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NUMERICAL DIFFERENTIATION USING ORTHOGONAL POLYNOMIALS

by

G. H. Jonas

November 1967

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BALLISTIC RESEARCH LABORATORIES

REPORT NO. 1380

NOVEMBER 1967

NUMERICAL DIFFERENTIATION USING ORTHOGONAL POLYNOMIALS

G. H. Jones

Terminal Ballistics Laboratory

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BALLISTIC RESEARCH LABORATORIES

REPORT NO. 1380

GHJonas/mba/ilm
Aberdeen Proving Ground, Md.
November 1967

NUMERICAL DIFFERENTIATION USING ORTHOGONAL POLYNOMIALS

ABSTRACT

The present method of reducing data by formulating a polynomial, taking the derivative and substituting experimental values, has been simplified for instances where the independent variable is evenly spaced and the derivative is desired at these given points only.

Orthogonal polynomials, resembling Legendre polynomials, are differentiated and tabulated in a useful form. When used in conjunction with available orthogonal polynomial tables, these new tables give derivatives more readily than the usual procedure. These tables present coefficients for obtaining first derivatives at 6 to 21 evenly spaced points. This report extends the available orthogonal polynomial tables to sixth and seventh degrees, when appropriate, and gives a demonstration of the method.

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INTRODUCTION

Numerical differentiation calculates approximate values of derivatives from given discrete values of the function itself. The functional values determine an interpolating function whose derivative approximates derivatives of the original function. Commonly, the interpolation function is a polynomial which fits the given points exactly, or it is a polynomial of lower degree determined by the method of least squares.

Numerical data obtained from experiments require polynomials of lower degree, since fitting given points exactly would lead to derivatives which are too sensitive to experimental errors in the original data. If the values of the independent variable are uniformly spaced, the method of orthogonal polynomials greatly simplifies the calculation. Tables of coefficients for using this method appear in Milne's Numerical Calculus.^{*} To use this method, the polynomial must be formulated and differentiated.

The study reported here avoids this laborious process when the quantities of interest occur at the given data points. Tables have been constructed which give the values of the derivative at the uniformly spaced points, avoiding the explicit conversion to the original independent variable and combining like powers; these tables are found in the Appendix. The successive contributions of the orthogonal polynomials of various degrees are calculated separately, thus providing a simple criterion for selecting the best degree to use. The degree must be high enough to follow the significant variation of the data but not so high as to destroy the particular direction of the original data.

The following sections present the approximating polynomial and the derivative, and they describe how the new tables are constructed for the derivative and how they are used.

^{*} William Edmond Milne, Numerical Calculus, 2nd ed., Princeton, New Jersey, Princeton University Press, 1949, Chapter 9, pp. 265-275.

FORMULATION

Let $(x_0, y_0), (x_1, y_1), \dots, (x_n, y_n)$ denote $(n + 1)$ corresponding pairs of values of any two variables x and y . If the values of x are free of error and equidistant (interval h), the independent variable x can be represented by the points, $0, 1, \dots, n$ in the transformation $s = (x - x_0)/h$ and $y(x) = f(s)$. The approximating polynomial of the m th degree can be written*

$$Q_m(s) = \sum_{k=0}^m \left[\frac{\sum_{s=0}^n P_{k,n}(s)f(s)}{\sum_{s=0}^n P_{k,n}^2(s)} \right] P_{k,n}(s) \equiv \sum_{k=0}^m \frac{c_k}{S_k} P_{k,n}(s)$$

where**

$$P_{k,n}(s) = \sum_{r=0}^k (-1)^r \binom{k}{r} \binom{k+r}{r} \frac{s^{(r)}}{n^{(r)}}.$$

The fractions c_k/S_k form a unique set of constants which yield the most probable equation through the least square technique. With the functional values from the data and available tables of orthogonal polynomial values,* these constants can readily be obtained.

Differentiating this approximating polynomial with respect to x , we have

$$Q'_m(x) = \frac{1}{h} Q'_m(s) = \frac{1}{h} \sum_{k=0}^m \frac{c_k}{S_k} P'_{k,n}(s).$$

Since the quantities $P'_{k,n}(s)$ do not depend on $f(s)$ they can be computed once and for all and presented in tabular form.

* Milne

** $s^{(r)} = s(s-1)(s-2) \dots (s-r+1)$

$\binom{k}{r} = \frac{k!}{r!(k-r)!}$

CONSTRUCTION AND USE OF THE TABLES

The tables in the Appendix of this report are constructed from a useful form of the derivatives. The entries in each column of the available orthogonal polynomial tables are numerators of numbers with their common denominator being the uppermost entry of its column.* Since the approximating polynomials take into account the denominators, the derivative tables had to be modified so that they could be used in conjunction with the available tables. The tabulated expressions $P'_{k,n}(s)$ are multiplied by the corresponding $s = 0$ value of the existing orthogonal polynomial tables. These products again are fractions, and since integral coefficients are desired in the derivative tables, a common denominator is determined for each column. These in turn are multiplied by h to take care of the change of the variable. These denominators, D , are indicated at the base of their respective columns in the derivative tables. It is also common practice to express $P_{k,n}(x)$ and $P'_{k,n}(x)$ in the tables as $P_k(x)$ and $P'_k(x)$ since n is indicated in the top line of each table. Also $P_0(x)$ is not shown since its entries are always unity.

To illustrate the use of the tables, consider the simple function $y = e^x$ where x takes on values 0 to 0.9 and the interval $h = 0.1$. Assume the data can be fitted with a fifth degree polynomial. It is well known that the derivation of $y = e^x$ is e^x . The appropriate orthogonal polynomial table and initial calculations can be seen in Example 1, Table 1a. The coefficients in row (C) are the quotients of (B) divided by (A). (A) is the sum of squares of the entries in the column above. (B) is the sum of the products of the entries in this column multiplied by their corresponding y 's. For example, the coefficient in row (C) at the base of the column $P_1(s)$ is

$$\left[\frac{(1.00000)(9) + (1.10517)(7) + \dots + (2.22554)(-7)}{330} + \frac{(2.45960)(-9)}{330} \right] = -0.0803467 .$$

* Milne

EXAMPLE 1

(Table 1a) Orthogonal Polynomial Table for $n = 9$ (10 points), (See Appendix)

| x | s | y | $P_0(s)$ | $P_1(s)$ | $P_2(s)$ | $P_3(s)$ | $P_4(s)$ | $P_5(s)$ |
|-----|-----|----------|------------|----------|------------|-----------|------------|----------|
| 0 | 0 | 1.00000 | 1 | 9 | 6 | 42 | 18 | 6 |
| 0.1 | 1 | 1.10517 | 1 | 7 | 2 | -14 | -22 | -14 |
| 0.2 | 2 | 1.22140 | 1 | 5 | -1 | -35 | -17 | 1 |
| 0.3 | 3 | 1.34980 | 1 | 3 | -3 | -31 | 3 | 11 |
| 0.4 | 4 | 1.49182 | 1 | 1 | -4 | -12 | 18 | 6 |
| 0.5 | 5 | 1.64372 | 1 | -1 | -4 | 12 | 18 | -6 |
| 0.6 | 6 | 1.82212 | 1 | -3 | -3 | 31 | 3 | -11 |
| 0.7 | 7 | 2.01375 | 1 | -5 | -1 | 35 | -17 | -1 |
| 0.8 | 8 | 2.22554 | 1 | -7 | 2 | 14 | -22 | 14 |
| 0.9 | 9 | 2.45960 | 1 | -9 | 6 | -42 | 18 | -6 |
| | (A) | 10 | 330 | 132 | 8580 | 2820 | 780 | |
| | (B) | 16.33798 | -26.51442 | 2.10577 | -1.36291 | 0.04529 | -0.00103 | |
| | (C) | 1.633798 | -0.0803467 | 0.015953 | -0.0001508 | 0.0000161 | -0.0000013 | |

(Table 1b) Numerical Differentiation Table for $n = 9$ (10 points), (See Appendix)

| s | $P_1'(s)$ | $P_2'(s)$ | $P_3'(s)$ | $P_4'(s)$ | $P_5'(s)$ |
|-------|-----------|-----------|-----------|-----------|-----------|
| 0 | -2 | -9 | -461 | -225 | -911 |
| 1 | -2 | -7 | -221 | -35 | 109 |
| 2 | -2 | -5 | -41 | 50 | 244 |
| 3 | -2 | -3 | 79 | 60 | 34 |
| 4 | -2 | -1 | 139 | 25 | -161 |
| ----- | | | | | |
| 5 | -2 | 1 | 139 | -25 | -161 |
| 6 | -2 | 3 | 79 | -60 | 34 |
| 7 | -2 | 5 | -41 | -50 | 244 |
| 8 | -2 | 7 | -221 | 35 | 109 |
| 9 | -2 | 9 | -461 | 225 | -911 |
| | (D) | 1h | 2h | 3h | 15h |

(Table 1c) Individual Contributions of the Various Degrees

| s | $P_1'(s)$ | $P_2'(s)$ | $P_3'(s)$ | $P_4'(s)$ | $P_5'(s)$ | |
|-------|-----------|-----------|-----------|-----------|-----------|-----------|
| 0 | 1.606934 | -0.717665 | 0.122165 | -0.012150 | 0.000011 | |
| 1 | 1.606934 | -0.558355 | 0.058565 | -0.001890 | -0.000109 | |
| 2 | 1.606934 | -0.398825 | 0.010005 | 0.002700 | -0.000244 | |
| 3 | 1.606934 | -0.239295 | -0.020935 | 0.003240 | -0.000034 | |
| 4 | 1.606934 | -0.079765 | -0.036835 | 0.001350 | 0.000161 | |
| ----- | | | | | | |
| 5 | 1.606934 | 0.079765 | -0.036835 | -0.001350 | 0.000161 | |
| 6 | 1.606934 | 0.239295 | -0.020935 | -0.003240 | -0.000034 | |
| 7 | 1.606934 | 0.398825 | 0.010005 | -0.002700 | -0.000244 | |
| 8 | 1.606934 | 0.558355 | 0.058565 | 0.001890 | -0.000109 | |
| 9 | 1.606934 | 0.717665 | 0.122165 | 0.012150 | 0.000011 | |
| | (D) | 0.1 | 0.2 | 0.6 | 0.3 | 1.5 |
| | (C/D) | -0.803467 | 0.079765 | -0.000265 | 0.000054 | -0.000001 |

EXAMPLE 1 (Cont'd)

(Table 1d) Cumulative Value of the Derivation For The Various Degrees

| s | $P_1'(s)$ | $P_2'(s)$ | $P_3'(s)$ | $P_4'(s)$ | $P_5'(s)$ |
|---|-----------|-----------|-----------|-----------|-----------|
| 0 | 1.606934 | 0.889049 | 1.011214 | 0.999064 | 0.999975 |
| 1 | 1.606934 | 1.048579 | 1.107144 | 1.105254 | 1.105145 |
| 2 | 1.606934 | 1.208109 | 1.218974 | 1.221674 | 1.221430 |
| 3 | 1.606934 | 1.367639 | 1.346704 | 1.349944 | 1.349910 |
| 4 | 1.606934 | 1.527169 | 1.490334 | 1.491684 | 1.491845 |
| 5 | 1.606934 | 1.686699 | 1.649864 | 1.648514 | 1.648675 |
| 6 | 1.606934 | 1.846229 | 1.825294 | 1.822054 | 1.822020 |
| 7 | 1.606934 | 2.005759 | 2.016624 | 2.013924 | 2.013680 |
| 8 | 1.606934 | 2.165289 | 2.223854 | 2.225744 | 2.225635 |
| 9 | 1.606934 | 2.324819 | 2.446984 | 2.459134 | 2.460045 |

For the derivative calculations, the coefficients in row (C) of the orthogonal polynomial tables are divided by the denominators of the respective columns in the derivative tables [row (D) in Table 1b]. The newly computed values [row (C) in Table 1a divided by row (D) in Table 1b] are presented in row (C/D) of Table 1c. Each of the tabulated values in the columns of Table 1c is obtained by multiplying the value in row (C/D) by the individual entries in the corresponding column of Table 1b. The products are entered in rows 0-9 in Table 1c. These are the individual contributions. Table 1d displays the cumulative values. For instance, the value in Table 1d corresponding to the entry in row $s = 4$, under P'_5 is

$$\frac{(-0.0803467)(-2)}{0.1} + \frac{(0.015955)(-1)}{0.2} + \frac{(-0.0001588)(139)}{0.6} \\ + \frac{(0.0000161)(25)}{0.3} + \frac{(-0.0000013)(-161)}{1.5} = 1.491845 .$$

The first three terms give the value for the entry in the same row, $s = 4$, under P'_3 . These values are indeed approximately the original values of $y = e^x$.

This example has shown the mechanics for using the tables to determine derivatives at the given points. Of course no local irregularities were noted in the fitting since the data was that of a smooth function and not experimentally obtained data.

SUMMARY

In the expectation that for many purposes derivatives at values of the independent variable in the original data would suffice, tables were constructed to facilitate this end. It is required that the data be evenly spaced. For convenience, extension of the orthogonal polynomial tables as in Milne were constructed to take into account fits of higher degrees.

ACKNOWLEDGMENT

We wish to express our thanks to Dr. R. F. Jackson of the University of Delaware for his helpful suggestions.

APPENDIX

Tables. Orthogonal Polynomials and Numerical Differentiation Tables for $n + 1$ Equally Spaced Points

$n = 5$ (6 points)

| s | $P_1(s)$ | $P_2(s)$ | $P_3(s)$ | $P_4(s)$ | $P_5(s)$ |
|-----|----------|----------|----------|----------|----------|
| 0 | 5 | 5 | 5 | 1 | 1 |
| 1 | 2 | -1 | -7 | -2 | -5 |
| 2 | 1 | -4 | -4 | 2 | 10 |
| 3 | -1 | -4 | 4 | 2 | -10 |
| 4 | -2 | -1 | 7 | -2 | 5 |
| 5 | -5 | 5 | -5 | 1 | -1 |
| A | 72 | 84 | 180 | 24 | 252 |

| s | $P_1'(s)$ | $P_2'(s)$ | $P_3'(s)$ | $P_4'(s)$ | $P_5'(s)$ |
|-----|-----------|-----------|-----------|-----------|-----------|
| 0 | -2 | -15 | -137 | -50 | -1756 |
| 1 | -2 | -9 | -17 | 12 | 504 |
| 2 | -2 | -3 | 43 | 11 | -181 |
| 3 | -2 | 3 | 43 | -11 | -181 |
| 4 | -2 | 9 | -17 | -12 | 504 |
| 5 | -2 | 15 | -137 | 50 | -1756 |
| D | $1h$ | $2h$ | $6h$ | $3h$ | $15h$ |

Tables. Orthogonal Polynomials and Numerical Differentiation Tables for $n + 1$
Equally Spaced Points

$n = 6$ (7 points)

| s | $P_1(s)$ | $P_2(s)$ | $P_3(s)$ | $P_4(s)$ | $P_5(s)$ | $P_6(s)$ |
|-----|----------|----------|----------|----------|----------|----------|
| 0 | 3 | 5 | 1 | 3 | 1 | 1 |
| 1 | 2 | 0 | -1 | -7 | -4 | -6 |
| 2 | 1 | -3 | -1 | 1 | 5 | 15 |
| 3 | 0 | -4 | 0 | 6 | 0 | -20 |
| 4 | -1 | -3 | 1 | 1 | -5 | 15 |
| 5 | -2 | 0 | 1 | -7 | 4 | -6 |
| 6 | -3 | 5 | -1 | 3 | -1 | 1 |
| A | 28 | 84 | 6 | 154 | 84 | 924 |

| s | $P_1'(s)$ | $P_2'(s)$ | $P_3'(s)$ | $P_4'(s)$ | $P_5'(s)$ | $P_6'(s)$ |
|-----|-----------|-----------|-----------|-----------|-----------|-----------|
| 0 | -1 | -6 | -20 | -177 | -1207 | -3479 |
| 1 | -1 | -4 | -5 | 22 | 368 | 924 |
| 2 | -1 | -2 | 4 | 53 | 53 | -413 |
| 3 | -1 | 0 | 7 | 0 | -262 | 0 |
| 4 | -1 | 2 | 4 | -53 | 53 | 413 |
| 5 | -1 | 4 | -5 | -22 | 368 | -924 |
| 6 | -1 | 6 | -20 | 177 | -1207 | 3479 |
| D | 1h | 1h | 6h | 6h | 3.h | 1.h |

Tables. Orthogonal Polynomials and Numerical Differentiation Tables for $n + 1$
Equally Spaced Points

$n = 7(8 \text{ points})$

| s | $P_1(s)$ | $P_2(s)$ | $P_3(s)$ | $P_4(s)$ | $P_5(s)$ | $P_6(s)$ | $P_7(s)$ |
|-----|----------|----------|----------|----------|----------|----------|----------|
| 0 | 7 | 7 | 7 | 7 | 7 | 1 | 1 |
| 1 | 5 | 1 | -5 | -13 | -23 | -5 | -7 |
| 2 | 3 | -3 | -7 | -3 | 17 | 9 | 21 |
| 3 | 1 | -5 | -3 | 9 | 15 | -5 | -35 |
| 4 | -1 | -5 | 3 | 9 | -15 | -5 | 35 |
| 5 | -3 | -3 | 7 | -3 | -17 | 9 | -21 |
| 6 | -5 | 1 | 5 | -13 | 23 | -5 | 7 |
| 7 | -7 | 7 | -7 | 7 | -7 | 1 | -1 |
| A | 168 | 168 | 264 | 616 | 2184 | 264 | 3432 |

| s | $P_1'(s)$ | $P_2'(s)$ | $P_3'(s)$ | $P_4'(s)$ | $P_5'(s)$ | $P_6'(s)$ | $P_7'(s)$ |
|-----|-----------|-----------|-----------|-----------|-----------|-----------|-----------|
| 0 | -2 | -7 | -55 | -287 | -4519 | -1029 | -75823 |
| 1 | -2 | -5 | -19 | 5 | 1151 | 305 | 16521 |
| 2 | -2 | -3 | 5 | 87 | 731 | -61 | -7643 |
| 3 | -2 | -1 | 17 | 43 | -739 | -127 | 2605 |
| 4 | -2 | 1 | 17 | -43 | -739 | 127 | 2605 |
| 5 | -2 | 3 | 5 | -87 | 731 | 61 | -7643 |
| 6 | -2 | 5 | -19 | -5 | 1151 | -305 | 16521 |
| 7 | -2 | 7 | -55 | 287 | -4519 | 1029 | -75823 |
| D | 1h | 1h | 3h | 6h | 30h | 10h | 7h |

Tables. Orthogonal Polynomials and Numerical Differentiation Tables for $n + 1$
Equally Spaced Points

$n = 8$ (9 points)

| s | $P_1(s)$ | $P_2(s)$ | $P_3(s)$ | $P_4(s)$ | $P_5(s)$ | $P_6(s)$ | $P_7(s)$ |
|-----|----------|----------|----------|----------|----------|----------|----------|
| 0 | 4 | 28 | 14 | 14 | 4 | 4 | 1 |
| 1 | 3 | 7 | -7 | -21 | -11 | -17 | -6 |
| 2 | 2 | -8 | -13 | -11 | 4 | 22 | 14 |
| 3 | 1 | -17 | -9 | 9 | 9 | 1 | -14 |
| 4 | 0 | -20 | 0 | 18 | 0 | -20 | 0 |
| 5 | -1 | -17 | 9 | 9 | -9 | 1 | 14 |
| 6 | -2 | -8 | 13 | -11 | -4 | 22 | -14 |
| 7 | -3 | 7 | 7 | -21 | 11 | -17 | 6 |
| 8 | -4 | 28 | -14 | 14 | -4 | 4 | -1 |
| A | 65 | 2772 | 990 | 2002 | 463 | 1980 | 858 |

| s | $P_1'(s)$ | $P_2'(s)$ | $P_3'(s)$ | $P_4'(s)$ | $P_5'(s)$ | $P_6'(s)$ | $P_7'(s)$ |
|-----|-----------|-----------|-----------|-----------|-----------|-----------|-----------|
| 0 | -1 | -24 | -181 | -436 | -1678 | -1076 | -39051 |
| 1 | -1 | -18 | -76 | -33 | 317 | 583 | 10096 |
| 2 | -1 | -12 | -1 | 118 | 392 | 52 | -3694 |
| 3 | -1 | -6 | 44 | 101 | -103 | -339 | -1496 |
| 4 | -1 | 0 | 59 | 0 | -358 | 0 | 3985 |
| 5 | -1 | 6 | 44 | -101 | -103 | 339 | -1496 |
| 6 | -1 | 12 | -1 | -118 | 392 | -52 | -3694 |
| 7 | -1 | 18 | -76 | 33 | 317 | -583 | 10096 |
| 8 | -1 | 24 | -181 | 436 | -1678 | 1076 | -39051 |
| D | 1h | 1h | 6h | 6h | 30h | 10h | 140h |

Tables. Orthogonal Polynomials and Numerical Differentiation Tables for $n + 1$
Equally Spaced Points

$n = 9(10 \text{ points})$

| s | $P_1(s)$ | $P_2(s)$ | $P_3(s)$ | $P_4(s)$ | $P_5(s)$ | $P_6(s)$ | $P_7(s)$ |
|-----|----------|----------|----------|----------|----------|----------|----------|
| 0 | 9 | 6 | 42 | 18 | 6 | 3 | 9 |
| 1 | 7 | 2 | -14 | -22 | -14 | -11 | -47 |
| 2 | 5 | -1 | -35 | -17 | 1 | 10 | 86 |
| 3 | 3 | -3 | -31 | 3 | 11 | 6 | -42 |
| 4 | 1 | -4 | -12 | 18 | 6 | -8 | -56 |
| 5 | -1 | -4 | 12 | 18 | -6 | -8 | 56 |
| 6 | -3 | -3 | 31 | 3 | -11 | 6 | 42 |
| 7 | -5 | -1 | 35 | -17 | -1 | 10 | -86 |
| 8 | -7 | 2 | 14 | -22 | 14 | -11 | 47 |
| 9 | -9 | 6 | -42 | 18 | -6 | 3 | -9 |
| A | 327 | 132 | 8587 | 2867 | 787 | 667 | 29172 |

| s | $P_1'(s)$ | $P_2'(s)$ | $P_3'(s)$ | $P_4'(s)$ | $P_5'(s)$ | $P_6'(s)$ | $P_7'(s)$ |
|-----|-----------|-----------|-----------|-----------|-----------|-----------|-----------|
| 0 | -2 | -7 | -461 | -225 | -911 | -1779 | -151399 |
| 1 | -2 | -7 | -221 | -35 | 109 | 483 | 42697 |
| 2 | -2 | -5 | -41 | 59 | 244 | 191 | -8466 |
| 3 | -2 | -3 | 79 | 69 | 34 | -278 | -16138 |
| 4 | -2 | -1 | 139 | 25 | -161 | -191 | 11981 |
| 5 | -2 | 1 | 139 | -25 | -161 | 191 | 11981 |
| 6 | -2 | 3 | 79 | -69 | 34 | -278 | -16138 |
| 7 | -2 | 5 | -41 | -59 | 244 | -191 | -8466 |
| 8 | -2 | 7 | -221 | 35 | 109 | -483 | 42697 |
| 9 | -2 | 7 | -461 | 225 | -911 | 1779 | -151399 |
| 0 | 1h | 2h | 5h | 3h | 15h | 20h | 140h |

Tables. Orthogonal Polynomials and Numerical Differentiation Tables for $n + 1$
Equally Spaced Points

$n = 10$ (11 points)

| s | $P_1(s)$ | $P_2(s)$ | $P_3(s)$ | $P_4(s)$ | $P_5(s)$ | $P_6(s)$ | $P_7(s)$ |
|-----|----------|----------|----------|----------|----------|----------|----------|
| 0 | 5 | 15 | 30 | 6 | 3 | 15 | 5 |
| 1 | 4 | 6 | -6 | -6 | -6 | -48 | -23 |
| 2 | 3 | -1 | -22 | -6 | -1 | 29 | 33 |
| 3 | 2 | -6 | -23 | -1 | 4 | 36 | -2 |
| 4 | 1 | -9 | -14 | 4 | 4 | -12 | -28 |
| 5 | 0 | -10 | 0 | 6 | 0 | -40 | 0 |
| 6 | -1 | -9 | 14 | 4 | -4 | -12 | 28 |
| 7 | -2 | -6 | 23 | -1 | -4 | 36 | 2 |
| 8 | -3 | -1 | 22 | -6 | 1 | 29 | -33 |
| 9 | -4 | 6 | 6 | -6 | 6 | -48 | 23 |
| 10 | -5 | 15 | -30 | 6 | -3 | 15 | -5 |
| A | 110 | 858 | 4290 | 286 | 156 | 11220 | 4852 |

| s | $P_1'(s)$ | $P_2'(s)$ | $P_3'(s)$ | $P_4'(s)$ | $P_5'(s)$ | $P_6'(s)$ | $P_7'(s)$ |
|-----|-----------|-----------|-----------|-----------|-----------|-----------|-----------|
| 0 | -1 | -10 | -286 | -125 | -1411 | -6765 | -139095 |
| 1 | -1 | -6 | -151 | -23 | 74 | 1424 | 40119 |
| 2 | -1 | -6 | -46 | 21 | 389 | 1033 | -159 |
| 3 | -1 | -4 | 20 | 34 | 154 | -648 | -20074 |
| 4 | -1 | -2 | 74 | 23 | -151 | -989 | 1549 |
| 5 | -1 | 0 | 89 | 0 | -236 | 0 | 17355 |
| 6 | -1 | 2 | 74 | -23 | -151 | 989 | 1549 |
| 7 | -1 | 4 | 20 | -34 | 154 | 648 | -20074 |
| 8 | -1 | 6 | -46 | -21 | 389 | -1033 | -159 |
| 9 | -1 | 6 | -151 | 23 | 74 | -1424 | 40119 |
| 10 | -1 | 10 | -286 | 125 | -1411 | 6765 | -139095 |
| D | 1h | 1h | 6h | 6h | 67h | 27h | 420h |

Tables. Orthogonal Polynomials and Numerical Differentiation Tables for $n + 1$ Equally Spaced Points

$n = 11$ (12 points)

| s | $P_1(s)$ | $P_2(s)$ | $P_3(s)$ | $P_4(s)$ | $P_5(s)$ | $P_6(s)$ | $P_7(s)$ |
|-----|----------|----------|----------|----------|----------|----------|----------|
| 0 | 11 | 55 | 33 | 33 | 33 | 11 | 55 |
| 1 | 9 | 25 | -3 | -27 | -57 | -31 | -225 |
| 2 | 7 | 1 | -21 | -33 | -21 | 11 | 251 |
| 3 | 5 | -17 | -25 | -13 | 29 | 25 | 83 |
| 4 | 3 | -29 | -19 | 12 | 44 | 4 | -204 |
| 5 | 1 | -35 | -7 | 28 | 27 | -27 | -147 |
| 6 | -1 | -35 | 7 | 28 | -27 | -27 | 147 |
| 7 | -3 | -29 | 19 | 12 | -44 | 4 | 204 |
| 8 | -5 | -17 | 25 | -13 | -29 | 25 | -83 |
| 9 | -7 | 1 | 21 | -33 | 21 | 11 | -251 |
| 10 | -9 | 25 | 3 | -27 | 57 | -31 | 225 |
| 11 | -11 | 55 | -33 | 33 | -33 | 11 | -55 |
| A | 572 | 12512 | 5148 | 8008 | 15912 | 4498 | 369512 |

| s | $P'_1(s)$ | $P'_2(s)$ | $P'_3(s)$ | $P'_4(s)$ | $P'_5(s)$ | $P'_6(s)$ | $P'_7(s)$ |
|-----|-----------|-----------|-----------|-----------|-----------|-----------|-----------|
| 0 | -2 | -33 | -139 | -1177 | -12581 | -9987 | -975225 |
| 1 | -2 | -27 | -79 | -333 | -131 | 1987 | 273855 |
| 2 | -2 | -21 | -31 | 133 | 3349 | 2121 | 55287 |
| 3 | -2 | -15 | 5 | 305 | 2179 | -445 | -146369 |
| 4 | -2 | -9 | 29 | 267 | -401 | -1711 | -58533 |
| 5 | -2 | -3 | 41 | 103 | -2231 | -877 | 170115 |
| 6 | -2 | 3 | 41 | -103 | -2231 | 877 | 170115 |
| 7 | -2 | 9 | 29 | -267 | -401 | 1711 | -58533 |
| 8 | -2 | 15 | 5 | -305 | 2179 | 445 | -146369 |
| 9 | -2 | 21 | -31 | -133 | 3349 | -2121 | 55287 |
| 10 | -2 | 27 | -79 | 333 | -131 | -1987 | 273855 |
| 11 | -2 | 33 | -139 | 1177 | -12581 | 9987 | -975225 |
| D | 1h | 1h | 3h | 12h | 63h | 63h | 427h |

Tables. Orthogonal Polynomials and Numerical Differentiation Tables for $n + 1$
Equally Spaced Points

$n = 12(13 \text{ points})$

| s | $P_1(s)$ | $P_2(s)$ | $P_3(s)$ | $P_4(s)$ | $P_5(s)$ | $P_6(s)$ | $P_7(s)$ |
|-----|----------|----------|----------|----------|----------|----------|----------|
| 0 | 6 | 22 | 11 | 99 | 22 | 22 | 33 |
| 1 | 5 | 11 | 0 | -66 | -33 | -55 | -121 |
| 2 | 4 | 2 | -6 | -96 | -18 | 8 | 103 |
| 3 | 3 | -5 | -8 | -54 | 11 | 43 | 75 |
| 4 | 2 | -10 | -7 | 11 | 26 | 22 | -65 |
| 5 | 1 | -13 | -4 | 64 | 20 | -20 | -100 |
| 6 | 0 | -14 | 0 | 84 | 0 | -47 | 0 |
| 7 | -1 | -13 | 4 | 64 | -20 | -20 | 100 |
| 8 | -2 | -10 | 7 | 11 | -26 | 22 | 65 |
| 9 | -3 | -5 | 8 | -54 | -11 | 43 | -75 |
| 10 | -4 | 2 | 6 | -96 | 18 | 8 | -103 |
| 11 | -5 | 11 | 0 | -66 | 33 | -55 | 121 |
| 12 | -6 | 22 | -11 | 99 | -22 | 22 | -33 |
| A | 182 | 2702 | 572 | 68068 | 6188 | 14712 | 92378 |

| s | $P_1'(s)$ | $P_2'(s)$ | $P_3'(s)$ | $P_4'(s)$ | $P_5'(s)$ | $P_6'(s)$ | $P_7'(s)$ |
|-----|-----------|-----------|-----------|-----------|-----------|-----------|-----------|
| 0 | -1 | -12 | -83 | -1542 | -7024 | -15424 | -409932 |
| 1 | -1 | -10 | -50 | -515 | -479 | 2235 | 107739 |
| 2 | -1 | -8 | -23 | 92 | 1726 | 3744 | 45033 |
| 3 | -1 | -6 | -2 | 363 | 1481 | 253 | -54045 |
| 4 | -1 | -4 | 13 | 382 | 256 | -2768 | -46660 |
| 5 | -1 | -2 | 22 | 233 | -899 | -2209 | 19063 |
| 6 | -1 | 0 | 25 | 0 | -1354 | 0 | 54189 |
| 7 | -1 | 2 | 22 | -233 | -899 | 2209 | 19063 |
| 8 | -1 | 4 | 13 | -382 | 256 | 2368 | -46660 |
| 9 | -1 | 6 | -2 | -363 | 1481 | -253 | -54045 |
| 10 | -1 | 8 | -23 | -92 | 1726 | -3744 | 45033 |
| 11 | -1 | 10 | -50 | 515 | -479 | -2235 | 107739 |
| 12 | -1 | 12 | -83 | 1542 | -7024 | 15424 | -409932 |
| D | 1h | 1h | 6h | 6h | 60h | 60h | 420h |

Tables. Orthogonal Polynomials and Numerical Differentiation Tables for $n + 1$
Equally Spaced Points

$n = 13(14 \text{ points})$

| s | $P_1(s)$ | $P_2(s)$ | $P_3(s)$ | $P_4(s)$ | $P_5(s)$ | $P_6(s)$ | $P_7(s)$ |
|----|----------|----------|----------|----------|----------|----------|----------|
| 0 | 13 | 13 | 143 | 143 | 143 | 143 | 143 |
| 1 | 11 | 7 | 11 | -77 | -137 | -219 | -473 |
| 2 | 9 | 2 | -66 | -132 | -132 | -11 | 297 |
| 3 | 7 | -2 | -98 | -92 | 28 | 227 | 353 |
| 4 | 5 | -5 | -95 | -13 | 139 | 185 | -95 |
| 5 | 3 | -7 | -67 | 63 | 145 | -25 | -375 |
| 6 | 1 | -8 | -24 | 108 | 60 | -200 | -200 |
| 7 | -1 | -8 | 24 | 108 | -60 | -200 | 200 |
| 8 | -3 | -7 | 67 | 63 | -145 | -25 | 375 |
| 9 | -5 | -5 | 95 | -13 | -139 | 185 | 95 |
| 10 | -7 | -2 | 98 | -92 | -28 | 227 | -353 |
| 11 | -9 | 2 | 66 | -132 | 132 | -11 | -297 |
| 12 | -11 | 7 | -11 | -77 | 137 | -219 | 473 |
| 13 | -13 | 13 | -143 | 143 | -143 | 143 | -143 |
| A | 910 | 728 | 97240 | 135136 | 235144 | 497420 | 1293292 |

| s | $P_1'(s)$ | $P_2'(s)$ | $P_3'(s)$ | $P_4'(s)$ | $P_5'(s)$ | $P_6'(s)$ | $P_7'(s)$ |
|----|-----------|-----------|-----------|-----------|-----------|-----------|-----------|
| 0 | -2 | -13 | -977 | -988 | -9794 | -81536 | -1328792 |
| 1 | -2 | -11 | -617 | -374 | -1184 | 8008 | 315872 |
| 2 | -2 | -9 | -317 | 9 | 2141 | 20937 | 205587 |
| 3 | -2 | -7 | -77 | 203 | 2281 | 5831 | -131197 |
| 4 | -2 | -5 | 103 | 250 | 916 | -9520 | -192440 |
| 5 | -2 | -3 | 223 | 192 | -694 | -13496 | -22032 |
| 6 | -2 | -1 | 283 | 71 | -1709 | -6027 | 150567 |
| 7 | -2 | 1 | 283 | -71 | -1709 | 6027 | 150567 |
| 8 | -2 | 3 | 223 | -192 | -694 | 13496 | -22032 |
| 9 | -2 | 5 | 103 | -250 | 916 | 9520 | -192440 |
| 10 | -2 | 7 | -77 | -203 | 2281 | -5831 | -131197 |
| 11 | -2 | 9 | -317 | -9 | 2141 | -20937 | 205587 |
| 12 | -2 | 11 | -617 | 374 | -1184 | -8008 | 315872 |
| 13 | -2 | 13 | -977 | 988 | -9794 | 81536 | -1328792 |
| D | 1h | 2h | 6h | 3h | 15h | 60h | 420h |

Tables. Orthogonal Polynomials and Numerical Differentiation Tables for $n + 1$ Equally Spaced Points

$n = 14(15 \text{ points})$

| s | $P_1(s)$ | $P_2(s)$ | $P_3(s)$ | $P_4(s)$ | $P_5(s)$ | $P_6(s)$ | $P_7(s)$ |
|----|----------|----------|----------|----------|----------|----------|----------|
| 0 | 7 | 91 | 91 | 1001 | 1001 | 143 | 13 |
| 1 | 6 | 52 | 13 | -429 | -1144 | -286 | -39 |
| 2 | 5 | 19 | -35 | -869 | -979 | -55 | 17 |
| 3 | 4 | -5 | -58 | -704 | -44 | 176 | 31 |
| 4 | 3 | -29 | -61 | -249 | 751 | 197 | 3 |
| 5 | 2 | -44 | -49 | 251 | 1000 | 50 | -25 |
| 6 | 1 | -53 | -27 | 621 | 675 | -125 | -25 |
| 7 | 0 | -56 | 0 | 756 | 0 | -200 | 0 |
| 8 | -1 | -53 | 27 | 621 | -675 | -125 | 25 |
| 9 | -2 | -44 | 49 | 251 | -1000 | 50 | 25 |
| 10 | -3 | -29 | 61 | -249 | -751 | 197 | -3 |
| 11 | -4 | -8 | 58 | -704 | 44 | 176 | -31 |
| 12 | -5 | 19 | 35 | -869 | 979 | -55 | -17 |
| 13 | -6 | 52 | -13 | -429 | 1144 | -286 | 39 |
| 14 | -7 | 91 | -91 | 1001 | -1001 | 143 | -13 |
| A | 280 | 37128 | 39780 | 6466460 | 10581480 | 426760 | 8398 |

| s | $P_1'(s)$ | $P_2'(s)$ | $P_3'(s)$ | $P_4'(s)$ | $P_5'(s)$ | $P_6'(s)$ | $P_7'(s)$ |
|----|-----------|-----------|-----------|-----------|-----------|-----------|-----------|
| 0 | -1 | -42 | -568 | -12425 | -119881 | -68306 | -94762 |
| 1 | -1 | -36 | -373 | -5190 | -20236 | 3592 | 19716 |
| 2 | -1 | -30 | -208 | -475 | 22499 | 17770 | 18022 |
| 3 | -1 | -24 | -73 | 2140 | 29114 | 8048 | -5519 |
| 4 | -1 | -18 | 32 | 3075 | 16019 | -4954 | -14682 |
| 5 | -1 | -12 | 107 | 2750 | -1756 | -11176 | -6772 |
| 6 | -1 | -6 | 152 | 1585 | -16561 | -3478 | 6486 |
| 7 | -1 | 0 | 157 | 0 | -22126 | 0 | 12597 |
| 8 | -1 | 6 | 152 | -1585 | -16561 | 3478 | 6486 |
| 9 | -1 | 12 | 107 | -2750 | -1756 | 11176 | -6772 |
| 10 | -1 | 18 | 32 | -3075 | 16019 | 4954 | -14682 |
| 11 | -1 | 24 | -73 | -2140 | 29114 | -8048 | -5519 |
| 12 | -1 | 30 | -208 | -475 | 22499 | -17770 | 18022 |
| 13 | -1 | 36 | -373 | 5190 | -20236 | -3592 | 19716 |
| 14 | -1 | 42 | -568 | 12425 | -119881 | 68306 | -94762 |
| 0 | 1h | 1h | 5h | 6h | 32h | 62h | 420h |

Tables. Orthogonal Polynomials and Numerical Differentiation Tables for $n + 1$
Equally Spaced Points

$n = 15(16 \text{ points})$

| s | $P_1(s)$ | $P_2(s)$ | $P_3(s)$ | $P_4(s)$ | $P_5(s)$ | $P_6(s)$ | $P_7(s)$ |
|-----|----------|----------|----------|----------|----------|----------|----------|
| 0 | 15 | 35 | 455 | 273 | 143 | 65 | 195 |
| 1 | 13 | 21 | 91 | -91 | -143 | -117 | -533 |
| 2 | 11 | 7 | -143 | -221 | -143 | -39 | 143 |
| 3 | 9 | -1 | -267 | -201 | -33 | 59 | 423 |
| 4 | 7 | -9 | -301 | -101 | 77 | 87 | 157 |
| 5 | 5 | -15 | -265 | 23 | 131 | 45 | -235 |
| 6 | 3 | -19 | -179 | 129 | 115 | -25 | -375 |
| 7 | 1 | -21 | -63 | 189 | 45 | -75 | -175 |
| 8 | -1 | -21 | 63 | 189 | -45 | -75 | 175 |
| 9 | -3 | -19 | 179 | 129 | -115 | -25 | 375 |
| 10 | -5 | -15 | 265 | 23 | -131 | 45 | 235 |
| 11 | -7 | -9 | 301 | -101 | -77 | 87 | -157 |
| 12 | -9 | -1 | 267 | -201 | 33 | 59 | -423 |
| 13 | -11 | 9 | 143 | -221 | 143 | -39 | -143 |
| 14 | -13 | 21 | -91 | -91 | 143 | -117 | 533 |
| 15 | -15 | 35 | -455 | 273 | -143 | 65 | -195 |
| A | 1360 | 5712 | 1007760 | 470288 | 201552 | 77520 | 1545232 |

| s | $P_1'(s)$ | $P_2'(s)$ | $P_3'(s)$ | $P_4'(s)$ | $P_5'(s)$ | $P_6'(s)$ | $P_7'(s)$ |
|-----|-----------|-----------|-----------|-----------|-----------|-----------|-----------|
| 0 | -2 | -15 | -1307 | -3075 | -15197 | -4435 | -577785 |
| 1 | -2 | -13 | -887 | -1391 | -3227 | 39 | 101551 |
| 2 | -2 | -11 | -527 | -253 | 2353 | 1133 | 125099 |
| 3 | -2 | -9 | -227 | 423 | 3703 | 687 | -9021 |
| 4 | -2 | -7 | 13 | 721 | 2623 | -119 | -85909 |
| 5 | -2 | -5 | 193 | 725 | 553 | -645 | -64695 |
| 6 | -2 | -3 | 313 | 519 | -1427 | -671 | 8871 |
| 7 | -2 | -1 | 373 | 187 | -2597 | -277 | 67839 |
| 8 | -2 | 1 | 273 | -187 | -2597 | 277 | 67839 |
| 9 | -2 | 3 | 313 | -519 | -1427 | 671 | 8871 |
| 10 | -2 | 5 | 193 | -725 | 553 | 645 | -64695 |
| 11 | -2 | 7 | 13 | -721 | 2623 | 119 | -85909 |
| 12 | -2 | 9 | -227 | -423 | 3703 | -687 | -9021 |
| 13 | -2 | 11 | -527 | 253 | 2353 | -1133 | 125099 |
| 14 | -2 | 13 | -887 | 1391 | -3227 | -39 | 101551 |
| 15 | -2 | 15 | -1307 | 3075 | -15197 | 4435 | -577785 |
| U | 1h | 1h | 3h | 6h | 30h | 21h | 210h |

Tables. Orthogonal Polynomials and Numerical Differentiation Tables for $n + 1$
Equally Spaced Points

$n = 16(17 \text{ points})$

| s | $P_1(s)$ | $P_2(s)$ | $P_3(s)$ | $P_4(s)$ | $P_5(s)$ | $P_6(s)$ | $P_7(s)$ |
|----|----------|----------|----------|----------|----------|----------|----------|
| 0 | 8 | 40 | 28 | 52 | 104 | 104 | 139 |
| 1 | 7 | 25 | 7 | -13 | -91 | -169 | -325 |
| 2 | 6 | 12 | -7 | -39 | -104 | -78 | 39 |
| 3 | 5 | 1 | -15 | -39 | -39 | 65 | 247 |
| 4 | 4 | -8 | -18 | -24 | 36 | 129 | 149 |
| 5 | 2 | -15 | -17 | -3 | 83 | 93 | -75 |
| 6 | 2 | -20 | -13 | 17 | 88 | 2 | -215 |
| 7 | 1 | -23 | -7 | 31 | 55 | -85 | -175 |
| 8 | 0 | -24 | 0 | 36 | 0 | -120 | 0 |
| 9 | -1 | -23 | 7 | 31 | -55 | -85 | 175 |
| 10 | -2 | -20 | 13 | 17 | -88 | 2 | 215 |
| 11 | -3 | -15 | 17 | -3 | -83 | 93 | 75 |
| 12 | -4 | -8 | 18 | -24 | -36 | 129 | -149 |
| 13 | -5 | 1 | 15 | -39 | 39 | 65 | -247 |
| 14 | -6 | 12 | 7 | -39 | 104 | -78 | -39 |
| 15 | -7 | 25 | -7 | -13 | 91 | -169 | 325 |
| 16 | -8 | 40 | -28 | 52 | -104 | 104 | -139 |
| A | 408 | 7752 | 3876 | 16796 | 100776 | 178296 | 579462 |

| s | $P_1'(s)$ | $P_2'(s)$ | $P_3'(s)$ | $P_4'(s)$ | $P_5'(s)$ | $P_6'(s)$ | $P_7'(s)$ |
|----|-----------|-----------|-----------|-----------|-----------|-----------|-----------|
| 0 | -1 | -16 | -149 | -536 | -9926 | -6192 | -214595 |
| 1 | -1 | -14 | -104 | -259 | -2571 | -203 | 30510 |
| 2 | -1 | -12 | -65 | -66 | 1204 | 1516 | 50348 |
| 3 | -1 | -10 | -32 | 55 | 2359 | 1135 | 5534 |
| 4 | -1 | -8 | -5 | 116 | 1954 | 104 | -28087 |
| 5 | -1 | -6 | 16 | 129 | 799 | -727 | -29410 |
| 6 | -1 | -4 | 31 | 106 | -476 | -988 | -7500 |
| 7 | -1 | -2 | 40 | 59 | -1421 | -669 | 17518 |
| 8 | -1 | 0 | 42 | 0 | -1766 | 0 | 28109 |
| 9 | -1 | 2 | 40 | -59 | -1421 | 669 | 17518 |
| 10 | -1 | 4 | 31 | -106 | -476 | 988 | -7500 |
| 11 | -1 | 6 | 16 | -129 | 799 | 727 | -29410 |
| 12 | -1 | 8 | -5 | -116 | 1954 | -104 | -28087 |
| 13 | -1 | 10 | -32 | -55 | 2359 | -1135 | 5534 |
| 14 | -1 | 12 | -65 | 66 | 1204 | -1516 | 50348 |
| 15 | -1 | 14 | -104 | 259 | -2571 | 203 | 30510 |
| 16 | -1 | 16 | -149 | 536 | -9926 | 6192 | -214595 |
| 0 | 1h | 2h | 6h | 6h | 30h | 10h | 140h |

Tables. Orthogonal Polynomials and Numerical Differentiation Tables for $n + 1$
Equally Spaced Points

$n = 17(18 \text{ points})$

| s | $P_1(s)$ | $P_2(s)$ | $P_3(s)$ | $P_4(s)$ | $P_5(s)$ | $P_6(s)$ | $P_7(s)$ |
|----|----------|----------|----------|----------|----------|----------|----------|
| 0 | 17 | 68 | 68 | 68 | 884 | 442 | 442 |
| 1 | 15 | 44 | 20 | -12 | -676 | -650 | -1014 |
| 2 | 13 | 23 | -13 | -47 | -871 | -377 | -13 |
| 3 | 11 | 5 | -33 | -51 | -429 | 169 | 715 |
| 4 | 9 | -15 | -42 | -36 | 156 | 481 | 585 |
| 5 | 7 | -22 | -42 | -12 | 588 | 439 | -31 |
| 6 | 5 | -31 | -35 | 13 | 733 | 145 | -563 |
| 7 | 3 | -37 | -23 | 33 | 583 | -209 | -651 |
| 8 | 1 | -40 | -8 | 44 | 220 | -440 | -280 |
| 9 | -1 | -40 | 8 | 44 | -220 | -440 | 280 |
| 10 | -3 | -37 | 23 | 33 | -583 | -209 | 651 |
| 11 | -5 | -31 | 35 | 13 | -733 | 145 | 563 |
| 12 | -7 | -22 | 42 | -12 | -588 | 439 | 31 |
| 13 | -9 | -15 | 42 | -36 | -156 | 481 | -585 |
| 14 | -11 | 5 | 33 | -51 | 429 | 169 | -715 |
| 15 | -13 | 23 | 13 | -47 | 871 | -377 | 13 |
| 16 | -15 | 44 | -20 | -12 | 676 | -650 | 1014 |
| 17 | -17 | 68 | -68 | 68 | -884 | 442 | -442 |
| A | 1938 | 23256 | 23256 | 28424 | 6953544 | 2941884 | 5794620 |

| s | $P_1'(s)$ | $P_2'(s)$ | $P_3'(s)$ | $P_4'(s)$ | $P_5'(s)$ | $P_6'(s)$ | $P_7'(s)$ |
|----|-----------|-----------|-----------|-----------|-----------|-----------|-----------|
| 0 | -2 | -51 | -337 | -323 | -38263 | -46597 | -623139 |
| 1 | -2 | -45 | -241 | -165 | -11023 | -3355 | 67453 |
| 2 | -2 | -39 | -157 | -52 | 3362 | 10712 | 153056 |
| 3 | -2 | -33 | -85 | 22 | 8672 | 9504 | 39880 |
| 4 | -2 | -27 | -25 | 63 | 8147 | 2631 | -65855 |
| 5 | -2 | -21 | 23 | 77 | 4487 | -3027 | -92399 |
| 6 | -2 | -15 | 59 | 70 | -148 | -7160 | -47872 |
| 7 | -2 | -9 | 83 | 49 | -4138 | -6368 | 23416 |
| 8 | -2 | -3 | 95 | 17 | -6403 | -2501 | 73725 |
| 9 | -2 | 3 | 95 | -17 | -6403 | 2501 | 73725 |
| 10 | -2 | 9 | 83 | -49 | -4138 | 6368 | 23416 |
| 11 | -2 | 15 | 59 | -70 | -148 | 7160 | -47872 |
| 12 | -2 | 21 | 23 | -77 | 4487 | 3027 | -92399 |
| 13 | -2 | 27 | -25 | -63 | 8147 | -2631 | -65855 |
| 14 | -2 | 33 | -85 | -22 | 8672 | -9504 | 39880 |
| 15 | -2 | 39 | -157 | 52 | 3362 | -10712 | 153056 |
| 16 | -2 | 45 | -241 | 165 | -11023 | 3355 | 67453 |
| 17 | -2 | 51 | -337 | 323 | -38263 | 46597 | -623139 |
| D | 1h | 2h | 3h | 3h | 15h | 2h | 140h |

Tables. Orthogonal Polynomials and Numerical Differentiation Tables for $n + 1$
Equally Spaced Points

$n = 18(19 \text{ points})$

| s | $P_1(s)$ | $P_2(s)$ | $P_3(s)$ | $P_4(s)$ | $P_5(s)$ | $P_6(s)$ | $P_7(s)$ |
|-----------|----------|----------|----------|----------|----------|----------|----------|
| 0 | 9 | 51 | 204 | 612 | 102 | 1326 | 306 |
| 1 | 8 | 34 | 68 | -68 | -68 | -1768 | -646 |
| 2 | 7 | 19 | -28 | -388 | -98 | -1222 | -96 |
| 3 | 6 | 6 | -89 | -453 | -58 | 234 | 411 |
| 4 | 5 | -5 | -120 | -354 | 3 | 1235 | 425 |
| 5 | 4 | -14 | -126 | -168 | 54 | 1352 | 97 |
| 6 | 3 | -21 | -112 | 42 | 79 | 729 | -267 |
| 7 | 2 | -26 | -83 | 227 | 74 | -214 | -427 |
| 8 | 1 | -29 | -44 | 352 | 44 | -1012 | -308 |
| 9 | 0 | -30 | 0 | 396 | 0 | -1320 | 0 |
| 10 | -1 | -29 | 44 | 352 | -44 | -1012 | 308 |
| 11 | -2 | -26 | 83 | 227 | -74 | -214 | 427 |
| 12 | -3 | -21 | 112 | 42 | -79 | 729 | 267 |
| 13 | -4 | -14 | 126 | -168 | -54 | 1352 | -97 |
| 14 | -5 | -5 | 120 | -354 | -3 | 1235 | -425 |
| 15 | -6 | 5 | 89 | -453 | 58 | 234 | -411 |
| 16 | -7 | 19 | 28 | -388 | 98 | -1222 | 96 |
| 17 | -8 | 34 | -68 | -68 | 68 | -1768 | 646 |
| 18 | -9 | 51 | -204 | 612 | -102 | 1326 | -306 |
| Λ | 570 | 13566 | 213180 | 2789132 | 89148 | 24515700 | 2451570 |

| s | $P_1'(s)$ | $P_2'(s)$ | $P_3'(s)$ | $P_4'(s)$ | $P_5'(s)$ | $P_6'(s)$ | $P_7'(s)$ |
|-----|-----------|-----------|-----------|-----------|-----------|-----------|-----------|
| 0 | -1 | -18 | -946 | -5391 | -16151 | -125241 | -374957 |
| 1 | -1 | -16 | -691 | -2889 | -5186 | -13632 | 27977 |
| 2 | -1 | -14 | -466 | -1057 | 889 | 26537 | 93987 |
| 3 | -1 | -12 | -271 | 186 | 3424 | 27296 | 36748 |
| 4 | -1 | -10 | -106 | 925 | 3589 | 11435 | -28625 |
| 5 | -1 | -8 | 29 | 1244 | 2374 | -6176 | -55547 |
| 6 | -1 | -6 | 134 | 1207 | 589 | -17267 | -40553 |
| 7 | -1 | -4 | 209 | 958 | -1136 | -18848 | -2508 |
| 8 | -1 | -2 | 254 | 521 | -2351 | -11889 | 33563 |
| 9 | -1 | 0 | 269 | 0 | -2786 | 0 | 48025 |
| 10 | -1 | 2 | 254 | -521 | -2351 | 11889 | 33563 |
| 11 | -1 | 4 | 209 | -958 | -1136 | 18848 | -2508 |
| 12 | -1 | 6 | 134 | -1207 | 589 | 17267 | -40553 |
| 13 | -1 | 8 | 29 | -1244 | 2374 | 6176 | -55547 |
| 14 | -1 | 10 | -106 | -925 | 3589 | -11435 | -28625 |
| 15 | -1 | 12 | -271 | -186 | 3424 | -27296 | 36748 |
| 16 | -1 | 14 | -466 | 1057 | 889 | -26537 | 93987 |
| 17 | -1 | 16 | -691 | 2889 | -5186 | 13632 | 27977 |
| 18 | -1 | 18 | -946 | 5391 | -16151 | 125241 | -374957 |
| 0 | $1h$ | $1h$ | $6h$ | $6h$ | $6h$ | $22h$ | $140h$ |

Tables. Orthogonal Polynomials and Numerical Differentiation Tables for $n + 1$
Equally Spaced Points

$n = 19(20 \text{ points})$

| s | $P_1(s)$ | $P_2(s)$ | $P_3(s)$ | $P_4(s)$ | $P_5(s)$ | $P_6(s)$ | $P_7(s)$ |
|-----|----------|----------|----------|----------|----------|----------|----------|
| 0 | 19 | 57 | 969 | 1938 | 1938 | 1938 | 646 |
| 1 | 17 | 39 | 357 | -102 | -1122 | -2346 | -1258 |
| 2 | 15 | 23 | -95 | -1122 | -1802 | -1870 | -306 |
| 3 | 13 | 9 | -377 | -1402 | -1222 | 6 | 702 |
| 4 | 11 | -3 | -539 | -1187 | -187 | 1497 | 891 |
| 5 | 9 | -13 | -591 | -687 | 771 | 1931 | 387 |
| 6 | 7 | -21 | -553 | -77 | 1351 | 1353 | -321 |
| 7 | 5 | -27 | -445 | 503 | 1441 | 195 | -777 |
| 8 | 3 | -31 | -287 | 948 | 1076 | -988 | -756 |
| 9 | 1 | -33 | -99 | 1188 | 396 | -1716 | -308 |
| 10 | -1 | -33 | 99 | 1188 | -396 | -1716 | 308 |
| 11 | -3 | -31 | 287 | 948 | -1076 | -988 | 756 |
| 12 | -5 | -27 | 445 | 503 | -1441 | 195 | 777 |
| 13 | -7 | -21 | 553 | -77 | -1351 | 1353 | 321 |
| 14 | -9 | -13 | 591 | -687 | -771 | 1931 | -387 |
| 15 | -11 | -3 | 539 | -1187 | 187 | 1497 | -891 |
| 16 | -13 | 9 | 377 | -1402 | 1222 | 6 | -702 |
| 17 | -15 | 23 | 85 | -1122 | 1802 | -1870 | 306 |
| 18 | -17 | 39 | -357 | -102 | 1122 | -2346 | 1258 |
| 19 | -19 | 57 | -969 | 1938 | -1938 | 1938 | -646 |
| A | 2652 | 17556 | 4903140 | 22881320 | 31201800 | 49031400 | 9806280 |

| s | $P_1'(s)$ | $P_2'(s)$ | $P_3'(s)$ | $P_4'(s)$ | $P_5'(s)$ | $P_6'(s)$ | $P_7'(s)$ |
|-----|-----------|-----------|-----------|-----------|-----------|-----------|-----------|
| 0 | -2 | -19 | -2111 | -31825 | -282629 | -165623 | -2093841 |
| 1 | -2 | -17 | -1571 | -17765 | -99299 | -23749 | 87303 |
| 2 | -2 | -15 | -1091 | -7275 | 6541 | 31765 | 525111 |
| 3 | -2 | -13 | -671 | 65 | 55051 | 37479 | 266783 |
| 4 | -2 | -11 | -311 | 4675 | 63871 | 20053 | -94781 |
| 5 | -2 | -9 | -11 | 6975 | 48121 | -2433 | -291621 |
| 6 | -2 | -7 | 229 | 7385 | 20401 | -19159 | -269837 |
| 7 | -2 | -5 | 409 | 6325 | -9209 | -25245 | -97189 |
| 8 | -2 | -3 | 529 | 4215 | -33149 | -20431 | 110823 |
| 9 | -2 | -1 | 589 | 1475 | -46379 | -7757 | 246399 |
| 10 | -2 | 1 | 589 | -1475 | -46379 | 7757 | 246399 |
| 11 | -2 | 3 | 529 | -4215 | -33149 | 20431 | 110823 |
| 12 | -2 | 5 | 409 | -6325 | -9209 | 25245 | -97189 |
| 13 | -2 | 7 | 229 | -7385 | 20401 | | -269837 |
| 14 | -2 | 9 | -11 | -6975 | 48121 | | -291621 |
| 15 | -2 | 11 | -311 | -4675 | 63871 | -20053 | -94781 |
| 16 | -2 | 13 | -671 | -65 | 55051 | -37479 | 266783 |
| 17 | -2 | 15 | -1091 | 7275 | 6541 | -31765 | 525111 |
| 18 | -2 | 17 | -1571 | 17765 | -99299 | 23749 | 87303 |
| 19 | -2 | 19 | -2111 | 31825 | -282629 | 165623 | -2093841 |
| 0 | 1h | 1h | 3h | 12h | 60h | 20h | 420h |

Tables. Orthogonal Polynomials and Numerical Differentiation Tables for $n + 1$
Equally Spaced Points

$n = 20(21 \text{ points})$

| s | $P_1(s)$ | $P_2(s)$ | $P_3(s)$ | $P_4(s)$ | $P_5(s)$ | $P_6(s)$ | $P_7(s)$ |
|----|----------|----------|----------|----------|-----------|-----------|-----------|
| 0 | 10 | 190 | 570 | 969 | 3876 | 6460 | 3230 |
| 1 | 9 | 133 | 228 | 0 | -1938 | -7106 | -5814 |
| 2 | 8 | 82 | -24 | -510 | -3468 | -6392 | -2006 |
| 3 | 7 | 37 | -196 | -680 | -2618 | -918 | 2754 |
| 4 | 6 | -2 | -298 | -615 | -788 | 3996 | 4266 |
| 5 | 5 | -35 | -340 | -406 | 1063 | 6075 | 2565 |
| 6 | 4 | -62 | -332 | -130 | 2354 | 5088 | -543 |
| 7 | 3 | -83 | -284 | 150 | 2819 | 2001 | -3087 |
| 8 | 2 | -98 | -206 | 385 | 2444 | -1716 | -3822 |
| 9 | 1 | -107 | -108 | 540 | 1404 | -4428 | -2548 |
| 10 | 0 | -110 | 0 | 594 | 0 | -5720 | 0 |
| 11 | -1 | -107 | 108 | 540 | -1404 | -4428 | 2548 |
| 12 | -2 | -98 | 206 | 385 | -2444 | -1716 | 3822 |
| 13 | -3 | -83 | 284 | 150 | -2819 | 2001 | 3087 |
| 14 | -4 | -62 | 332 | -130 | -2354 | 5088 | 543 |
| 15 | -5 | -35 | 340 | -406 | -1063 | 6075 | -2565 |
| 16 | -6 | -2 | 298 | -615 | 788 | 3996 | -4266 |
| 17 | -7 | 37 | 196 | -680 | 2618 | -918 | -2754 |
| 18 | -8 | 82 | 24 | -510 | 3468 | -6392 | 2006 |
| 19 | -9 | 133 | -228 | 0 | 1938 | -7106 | 5814 |
| 20 | -10 | 190 | -570 | 969 | -3876 | 6460 | -3230 |
| A | 772 | 201894 | 1730520 | 5720330 | 121687020 | 514829700 | 223092870 |

| s | $P'_1(s)$ | $P'_2(s)$ | $P'_3(s)$ | $P'_4(s)$ | $P'_5(s)$ | $P'_6(s)$ | $P'_7(s)$ |
|----|-----------|-----------|-----------|-----------|-----------|-----------|-----------|
| 0 | -1 | -60 | -1171 | -7450 | -523756 | -1511160 | -9341610 |
| 1 | -1 | -54 | -886 | -4311 | -198571 | -265587 | 91149 |
| 2 | -1 | -48 | -631 | -1928 | -4006 | 257376 | 2306691 |
| 3 | -1 | -42 | -476 | -217 | 92069 | 347459 | 1421821 |
| 4 | -1 | -36 | -211 | 906 | 118004 | 220472 | -132186 |
| 5 | -1 | -30 | -46 | 1525 | 98369 | 27545 | -1159135 |
| 6 | -1 | -24 | 89 | 1724 | 53954 | -135632 | -1307521 |
| 7 | -1 | -18 | 194 | 1587 | 1769 | -219569 | -740199 |
| 8 | -1 | -12 | 269 | 1198 | -44956 | -211736 | 135886 |
| 9 | -1 | -6 | 314 | 641 | -76771 | -127323 | 879909 |
| 10 | -1 | 0 | 329 | 0 | -88006 | 0 | 1167315 |
| 11 | -1 | 6 | 314 | -641 | -76771 | 127323 | 879909 |
| 12 | -1 | 12 | 269 | -1198 | -44956 | 211736 | 135886 |
| 13 | -1 | 18 | 194 | -1587 | 1769 | 219569 | -740199 |
| 14 | -1 | 24 | 89 | -1724 | 53954 | 135632 | -1307521 |
| 15 | -1 | 30 | -46 | -1525 | 98369 | -27545 | -1159135 |
| 16 | -1 | 36 | -211 | -906 | 118004 | -220472 | -132186 |
| 17 | -1 | 42 | -476 | 217 | 92069 | -347459 | 1421821 |
| 18 | -1 | 48 | -631 | 1928 | -4006 | -257376 | 2306691 |
| 19 | -1 | 54 | -886 | 4311 | -198571 | 265587 | 91149 |
| 20 | -1 | 60 | -1171 | 7450 | -523756 | 1511160 | -9341610 |
| D | 1h | 1h | 3h | 6h | 6h | 6h | 420h |

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| 13. ABSTRACT <p>The present method of reducing data by formulating a polynomial, taking the derivative and substituting experimental values, has been simplified for instances where the independent variable is evenly spaced and the derivative is desired at these given points only.</p> <p>Orthogonal polynomials, resembling Legendre polynomials, are differentiated and tabulated in a useful form. When used in conjunction with available orthogonal polynomial tables, these new tables give derivatives more readily than the usual procedure. These tables present coefficients for obtaining first derivatives at 6 to 21 evenly spaced points. This report extends the available orthogonal polynomial tables to sixth and seventh degrees, when appropriate, and gives a demonstration of the method.</p> | | | |

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