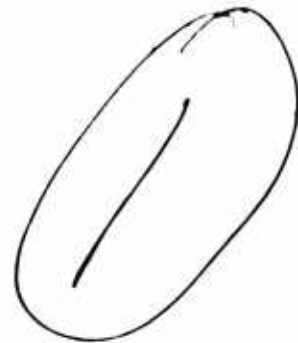


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SYSTEMS RESEARCH MEMORANDUM No. 183

The Technological Institute

The College of Arts and Sciences

Northwestern University

CHANCE CONSTRAINED MODELS
FOR TRANSPORT PRICING
AND SCHEDULING
UNDER COMPETITION

by

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SYSTEMS RESEARCH GROUP

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1. Introduction

To date, little work has been done on the problem of developing management models for analyzing the pricing policy of a railroad. There are many reasons for this: One is that in many countries, including the United States, railroad management has little or no freedom in setting its price for a particular service. Instead, prices are fixed by a federal regulatory body, so that the only form of competition between railroads is through better schedules, faster trains, more direct service, etc. Also, in many countries there is only a single railroad and so the need for a competitive model to analyze pricing policy vis-a-vis another railroad does not exist. In such a case, however, the railroad is almost certain to be in competition with one or more trucking or shipping companies, so that a competitive model in which the railroad's opponent is, say, a trucker would be very useful in analyzing the railroad's price structure.

Another reason why such models have not been employed is that they appear to involve mathematical complexities even for simple cases. The reason for this is two-fold. First, because the model involves competition, the notions of game theory and constrained games appear to be needed. Second, and most important, any realistic model must deal with quantities (e. g. , future demand) whose values are not known in advance with certainty. This means that the model requires some combination of stochastic or chance-constrained elements together with game theoretic elements. Relatively little work has been done on such models. For a survey of the field, see references [1], [2], [3] and perhaps cited therein.

There are, however, countries where railroads do compete directly with each other as well as with truckers. Indeed, under the new transport legislation in Canada such competition has become possible as never before. With

government subsidies declining and costs increasing in the years ahead, it has become clear that railroads in Canada must do more than just hold their own in the competitive struggle for freight traffic. They must capture a sufficient share of the market to secure the revenue needed to remain economically self-sufficient in a competitive transportation environment. As Mr. N. R. Crump, President of the Canadian Pacific Railway, has said,

"This challenge can only be met by providing the facilities and equipment desired by our customers, by offering a dependable transportation service, and by following a pricing policy which leads the customer to choose our products in preference to those of our competitors. . . It will be necessary to adopt an aggressive marketing and pricing policy to realize the potential economies which are available. Our competitors may be surprised at what the railways can accomplish when the rules of the game are the same for all players."

It is clear from Mr. Crump's remarks that a game-like competitive model may be a worthwhile tool for analyzing competition between Canada's railroads. However, in the model developed below, we do not restrict a railroad's competitor to be another railroad. Instead, we allow the possibility that the opponent may be any other shipping firm, and that the railroad's competitor may be a composite one comprised of different firms on some of the routes.

We begin with a detailed description of the assumptions and main mathematical features of the model. We obtain a game-like formulation which involves each player in a chance-constrained programming problem which is partly controlled by the play of his opponent. In section 3 we show how to obtain deterministic equivalents which are convex programming problems and therefore amenable to solution by existing methods. Section 4 introduces a simple 3 node example used in later sections for illustration, where we are able to derive price and scheduling policies as explicit functions of the opponent's prices. In section 5 we explore the implications of a welfare economic

approach, and are able to obtain a set of optimal prices which reflect the complex joint cost and constraint structure. The relation of these prices to capacity costs and social costs is made apparent. Sections 6 and 7 consider two main notions of optimal play and of equilibrium. The first approach is to assume one's opponent's prices fixed, and to derive a reaction curve for each player. Section 7 suggests several possibilities of a game theoretic approach.

2. Assumptions and Description of the Model

The model we will develop is designed to analyze the competitive pricing and scheduling of services provided by a railroad and its competitor(s). We will assume that the location of the terminals and tracks of the railroad system and the analogous facilities of its competitor are given, so we will not initially be concerned with the problem of determining whether to expand existing facilities or build new ones. However, these questions may be approached through devices such as sensitivity analyses of our models.

In our formulation, we shall find it convenient to distinguish between the actual rail or highway network of the firm, and a descriptive network. The latter has the following properties: that a direct link exists between any two nodes (i. e., origins or destinations) over which some demand occurs, and that at least one train covers that link. Thus, "descriptive" passengers or goods travelling between an origin and a destination on the descriptive network never have to change "descriptive" trains, and indeed there are no intermediate nodes. (Of course, the actual route may pass through actual intermediate nodes).

There are several ways in which one might formulate the alternatives in carrying out the transportation tasks demanded of the railroad and its competitor. One might use a network formulation, assigning to each link a variable which represents the number of "freight cars" traversing the link. While it is possible to formulate our model in this way (indeed, this is what we have

done in the trucker section of our model given below) we have not done so for the railroad because of the nature of a railroad's operations and the manner in which costs of system operation are incurred. For example, trains are not normally scheduled to run along a single link; usually they are scheduled along a group of links or a route. These routes are sometimes circular in that the end of the route coincides with the beginning. Sometimes, too, a train is required to return to its point of origin at the end of a work shift (see [4], [5] where the problem of optimally allocating train crews to satisfy such constraints is discussed in detail). To put these "route requirements" into the link-node format of a network problem would greatly complicate the mathematics of the model and, as we shall see, unnecessarily.

A link-node formulation with variables such as the number of "freight cars" assigned to each link causes difficulties since the costs are joint and are per train rather than per car or per item transported. A major element in the cost of moving a train is the cost of paying the crew. Adding a few more cars to the train when they are available costs comparatively little, so that a cost is not linear in the number of freight cars. It is convenient therefore to employ variables such as train loads rather than the number of freight cars, since the variable cost is approximately linear in the number of train loads shipped. In fact, to capture also the notion of a route referred to above, we will use as variables (x_j) the number of trains of fixed size k_j assigned to a particular route j , where each route consists of a number of consecutive links (see [4], [5]).^{1]} Although at first sight one might conjecture that the number of routes would be hopelessly large, this is not the case generally because of crew restrictions and other con-

^{1]} It should be noted that possibilities of trains of different sizes or type on the same actual route j can be handled by introducing additional phony "points" between the same routes, one for each size or type.

siderations involved in a railroad's day to day operations. Costs are given by c_j , the cost of operating a train of size k_j over route j .

We assume that over each descriptive network link (r, s) joining node r to node s the railroad is faced with competition from at most one other shipping firm. The competitive aspect of the model is embodied in the demand structure over each link: the demand which each firm receives depends upon the prices charged by both firms.

We allow for the possibilities that there is no competition over certain links and that competition comes from different firms (railroads or truckers) over different links.

Specifically, let d_{rs} be the demand upon the railroad per time period for transportation from node r to node s , measured in terms of persons or units of goods. Then d_{rs} is the sole demand upon the railroad on descriptive link (r, s) ,¹ and we assume that this demand is distributed with mean μ_{rs} and variance σ_{rs}^2 , where μ_{rs} is given by

$$(1) \quad \mu_{rs} = a_{rs} - b_{rs}\pi_{rs} + g_{rs}\Delta_{rs}$$

In (1), π_{rs} and Δ_{rs} are the prices (per person or per unit-goods) charged by the railroad and its competitor, respectively, over link (r, s) ; a_{rs} , b_{rs} and g_{rs} are given non-negative constants. We will assume that σ_{rs}^2 is known and fixed, although an interesting extension of the model would be to allow σ_{rs}^2 to vary with π_{rs} and Δ_{rs} . (For example, σ_{rs}^2 might be large when π_{rs} , Δ_{rs} are nearly equal, and small in the contrary case.)

The opponent's demand on the same link is similarly given:

¹ There may be more than one route serving this link.

$$(1)' \quad d'_{rs} = a'_{rs} - b'_{rs} \Delta_{rs} + g'_{rs} \pi_{rs}$$

Equations (1) and (1)' embody the assumption that demand on a link is higher, the lower is one's own price on that link and the higher is one's opponent's price. Since our descriptive network provides a separate link for each desired journey, it is not unreasonable to ignore prices on other links. But two points should be noted:

- (a) the actual network would no doubt include feeder links into the major nodes, and demand on the major links would be affected by prices on the feeder links. The model will enable us to evaluate demand increases due to improved access facilities.
- (b) Reduction of transport charges on one link might enable manufacturers to realize economies of scale and find it profitable to increase shipments on other routes. We have ignored this possibility; demand distributions along each link are assumed independent of prices charged and quantities shipped on other routes.

We have also assumed the demands as random variables are stochastically independent, though this could be relaxed without difficulty. We turn now to a discussion of the constraints of the model. The simplest of these constraints prescribes upper and lower bounds on the prices that can be charged for transportation over each link. In Canada, for example, a railroad cannot charge a below-cost price, thus putting a lower bound on π_{rs} . An upper bound may be established by the government in order to prevent a railroad from charging an excessive price, though "below-cost" and "excessive" are elusive terms and cannot really be defined independently of the model. The railroad itself, bearing in mind long-run considerations,

might want to maintain or encourage certain demands. Thus we will assume that constants L_{rs} , L'_{rs} , U_{rs} , U'_{rs} are prescribed such that one must have

$$(2) \quad L_{rs} \leq \pi_{rs} \leq U_{rs} \quad \text{for all } (r, s)$$

and

$$(2)' \quad L'_{rs} \leq \Delta_{rs} \leq U'_{rs} \quad \text{for all } (r, s).$$

We allow the possibility that L_{rs} or L'_{rs} may be zero, and U_{rs} or U'_{rs} may be infinite.

Since, in general, demands between nodes are not the same in both directions, there is no reason to expect that at the optimum $\pi_{rs} = \pi_{sr}$. But this may be desirable for ease of administration or it may be a government restriction. Our model can evaluate the cost of such a constraint (and similarly with constraints such as: the price of a journey from A to C via B (on the actual network) should be equal to the sum of the prices of journeys from A to B and from B to C; or again, a constant price per mile should be charged). ¹

In many instances a railroad must provide a certain minimum frequency of service over each link (r, s) . That is, operating policy or government regulation may require that the total number of trains scheduled over link (r, s) be at least R_{rs} . This would give constraints

$$(3) \quad \sum_{j \in D_{rs}} x_j \geq R_{rs} \quad \text{for all } (r, s),$$

where $D_{rs} = \{j : \text{route } j \text{ includes link } (r, s) \text{ as one of its links}\}$.

¹[cf.]Footnote 13, page 90, where, in one example, contrary to the implications of traditional analysis, marginal costs in the empty, back-hand direction are actually higher than in the main direction.

In order to maintain a good quality of service on all trains operating in or out of node i , it may be necessary to limit the number of trains that can use node i to be less than or equal to a given number, M_i . This limitation may also be imposed for physical reasons, since node i may not have enough facilities to handle more than M_i train loads in a given period. If this constraint is binding in an optimal solution, sensitivity techniques can be used to determine what the increase in profit will be if M_i is increased. In this way we can evaluate the profitability of expanding the existing facilities at node i . Hence we include the constraint

$$(4) \quad \sum_{j \in H_i} x_j \leq M_i, \quad \text{for all } i,$$

where $H_i = \{j : \text{route } j \text{ includes node } i \text{ as one of its nodes}\}$.

We turn now to a discussion of the chance constraints involved in the model. Since each competing firm sets its schedule and price independently, and before the variable demand is known, the railroad cannot guarantee that the number of trains it has scheduled to run from node r to node s will be sufficient to meet every possible volume of demand. The railroad can, however, have a policy which states that it wants to set a schedule which will meet demand over link (r, s) with at least probability β_{rs} . A similar type of chance constraint which required that demand be met with at least a specified probability was employed in [7]. There the company involved had a rule of "always" meeting customer demand. However, upon analysis of past data to see if "always" meant "with probability one", it became clear that the real meaning of their rule was "as often as we possibly can without incurring unbearable cost." Such a "policy" was mirrored in the model by requiring that the chance constraints hold with a suitably high level of probability.

This distinction between a policy and a rule should, perhaps, be explained further. If a firm requires that a constraint hold with probability one (i. e., if the constraint represents a hard and fast operating rule) and if the constraint involves random variables, then the firm must operate so as to protect itself against all possible values of the random variables, no matter how extreme they may be or how little probability they have of occurring. On the other hand, if a chance constraint is employed with probability close to but less than one (thus making the operating procedure a policy rather than a rule), the firm can still plan to follow the rule except in the case of extreme events which are very unlikely to occur. We will employ such chance constraints in the model being discussed here, because it seems reasonable that a railroad will want to plan to meet demands with high probability, but will not want to plan to meet demands all the time, since unusual events may require emergency (perhaps external) operations whose precise character cannot be delineated in advance.

In our model the chance constraints on meeting demand with a least a specified probability are

$$(5) \quad P \left\{ \sum_{j \in D_{rs}} k_j x_j \geq d_{rs} \right\} \geq \beta_{rs}, \quad \text{for all } (r, s)$$

where, to repeat, k_j is the fixed capacity of a train on route j .

The railroad may wish to ensure a certain minimal level of profit on a particular route or set of routes, or set of links. Because demand is random, this cannot be guaranteed with certainty, but for any such subset of routes α we may impose a chance constraint of the form:

$$(6) \quad P \left(\sum_{(r,s) \in V_\alpha} \pi_{rs} d_{rs} - \sum_{j \in \alpha} c_j x_j \geq \delta_\alpha \right) \geq \gamma_\alpha$$

where δ_α is a specified level of profit, γ_α is a specified probability level, c_j is the cost of operating a train over route j , γ is the subset of routes and $V_\gamma = \{ (r, s) : \text{link } (r, s) \text{ is a link on one of the subset of routes } \alpha \}$. In the case where some of the subset α overlap with other routes, a method of revenue allocation must be decided upon. If the subset is defined by links, a cost allocation is necessary. We shall not pursue here the possibility of introducing decentralization by dividing the routes into disjoint divisions: on this, see []. Note, however, that if the divisions of the opponent are not the same as, or contained within, those of the railroad company, then we shall see that pricing in one division may affect the opponent's price, and hence one's own revenue, in another division.

We may consider a variety of different possible objective functions for our model. One might be the expected profit earned over the whole system. This can be written as

$$(7) \quad E \left(\sum_{r,s} \pi_{rs} d_{rs} - \sum_j c_j x_j \right) .$$

Another might be the probability of earning at least a specified amount δ over the whole system. This can be expressed as

$$(8) \quad P \left(\sum_{r,s} \pi_{rs} d_{rs} - \sum_j c_j x_j \geq \delta \right)$$

A third choice of objective function might be the expected difference between the revenue earned by the railroad and that earned by its opponent. That is

$$(9) \quad E \left(\sum_{r,s} \pi_{rs} d_{rs} - \sum_{r,s} \Delta_{rs} d'_{rs} \right)$$

Still another possibility might be the expectation of the ratio of the railroad's revenue to that of its competition.

In the literature of chance-constrained programming, models whose objective function is the expected value of a linear function are called E-models, while models whose objective function involves optimizing the probability of a certain event are called P-models. Properties of solutions to these models have been presented in [8, 9, 10, 11].

If the railroad's opponent is another railroad, its own model is analogous to that given above. However, if the opponent is a trucker the cost and operating structure for the trucking firm makes the link-node formulation preferable to the route formulation. The chief reasons for this are twofold: First, truck schedules are normally given over specific individual links and not over a set of links. Second, and most important, the variable cost of transporting a load over a link is approximately linear in the number of trucks used. Thus, if c'_{rs} is the cost of moving a single truck of fixed capacity k'_{rs} along (r, s) , and if y_{rs} is the number of trucks which are to be allocated to link (r, s) , the constraints for the trucker in terms of the variables y_{rs} , and Δ_{rs} , are:

$$(2)' \quad L'_{rs} \leq \Delta_{rs} \leq U'_{rs} \quad , \quad \text{for all } (r, s)$$

$$(3)' \quad y_{rs} \geq R'_{rs} \quad , \quad \text{for all } (r, s)$$

$$(4)' \quad \sum_r y_{rs} + \sum_t y_{st} \leq N'_s \quad , \quad \text{for all nodes } s$$

$$(5)' \quad P(k'_{rs} y_{rs} \geq d'_{rs}) \geq \beta'_{rs} \quad , \quad \text{for all } (r, s)$$

$$(6)' \quad P \left(\sum_{(r,s) \in V'_\alpha} \Delta_{rs} d'_{rs} - \sum_{(r,s) \in V'_\alpha} c'_{rs} y_{rs} \geq \delta'_\alpha \right) \geq \gamma'_\alpha \text{ for all } \alpha.$$

There is no difficulty in allowing the links to be divided (disjointly) among several opponent firms, possibly of different type. To avoid constraint inter-dependencies, the node constraints should be interpreted to refer to separate terminals for each opponent. It would be reasonable to assume no collaboration between the firms with regard to price setting, but there might well be indirect interactions via the responses of the railroad.

3. The Deterministic Equivalents

We can summarize the constraints faced by the railroad as

$$(10) \quad \begin{aligned} & L_{rs} \leq \pi_{rs} \leq U_{rs} && \text{for all } (r,s) , \\ & \sum_{j \in D_{rs}} x_j \geq R_{rs} && \text{for all } (r,s) , \\ & \sum_{j \in H_i} x_j \leq M_i && \text{for all } i , \\ & P \left\{ \sum_{j \in D_{rs}} k_j x_j \geq d_{rs} \right\} \geq \beta_{rs} && \text{for all } (r,s) \\ & P \left\{ \sum_{(r,s) \in V'_\alpha} \pi_{rs} d_{rs} - \sum_{j \in \alpha} c_j x_j \geq \delta_\alpha \right\} \geq \gamma_\alpha && \text{for all relevant } \alpha \\ & x_j \geq 0 . \end{aligned}$$

In (10) we will assume that the d_{rs} are a set of independent normal random variables. The extension of subsequent results to cover situations wherein such is not the case is straightforward and is accomplished using the same technique which was demonstrated in [8]. In any case, since the π_{rs} and x_j are chosen before the random demand is observed, the problem

is one involving "zero-order" or constant decision rules in the terminology of chance-constrained programming.

We can use methods similar to those introduced in [8] in order to obtain deterministic constraints equivalent to the chance constraints in (10). By subtracting the mean μ_{rs} and dividing by the standard deviation σ_{rs} , we see that

$$P \left\{ \sum_{j \in D_{rs}} x_j \geq d_{rs} \right\} = P \left\{ \frac{d_{rs} - \mu_{rs}}{\sigma_{rs}} \leq \frac{\sum_{j \in D_{rs}} x_j - \mu_{rs}}{\sigma_{rs}} \right\}.$$

If we let $\Phi(z) = P(Z \leq z)$, where Z is a $N(0,1)$ random variable, and let Φ^{-1} be the inverse of Φ , the first chance constraints in (10) can then be seen to be equivalent to

$$(11) \quad \sum x_j + b_{rs} \pi_{rs} \geq a_{rs} + g_{rs} \Delta_{rs} + \sigma_{rs} \Phi^{-1}(\beta_{rs}) \quad \text{for all } (r, s).$$

It should be noted that (11) is a linear inequality. The excess capacity necessitated by the uncertain demand is quite apparent.

To obtain deterministic equivalents for the other chance constraints in (10) is somewhat more involved. Omitting subscripts for clarity, the process of normalizing both sides of the inequality in brackets yields

$$P \left\{ \sum \pi d \geq \delta + \sum c x \right\} = P \left\{ \frac{\sum \pi d - \sum \pi \mu}{\sum \pi^2 \sigma^2} \geq \frac{\delta + \sum c x - \sum \pi \mu}{\sum \pi^2 \sigma^2} \right\}$$

where the term on the left in the rightmost brackets is a $N(0,1)$ random variable. The chance constraints are therefore equivalent to

$$\sum \pi \mu - \sum c x \geq \delta - [\Phi^{-1}(1 - \gamma)] \sqrt{\sum \pi^2 \sigma^2}.$$

Now following the technique of [8] and remarking that $\phi^{-1}(1-\gamma) < 0$ for $\gamma > \frac{1}{2}$, we can introduce the "spacer variable" w to rewrite the above relation as

$$\sum \pi_{rs} \mu_{rs} - \sum c_j x_j - \delta \geq w$$

$$w^2 \geq [\phi^{-1}(1-\gamma)]^2 \sum \pi_{rs}^2 \sigma_{rs}^2$$

$$w \geq 0 .$$

w is in fact a measure of the allowance necessary for uncertainty. Replacing subscripts, we can write the following as deterministic equivalents for the (divisional profit) chance constraints in (10), for each α :

$$(12) \quad \sum_{(r,s)} \pi_{rs} (a_{rs} - b_{rs} \pi_{rs} + g_{rs} \Delta_{rs}) - \sum_{j \in \alpha} c_j x_j \geq w_\alpha$$

$$w_\alpha^2 - [\phi^{-1}(1-\gamma_\alpha)]^2 \sum_{\substack{(r,s) \\ \in V_\alpha}} \pi_{rs}^2 \sigma_{rs}^2 \geq 0$$

$$w_\alpha \geq 0$$

Finally, the deterministic equivalent for the objective function (7) is easily obtained since π_{rs} and x_j are zero-order:

$$(13) \quad E \left(\sum_{(r,s)} \pi_{rs} d_{rs} - \sum_j c_j x_j \right) = \sum_{(r,s)} \pi_{rs} \mu_{rs} - \sum_j c_j x_j$$

$$= \sum_{(r,s)} \pi_{rs} (a_{rs} + g_{rs} \Delta_{rs} - b_{rs} \pi_{rs}) - \sum_j c_j x_j .$$

which is quadratic in π_{rs} and linear in x_j .

4. An Example

In this section we shall introduce a simple non-stochastic 3-node example to give some feeling for the working of the model. In particular, we shall examine the effects on prices and capacities of changes in demand, as a prelude to looking at competitive interactions.

Suppose we have 3 nodes ($i = 1, 2, 3$) with demand along 2 links (1, 2) and (2, 3). There are 3 routes: $j = 1 : (1, 2)$, $j = 2 : (2, 3)$ and $j = 3 : (1, 2, 3)$. Train sizes are the same ($k_j = k$) and costs are such that

$$(14) \quad 0 \leq c_1, c_2 < c_3 < c_1 + c_2$$

Demand, measured now in train-loads to simplify notation, is given deterministically by

$$d_{rs} = a_{rs} - b_{rs} \pi_{rs} + g_{rs} \Delta_{rs} \quad (r, s) = (1, 2), (2, 3) ,$$

where $b_{rs} > 0$.

We shall suppose that the opposing prices Δ_{rs} are known and fixed, so that to maximize profit the railroad must solve the problem:

$$(15) \quad \begin{aligned} &\text{Maximize } \pi_{12} d_{12} + \pi_{23} d_{23} - c_1 x_1 - c_2 x_2 - c_3 x_3 \\ &\text{subject to } \quad x_1 \quad \quad + x_3 \quad \geq d_{12} \\ &\quad \quad \quad x_2 \quad + x_3 \quad \geq d_{23} \\ &\quad \quad \quad x_1, x_2, x_3 \quad \geq 0 . \end{aligned}$$

We have analyzed this example in more detail elsewhere []. We give here some idea of its properties. The route and cost structure is such that, when service on link 1 exceeds that on link 2 (i. e. $x_1 + x_3 > x_2 + x_3$) the cost of increasing service is c_1 , the cost of increasing service on route 1. When service on link 1 is lower than on link 2, service can be increased by substituting a train on route 3 for one on route 2, at cost of $c_3 - c_2 < c_1$. If it is optimal to have equal service ($x_1 + x_3 = x_2 + x_3$, where in fact $x_1 = x_2 = 0$) the sum of the marginal revenues on the two links will just cover the marginal cost c_3 of expanding service on both.

It can be shown [] that the prices π_{12}, π_{23} and the levels of service x_1, x_2, x_3 are continuous and piecewise linear functions of the opponent's prices Δ_{12}, Δ_{23} . Further, where demands on the two links are sufficiently disparate, prices and levels of service on the two links are independent. However, where similar levels of service are provided on the two links, a rise in demand on one link will make it profitable to increase service (as well as price) on that link, thereby reducing marginal costs on the other link, because of the joint cost structure. It is then profitable to reduce price on the second link and expand service at the same rate as on the first link.

Generally speaking, price, frequency, output and mode constraints all have such "complementary" effects within certain ranges. Thus, a minimum frequency constraint on one link, by expanding output on that link, may, through the joint cost structure, induce expansion on another link.

On the other hand, constraints also have "substitution" effects. A minimum-frequency constraint, for example, induces a shift to routes which are cheap per trainload, rather than per unit capacity. Node constraints

work in the opposite direction. These in turn may induce "complementary" effects of the first type.

5. Welfare Economics and Marginal Cost Pricing

Suppose that the railroad were government-run, or regulated with "social benefit", rather than profits, as the objective function to be maximized. What price policy would lead to the "correct" provision of services and its efficient utilization? Some form of marginal cost pricing would usually be thought of as appropriate, but the complications of joint costs, resulting from the route-cost and the link-demand structures, and of various other operating, managerial and governmental constraints render marginal cost an ill-defined concept.

Elsewhere [], we have analyzed our example using Steiner's approach [] to operating and capacity costs. However, his diagrammatic (and mathematical) treatment is not adequate for dealing with the full complexities of our problem. By formulating the appropriate programming model we can derive a set of optimal prices which reflect the marginal system cost of the various constraints imposed. Again, we merely sketch out the results here, referring the reader to [] for more details.

Take as the criterion of social benefit the sum of the areas under the demand curves up to the level of output on each link, less the costs of providing these outputs. Assume a non-stochastic demand structure

$$d_{rs} = a_{rs} + g_{rs} \Delta_{rs} - b_{rs} \pi_{rs} ,$$

and define K_{rs} as the number of people to be transported on link (r, s) , so that, by the inverse demand function, price charged on that link is

$$\pi_{rs} = \frac{a_{rs} + g_{rs} \Delta_{rs} - K_{rs}}{b_{rs}}$$

The area under the demand curve up to K_{rs} is given by

$$\frac{K_{rs}}{b_{rs}} \left(a_{rs} + g_{rs} \Delta_{rs} - \frac{K_{rs}}{2} \right) .$$

Assuming the opponent's prices Δ_{rs} constant, and ignoring for the moment all further constraints, the problem is to

$$(16) \quad \max_{K_{rs}, x_{rs}} \sum_{(r,s)} \frac{K_{rs}}{b_{rs}} \left(a_{rs} + g_{rs} \Delta_{rs} - \frac{K_{rs}}{2} \right) - \sum_j c_j x_j$$

$$\text{subject to} \quad \sum_{j \in D_{rs}} x_j \geq K_{rs}$$

$$x_j \geq 0$$

Using Kuhn-Tucker conditions for optimality we can show that (as we might expect from the formulation) price on link (r, s) is precisely equal to the marginal cost of increasing capacity on that link. In the unconstrained case, revenues just cover costs. Further constraints, as in (10), can be introduced without difficulty. Now, price on any link is set equal to the marginal system cost of expanding capacity on that link. By this we mean not just the cost of running another train but if, for example, a node constraint were binding, then the cost in terms of consumer benefits of restricting other routes using the critical node. We can thereby make an assessment of the cost to one set of consumers (within the model) of (e. g.) governmental constraints designed to protect the interests of another set of consumers.

6. Notions of Optimal Competitive Action

The equations and inequalities we have derived as deterministic equivalents for each firm's constraints describe the joint ranges of prices and scheduling alternatives available to the two competitors. These ranges are joint since the price variables of a firm appear in the constraints of its competitor. For ease of reference these constraints are reproduced below:

Railroad Constraints

$$L_{rs} \leq \pi_{rs} \leq U_{rs} , \quad \text{for all } (r, s)$$

$$\sum_{j \in D_{rs}} x_j \geq R_{rs} , \quad \text{for all } (r, s)$$

$$\sum_{j \in H_i} x_j \leq M_i , \quad \text{for all } i$$

$$(17) \quad \sum_{j \in D_{rs}} k_j x_j + b_{rs} \pi_{rs} \geq a_{rs} + g_{rs} \Delta_{rs} + \sigma_{rs} \delta^{-1}(\beta_{rs}) \quad \text{for all } (r, s)$$

$$\sum_{(r,s) \in v_\alpha} \pi_{rs} (a_{rs} + g_{rs} \Delta_{rs} - b_{rs} \pi_{rs}) - \sum_{j \in \alpha} c_j x_j - \delta_\alpha \geq w_\alpha$$

$$w_\alpha^2 \geq 0$$

$$x_j \geq 0$$

and

Competitor Constraints

$$L'_{rs} \leq \Delta_{rs} \leq U'_{rs} , \quad \text{for all } (r, s)$$

$$(18) \quad y_{rs} \geq R'_{rs} , \quad \text{for all } (r, s)$$

$$\sum_r y_{rs} + \sum_t y_{st} \leq M'_s , \quad \text{for all } s$$

$$k'_{rs} y_{rs} + b'_{rs} \Delta_{rs} \geq a'_{rs} + g'_{rs} \Pi_{rs} + \sigma'_{rs} \phi^{-1}(\beta'_r) \quad \text{for all } (r, s)$$

$$y_{rs} \geq 0$$

$$\sum_{(r, s) \in v'_\alpha} \Delta_{rs} (a'_{rs} - b'_{rs} \Delta_{rs} + g'_{rs} \Pi_{rs}) - \sum_{(r, s) \in v'_\alpha} c'_{rs} y_{rs} \geq \delta'_\alpha + w'_\alpha$$

$$w'_\alpha{}^2 \leq [\phi^{-1}(1 - \gamma'_\alpha)]^2 \sum_{(r, s) \in v'_\alpha} \Delta_{rs}^2 \sigma'_{rs}{}^2$$

$$w'_\alpha \geq 0$$

for all relevant α .

It is important to emphasize that the above conditions do not delineate independent constraint sets or regions of alternatives. Rather, the actions and decisions of the two firms are intertwined in an essential way. The feasibility or desirability of any set of prices or any scheduling pattern for one party cannot be evaluated by reference to only one of (17) or (18). Indeed, the interrelations require that each firm evaluate its decisions in terms which involve the competitor's decisions as well. These interconnections and interdependencies occur also in the objectives or payoffs of the competitors, several possibilities for which we describe in (7), (8) and (9). This type of intertwining already takes our model out of the classes which have thus far been analyzed in the theory of games.¹¹

In this and the next section we mention two main approaches to the questions of optimal play and equilibrium solutions. The next section considers the possibilities of a game theoretic formulation. The approach of this section is familiar in the economic literature (for a discussion see []): that of reaction curves representing the optimal decision for each firm on the assumption that the opponent's price is known and fixed.

Consider the constraint set (17) together with the objective function

¹¹ But see [] for the "Advertising Demand-Capture Games" introduced by Charnes and Cooper.

$$(19) \quad \text{Max}_{(r,s)} \sum_{(r,s)} \Pi_{rs} (a_{rs} + g_{rs} \Delta_{rs} - b_{rs} \Pi_{rs}) - \sum_j c_j x_j$$

where $a_{rs} + g_{rs} \Delta_{rs} - b_{rs} \Pi_{rs} = \mu_{rs} = E d_{rs}$.

Now, for the opponent's prices Δ_{rs} held fixed, under very mild conditions the set of prices Π_{rs} and activities x_j satisfying (17) is a convex set and the (quadratic) objective function (19) is a concave function of the Π_{rs} . It is apparent that the conditions $\gamma_\alpha > \frac{1}{2}$ for all α (i. e. more than a 50-50 chance of satisfying the requirement is specified) and $b_{rs} \geq 0$ (i. e. increasing own price does not increase own demand) for all (r, s) will suffice, since the latter ensures that $-\sum_{(r,s)} b_{rs} \Pi_{rs}^2$ is concave as a function of Π_{rs} and the former (together with $w_\alpha \geq 0$) restricts w_α and Π_{rs} to lie in one nappe of an elliptic hyperboloid. Hence the problem of maximizing (19) subject to the constraint set (17) with the Δ_{rs} held fixed is therefore under these assumptions a convex programming problem and as such yields theoretically and computationally to a variety of modern methods. Likewise, the opponent's problem is convex for fixed Π_{rs} .

In principle, then, we can build up an optimal decision vector, or reaction function, by sensitivity analysis on the opponent's price vector. An equilibrium solution to the whole problem is a pair of price (and output) vectors such that unilateral departure by either firm is not advantageous to that firm. Further considerations of interest are: whether or not the equilibrium is unique, and whether it is stable, i. e., from any pair of price vectors the sequence of optimal reactions leads to the equilibrium point. Stability may be local or global. We shall not consider here the conditions on the general problem which ensure the existence of equilibria, but we might mention that in the example introduced earlier, a stable

equilibrium exists, but is not necessarily unique.

The above notion of equilibrium via reaction curves has been criticized on certain grounds:

"...ultimately they prove to be 'right' for the wrong reasons. Each assumes that his rival follows a policy of fixed output while in reality each follows a policy of adjusting his own output to the requirement of profit maximization, on the assumption that the other follows a policy of fixed output. But, if, on this incorrect assumption, they both have actually adjusted their output to the simultaneous output of the other, then (from there on) the assumption they make with respect to one another is 'quasi-correct', as we might say. It has become true that the other producer goes on producing a fixed output, although the reason is not (as is mutually assumed) that he follows a policy of producing a fixed output disregarding his rival's behavior. This is what we meant by saying that they are right for the wrong reason." ¹

"...that firms should assume of one another that the other follows a policy of fixed output is conceivable, but on the way to the Cournot equilibrium* they would necessarily realize that their assumptions were incorrect and they would change their assumptions. This would, of course, destroy the validity of the Cournot reaction functions and of any analysis based on them. Moreover, such approaches to the equilibrium--during which the Cournot assumptions concerning rival behavior are patently incorrect--would have to take place more or less 'all the time' because, in consequence of shifting demand and cost functions in the actual world, no single equilibrium position would stay established for long." ²

The logical objections can be overcome if we assume a leader-follower model, where one firm (the leader) can assume that the opponent (the follower) will keep to his reaction curve, then the first firm will find it optimal to select a price and output vector which will maximize his profits when taking into account what the second firm's reaction will be. This might be appropriate in the following circumstance: a competitive trucking industry wherein each firm would have to take market conditions as given, in particular the railroad's prices. The railroad could assume a more or less predictable reaction from the competitive trucking

¹ Fellner, W., *Competition Among the Few*, Augustus M. Kelley, N. Y. 1965, p. 58.

² *ibid*, p. 65 * That is, on the way to the intersection point, starting from any section of the graph.

industry, and set its own prices accordingly.

The actual reaction functions derived from the initial problem need have no simple form. Examination of the example suggests that price on any link would be a non-decreasing function of the opponent's price on that link, and, within certain ranges, a decreasing function of price on another link.

Suppose the firm was able to locally approximate the opponent's reaction by a linear relation

$$\Delta_{rs} = p_{rs} + q_{rs} \Pi_{rs} ,$$

so that

$$d_{rs} = (a_{rs} + g_{rs} p_{rs}) - (b_{rs} - g_{rs} q_{rs}) \Pi_{rs} = \tilde{a}_{rs} - \tilde{b}_{rs} \Pi_{rs} .$$

Omitting constraints, the problem now reduces to essentially

$$\text{Max } \sum_{rs} \Pi_{rs} (\tilde{a}_{rs} - \tilde{b}_{rs} \Pi_{rs}) - \sum_j c_j x_j$$

(20)

$$\text{s. t. } \sum_{j \in D_{rs}} x_j \geq \tilde{a}_{rs} - \tilde{b}_{rs} \Pi_{rs}$$

Notice that it is separate from the opponent's problem, and that it remains a convex programming problem both in this and the stochastic case.

Nevertheless, the leader-follower notion is not entirely satisfactory. If both firms attempt to be leaders, the reaction curves cease to be valid. They also imply knowledge about the opponent which is not usually available except possibly by experience which would imply an adjustment process over time. We therefore turn to a game-theoretic approach as an alternative means of separating the problems of the firm and its opponent.

7. Game Theoretic Approaches

In the preceding section we have considered two approaches to optimal action: first, the derivation of a reaction curve for each player, giving his optimal prices and schedules for each set of prices that his opponent might charge; second, the choice of optimal prices and schedules by one player who can assume the other will obey a known reaction curve.

But often such information as to the opponent's decisions is not available and we turn to a game theoretic approach to this situation of uncertainty. Take as the objective function the familiar concept of minima x (or rather maximin) subject to satisfying his own constraint set, each player chooses prices and schedules to maximize his expected profit, on the assumption that, for any choice, his opponent will choose his own prices to minimize the first player's profit.

Immediately one difficulty arises: for any choice of π_{rs} and (x_j) , the opponent can choose Δ_{rs} high enough to violate the first player's capacity constraint (5), (11). Therefore, unless we assume known and finite bounds on the opponent's prices, a player cannot make any decision which will ensure that his constraint set is satisfied for all possible actions of his opponent. Accordingly, let us assume that the upper and lower bounds on prices (3), (3)' are finite and known to the opponent. This yields the problem

$$\begin{aligned}
 & \text{Max}_{\pi_{rs}, x_j} && \text{Min}_{L'_{rs} \leq \Delta_{rs} \leq U'_{rs}} && \sum_{(r,s)} \pi_{rs} (a_{rs} + g_{rs} \Delta_{rs} - b_{rs} \pi_{rs}) - \sum_j c_j x_j \\
 (21) \quad & \text{s. t.} && && L_{rs} \leq \pi_{rs} \leq U_{rs} \\
 & && && x_j \geq 0 \\
 & && && P_r \left\{ \sum_j k_j x_j \geq d_{rs}(\pi_{rs}, \Delta_{rs}) \right\} \geq \beta_{rs} \\
 & && && \text{for each } L'_{rs} \leq \Delta_{rs} \leq U'_{rs}
 \end{aligned}$$

where the other constraints of (17) have been omitted for convenience. The opponent faces an analogous problem.

The deterministic equivalent of the capacity constraint can be written as (11); clearly, if it is satisfied for the maximum value of Δ_{rs} it will be satisfied for all Δ_{rs} in the range $[L'_{rs}, U'_{rs}]$. Note also that for any π_{rs} , the choice of $\Delta_{rs} = L'_{rs}$ by the opponent minimizes the payoff. Hence we can write the simpler equivalent to (21)

$$\begin{aligned}
 (22) \quad & \text{Max}_{\pi_{rs}, x_j} \quad \sum_{r,s} \pi_{rs} (a_{rs} + g_{rs} L'_{rs} - b_{rs} \pi_{rs}) - \sum_j c_j x_j \\
 & \text{s. t.} \quad L_{rs} \leq \pi_{rs} \leq U_{rs} \\
 & \quad \sum_{j \in D_{rs}} k_j x_j + b_{rs} \pi_{rs} \geq a_{rs} + g_{rs} U'_{rs} + \sigma_{rs} \phi^{-1}(\beta_{rs}) \\
 & \quad x_j \geq 0
 \end{aligned}$$

In (22) we have a deterministic strategic equivalent for (21) which does not involve interactions with the opponent's decisions.

In (21), and indeed in the preceding analysis, we have implicitly assumed that a possible solution exists. In particular, we have assumed that the node constraints on x_j and price constraints on π_{rs} do not prevent the player from meeting maximum demand that the opponent can generate (i. e. by putting $\Delta_{rs} = U'_{rs}$). But obviously, the more restrictions upon any player, and the more latitude the opponent has to set prices, the less likely is this feasibility to be met.

A second observation is that a player might find it very unreasonable, although feasible, to provide a schedule catering for the highest price his opponent might charge--either because he thinks such a price unlikely, or because of the high cost of catering for it.

In view of these observations, we might reconsider the capacity constraints (5). They were originally formulated to treat the random element of demand; they are perhaps insufficiently flexible to deal with the uncertainty surrounding the opponent's price. An appropriate reformulation might therefore be to require demand be met up to a certain specified level ξ_{rs} , e. g.

$$(23) \quad \sum_{j \in D_{rs}} k_j x_j \geq \min [\xi_{rs}, a_{rs} + g_{rs} U'_{rs} - b_{rs} \pi_{rs} + \sigma_{rs} \phi^{-1}(\beta_{rs})] .$$

Another means of relaxing the problem faced by the players (which is not incompatible with the previous suggestion) follows from the following observation: If a player has to satisfy a set of constraints then he will not necessarily be able to range over the full set of prices

$\otimes_{(r,s)} [L_{rs}, U_{rs}]$ open to him. His feasible set of prices will be even further reduced if he has to meet the inter-twined constraints for a large range of his opponent's prices. Hence, any player need not be prepared to meet any set of prices from $\otimes_{(r,s)} [L'_{r,s}, U'_{r,s}]$, but rather from his opponent's smaller feasible set which depends, moreover, upon the range of prices which he himself may choose. This emphasizes the essential intertwining of the constraint sets. By introducing suitable notation, we can show how three different types of possible regions can be constructed from the intertwined constraint sets, such that a player need not consider the prices chosen by his opponent.¹⁾

Denote by S_o, S'_o the range within which prices must lie, i. e.

¹⁾ The constraint set may include chance constraints.

$$(24) \quad S_0 \equiv [L, U], \quad S'_0 \equiv [L', U']$$

where L, U, L', U' are vectors with as many components as there are links (r, s) . In a more general formulation the sets S_0, S'_0 might refer to constraints on prices known to the opponent and independent of the latter's actions.

The sets S_j, S'_j ($j = 1, 2, \dots$) are then defined inductively:

$$(25) \quad S_j \equiv \{ \pi \text{ feasible for } \forall \Delta \in S'_{j-1} \}$$

$$S'_j \equiv \{ \Delta \text{ feasible for } \forall \pi \in S_{j-1} \} \quad j = 1, 2, \dots$$

Thus, S_1 is the set of feasible prices open to the first player when the opponent's prices may range over $[L', U']$; S_2 ditto when the opponent's prices have to satisfy his constraints for any price that the first player might set in the range $[L, U]$, and so on. In the present model, the feasibility of prices depends also upon schedules x_j, y_{rs} , but henceforth the latter are ignored for convenience, without lack of generality.

It follows immediately from the definitions that $S_j \subseteq S_0, S'_j \subseteq S'_0, \forall j$, since for feasibility a price must satisfy the bounds $[L, U], [L', U']$ ¹. Further, $S_1 \subseteq S_j, S'_1 \subseteq S'_j, \forall j$, since the sets S_1, S'_1 have to be feasible under the widest range of opponent's prices, namely S_0, S'_0 . It can further be shown that

¹In general, S_0, S'_0 are not feasible.

$$(26) \quad S_1 \subseteq S_3 \subseteq S_5 \subseteq \dots \subseteq \underline{S}, \text{ and}$$

$$S_0 \supseteq S_2 \supseteq S_4 \supseteq \dots \supseteq \bar{S}, \text{ where}$$

$\underline{S} \subseteq \bar{S}$ and as a special case $\underline{S} = \bar{S}$. Analogous statements hold for primed sets.

We can now distinguish the three types of feasible regions referred to earlier.

- I $\underline{S}_1 \otimes \underline{S}'_1$: \underline{S}_1 is the set of prices open to the first player which ensure that his constraint set will be satisfied for any set of prices in the opponent's known range $[L', U']$. Analogously for \underline{S}'_1 .
- II $\underline{S} \otimes \underline{S}'$: In fact, the first player's feasible region is $\underline{S} \subseteq \underline{S}_1$, since the opponent must satisfy further constraints (unknown to the first player, and depending upon the latter's prices). \underline{S} is the largest set available to the first player such that the opponent cannot choose a feasible price (for him) which violates the first player's constraint set.
- III $\bar{S} \otimes \underline{S}'$ (or $\underline{S} \otimes \bar{S}'$): If a player can assume that his opponent will play safe by choosing a price in \underline{S}' , then he can choose any price $\bar{S} \subseteq \underline{S}$ and still be sure of satisfying his constraints. This concept is analogous to that of the leader-follower, but applying to simultaneous decisions i. e. without knowledge of the opponent's decision. Notice that, for any given π in \bar{S} but not in \underline{S} , the opponent can choose Δ not in \underline{S}' to satisfy his own constraints while ensuring that his opponent's are violated, but for any unspecified π in \bar{S} and not \underline{S} he runs the risk of violating his own constraints unless he chooses Δ in \underline{S}' . The cost of constraint violation, which is exterior to the model, is obviously important for the determination of the leader or the follower.

The first player's problem, corresponding to the three cases

above, may be represented:

$$(27) \quad \begin{array}{l} \text{I} \quad \text{Max} \quad \text{Min} \quad (\quad) \\ \pi \in S_1 \quad \Delta \in S'_0 \end{array}$$

$$(28) \quad \begin{array}{l} \text{II} \quad \text{Max} \quad \text{Min} \quad (\quad) \\ \pi \in \underline{S} \quad \Delta \in S'_2 \end{array}$$

$$(29) \quad \begin{array}{l} \text{III} \quad \text{Max} \quad \text{Min} \quad (\quad) \\ \pi \in \bar{S} \quad \Delta \in \underline{S}' \end{array}$$

or

$$\begin{array}{l} \text{Max} \quad \text{Min} \quad (\quad) \\ \pi \in \underline{S} \quad \Delta \in \bar{S}' \end{array}$$

Where the decisions are not made simultaneously, the first player's problem is to choose $\pi = \pi_0$ to

$$(30) \quad \begin{array}{l} \text{Max} \quad \text{Min} \quad (\quad) \\ \pi \in \underline{S} \quad \Delta \in S'(\pi_0) \end{array}$$

where $S'(\pi_0) \equiv \{ \text{feasible } \Delta \text{ for } \pi = \pi_0 \}$

We have $S'_0 \supseteq S'(\pi_0) \supseteq \bar{S}'$, but not necessarily $S'(\pi_0) \subseteq S'_2$, since π_0 need not be in S_1 . The second player is no longer faced with a programming problem. However, this line is worth pursuing in the case where each player believes the opponent might discover his prices before deciding upon his (the opponent's) own.

A line for future research is the existence of equilibrium strategies within the various feasible regions. The overall problem is a non-zero sum chance-constrained game similar to those introduced in [2], [3] but, as we have pointed out, the players' strategy sets are intertwined and the joint strategy set is not a simple Cartesian product of the individual sets.

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<p>This paper is concerned with the development of a model for planning shipping prices over various routes of a transportation network. It is assumed that over each route a "railroad" (or other transport firm) competes directly with one other shipping firm (e. g., another railroad, a trucker) for the volume of business which is to be shipped over the route. We allow the possibility that the railroad's competition will come from different firms over different routes.</p> <p>The model we consider is a multi-route problem which involves both institutional and physical operating constraints. Each firm's demand depends linearly upon its own and its opponent's price. Since each competing firm sets its price policy independently and before the variable demand is known, some of these constraints cannot be guaranteed to hold with certainty. Thus they are best expressed as chance constraints. The objective of the "railroad" is to maximize its profit subject to the constraints of the model.</p> <p>The solution of the model synthesizes a pricing policy for the "railroad;" we also indicate how variations of the model can be employed in a sensitivity analysis of the suggested policy and discuss the relationship between our model and other notions in game theory and chance-constrained games. An example is outlined, and the implications for a welfare-economic pricing policy considered.</p>		

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