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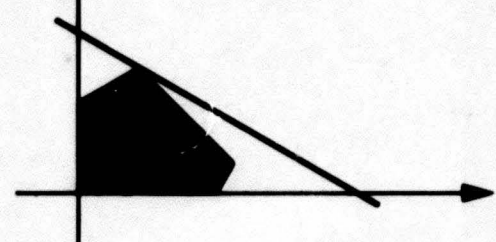
by

Peter J. Bickel and Kjell A. Doksum

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TESTS FOR MONOTONE FAILURE RATE
BASED ON NORMALIZED SPACINGS

by

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ABSTRACT

Let $X_{(1)} < \dots < X_{(n)}$ be the order statistics of a random sample from a population with density f and distribution function F such that $F(0) = 0$. Let $q(t) = f(t)(1 - F(t))^{-1}$ be the failure rate of F . In testing $H_0: q(t) = \lambda$ vs. $H_1: q(t) \neq \lambda$, Proschan and Pyke (Vth Berk. Symp.) considered certain statistics based on R_1, \dots, R_n , the ranks of the normalized sample spacings $D_i = (n - i + 1)(X_{(i)} - X_{(i-1)})$, $1 \leq i \leq n$, $X_{(0)} = 0$. They show that these statistics are asymptotically normal for fixed F and compute the efficacy of one of these statistics for selected distributions. We show that asymptotic normality holds also for sequences of alternatives approaching H_0 as $n \rightarrow \infty$ and conclude that the above efficacies yield Pitman efficiencies. The statistic $V = -\sum_{i=1}^n i R_i$ is asymptotically equivalent to the one considered by Proschan and Pyke. If $S = -\sum_{i=1}^n i \log [1 - R_i(n + 1)^{-1}]$ then the Pitman efficiency of V to S is $3/4$. If $\{f_{\theta_n}\}$ is a sequence of alternative densities, let $h(t) = [\partial \log f_{\theta}(t) / \partial \theta]_{\theta=0}$, $c_i = [\int_{a_1}^{\infty} h'(t) \exp(-t) dt] [1 - i(n + 1)^{-1}]^{-1}$, $a_1 = -\log \{1 - i(n + 1)^{-1}\}$. Then $\sum_{i=1}^n c_i D_i$ is asymptotically most powerful for $\{f_{\theta_n}\}$ if the scale parameter λ is known. However, the statistic S is nowhere asymptotically most powerful, although it is the asymptotically most powerful linear rank statistic for a suitable parametric family $\{f_{\theta_n}\}$. A comparative study of these and other statistics is given in terms of Pitman efficiencies, and Monte Carlo power.

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1. Introduction and Summary

Let F be a distribution with density f , and let $q(t) = f(t)/[1 - F(t)]$ be the failure rate of F . Tests for constant versus monotone increasing failure rate based on the ranks of the normalized spacings between the ordered observations have been considered by Proschan and Pyke [10]. They show that these statistics are asymptotically normally distributed for fixed alternatives F and compute the ratios of the efficacies of one of their rank tests to the best statistics for Weibull and Gamma alternatives.

In this paper, it is shown that asymptotic normality holds also for sequences of alternatives $\{F_{\theta_n}\}$ that approach the H_0 distribution $1 - \exp(-\lambda t)$, $t \geq 0$, as $n \rightarrow \infty$; and that the above mentioned ratios of efficacies are in fact Pitman efficiencies.

Let R_1, \dots, R_n be the ranks of the normalized spacing, $T_1 = \sum_{i=1}^n i R_i$ and $T_2 = -\sum_{i=1}^n i \log[1 - R_i/(n+1)]$. Then T_1 is asymptotically equivalent to the Proschan Pyke statistic. It is shown that the Pitman efficiency satisfies

$$e(T_1, T_2) \equiv 3/4 \quad (1.1)$$

for all sequences of alternatives $\{F_{\theta_n}\}$ and thus T_1 is asymptotically inadmissible.

Statistics that are linear in the normalized spacings and asymptotically most powerful for parametric alternatives $\{F_{\theta_n}\}$, if the scale parameter λ is known, are derived, and it is shown that the rank statistics that are

asymptotically most powerful in the class of linear rank tests, are nowhere most powerful in the class of all tests, when λ is known.

If λ is unknown, studentizing of the linear normalized spacing tests which are asymptotically most powerful for λ known lead to procedures which have only the same asymptotic power as the most powerful linear rank tests.

Unbiasedness is shown for tests that are monotone in the normalized spacings, and Monte Carlo power estimates are used to compare the various statistics with the likelihood ratio tests considered by Barlow [1].

2. Tests Monotone in the Normalized Spacings

Let X_1, \dots, X_n be a random sample from a population with a continuous distribution F satisfying $F(0) = 0$, and let $0 = X_{(0)} < X_{(1)} < \dots < X_{(n)}$ be the order statistics. The normalized sample spacings D_1, \dots, D_n are defined by $D_i = (n - i + 1)(X_{(i)} - X_{(i-1)})$, $i = 1, \dots, n$; and R_i denotes the rank of D_i among D_1, \dots, D_n . The problem is to test H_0 : " $-\log[1 - F(x)] = \lambda x$ on $[0, \infty)$ for some positive constant λ " against H_1 : " $-\log[1 - F(x)]$ is convex and not of the form λx on $[0, \infty)$." Note that the only distributions satisfying H_0 are

$$K_\lambda(x) = 1 - e^{-\lambda x}; \quad x \geq 0, \lambda > 0. \quad (2.1)$$

Under H_0 it is well known that (D_1, \dots, D_n) has the same distribution as a random sample from a population with distribution K_λ , while under the alternative there is a downward trend in the sense that $P(D_i \geq D_j) < \frac{1}{2}$ whenever $i > j$ (see [10]). One thus defines a test $\phi = \phi(D_1, \dots, D_n)$ to be *monotone in the D's* if

$$\begin{aligned} \phi(D'_1, \dots, D'_n) &\leq \phi(D_1, \dots, D_n) \text{ for} \\ \text{all } (D_1, \dots, D_n) \text{ and } (D'_1, \dots, D'_n) \text{ such} & \\ \text{that } i < j \text{ and } D'_i \geq D'_j \text{ implies } D_i \geq D_j. & \end{aligned} \quad (2.2)$$

Following van Zwet [9], one defines a distribution F_1 to have a more slowly increasing failure rate than F , written $F_1 \succeq F$, if $F_1^{-1} F$ is convex. Here $F_1^{-1} F$ is defined by $P(F_1^{-1} F(X) \leq x | F) = F_1(x)$, $x \geq 0$.

Theorem 2.1:

Monotone tests have monotone power, i.e., if ϕ is a monotone test and if

$F_1 \stackrel{c}{\geq} F$, then

$$E(\phi \mid F_1) \leq E(\phi \mid F). \quad (2.3)$$

Proof:

Since $F_1^{-1} F$ is increasing, $X'_{(i)} = F_1^{-1} F(X_{(i)})$ is the i^{th} order statistic in a random sample from a population with distribution F_1 . Let $D'_i = (n - i + 1) (X'_{(i)} - X'_{(i-1)})$, $i = 1, \dots, n$. Since $F_1^{-1} F$ is convex, $i < j$ and $D'_i \geq D'_j$ implies $D_i \geq D_j$. From (2.2) one obtains

$$\phi(D'_1, \dots, D'_n) \leq \phi(D_1, \dots, D_n), \quad (2.4)$$

and (2.3) follows upon taking expectations in (2.4).

Note that a test ϕ is *similar* if $E(\phi \mid K_\lambda)$ is independent of λ .

Thus all rank tests are similar.

Corollary 2.1:

All monotone tests are similar and unbiased.

Proof:

Similarity follows since (2.2) implies $\phi(D_1, \dots, D_n) = \phi(\lambda D_1, \dots, \lambda D_n)$.

Unbiasedness follows by letting $F_1 = K_\lambda$ in Theorem 2.1.

Corollary 2.2:

If $-c_n(i)$ and $J_n(i)$ are nondecreasing in $i = 1, \dots, n$, then the test that rejects when

$$\sum_{i=1}^n c_n(i) J_n(R_i) \geq C_1$$

has monotone power and is unbiased.

Proof:

Using the notation of Theorem 2.1, define R'_i to be the rank of D'_i among D'_1, \dots, D'_n . Then $i < j$ and $R'_i \geq R'_j$ implies $R_i \geq R_j$. From Corollary 2 of Lehmann [9] one obtains

$$\sum_{i=1}^n c_n(i) J_n(R'_i) \leq \sum_{i=1}^n c_n(i) J_n(R_i). \quad (2.6)$$

It follows that the test is monotone, and Theorem 2.1 and Corollary 2.1 apply.

Remark 2.1:

The results of this section can be used to obtain bounds on the power of monotone tests. For instance, if $\beta(V_n, F_2)$ denotes the power of the Proschan Pyke statistic $V_n =$ "number of pairs (i, j) with $i < j$ and $D_i \geq D_j$ " for the Weibull distribution F_2 , then the power satisfies $\beta(V_n, F) \geq \beta(V_n, F_2)$ for all distributions F such that $F_2^{-1}F$ is convex. $\beta(V_n, F_2)$ is given in [1].

Remark 2.2:

Barlow and Proschan [13] have shown that if $c_n(1) \geq \dots \geq c_n(n)$, the test that rejects when

$$\frac{\sum_{i=1}^n c_n(i) D_i}{\sum_{i=1}^n D_i} \geq C_2$$

is similar and unbiased.

3. Asymptotically Most Powerful Tests

Let $\{f_{\theta,\lambda} : \theta \geq 0, \lambda > 0\}$ be a class of densities such that $f_{0,\lambda}(x) = \lambda \exp(-\lambda x)$, $x \geq 0$, and such that

$$h_{\lambda}(x) = \frac{\partial}{\partial \theta} \log f_{\theta,\lambda}(x) \Big|_{\theta=0} \quad (3.1)$$

exists.

We suppose λ is a scale parameter, i.e.,

$$P_{\theta,\lambda}(\lambda X \leq t) = P_{\theta,1}(X \leq t) . \quad (3.2)$$

For test " $\theta = 0$ " versus " $\theta > 0$ ", the locally most powerful test rejects " $\theta = 0$ " for large values of

$$T_n = n^{-1/2} \sum_{i=1}^n h_{\lambda}(x_i) . \quad (3.3)$$

We will consider sequences of alternatives $\{f_{\theta_n,\lambda}\}$ such that

$$\lim n^{1/2} \theta_n = b \text{ for some } 0 \leq b < \infty . \quad (3.4)$$

A sequence $\{f_{\theta_n,\lambda}\}$ is said to be *contiguous* to $f_{0,\lambda}$ (in the sense of LeCam - Hájek) if for any sequence of random variables $R_n(X_1, \dots, X_n)$, $R_n \rightarrow 0$ in P_0 probability implies $R_n \rightarrow 0$ in P_{θ_n} probability, where P_{θ} denotes the probability distribution of X_1, \dots, X_n if $f_{\theta,\lambda}$ is true. The following condition implying contiguity for sequences as in (3.4) can be obtained from LeCam [8].

$$(a) \quad \partial f_{\theta,\lambda}(x) / \partial \theta \neq 0 \text{ whenever } f_{\theta,\lambda}(x) > 0 . \quad (3.5)$$

$$(b) \quad \text{For some } \delta > 0 \text{ and all } \theta \in [0, \delta], \quad 0 < H(\theta) = \int_0^{\infty} [\partial f_{\theta,\lambda}(x) / \partial \theta]^2 [f_{\theta,\lambda}(x)]^{-1} dx < \infty, \text{ and } H(\theta) \text{ is continuous in } \theta .$$

and $H(\theta)$ is continuous in θ

$$(c) \lim_{|h| \rightarrow 0} \int_0^{\infty} \left\{ h^{-1} \left[f_{\theta+h, \lambda}^{1/2}(x) - f_{\theta, \lambda}^{1/2}(x) \right] - \frac{1}{2} \left[\frac{\partial f_{\theta, \lambda}(x)}{\partial \theta} \right] f_{\theta, \lambda}^{-1/2}(x) \right\}^2 dx = 0 \text{ for } \theta \in [0, \delta].$$

An easy sufficient condition implying (3.5) (b) and (c) is

$$\int_0^{\infty} \sup \left\{ \left[\frac{\partial f_{\theta, \lambda}(x)}{\partial \theta} \right]^2 [f_{\theta, \lambda}(x)]^{-1} : 0 \leq \theta \leq \delta \right\} dx < \infty \quad (3.6)$$

for some $\delta > 0$.

It also follows from LeCam's work that under condition (3.5) we have

$$\begin{aligned} & \left[b n^{-1/2} \sum_{i=1}^n h_{\lambda}(X_i) - \frac{b^2}{2} E_{0, \lambda}(h_{\lambda}^2(X_1)) \right] \\ & - \sum_{i=1}^n \left\{ \log f_{\theta_n, \lambda}(X_i) - \log f_{0, \lambda}(X_i) \right\} \rightarrow 0 \end{aligned} \quad (3.7)$$

in P_0 and hence in P_{θ_n} probability.

According to Wald [12], a sequence of level α tests $\{\phi_n\}$ is said to be asymptotically most powerful (λ known) if

$$\lim_n \sup \{ E_{\theta, \lambda}(\psi_n) - E_{\theta, \lambda}(\phi_n) : \theta > 0, E_{0, \lambda}(\psi_n) \leq \alpha \} = \alpha.$$

Using contiguity we can prove Wald's [12] main theorem under weaker conditions.

Theorem 3.1:

Suppose (3.5) holds and,

$$n^{-1/2} \sum_{i=1}^n h_{\lambda}(X_i) \rightarrow \infty \quad (3.9)$$

in $P_{\theta_n, \lambda}$ probability if $n^{1/2} \theta_n \rightarrow \infty$. Then the sequence of level α tests

ϕ_n which reject for suitably large values of $\sum_{i=1}^n h_\lambda(X_i)$ is asymptotically most powerful.

Proof:

The result is an easy consequence of the Neyman - Pearson Lemma, (3.7) and (3.9) . ||

It is easy to see that the ϕ_n are locally most powerful.

Our next result employs Theorem 4.1 of the following section. We need the notation

$$\sigma^2(h_1) = \int_0^{\infty} h_1^2(x) e^{-x} dx, \text{ and} \quad (3.10)$$

$$S_n = n^{-\frac{1}{2}} \sum_{i=1}^n a\left(\frac{i}{n+1}\right) (D_i - \lambda^{-1}), \text{ where} \quad (3.11)$$

$$a(u) = (1-u)^{-1} \int_{-\log(1-u)}^{\infty} h_1'(x) e^{-x} dx, \quad 0 < u < 1. \quad (3.12)$$

Theorem 3.2:

If (3.4) and (3.5) hold, and if h_λ of (3.1) satisfies the conditions of Theorem 4.1, then $\lambda S_n \sigma^{-1}(h_1)$ has under H_0 asymptotically a standard normal distribution. Moreover, the test that rejects for large values of S_n is asymptotically most powerful.

Proof:

Since λ is scale parameter, and since S_n is linear, one obtains $P_{\theta, \lambda}(\lambda S_n \leq x) = P_{\theta, 1}(S_n \leq x)$. Thus one may assume without loss of generality that $\lambda = 1$. Since $\sigma^2(h_1)$ is the asymptotic variance of T_n , the asymptotic normality follows at once from Theorem 4.1. By Theorem 4.1, the test rejecting for large values of T_n is asymptotically most powerful. Thus

it is enough to show that $T_n - S_n \rightarrow 0$ in P_{θ_n} probability. However, this follows at once from the contiguity condition (3.5) and Theorem 4.1. ||

Remark:

Theorem 3.2 and the contiguity condition implies the asymptotic normality of S_n also under sequences of alternatives for which (3.4) is satisfied, i.e., if

$$\mu_n(h_1) = n^{1/2} \int_0^{\infty} h_1(x) f_{\theta,1}(x) dx, \quad (3.13)$$

then $[\lambda S_n - \mu_n(h_1)]/\sigma(h_1)$ has asymptotically a standard normal distribution.

4. Linear Approximations to Locally and Asymptotically most Powerful Statistics

As we have seen, locally and asymptotically most powerful test statistics for parametric alternatives are of the form

$$T_n = n^{-1/2} \sum_{i=1}^n h(X_i) = n^{-1/2} \sum_{i=1}^n h(X_{(i)})$$

for some function h on $[0, \infty)$. In order to compute asymptotic efficiencies, we will need the following result which gives conditions under which T_n can be approximated by a statistic linear in the D_i 's, and given by,

$$S_n = n^{-1/2} \sum_{i=1}^n a\left(\frac{i}{n+1}\right) (D_i - 1) \quad (4.2)$$

where,

$$a(u) = (1 - u)^{-1} \int_{-\log(1-u)}^{\infty} h'(x) e^{-x} dx, \quad 0 < u < 1.$$

Theorem 4.1:

Let h be any function such that

- (i) h' is continuous on $(0, \infty)$,
- (ii) $\int_0^{\infty} h(t) e^{-t} dt = 0$, $0 < \int_0^{\infty} h^2(t) e^{-t} dt < \infty$,
- (iii) either one of the following holds

- (a) $h'(-\log(1 - u))$, $0 < u < 1$, satisfies assumption E of Chernoff, Gastwirth and Joh. (1967) and

$$\int_0^{\infty} |h'(v)| e^{-\frac{v}{2}} (1 - e^{-v})^{\frac{1}{2}} dv < \infty.$$

(b) $h'(t)$ changes sign only a finite number of times as $t \rightarrow 0$ or ∞ and h' does not vanish infinitely often.

Moreover, suppose that the X 's have the exponential density $\exp(-x)$, $x \geq 0$.

Then $T_n - S_n$ tends to zero in probability.

The proof is deferred to the appendix; Section 7.

Remark:

It may also be shown that if (i), (ii) and (iii) (b) hold and $\lambda = 1$, then $E_{0,1}(T_n - S_n)^2 \rightarrow 0$.

5. Asymptotic Normality and Inefficiency of Rank Statistics

Let Pitman asymptotic efficiency be defined as usual (e.g., Hodges and Lehmann [6]). In this section we shall show the asymptotic normality of statistics of the form

$$W_n = n^{-1/2} \sum_1^n (c_i - \bar{c}) J\left(\frac{R_i}{n+1}\right), \quad \bar{c} = n^{-1} \sum_1^n c_i, \quad (5.1)$$

and compute their Pitman efficiencies with respect to the asymptotically most powerful statistics of Section 4. Let h_λ be as defined in (3.1) and let

$$a_i = a\left(\frac{i}{n+1}\right) \quad \text{where} \quad a(u) = (1-u)^{-1} \int_{-\log(1-u)}^{\infty} h_1'(x) e^{-x} dx. \quad \text{Then} \quad (5.2)$$

$$S_n = n^{-1/2} \sum_1^n a_i (D_i - \lambda^{-1}) \quad (5.3)$$

is the asymptotically most powerful statistic of Section 3.

Theorem 5.1:

If (3.4) and (3.5) hold, if h_λ satisfies the conditions of Theorem 4.1 of this paper, and if $\{c_i\}$ and $\left\{J\left(\frac{i}{n+1}\right)\right\}$ satisfy the conditions of Theorem 4.1 of Hájek [5], then for the sequence of alternatives given in (3.4)

$[W_n - \mu(W_n)]/\sigma(W_n)$ has asymptotically a standard normal distribution, where

$$\mu(W_n) = bn^{1/2} \left[\frac{1}{n} \sum_1^n a_i (c_i - \bar{c}) \right] \int_0^1 J(u) [-\log(1-u) - 1] du, \quad \text{and} \quad (5.4)$$

$$\sigma^2(W_n) = \left[\frac{1}{n} \sum_1^n (c_i - \bar{c})^2 \right] \left[\int_0^1 J^2(u) du - \left(\int_0^1 J(u) du \right)^2 \right] \quad (5.5)$$

Proof:

Since the ranks are scale invariant, assume without loss of generality that $\lambda = 1$. The results of Hajek [5] and the contiguity condition imply that

$$W_n - Q_n \rightarrow 0 \text{ in } P_{\theta_n, 1} \text{ probability where,} \quad (5.6)$$

$$Q_n = n^{-1/2} \sum_1^n (c_i - \bar{c}) J(1 - \exp(-\lambda D_i)) . \quad (5.7)$$

It follows from the Lindeberg-Feller Theorem that under H_0 , the joint limiting distribution of $bS_n/\sigma_n(bS_n)$ and $Q_n/\sigma(W_n)$ is the bivariate normal distribution with means zero, variances one, and covariance

$$b \left[n^{-1} \sum_1^n a_i (c_i - \bar{c}) \right] \left[\int_0^1 J(u) [-\log(1 - u) - 1] du \right]$$

$$\text{where } \sigma_n^2(bS_n) = b^2 n^{-1} \sum_1^n a_i^2 .$$

The result now follows from LeCam's third lemma (see Hájek and Sidák ([4] p.208)).

For each vector $c = (c_1, \dots, c_n)$, define

$$V_n(c) = \frac{1}{n} \sum_1^n (c_i - \bar{c})^2 \quad (5.8)$$

and for two vectors c and a define

$$\text{Cor}_n^2(a, c) = \frac{\left[\frac{1}{n} \sum_1^n a_i (c_i - \bar{c}) \right]^2}{V_n(a) V_n(c)} . \quad (5.9)$$

Then Theorem 5.1 yields

Corollary 5.1:

Under the conditions of Theorem 5.1, the Pitman asymptotic relative efficiency of W_n to S_n is

$$e(W, S) = \text{Cor}^2(J(U), -\log(1 - U)) \lim_{n \rightarrow \infty} \left[\text{Cor}_n^2(a, c) \frac{V_n(a)}{\frac{1}{n} \sum_{i=1}^n a_i^2} \right] \quad (5.10)$$

where U is an uniform $(U(0, 1))$ random variable.

It is clear from (5.9) that one obtains the most efficient linear rank statistic W_n by taking $J(u) = -\log(1 - u)$ and $c_i = a_i$. Note that the choice of J is independent of the alternative densities $\{f_{\theta, \lambda}\}$. Moreover, $e(W, S)$ is less than one when

$$\text{lim}(\bar{a})^2 = \left[\int_0^{\infty} x h_1(x) e^{-x} dx \right]^2 = \text{Cor}^2(X_1, h_1(X_1)) \neq 0. \quad (5.11)$$

This is equivalent to having $\bar{X}_n = n^{-1} \sum_{i=1}^n X_i$ correlated with the locally most powerful statistic T_n . It will be shown in the proof of the next result that $\text{Cor}(\bar{X}_n, T_n) < 0$ when the failure rate is increasing. Let

$$q_{\lambda}(x, \theta) = f_{\theta, \lambda}(x) [1 - F_{\theta, \lambda}(x)]^{-1}, \quad x \geq 0, \quad (5.12)$$

denote the failure rate of $F_{\theta, \lambda}$. We assume in what follows that $\partial q_{\lambda}(x, \theta) / \partial \theta$ exists and is continuous in (x, θ) for $0 \leq \theta \leq \delta$, $x > 0$, some $\delta > 0$. Let

$$L(x) = \partial q_{\lambda}(x, \theta) / \partial \theta \Big|_{\theta = 0}. \quad (5.13)$$

Increasing failure rate clearly implies

$$L(x) \geq 0 \quad (5.14)$$

for all $x > 0$. We have

Theorem 5.2:

Suppose the conditions of Theorem 5.1 hold, that (5.14) is satisfied with strict inequality for x in some set of positive measure and that,

$$\int_0^{\infty} t e^{-t} L(t) dt < \infty. \quad (5.15)$$

Then each linear rank statistic of the form (5.1) is inefficient, i.e.,

$$e(W, S) < 1. \quad (5.16)$$

Proof:

Without loss of generality let $\lambda = 1$. Now,

$$f_{\theta, 1}(x) = q_1(x, \theta) \exp \left\{ - \int_0^x q_1(t, \theta) dt \right\}. \quad (5.17)$$

Since $\partial q_1(x, \theta) / \partial \theta$ is continuous in (x, θ) we have differentiating under the integral sign,

$$h_1(x) = L(x) - \int_0^x L(t) dt. \quad (5.18)$$

Using Fubini's theorem and (5.15) we get if $\lambda = 1$,

$$\text{cov}(X_1, h_1(X_1)) = \int_0^{\infty} x h(x) e^{-x} dx = - \int_0^{\infty} e^{-x} L(x) dx < 0 \quad (5.19)$$

by our assumptions. ||

Example 5.1:

An alternative for which there is an efficient rank test is provided by

$$f_{\theta,1}(x) = (1 - \theta)^{-1} [1 + \theta(-2x + \frac{1}{2} x^2)] e^{-x}, \theta < \frac{1}{2}. \quad (5.20)$$

For this density, the failure rate is not monotone. It is easy to see that $\text{Cor}(X_1, h(X_1)) = 0$. The efficient rank test rejects for large values of

$$n^{-\frac{1}{2}} \sum_1^n \left[\log \left(1 - \frac{i}{n+1} \right) + 1 \right] \left[-\log \left(1 - \frac{R_i}{n+1} \right) \right] \quad (5.21)$$

We conjecture that under the hypotheses of our theorem there is no sequence of rank tests which is asymptotically most powerful in the sense of Wald. Our best approximation to this result is Theorem 5.3.[†]

Definition:

We say a sequence of statistics $M_n(X_1, \dots, X_n)$ is *efficient* if

$$\text{as. cor.} \left(M_n, \sum_{i=1}^n \left\{ \log f_{\theta_n, \lambda}(X_i) - \log f_{0, \lambda}(X_i) \right\} \right) = 1 \quad (5.22)$$

for every sequence of alternatives θ_n satisfying (3.4). By as. cor. we refer to the asymptotic correlation computed under H_0 .

It is easy to see that if (4.15) holds and

$$R_n \rightarrow \infty \quad (5.23)$$

in P_{θ_n} probability if $n^{\frac{1}{2}} \theta_n \rightarrow \infty$, then an efficient sequence of M_n 's can be used to construct asymptotically most powerful tests for every level α .

[†]E. Torgersen, using general considerations, has proved that our conjecture is implied by Theorem 5.3. A more direct proof using the special structure of this problem may also be given.

Theorem 5.3:

Under the assumption of Theorem 5.2 no efficient sequence of rank statistics exists.

We need an elementary lemma.

Lemma 5.4:

Let U_n be a sequence of random variables such that $U_n \xrightarrow{P} 0$.

Denote by $U_n^{(k)}$ the random variable equal to U_n if $|U_n| \leq k$ and equal to $k \operatorname{sgn} U_n$ otherwise. Then there exists $k_n \uparrow \infty$, such that,

$$E \left[U_n^{(k_n)} \right]^2 \rightarrow 0. \quad (5.24)$$

Proof:

Let k_n be the largest k such that $E \left[U_n^{(k)} \right]^2 \leq \frac{1}{n}$.

Clearly $k_n \uparrow \infty$. ||

Proof of Theorem:

Suppose $M_n(R_1, \dots, R_n)$ is an efficient sequence of rank statistics. Let $\theta_n = n^{-\frac{1}{2}}$, $\lambda = 1$. Without loss of generality suppose,

$$M_n - \sum_{i=1}^n \left[\log f_{\theta_n, 1}(X_i) - \log f_{0, 1}(X_i) \right] - \frac{1}{2} E_{0, 1}(h_1^2(X_1)) \rightarrow 0 \quad (5.25)$$

in P_{θ_n} probability. By (3.7) and (3.12)

$$M_n - S_n \rightarrow 0 \quad (5.26)$$

in P_0 probability. By the lemma there exist k_n such that

$$E_{0, 1} \left[(M_n - S_n)^{(k_n)} \right]^2 \rightarrow 0. \quad (5.27)$$

But

$$E_{0,1} \left[(M_n - S_n)^{(2k_n)} \right]^2 \geq E_{0,1} M_n^{(k_n)} - S_n^{(k_n)} \quad (5.28)$$

and

$$E_{0,1} \left(S_n - S_n^{(k_n)} \right)^2 \rightarrow 0. \quad (5.29)$$

From (5.27) - (5.29) we conclude that

$$E_{0,1} (S_n - E_{0,1}(S_n | R_1, \dots, R_n))^2 \rightarrow 0. \quad (5.30)$$

But,

$$U_n = E_{0,1}(S_n | R_1, \dots, R_n) = n^{-\frac{1}{2}} \sum a\left(\frac{i}{n+1}\right) E_{0,1}(X_{(R_i)}) \quad (5.31)$$

is a statistic of the form (5.1). Theorem 5.2 and (5.30) are therefore incompatible. ||

We conjecture that rank tests which reject for large values of $E_{0,1}(S_n | R_1, \dots, R_n)$ are asymptotically most powerful among all rank tests for the given model whatever be λ .

It is not difficult to see by using the method of Hoeffding [7] that the locally most powerful rank test rejects the null hypothesis for large values of the statistic

$$\begin{aligned} W_n &= E_{0,1}(T_n | R_1, \dots, R_n) \\ &= n^{-\frac{1}{2}} \sum_{i=1}^n E_{0,1} \left[h_i \left(\sum_{j=1}^i X_{(R_j)} (n-j+1)^{-1} \right) \right]. \end{aligned} \quad (5.32)$$

By the remark following Theorem 4.1 since

$$E_{0,1}(U_n - W_n)^2 \leq E_{0,1}(S_n - T_n)^2 \quad (5.33)$$

U_n and W_n are asymptotically equivalent. This lends further support to our conjecture.

These rank statistics are of course usable even if λ is unknown, the situation which primarily concerns us, while the optimal statistics of Section 4 depend on λ and do not lead to similar tests. If we use the method of Barlow [1] and Nadler and Eilbott [14] and consider studentized statistics of the form

$$S_n^* = \frac{n^{1/2} \sum_{i=1}^n a\left(\frac{i}{n+1}\right) D_i}{\sum_{i=1}^n D_i} - n^{1/2} \sum_{i=1}^n a\left(\frac{i}{n+1}\right), \quad (5.34)$$

it is not difficult to show that for any fixed λ under the null hypothesis,

$$S_n^* - n^{-1/2} \lambda \sum_{i=1}^n \left(a\left(\frac{i}{n+1}\right) - \bar{a} \right) (D_i - \lambda^{-1}) \rightarrow 0 \quad (5.35)$$

in probability and hence by (5.6) the best studentized test of this form is *asymptotically equivalent* under contiguous alternatives to the rank test based on $M_n(R_1, \dots, R_n)$ where,

$$M_n(R_1, \dots, R_n) = - \sum_{i=1}^n a\left(\frac{i}{n+1}\right) \log\left(1 - \frac{R_i}{n+1}\right). \quad (5.36)$$

We conjecture that the asymptotically most powerful linear rank tests, asymptotically equivalent to the studentized asymptotically most powerful linear spacings tests, are in fact asymptotically most powerful among all level α tests which are similar for the hypothesis H_0 , λ unknown.

Finally we remark that the rank statistic whose Monte Carlo power is studied

in Section 6 is not U_n but an asymptotically equivalent simpler form given by,

$$U_n^* = n^{-\frac{1}{2}} \sum_{i=1}^n -L\left(-\log 1 - \left(\frac{i}{n+1}\right)\right) - \log\left(1 - \frac{R_i}{n+1}\right).$$

suggested by the discussion following Corollary 5.1.

6. Applications of the Theory and Monte Carlo Results

The results of the previous sections will now be applied to specific alternatives and specific statistics. The weights c_i of (5.1) will be of the form

$$c_i = c\left(\frac{i}{n+1}\right) \text{ for some function } c \text{ on } (0, 1) \quad (6.1)$$

and the efficiency (5.10) will be

$$e(W, S) = \text{Cor}^2(J(U), -\log(1 - U)) \text{Cor}^2(a(U), c(U)) \frac{\text{Var}(a(U))}{E(a^2(U))} \quad (6.2)$$

where U is an uniform $(U(0, 1))$ random variable.

The statistics to be considered are:

$$W_0 = \sum_1^n -\left(\frac{i}{n+1}\right)\left(\frac{R_i}{n+1}\right)$$

$$W_1 = \sum_1^n -\left(\frac{i}{n+1}\right) \left[-\log\left(1 - \frac{R_i}{n+1}\right)\right]$$

$$W_2 = \sum_1^n \left[-\log\left(1 - \frac{i}{n+1}\right)\right] \left[-\log\left(1 - \frac{R_i}{n+1}\right)\right]$$

$$W_3 = \sum_1^n -\left\{\log\left[-\log\left(1 - \frac{i}{n+1}\right)\right]\right\} \left[-\log\left(1 - \frac{R_i}{n+1}\right)\right]$$

$$W_4 = \sum_1^n g\left(\frac{i}{n+1}\right) \left[-\log\left(1 - \frac{R_i}{n+1}\right)\right]$$

$$S_1 = \sum_1^n -\left(\frac{i}{n+1}\right) D_i$$

$$S_2 = \sum_1^n \left[\log\left(1 - \frac{i}{n+1}\right)\right] D_i$$

$$S_3 = \sum_{i=1}^n -\left\{ \log \left[-\log \left(1 - \frac{j}{n+1} \right) \right] \right\} D_i$$

$$S_4 = \sum_{i=1}^n g\left(\frac{j}{n+1}\right) \left[-\log \left(1 - \frac{R_j}{n+1} \right) \right]$$

$$\text{where } g(t) = (1-t)^{-1} \int_{-\log(1-t)}^{\infty} x^{-1} e^{-x} dx .$$

Large values are significant.

The alternative densities to be considered are listed below for $\lambda = 1$;
to obtain the general form, make the transformation $f(x) \rightarrow \lambda f(\lambda x)$.

$$f_{\theta}^{(1)}(x) = [1 + \theta(1 - e^{-x})] \exp \{-[x + \theta(x + e^{-x} - 1)]\} \quad (\text{Makeham})$$

$$f_{\theta}^{(2)}(x) = (1 + \theta x) \exp\{-(x + \frac{1}{2}\theta x^2)\} \quad (\text{linear F.R.})$$

$$f_{\theta}^{(3)}(x) = (1 + \theta) x^{\theta} \exp\{-x^{(1+\theta)}\} \quad (\text{Weibull})$$

$$f_{\theta}^{(4)}(x) = [x^{\theta} e^{-x}] / \Gamma(1 + \theta) \quad (\text{Gamma})$$

for each density, $x \geq 0$, $\theta \geq 0$; and the null hypothesis is obtained for $\theta = 0$. Each of these densities have increasing failure rates for $\theta > 0$. Note that $f_{\theta}^{(2)}$ has the linear F.R. (failure rate) $1 + \theta x$, while $f_{\theta}^{(1)}$ has the failure rate $1 + \theta(1 - e^{-x})$.

From Theorem 3.1 it follows that S_i is asymptotically most powerful for $f_{\theta}^{(i)}$ when λ is known, $i = 1, 2, 3, 4$. Theorem 5.1 implies that W_1 is asymptotically most powerful for $f_{\theta}^{(i)}$ in the class of linear rank statistics, $i = 1, 2, 3, 4$.

W_0 is asymptotically equivalent to the Proschan-Pyke statistic V_n (see Remark 3.1). It is uniformly improved asymptotically by W_1 . We now list the efficiencies of W_1 to the asymptotically most powerful statistic of Section 3.

The efficiencies are given in general as functions of $h(x) = h_1(x) = [\partial \log f_{\theta,1}(x)/\partial \theta]_{\theta=0}$. They are always independent of λ .

$$e(W_0) = \frac{9 \left[\int_0^{\infty} h'(x) e^{-x} (\frac{1}{2}x - 1 + e^{-x}) dx \right]^2}{\int_0^{\infty} h^2(x) e^{-x} dx}$$

$$= \frac{9 \left[\int_0^{\infty} h(x) e^{-x} (\frac{1}{2}x - 3/2 + 2e^{-x}) dx \right]^2}{\int_0^{\infty} h^2(x) e^{-x} dx}$$

$$e(W_1) = \frac{4}{3} e(W_0)$$

$$e(W_2) = \frac{\left[\int_0^{\infty} h^1(x) e^{-x} (\frac{1}{2}x^2 - x) dx \right]^2}{\int_0^{\infty} h^2(x) e^{-x} dx}$$

$$e(W_3) = \frac{\left[\int_0^{\infty} h^1(x) e^{-x} x(\log x + \gamma - 1) dx \right]^2}{\left(\frac{\pi^2}{6} - 2\gamma + 1 \right) \int_0^{\infty} h^2(x) e^{-x} dx}$$

$$e(W_4) = \frac{\int_0^{\infty} x^{-1} e^{-x} \int_0^{1-e^{-x}} (1-t)^{-1} (a(t) - \bar{a}) dt dx}{\left(\frac{\pi^2}{6} - 1 \right) \int_0^{\infty} h^2(x) e^{-x} dx}$$

where $\gamma = .5772$ is Euler's constant,

$$a(t) = (1-t)^{-1} \int_{-\log(1-t)}^{\infty} h'(t) e^{-t} dt, \text{ and } \bar{a} = \int_0^1 a(t) dt.$$

As remarked in Section 5, if

$$S_i^* = S_i/n^{-1} \sum_{j=1}^n D_j - n^{1/2} \sum_{j=1}^n a\left(\frac{j}{n+1}\right), \quad i = 1, 2, 3, 4 \quad (6.3)$$

are the studentized linear spacings statistics, then S_i^* have the same efficiency as W_i , i.e., $e(S_i^*) = e(W_i)$ $i = 1, 2, 3, 4$. These efficiencies are given in Table 6.1.

Tables 6.2 deal with a Monte Carlo study of the powers of the W_i and S_i^* statistics for the linear F.R., Weibull and Gamma distributions.

From the tables, the following conclusions are apparent: 1) The rank tests are uniformly less powerful than the corresponding studentized linear spacings tests. 2) Of all the tests considered in the above tables, the total time on test statistic S_i^* is generally best both on the basis of asymptotic efficiency and Monte Carlo power. 3) Barlow [1] has shown that for Weibull and Gamma alternatives, S_i^* is much better than the IFR likelihood ratio test and slightly worse than the IFRA likelihood ratio test.

Table 6.2 also gives the Monte Carlo power of the test which rejects for large values of

$$T_3 = \frac{\sum_{i=1}^n (1 - x_i) \log x_i}{\sum x_i}.$$

Using the results of Section 3, we see that this test is the studentized version of the locally most powerful test for the Weibull distribution. The table shows the nonrobustness of this test.

The last row in Table 6.2 gives the factor by which the efficiencies have to be multiplied in order to obtain the efficiencies with respect to the best linear rank tests (or the best studentized linear spacings tests).

TABLE 6.1 $e(W_i) = e(S_i^*)$

Statistic $f_{\theta}^{(i)}$	Makeham	Linear F.R.	Weibull	Gamma
W_0	$\frac{3}{16} = .1875$	$\frac{9}{32} = .2813$	$\frac{9(\log 2)^2}{4\gamma_0} = .5927$	$\frac{54(\log 2 - \frac{1}{2})^2}{\pi^2} = .2041$
W_1	$\frac{1}{4} = .25$	$\frac{3}{8} = .375$	$\frac{3(\log 2)^2}{\gamma_0} = .7903$	$\frac{72}{\pi^2}(\log 2 - \frac{1}{2})^2 = .2721$
W_2	$\frac{3}{16} = .1875$	$\frac{1}{2} = .50$	$\frac{1}{\gamma_0} = .5483$	$\frac{3}{2\pi^2} = .1520$
W_3	$\frac{3(\log 2)^2}{4(\gamma_0 - \gamma^2)} = .2416$	$\frac{1}{2(\gamma_0 - \gamma^2)} = .3355$	$\frac{1 - \gamma^2}{\gamma_0} = .8173$	$\frac{6\gamma^2(\pi^2 - 6)}{\pi^4 \gamma_0} = .0435$
W_4	$\frac{18(\log 2 - \frac{1}{2})^2}{\pi^2 - 6} = .1735$	$\frac{3}{4(\pi^2 - 6)} = .1938$	$\frac{6\gamma^2(\gamma_0 - \gamma^2)}{\pi^2 \gamma_0} = .0908$	$1 - \frac{6}{\pi^2} = .3921$
Rank efficiency factor	4	2	$\frac{\gamma_0}{\gamma_0 - \gamma^2} = 1.2235$	$\frac{\pi^2}{2} = 2.5505$

$$\gamma = .5772 \dots = \text{Euler's constant}, \quad \gamma_0 = (\gamma - 1)^2 + \frac{\pi^2}{6} = 1.8237$$

TABLE 6.2: Monte Carlo power based 2,000 trials for samples of size 10 from Linear F. R., Weibull and Gamma distributions for significance level $\alpha = .01, .05$ and $.10$.

LINEAR F. R.

α	.01	.05	.10	.01	.05	.10	.01	.05	.10	.01	.05	.10						
θ	.50			1.00			2.50			4.00			6.00			10.00		
W_1	.022	.104	.198	.032	.136	.263	.061	.211	.359	.068	.248	.415	.085	.300	.478	.094	.334	.518
W_2	.022	.105	.190	.032	.126	.236	.053	.201	.342	.059	.222	.387	.074	.263	.435	.077	.294	.482
W_3	.016	.109	.200	.022	.153	.261	.047	.266	.361	.054	.259	.420	.066	.322	.470	.070	.350	.516
S_1^*	.026	.105	.198	.046	.159	.265	.082	.262	.394	.101	.302	.453	.121	.350	.522	.152	.414	.582
S_2^*	.019	.112	.179	.036	.157	.249	.064	.263	.369	.080	.295	.421	.090	.341	.485	.117	.405	.548
S_3^*	.028	.109	.206	.048	.156	.271	.079	.249	.404	.106	.306	.468	.120	.342	.523	.164	.408	.579
T_3	.020	.084	.143	.019	.067	.116	.005	.024	.042	.001	.002	.006	.0	.001	.002	.0	.0	.0

WEIBULL

α	.01	.05	.10	.01	.05	.10	.01	.05	.10	.01	.05	.10	.01	.05	.10	.01	.05	.10						
θ	1.25			1.50			1.75			2.00			2.50			3.00			3.50			4.50		
W_1	.029	.136	.251	.073	.280	.477	.130	.420	.621	.217	.562	.757	.318	.727	.880	.405	.805	.932	.482	.832	.943	.538	.863	.954
W_2	.021	.112	.227	.055	.255	.426	.106	.340	.538	.174	.477	.670	.252	.605	.791	.317	.675	.846	.369	.731	.869	.403	.774	.895
W_3	.023	.150	.252	.059	.318	.506	.107	.470	.672	.188	.617	.800	.286	.797	.924	.388	.879	.963	.470	.702	.971	.529	.827	.985
S_1^*	.039	.149	.269	.112	.337	.507	.238	.546	.719	.400	.734	.869	.721	.941	.982	.905	.994	.999	.977	.999	1.00	1.00	1.00	1.00
S_2^*	.025	.146	.226	.079	.306	.448	.168	.513	.657	.310	.686	.813	.621	.916	.967	.837	.989	.997	.946	.998	1.00	.999	1.00	1.00
S_3^*	.043	.161	.277	.124	.352	.533	.270	.566	.736	.434	.754	.883	.740	.945	.986	.909	.992	.999	.977	.999	1.00	.999	1.00	1.00
T_3	.053	.178	.288	.146	.355	.501	.272	.526	.658	.425	.694	.816	.694	.881	.935	.868	.969	.983	.946	.993	.999	.994	1.00	1.00

GAMMA

	.01 .05 .10	.01 .05 .10	.01 .05 .10	.01 .05 .10	.01 .05 .10	.01 .05 .10	.01 .05 .10	.01 .05 .10	.01 .05 .10	.01 .05 .10
θ	1.50	2.00	2.50	3.00	3.50	4.00	4.50	5.50		
λ_1	.036 .162 .288	.068 .275 .455	.110 .375 .583	.159 .447 .665	.195 .553 .733	.214 .576 .775	.232 .603 .797	.264 .628 .810		
λ_2	.036 .144 .249	.055 .225 .386	.084 .287 .482	.116 .356 .555	.142 .410 .600	.157 .499 .635	.158 .454 .664	.184 .498 .678		
λ_3	.023 .178 .312	.055 .323 .497	.093 .452 .659	.138 .548 .747	.176 .635 .814	.204 .691 .876	.221 .720 .879	.258 .746 .896		
λ_1^*	.045 .175 .298	.096 .328 .492	.203 .502 .680	.321 .647 .809	.434 .775 .893	.541 .864 .953	.651 .905 .960	.803 .971 .993		
λ_2^*	.033 .161 .251	.063 .283 .415	.131 .434 .577	.226 .570 .718	.300 .695 .803	.395 .782 .886	.470 .841 .915	.640 .922 .962		
λ_3^*	.052 .190 .323	.121 .365 .553	.238 .559 .748	.364 .707 .859	.509 .834 .929	.622 .906 .974	.719 .934 .983	.864 .987 .998		
Γ_3	.043 .174 .315	.021 .112 .237	.004 .036 .091	.0 .007 .024	.0 .001 .004	.0 .0 .0	.0 .0 .0	.0 .0 .0	.0 .0 .0	.0 .0 .0

Table 6.2 shows that for $n = 10$ and the distributions considered, the rank statistics W_1 are consistently less powerful than the corresponding studentized linear spacings statistics S_1^* . This led us to compute Table 6.3 in which $n = 30$ and the Weibull distribution is considered. This table shows that for small θ , there is no difference in the powers between the total time on test statistic S_1^* and its rank counter part W_1 , while for larger θ , the total time on test statistic is again better. However, as expected from the asymptotic theory, the difference in power seems to be decreasing.

7. Appendix

We use the notation of Section 4. Heuristically our argument is very simple and is essentially the same as that used in [3].

$$\begin{aligned}
 T_n &= n^{-1/2} \left\{ \sum_{i=1}^n [h(X_{(i)}) - E\{h(X_{(i)})\}] \right\} \\
 &= n^{-1/2} \left\{ \sum_{i=1}^n h(X_{(i)}) - h(E\{X_{(i)}\}) \right\} \\
 &= n^{-1/2} \left\{ \sum_{i=1}^n h'(E\{X_{(i)}\}) [X_{(i)} - E\{X_{(i)}\}] \right\} \\
 &= n^{-1/2} \sum_{i=1}^n h' \left(-\log \left(1 - \frac{i}{n+1} \right) \right) \sum_{j=1}^i (D_j - 1)(n - j + 1)^{-1}
 \end{aligned} \tag{7.1}$$

under H_0 . From the last approximate identity our result follows. The justification of these approximations poses some minor technical difficulties. We proceed with the proof of the theorem. We show that i, ii and iii b) suffice. Let,

$$\begin{aligned}
 J_\delta(t) &= 1 & \delta \leq t \leq 1 - \delta \\
 &= 0 & \text{otherwise}
 \end{aligned} \tag{7.2}$$

From Theorem 3 of [3] it follows that, if $T_n^\delta = n^{-1/2} \sum_{i=1}^n J_\delta \left(\frac{i}{n+1} \right) h(X_{(i)})$ then,

$$T_n^\delta = n^{1/2} \mu_n^\delta + n^{-1/2} \sum_{j=1}^n a_j (j(n+1)^{-1})(D_j - 1) + o_p(1) \tag{7.3}$$

where

$$a_\delta(s) = (1-s)^{-1} \int_s^1 J_\delta(t) h'(-\log(1-t)) dt \tag{7.4}$$

$$\mu_n^\delta = n^{-1} \sum_{i=1}^n J_{\delta} \left(\frac{i}{n+1} \right) h(-\log(1 - i(n+1)^{-1})) \quad (7.5)$$

and $o_p(1)$ as usual denotes a remainder converging to 0 in probability.

Now let

$$\gamma_n^\delta = -n^{-1/2} \sum_{j=1}^n J_{\delta} \left(\frac{j}{n+1} \right) E(h(X_{(j)})) \quad (7.6)$$

$$R_{n1}^\delta = n^{-1/2} \sum_{j < (n+1)\delta} \{h(X_{(j)}) - E(h(X_{(j)}))\} \quad (7.7)$$

and

$$R_{n2}^\delta = n^{-1/2} \sum_{j > (n+1)(1-\delta)} \{h(X_{(j)}) - E(h(X_{(j)}))\} \quad (7.8)$$

Then,

$$T_n = T_n^\delta + R_{n1}^\delta + R_{n2}^\delta + \gamma_n^\delta. \quad (7.9)$$

We begin with,

Lemma 7.1:

If $\delta_n \rightarrow 0$ as $n \rightarrow \infty$

$$n^{-1/2} \sum_{i=1}^n \left[a_{\delta_n} \left(\frac{i}{n+1} \right) - a \left(\frac{i}{n+1} \right) \right] (D_i - 1) \rightarrow 0 \quad (7.10)$$

in probability.

Proof:

$$\begin{aligned} & E \left\{ n^{-1/2} \sum_{i=1}^n \left[a_{\delta_n} \left(\frac{i}{n+1} \right) - a \left(\frac{i}{n+1} \right) \right] (D_i - 1) \right\}^2 \\ &= n^{-1} \sum_{i=1}^n \left[a_{\delta_n} \left(\frac{i}{n+1} \right) - a \left(\frac{i}{n+1} \right) \right]^2 \end{aligned} \quad (7.11)$$

$$| \alpha_\delta(s) |^2 \leq (1-s)^{-2} \left[\int_s^1 | h'(-\log(1-t)) | dt \right]^2 \quad \text{for } \delta \geq 0. \quad (7.12)$$

Moreover,

$$\int_0^1 (1-s)^{-2} \left(\int_s^1 | h'(-\log(1-t)) | dt \right)^2 ds < \infty. \quad (7.13)$$

To see (7.13) note that the left-hand side equals

$$\begin{aligned} & \int_0^1 (1-s)^{-2} \int_s^1 | h'(-\log(1-t)) | dt \int_s^1 | h'(-\log(1-v)) | dv ds \\ & = 2 \int_0^\infty \int_0^u | h'(u) | | h'(r) | e^{-u}(1-e^{-r}) dr du \end{aligned} \quad (7.14)$$

after some standard arguments. Now by c) there exists $\delta > 0$ such that $h'(x)$ has constant sign for $x < \delta$, $x > \delta^{-1}$ and $\sup_{\delta \leq x \leq \delta^{-1}} | h'(x) | < \infty$. Then

$$\int_\delta^{\delta^{-1}} \int_\delta^u | h'(u) | | h'(r) | e^{-u}(1-e^{-r}) dr du < \infty.$$

Suppose for simplicity $h'(x) \geq 0$ $x < \delta$, $h'(x) \leq 0$ $x > \delta^{-1}$. Then

$$\begin{aligned}
& \int_{\delta}^{\delta^{-1}} \int_0^{\delta} |h'(u)| |h'(r)| e^{-u}(1 - e^{-r}) du dr \\
&= \int_{\delta}^{\delta^{-1}} |h'(u)| e^{-u} \int_0^{\delta} h'(r)(1 - e^{-r}) dr du \\
&= \int_{\delta}^{\delta^{-1}} |h'(u)| \left\{ \int_0^{\delta} h(r) e^{-r} dr + h(\delta)(1 - e^{-\delta}) \right\} du < \infty \\
&\text{since } \int_0^{\infty} e^{-r} |h(r)| dr < \infty .
\end{aligned} \tag{7.15}$$

Continuing,

$$\begin{aligned}
& 2 \int_0^{\delta} \int_0^u |h'(u)| |h'(r)| e^{-u}(1 - e^{-r}) du dr \\
&= \lim_{\lambda \rightarrow 0} 2 \int_{\lambda}^{\delta} h'(u) e^{-u} \{h(u) - h(\lambda)\} du - \int_{\lambda}^{\delta} \frac{\left[\int_{\lambda}^u e^{-v} h'(v) dv \right]^2}{du} du \\
&= \lim_{\lambda \rightarrow 0} \int_0^{\delta} e^{-u} dh^2(u) - 2h(\lambda) \int_{\lambda}^{\delta} h'(u) e^{-u} du - \left[\int_{\lambda}^{\delta} h'(u) e^{-u} du \right]^2 \\
&= \int_0^{\delta} h^2(u) e^{-u} du - h^2(\delta) e^{-\delta} - 2 \left[\int_0^{\delta} e^{-u} h(u) du \right] h(\delta) e^{-\delta}
\end{aligned} \tag{7.16}$$

$$+ \lim_{\lambda \rightarrow 0} - \left\{ h^2(\lambda)(1 - e^{-\lambda}) + 2h(\lambda)(1 - e^{-\lambda}) \int_{\lambda}^{\delta} h(u) e^{-u} du + \right. \\ \left. + 2h(\delta) e^{-\delta} h(\lambda)(1 - e^{-\lambda}) \right\} < \infty$$

as long as $h^2(\lambda)(1 - e^{-\lambda})$ is bounded as $\lambda \rightarrow 0$. But this, of course, holds if

$$\int_0^{\epsilon} h^2(u) e^{-u} du < \infty, \text{ and } h \uparrow \text{ in } (0, \delta).$$

One can dispose of the other pieces of the integral by similar arguments.

From (7.11) and (7.12) it readily follows that

$$a_{\delta_n}(s) \rightarrow a(s) \quad (7.17)$$

for every $0 < s < 1$ and again by (7.12), (7.16) and the dominated convergence theorem the lemma follows. Suppose we can show

$$\lim_{\delta \rightarrow 0} \sup_n \text{Var } R_{ni}^{\delta} = 0 \quad (7.18)$$

for $i = 1, 2$. We claim the theorem follows under our assumptions. To see this note that (7.9) and (7.18) imply that for every $\epsilon > 0$, there exists a δ such that

$$\lim_{\delta \rightarrow 0} \sup_n d(T_n, T_n^{\delta} + Y_n^{\delta}) \leq \epsilon \quad (7.19)$$

where

$$d(X, Y) = E \{ |X - Y| (1 + |X - Y|)^{-1} \} \quad (7.20)$$

is the usual metric for convergence in probability. Now, (7.3) yields for δ as

above,

$$\limsup_n d\left(T_n, n^{-\frac{1}{2}} \sum_{i=1}^n a_\delta(i(n+1)^{-1})(D_i - 1) + n^{\frac{1}{2}} \mu_n^\delta + \gamma_n^\delta\right) \leq \epsilon \quad (7.21)$$

and by Lemma 7.1, for δ sufficiently small.

$$\limsup_n d\left(T_n, S_n + n^{\frac{1}{2}} \mu_n^\delta + \gamma_n^\delta\right) \leq \epsilon \quad (7.22)$$

This, of course, implies that there exists a sequence of constants K_n such that,

$$\lim_n d(T_n, S_n + K_n) = 0. \quad (7.23)$$

Since S_n and T_n are both asymptotically normal with mean 0 by Lemma 4.1 of [2] and (ii) and the central limit theorem, it follows that $K_n \rightarrow 0$. We prove (7.18) for $i = 1$; $i = 2$ is proved analogously. Let δ be defined as in the discussion preceding (7.15). Define,

$$\begin{aligned} Z_1 &= h(X_1) \quad \text{if } 0 \leq X_1 < -\log(1 - \delta) \\ &= U_1 \quad \text{otherwise} \end{aligned} \quad (7.24)$$

where the U_1 are uniformly distributed on $(-\log(1 - \delta), B)$ and independent of each other and the X_1 . B is so chosen that the density of Z_1 at $h(-\log(1 - \delta))$ is the same as the density of $h(X_1)$ at $h(-\log(1 - \delta))$.

We argue as in [2]. Let $Z_{(1)} < \dots < Z_{(n)}$ be the order statistics of the Z_i 's. Then,

$$\begin{aligned}
& E \left(\sum_{i < \frac{\delta n}{2}} (h(X_{(i)}) - Z_{(i)}) \right)^2 \\
&= E \left(\sum_{i < \frac{\delta n}{2}} (h(X_{(i)}) - Z_{(i)}) I_{[X_{(i)} \geq -\log(1 - \delta)]} \right)^2 \\
&\leq 2 E \left(\sum_{i < \frac{\delta n}{2}} Z_{(i)} I_{[X_{(i)} \geq -\log(1 - \delta)]} \right)^2 \\
&+ 2 \sum_{i < \frac{\delta n}{2}} \left[E \left(h^2(X_{(i)}) I_{[X_{(i)} \geq -\log(1 - \delta)]} \right) \right]^{1/2}.
\end{aligned} \tag{7.25}$$

The first identity follows by definition of the Z_i , the second inequality from the c_r and Minkowski inequalities. Now,

$$E \left(\sum_{i < \frac{\delta n}{2}} Z_{(i)} I_{[X_{(i)} \geq -\log(1 - \delta)]} \right)^2 \leq B^2 \frac{\delta^2 n^2}{4} P \left[X_{\left[\frac{\delta n}{2} \right]} > -\log(1 - \delta) \right] \rightarrow 0 \tag{7.26}$$

since by [2] Lemma 2.2, $P \left[X_{\left[\frac{\delta n}{2} \right]} \geq -\log(1 - \delta) \right] \rightarrow 0$ exponentially.

On the other hand,

$$\begin{aligned}
& E \left(h^2(X_{(i)}) I_{[X_{(i)} \geq -\log(1 - \delta)]} \right) \\
&= \int_{v \geq -\log(1 - \delta)} h^2(v) n! [(i - 1)! (n - i)!]^{-1} e^{-(n-i+1)v} \\
&\quad (1 - e^{-v})^{i-1} dv \\
&\leq \sup_{v \geq -\log(1 - \delta)} e^{-(n-1)v} (1 - e^{-v})^{i-1} n \binom{n-1}{i-1} E(h^2(X_1)).
\end{aligned} \tag{7.27}$$

Now,

$$\begin{aligned} & \sup_{v > -\log(1-\delta)} e^{-(n-1)v} (1 - e^{-v})^{i-1} n \binom{n-1}{i-1} \\ & = \delta^{i-1} (1 - \delta)^{n-i} n \binom{n-1}{i-1} \quad \text{for } i \leq \frac{\delta}{2} n. \end{aligned} \tag{7.28}$$

Finally,

$$\begin{aligned} & \sup_{i \leq \frac{\delta n}{2}} \delta^{i-1} (1 - \delta)^{n-i} n \binom{n-1}{i-1} \\ & = \delta \left[\frac{\delta n}{2} \right]^{-1} (1 - \delta)^{n - \left[\frac{\delta n}{2} \right]} n \binom{n-1}{i-1} \end{aligned} \tag{7.29}$$

by an easy induction argument on i . By Lemma 2.2 of [2] the right-hand side of (7.29) $\rightarrow 0$ exponentially. Then, (7.27), (7.28) and (7.29) imply that

$\lim_{\delta \rightarrow 0} \limsup_n \text{Var } R_{n1}^{\delta'} = 0$ if and only if

$$\lim_{\delta \rightarrow 0} \limsup_n \text{Var} \left[\sum_{i \leq \delta' n} Z_{(i)} \right] = 0 \tag{7.30}$$

where Z_1 are defined for fixed δ . Since the Z_1 's are independently and identically distributed with a density positive on its convex support and $E(Z_1^2) < \infty$, (7.30) follows from [2]. The sufficiency of (i), (ii) and (iii) b) for the conclusion of the theorem follows. The sufficiency of (i), (ii), and (iii) a) is an easy consequence of theorem of [3].

Using the methods of [2] on T_n^δ one can show that $E(S_n - T_n)^2 \rightarrow 0$.

As in [3] the smoothness conditions on h may be relaxed.

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