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ON THE DEFECT INDICES OF LINEAR OPERATORS IN BANACH SPACE AND ON SOME GEOMETRIC QUESTIONS

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by

Nathan Rubinstein

Summary

The authors show that the theory of defect indices of Hermitian operators in a Hilbert space can be extended to the case of linear operators in a Banach space. For this purpose they use the idea of aperture (gap) of two subspaces with whose help it is convenient to establish in many cases the equality of the dimensions of the subspaces of the Banach space. The authors also arrive at several geometric conjectures.

ON THE DEFECT INDICES OF LINEAR OPERATORS IN BANACH SPACE AND ON SOME GEOMETRIC QUESTIONS*

by

M. G. Krein, M. A. Krasnosel'skii, D. P. Milman

It has been shown not long ago [1,2] that the Carleman-Neumann theory of defect indices of Hermitian operators in a Hilbert space in its basic state can be extended to the case of arbitrary linear operators in that space.

In this paper we shall show that this theory permits the extension to the case of linear operators in a Banach space.

Just as in [2], we shall use for this purpose the idea of aperture of two subspaces with whose help it is convenient to establish in many cases the equality of the dimensions of the subspaces of the Banach space.

In connection with this concept the authors came to several geometric conjectures, which possibly present an interest in themselves.

Detailed examination of this concept, in the case of a Hilbert space led to the study of a spectral reflection operator generated by two subspaces, and various characteristic slopes of one subspace with respect to the other.

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I. On the dimension number of a Banach Space

1. Let M be a subset of a Banach space E (real or complex) and suppose that the linear span M is dense in E. We shall call such subsets $M \subset E$ generators of E. Let a_M be the cardinal number of the subset M.

The smallest cardinal number a_M, where M is the set in the aggregate generating E, is called the dimension number of E and is denoted by dim E.

If the dimension number of E is finite, then it is easy to see that it coincides with the maximum number of linearly independent elements belonging to E.

If dim E is infinite, then it coincides with the minimum of all the cardinal numbers of sets which are dense in E. In order to show this, it suffices to notice that the set of linear combinations of elements with rational coefficients contained in the set which generates E is dense in E.

We shall call a subset A of a set $B \subset E$ an α-lattice B, if for arbitrary $x, y \in A$ we have $\|x-y\| \geq \alpha$. We say that the α-lattice is maximal, if the α-lattice A of the set B does not preserve the inequality for any other α-lattice of B.

Thus, if A is a maximal α-lattice of the set B, then for every $z \in B$

$$\rho(z,A) = \inf_{x \in A} \|z-x\| < \alpha .$$

Any α-lattice can be extended to be maximal by the transfinite process; from this, in particular, follows the existence of a maximal lattice.

Lemma 1. Let E be an infinite dimensional Banach space. Then the cardinal number of every maximal α-lattice A of the unit hyperball K with $0 < \alpha < 1$ coincides with dim E.

Proof: Let A be some maximal α-lattice $(0 < \alpha < 1)$ of the unit hyperball K.

Assume that its cardinal number a_A is less than dim E. Then the set L of linear combinations of elements of A with rational coefficients is not dense in E. It follows by a well-known lemma of Riesz that for arbitrary $\epsilon > 0$ we can find an element $x \in E$ $(\|x\| = 1)$ such that, $\rho(x,L) \geq 1 - \epsilon$. Choosing $\epsilon < 1 - \alpha$ leads to a contradiction, since A is a maximal α-lattice. Thus a_A is not smaller than dim E.

On the other hand, a_A cannot be larger than dim E, since in the neighborhood of radius $\alpha/2$ the elements of the α-lattice don't intersect, but in each of them there is at least one element of every set which is dense in E.

This concludes the proof of the lemma.

Remark: We can easily convince ourselves that Lemma 1 holds when the hyperball K is replaced by the hypersphere S $(\|x\| = 1)$.

2. The following assertion is easily shown using Lemma 1.

Theorem 1. The dimension of the Banach space E is not larger than the dimension of the conjugate space E^*.

In particular, if the space E can be reflected, then

$$\dim E = \dim E^* .$$

Proof: If dim E is finite, then the assertion of the theorem is obvious.

Suppose dim E is infinite. Choose a transfinite maximal sequence of elements $\{x_\alpha\}$ ($\|x_\alpha\| = 1$) in E such that

$$\rho(x_\beta, L_\beta) > 1/2 ,$$

where L_β is the linear span of elements x_α for which $\alpha < \beta$. The closure of the linear span of elements of the sequence is E, and the cardinal number of the set of indexes α of the sequence is dim E.

We now construct a sequence of functionals $\{f_\alpha\}$ ($\|f_\alpha\| = 1$) such that

$$|f_\alpha(x_\alpha)| > 1/2, \; f_\alpha(x) = 0 \text{ for } x \in L_\alpha .$$

Suppose $f_{\alpha'}$ and $f_{\alpha''}$ ($\alpha' < \alpha''$) are elements of the constructed sequence of functionals. Then

$$\|f_{\alpha'} - f_{\alpha''}\| \geq \frac{|(f_{\alpha'} - f_{\alpha''})(x_{\alpha'})|}{\|x_{\alpha'}\|} = |f_{\alpha'}(x_{\alpha'})| > 1/2 .$$

Consequently, the constructed transfinite sequence of functionals $\{f_\alpha\}$ is a part of some maximal 1/2-lattice of a unit hyperball of the space E* .

This concludes the proof of the theorem.

II. The aperture of two subspaces

1. The aperture of two linear manifolds L_1 and L_2 of a Banach space E, designated by $\theta(L_1, L_2)$ (sec [2]), is defined satisfying the following equation

$$\theta(L_1, L_2) = \max \{\sup_{\substack{x \in L_1 \\ \|x\|=1}} \rho(x, L_2), \sup_{\substack{y \in L_2 \\ \|y\|=1}} \rho(y, L_1)\} . \tag{1}$$

Several evident properties of the aperture can be noted. First, we always have

$$0 \leq \theta(L_1, L_2) \leq 1 ,$$

and second,

$$\theta(\bar{L}_1, \bar{L}_2) = \theta(L_1, L_2) ,$$

where $\bar{L}_1$, $\bar{L}_2$ are the closures of L_1 and L_2 respectively.

<u>Theorem 2</u>. Let L_1 and L_2 be linear sets of a Banach space E and suppose

$$\theta(L_1, L_2) = a < 1 .$$

If one of the numbers dim L_1, dim L_2*) is finite, then

$$\dim L_1 = \dim L_2 .$$

Proof: The theorem is equivalent to showing that if dim $L_1 = n$, dim $L_2 > n$, then $\theta(L_1, L_2) = 1$. In order to show the latter it suffices to show that there exists a unit vector $y(\|y\| = 1)$, in L_2 which is orthogonal to L_1, i.e., such that $\rho(y, L_1) = 1$; in this case we can assume without loss of generality that dim $L_1 = n + 1$. Let E denote the linear span of either L_1 or L_2. Assume first that the unit sphere $\|z\| = 1$ $(z \in E)$ is strictly convex, i.e., it doesn't contain segments. Then for each $z \in E$ there will exist only one projection in L_1, i.e., only one $x \in L_1$ through which the distance from z to $L_1 : \rho(z, L_1) = \|z - x\|$ will be attained. It's easily seen that the projection operator $x = \varphi(z)$ $(z \in E)$ is continuous and satisfies $\varphi(-z) = -\varphi(z)$. The orthogonality of z to L_1 shows that $\rho(z, L_1) = \|z - \varphi(z)\| = \|z\|$, i.e., $\varphi(z) = 0$ (by virtue of the uniqueness of projection).

*) By dim L it's understood dim $\bar{L}$.

If we now assume that on the unit sphere $S_2(\|y\| = 1)$ of the space L_2, the operator $\varphi(y)$ is different from zero, then by virtue of the compactness of S_2, it will be possible to assert that the operator

$$\psi(y) = \frac{\varphi(y)}{\|\varphi(y)\|}$$

is also continuous on S_2. But this operator continuously maps the n-dimensional sphere S_2 into the (n - 1)-dimensional sphere $S_1(\|x\| = 1)$ of the space L_1, and since in addition centrally symmetric points map into centrally symmetric points $(\varphi(-y) = -\psi(y))$, this is impossible)*.

Thus, the theorem is shown assuming that the sphere is strictly convex in E.

We shall reduce the general case to the one considered, by showing that for an arbitrary $\varepsilon > 0$ one can always construct a new norm $\|z\|_o$ in E such that

$$\|z\| \leq \|z\|_o \leq (1 + \varepsilon)\|z\| \quad (z \varepsilon E) \tag{2}$$

and such that the new sphere $\|z\|_o = 1$ which is strictly convex, i.e., such that for any two vectors $z_1, z_2 \varepsilon E$ with different directions

$$\|z_1 + z_2\|_o < \|z_1\|_o + \|z_2\|_o . \tag{3}$$

)* We clarify, that the n-dimensional sphere in the Banach space E can be homeomorphically mapped into an n-dimensional (if E is real) or 2 n-dimensional (if E is a complex space) sphere of ordinary Euclidean space preserving the central symmetry of images of symmetric points. On the other hand, using a theorem of L. A. Lusternik and L. G. Shpirelman on categories of projection spaces (see [7]), it can easily be shown that if a k-dimensional Euclidean sphere is continuously mapped preserving the central symmetry into a m-dimensional Euclidean sphere, then $m \geq k$.

In fact, inequality (2) evidently implies the following inequality

$$\theta_o(L_1,L_2) \leq (1+\varepsilon)\,\theta(L_1,L_2) ,$$

where $\theta_o(L_1,L_2)$ is the aperture between L_1 and L_2 corresponding to the norm $\|z\|_o$. Consider L_1 and L_2, then by showing $\theta_o(L_1,L_2) = 1$ we'll have by virtue of condition (3) that $\theta(L_1,L_2) = 1$ since $\varepsilon > 0$ is arbitrary.

We shall show now how one can proceed constructing the norm $\|z\|_o$.

Let $\|z\|_1$ be a norm in E, and corresponding to it the strictly convex sphere $\|z\|_1 = 1$; for example this norm can be determined by setting

$$\|\xi_1 e_1 + \xi_2 e_2 + \dots + \xi_m e_m\|_1 = \sqrt{\xi_1^2 + \xi_2^2 + \dots + \xi_m^2} ,$$

where $\{e_1, e_2, \dots, e_m\}$ is some basis in E.

Let $K(>0)$ be the maximum $\|z\|_1$ on the unit sphere $\|z\| = 1$, then we can write

$$\|z\|_1 \leq K\|z\| \quad (z \in E) .$$

Since $\|z\|_1$ satisfies condition (3), it follows that for arbitrary $\delta > 0$ the following norm will also satisfy this condition

$$\|z\|_o = \|z\| + \delta\,\|z\|_1 .$$

For this norm inequality (2) will hold with $\varepsilon = \delta K$.

Consequently since $\delta > 0$ is arbitrary the required construction is achieved and the theorem is shown.

Theorem 3. Let L_1 and L_2 be linear manifolds of a Banach space E and let

$$\theta(L_1, L_2) < 1/2 .$$

Then

$$\dim L_1 = \dim L_2 .$$

Proof: By virtue of theorem 2, it follows that we need only consider the case when $\dim L_1$ and $\dim L_2$ are infinite.

Let $\theta(L_1, L_2) = 1/2 - b$ $(b > 0)$. In the unit hyperball K of the linear manifold L_1, construct a maximal α-lattice A, where for instance $\alpha = 1 - b/2$. By this condition we have that for each element $x \in A \subset L_1$ there exists an element $y_x \in L_2$ such that $\|x - y_x\| \leq 1/2 - b/2$. If $x_1, x_2 \in A$ $(\|x_1 - x_2\| > 1 - b/2)$, then

$$\|y_{x_1} - y_{x_2}\| \geq \|x_1 - x_2\| - \|x_1 - y_{x_1}\| - \|x_2 - y_x\| > b/2 .$$

From this inequality it follows that the elements y_x, which correspond to elements x of the maximal $(1 - b/2)$-lattice A of the hyperball K, are contained in some $b/2$-lattice of a hyperball of radius 3/2 $(\|y_x\| \leq \|y_x - x\| + \|x\| < 3/2)$ of the linear set L_2.

It follows by virtue of lemma 1 that

$$\dim L_1 \leq \dim L_2 .$$

Interchanging the roles of L_1 and L_2, we come to the conclusion of the theorem.

2. Let L be a linear subset of E. The subspace $L^* \subset E^*$, consisting of all functionals which vanish for all elements of L, is called the orthogonal complement of L in E^*.

Theorem 4. Let L_1 and L_2 be two linear subsets of E, and let L_1^*, L_2^* be their orthogonal complements in E^*.

Then the following holds:

$$\theta(L_1, L_2) = \theta(L_1^*, L_2^*).$$

Proof: By formula (1) we can write:

$$\theta(L_1, L_2) = \sup\{|g(x)|, |f(y)|\},$$

where the supremum is taken over all

$$x \in L_1,\ y \in L_2,\ f \in L_1^*,\ g \in L_2^*;\ \|x\| = \|y\| = \|f\| = \|g\| = 1.$$

Recall, that if the subspace $L^* \subset E^*$ is regularly closed (i.e., for every functional $f_o \overline{\in} L^*$, $f_o \in E^*$ there exists an element $x_o \in E$ such that $f_o(x_o) \neq 0$, but $f(x_o) = 0$ for all $f \in L^*$), then by a known theorem of Banach (see [3]) we have that for an arbitrary $f_o \overline{\in} L^*$ and $\varepsilon > 0$ there exists an element $x_o \in L$ ($\|x_o\| = 1$) such that

$$|f_o(x_o)| \geq \rho(f_o, L^*) - \varepsilon,\ f(x_o) = 0 \text{ for all } f \in L^*.$$

In this connection it's obvious that $|f_o(x_o)| \leq \rho(f_o, L^*)$ always holds.

Since the orthogonal complement L^* of the linear set L is, obviously, regularly closed, it follows on the basis of this theorem that

$$\rho(f, L^*) = \sup_{x \in L, \|x\|=1} |f(x)| .$$

This means that

$$\theta(L_1^*, L_2^*) = \sup \{|g(x)|, |f(y)|\} ,$$

where the supremum is taken over all

$$x \in L_1, \ y \in L_2, \ f \in L_1^*, \ g \in L_2^*; \ \|x\| = \|y\| = \|f\| = \|g\| = 1 .$$

This concludes the proof of the theorem.

The following assertion follows directly from theorems 3 and 4.

<u>Theorem 5</u>. Let L_1 and L_2 be two linear subsets in a Banach space E, let L_1^* and L_2^* be their orthogonal complements in E^* .

Then, if

$$\theta(L_1, L_2) < 1/2,$$

then

$$\dim L_1^* = \dim L_2^* .$$

3. Let L be a subspace of a Banach space E. It is known that the subspace conjugate with the quotient-space E/L is equivalent to the subspace $E^* \subset L^*$.

If the space E/L is reflexive, (which occurs if, for example, E is reflexive, see [4]), then by theorem 1 dim E/L = dim L* .

With $\theta(L_1,L_2) < 1/2$ $(L_1,L_2 \subset E)$ it follows by virtue of theorem 5 that

$$\dim E/L_1 = \dim E/L_2 .$$

It turns out that this fact also holds for nonreflexive spaces.

<u>Theorem 6</u>. Let L_1 and L_2 be subspaces of a Banach space E and let

$$\theta(L_1,L_2) < 1/2 .$$

Then

$$\dim E/L_1 = \dim E/L_2 . \tag{4}$$

In case one of $\dim E/L_1$, $\dim E/L_2$ is finite, equation (4) follows from

$$\theta(L_1,L_2) < 1.$$

Proof: The second assertion of the theorem follows from the fact that a finite dimensional space is reflexive, and from theorems 2 and 4.

Assume now that $\dim E/L_1$ and $\dim E/L_2$ are infinite.

Denote by x the image of the element $x \in E$ under the mapping into the quotient space E/L_1, denote the image in E/L_2 by $\tilde{x}$.

Let us assume that dim $E/L_1 >$ dim E/L_2.

In the unit sphere S_1 of the space E/L_1 construct a maximal $(1 - \beta)$-lattice A_1 (where β is an arbitrary positive number), i.e., the totality of elements $x \in E$, whose cardinal equals dim E/L_1 (see remark 1), for which

$$\|\hat{x} - \hat{y}\| \geq 1 - \beta, \ \|\hat{x}\| = \|\hat{y}\| = 1, \ \hat{x}, \ \hat{y} \in A_1 \subset E/L_1 \ .$$

From the definition of the quotient space it follows that in every co-set $\hat{x} \in A_1$ we can choose an element $x \in E$, such that if γ is an arbitrary positive number then $\|x\| \leq 1 + \gamma$.

Then from $\|\hat{x} - \hat{y}\| > 1 - \beta$ also follows that $\|x - y\| > 1 - \beta$. By assuming dim $E/L_1 >$ dim E/L_2, and also by lemma 1 we have that the elements x cannot be contained in any α-lattice of a hypersphere of radius $1 + \gamma$ $(\|\tilde{x}\| < 1 + \gamma)$ of the space E/L_2. This means that for an arbitrary positive α there exist $x, y \in E$ $(\|x\|, \|y\| < 1 + \gamma; \ \hat{x}, \ \hat{y} \in A_1)$ such that $\|\hat{x} - \hat{y}\| < \alpha$. The last inequality means that there exists an element $z \in L_2$ such that $\|x - y - z\| < \alpha$, whence

$$1 - \beta - \alpha < \|z\| < 2 + 2\gamma + \alpha \ .$$

Then

$$\rho\left(\frac{z}{\|z\|}, \ L_1\right) \geq \frac{\rho(x-y, L_1) - \|x-y-z\|}{\|z\|} > \frac{1 - \beta - \alpha}{2 + 2\gamma + \alpha} \ .$$

By virtue of the arbitrariness of α, β, γ it follows from the inequality above that

$$\theta(L_1, L_2) \geq 1/2 \ .$$

This concludes the proof of the theorem.

III. Several Questions of the Geometry of a Hilbert Space

1. It is not clear whether the conclusion of theorem 2 holds without assuming finiteness of dimension of one of the linear manifolds (i.e., can we replace the condition $\theta(L_1,L_2) < 1/2$ in theorem 3 by the condition $\theta(L_1,L_2) < 1$?).

In the case when E is a Hilbert space, the posed question, as already remarked (see [2]), is trivially solvable.

If L_1,L_2 are two subspaces of arbitrary dimension in a Hilbert space E (we shall only consider such in this paragraph) which satisfy the following condition $\theta(L_1,L_2) < 1$, then in any one of them there does not exist a non-zero vector which is orthogonal to the other space.

We shall say that any two subspaces L_1,L_2 of a Hilbert space E, which satisfy the last condition, form a proper pair.

One can easily be convinced of the validity of the following proposition.

The subspaces $L_1,L_2 \subset E$, which form a proper pair, have identical dimensions.

In fact, $P_{L_2} L_1$ [1]) is dense in L_2, since otherwise we can find a vector $h \neq 0$ in L_2 which is orthogonal to $P_{L_2} L_1) \leq \dim L_1$. Analogously, $\dim L_1 \leq \dim L_2$ and it therefore follows that

$$\dim L_1 = \dim L_2 .$$

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[1]) If L is a subspace of E, then P_L denotes an operator which projects orthogonally on L.

2. The aperture in a Hilbert space is only a characteristic ordering relative to two subspaces. Below, we shall derive a more complete characteristic ordering relative to subspaces.

Let M be a subspace of E. An element $x^* \in E$ is said to be the spectral reflection of an element $x \in E$ with respect to M, if $x^* = 2P_M x - x$. If we introduce into consideration the orthogonal complement N of M

$$E = M \oplus N$$

then x^* can also be denoted by

$$x^* = P_M x - P_N x \ .$$

Thus $x^* = Sx$, where $S = P_M - P_N$ (from which, in particular, $(x^*)^* = x$).

We see that the spectral reflection operator S with respect to M, is first unitary, and second possesses an inverse: $S^2 = I$ (i.e., $(x^*)^* = x$).

On the other hand, it's easy to see that any operator S which is unitary and possesses an inverse ($S^2 = I$) is a spectral operator - a spectral reflection operator with respect to some subspace M.

In fact for such an operator we have

$$S = P - Q, \ P + Q = I$$

where

$$P = \frac{I + S}{2} \quad , \quad Q = \frac{I - S}{2}$$

are orthogonal projection operators ($P^2 = P = P^*$, $Q^2 = Q = Q^*$).

Suppose $L_1, L_2 \subset E$ are two subspaces which are spectral reflections of each other with respect to M, i.e., $SL_1 = (P_M - P_N)L_1 = L_2$, then $(P_N - P_M)L_1 = L_2$, i.e., L_1 and L_2 are also spectral reflections of each other with respect to N.

Since, if $x \in L_1$, the projection $P_M x = \frac{1}{2}(x + Sx) \in L_1 + L_2$, we conclude that if L_1, L_2 are spectral reflections of each other with respect to M, then they are also spectral reflections of each other with respect to the intersection $M_1 = M \quad \overline{L_1 + L_2}$ and vice versa.

In order that the subspaces L_1 and L_2 be spectral reflections of each other, it is evidently necessary that $\dim L_1 = \dim L_2$. However, this condition is not sufficient. In fact, suppose the infinite dimensional subspaces L_1 and L_2 form a proper pair.

By virtue of the following theorem, L_1 and L_2 will be spectral reflections with respect to some M.

Let e be a vector, orthogonal to L_1, and L_2' the linear span of L_2 and e. Then as before $\dim L_1 = \dim L_2'$; moreover L_1 cannot be the spectral reflection of L_2' with respect to any M', since by assuming the contrary, we would have vector e^* in L_2 orthogonal to L_2', and hence to L_2, where e is the spectral reflection of e^* $(\|e^*\| = \|e\| > 0)$.

As a consequence of all this the following assertion is of interest.

Theorem 7. If the subspaces $L_1, L_2 \subset E$ form a proper pair, then they are spectral reflections of each other with respect to some $M \subset \overline{L_1 + L_2}$.

Proof. Consider the operator $P_{L_1} P_{L_2}$ in the invariant subspace L_1 .

This operator is self-adjoint, and furthermore, strictly positive:

$$(P_{L_1} P_{L_2} x, x) = (P_{L_2} x, x) = \|P_{L_2} x\|^2 > 0 \qquad (x \in L_1, x \neq 0),$$

since by the condition, there are no vectors ($\neq 0$) in L_1 which are orthogonal to L_2. Consequently, there exists a self-adjoint operator H (which is not uniquely determined) in L_1, such that

$$H^2 x = P_{L_1} P_{L_2} x \qquad (x \in L_1) .$$

This operator satisfies

$$\|Hx\|^2 = (H^2 x, x) = \|P_{L_2} x\|^2 > 0 \text{ for } x \neq 0 \qquad (x \in L_1) .$$

Therefore the self-adjoint operator H^{-1} has a meaning in L_1; its domain of definition D (the set of all values of the operator H) is dense in L_1 .

Let us introduce into consideration the operator S which acts from D to L_2, by setting:

$$Sx = P_{L_2} H^{-1} x \qquad (x \in D) .$$

We have

$$(Sx, y) = (P_{L_1} P_{L_2} H^{-1} x, y) = (P_{L_1} P_{L_2} H^{-1} x, y) = (Hx, y) \qquad (x \in D,\ y \in L_1) .$$

Since $(Hx,y) = (x,Hy)$, then for any $x,y \in D$

$$(Sx,y) = (x,Sy) = (Hx,y) \; . \tag{5}$$

It follows that, for any $x,y \in D$:

$$(Sx,Sy) = (P_{L_2} H^{-1}x,Sy) = (H^{-1}x,Sy) = (HH^{-1}x,y) = (x,y) \; .$$

Thus we conclude that S isometrically maps D into L_2. Since $SD = P_{L_2} H^{-1} D = P_{L_2} L_1$, then SD is dense in L_2.

We extend the operator S to all of L_1 preserving the isometry, we use the same notation S for the extended operator. The extended operator maps isometrically all of L_1 onto a closed part of L_2, and hence onto all L_2, and equation (5) holds now for all $x,y \in L_1$.

Relation (5) is equivalent to the following

$$(Sx_1, S^{-1}x_2) = (x_1,x_2) \qquad (x_1 \in L_1, x_2 \in L_2)$$

and therefore, for any $x_1 \in L_1, x_2 \in L_2$:

$$\|Sx_1 + S^{-1}x_2\| = \|x_1 + x_2\| \; .$$

From this we conclude that the operator S admits an isometric extension $\widetilde{S}$ to all $E = \overline{L_1 + L_2}$, which is defined on $L_1 + L_2$ by the equation

$$\widetilde{S}(x_1 + x_2) = Sx_1 + S^{-1}x_2 \; .$$

Since $\tilde{S}x_1 = y_2 \in L_2$ and $\tilde{S}x_2 = y_1 \in L_1$, then

$$\tilde{S}^2(x_1 + x_2 = \tilde{S}(y_1 + y_2) = x_1 + x_2 .$$

Thus $\tilde{S}^2 = I$; since, in addition, $\tilde{S}L_1 = L_2$, $\tilde{S}L_2 = L_1$, then $\tilde{S}E \supset L_1 + L_2$ and consequently $\tilde{S}E \dot{+} E$.

Thus $\tilde{S}$ is a spectral reflection operator in E, which maps L_1 onto L_2. This proves the theorem.

Remark. We leave it to the reader to show that the indicated construction of the spectral operator $\tilde{S}$ in $E = \overline{L_1 + L_2}$ exhausts all spectral operators reflecting one onto the other L_1 and L_2.

There exists among these operators an operator S_+, which corresponds to a positive Hermitian operator H_+ and which characterizes the geometric conditions that for this operator the angle between any $x \in L_1$ $(x \neq 0)$ and its image $x^* = S_+ x \in L_2$ is not greater than $\pi/2$:

$$(x,x^*) = (x,S_+x) = (x,H_+x) > 0 .$$

Any other Hermitian operator H in our class can be given by $H = 0^{-1}H_+$, where 0 is a unitary operator in L_1. This follows since in our class the condition $H^2 = H_+^2$, means that $\|Hx\| = \|H_+x\| \quad (x \in L_1)$.

Since $H^* = H$, then also

$$H = 0^{-1}H_+ = H_+0 . \qquad (6)$$

Consequently,

$$H_+^2 = H^2 = 0^{-1} H_+^2 \, 0 \ ,$$

i.e., 0 commutes with H_+^2, and consequently, with H_+ .

But then by (6): $0^2 = I$.

Thus, the general form of the spectral reflection $x \rightarrow x^*$ of the spaces L_1 onto L_2 is given by the formula

$$x^* = \overline{P_{L_2} H_+^{-1} 0 \ x} \qquad (x \in L_1) \ ,$$

where H_+ is a positive self-adjoint operator acting in L_1 which when squared yields $P_{L_1} P_{L_2}$ and 0 is an arbitrary reflection operator in L_1 which commutes with H_+ .

2. Now let L_1, L_2 be arbitrary subspaces in E. As previously, we denote by H_+, which is given by the formula $H_+^2 x = P_{L_1} P_{L_2} x \quad (x \in L_1)$, a positive Hermitian operator in L_1, which we shall also denote by H_{12}. Since the spectrum of the operator H_{12} lies in the interval (0,1):

$$0 \le (H_{12}^2 x, x) = (P_{L_2} x, x) = \|P_{L_2} x\|^2 \le \|x\|^2 \qquad (x \in L_1),$$

then, there exists one and only one self-adjoint operator Φ_{12} in L_1 with spectrum in the interval $(0, {}^{\pi}/2)$, such that

$$\cos \Phi_{12} = H_{12} \ .$$

It is natural to call the spectrum of the operator Φ_{12} as the spectrum of the angles of inclination of L_1 and L_2. Let $\varphi_{12}^{(M)}$ and $\varphi_{12}^{(m)}$ $(0 \le \varphi_{12}^{(m)} \le {}^{\pi}/2)$ correspond to the biggest and smallest points of this spectrum.

For any unit vector x ($\|x\| = 1$) the angle of inclination to L_2 $\varphi (0 \le \varphi \le \pi/2)$, is determined by the equation

$$\sin \varphi = \rho(x, L_2) = \sqrt{1 - \|P_{L_2} x\|^2} \quad .$$

Obviously,

$$\sin^2 \varphi = 1 - (H_+^2 x, x) = (x,x) - (\cos^2 \Phi_{12} x, x) = (\sin^2 \Phi_{12} x, x) \quad .$$

Thus for the magnitude $\theta_{12} = \sup\limits_{\substack{x \in L_1 \\ \|x\| = 1}} \rho(x, L_2)$ we have the following interpretation:

$$\theta_{12} = \sup \sin \varphi = \sin \varphi_{12}^{(M)} \quad .$$

Interchanging the roles of L_1 and L_2, we obtain, correspondingly, the operators H_{21} and Φ_{21} and the magnitudes θ_{21}, $\varphi_{21}^{(M)}$, $\varphi_{21}^{(m)}$.

In the case if the subspaces L_1 and L_2 form a proper pair, it follows by theorem 7 that they are spectral reflections of each other and, consequently,

$$\theta_{12} = \theta_{21}; \; \varphi_{12}^{(M)} = \varphi_{21}^{(M)} = \varphi^{(M)} \quad . \tag{7}$$

Thus the aperture of a proper pair of subspaces L_1 and L_2 can be interpreted, as the sine of the maximal angle $\varphi^{(M)}$ between the subspaces L_1 and L_2:

$$\theta(L_1, L_2) = \sin \varphi^{(M)} \quad .$$

The equalities in (7) are consequences of a more general proposition.

The angular operators Φ_{12} and Φ_{21} are unitarily-equivalent when L_1 and L_2 are a proper pair of subspaces.

In fact, let S be a spectral reflection operator ($SL_1 = L_2$). Let $x^* = Sx$ ($x \in L_1$) and $y = P_{L_2} x$, then $y^* = P_{L_1} x^*$, since if $x - y$ is orthogonal to L_2, $x^* - y^*$ is orthogonal to L_1. Thus, if $z = P_{L_1} P_{L_2} x$, then $z^* = P_{L_2} P_{L_1} x^*$, i.e.,

$$SH_{12}^2 S^{-1} = H_{21}^2, \quad SH_{12} S^{-1} = H_{21}$$

and

$$S\Phi_{12} S^{-1} = \Phi_{21} .$$

We remark that in case L_1 and L_2 are a proper pair the spectral reflection operator S can be written in the following form:

$$Sx = \overline{P_{L_2}(\cos \Phi_{12})^{-1} x} \qquad (x \in L_1) \quad .$$

In case the subspaces L_1 and L_2 do not form a proper pair, then they can be decomposed in a natural fashion into orthogonal sums

$$L_1 = L_1' \oplus L_{12} \ ; \quad L_2 = L_2' \oplus L_{21} \ ,$$

where L_{12} - (correspondingly L_{21}) is a subspace of elements of $L_1(L_2)$, orthogonal to $L_2(L_1)$. Then it is easily seen that L_1 and L_2' form a proper pair, and in order that there exists a spectral reflection of L_1 onto L_2 it is necessary and sufficient that $\dim L_{12} = \dim L_{21}$.

We remark that the subspace L_{12} for the operator $H_{12}(\Phi_{12})$ is an eigen-subspace, corresponding to the eigenvalue $0(\pi/2)$.

The intersection $L_1 \quad L_2$ consists of the subspace of fixed vectors of the operators H_{12} and H_{21} and the null subspace of the operators Φ_{12} and Φ_{21}.

In the case of Hilbert spaces, theorem 4 of the previous paragraph is a partial consequence of the more complete proposition:

Theorem 8. Let the subspaces L_1 and L_2, whose intersection is the empty set, form a proper pair and let $E = \overline{L_1 + L_2}$. Furthermore, let Φ - be the inclination operator of L_1 onto L_2, and ψ - the inclination operator of N_1 and N_2, where $N_1 = E \quad L_1$, $N_2 = E \quad L_2$. Then dim N_1 = dim N_2 on the operators Φ and ψ are unitarily equivalent.

<u>Proof</u>. First of all, we remark that L_1 and N_2 form a proper pair of subspaces. In fact, the orthogonal complement of N_2, i.e., L_2 and L_1 do not intersect. On the other hand, if $x \in N_2$ is orthogonal to L_1, then x is orthogonal to $L_1 + L_2$, i.e., $x = 0$.

It follows from theorem 7 that there exists a spectral operator S which maps L_1 and N_2 onto each other. Since the operator S is unitary, it maps N_1 and L_2 onto each other.

By virtue of this unitary property, if $x \in L_1$ and $y = P_{L_2} x$, then $Sx \in N_2$ and $Sy \in N_1$, where

$$Sy = P_{N_1} Sx, \quad SP_{L_2} x = P_{N_1} Sx \qquad (x \in L_1) \ .$$

Analogously,

$$SP_{L_1} z = P_{N_2} Sz \qquad (z \in L_2) \ .$$

Consequently,

$$SP_{L_1}P_{L_2}x = P_{N_2}P_{N_1}Sx \qquad (x \in L_1) ,$$

i.e., the operators $\cos^2 \Phi$ and $\cos^2 \psi$ are unitarily equivalent, and therefore the inclination operators Φ and ψ are also unitarily equivalent.

This concludes the proof of the theorem.

On the basis of the proven theorem, it is easy to point out the relation between the operators Φ and ψ when L_1 and L_2 are arbitrary subspaces.

We leave it to the reader to show, with the help of a theorem of Banach [1]), the following two propositions:

1. If the subspaces L_1 and L_2 form a proper pair, then $PL_1 = L_2$ in the case and only in the case when the maximum angle $\varphi^{(M)}$ between the subspaces L_1 and L_2 is less than $\pi/2$, i.e., $\theta(L_1, L_2) < 1$.

2. In order that the direct sum of the non-intersecting subspaces L_1 and L_2 be a complete space, it is necessary and sufficient that the minimum angle $\varphi^{(m)}$ be greater than zero.

We clarify that the minimum angle $\varphi^{(m)} (0 \leq \varphi \; \pi/2)$ is determined by the formula

$$\cos \varphi^{(m)} = \sup_{\substack{x \in L_1, y \in L_2 \\ \|x\| = \|y\| = 1}} |(x,y)|$$

---

[1]) If a one-to-one mapping of a complete linear subspace into a complete one is continuous, then the inverse mapping is also continuous.

and, since

$$\sup_{y\in L_2,\|y\|=1} |(x,y)| = \|P_{L_2} x\| = \sqrt{(\cos^2 \Phi_{12} x,x)} \qquad (x\in L_1) ,$$

$$\sup_{x\in L_1,\|x\|=1} |(x,y)| = \|P_{L_1} y\| = \sqrt{(\cos^2 \Phi_{21} y,y)} \qquad (y\in L_2) ,$$

then

$$\varphi^{(m)} = \varphi_{12}^{(m)} = \varphi_{21}^{(m)} .$$

3. The whole spectrum of the self-adjoint operator $\Phi = \Phi_{12}$ lies in the interval $(0,{}^{\pi}/2)$, i.e.,

$$0 \leq (\cos \Phi x,x) \leq (x,x) \qquad (x\in L_1) .$$

There naturally arises the question of whether any self-adjoint operator Φ acting in some subspace $L_1 \subset E$ and satisfying the conditions derived above can be an inclination operator of L_1 to some subspace $L_2 \subset E$, of course under the condition that dim $(E \ominus L_1)$ is sufficiently large.

The affirmative answer to this question follows directly from a theorem of M. A. Naymark [5,6], according to which we have that, in order that the self-adjoint operator H, which acts on the subspace $L_1 \subset E$ and satisfies the condition

$$0 \leq (Hx,x) \leq (x,x) \qquad (x\in L_1) ,$$

admits the representation

$$Hx = P_{L_1} P_{L_2} x \qquad (x\in L_1) ,$$

where L_2 - is some subspace of E, it is necessary, it is necessary and sufficient that

$$\dim (E \ominus L_1) \geq \dim L_1' ,$$

where L_1' - is the collection of vectors in L_1, which are orthogonal to all null and fixed vectors of the operator H.

4. Defective indexes of additive and homogeneous operators.

We denote by $\mathcal{D}(A)$ the domain of definition and by $\mathcal{R}(A)$ the range of the additive and homogeneous operator A, acting in a Banach space E.

The point λ_o in the complex plane is said to be of the regular type for the operator A, if there exists a positive number k_{λ_o} such that

$$\|(A - \lambda_o I)f\| \geq k_{\lambda_o} \|f\| \qquad (f \epsilon \quad (A)) ,$$

where I denotes the identity operator.

The points of regular type for the operator A form an open set, since if λ_o is a regular type point for the operator A, then if $|\lambda - \lambda_o| < k_{\lambda_o}$, then

$$\|(A - \lambda I)f\| \geq \|(A - \lambda_o I)f\| - |\lambda - \lambda_o| \cdot \|f\| \geq k_\lambda \|f\| \qquad (f \epsilon \mathcal{D}(A))$$

where

$$k_\lambda = k_{\lambda_o} - |\lambda - \lambda_o| \quad .$$

<u>Theorem 9</u>. Let G be a region in the complex plane, consisting of regular type points for the operator A.

Then the dimensions of all orthogonal complements N_λ in E* of $\mathcal{R}(A - \lambda I) c E$ are identical for all $\lambda \epsilon G$.

Proof. We shall show that for every point $\lambda_o \varepsilon G$ there exists a neighborhood W such that for all $\lambda \varepsilon W$

$$\dim N_\lambda = \dim N_{\lambda_o} ,$$

from which the theorem follows.

Let W be a neighborhood of the point λ_o of radius $1/4\ k_{\lambda_o}$; then for all $\lambda \varepsilon W$

$$\|(A - \lambda I)f\| \geq \|(A - \lambda_o I)f\| - |\lambda - \lambda_o|\ \|f\| > 3/4\ k_{\lambda_o} \|f\| \qquad (f \varepsilon \mathfrak{D}(A)) ,$$

and

$$\|(A - \lambda I)f - (A - \lambda_o I)f\| = |\lambda - \lambda_o| \cdot \|f\| < 1/3\ \|(A - \lambda I)f\| ,$$

$$\|(A - \lambda_o I)f - (A - \lambda I)f\| < 1/4\ \|(A - \lambda_o I)f\| .$$

Consequently,

$$\theta[\mathfrak{R}(A - \lambda I),\ \mathfrak{R}(A - \lambda_o I)] \leq 1/3 \qquad (\lambda \varepsilon W) .$$

By virtue of theorem 5

$$\dim N_\lambda = \dim N_{\lambda_o} .$$

Thus the theorem is proven.

An analogous reasoning, using theorem 6, leads us to the following assertion:

The dimensions of the quotient space $E/\mathfrak{R}(A - \lambda I)$ are identical for all λ, where λ is an element of the connected set of points of regular type for the operator A.

This assertion and theorem 9 give the reason for establishing the concept of defective indices of two types, for each component G of the set of regular type points, for linear operators in a Banach space:

the defective index m^*, which equals dim $N_\lambda (\lambda \epsilon G)$, and

the defective index $\hat{m}$, which equals dim $E/\mathfrak{R}\ (A - \lambda I)$.

In case the space is reflective, or in particular when it's a Hilbert space, the defective indices m^* and $\hat{m}$ will coincide.

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The authors show that the theory of defect indices of Hermitian operators in a Hilbert space can be extended to the case of linear operators in a Banach space. For this purpose they use the idea of aperture (gap) of two sub-spaces with whose help it is convenient to establish in many cases the equality of the dimensions of the subspaces of the Banach space. The authors also arrive at several geometric conjectures.

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