1878-1-R336

ON THE MERGING OF UNIFORM SHEAR FLOWS AT A TRAILING EDGE

APRIL 1967

DAC-60862

PREPARED BY
R.J. HAKKINEN
CHIEF SCIENTIST
PHYSICAL SCIENCES DEPARTMENT
RESEARCH AND DEVELOPMENT

ELIZABETH J. O'NEIL
(PRESENTLY WITH THE COURANT INSTITUTE
OF MATHEMATICAL SCIENCES,
NEW YORK UNIVERSITY)

PREPARED UNDER THE SPONSORSHIP OF
DOUGLAS AIRCRAFT COMPANY
INDEPENDENT RESEARCH AND DEVELOPMENT
PROGRAM, ACCOUNT No. 88080-321

R. M. Wood

APPROVED BY
R.M. WOOD
DEPUTY DIRECTOR
RESEARCH AND DEVELOPMENT

DOUGLAS
MISSILE & SPACE SYSTEMS DIVISION

43

ABSTRACT

The merging of two uniform, infinite, incompressible shear flows downstream of the trailing edge of a semi-infinite flat plate is studied by means of an asymptotic inner-outer expansion procedure applied to the Navier-Stokes equations. The flow field is divided into three regions; an outer flow, where inertia effects dominate and vorticity is primarily convected along streamlines; a wake, where inertia and viscous effects are of the same order and vorticity is diffused across streamlines; and a second-order vorticity-diffusing layer along the plate, generated by the upstream effect of the wake. The expansions appropriate to each region match term by term in the overlap regions; however, coefficients are determined numerically only to the third and fifth orders, in the inner and outer regions, respectively. The results indicate the existence of an induced pressure field. The significance of the present study in relation to the merging of two laminar boundary layers at a trailing edge is evaluated.

TABLE OF CONTENTS

		Page
1.	Introduction	1
1.1	General Outline	1
1.2	The Expansion Procedure	3
2.	Wake Expansion	6
3.	Upstream Boundary Layer	9
4.	Outer Flow and Matching With Inner Regions	10
5.	Description of Flow Field - Summary of Results	22
5.1	Stream Function and Streamlines	22
5.2	Velocity	23
5.3	Skin Friction	23
5.4	Pressure Field	24
6.	Effects of Infinite Extent of the Shear Flow	26
7.	Concluding Remarks	27
	References	47

LIST OF ILLUSTRATIONS

		Page
Table I	Matching of Leading Orders of Inner (a_{ij}, b_{ij}) and Outer (A_{ij}, B_{ij}) Expansions	31

Figure

1.	Flow Configuration and Velocity Profiles at $x = -\infty$, 0, 10 and 20	33
2.	Streamlines of Outer Flow Near Trailing Edge.	35
3.	Centerline Velocity	37
4.	Skin Friction on Plate	39
5.	Pressure on Upstream Plate	41
6.	Pressure on Centerline of Wake, Including Extrapolation of Outer Solution	43
7.	Isobars of Induced Pressure Field	45

BLANK PAGE

1. INTRODUCTION

1.1 General Outline

The present investigation concerns the merging of two incompressible, uniform shear flows of infinite extent behind the trailing edge of a semi-infinite flat plate (figure 1). An inner-outer expansion procedure, based on the full Navier-Stokes equations and asymptotic in the radial distance from the trailing edge, is carried out for the symmetrical case. The expansion procedure divides the flow field into three parts according to the type of the dominating physical phenomenon. In the wake, an expansion is selected whose leading members are governed by equations containing inertia and viscous terms of equal orders of magnitude. The wake solution is matched, in the overlap region at the boundary of the viscous wake, to an outer expansion consistent with the essentially inviscid nature of the slightly deflected free stream.

It is possible to satisfy the boundary conditions at upstream infinity by means of the two expansions described so far. For finite distances from the trailing edge, however, the essentially inviscid outer expansion produces an unavoidable velocity slip along the plate. To correct for the velocity slip, a third expansion is introduced for a second-order boundary layer along the plate. This expansion is identical in form to that for the downstream wake, except that the leading member is the undisturbed free stream, and the first nontrivial similarity solution occurs as the second member.

Application of boundary conditions and matching between these three expansions in their appropriate overlap regions yields a consistent asymptotic description in the entire flow field. While it appears that the matching of the expansions could be continued to higher orders, the numerical values of the coefficients have been calculated only for the leading three and five members of the inner and outer expansions, respectively. The results indicate the existence of an asymptotically unbounded first order induced pressure field.

A shear flow of infinite lateral extent cannot have a physical counterpart; therefore, the present results should be expected to give a reasonable physical description in the interior of the wake following thick laminar boundary layers when the merging region has not yet penetrated the strongly nonlinear part of the initial velocity profile. The results of an approximate analysis of the flow field if the uniform shear adjoins a uniform velocity at some finite lateral distance, in analogy to a corresponding study of flat plate boundary layer (Toomre and Rott 1964), are discussed briefly.

In principle, the present downstream inner expansion is analogous to the Blasius solution with its higher approximations for a flat plate boundary layer in uniform flow. In the higher approximations to the Blasius solution, however, a logarithmic sequence must be introduced and certain numerical coefficients cannot be determined uniquely (Goldstein 1960). It is of interest to note that in the merging of uniform shear flows, members logarithmic in the expansion parameter are not needed in the inner expansion. However, the lateral asymptotic forms of the coefficient functions include logarithmic terms, making a logarithmic sequence necessary in the outer expansion.

Through the use of the complete Navier-Stokes equation for determining the outer expansion, an extra sequence of terms is obtained which would match formally with those terms in the inner expansion displaying algebraically decreasing vorticity in the lateral direction. A partial sequence of such terms is excited by the upstream second-order boundary layer in the present problem.

1.2 The Expansion Procedure

Using the vorticity Ω of the free stream (figure 1), the following dimensionless variables are adopted:

$$\psi = \bar{\psi}/\nu; \; x = \bar{x}\sqrt{\frac{\Omega}{\nu}}\; ; \; y = \bar{y}\sqrt{\frac{\Omega}{\nu}} \tag{1.1}$$

where $\bar{\psi}$ is the stream function, ν the kinematic viscosity, and $\bar{x}$, $\bar{y}$ the physical coordinates.

The dimensionless coordinates x and y can be considered "local Reynolds numbers" based on the free stream "friction velocity" $\sqrt{\nu\Omega}$. In these dimensionless variables, the full incompressible, two-dimensional Navier-Stokes equation in the stream function form is

$$\psi_y\,(\Delta\psi)_x - \psi_x\,(\Delta\psi)_y = \Delta\Delta\psi \tag{1.2}$$

with boundary conditions (considering only the upper halfplane)

$$\begin{array}{lll} \text{At} & x > 0,\; y = 0 & \psi = 0,\; \psi_{yy} = 0 \\ \text{At} & x < 0,\; y = 0 & \psi = 0,\; \psi_y = 0 \\ \text{At} & x \to -\infty,\; y > 0 & \psi_y \to y,\; \psi_x \to 0 \end{array} \tag{1.3}$$

The inner expansion for the merging layer is chosen as

$$\psi^{v}(x,y) = x^{2/3} f_o(\eta) + f_1(\eta) + x^{-2/3} f_2(\eta) + \cdots \qquad (1.4)$$

where $\eta = yx^{-1/3}$. The fractional exponents have been so determined that, asymptotically at large η, vorticity becomes independent of x (the initial vorticity can only be changed along a streamline by viscous action, which is presumed to be of higher order when the outer flow is approached). The governing equation is then reduced to a set of ordinary differential equations for the f_j in terms of the similarity variable η. Formally, however, the choice of the asymptotic expansion (1.4) must be justified a posteriori by the successful accomplishment of a consistent solution for the entire flow field.

The inner expansion chosen above must be matched in the overlap zone at the edge of the merging region with an appropriately formed outer expansion

$$\psi^{o}(r,\theta) = r^2 G_0(\theta) + r^{4/3} G_1(\theta) + r^{2/3} G_2(\theta) + G_3(\theta) + r^{-2/3} G_4(\theta)$$
$$+ \cdots + (\log r) H_3(\theta) + \frac{\log r}{r^2} H_6(\theta) + \cdots \qquad (1.5)$$

where $r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}(y/x)$. It is evident that the derivation of the differential equations governing each $G_k(\theta)$ or $H_\ell(\theta)$ is particularly simple in view of the separation of variables in (1.5). The matching with the inner expansion is accomplished by expanding first (1.5) for small θ, and writing the result in terms of x and y. A comparison with the asymptotic form for large η of (1.4), written in terms of the same variables, then allows the identification of corresponding terms of the same functional form (excluding the decaying exponentials in the inner expansion) and, thereby, the matching of their coefficients. The details of the matching procedure are best illustrated in tabular form (Table I); however, a general matching condition may be stated as $\psi^o - \psi^w \to 0\ [\exp(-\eta)]$ in the overlap zone.

The remaining inner expansion for the upstream boundary layer is taken as

$$\psi^u(x,y) = x^{2/3}h_0(\zeta) + h_1(\zeta) + x^{-2/3}h_2(\zeta) + \cdots \tag{1.6}$$

where $\zeta = yx^{-1/3}$. Obviously, the differential equations governing $h_j(\zeta)$ will be identical to those for $f_j\ (\eta)$; however, since $x < 0$ in the upstream region, $yx^{-1/3} < 0$, and a different notation, ζ, is used to distinguish this region from the $\eta < 0$ range pertaining to the lower half of the merging wake. It may at first appear unrealistic to attempt a description of the upstream boundary layer, which is intuitively expected to vanish as $x \to -\infty$, in terms of a similarity variable $\zeta = yx^{-1/3}$, when ζ = constant lines diverge towards $x = -\infty$. It will turn out, however, that the first term, $x^{2/3}h_0$, reduces simply to the expression for the free stream, $1/2\ y^2$, so that the first nontrivial correction is relegated to the <u>second</u> term which is of a higher order as $x \to -\infty$. The resulting "second-order boundary layer" therefore behaves in a physically acceptable manner. Were the problem treated in terms of the two expansions (1.4) and (1.5) only, a slip velocity asymptotically proportional to $|x|^{-1/3}$ would be required along the upstream plate; consequently, the second-order boundary layer described by the third expansion can be thought of as being induced by this slip velocity. The concept of a "backward" boundary layer has recently also been discussed by Goldstein (1965).

It would now appear logical to derive the differential equations and general solutions for the leading members of the inner and outer expansions. The subsequent step-by-step matching process, however, provides such considerable simplifications in the outer expansion that it is best carried out simultaneously with the determination of the individual solutions. This approach, moreover, serves to emphasize the essence of the inner-outer expansion procedure: the alternating use, at the "plate" and "wake" limits of the outer region, of the inner limit of a member of the outer expansion to furnish the required asymptotic conditions for members of the corresponding inner expansion, or vice versa.

A more detailed account of this analysis is available upon request (Hakkinen and O'Neil 1965).

2. WAKE EXPANSION

Upon substitution of (1.4) into the governing equations (1.2), a sequence of ordinary differential equations is obtained for the f_j. These equations, boundary conditions and asymptotic forms of their solutions are derived in the following for the leading members of the expansion.

The function $f_0(\eta)$, a counterpart of the classical Blasius similarity solution, is governed by

$$f_0'''' + \frac{2}{3} f_0 f_0''' = 0 \qquad (2.1)$$

with initial conditions from (1.3), $f_0(0) = 0$, $f_0''(0) = 0$. Equation (2.1) integrates directly into

$$f_0''' + \frac{2}{3} f_0 f_0'' - \frac{1}{3} f_0'^2 = C_0 \tag{2.2}$$

where the constant C_0 relates to a pressure field, $dp/dx = C_0\, x^{-1/3}$, in the merging region. The similarity solution $f_0(\eta)$ has been discussed extensively elsewhere (Rott and Hakkinen 1962, 1965 and Hakkinen and Rott, 1965). Only the asymptotic behavior will be of specific interest here:

$$f_{0_\infty} = a_{00}\, \eta^2 + a_{01}\, \eta + a_{02} + a_{0e}\, O(-\exp) \tag{2.3}$$

where $a_{02} = \frac{1}{2}(a_{01}{}^2 + 3C_0)$. The coefficients a_{00}, a_{01}, C_0 and a_{02} depend on the initial values of $f_0(0)$, $f_0'(0)$, $f_0''(0)$ and $f_0'''(0)$ in a manner which can be determined only through numerical integration. Since $f_0(0)$ and $f_\theta''(0) = 0$, two asymptotic matching conditions will be required, and they will turn out to determine a_{00} and a_{01}. The exponential coefficient a_{0e} does not appear in the matching and its value is of no direct interest. The parameter C_0 (and a_{02}) and the remaining initial values $f_0'(0)$ and $f_0'''(0)$ are determined through their implicit dependence on a_{00} and a_{01}.

All f_j:s for $j \neq 0$ are governed by fourth-order linear differential equations, the one for $f_1(\eta)$ being

$$f_1'''' + \frac{2}{3} f_0 f_1''' + \frac{2}{3} f_0' f_1'' = 0 \tag{2.4}$$

with initial conditions $f_1(0) = 0$, $f_1''(0) = 0$. Again, (2.4) integrates directly into

$$f_1''' + \frac{2}{3} f_0 f_1'' = C_1 \tag{2.5}$$

Substitution of f_{0_∞} from (2.3) yields the asymptotic behavior

$$f_{1_\infty} = a_{12}\,\eta + a_{13} + C_1\,[-\frac{3}{2a_{00}}\log\eta - \frac{3}{4}(\frac{a_{01}}{a_{00}^2})\frac{1}{\eta} + (\frac{a_{01}^2 - a_{00}a_{02}}{4a_{00}^2})\frac{1}{\eta^2} + (\frac{3a_{00}^2 + 2a_{00}a_{01}a_{02} - a_{01}^3}{8a_{00}^4})\frac{1}{\eta^3} + \cdots\cdots] + a_{1e}\,O(\exp(-\eta^3)) \qquad (2.6)$$

where the constants a_{12}, a_{13}, C_1 and a_{1e} contain two degrees of freedom to be specified by matching conditions. Thereafter, the determination of the remaining constants and initial values follows through relationships implicit in the numerical integration procedure.

The function $f_2(\eta)$ is governed by

$$f_2'''' + \frac{2}{3}f_0 f_2''' + \frac{4}{3}f_0' f_2'' - \frac{2}{3}f_0''' f_2 = \frac{1}{27}[12\,f_0 f_0' - 4\eta\,f_0 f_0'' - 2\eta^2\,f_0 f_0''' - 4\eta^2\,f_0' f_0'' - 24\eta\,f_0''' - 6\eta^2\,f_0''''] - \frac{2}{3}f_1' f_1'' \qquad (2.7)$$

with initial conditions $f_2(0) = 0$, $f_2''(0) = 0$. Substitution of f_{0_∞} from (2.3) and f_{1_∞} from (2.6) into (2.7) yields the asymptotic behavior

$$f_{2_\infty} = \frac{1}{27}a_{01}\eta^3 + \frac{1}{9}a_{02}\eta^2 + a_{23}\eta + a_{24} - (\frac{3C_1 a_{12}}{4a_{00}^2})\frac{1}{\eta} + (\frac{3C_1^2}{8a_{00}^3})\frac{\log\eta}{\eta^2} + a_{26}\frac{1}{\eta^2} + \cdots + a_{2e}\,O(\exp(-\eta^3)) \qquad (2.8)$$

where the four coefficients a_{23}, a_{24}, a_{26} and a_{2e} are related to the initial conditions $f_2(0)$, $f_2'(0)$, $f_2''(0)$ and $f_2'''(0)$ by an "influence matrix" whose elements can be determined by straightforward numerical integration. Since $f_2(0)$ and $f_2''(0) = 0$, two conditions are required to specify the solution uniquely. The integration must be continued far enough to allow identification of the leading terms in the asymptotic expansion (2.8) with sufficient accuracy.

The equations for higher-order $f_j(\eta)$ are quite similar to those for f_2, i.e., linear nonhomogeneous fourth-order differential equations with initial conditions $f_j(0)$ and $f_j''(0) = 0$. The leading term in the asymptotic form f_{j_∞} increases in power with increasing j, so that the entire "outer" asymptotic form of the wake expansion

$$\psi_\infty^w = x^{2/3} f_{o_\infty}(\eta) + f_{1_\infty}(\eta) + x^{-2/3} f_{2_\infty}(\eta) + \ldots \qquad (2.9)$$

is clearly divergent for large η. Except for the rapidly decaying exponential terms, it may be interpreted as a series development of the outer solution about $y = 0$.

3. UPSTREAM BOUNDARY LAYER

The substitution of (1.6) into (1.2) yields a set of equations and general solutions for the $h_j(\zeta)$ which, except for replacing f_j by h_j, η by ζ, a_{jk} by b_{jk}, and C_j by K_j, are identical to those for the $f_j(\eta)$. The initial conditions, however, are different; now $h_j(0) = 0$, $h_j'(0) = 0$. Also, since $-\infty < \zeta < 0$, the asymptotic behavior is considered as $\zeta \to -\infty$, and the exponential terms are now diverging. Otherwise, earlier statements concerning the higher $f_j(\eta)$ carry over to the $h_j(\zeta)$.

4. OUTER FLOW AND MATCHING WITH INNER REGIONS

Substitution of (1.5) into the polar coordinate form of (1.2)

$$2\psi_\theta \psi_{\theta\theta} + r(\psi_r \psi_{\theta\theta\theta} + \psi_\theta \psi_r - \psi_\theta \psi_{r\theta\theta}) + r^2 (\psi_r \psi_{r\theta} - \psi_\theta \psi_{rr})$$

$$+ r^3 (\psi_r \psi_{rr\theta} - \psi_\theta \psi_{rrr}) = \psi_{\theta\theta\theta\theta} + 4\psi_{\theta\theta} + r(\psi_r - 2\psi_{r\theta\theta})$$

$$+ r^2 (2\psi_{rr\theta\theta} - \psi_{rr}) + r^3 (2\psi_{rrr} + r\,\psi_{rrrr}) \qquad (4.1)$$

yields now three sequences of equations involving products of squares of logarithms and algebraic powers, products of single logarithms and algebraic powers, and simple algebraic powers of the radius, respectively. Generally speaking, the algebraic sequence provides third-order equations for the $G_k(\theta)$, while the logarithmic sequences yield equations for $H_\ell(\theta)$ and, under certain conditions, also eliminate one degree of freedom from the $G_k(\theta)$. No contradictions are introduced by the multiplicity of the equations, however.

The derivation of general solutions for the G_k and H_ℓ would become extremely cumbersome and be of little use since these functions are completely determined by matching with the inner expansions as $\theta \to \pi$ and $\theta \to 0$. The matching procedure is, therefore, carried out simultaneously with the determination of the individual G_k and H_ℓ. At the same time, of course, the matching leads to the successive unique determination of the inner solutions.

The entire matching procedure is illustrated in Table I, where the nonexponential terms of the inner expansions, $x^{2/3(1-j)} f_{j_\infty}$ and $x^{2/3(1-j)} h_{j_\infty}$, are listed on the horizontal rows and those of the outer expansion, $r^{2-\frac{2}{3}k} G_k(\theta)$ and $r^{2(1-k)} (\log r) H_{3(k-1)}(\theta)$ developed in powers of x and y as $\theta \to 0$ and π, on the vertical columns. No explicit rules are required since a complete term-by-term correspondence exists between the inner and outer expansions, except for the exponential solutions of the inner expansions which either decay very rapidly (wake) or are determined to have zero coefficients in the matching process (upstream boundary layer).

Collecting terms of order r^4 in (4.1) yields

$$G_0 \, (G_0''' + 4G_0') = 0 \qquad (4.2)$$

Aside from the trivial solution $G_0 \equiv 0$, (4.2) has the general solution

$$G_0(\theta) = A_{00} \sin^2 \theta + A_{01} \cos 2\theta + A_{02} \sin 2\theta \qquad (4.3)$$

Since $2 \sin^2 \theta = 1 - \cos 2\theta$, the first term could be replaced by an arbitrary constant; however, for reasons of consistency with other G_k, the form (4.3) will be retained.

The coefficients A_{00}, A_{01} and A_{02} are determined entirely by matching with the upstream boundary layer and wake expansions. Consider the leading members of the upstream layer and outer expansions, respectively:

$$\psi_0^u = x^{2/3} \, h_0(\zeta)$$

$$\psi_0^o = r^2 \, (A_{00} \sin^2 \theta + A_{01} \cos 2\theta + A_{02} \sin 2\theta)$$

Matching requires that as $\zeta \to -\infty$ and $\theta \to \pi$, a term of a common functional form be found in each expansion. Writing $\psi^u_{0_\infty}$ in terms of x and y, one obtains

$$\psi^u_{0_\infty} = b_{00}\, y^2 + b_{01}\, y\, x^{1/3} + b_{02}\, x^{2/3} + b_{0e}\, x^{2/3}\, (\exp(- \frac{y^3}{x})) \tag{4.4}$$

while

$$\psi^o_0 \xrightarrow[\theta\to\pi]{} A_{00}\, y^2 - A_{01}\, x^2 - 2A_{02}\, xy + \ldots \tag{4.5}$$

Referring to Table I, it is evident that the choices of

$$b_{00} = A_{00} = \frac{1}{2}\ ;\ b_{01} = 0;\ A_{01} = 0,\ A_{02} = 0,\ \text{i.e. } \psi^o_0 = \frac{1}{2}\, r^2 \sin^2 \theta,$$

not only facilitiate the first step in matching but also satisfy the required boundary conditions at $x = -\infty$, i.e., $\psi_y = y$ and $\psi_x = 0$.

The asymptotic form of the leading member of the inner expansion, (4.4), contains a strongly growing (ζ, $x < 0$) exponential term for which no counterpart exists in the outer flow, neither on physical grounds nor within the trigonometric general form of the outer solution. If one therefore requires $b_{0e} = 0$, the remainder $h_0(\zeta) = \frac{1}{2}\zeta^2 + b_{02}$ is valid in the <u>entire</u> inner region. In fact, if b_{02} is also chosen equal to zero, it satisfies the boundary conditions on the plate, $h_0(0)$ and $h_0'(0) = 0$. Thus,

$$h_0\,(\zeta) = \frac{1}{2}\,\zeta^2 \tag{4.6}$$

is a solution consistent with all requirements at this time. This result also means that the leading member of the upstream boundary layer expansion is identical to the undisturbed free stream. Therefore, any higher members of the inner expansion must satisfy

$$(\psi^u_k)_y = 0,\ (\psi^u_k)_x = 0,\ k \neq 0 \tag{4.7}$$

as $x \to -\infty$, and those of the outer expansion, ψ_k^o, must match with the appropriate terms of the ψ_k^u as $\theta \to \pi$.

At $\theta \to 0$ the leading member of the asymptotic wake expansion

$$\psi_{0_\infty}^w = x^{2/3} f_{0_\infty} = a_{00} y^2 + a_{01} yx^{1/3} + a_{02} x^{2/3}$$

$$+ a_{oe} x^{2/3} (\exp(- \frac{y^3}{x}))$$

is now seen to match its first term with $\psi_0^o = \frac{1}{2} y^2$ if $a_{00} = \frac{1}{2}$.

With reference to the uppermost left-hand square in Table I, the results of applying boundary conditions and the first step in the matching process have yielded

$$h_o(\zeta) = \frac{1}{2} \zeta^2$$

$$G_o(\theta) = \frac{1}{2} \sin^2 \theta$$

$$f_{o_\infty}(\eta) = \frac{1}{2} \eta^2 + a_{01} \eta + a_{02}$$

The next step is to determine $G_1(\theta)$ and match it with h_0, f_0, h_1, f_1, h_2, and f_2.

Terms of order $r^{10/3}$ in (4.1) yield

$$2G_0 G_1''' + \frac{2}{3} G_0' G_1'' + \frac{32}{9} G_0 G_1' + \frac{32}{27} G_0' G_1 = 0 \qquad (4.8)$$

Substitution of $G_0(\theta) = \frac{1}{2} \sin^2 \theta$ yields the solution

$$G_1(\theta) = A_{10} \sin^{4/3} \theta + A_{11} \cos \frac{4\theta}{3} + A_{12} \sin \frac{4\theta}{3} \tag{4.9}$$

Writing $r^{4/3}\, G_1$ in powers of x and y as $\theta \to \pi$ yields

$$\psi_1^o \xrightarrow[\theta\to\pi]{} A_{10}\, y^{4/3} + (A_{11} \cos \frac{4\pi}{3} + A_{12} \sin \frac{4\pi}{3})\, x^{4/3}$$

$$+ \frac{4}{3} (A_{12} \cos \frac{4\pi}{3} - A_{11} \sin \frac{4\pi}{3})\, yx^{1/3} + \ldots \tag{4.10}$$

With reference to Table I, it is obvious that neither A_{10} nor $(A_{11} \cos \frac{4\pi}{3} + A_{12} \sin \frac{4\pi}{3})$ match with the inner expansion, and are therefore required to be zero. However, in comparing (4.10) with (4.4), it is also required that $\frac{4}{3} (A_{12} \cos \frac{4\pi}{3} - A_{11} \sin \frac{4\pi}{3}) = b_{01} = 0$. It is impossible to satisfy these two conditions for A_{11} and A_{12} unless both are zero. Thus, the free stream boundary condition and matching conditions available so far require $G_1(\theta) \equiv 0$. This result further implies, through matching with $\psi^w_{0_\infty} = x^{2/3}\, f_{0_\infty}$, the condition $a_{01} = 0$, which in turn eliminates the remaining degree of freedom in f_0 and allows its final determination. With reference to the more detailed information given elsewhere (Rott and Hakkinen 1965), the final results are

$$f_0(0) = 0, \qquad f_0'(0) = 0.89914,$$

$$f_0''(0) = 0, \qquad f_0'''(0) = 0.67838,$$

$$a_{01} = 0, \quad a_{02} = 0.61334, \quad C_0 = 0.40889.$$

As discussed in the reference, these values are consistent with the "Li-Murray" condition, i.e., with no shift of the asymptotic velocity profile relative to free stream, and the existence of a first-order induced pressure gradient in the merging region.

Terms of order $r^{8/3}$ in (4.1) yield

$$2G_0 G_2''' + \frac{4}{3} G_0 G_2'' + \frac{8}{9} G_0 G_2' + \frac{16}{27} G_0' G_2$$

$$= \frac{4}{3} (2G_1 G_1''' + G_1' G_1'' + \frac{16}{3} G_1 G_1') \qquad (4.11)$$

Substitution of $G_0 = \frac{1}{2} \sin^2 \theta$ and $G_1 \equiv 0$ produces the general solution

$$G_2(\theta) = A_{20} \sin^{2/3} \theta + A_{21} \cos \frac{2\theta}{3} + A_{22} \sin \frac{2\theta}{3} \qquad (4.12)$$

As $\theta \to \pi$, $\psi_2^o = r^{2/3} G_2(\theta)$ must be matched with the asymptotic form of $\psi_{o_\infty}^u = x^{2/3} h_{0_\infty}$. With reference to Table I, the following conditions are established: $A_{20} = 0$; $A_{21} \cos \frac{2\pi}{3} + A_{22} \sin \frac{2\pi}{3} = b_{02} = 0$, i.e. $A_{22} = \frac{1}{\sqrt{3}} A_{21}$. As $\theta \to 0$, matching ψ_2^o with $\psi_{0_\infty}^w$ yields $A_{21} = a_{02} = 0.61334$, and, hence, $A_{22} = 0.35411$.

Comparing h_{1_∞} with ψ_2^o as $\theta \to \pi$ yields $b_{12} = -\frac{1}{\sqrt{3}} A_{21} - \frac{1}{3} A_{22} = -0.47215$. It is now possible to determine $h_1(\zeta)$ uniquely, since b_{12} is known and the positive exponential in h_{1_∞} must vanish in analogy to that in h_{0_∞}, requiring $b_{1e} = 0$. Taking $h_0 = \frac{1}{2} \zeta^2$, the equation for $h_1(\zeta)$ becomes

$$h_1''' + \frac{1}{3} \zeta^2 h_1'' = K_1 \qquad (4.13)$$

which can be integrated directly into

$$h_1(\zeta) = K_1 \int_\zeta^0 \int_{\bar{\zeta}}^0 e^{-\frac{1}{9}\bar{\zeta}^3} \int_{-\infty}^{\bar{\zeta}} e^{\frac{1}{9}\bar{\bar{\zeta}}^3} \, d\bar{\bar{\zeta}}\, d\bar{\zeta}\, d\zeta \tag{4.14}$$

where both initial conditions, $h_1(0) = 0$ and $h_1'(0) = 0$, and the asymptotic condition $b_{1e} = 0$ have been satisfied. The constants b_{12} and b_{13} can be expressed explicitly:

$$b_{12}/K_1 = -2^{-2/3}\, 3^{1/12}\, \Gamma\left(\tfrac{2}{3}\right) F\left(\cos^{-1} \frac{1-\sqrt{3}}{1+\sqrt{3}} \Big\backslash 75^\circ\right)$$

$$= -3.44878$$

$$b_{13}/K_1 = \ln 2 + \frac{\pi}{2\sqrt{3}} - (3 + \mathcal{E})$$

$$= -2.12101$$

where F is an elliptic function of the first kind (Abramowitz and Stegun 1965, p. 589) and $\mathcal{E} = 0.57721$ is Euler's constant. Thus, one must have $K_1 = \frac{b_{12}}{-3.44878} = 0.13690$ which, in turn, specifies $b_{13} = -0.29047$, $h_1''(0) = 3^{-1/3}\, \Gamma(1/3)\, K_1 = 0.25429$, and $h_1'''(0) = K_1 = 0.13690$.

Matching between f_{1_∞} and ψ_2^o as $\theta \to 0$ yields

$$a_{12} = \frac{2}{3} A_{22} = 0.23607$$

However, since no requirement can be established for the coefficient of the new <u>decaying</u> exponential in (2.6), complete determination of $f_1(\eta)$ must await the next step in the matching process.

Grouping of terms of order r^2 in (4.1) yields

$$2\, G_0 G_3''' + 2\, G_0' G_3'' = 0$$

where the right hand side, which otherwise would consist of a sum of products of derivatives of G_1 and G_2, vanishes in the present case since $G_1 \equiv 0$. With $G_0 = \frac{1}{2} \sin^2 \theta$,

$$G_3(\theta) = A_{30} \log |\sin \theta| + A_{31} + A_{32}\theta \tag{4.15}$$

In order to determine $H_3(\theta)$, one must group the terms of order $(\log r)^2$ in (4.1), yielding

$$2H_3' H_3'' = 0 \tag{4.16}$$

or

$$H_3(\theta) = B_{31} + B_{32}\,\theta \tag{4.17}$$

Comparison of G_3 and $H_3 \log r$ with h_{1_∞} as $\theta \to \pi$ yields the matching conditions $A_{30} = -3K_1 = -\,0.41071$; $-A_{30} + B_{31} + B_{32}\pi = K_1 = 0.13690$; and $A_{31} + A_{32}\pi = b_{13} = -\,0.29037$. Matching with f_{1_∞} yields $A_{30} = -3C_1$; $-A_{30} + B_{31} = C_1$; and $A_{31} = a_{13}$. These two sets of matching conditions require that $A_{30} = -3K_1 = -0.41071$; $b_{31} = -2K_1 \simeq -\,0.27381$; $B_{32} = 0$; and $C_1 = K_1 = 0.13690$.

At this point, $f_1(\eta)$ can be completely determined by performing a numerical integration of (2.5), with $C_1 = +0.13690$, $f_1(0) = 0$ and $f_1''(0) = 0$, and choosing $f_1'(0)$ so that $a_{12} = +0.23607$. The results are $f_1'(0) = -0.16544$ and $f_1'''(0) = C_1 = 0.13690$. In order to calculate

a_{13}, it is necessary to integrate f_1 from the known initial values until the leading terms of the asymptotic series for f_{1_∞} in (2.6) yield satisfactory accuracy. The result is $a_{13}/C_1 = -2.79091$, i.e., $a_{13} = -0.38209$. Thus, $A_{31} = a_{13} = -0.38209$ and $A_{32} = \frac{1}{\pi}(b_{13} - A_{31}) = 0.029193$.

It is appropriate to note here that assembling terms of orders $r^2 \log r$, $r^{4/3} \log r$, $r^{2/3} \log r$ and $\log r$ also yield equations for G_0, G_1, G_2, and G_3, respectively. Two cases can be distinguished: first, $H_3' = B_{32} = 0$, a trivial case which yields no other information; and second, $H_3' = B_{32} \neq 0$, in which case the solutions of the resulting second-order linear equations are identical to (4.3), (4.9), (4.12), and (4.15), with the restriction that A_{00}, A_{10}, A_{20} and $A_{30} = 0$. In the present case, however, $H_3' = B_{32} = 0$, so that some of these additional degrees of freedom cannot be eliminated.

In complete analogy to G_1 and G_2, the higher G_k $(k > 3)$ have a homogeneous solution of the form

$$(G_k)_h = A_{ko} \sin^{2-\frac{2}{3}k} \theta + A_{k1} \cos(2-\tfrac{2}{3}k)\theta + A_{k2} \sin(2-\tfrac{2}{3}k)\theta \qquad (4.18)$$

In addition, there will be a particular solution, since products of derivatives of lower G_k:s and of H_3 appear on the right-hand side of the governing equation. It can be shown that at $\theta \to \eta$ and $\theta \to 0$ such particular solutions match higher-order terms in the asymptotic forms of the inner expansions, as indicated in more detail in Table I. Only

G_4 and G_6 will be of further interest; determination of the homogeneous part of G_5 will already involve matching with h_3 and f_3. Also, H_6 will be the only H_ℓ of concern in the following.

The outer expansion members $r^{-2}\, G_6(\theta)$ and $(\log r/r^2)\, H_6(\theta)$ occupy a special position because of their overlap with a free coefficient (i.e., a term not determined as part of a particular solution) in $\psi_{2_\infty}^{v}$ and $\psi_{2_\infty}^{u}$. The equation for G_6 results by collecting terms of order r^o in (4.1); for $G_1 \equiv 0$ one obtains

$$2\, G_0G_6''' + 4\, G_0'G_6'' + 8\, G_0G_6' + 16\, G_0'G_6$$

$$= -\frac{8}{3}\, G_2'G_4'' - \frac{4}{3}\, G_2''G_4' - \frac{2}{3}\, G_2G_4''' + \frac{2}{3}\, G_2'''G_4 \qquad (4.19)$$

$$- 2\, G_3'G_3'' - H_3G_3''' + H_3'G_3'' + G_3'''' + 4G_3''$$

The particular solution will again match higher-order terms in the inner asymptotic expansions involving h_{1_∞}, f_{1_∞}, h_{2_∞}, f_{2_∞}, as indicated in Table I. For matching with f_{2_∞}, only the homogeneous solution need be considered

$$(G_6)_h(\theta) = \frac{A_{60}}{\sin^2\theta} + A_{61}\cos 2\theta + A_{21}\sin 2\theta \qquad (4.20)$$

As $\theta \to \pi$, or $\theta \to 0$,

$$\psi_6^o \to \frac{A_{60}}{y^2} + A_{61}\, x^{-2} + 2A_{62}\, yx^{-3} + \dots \qquad (4.21)$$

If $H_3' \neq 0$, the equation for H_6 is obtained from terms of order $r^{-2}(\log r)^2$ in (4.1); in the present case, it results from assembling terms of order log r:

$$(H_6''' + 4H_6')\, G_0 + 2(H_6'' + 4H_6)\, G_0' = -H_3'G_3'' \qquad (4.22)$$

Since $H_3' \equiv 0$, (4.22) integrates into

$$H_6'' + 4H_6 = \text{const.} \times (G_0)^{-2} \qquad (4.23)$$

which, for $G_0 = \frac{1}{2}\sin^2\theta$, has the general solution

$$H_6(\theta) = \frac{B_{60}}{\sin^2\theta} + B_{61}\cos 2\theta + B_{62}\sin 2\theta \qquad (4.24)$$

Considering h_{2_∞}, all coefficients related to the nonhomogeneous solution have been determined by the specification of b_{00}, b_{01}, b_{02} and K_1. Of the four independent coefficients b_{23}, b_{24}, b_{26} and b_{2e}, matching with G_3 yields $b_{23} = A_{32} = 0.029193$ and, since the positive exponential is again required to vanish, $b_{2e} = 0$. Thus, with the initial conditions $h_2(0) = 0$ and $h_2'(0) = 0$, all prerequisites now exist for the complete determination of h_2. This can be done, in principle, by varying $h_2''(0)$ and $h_2'''(0)$ in numerical integration until the required asymptotic coefficients are reached, making use of an "influence matrix" similar to that for $f_2(\eta)$. However, in view of the strongly diverging exponential, it is now necessary to invert the process, i.e., to start integrating from a large $|\zeta|$, with $b_{23} = +0.029193$ and $b_{2e} = 0$, and vary b_{24} and b_{26} until the proper initial conditions $h_2(0) = 0$ and $h_2'(0) = 0$ have been reached. Such an integration, started from initial values of h_2, h_2', h_2'' and h_2''' calculated at $\zeta = -10$, yields $b_{24} = 0.10110$ and $b_{26} = 0.10898$. The corresponding values of other initial derivatives are $h_2''(0) = -0.0042734$ and $h_2'''(0) = 0.0069540$. Matching now $\psi_{2_\infty}{}^u = x^{2/3}h_{2_\infty}(\zeta)$ with (4.20) and (4.24) as $\theta \to \pi$ yields $A_{60} = b_{26} = 0.10898$. The other matching terms in $\psi_{2_\infty}{}^u$ of orders $x^{-1/3}y^{-1}$ and $(\log y)/y^2$ can easily be seen to arise from particular solutions of G_5 and G_6, respectively, with complete agreement between numerical coefficients derived from either inner or outer expansions.

Since G_6 and H_6, as such, are outside the objective of this study, the only result of significance here pertains to further matching of G_6 and H_6 with $\psi_{2_\infty}{}^w = x^{2/3} f_{2_\infty}$. Using (2.8), one obtains $B_{60} = -0.03749$; $a_{26} = A_{60} = b_{26} = 0.10898$; and the above comments regarding h_{2_∞} apply to the remaining terms. Since matching with G_3 as $\theta \to 0$ has already yielded $a_{32} = A_{32} = 0.29193$, two of the four independent coefficients a_{23}, a_{24}, a_{26} and a_{2e} have been determined, and, thus, the entire $f_2(\eta)$ has been specified. The proper initial conditions and a_{24} can now be calculated by numerical integration, the results being $f_2'(0) = 0.885$, $f_2'''(0) = 1.176$, and $a_{24} = -1.079$. The coefficient a_{2e}, although nonzero, now multiplies a rapidly decaying exponential and is of no specific interest.

From (4.18),

$$G_4(\theta) = A_{40} \sin^{-2/3}\theta + A_{41} \cos \frac{2\theta}{3} + A_{42} \sin \frac{2\theta}{3} \qquad (4.25)$$

Matching with $\psi_{2_\infty}^{u}$ and $\psi_{2_\infty}^{w}$ yields $(-\frac{1}{2}A_{41} + \frac{\sqrt{3}}{2}A_{42}) = b_{24} = 0.1011$; $A_{41} = a_{24} = -1.079$; and $A_{42} = -0.506$.

5. DESCRIPTION OF FLOW FIELD - SUMMARY OF RESULTS

5.1 Stream Function and Streamlines:

In the upstream second-order boundary layer

$$\psi^u(x,y) = \frac{1}{2}y^2 + h_1(yx^{-1/3}) + x^{-2/3}h_2(yx^{-1/3}) + \ldots \qquad (5.1)$$

In the wake

$$\psi^w(x,y) = x^{2/3}f_o(yx^{-1/3}) + f_1(yx^{-1/3}) + x^{-2/3}f_2(yx^{-1/3}) + \ldots \qquad (5.2)$$

In the outer region

$$\begin{aligned}\psi^o(r,\theta) = \frac{1}{2}r^2\sin^2\theta + r^{2/3}\,&(0.70822\sin\frac{2\theta+\pi}{3})\\ &- 0.27381\log r - 0.41071\log\sin\theta\\ &- 0.38209 + 0.029193\,\theta\\ &- r^{-2/3}(1.079\cos\frac{2\theta}{3} + 0.506\sin\frac{2\theta}{3}) + \ldots\end{aligned} \qquad (5.3)$$

A uniformly valid composite expansion can be formed from the above results by noting that $\lim_{\theta\to\pi}\psi^o = \psi_\infty^u$ and $\lim_{\theta\to o}\psi^o = \psi_\infty^w$:

$$\psi^c = \psi^o + \delta(\psi^w - \psi_\infty^w) + (1-\delta)(\psi^u - \psi_\infty^u) \qquad (5.4)$$

where $\delta(x < 0) = 0$, $\delta(x > 0) = 1$. The resulting streamline pattern is presented in figure 2. It is reassuring to note the coincidence of the ψ = constant streamlines derived from inner and outer expansions at the edge of the merging zone, especially in view of the rather small values of the dimensionless radius in this region.

5.2 Velocity

Velocity profiles calculated from (5.1) through (5.3) are presented in figure 1. Of particular interest is the velocity along the centerline of the wake

$$u_o = u(x,0) = f_0'(0)\, x^{1/3} + f_1'(0)\, x^{-1/3} + f_2'(0)\, x^{-1} + \ldots.$$

$$= 0.899\, x^{1/3} - 0.165\, x^{-1/3} - 0.885\, x^{-1} + \ldots \qquad (5.5)$$

which is plotted in figure 3. A similar series has been derived by Imai (1966) as the asymptotic form of his trailing-edge solution, which was obtained from an equation linearized with respect of an infinite uniform shear flow. Imai's approach yields a centerline velocity

$$u(x,0) = \frac{3^{7/6}}{2\pi} [\Gamma(\tfrac{2}{3})]^2\, x^{1/3} + O(x^{-1/3})$$

$$= 1.051\, x^{1/3} + O(x^{-1/3})$$

In view of the different assumptions made in Imai's analysis, it is significant to note that the coefficient of the leading term can be considered in satisfactory agreement with that of (5.5). Both values are quite different from the "classical" one for zero induced pressure field, 1.611. (e.g., Hakkinen and Rott, 1965.)

5.3 Skin Friction

Since the uniform shear flow $\psi = \frac{1}{2} y^2$ is an exact solution of the Navier-Stokes equations, a flat plate extending from $x = -\infty$ to $x = +\infty$ would have a surface friction $\tau_{-\infty} = \rho\nu$, independent of x. Absence of the downstream half of the plate allows the build-up of velocity on the centerline for $x > 0$, which in turn accelerates the flow near the upstream plate and produces, at least sufficiently far from the trailing edge, a region of increased skin friction, as given by the result

$$\tau/\tau_{-\infty} = 1 + h_1''(0)\ |x|^{-2/3} + h_2''(0)\ |x|^{-4/3} + \ldots$$

$$= 1 + 0.2543\ |x|^{-2/3} - 0.0043\ |x|^{-4/3} + \ldots \qquad (5.6)$$

A plot of this result is given in figure 4. Again, a similar asymptotic series has been derived by Imai (1966). His latest result shows that the coefficient of the $O(x^{-2/3})$ term is positive, and the full analysis actually indicates the presence of a shear singularity at the trailing edge. However, the exclusively asymptotic nature of the present investigation does not allow any definite conclusions to be drawn from it in support of or against the somewhat controversial existence of the trailing edge singularity.

5.4 Pressure Field

In each region, the pressure distribution can be determined by integrating separately the x and y (or r and θ) momentum equations, and determining the remaining arbitrary functions (except for a constant reference pressure p_o) by comparing the two results. For the upstream boundary layer, one obtains

$$p(x,\zeta) - p_o^u = K_1 \log |x| - \frac{2}{3} x^{-2/3} (h_2''' + \frac{1}{3}\zeta^2 h_2'' + \frac{2}{3}\zeta h_2'$$

$$- \frac{2}{3} h_2 + \frac{1}{3} h_1'^2) \qquad (5.7)$$

which is equivalent to

$$p(x,\zeta) - p_o^u = K_1 \log |x| - \frac{3}{2} h_2'''(0)\ |x|^{-2/3} + \ldots$$

$$= 0.13690 \log |x| - 0.0104\ |x|^{-2/3} + \ldots \qquad (5.8)$$

Thus, within the present approximation, there is no lateral pressure variation in the upstream boundary layer. A plot of (5.8) is presented in figure 5. In the wake

$$p(x,\eta) - p_0^w = \frac{3}{2} C_0 x^{2/3} - C_1 \log x - \frac{3}{2} x^{-2/3} [f_2''' + \frac{2}{3} f_0 f_2'' + \frac{2}{3} f_0'' f_2 + \frac{1}{3} f_1'^2 - \frac{2}{9} f_0 + \frac{2}{9} \eta(f_0'' - \frac{2}{3} f_0 f_0') + \frac{1}{9} \eta^2 (2f_0''' + \frac{2}{3} f_0 f_0'' + \frac{1}{3} f_0'^2)] + \ldots \tag{5.9}$$

which on the centerline reduces to

$$p(x,0) - p_0^w = 0.61334\, x^{2/3} + 0.13690 \log x - 0.67787\, x^{-2/3} + \ldots , \tag{5.10}$$

plotted in figure 6. Finally, in the outer region

$$\begin{aligned} p(r,\theta) - p_0^o = {} & \frac{3}{2} C_0 r^{2/3} [\frac{4}{3\sqrt{3}} (\sin \frac{2\theta+\pi}{3})(1 + \cos \frac{2\theta+\pi}{3})] \\ & + C_1(\log r - \sin^2 \theta) + A_{32}(\theta - \frac{1}{2} \sin 2\theta) \\ & + r^{-2/3} \{[A_{41} + \frac{2}{3} \sin \theta(A_{41} \sin \theta - A_{42} \cos \theta)] \cos \frac{2\theta}{3} \\ & + [A_{42} + \frac{2}{3} \sin \theta(A_{42} \sin \theta + A_{43} \cos \theta)] \sin \frac{2\theta}{3} \\ & - \frac{8}{27} A_{21}^2\} + \ldots \qquad (5.11) \\ = {} & 0.4721\, r^{2/3} (\sin \frac{2\theta+\pi}{3})(1 + \cos \frac{2\theta+\pi}{3}) \\ & + 0.1369 (\log r - \sin^2 \theta) + 0.02919 (\theta - \frac{1}{2} \sin 2\theta) \\ & - r^{-2/3} \{[1.079 + \frac{2}{3} \sin \theta(1.079 \sin \theta - 0.506 \cos \theta)] \cos \frac{2\theta}{3} \\ & + [0.506 + \frac{2}{3} \sin \theta(0.506 \sin \theta \\ & + 1.079 \cos \theta)] \sin \frac{2\theta}{3} + 0.111\} + \ldots . \end{aligned}$$

At the limit $\theta \to \pi$, (5.11) reduces to (5.8), except for a constant $A_{32}\pi = 0.091713$. Since (5.8) contains an undetermined reference pressure level, one must conclude that, in order to match the pressure fields, the proper reference pressure for the upstream boundary layer is $A_{32}\pi$ higher than in the outer flow, i.e., $p_o^u = p_o^o + A_{32}\pi$. At the limit $\theta \to 0$, (5.11) reduces to

$$p(x,0) - p_o^o = 0.61334\, x^{2/3} + 0.13690 \log x - 1.190\, x^{-2/3} + \ldots \qquad (5.12)$$

which deviates from (5.9) at $O(x^{-2/3})$. This difference represents lateral pressure variation in the merging layer (figures 6 and 7). It is easily verified that the asymptotic form of (5.9) for $\eta \to \infty$ matches with the expansion of (5.11) for small θ, provided that the same reference pressures are used, i.e., $p_o^o = p_o^w$. Thus, within the approximation considered, a complete consistency is achieved of the pressure fields in the three regions. Some typical isobars are presented in figure 7. The pressure variation corresponding to (5.12) is illustrated by continuing the outer solution (5.11) across the wake.

6. EFFECTS OF INFINITE EXTENT OF THE SHEAR FLOW

While all manifestations of pressure gradients vanish as $r \to \infty$, the pressure level induced by the absence of the downstream half of the plate grows as $r^{2/3}$, except along the plate $\theta = \pi$, where the growth is proportional to log r. It is clear that this asymptotically unbounded pressure field is related to the infinite extent of the shear flow, which after all implies asymptotically unbounded velocities everywhere except at $x \to -\infty$. In reality, shear flows adjoin regions of uniform, or at least asymptotically bounded, velocity fields. In order to clarify the questions raised by the asymptotically unbounded pressure field, an approximate analysis was carried out to investigate the pressure induced by the merging layer in the outer flow provided that the shear flow now adjoins a uniform-velocity field some distance away from the centerline. The analysis is essentially analogous to that of Toomre and Rott (1964) for the effects of a finite shear layer in the pressure field induced by a flat plate in a vortical free stream. The lateral velocity distribution at a given small distance away from the centerline was assumed to be that given by the result for infinite shear flow (equation 5.3), and the pressure field induced in the shear flow and in the uniform stream bounding it was then calculated analytically. The results show a rapid decay of the induced pressure field ($\sim |x|^{-1/3}$) within a radius of the order of the shear layer thickness from the trailing edge, in accordance with the physical reasoning of Hakkinen and Rott (1965). Further details of this analysis can be found in Hakkinen and O'Neil (1965).

7. CONCLUDING REMARKS

The purpose of the present investigation has been twofold: first, to study the merging of uniform shear flows as a model for the wake of two boundary layers near the trailing edge, with special emphasis on the consequences of treating shear flows of infinite extent; and second, to explore a particular formulation of the inner-outer-expansion technique applied to the complete Navier-Stokes equations.

While most of the results, for example the streamline pattern and skin friction, appear quite reasonable, the calculated pressure field does indeed show a physically unacceptable character far away from the trailing edge. This is not really surprising since even the "free stream" velocity field already is unbounded in the lateral direction. It is evident, however, that the analysis of vorticity diffusion phenomena in infinite shear layers will hardly yield new information meaningful within the context of physically observable, high-Reynolds number boundary layers. On the other hand, the asymptotic expansion procedure limits the analysis to the region beyond some multiple of the "Stokes radius" ($\sqrt{\nu/\Omega}$) from the trailing edge, so that it actually applies in an annular region, the limits of which depend on the Reynolds number (Hakkinen and Rott 1965). The proper significance of the present study is, therefore, most likely as an intermediate step between the "classical" vorticity diffusion analyses, e.g., boundary and merging layers, and the viscosity-dominated (Stokes) flow which must prevail in the immediate vicinity of the trailing edge. It appears reasonable to expect that the present "intermediate" flow field would be the one "felt" by the innermost viscous flow even in the case of the physically acceptable outer shear flow of finite extent. It is of interest to note in this connection that the shear stress distribution along the upstream plate is formally similar to the asymptotic form of Imai's (1966) approximate solution which is, in principle, valid all the way to the trailing edge. It is, however, re-emphasized at this point that, unlike the investigations of Imai, and of Goldburg and Cheng (1961), the present study is not concerned with the flow very close to the trailing edge, and consequently does not yield information on the possible singular behavior there.

Hakkinen and Rott (1965) evaluated the available experimental evidence for any manifestation of the induced pressure field in the merging of two laminar boundary layers at a trailing edge. Such effects are probably observable only in a limited region near the trailing edge where the merging layer has not yet penetrated far into the outer, nonlinear part of the velocity profile. Inasmuch as this region was not covered by the experiments, which showed no evidence of the phenomenon elsewhere, the empirical verification remains inconclusive.

The inner-outer expansion technique used in the present study is based on the full Navier-Stokes equations, which are expanded in the three regions using the inverse two-thirds power of the dimensionless radius as the characteristic small quantity. In the overlap zones, it has been shown that the asymptotic form of the appropriate inner expansion is identical to the series development of the outer expansion in terms of the lateral distance (except for the exponentially decaying asymptotic solutions of the wake expansion). This term-by-term (not approximation-by-approximation) correspondence is easily expressed in a tabular form without need to state formal matching rules.

The hierachy of terms evolving from the expansion procedure also suggests that the outer expansion could possibly be extended to free stream configurations with nonuniform initial vorticity. This is solely the consequence of basing the outer expansion on the complete Navier-Stokes equation instead of the Laplace equation often used in connection with uniform-velocity free streams. Further research on this possibility is contemplated.

The terms of infinite shear flows, it appears possible to analyze the asymmetric merging of two (dissimilar) shear flows, and thus eliminate both of the two degrees of freedom possessed in that case by the leading similarity solutions (Rott and Hakkinen 1962). An initial study has also been made of infinite shear flow along a solid surface, subjected to a pressure disturbance; one family of the leading similarity solutions exhibits reverse flow for positive pressure gradients. Whether or not higher approximations to this problem would illuminate boundary layer separation phenomena, remains also a subject to further study.

It is also possible to formulate the higher approximation problem for a flat plate in uniform stream in a manner entirely analogous to the present study. It has been verified that the results are equivalent to those discussed by Goldstein (1960), including the need to introduce logarithmic terms at the third order.

This work was carried out under the sponsorship of the Douglas Independent Research and Development Program. Mrs. O'Neil, who contributed the upstream boundary layer concept during her stay at the Douglas Missile and Space Systems Division in the summer of 1964, graciously participated in the subsequent completion of the study. Special gratitude is extended to Professor Isao Imai for making his recent results available to the authors and for reading an earlier version of this paper. The many stimulating discussions with Drs. N. Rott, Alar Toomre, and T. Kubota, as well as the conscientious help received from Messrs. A. S. Hersh, W. A. Anderson and W. W. Tieman in carrying out the numerical calculations, are gratefully acknowledged.

TABLE I

MATCHING OF LEADING ORDERS OF INNER (a_{ij}, b_{ij},) AND OUTER (A_{ij}, B_{ij}) EXPANSIONS

<table>
<tr><th>$\theta \to 0,\pi$
$\lvert\zeta\rvert, \eta \to \infty$</th><th>$r^2 G_0$</th><th>$r^{4/3} G_1$</th><th>$r^{2/3} G_2$</th><th colspan="6">$(\log r) H_3 + G_3$</th><th>$r^{-2/3} G_4$</th><th>$r^{-4/3} G_5$</th><th colspan="6">$r^{-2}(\log r) H_6 + r^{-2} G_6$</th></tr>
<tr><td>$h_{0\infty}$
$\times\left(x^{2/3}\right)$
$f_{0\infty}$</td><td>b_{00}
A_{00}
y^2
A_{00}
a_{00}</td><td>$b_{01} = 0$
$2/3\,(\sqrt{3}A_{11} - A_{12})$
$x^{1/3}y$
$4/3\,A_{12}$
$a_{01} = 0$</td><td>$b_{02} = 0$
$1/2\,(\sqrt{3}A_{22} - A_{21})$
$x^{2/3}$
A_{21}
a_{02}</td><td colspan="6"></td><td></td><td></td><td colspan="6"></td></tr>
<tr><td>$h_{1\infty}$
$\times$ (1)
$f_{1\infty}$</td><td></td><td></td><td>b_{12}
$-1/3\,(\sqrt{3}A_{21} + A_{22})$
$x^{-1/3}y$
$2/3\,A_{22}$
a_{12}</td><td colspan="2">K_1
$B_{31} + B_{32} = -A_{30}$
$\log\lvert x\rvert$
$B_{31} - A_{30}$
C_1</td><td colspan="2">$-3K_1$
A_{30}
$\log y$
A_{30}
$-3C_1$</td><td colspan="2">b_{13}
$A_{31} + A_{32} =$
1
A_{31}
a_{13}</td><td>$-3b_{01}K_1 = 0$
$x^{1/3}y^{-1}$
$-3a_{01}C_1 = 0$</td><td>$K_1(2b_{01}^2 - b_{02}) = 0$
$1/6\,A_{30}(\sqrt{3}A_{22} - A_{21})$
$x^{2/3}y^{-2}$
$1/3\,A_{30}A_{21}$
$C_1(2a_{01}^2 - a_{02})$</td><td colspan="3"></td><td colspan="3">$3/2\,K_1 + (-b_{01})$
$-1/2\,A_{30}$
$x y^{-3}$
$-1/2\,A_{30}$
$3/2\,C_1 + (-a_{01})$</td></tr>
<tr><td>$h_{2\infty}$
$\times\left(x^{-2/3}\right)$
$f_{2\infty}$</td><td></td><td>$1/27\,b_{01} = 0$
$2/81\,(\sqrt{3}A_{11} - A_{12})$
$x^{-5/3}y^3$
$4/81\,A_{12}$
$1/27\,a_{01} = 0$</td><td>$1/9\,b_{02} = 0$
$1/18\,(\sqrt{3}A_{22} - A_{21})$
$x^{-4/3}y^2$
$1/9\,A_{21}$
$1/9\,a_{02}$</td><td colspan="3">$B_{32} = 0$
$x^{-1}y\log\lvert x\rvert$
$B_{32} = 0$</td><td colspan="3">b_{23}
A_{32}
$x^{-1}y$
A_{32}
a_{23}</td><td>b_{24}
$1/2\,(\sqrt{3}A_{42} - A_{41})$
$x^{-2/3}$
A_{41}
a_{24}</td><td>$-3K_1b_{12}$
$-1/3\,A_{30}(\sqrt{3}A_{21} + A_{22})$
$x^{-1/3}y^{-1}$
$2/3\,A_{30}A_{22}$
$-3C_1a_{12}$</td><td colspan="2">$-K_1^2 + (-b_{01})$
B_{60}
$y^{-2}\log\lvert x\rvert$
B_{60}
$-C_1^2 + (-a_{01})$</td><td colspan="2">$3K_1^2 + (-b_{01})$
$1/3\,A_{30}^2$
$y^{-2}\log y$
$1/3\,A_{30}^2$
$3C_1^2 + (-a_{01})$</td><td colspan="2">b_{26}
A_{60}
y^{-2}
A_{60}
a_{26}</td></tr>
<tr><td>$h_{3\infty}$
$\times\left(x^{-4/3}\right)$
$f_{3\infty}$</td><td></td><td></td><td>$-2/27\,b_{12}$
$2/81\,(\sqrt{3}A_{21} + A_{22})$
$x^{-7/3}y^3$
$-4/81\,A_{22}$
$-2/27\,a_{12}$</td><td colspan="3"></td><td colspan="3">$1/2\,K_1 + (-b_{01})$
$1/2\,(B_{31} + B_{32} = -A_{30})$
$x^{-2}y^2$
$1/2\,(B_{31} - A_{30})$
$1/2\,C_1\ (-a_{01})$</td><td>b_{34}
$2/3\,(\sqrt{3}A_{41} - A_{42})$
$x^{-5/3}y$
$4/3\,A_{42}$
a_{34}</td><td>.....
.....
$x^{-4/3}$
.....
.....</td><td colspan="3">.....
.....
$x^{-1}y^{-1}\log\lvert x\rvert$
.....
.....</td><td colspan="3">.....
.....
$x^{-1}y^{-1}$
.....
.....</td></tr>
</table>

PRECEDING PAGE BLANK

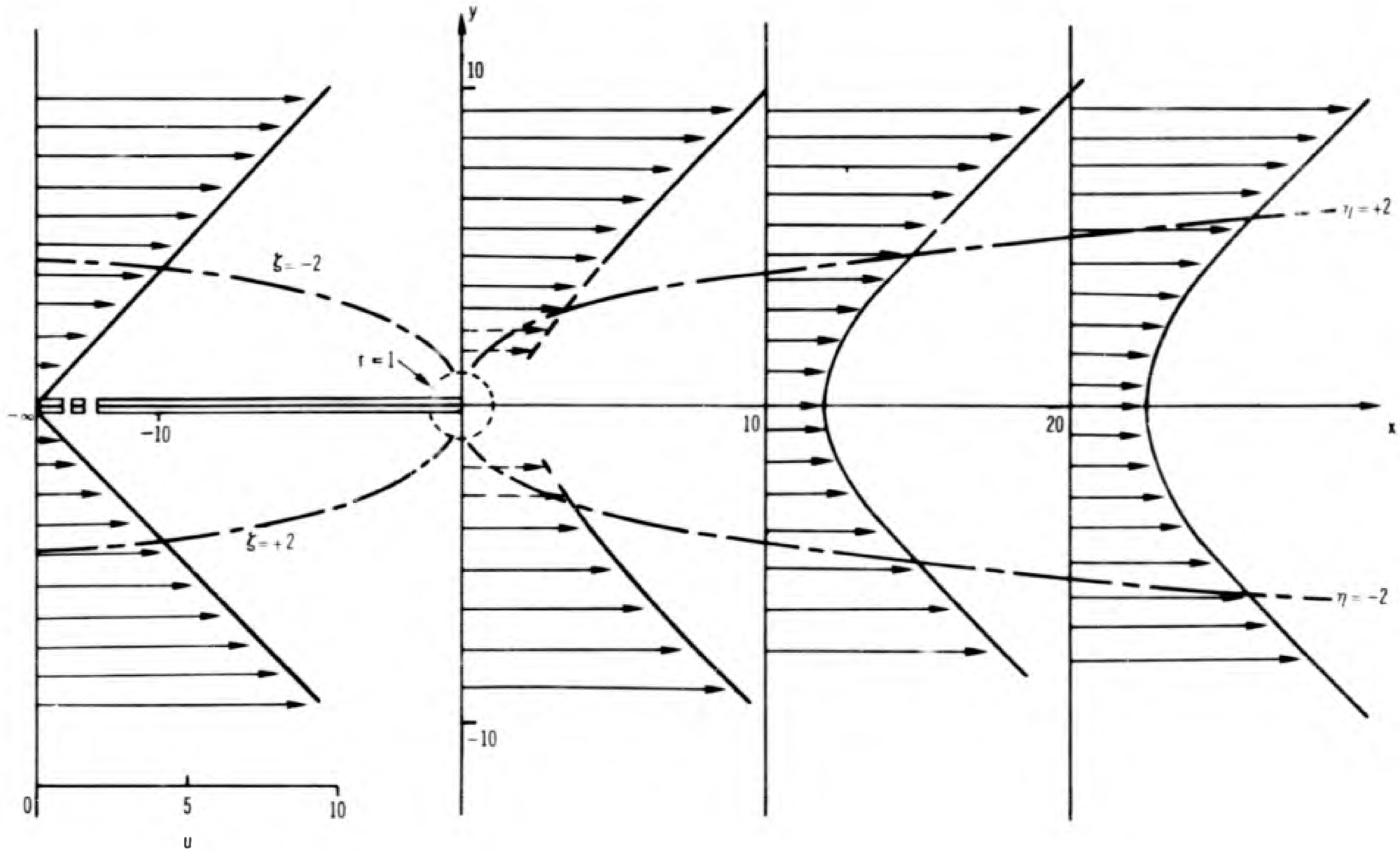

Figure 1. Flow Configuration and Velocity Profiles at $x = -\infty$, 0, 10 and 20

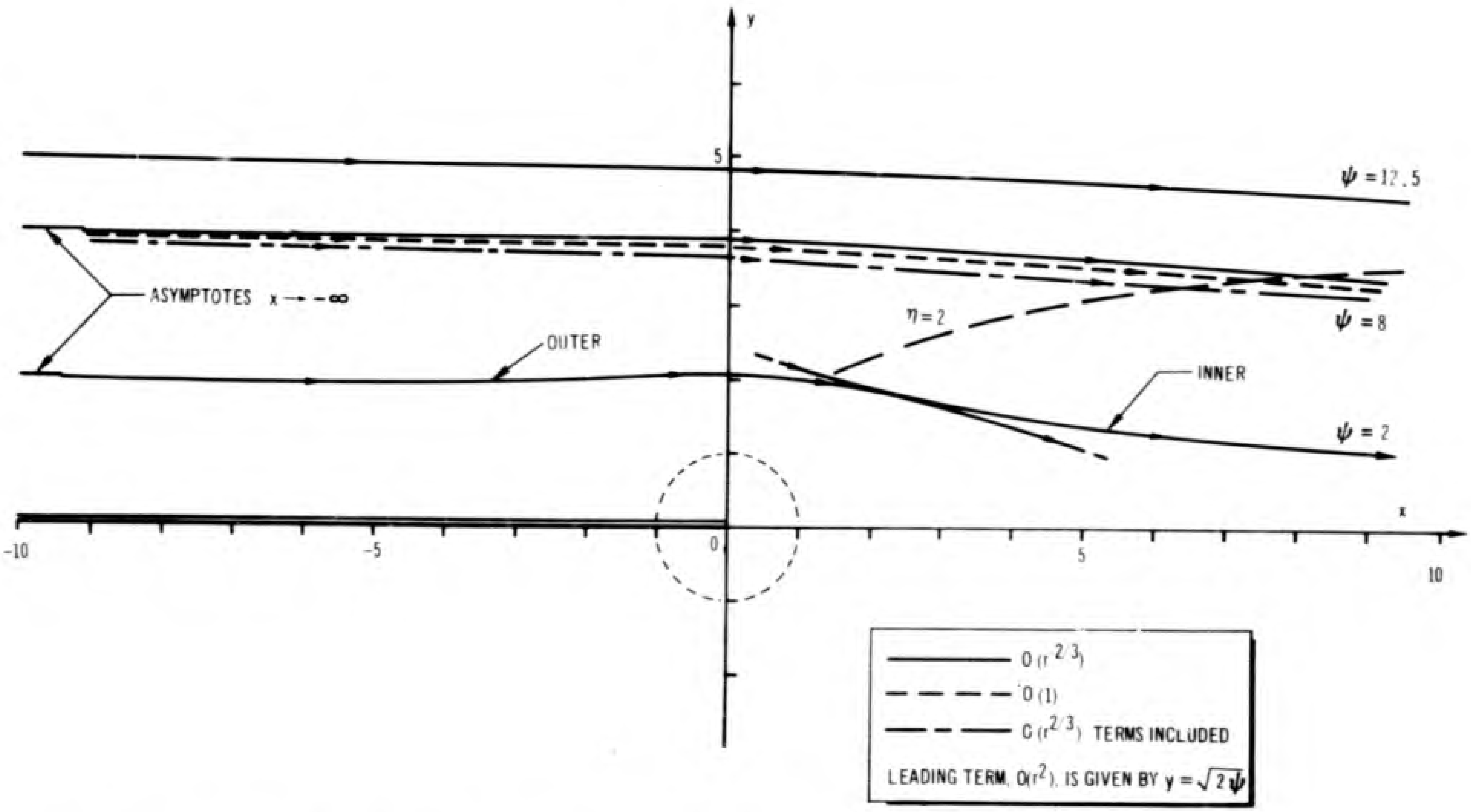

Figure 2. Streamlines of Outer Flow Near Trailing Edge

PRECEDING PAGE BLANK

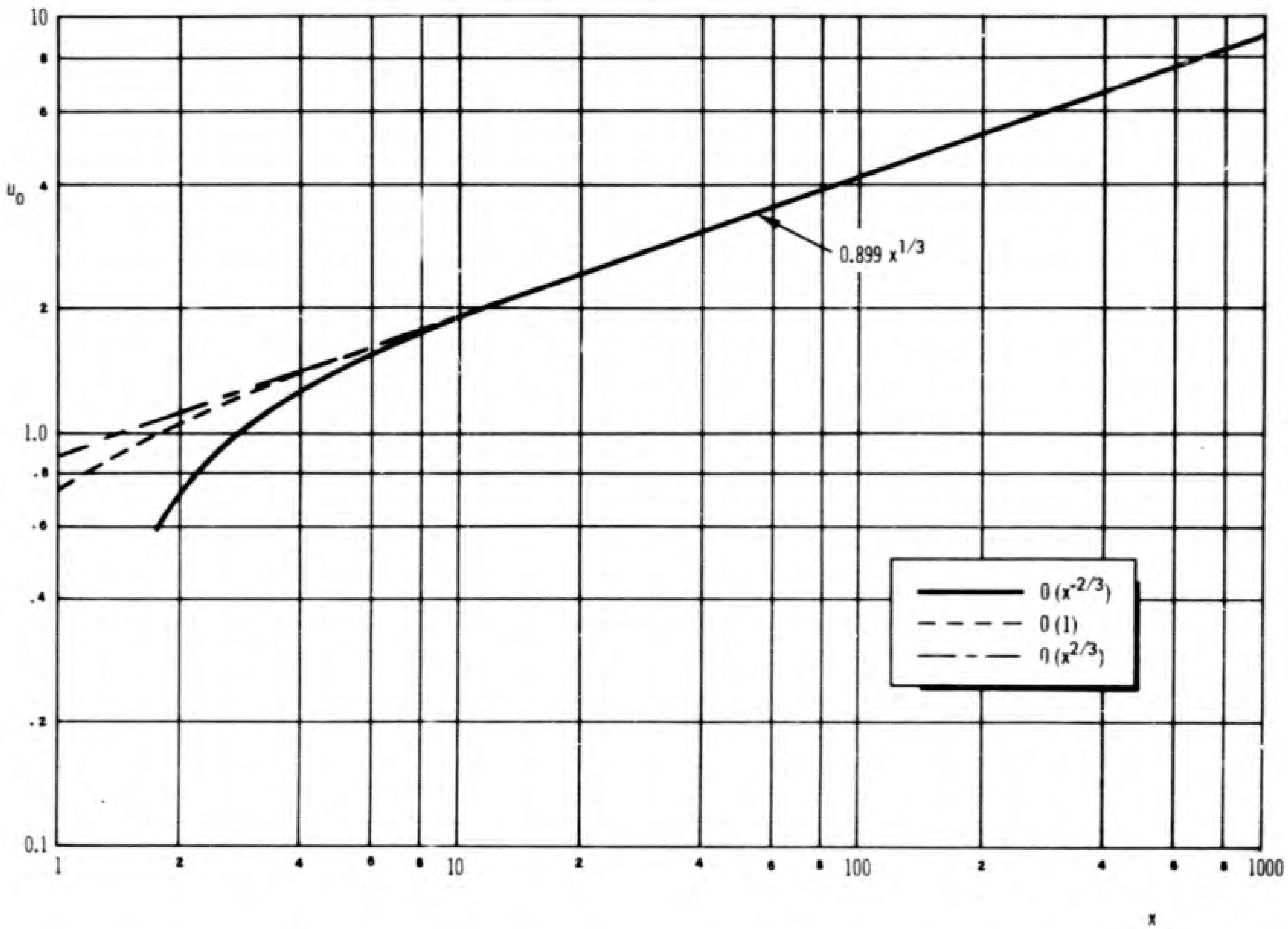

Figure 3. Centerline Velocity

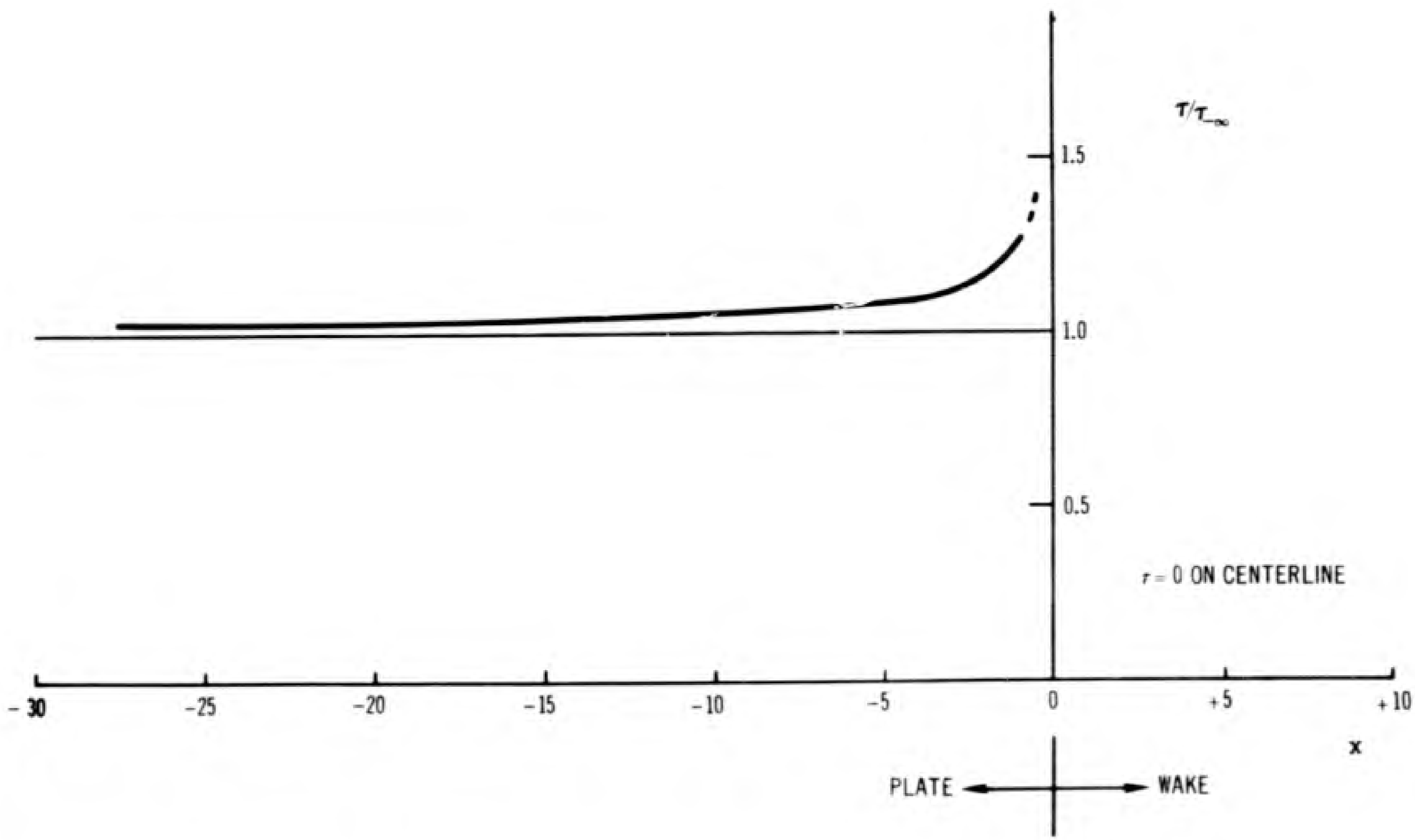

Figure 4. Skin Friction on Plate

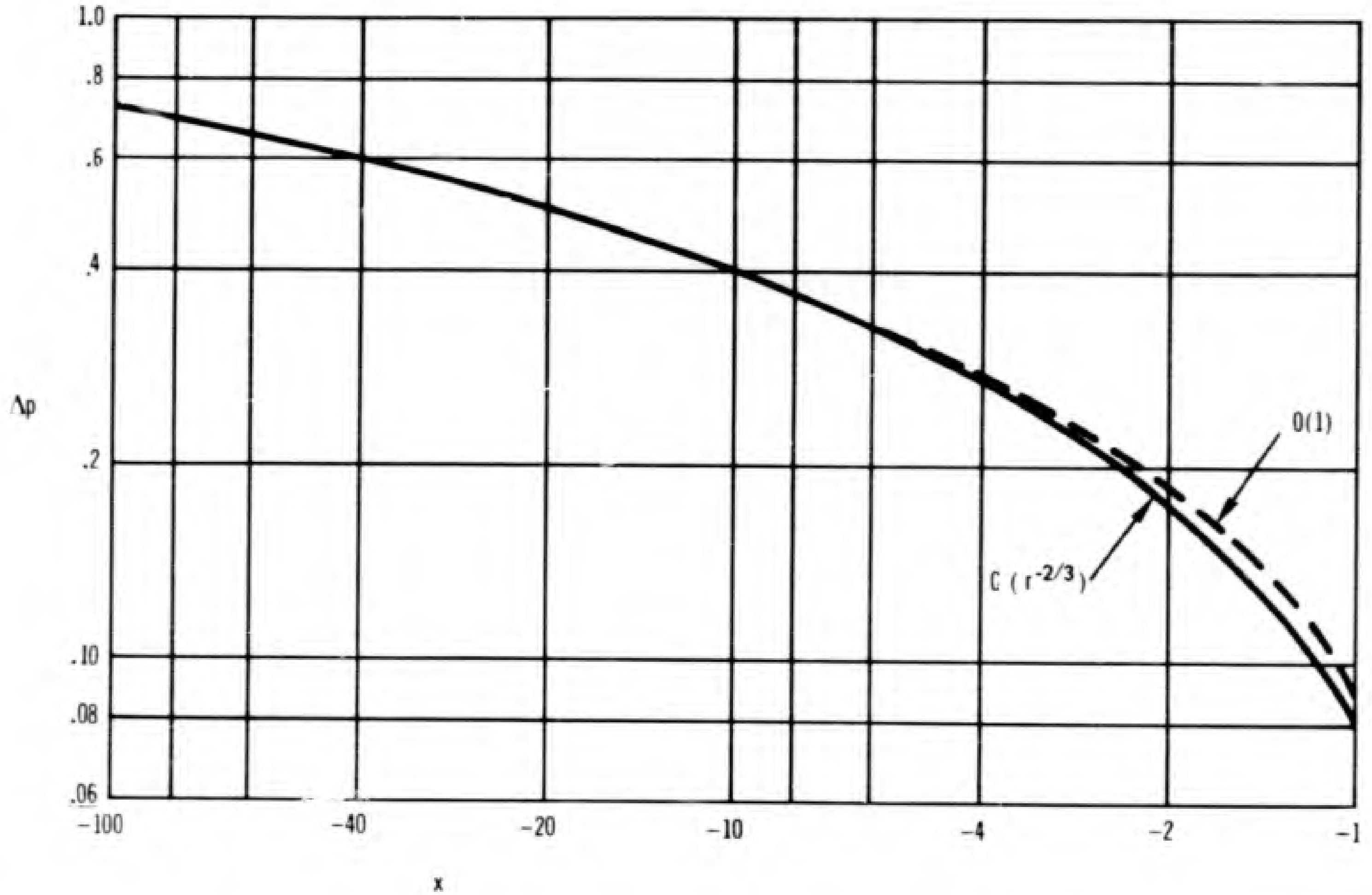

Figure 5. Pressure on Upstream Plate

PRECEDING PAGE BLANK

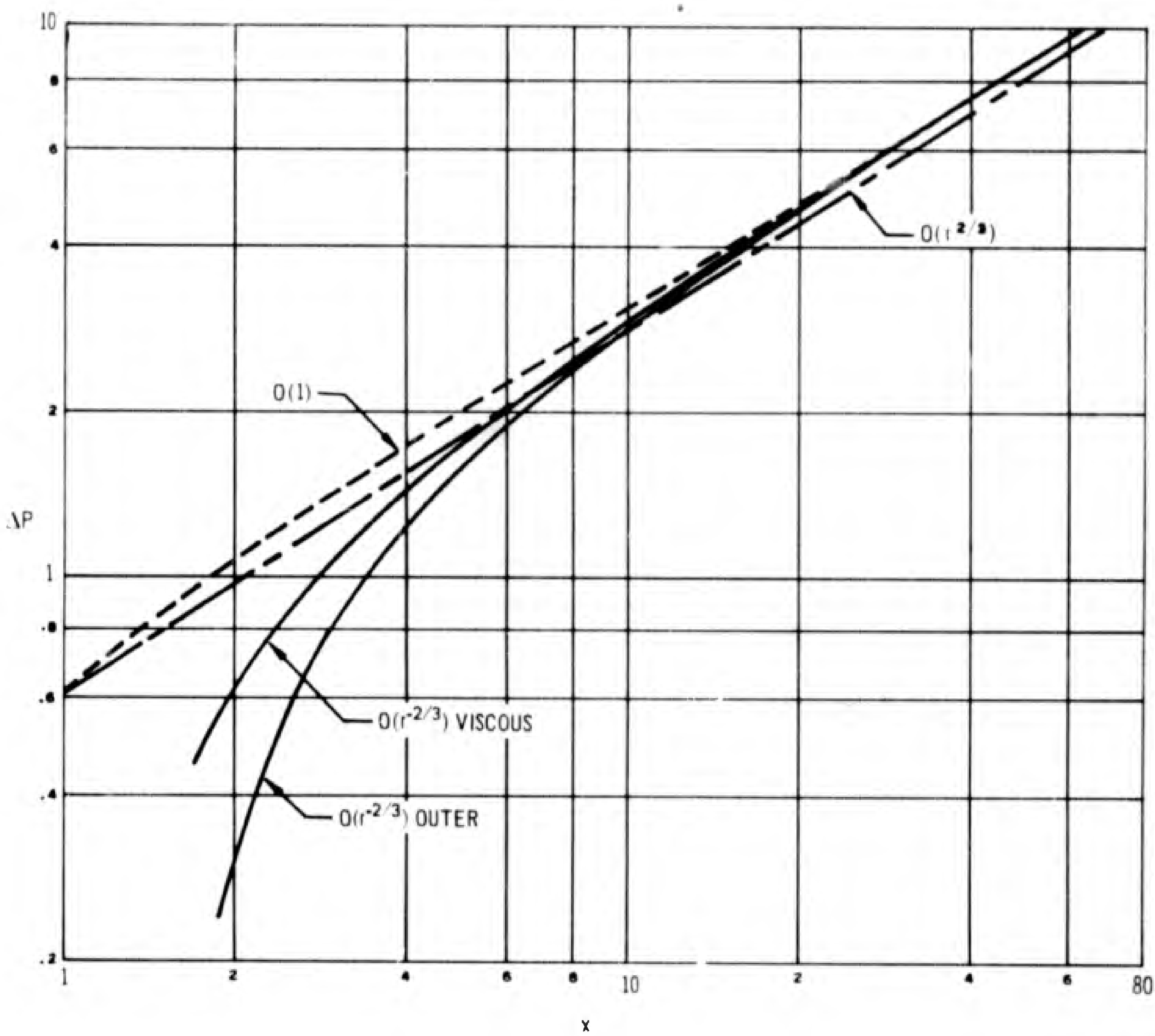

Figure 6. Pressure on Centerline of Wake, Including Extrapolation of Outer Solution

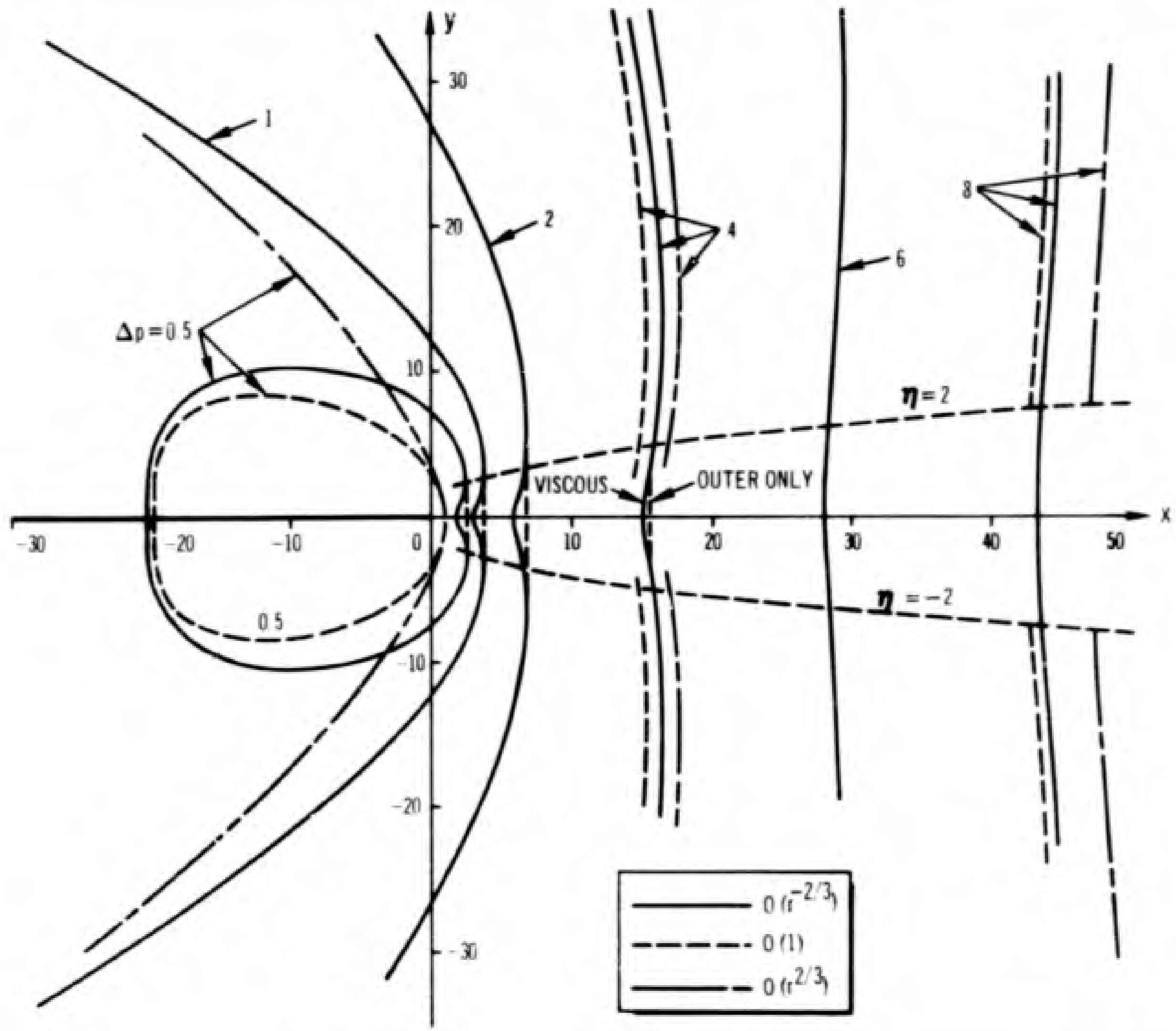

Figure 7. Isobars of Induced Pressure Field

REFERENCES

1. Abramowitz, M. and I. A. Stegun (editors). 1965 Handbook of Mathematical Functions. National Bureau of Standards Applied Mathematics Series 55, U.S. Government Printing Office, 3rd printing, Washington, D.C.

2. Goldburg, A. and S. Cheng, 1961. "The Anomaly in the Application of PLK and Parabolic Coordinates to the Trailing Edge Boundary Layer," Journal of Mathematics and Mechanics, Transactions, Vol. 10, pp. 529-536.

3. Goldstein, S. 1960. Lectures on Fluid Mechanics, John Wiley and Sons (Interscience), New York.

4. Goldstein, S. 1965. "On Backward Boundary Layers and Flow in Converging Passages," Journal of Fluid Mechanics, Vol. 21, Pt. 1, pp. 33-45.

5. Hakkinen, R. J. and Elizabeth J. O'Neil, 1965 (revised 1966). Higher Approximations for Merging of Uniform Shear Flows, Douglas Aircraft Company Paper No. 3050. Presented to the American Physical Society, Fluid Dynamics Division, Honolulu, Hawaii, Sept. 2, 1965; abstract No. BA8, Bull. Am. Phys. Soc., Ser. II, Vol. 10, No. 6, p. 685.

6. Hakkinen, R. J. and N. Rott, 1965. "Similar Solutions for Merging Shear Flows II," AIAA Journal, Vol. 3, No. 8, pp. 1553-1554.

7. Imai, I., 1966. "On the Viscous Flow Near the Trailing Edge of a Flat Plate," Proceedings of the XIth International Congress of Applied Mechanics (1964), Julius Springer-Verlag, Berlin, pp. 663-671 and subsequent private communications.

8. Li, T. Y., 1962. "Effects of Free Stream Vorticity on the Behavior of a Viscous Boundary Layer," Journal of Aeronautical Sciences, 23, pp. 1128-1129.

9. Murray, T. D. 1961. "The Boundary Layer on a Flat Plate in a Stream With Uniform Shear," Journal of Fluid Mechanics, II, Pt. 2, pp. 309-316.

10. Rott, N. and R. J. Hakkinen, 1962. "Similar Solutions for Merging Shear Flows," Journal of Aerospace Sciences, 29, pp. 1134-1135.

11. Rott, N. and R. J. Hakkinen, 1965. Numerical Solutions for Merging Shear Flows. Douglas Report No. SM-47809.

12. Toomre, A. and N. Rott, 1964. "On the Pressure Induced by the Boundary Layer on a Flat Plate in Shear Flow," Journal of Fluid Mechanics, Vol. 19, Pr. 1, pp. 1-10.