{i^2}\varphi_2} \approx 1 + \frac{i}{2k}\int^{z}\frac{1}{\sqrt{n}}\frac{d^2}{dz^2}\frac{1}{\sqrt{n}}\,dz = 1 + \frac{\mu A_1}{A_0}$$

and arrive at the previous result. If, however, the absolute value $|\mu\varphi_2|$ is not small, we may not expand the exponential factor, and it becomes possible to consider large distortions of amplitude and phase. Actually, the condition for applicability of the method is here again the admissibility of the expansion, in this case the expansion of (61.18). It would appear necessary to require that $|\mu\varphi_2| \ll \varphi_1$. But $\varphi_1 = -\ln\sqrt{n}$ has a singular form that distinguishes φ_1 from all other

φ_1: the size L of the entire region does not appear anywhere in φ_1. Since this term is separated at once into the common factor $\exp(-\ln\sqrt{n}) = n^{-\frac{1}{2}}$, it is convenient to drop it and adopt as conditions

$$|\mu^2\varphi_2| \ll |\varphi_0|, \qquad (61.21)$$

$$|\mu\varphi_3| \ll |\varphi_2| \qquad (61.21a)$$

and so forth. But $\frac{1}{\mu}\varphi_0 \sim nkL$, $\varphi_2 \sim n\frac{L}{l}$. Using the same evaluation for φ_2 as in the formulas in (61.9a), (61.10) and (61.10a), we arrive at the conditions

$$\frac{L}{nk^2l^3} \ll \frac{nL}{l}, \text{ i.e. } k^2l^2n^2 \gg 1 \text{ for } n-1 \gtrsim 1; \qquad (61.22)$$

$$\left.\begin{aligned} &\frac{\overline{|\delta n|}}{4k^2l^2} \ll \frac{L}{l}, \text{ i.e. } \frac{L}{l} \gg \frac{\overline{|\delta n|}}{4k^2l^2}, \qquad (61.22a)\\ &\frac{\overline{(\delta n)^2}\cdot L}{8k^2l^2} \ll \frac{L}{l}, \text{ i.e. } 8k^2l^2 \gg \overline{(\delta n)^2} \qquad (61.22b)\end{aligned}\right\} \text{ for } n-1=\delta n \ll 1.$$

Condition (61.22) agrees for dense media with the corresponding condition (61.12) for the method using the amplitude expansion. Further, Condition (61.22a) is practically nonessential for the atmosphere: even with $L \sim l$, it implies $k^2l^2 \gg |\delta n|$. On the other hand, Condition (61.22b) agrees with Condition (61.13). As concerns Condition (61.21a) and subsequent constraints of the same type, it is readily seen that they can add nothing new. Actually, if $\delta n = n - 1 = 0$, then, according to Formula (61.19), $\varphi_0' = 1$ and all the other φ_i' are equal to zero. Hence the product $\mu\varphi'$ is proportional at least to the first power of δn and, like Formula (61.20b), $\mu\varphi_3$ may also contain L/l. But since this same quantity L/l is contained in φ_2, the condition $|\mu\varphi_3| \ll \varphi_2$ cannot be more rigid than $\mu\delta n \ll 1$; this agrees with Condition (61.13). Thus we arrive at the conclusion that the conditions for validity of Solution (61.16), (61.18), where φ_0, φ_1, φ_2, etc., are determined by Formulas (61.20)-(61.20b) are as follows:

$$kln \gg 1 \text{ for } |n-1| \geqslant 1; \qquad (61.23a)$$
$$kl \gg |\delta n| \text{ for } |n-1| = |\delta n| \ll 1. \qquad (61.23b)$$

It follows from this that in dense media, the expansion proceeds essentially in powers of $1/kln = \mu/n$, while in a medium with $\delta n \ll 1$ it uses powers of $(\delta n/kl) \sim \mu\nu$, as spoken of earlier.

If the product $\mu\varphi_2$ is not small as compared to unity but $\mu^2\varphi_3$ is small and can be dropped,

$$u = \frac{1}{\sqrt{n}} e^{ik\int^z \left(n + \frac{1}{8k^2\sqrt{n}} \frac{d^2}{dz^2} \frac{1}{\sqrt{n}}\right) dz}. \qquad (61.24)$$

However, in the problem of tropospheric radio wave propagation, the additional term in the exponent is, according to Formula (61.15), so small as to be unnecessary. The two forms of the geometrical-optical approximation considered here are fully equivalent, and their applicability conditions differ only by the additional requirement (61.14) in the first of the methods. However, the remaining conditions (61.23a) and (61.23b) are the same as before.

c) Let us now consider the so-called method of "smooth perturbations" [6] (see also the surveys [1, 2]), which is specifically adapted to the case of small δn and uses this smallness at the very outset. That is to say, if we seek the solution in the form (61.16), we may assume at once that

$$\varphi = \Phi_0 + \Psi_1. \qquad (61.25)$$

Here Φ_0 is the solution of the equation for a homogeneous medium, i.e., for the case $\delta n = 0$, and hence we have simply

$$\Phi_0 = \mu\xi, \quad u_0 = e^{\frac{i}{\mu}\Phi_0} = e^{i\xi} = e^{ikz}, \quad \mu\xi = \frac{z}{l}. \qquad (61.25a)$$

Substituting the value of (61.25) into Eq. (61.17) and remembering in accordance with Formula (61.25a) that $\Phi_0' = 1$ (as before, the prime signifies differentiation with respect to the argument $\mu\xi$), we have

$$i\mu\Phi_1'' - 2\Phi_1' + 2\delta n - (\Phi_1')^2 + (\delta n)^2 = 0. \qquad (61.26)$$

The method is based on the remark [6] that with small δn, as is almost obvious, $(\Phi_1')^2 \sim (\delta n)^2$, so that the last two terms can be dropped. The remaining equation is easily solved. With a view to estimation of the error, however, we shall proceed systematically. We expand Φ_1 in powers of ν (the expansion begins with ν in the first power, since Φ_1 vanishes in a homogeneous medium):

$$\Phi_1 = \Phi_1^{(1)} + \Phi_1^{(2)} + \ldots = \nu\chi_1 + \nu^2\chi_2 + \ldots, \qquad (61.27)$$

and, substituting this expression in Formula (61.26), we equate the sums of the terms with the same powers of ν to zero:

$$-i\mu\chi_1'' + 2\chi_1' = 2f, \qquad (61.28a)$$

$$-i\mu\chi_2'' + 2\chi_2' = f^2 - (\chi_1')^2, \qquad (61.28b)$$

. . . .

After simple integration, we find the solution from Eq. (61.28a); it vanishes for $\xi = 0$:

$$\chi_1' = 2i\int_0^{\xi} f(\mu\xi')\, e^{-2i\mu(\xi-\xi')}\, d\xi' = 2ik\int_0^{z} f(z')\, e^{-2ik(z-z')}\, dz'. \qquad (61.29)$$

For $kl \gg 1$, the exponential factor oscillates rapidly, while $f(z')$ varies relatively slowly. The effective integration region lies at $z' \approx z$ and is determined by the relation $k(z - z') \lesssim 1$. Therefore, expanding $f(z')$ in series in powers of $\zeta = z - z'$, introducing temporarily the factor $\exp(-\alpha\zeta)$, which ensures convergence, and then letting α tend to zero, we obtain (the integral over ζ may be extended to infinity)

$$\frac{d\Phi_1^{(1)}}{d\mu\xi} - \nu\chi_1' = 2ik\int_0^{\infty}\left(f(z) + f'(z)\zeta + \frac{1}{2}f''(z)\zeta^2 + \ldots\right)e^{(2ik-\alpha)\zeta}\, d\zeta \to$$

$$\to \delta n(z) + \frac{i}{2k}\frac{d}{dz}\delta n(z) - \frac{1}{4k^2}\frac{d^2\,\delta n(z)}{dz^2} + \ldots. \qquad (61.29a)$$

Thus, $\Phi_1^{(1)\prime} \approx \delta n$. Consequently, we obtain the exact value of $\Phi_1^{(1)}$ from Solution (61.29) (here we reverse the order of integration and carry out one integration):

$$\Phi_1^{(1)} = \mu \int_0^{\xi} \delta n \left(1 - e^{-2\mu(\xi - \xi')}\right) d\xi', \qquad (61.30)$$

and from Formula (61.29a) the expansion in powers of μ (only the first term is written out):

$$\frac{1}{\mu}\Phi_1^{(1)} \approx \int_0^{\xi} \delta n \, d\xi' = k \int_0^{z} \delta n(z')\, dz'. \qquad (61.30a)$$

Formula (61.30a) can be obtained directly from Eq. (61.28a) if we consider the smallness of μ and drop the term $\mu\chi_1''$.

The second approximation, $\Phi_1^{(2)}$, is obtained by exactly the same method, solving Eq. (61.28b), if the value of (61.29a) is substituted in the right member. The solution reduces to a formula of the form (61.29) or directly to (61.30a), in which $2f$ is replaced by $f^2 - (\chi_1')^2$, i.e., according to Formula (61.29a), by

$$f^2 - (\chi_1')^2 = -\frac{l}{2k}\frac{df^2}{dz} + \frac{1}{4k^2}\frac{d}{dz}\left(f\frac{df}{dz}\right) + \frac{1}{4k^2} f \frac{d^2 f}{dz^2} + \ldots. \qquad (61.30b)$$

With a view to estimation of the error, we may limit ourselves to calculation of the average value $\overline{\Phi_1^{(2)}}$. Averaging Eq. (61.28b) for this purpose, we see that we must use the average value of Expression (61.30b). But the average of the first two terms in the formula vanishes. Hence, in analogy to Formula (61.30a),

$$\frac{1}{\mu}\overline{\Phi_1^{(2)}} \equiv \frac{\nu^2}{\mu}\overline{\chi_2} \approx \frac{1}{8k}\int_0^{z} \overline{\delta n \frac{d^2\delta n}{dz^2}}\, dz' - \frac{z\mu}{8l}\overline{\delta n \frac{d^2 \delta n}{d\left(\frac{z}{l}\right)^2}} + O(\mu^2) \sim \mu\nu^2 \frac{z}{l}. \qquad (61.30c)$$

Thus, sources in the right member of Eq. (61.28b) nullify one another to a high degree. The difference $\nu^2 f^2 - \nu^2(\chi_1')^2$ has an average of the order of only $\mu^2\nu^2$, i.e., smaller by a factor of μ^2 than each of the terms. This is what makes the method useful. However, since the average value of the difference does not vanish, $\overline{\Phi_1^{(2)}}$ increases with distance z and may ultimately become larger. The condition for applicability of the method consists of the requirements that

$$|\nu\chi_1| \ll \infty,$$

$$|\nu^2\chi_2| \ll |\nu\chi_1| \approx \left|\mu\int_0^z \delta n\, dz'\right|. \qquad (61.31)$$

We may evaluate the modulus of $\nu\chi_1$ from the root-mean-square value of this quantity. Setting $z' - z = \zeta$, $z + z' = 2Z$ for integration and integrating over Z from 0 to z and over ζ indefinitely, we obtain

$$\overline{\left|\mu\int_0^z \delta n\, dz'\right|^2} = \frac{1}{\mu}\int_0^z dz'\int_0^z dz''\,\overline{\delta n(z')\,\delta n(z'')}\,dz'\,dz'' \approx$$

$$\approx \frac{\nu^2}{l}\int_0^z dZ\int_{-\infty}^{+\infty} F\left(\frac{\zeta}{l}\right)\frac{d\zeta}{l} \sim \frac{\nu^2 z}{l}. \qquad (61.31a)$$

Thus, according to Formulas (61.25a), (61.30c) and (61.31), the conditions of applicability of the method read as follows (we denote the distance z by L):

$$4L \gg \nu^2, \quad P = \frac{L}{kl^2} \gg \nu^2\mu^2; \qquad (61.32)$$

$$L \ll 64\frac{l}{\mu^4\nu^4}, \quad P \ll \frac{64}{\nu^2\mu^2}. \qquad (61.32a)$$

These conditions are compatible if

$$\nu^2\mu^2 \ll 1. \qquad (61.32b)$$

Together with Formulas (61.25)-(61.25a), the first approximation (61.30a) gives

$$u = e^{i\left(kz + k\int_0^z \delta n(z')dz'\right)} = e^{ik\int_0^z n(z')dz'} \qquad (61.33)$$

– a result in perfect agreement with that given by other forms of the geometrical-optical approximation. We may limit ourselves to it if the second-order correction is small in absolute value, i.e., if

$\left|\frac{1}{\mu}\Phi_1^{(2)}\right| \ll 1$. Considering the necessary condition, we may use the average value (61.30c), thus requiring that (see Formula (61.30c))

$$\mu\nu^2 \frac{L}{l} \equiv \nu^2 P \ll 1. \tag{61.34}$$

Moreover, to the extent that we are assuming $\mu \ll 1$, it must at any rate be true that (see Formula (61.32b))

$$\mu\nu \ll 1. \tag{61.35}$$

In the one-dimensional case under consideration, therefore, all three forms of the geometrical-optical approximation (points a), b) and c) above) lead us to the same result (61.33). The applicability condition takes the form $kl \gg \delta n$ for all of them. Further, additional conditions must be observed: (61.14) in the first form of the method (point a)) and (61.32)-(61.32b) and (61.34) in the third (point c)). It is evident from this that as regards region of applicability, the method of smooth perturbations has no advantages – at least in the one-dimensional problem – over other forms of the geometrical-optical approximation. In all cases the expansion is actually in the parameter $\mu\nu \sim (\delta n/kl)$.

3. Let us pass to the geometrical-optical approximation in the <u>three-dimensional</u> case for the <u>scalar</u> equation

$$(\nabla^2 + p^2)u = 0, \quad p^2 = k^2\varepsilon(r) \tag{61.36}$$

(S.M. Rytov [13] examined the geometrical-optical approximation for the Maxwell equations and arbitrary ε, with particular attention to the variation of polarization along the ray).

We shall first consider arbitrary n. Presenting u in the following form in accordance with the first of the forms taken by the method:

$$u = A(\mu\xi)\, e^{\frac{i}{\mu}\varphi(\mu\xi)}, \tag{61.37}$$

with

$$\nabla_\xi^2 u + n^2 u = 0,$$
$$\xi = kr, \quad \nabla_\xi^2 = \frac{\partial^2}{\partial\xi_x^2} + \frac{\partial^2}{\partial\xi_y^2} + \frac{\partial^2}{\partial\xi_z^2}. \tag{61.38}$$

we arrive at the equation

$$\mu^2\nabla_0^2 A + i\mu\,[2(\nabla_0 A\nabla_0\varphi) + A\nabla_0^2\varphi] - A\,[(\nabla_0\varphi)^2 - n^2] = 0. \qquad (61.39)$$

Here the operators ∇_0 and ∇_0^2 should be understood as containing the <u>differentiation with respect to the argument</u> $\mu\xi$:

$$\nabla_0 = \frac{1}{\mu}\nabla_\xi.$$

Substituting, as in Formula (61.6),

$$A = A_0 + \mu A_1 + \dots \qquad (61.40)$$

and equating terms of the same order in μ, we obtain instead of Eqs. (61.7)

$$(\nabla_0\varphi)^2 = n^2, \qquad (61.41)$$

$$i\mu\,[A_0\nabla_0^2\varphi + 2(\nabla_0 A_0\cdot\nabla_0\varphi)] \equiv \frac{i\mu}{A_0}\operatorname{div}_0(A_0^2\nabla_0\varphi) = 0, \qquad (61.41a)$$

$$i\mu^2\,[A_1\nabla_0^2\varphi + 2(\nabla_0 A_1\cdot\nabla_0\varphi)] + \mu^2\nabla_0^2 A_0 \equiv \mu^2\left(\frac{i}{A_1}\operatorname{div}_0(A_1^2\nabla_0\varphi) + \nabla_0^2 A_0\right) = 0.$$

$$(61.41b)$$

The basic equation (61.41), the eiconal equation (the phase φ/μ is known as an eiconal), is a second-order partial differential equation and, in contrast to the one-dimensional case, cannot be reduced to a quadrature containing a single constant; the solution depends on an arbitrary function, on the value of φ on a certain surface S_0, for example, on that on which φ is constant: $\varphi(r(S_0)) = \varphi_0$. Then, in accordance with Eq. (61.41), the gradient of φ on this same surface is also assigned: it is directed along the normal to S_0 and has the absolute value $|\nabla_0\varphi| = n$. Therefore, knowing n over the entire space, we may, in principle, construct all surfaces of constant eiconal and find the complete solution φ. It will be a functional of the initial surface S_0 and hence of the field of normals to this S_0, of the unit vectors $\vec{s}(S_0)$. The wave vector of the wave at a given point $\vec{r}$ is the vector

$$p = \operatorname{grad}\left(\frac{\varphi}{\mu}\right) = k\nabla_0\varphi, \quad \nabla_0\varphi = ns, \qquad (61.42)$$

where $\vec{s}$ is the unit vector of the normal to the constant-phase surface passing through the point in question. Here

$$p = ksn = ps. \tag{61.43}$$

Moreover, the solution φ depends on one constant – on the assigned value of $\varphi = \varphi_a$ on S_0. However, this constant gives only the common multiplier $\exp(ik\varphi_a)$ before u.

In geometrical optics, it is customary to characterize the field not so much by the function φ as directly by the field of unit vectors $\vec{s}$. It indicates the direction of the rays at each point. It is important that, according to Formula (61.42),

$$\text{rot}\,(ns) = \text{rot grad}\,\varphi = 0. \tag{61.44}$$

In a conceptual representation, therefore, we might liken φ to electrostatic potential and the field of vectors $\vec{p}$ or $n\vec{s}$ to lines of force. It follows from Formula (61.44) that this field is nonvortical, so that the line integral of $n\vec{s}$ between two points A and B does not depend on distance. The increment of the eiconal φ/μ is ($\mu k d\vec{l} = d\vec{\xi}$)

$$\frac{\varphi(B) - \varphi(A)}{\mu} = \int_A^B kns\,dl = k\int_A^B n\cos(s,\,dl)\cdot dl.$$

If, however, we take under the integral not the dot product of the unit vector $\vec{s}$ by $d\vec{l}$, but instead simply the distance dl, i.e., if we drop the multiplier $\cos(\vec{s},\, d\vec{l})$, then, generally speaking, we obtain a larger value. The correct value of the integral is preserved only when the path of integration is laid along the ray – along $\vec{s}$ –, when the cosine actually does become unity. Thus, if integration is conducted along the true ray, the integral

$$\int_A^B n\,dl = \min \tag{61.45}$$

assumes the smallest value among all of those obtained for possible neighboring trajectories. This relationship – the Fermat principle –

may serve as a starting point for finding (by the variational method) the ray paths (see, for example, its application in §56, Subsection 3). The differential equation for $\vec{s}$, which proceeds from this principle, may also be found from Formula (61.44) as the Euler equation for this problem (see the book [1], §3).

Fig. 61.1. Illustrating derivation of ray equation.

On passing from the surface $\varphi = \varphi_1 = \text{const}$ to the surface $\varphi = \varphi_1 + d\varphi$ over a distance $d\sigma$, the vector $\vec{s}$, generally speaking, rotates, unless ∇n coincides with the direction of $\vec{s}$. We introduce rectangular coordinates characterized by the unit vectors $\vec{\sigma}$ and $\vec{t}$ (see Fig. 61.1) at the beginning of the segment $d\sigma$, by which we have moved along $\vec{s}$ in the plane in which the vectors ∇n and $\vec{s}$ lie. It follows from Formula (61.44), for example, that

$$\frac{\partial (ns_t)}{\partial s} - \frac{\partial (ns_\sigma)}{\partial t} \equiv \frac{\partial n}{\partial s} s_t + \frac{\partial s_t}{\partial s} n - \frac{\partial n}{\partial t} s_\sigma - \frac{\partial s_\sigma}{\partial t} n = 0.$$

But s_t and $\partial s_\sigma / \partial t$ are by definition zero. Further,

$$\frac{\partial s_t}{\partial s} = t \frac{d\varphi}{ds}, \quad \frac{\partial n}{\partial t} = (t \nabla n), \quad s_\sigma \equiv 1.$$

Therefore (we also subject the entire relationship to scalar multiplication by $\vec{t}$)

$$n \frac{d\varphi}{ds} = t(\nabla n \cdot t).$$

Further, we have the obvious formula

$$\frac{dn}{ds} = (\nabla n \cdot s).$$

Multiplying it by $\vec{s}$ and adding to the preceding formula, we obtain

$$s\frac{dn}{ds} + n\frac{ds}{ds} = s(\nabla n \cdot s) + t(\nabla n \cdot t),$$

or, finally,

$$\frac{d(ns)}{ds} = \nabla n. \tag{61.46}$$

This is the differential equation of the rays. Here σ is a scalar quantity reckoned along the given – generally bent – ray. In other words, it is assumed that the ray equation is sought in parametric form, $\vec{r} = \vec{r}(\sigma)$.

4. The partial differential equations cannot be solved for arbitrary n as in the one-dimensional case. Hence not all forms of the geometrical-optical approximation analyzed in Subsection 2 are of the same effectiveness. General expressions can be obtained only if the smallness of δn is taken into account.

The first and simplest form of the method consists in limiting ourselves to the first two terms of the expansion in μ as with Formulas (61.7)-(61.9), i.e., to Eqs. (61.41) and (61.41a). In order to solve them, it will be helpful to apply the expansion in ν at once [7a], i.e., to set

$$\varphi = \varphi_0 + \nu\varphi_1 + \nu^2\varphi_2 + \dots, \tag{61.47a}$$

$$A_0 = 1 + \nu A_{01} + \nu^2 A_{02} + \dots. \tag{61.47b}$$

In the case of incidence of a plane wave, we have $\varphi = \varphi_0 = \mu s_0 \xi = \mu k_0 r$, where $\vec{s}_0$ is the unit vector of the direction of ray incidence. Therefore $\nabla_0\varphi_0 = s_0, \nabla_0^2\varphi_0 = 0$. From Eq. (61.41), in the first approximation with respect to ν, we obtain

$$s_0 \cdot \nabla_0\varphi_1 = \frac{\delta n}{\nu}; \tag{61.48a}$$

it follows further from Eq. (61.41a) that

$$\nabla_0^2\varphi_1 + 2(\mathbf{s}_0\cdot\nabla_0 A_{01}) = 0. \tag{61.48b}$$

If the wave is incident along the z-axis, Eq. (61.48a) contains only the derivative with respect to $\mu\xi_z$ and is integrated at once:

$$\nu\varphi_1(x, y, z) = \int \delta n\,\mu\,d\xi_z = k\int_0^z \delta n(x,y,z')\,dz'. \tag{61.49}$$

In this approximation, therefore, the phase advance at point (x, y, z) is determined only by the optical thickness of the layer traversed by the rectilinear ray. The medium acts as a flat screen of variable density. This result therefore leads us to the so-called "equivalent screen method." This also implies

$$\nabla_0^2\varphi_1 = \frac{\partial\delta n}{\partial\mu\xi_z} - \int\left(\frac{\partial^2\delta n}{\partial(\mu\xi_x)^2} + \frac{\partial^2\delta n}{\partial(\mu\xi_y)^2}\right)\mu\,d\xi_z = \frac{1}{k}\left\{\frac{\partial\delta n}{\partial z} + \int_0^z \nabla_\perp^2\,\delta n(x,y,z')\,dz'\right\},$$

where we have passed from differentiation with respect to $\mu\xi$ to differentiation with respect to the coordinates, $\nabla_0 = \frac{1}{k}\nabla$, and introduced the notation $\nabla_\perp^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ for the "transverse laplacian." Solving Eq. (61.48b), we now obtain

$$A_0(x, y, z) = 1 - \frac{1}{2}\left\{\delta n(x,y,z) + \int_0^z (z-z')\nabla_\perp^2\delta n(x,y,z')\,dz'\right\} \tag{61.50}$$

(Here it is assumed that $\delta n(x, y, 0) = 0$ and the order of integration has been changed). The solution found above is valid if the expansion in powers of ν is admissible, and in particular if A_0 differs little from unity. For this, first the value of δn itself and, secondly, the root-mean-square value of the integral must be small. That is to say, if we denote $A_0 - 1 = \delta A$ and assume that observation is conducted from a point that is now outside the layer, $\delta n(x, y, z) = 0$, then the mean square of the amplitude fluctuation must be small:

$$\overline{(\delta A)^2} = \frac{1}{4}\int_0^z dz'\int_0^z dz''(z-z')(z-z'')\overline{\nabla_\perp^2\delta n(x,y,z')\cdot\nabla_\perp^2\delta n(x,y,z'')} \ll 1. \tag{61.51}$$

If the correlation is assigned by a single-parameter function,

$$\overline{\nabla_{\perp}^{2}\delta n(x, y, z')\nabla_{\perp}^{2}\delta n(x, y, z'')} = \nu^{2}\nabla_{\perp}^{2}\cdot\nabla_{\perp}^{2}F\left(0, 0, \frac{z'-z''}{l}\right).$$

Let us set $z' - z'' = \rho_z l$. Disregarding the terminal effect, we extend the interval over ρ_z to $\pm\infty$ and substitute $z - z'$ for $z - z''$. This yields

$$\overline{(\delta A)^{2}} = \frac{l\nu^{2}z^{3}}{12}\int_{-\infty}^{+\infty}\nabla_{\perp}^{2}\cdot\nabla_{\perp}^{2}F(0, 0, \rho_z)\,d\rho_z.$$

If the transverse Laplace operators are understood as applied to the dimensionless variables ρ_x, ρ_y, i.e., $\nabla_{\perp}^{2}\to\frac{1}{l^{2}}\nabla_{\rho\perp}^{2}$, then

$$\overline{(\delta A)^{2}} = \frac{\nu^{2}L^{3}}{12l^{3}}\int_{-\infty}^{\infty}\nabla_{\rho\perp}^{2}\cdot\nabla_{\rho\perp}^{2}F(0, 0, \rho_z)\,d\rho_z\sim\frac{\nu^{2}L^{3}}{l^{3}}, \qquad (61.51a)$$

where $L \equiv z$ is the longitudinal dimension of the inhomogeneous region. Thus, one of the conditions for applicability of the method is

$$\nu^{2}L^{3}/l^{3}\ll 1. \qquad (61.52)$$

For $\nu^2 \sim 10^{-10}$-10^{-12}, this is not a very rigid condition: $L \ll 10^3$-$10^4 \cdot l$.

But the quadruple differentiation under the integral in (61.51) stresses the role of short distances. In the case of a two-parameter correlation function (59.11), this means that l_s^{-4} appears rather than l^{-4}, and the integral over $z' - z''$ will give l_0. Thus we obtain $\nu^2 L^3 l_0 l_s^{-4}$ on the right in (61.51a) and instead of Condition (61.52), we arrive at the much more difficult condition $\nu^2 L^3 l_0 l_s^{-4} \ll 1$. Further, the terms $\nu^2\varphi_2$, etc., must be small. Without going into detail, we may state at once that validity of the equivalent-screen method must signify that three-dimensional inhomogeneities act as one-dimensional inhomogeneities. This is possible only if a small part of one inhomogeneity decreases within the limits of the essential zone – the first Fresnel zone. Consequently, the much stronger condition

$$L \ll kl^2, \quad P \ll 1 \tag{61.53}$$

must be satisfied (compare also the end of Subsection 4, below).

It is impossible in the three-dimensional case to find a general solution in a form corresponding to the second form of the geometrical-optics method (see Subsection 2b above). Hence we pass at once to its third form, to the <u>method of smooth perturbations</u> [6]. Here it is a matter of <u>first</u> expanding in ν and <u>then</u> in μ. (This procedure was first proposed and carried through consistently for the two-dimensional problem in [5].)

As with Formula (61.16), we set

$$u = e^{\frac{i}{\mu}\varphi(\mu,\xi)}. \tag{61.54}$$

Then we have for φ an equation analogous to (61.17), obtained from Formula (61.39) for A = 1:

$$i\mu\nabla_0^2\varphi - (\nabla_0\varphi)^2 + n^2 = 0. \tag{61.55}$$

For $\delta n = 0$ in the case of a plane wave $\varphi = \Phi_0 = \mu s_0\xi \equiv \mu k_0 r$. In the case of a point source, the incident field takes the form

$$u_0 = \frac{\text{const}}{kr} e^{ikr} = e^{\frac{i}{\mu}(\mu\xi + i\mu\ln\xi)}, \quad \xi = |\xi|.$$

so that $\Phi_0 = \mu\xi + i\mu\ln\xi$. Φ_0 always satisfies the condition that proceeds from the wave equation in a homogeneous ($\delta n = 0$) medium,

$$i\mu\nabla_0^2\Phi_0 - (\nabla_0\Phi_0)^2 + 1 = 0.$$

Hence for the "scattered field" (compare Formula (61.25) in the one-dimensional case

$$\Phi_1 = \varphi - \Phi_0 \tag{61.56}$$

Eq. (61.55) reduces to the equation

$$i\mu\nabla_0^2\Phi_1 - 2(\nabla_0\Phi_0\cdot\nabla_0\Phi_1) - (\nabla_0\Phi_1)^2 + n^2 - 1 = 0; \tag{61.57}$$

or, more conveniently, for the auxiliary function χ

$$\Phi_1 = \chi \cdot e^{-\frac{i}{\mu}\Phi}. \tag{61.58}$$

to an equation that is as yet rigorous:

$$\mu^2\nabla_0^2\chi + \chi = \frac{i}{\mu}(\chi\cdot\nabla_0\Phi_0 + i\mu\nabla_0\chi)^2 e^{-\frac{i}{\mu}\Phi_0} + i\mu(2\delta n + (\delta n)^2)e^{\frac{i}{\mu}\Phi_0} \quad (61.59)$$

We need a solution that vanishes when $\delta n = \nu f(r) \to 0$. We expand χ in series in powers of ν:

$$\chi = \nu\chi_1 + \nu^2\chi_2 + \ldots; \quad \Phi_1 = \Phi_1^{(1)} + \Phi_1^{(2)} + \ldots. \quad (61.60)$$

Substituting Φ_1 in Formula (61.57), we obtain the necessary equation system, in writing which we return to the variable $\vec{r}$ – the real radius vector of the point ($\mu\vec{\xi} = \vec{r}/l$):

$$\nabla^2\chi_1 + k^2\chi_1 = \frac{2i\mu k^2}{\nu}\delta n e^{\frac{i}{\mu}\Phi_0}, \quad (61.61a)$$

$$\nabla^2\chi_2 + k^2\chi_2 = \frac{i}{\mu}\left(\frac{1}{\mu}\nabla\Phi_0\cdot\chi_1 + i\nabla\chi_1\right)^2 e^{-\frac{i}{\mu}\Phi_0} + \frac{i\mu k^2}{\nu^2}(\delta n)^2 e^{\frac{i}{\mu}\Phi_0}, \quad (61.61b)$$

where ∇ denotes differentiation with respect to the conventional coordinates, $\nabla = \mu k\nabla_0$.

The first approximation $\Phi_1^{(1)}$ for Φ_1 is obtained from Eq. (61.61a). According to Formula (5.10), we have

$$\Phi_1 = \Phi_1^{(1)} = \nu\chi_1 e^{-\frac{i}{\mu}\Phi_0} = -i\mu\psi, \quad (61.62)$$

$$\psi = \frac{k^2}{2\pi}\int\frac{dr'}{|r-r'|}e^{\frac{i}{\mu}(\Phi_0(r')-\Phi_0(r))+ik|r-r'|}\delta n(r'), \quad (61.62a)$$

$$u = e^{\frac{i}{\mu}\Phi_0 + \psi(r)}. \quad (61.63)$$

Here the integral is taken formally over the entire space. Actually, of course, the first Fresnel zone is effective (see §11). Hence we may set

$$|r - r'| \approx z - z' + \frac{(x-x')^2 + (y-y')^2}{2(z-z')}$$

and extend the integration over z' from 0 to z.

For a plane wave $\Phi_0(\vec{r}) = \mu kz$ and

$$\Phi \approx \frac{\mu k}{2\pi} \int_0^z \frac{dz'}{z-z'} \int_{-\infty}^{+\infty} dx'\,dy'\,e^{i\frac{k}{2(z-z')}\left((x-x')^2+(y-y')^2\right)} \cdot \delta n(x', y', z'). \qquad (61.64)$$

We note that the same result follows from Eq. (61.57) if we drop terms of the second order in δn and consistently apply the smallness of μ. Developing the sense of the operator ∇_0^2, we note that $\partial^2\Phi_1/\partial(\mu\xi_z)^2$ appears with the coefficient μ^2 and can therefore be dropped as a quantity that is small by comparison with $\nabla_0\Phi_0 \cdot \nabla_0\Phi_1 = -\frac{\partial\Phi_1}{\partial(\mu\xi_z)}$. (We may not drop $\mu^2\left(\frac{\partial^2}{\partial(\mu\xi_x)^2} + \frac{\partial^2}{\partial(\mu\xi_y)^2}\right)$ because the scale of variation along the ξ_x- and ξ_y-axes may be different from that along ξ_z). We obtain an equation of the parabolic:

$$\frac{\partial^2\Phi_1}{\partial x^2} + \frac{\partial^2\Phi_1}{\partial y^2} + 2ik\frac{\partial\Phi_1}{\partial z} = -2k^2\mu\delta n. \qquad (61.65)$$

It can be identified, for example, with the equation of heat conduction in two dimensions x and y when the thermal-conductivity coefficient is equal to $a^2 = 1$ and the quantity $-z'/2ik$ plays the role of time, with the sources distributed in "time" and space with a density $2k^2\mu\delta n$. In this case, a point source of unit intensity placed at a point (x', y') at the "time" $-z'/2ik = 0$, gives the temperature field

$$u = \frac{2ik}{4\pi(z-z')} e^{-\frac{(x-x')^2+(y-y')^2}{4\frac{z-z'}{2ik}}}, \qquad (61.65a)$$

and the distributed sources give Solution (61.64) exactly. Thus, conversion from Expression (61.62a) to Solution (61.64) signifies that we are making the expansion in μ after the expansion in δn.

An important particular case arises when the distances are so small that the width of the first Fresnel zone is small by comparison with the inhomogeneity length l, i.e., if the wave parameter P is small: $l^2 \gg L\lambda$. In this case, the dependence of $\delta n(x', y', z')$ on x' and y' in Formula (61.64) is slower than the dependence of the exponential multiplier, which is of the nature of the product of the delta-functions $\delta(x - x')\delta(y - y')$. Taking out the value of δn taken at the point $x = x'$, $y = y'$ and performing integration over x' and y' (see Formula (10.13d)), we again obtain Formula (61.49), as would be expected. We thereby satisfy ourselves that it is valid with Condition (61.53).

5. The limits of applicability of the solution obtained in this manner for a three-dimensionally inhomogeneous medium have been discussed on more than one occasion. Since, however, the pronouncements on this matter disagree very sharply, we shall consider the problem again from the beginning.

In the derivations set forth above, the question as to the validity of Solution (61.63) reduces to evaluation of the higher-order approximations that are dropped. Obviously, we must have first of all

$$|\nabla^2\chi_2| \ll |\nabla\chi_1|. \qquad (61.66)$$

Solving Eq. (61.61b) in the same way as Eq. (61.61a), we can provisionally replace the first term in the right member by the value found for $-\frac{i}{\mu}(\nabla\Phi_1^{(1)})^2 \exp\left(\frac{i}{\mu}\Phi_0\right)$. As a result, we find, according to (5.10),

$$\frac{1}{\mu}\Phi_1^{(2)} = \frac{\nabla^2}{\mu} e^{-\frac{i}{\mu}\Phi_0(r)} \chi_2 =$$

$$= \frac{k^2}{4\pi}\int\left[(\delta n)^2 - \frac{1}{k^2\mu^2}(\nabla\Phi_1^{(1)})^2\right] e^{\frac{i}{\mu}(\Phi_0(r') - \Phi_0(r) + \mu k|r - r'|)} \frac{dv'}{|r - r'|}. \qquad (61.67)$$

This quantity must be small by comparison with ψ (61.62a) for the expansion and the entire method of successive approximations to be applicable. It must be small in absolute magnitude so that we may limit ourselves in the solution to the terms taken into account in Formula (61.62). Comparison with (61.62a) indicates that we are speaking of additional "sources" that are added to δn. Since the function δn is sign-changing and its square $(\delta n)^2$ is positive, the cumulative effect of the second-order term may become significant even for $\delta n \ll 1$. This results in an integral condition indicating the limiting distances L at which the method is still valid. We satisfied ourselves earlier in the case of the one-dimensional problem that two second-order sources compensate one another to a major degree, so that their sum differs by a factor μ^2 from either of them. As we shall now see, this is not the case in three dimensions.

Let us consider the average value of a second-order source $\overline{(\delta n)^2} - k^{-2}\mu^{-2}\overline{(\nabla\Phi_1^{(1)})^2}$ in Formula (61.67) or in Formula (61.57) (in fact, the average value of the right member in Eq. (61.61b)) for incidence of a plane wave $\Phi_0 = \mu k_0 r$. According to Formulas (61.62), (61.62a) and (61.64), we obtain on introduction of the operator ∇ under the integral sign, remembering that $\vec{r}$ appears only in the form of the difference $\vec{r} - \vec{r}'$ and integrating by parts

$$\frac{1}{k^2\mu^2}\overline{(\nabla\Phi_1^{(1)})^2} - \frac{1}{k^2}\overline{(\nabla\psi)^2} =$$

$$= -\frac{k^2}{4\pi^2}\iint \overline{\nabla\delta n(r')\nabla\delta n(r'')}\, e^{\frac{ik}{2}\left\{\frac{(\rho-\rho')^2}{z-z'} + \frac{(\rho-\rho'')^2}{z-z''}\right\}} \frac{dr'\,dr''}{(z-z')(z-z'')}, \quad (61.68)$$

where the two-dimensional vectors $\rho(x, y)$, $\rho'(x', y')$, $\rho''(x'', y'')$ have been introduced. The integrations over $\vec{\rho}'$ and $\vec{\rho}''$ are extended over the entire x'y'- and x"y"-planes, respectively, and those over z' and z" from 0 to z. For the inhomogeneities, we shall assume a concrete form of the correlation function (59.16):

$$\overline{\nabla\delta n(r')\nabla\delta n(r'')} = \nabla_{r'}\nabla_{r''}\overline{(\delta n)^2}\, e^{-\frac{|r'-r''|^2}{l^2}} =$$

$$= -\nu^2\nabla_{r'}^2 e^{-\frac{|r'-r''|^2}{l^2}} = \frac{2\nu^2}{l^2}\left(3 - 2\frac{(\rho'-\rho'')^2}{l^2} -\right.$$

$$\left. - 2\frac{(z'-z'')^2}{l^2}\right) e^{-\frac{(\rho'-\rho'')^2 + (z'-z'')^2}{l^2}}. \quad (61.68a)$$

After this, integration can be carried out (see [21]) if we set $\vec{r}' - \vec{r}'' = \vec{R}$, $\vec{r}' + \vec{r}'' = 2\vec{R}_0$ and the limits of the integrals $\vec{R}$ and over $\vec{R}_0$ are regarded as independent and indefinite. This method of integration (like the transition from Solution (61.62a) to Solution (61.64) itself) is admissible to the extent that all distances are large as compared with the dimension l of the inhomogeneity, provided that the region around the observation point does not play a prominent role. In this case, the essential zone coincides with the fundamental ellipsoid – the first Fresnel zone – and such an approximation is justified. In the one-dimensional case, however, this approach, for example, as applied to Formula (61.30), would lead to an incorrect result, since in that case (in the second term in the integrand), the essential region is a segment with a length of the order of $1/k$ near the point $z' = z$.

In addition to the elementary integrals, an integral

$\int_0^\infty R\,dR\,J_0(\alpha R)\exp(-\beta^2R^2) = (2\beta^2)^{-1}\exp\left(-\frac{\alpha^2}{4\beta^2}\right)$ [11,11] is encountered in the integration process. The final result is as follows:

$$\nu^2 - \frac{1}{k^2}\overline{(\nabla\varphi)^2} = \nu^2 - 4\sqrt{\pi}\,\nu^2 kl\,\frac{P}{1+4P^2}. \qquad (61.69)$$

As is clear from the above, however, this result is incorrect if $P \ll 1$ and the problem is essentially one-dimensional, like that studied above. With interest in the three-dimensional case, therefore, we must assume $R \gtrsim 1$. Then, as we see, mutual compensation of the second-order sources does not occur, as it does in the one-dimensional problem, and the term ν^2 is relatively small by comparison with $k^{-2}\overline{(\nabla\varphi)^2}$ and may be dropped.

It must be emphasized that this difference is actually related to the three-dimensional nature of the problem. If in Formula (61.68a) we drop the "transverse" derivatives, i.e., we calculate $\overline{\left(\frac{\partial\Phi_1^{(1)}}{\partial z}\right)^2}$ instead of $\overline{(\nabla\Phi_1^{(1)})^2}$ in (61.68), we obtain zero instead of Expression (61.69) (this also implies the need to take the following terms of the expansion in μ into account; compare Formulas (61.29a) and (61.30c)). In fact, Formula (61.69) gives $\nu^2 \frac{1}{k^2}\left(\overline{\left(\frac{\partial\varphi}{\partial x}\right)^2}+\overline{\left(\frac{\partial\varphi}{\partial y}\right)^2}\right)$. The transverse phase-perturbation derivatives are much larger than the longitudinal ones.

Using Formula (61.69), we can calculate the average value of the second-order term $\overline{\phi_1^{(2)}}$ (in any case, it will not be larger than the root-mean-square value, so that the condition derived below will be necessary but perhaps insufficient). Averaging Eq. (61.61b), we obtain for $\overline{x^2}$ and, consequently, for $\overline{\phi_1^{(2)}}$ (see Formula (61.60)) a solution in the same form as for $\phi_1^{(1)}$ (61.62), (61.62a) or (61.64), differing from them only in the substitution of (61.69) for δn. Then the integral is easily evaluated and we obtain a second-order correction in phase (see [21]):

$$\frac{1}{\mu}\overline{\Phi_1^{(2)}} = k^2 P \nu^2 \left\{ P\left[1 - 4\sqrt{\pi}\, M\left(1 - \frac{\operatorname{arctg} 4P}{4P}\right)\right] - \right.$$

$$\left. - \frac{l\sqrt{\pi}}{2} M \ln(1 + 16P^2)\right\} \approx -4\sqrt{\pi}\, \nu^2 k^2 l^2 \left(P + \frac{l}{8}\ln(1+16P^2)\right). \quad (61.70)$$

Let us now find the root-mean-square value of $\phi_1^{(1)}$. Using Formulas (61.62), (61.64) and proceeding in the same way as in passing from Expression (61.68) to the result (61.69), we obtain after simple calculations ($L \equiv z$ is the total distance along the z-axis)

$$\overline{|\Phi_1^{(1)}|^2} \approx \frac{\sqrt{\pi}}{2}\nu^2 \frac{L}{l} \sim \nu^2 klP. \quad (61.71)$$

For the method of expanding in the parameter ν (with subsequent consideration of the smallness of μ) to be correct, it is necessary in any event that this quantity be small by comparison with $|\phi_0|^2 = k^2L^2 = k^4 l^4 P^2$, which is the case if $k^2 lL \gg \nu^2$, $P \gg \nu^2\mu^3$, and also that it be large by comparison with the square of $\overline{\phi_1^{(2)}}$. Thus, according to

Formula (61.10), we arrive at the final condition

$$\nu^2 k^2 l^2 P < 1. \tag{61.72}$$

To permit the use of the simple formula (61.62)-(61.64), it is necessary that the correction in phase (61.70) be small in absolute magnitude. This again takes us to Condition (61.72), which represents a very rigid limitation on the permissible distances L. Comparison of Formulas (61.71) and (61.72) indicates that within the limits of applicability of the method (i.e., when Condition (61.72) is satisfied), the complex phase Φ_1 of the "scattered" field calculated by the method is small:

$$\left|\frac{1}{\mu}\Phi_1\right| \sim \frac{1}{\mu}\sqrt{\overline{|\psi_1|^2}} < 1. \tag{61.73}$$

This result is consistent with the conclusions drawn in [20]. It is at variance with the valuations in other papers, in which the limits of applicability were derived from analysis of a single isolated inhomogeneity [2, 6] or else correct calculations [21] were used to draw a different conclusion.

§62. DIFFUSION OF RAYS

1. According to Formula (60.21), the method of perturbations is inapplicable for large L, small λ and very small ϑ. It becomes essential to take secondary, tertiary, etc., scattering of the rays into account. There is a simple method based on the assumption that <u>multiple</u> scattering is a major factor, i.e., that

$$\nu^2 k^2 l L \gg 1, \quad \nu^2 \sim \overline{(\delta\varepsilon)^2} \tag{62.1}$$

or

$$\nu^2 k^2 l^2 P \gg 1, \quad P = \frac{L}{kl^2}. \tag{62.1a}$$

Thus, this method is, in a certain sense, the opposite of the perturbation method. However, the question as to exact criteria for its validity is not exhausted by Inequality (62.1); this is by no means a simple problem and will be considered below in Subsection 4.

We shall assume (as is indeed the case for the atmosphere in real problems) that the deviation ϑ of the propagation direction from the direction of the incident rays remains small at all times. Indeed, for $kl \gg 1$, multiple scattering is particularly likely in the region of small angles, as will be seen, for example, from Formula (60.16). As it passes through the medium, the incident pencil is spread out directionally (from the standpoint of the wave field, this means that the direction of the normal to the wave front at each point fluctuates through small angles ϑ around the original direction; it is the distribution of these fluctuations that is the distribution of ray directions) and in width. If Inequality (62.1) is valid, the major part of the radiation that has traversed a layer of thickness L remains in the region of angles ϑ and distances from the axis of the incident beam corresponding to multiple scattering. This radiation is characterized, in particular, by the values of the statistical averages $\vartheta_L = \sqrt{\overline{\vartheta^2(L)}}$, $r_L = \sqrt{\overline{r^2(L)}}$ and the correlations $(\overline{r\vartheta})$ between the displacement $\vec{r}$ in the plane perpendicular to the flux axis and the direction vector $\vec{\vartheta}$.

In the range of angles ϑ much larger than ϑ_L, there is no multiple scattering, and the diffusion method is incorrect here. Scattering through such large angles (they may still be small by comparison with unity) is improbable; scattering is single and determined by Formulas (60.16) (in computing this scattering, we may disregard dispersion within the limits of the narrow diffusion cone, $\vartheta \lesssim \vartheta_L$, assuming that the basic mass of the radiation proceeds in the original unperturbed direction).

Thus, in addition to the region of multiple scattering, a faint "tail" of single scattering must be present at angles $\vartheta \gg \vartheta_L$.

Let us subdivide the entire medium into "elementary volumes" with longitudinal (along the original propagation direction) dimensions L_0 and transverse dimensions r_0. The entire method is based on the assumption that it is possible to select these dimensions such that the propagation of the wave field will reduce to multiple scattering of individual rays passing from volume to volume. Here the change in the field on passage through one volume is small and is calculated exactly, with consideration of the wave properties, by the perturbation method, and the wave effects may be disregarded as we pass to the next volume. In other words, the medium is replaced by a set of independent elements the scattering effect of each of which is calculated exactly by the method of perturbations. Passage of the field through the aggregate of these elements, however, takes place in accordance with the scattering laws for single rays.

Thus, we are dealing with diffusion of rays each of which experiences successive scatterings through angles $\Delta\vec{\vartheta}$, in each case independent of the accidental values of the parameters of the media at the point in question and of the incidence angle of the ray on the inhomogeneity in question. Thus, for example, $\Delta\vec{\vartheta}$ and the total deviation $\vec{\vartheta}$ after the ray has traversed (on a straight line) a distance z are random quantities. Hence the physical characteristic of the field that must be calculated is the probability $W(\vec{R}, \vec{\vartheta})$ that a ray that entered the medium at a point $\vec{R} = 0$ at an angle $\vartheta = 0$ to the direction taken as the polar axis will pass through an arbitrary point $\vec{R}(x, y, z)$ in a direction characterized by the unit vector $\vec{\vartheta}$. This vector is given by the angles ϑ and φ in the spherical coordinate system or by the components $\vartheta_x = \vartheta\cos\varphi$ and $\vartheta_y = \vartheta\sin\varphi$ in a rectangular system. It is convenient to introduce in place of $\vec{R}$ the cylindrical coordinates $R = R(z, x, y) = R(z, r), x = r\cos\chi, y = r\sin\chi,$ where χ is the azimuth angle in the plane perpendicular to the propagation axis z. The direction angles ϑ, φ of the ray are also reckoned from the z-axis.

Leaving aside for the moment the transverse displacement of the ray and considering the integral of W over $\vec{r}$, which we shall also denote by W, $W(z, \vec{\vartheta})$. By virtue of the cylindrical symmetry, the result cannot depend on φ. Thus, $W = W(z, \vartheta)$. The problem that arises now is the same as the problem of random rotations (through an angle ϑ) in rotational Brownian motion, where z plays the part of time. Let us envisage a sphere of unit radius for which the polar axis is the initial direction of the ray and any instantaneous direction of propagation $\vec{\vartheta}$ is represented by a point on it. In the process of propagation and scattering of the ray, the representing point executes Brownian motion, moving away from the pole on the average. If the total scattering and, consequently, the recession from the pole are small, $\vec{\vartheta} \ll 1$, then diffusion takes place on a small segment of the sphere around the pole. This segment can be replaced by a tangential plane. Hence we are concerned with the diffusion of a Brownian particle in a plane. Using the familiar formula for this case (see, for example, the textbook [11]), we may at once write the distribution of the representative $W(z, \vartheta)$ at the "moment of time" z:

$$W(z, \vartheta)\,d\Omega = \frac{1}{4\pi\mathcal{D}z} e^{-\frac{\vartheta^2}{4\mathcal{D}z}}\,d\Omega. \qquad (62.2)$$

where $\mathcal{D}$ is the diffusion coefficient, which must still be found specifically for our case. This function is normalized in accordance with the relation

$$\int W(z, \vartheta)\,d\Omega = \int_0^{2\pi} d\varphi \int_0^{\infty} W(z, \vartheta)\,\vartheta\,d\vartheta = 1, \qquad (62.3)$$

and satisfies the condition $W(0, \vartheta) = \frac{1}{\pi\vartheta}\delta(\vartheta)$, where $\delta(\vartheta)$ is the delta function. From Formula (62.2), we find the mean square of the ray angle of rotation at a distance z:

$$\overline{\vartheta^2} = \int_0^\infty \frac{\vartheta^2}{4\pi\mathcal{D}z} e^{-\frac{\vartheta^2}{4\mathcal{D}z}} d\Omega = 4\mathcal{D}z. \qquad (62.4)$$

Thus, if the pencil passes through a layer with a small thickness Δz, the mean square of the angle $\overline{\vartheta^2} = 4\mathcal{D}\cdot\Delta z$. This will enable us to connect the result to microscopic theory and determine $\mathcal{D}$. Indeed, in the case of passage through a thin layer (when its thickness Δz may nevertheless be large as compared with the dimension l of the inhomogeneity), we may calculate scattering by the method of perturbations. Hence $\overline{\vartheta^2}$ is determined by Formula (60.19), in which it is necessary only to substitute Δz for L. Comparing the resulting two expressions for $\overline{\vartheta^2}$, we find

$$\mathcal{D} = -\frac{\overline{(\delta\varepsilon)^2}}{4l}\int_0^\infty \frac{1}{\rho}\frac{dF}{d\rho}d\rho = -\frac{\overline{(\delta n)^2}}{l}\int_0^\infty \frac{1}{\rho}\frac{dF}{d\rho}d\rho. \qquad (62.5)$$

The important fact is the independence of $\mathcal{D}$ of the wavelength of the radiation (if ε does not depend on it). It is a consequence of the fact that the refraction of a plane wave on the quasiplanar boundary of the inhomogeneity does not depend on λ. The same result can be obtained by analysis of the ray paths [1]. It must be remembered, however, that the formulas obtained are themselves valid only as long as $l \gg \lambda$. When we have $\lambda \gg l$, as discussed in §60, the effects of the various inhomogeneities that fit into one wavelength are compensated to a considerable degree and there is less scattering. If the characteristic l are of the order of tens or hundreds of meters, radio waves shorter than about 10 m must be scattered with the same $\mathcal{D}$, while radio waves with λ > 10-100 m should not be influenced noticeably by air turbulence. If, however, the fluctuations are described by the two-parameter function (59.11), then, as we stated in connection with Formula (60.18), much smaller l may be effective. Formulas (62.2)-(62.4) can also be generalized for the case of angles that are not small [1].

2. Let us turn now to the more general problem – that of diffusion of a pencil both in direction and in space, limiting ourselves, as before, to the case in which the resultant deflection angles ϑ and the resultant displacements r are small.

Since we know the elementary law of scattering in a thin layer (60.17) and the diffusion coefficient that it implies (62.5), the latter being uniquely related to the mean-square scattering angle (62.4), our problem is the same as that of scattering of a pencil of particles in the medium. This problem was solved by Fermi in the approximation of small ϑ that we require (see the monograph [12], §27). The distribution function $W(z; \vec{r}, \vec{\vartheta})$ that we seek satisfies the kinetic equation. The number of particles per unit volume of the five-coordinate space z, $\vec{r}$, $\vec{\vartheta}$ varies firstly because some rays (particles) leave the element of the real space $(z, \vec{r})$, while others enter it, i.e., generally speaking, the spatial divergence is nohzero; secondly, because scattering changes the content of rays (particles) in the two-dimensional volume element of the space ϑ_x, ϑ_y. To write the kinetic equation, we first take the spatial motion into account. The same rays will pass through the point $(z + \Delta z, x + \Delta x, y + \Delta y)$ that passed through the point (z, x, y) and were moving at angles $\vartheta_x = \Delta x/\Delta z$ and $\vartheta_y = \Delta y/\Delta z$. If there were no scattering on this path, the quantity

$$W(z+\Delta z, x+\Delta x, y+\Delta y, \vartheta)\approx W(z, x, y, \vartheta)+\frac{\partial W}{\partial z}\Delta z+\frac{\partial W}{\partial x}\Delta x+\frac{\partial W}{\partial y}\Delta y=$$
$$=W(z, x, y, \vartheta)+\left(\frac{\partial W}{\partial z}+\frac{\partial W}{\partial x}\vartheta_x+\frac{\partial W}{\partial y}\vartheta_y\right)\Delta z$$

would necessarily coincide with $W(z, x, y, \vec{\vartheta})$. Consequently, the last parenthesis would be equal to zero. However, the number of these rays also varies as a result of scattering: the ray loss is equal to the integral $\int W(z, x, y; \vartheta)\, dw_{\Delta z}(\theta)$ over all $\vec{\theta}$, where $dw_{\Delta z}(\vec{\theta})$ is the probability of scattering by an angle $\vec{\theta}$ in the layer Δz, with this angle reckoned from the direction $\vec{\vartheta}$. On the other hand, the same scattering increases the number of rays with a given $\vec{\vartheta}$ if they had the direction $\vec{\vartheta} + \vec{\theta}$ before scattering and were scattered by an angle $\vec{\theta}$. Thus, we obtain the equation

$$\frac{\partial W}{\partial z} + \vartheta_x \frac{\partial W}{\partial x} + \vartheta_y \frac{\partial W}{\partial y} = \int (W(z, r; \vartheta + \theta) - W(z, r; \vartheta)) \frac{dw_{\Delta z}(\theta)}{\Delta z}, \quad (62.6)$$

where $dw_{\Delta z}(\vec{\theta})$ is the function (60.17), in which it is necessary to substitute $\vec{\theta}$ for $\vec{\vartheta}$, Δz for L and $\theta d\theta d\Phi$ for $d\Omega$. We note that θ and Φ define the unit vector $\vec{\theta}$, which must also be taken in cartesian components, $\theta_x = \theta \cos \Phi$, $\theta_y = \theta \sin \Phi$.

This may be reduced to a differential equation if we consider that scattering takes place preferentially by small angles θ. We expand the integrand in powers of θ and limit ourselves to terms up to and including the second order:

$$W(R; \vartheta + \theta) = W(R; \vartheta) + \frac{\partial W}{\partial \vartheta_x}\theta_x + \frac{\partial W}{\partial \vartheta_y}\theta_y + \frac{1}{2}\left(\frac{\partial^2 W}{\partial \vartheta_x^2}\theta_x^2 + 2\frac{\partial^2 W}{\partial \vartheta_x \partial \vartheta_y}\theta_x\theta_y + \frac{\partial^2 W}{\partial \vartheta_y^2}\theta_y^2\right).$$

Since $dw/d\Omega$ depends only on θ and not on Φ, the terms containing θ_x or θ_y in odd powers give zero on integration over Φ after substitution of this expansion into the integral; the integrals of θ_x^2 and θ_y^2 are equal to one another and to the integral of $\theta^2/2$.

Therefore,

$$\int (W(R; \vartheta + \theta) - W(R; \vartheta)) \frac{dw_{\Delta z}(\theta)}{\Delta z} = \frac{1}{4}\left(\frac{\partial^2 W}{\partial \vartheta_x^2} + \frac{\partial^2 W}{\partial \vartheta_y^2}\right)\int \theta^2 \frac{dw_{\Delta z}(\theta)}{\Delta z}. \quad (62.7)$$

The remaining integral gives the mean-square scattering angle divided by the layer thickness, i.e., according to Formula (62.4), it is equal to $4\mathcal{D}$. Denoting by Δ_ϑ the two-dimensional laplacian in the $\vec{\vartheta}$-space, we finally obtain a differential equation that is valid as long as the ϑ are small:

$$\frac{\partial W}{\partial z}+\frac{\partial W}{\partial x}\vartheta_x+\frac{\partial W}{\partial y}\vartheta_y-\mathcal{D}\Delta_\vartheta W=\mathcal{D}\left(\frac{\partial^2 W}{\partial\vartheta_x^2}+\frac{\partial^2 W}{\partial\vartheta_y^2}\right). \qquad (62.8)$$

As is easily seen on substitution, the solution of this equation is the function

$$W(z,r,\vartheta)=\frac{3}{4\pi^2\mathcal{D}^2z^4}e^{-\frac{1}{\mathcal{D}}\left(\frac{3}{z^3}r^2-\frac{3}{z^2}\vartheta r+\frac{\vartheta^2}{z}\right)}. \qquad (62.9)$$

where $\vartheta r=\vartheta_x x+\vartheta_y y$.

This function is normalized so that

$$\int W(z,r;\vartheta)\,dr\,d\vartheta=\int W(z,r;\vartheta)\,dx\,dy\,d\vartheta_x d\vartheta_y=1 \qquad (62.10)$$

and corresponds to the condition that the ray moves along the z-axis at $z = 0$.

Integrating the general formula (62.9) over x and y, we can obtain a distribution over ϑ that agrees with the distribution (62.2), while integration over ϑ_x and ϑ_y gives the distribution over r,

$$W(z,r)=\frac{3}{4\pi\mathcal{D}z^3}\exp\left[-\frac{3r^2}{4\mathcal{D}z^3}\right]. \qquad (62.11)$$

Further, the following average values proceed from Formula (62.9):

$$\overline{\vartheta^2}=4\mathcal{D}z;\quad \overline{(\vartheta\cdot r)}=2\mathcal{D}z^2;\quad \overline{r^2}=\frac{4}{3}\mathcal{D}z^3. \qquad (62.12)$$

Thus, the root-mean-square angle increases in proportion to the square root of distance, and the root-mean-square lateral displacement in proportion to the distance covered to the 3/2 hour. Obviously, the relation

$$\sqrt{\overline{\vartheta^2}}=\sqrt{\overline{r^2}}\cdot\frac{z}{\sqrt{3}}$$

may be interpreted as indicating that the rays are scattered effectively over a given angle $\sqrt{\overline{\vartheta^2}}$ at a distance of the order of $1/\sqrt{3}$ of the distance traversed from the observation point. The result obtained is valid primarily when $\overline{\vartheta^2} \ll 1$. Without leaving the range of applicability of the geometrical-optical approximation, we can also consider diffusion over larger angles [1]. However, as we shall see shortly, this is by no means necessary in the case of radio waves and the formulas obtained above virtually exhaust the problem.

Let us consider a numerical example. As we have noted more than once before, we may assume an order of magnitude of 10^{-11}-10^{-10} for $\overline{(\delta\varepsilon)^2}$ in the troposphere, and 10^4 cm for l. In this case – as also indicated by direct data for $\overline{(\delta n)^2}/l$ at an altitude of several kilometers – we may assume that $\mathcal{D}$ reaches values

$$\mathcal{D} \sim 10^{[illegible]} - 10^{[illegible]}\ \mathrm{cm}^{-1} \qquad (62.13)$$

but may also be considerably smaller, for example, $\sim 10^{-18}$). Assuming $\mathcal{D} = 10^{-16}$ cm^{-1}, we obtain at distances of 100 and 1000 km, respectively,

z, km	$\sqrt{\overline{\vartheta^2}}$	$\sqrt{\overline{r^2}}$, cm
100	$0{,}6 \cdot 10^{-4}$	$4 \cdot 10^{2}$
1000	$2 \cdot 10^{-4}$	10^{4}

Thus, the dispersion of propagation directions is reckoned in minutes and the width of dispersion of the pencil in tens of meters. It also follows from this that in the problems considered here, we encounter only the case $\mathcal{D}z \ll 1$. As is seen from Equality (62.12), numerical estimates are sensitive to the values of $\overline{(\delta\varepsilon)^2}$, which depend strongly on meteorological conditions. Moreover, if l close to l_s figure in the correlation function (59.11), the value of $\mathcal{D}$ indicated above is found to be much too low.

Multiple scattering may be of great importance in optical observation of astronomical objects, when chaotic distortion of the directions of arrival of the radiation, even over fractions of a minute, causes wavering of the source position as seen in the telescope. In this case, the path s traversed in the turbulent atmosphere is much shorter than in the numerical example cited above. Nevertheless, the effect is substantial (this problem was analyzed in detail by the method of smooth perturbations, with consistent consideration of the correlation function (59.11), by V.I. Tatarskiy [2]; see §61, Subsection 4).

3. The general formulas (62.2), (62.9) and (62.11) enable us to obtain a number of other results. Let us calculate, for example, the fluctuations $\Delta\varphi$ of the phase φ of an arriving ray. In the geometrical-optical approximation, the phase is

$$\varphi = \int_0^\sigma k(\sigma')\,d\sigma', \quad k(s') = k_0(1 + \delta n(s')), \tag{62.14}$$

where, strictly speaking, the integral is taken along the distorted path σ of the ray. At each point we may assume $ds' = \frac{dz'}{\cos\vartheta}$, where z' is reckoned along the original direction and ϑ is the angle formed by the ray with this direction at the point of interest. Since ϑ is everywhere small,

$$\varphi = k_0\int_0^\sigma (1 + \delta n(s'))\,ds' \approx k_0\int_0^z (1 + \delta n(z'))\left(1 + \frac{\vartheta^2(z')}{2}\right)dz' \approx k_0\int_0^z \Big(1 + \delta n(z') + \frac{\vartheta^2(z')}{2}\Big)dz'.$$

But according to Formulas (62.4) and (62.5), since $z/l \lesssim 10^4$ even for $z \sim 1000$ km, while $\delta n \lesssim 10^{-5}$–$10^{-6}$, we have

$$\vartheta^2 \sim (\delta n)^2 \cdot \frac{z}{l} \ll |\delta n|.$$

Consequently, ϑ^2 may be dropped, i.e., the difference between s and σ gives an infinitesimal effect. Therefore,

$$\varphi = \varphi_0 + \Delta\varphi, \quad \varphi_0 = k_0 z, \quad \Delta\varphi = k_0\int_0^z \delta n(z')\,dz'. \tag{62.15}$$

and, consequently,

$$\overline{(\Delta\varphi)^2} = k_0^2\int_0^z\int_0^z dz'dz''\,\overline{\delta n(z')\,\delta n(z'')}.$$

Converting to the variables $z' - z'' = l\rho_z$ and z'', we may integrate over z'' to obtain z, and introduce for the average of the product δn at two points separated from one another by a distance $l\rho_z$ on the z-axis ($x' = x'' = 0$, $y' = y'' = 0$) its value $\overline{(\delta n)^2}F(0, 0, \rho_z)$. We obtain as a result (the integral over ρ_z can be extended from $-\infty$ to $+\infty$, since $z \gg l$)

$$\overline{(\Delta\varphi)^2} = k_0^2 z l\,\overline{(\delta n)^2}\int_{-\infty}^{+\infty} F(0, 0, \rho_z)\,d\rho_z. \tag{62.16}$$

The dimensionless integral is of the order of unity. Hence $\overline{(\Delta\varphi)^2} \sim k_0^2 z l^2 \mathscr{D} \sim k_0^2 z l\,\overline{\delta^2}$. Assuming, for example, that $\overline{(\delta n)^2} \sim 10^{-11}$, $l \sim 10^4$, and $z \sim 100$ km $= 10^7$, we find the root-mean-square phase fluctuation:

$$\sqrt{\overline{(\Delta\varphi)^2}} \sim \frac{2\pi}{\lambda_{cm}}, \tag{62.16a}$$

where λ is to be taken in centimeters. It is large (2π) for $\lambda_{cm} = 1$, and for $\lambda = 1$ m it amounts to ~ 0.1 radian. Phase differences between the rays at various points on the antenna may interfere with its operation. In actuality, it would appear that the harm done by this effect in practice has been found to be less than was originally feared [10].

4. Only with a more complete solution of the problem would it be possible to ascertain the range of applicability of the ray-diffusion method rigorously. Due to the lack of a rigorous theory (as was indicated in §61, the theory based on the smooth-perturbation method cannot always be regarded as rigorous), we are obliged to limit ourselves to a few brief remarks.

First it is necessary to determine whether (and within what limits) it is possible to select the parameters L_0 and r_0 of the elementary scattering centers mentioned at the beginning of this subsection in such a way that Formula (60.19), which consistently takes the wave nature of scattering in the elementary volume into account and was used in Relationship (62.5) can be applied to them.

It is quite obvious that $r_0 \gg l$ and $L_0 \gg l$ must hold in any event. Otherwise scattering events in successive volumes will not be statistically independent. On the other hand, scattering within the limits of each volume must be small enough to permit application of the perturbation-method formulas to each event. According to Formula (60.21), this necessitates

$$L_0 < \frac{1}{\nu^2 k^2 l}.$$

Consequently, for $l \ll L_0$ to apply simultaneously, it is necessary that the condition

$$\nu^2 k^2 l^2 \ll 1 \qquad (62.17)$$

be observed in any case. For $l \sim 10^4$ cm, $\nu^2 \sim 10^{-10}$-10^{-12} this means that $k < 10$-10^2 cm^{-1}, $\lambda > 1$-0.1 cm.

On the other hand, many scattering events must take place over the entire path. Consequently, according to Formula (62.1), it is necessary that $\nu^2 k^3 l^3 P \gg 1$. In itself, this condition enables us to apply the diffusion method in any case with very large P. But with $P \lesssim 1$, it is applicable only when $\nu^2 k^3 l^3 \gg 1$, i.e., by virtue of Formula (62.17), we must have $kl \gg 1$.

Thus, at distances L at which the wave parameter $P = L/kl^2$ is small, $P \lesssim 1$, the method is correct if

$$\frac{1}{\nu^{2/3}} < kl < \frac{1}{\nu}.$$

In general, however, we must have

$$\frac{1}{\nu}\sqrt{\frac{l}{L}} < kl < \frac{1}{\nu}. \qquad (62.18)$$

i.e., for example, with $\nu^2 \sim 10^{-12}$, $l \sim 10^4$, we must have $1 < \lambda_{cm} <$ $< \sqrt{10 \cdot L_{km}}$, where λ is taken in centimeters and L in kilometers. It must be remembered, however, that with $\lambda \lesssim 1$ cm we enter a region in which the turbulence in the atmosphere is characterized by two length parameters, since the internal parameter l_s is of the order of 1 cm and the considerations brought forth above, in which a single (large) parameter is taken into account, become quantitatively unreliable.

Thus, the range of applicability of the multiple-scattering formulas (62.2), (62.9), as determined by Conditions (62.1) and (62.18), is rather narrow (not to mention the fact that these conditions are necessary but perhaps not sufficient). But this does not imply that the range of applicability of more particular formulas, such as Formula (62.4) for $\overline{\vartheta^2}$, is the same. In fact, this formula (as was, after all, taken into account in its derivation) is also valid in the region of single scattering, for which Condition (62.1) is invalid.

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In connection with Chapter 9, see also I, 2; I, 9; I, 11; I, 12; I, 13.

X. PROPAGRATION OF RADIOWAVES IN A TURBULENT TROPOSPHERE

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In connection with Chapter 10, see also I, 7; I, 8; I, 12; III, 1; III, 2.