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AN EXPLICIT SOLUTION OF A SPECIAL
CLASS OF LINEAR PROGRAMMING
PROBLEMS

by

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SYSTEMS RESEARCH GROUP

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Introduction

The linear programs considered here are of the form:

$$\begin{array}{ll} \text{Maximize } (c, x) \\ \text{(LP)} & \text{subject to} \\ & a \leq Ax \leq b \end{array}$$

where A is of full row rank, and (LP) is feasible with bounded optimal solutions.

The main result, equation (24), is an explicit representation of the general optimal solution of (LP), in terms of a generalized inverse of $A^{1/}$. This explicit solution of (LP) -explicit in the sense that $A^{-1}b$ is an explicit solution of $Ax = b$ - has obvious theoretical (and possibly computational) advantages over the well known iterative methods of linear programming, e. g. [5], [8].

The results are illustrated by a simple example, and extensions to general linear programs are discussed.

Preliminaries and notations

We denote by:

R^n the n -dimensional real vector space

For any two vectors x, y in R^n :

$x \geq y$ denotes $x_i \geq y_i \quad (i = 1, \dots, n)$

^{1/} For other applications of generalized inverses in linear programming see [14], [7] and [6].

For any subspace L of R^n :

$L^\perp$ denotes the orthogonal complement of L

P_L denotes the perpendicular projection on L

For any $m \times n$ real matrix A :

A^T -- the transpose of A

$R(A)$ -- the range space of A

$N(A)$ -- the null space of A .

For the fixed $m \times n$ real matrix A consider the 4 matrix equations:

- (1) $AXA = A$
- (2) $XAX = X$
- (3) $(AX)^T = AX$
- (4) $(XA)^T = XA$

We denote by $A\{i, j, \dots, k\}$ the set of $n \times m$ real matrices X satisfying equations (i), (j), $\dots$, (k), ($1 \leq i, j, \dots, k \leq 4$). These sets $A\{i, j, \dots, k\}$, ($1 \leq i, j, \dots, k \leq 4$), are nonempty because $A\{1, 2, 3, 4\}$ is nonempty, e.g. [13].

A matrix $X \in A\{i, j, \dots, k\}$ is called an $\{i, j, \dots, k\}$ - g.i. (generalized inverse) of A . The $\{1, 2, 3, 4\}$ - g.i. of A is unique, and is the well known Moore-Penrose generalized inverse, e.g. [13], [12], denoted by A^+ . For some applications a weaker g.i. will do, e.g. [3], [4], [11] and [10]. Thus for solving linear equations (and for the purpose of this paper) $\{1\}$ - g.i.'s are sufficient, as shown by the following:

Lemma 1 ([3], [13]): The linear equations

$$(5) \quad Ax = b$$

are solvable iff for any $T \in A\{1\}$

(6) $ATb = b,$

in which case the general solution of (5) is:

(7) $x = Tb + (I - TA)y, y$ arbitrary □

The set $A\{1\}$ is represented in terms of one of its elements as follows:

Lemma 2 ([3]): Let R be any $\{1\}$ - g. i. of A . Then

(8) $A\{1\} = \{RAR + Y - RAYAR : Y$ arbitrary

$n \times m$ real matrix $\}$

Projections associated with g. i.'s are given in: □

Lemma 3 ([1]):

(a) If $S \in A\{1,3\}$ then

(9) $AS = P_{R(A)}$

(b) If $T \in A\{1,2,4\}$ then

(10) $TA = P_{R(A^T)}$

A recipe for computing a $\{1\}$ - g. i. and for constructing a basis of $N(A)$ is given in:

Lemma 4: Let A be an $m \times n$ real matrix of rank r , and let E be a nonsingular $m \times m$ real matrix such that:

$$(11) \quad EA = \left(\begin{array}{c|c} I_r & \Delta \\ \hline O_{(m-r) \times n} & \end{array} \right) P$$

where P is a permutation matrix.

Conclusions:

(a) Let E be partitioned

$$(12) \quad E = \left(\begin{array}{c|c} E_{11} & E_{12} \\ \hline E_{21} & E_{22} \end{array} \right) \quad \text{where } E_{11} \text{ is } r \times r$$

then the $n \times m$ matrix

$$(13) \quad T = P^T \left(\begin{array}{c|c} E_{11} & E_{12} \\ \hline O_{(n-r) \times m} & \end{array} \right)$$

is a $\{1,2\}$ - inverse of A .

(b) The columns of the $n \times (n-r)$ matrix

$$(14) \quad N = P^T \left(\begin{array}{c} -\Delta \\ \hline I_{n-r} \end{array} \right)$$

form a basis of $N(A)$.

Proof:

(a) Consider the $n \times n$ nonsingular matrix

$$(15) \quad F = P^T \left(\begin{array}{c|c} I_r & -\Delta \\ \hline O & I_{n-r} \end{array} \right)$$

From (11), (15) and $PP^T = I$ we get

$$EAF = \left(\begin{array}{c|c} I_r & O \\ \hline O & O \end{array} \right)$$

or

$$(16) \quad A = E^{-1} \left(\begin{array}{c|c} I_r & O \\ \hline O & O \end{array} \right) F^{-1}$$

Now the matrix

$$(17) \quad T = F \left(\begin{array}{c|c} I_r & O \\ \hline O & O \end{array} \right) E$$

is a $\{1, 2\}$ - g. i. of A , as shown by substituting (16) and (17) in (1) and (2).

Substituting (12) and (15) in (17) we get

$$\begin{aligned} T &= P^T \left(\begin{array}{c|c} I_r & -A \\ \hline O & I_{n-r} \end{array} \right) \left(\begin{array}{c|c} I_r & O \\ \hline O & O \end{array} \right) \left(\begin{array}{c|c} E_{11} & E_{12} \\ \hline E_{21} & E_{22} \end{array} \right) \\ &= P^T \left(\begin{array}{c|c} E_{11} & E_{12} \\ \hline O & O \end{array} \right) \end{aligned}$$

which proves (13).

(b) Obvious from (11) and the facts : $PP^T = I$ and E nonsingular. $\square$

For other results and references on $\{1\}$ - g. i's see [4], [10] and [1].

Results:

Consider the linear programming problem:

(LP) Maximize (c, x)
subject to

$$(18) \quad a \leq Ax \leq b,$$

for given

$$A = (a_{ij}), \quad a = (a_i), \quad b = (b_i), \quad c = (c_j)$$

$$(i = 1, \dots, m; \quad j = 1, \dots, n),$$

and assume:

Assumption 1: (LP) is feasible, i. e.

$$(19) \quad S = \{x \in R^n : a \leq Ax \leq b\} \neq \emptyset$$

The following properties of S are obvious:

Lemma 5: If (19) then

$$(20) \quad S = S + N(A) = \{x + y : x \in S, y \in N(A)\}$$

and the set

$$(21) \quad P_{R(A^T)} S = \{P_{R(A^T)} x : x \in S\} \text{ is bounded.}$$

□

The case when (LP) has a finite maximum is characterized in:

Lemma 6: (LP) has a bounded optimal solution iff

$$(22) \quad c \perp N(A)$$

Proof: If: From (22) and the fact

$$N(A) = R(A^T)^\perp$$

it follows that

$$\{(c, x) : x \in S\} = \{(c, x) : x \in P_{R(A^T)} S\}$$

and is a bounded interval, since (21) is a bounded set.

Only if: Suppose $P_{N(A)} c \neq 0$. Then the interval

$$\{(c, x) : x \in S\}$$

is the entire real line, by (20)

□

We make now 2 additional assumptions:

Assumption 2: (LP) has a bounded optimal solution, i. e. (22) holds.

Assumption 3: The matrix A is of full row rank:

$$(23) \quad \text{rank } A = m$$

An explicit representation of the general optimal solution of (LP) is now given:

Theorem: Let (LP) satisfy assumptions 1, 2 and 3, and let the $n \times m$ matrix

T with columns $(t_1, t_2, \dots, t_m)$ be a $\{1\}$ -g.i. of A. Then the optimal solutions of (LP) form the manifold:

$$(24) \quad x = \sum_{i \in I_-} t_i a_i + \sum_{i \in I_+} t_i b_i + \sum_{i \in I_0} t_i (\theta_i b_i + (1 - \theta_i) a_i) + N(A)$$

where

$$(25) \quad \begin{aligned} I_- &= \{i : (c, t_i) < 0\} \\ I_+ &= \{i : (c, t_i) > 0\} \\ I_0 &= \{i : (c, t_i) = 0\} \end{aligned}$$

and $0 \leq \theta_i \leq 1, \quad i \in I_0.$

Proof: From (23) it follows that

$$R(A) = R^m$$

so that for any z in R^m we have

$$(26) \quad z = A x$$

where

$$(27) \quad x = T z + N(A), \text{ by lemma 1.}$$

Substituting (26), (27) in (LP) we conclude from (22) that (LP) is

equivalent to the following linear program over a parallelopiped:

(LPP) Maximize (c, Tz)
 subject to

$$(28) \quad a \leq z \leq b$$

whose optimal solution is obviously:

$$z_i = \begin{cases} a_i & i \in I_- \\ b_i & i \in I_+ \\ \theta_i b_i + (1 - \theta_i) a_i & i \in I_0 \end{cases}$$

for any $0 \leq \theta_i \leq 1$, $i \in I_0$.

From (27) and (29) it follows that the manifold (24) is the set of optimal solutions of (LP). . □

Remark: It can be shown directly that the set (24) is independent of the particular $\{1\}$ - g. i. used in its definition. We need:

Lemma 7: Let A be an $m \times n$ matrix of rank m , and let T be any $\{1\}$ - g. i. of A. Then

$$(30) \quad T = A^+ + W$$

where W is a matrix whose columns lie in $N(A)$.

Proof: Lemma 2, with $R = A^+$, the unique $\{1, 2, 3, 4\}$ - g. i. of A, gives:

$$(31) \quad A\{1\} = A^+ A A^+ + Y - A^+ A Y A A^+ , \quad Y \text{ arbitrary}$$

$$= A^+ + P_{N(A)} Y , \quad Y \text{ arbitrary}$$

This follows from $A^+AA^+ = A^+$ by (2),

$$\begin{aligned} AA^+ &= P_{R(A)} \quad , \text{ by (9) ,} \\ &= I \quad , \text{ since rank } A = m \end{aligned}$$

and

$$\begin{aligned} I - A^+A &= I - P_{R(A^T)} \quad , \text{ by (10) ,} \\ &= P_{N(A)} . \end{aligned}$$

Now (30) follows from (31) with $W = P_{N(A)} Y$. □

From lemma 7 and (22) it follows that the sets I_- , I_+ , I_0 defined by (25) and the general solution (24) are independent of the $\{1\}$ - g. i. used in the theorem.

Example:

The problem, of class (LP) , is:

$$(32) \quad \text{Maximize } 2x_1 - x_2 - x_3 + 3x_4$$

subject to:

$$\begin{aligned} 0 &\leq x_1 + 2x_2 - x_3 && \leq 1 \\ -3 &\leq -x_1 + x_3 - x_4 && \leq 0 \\ 1 &\leq 2x_1 + x_2 - 3x_3 + x_4 && \leq 3 \end{aligned}$$

We use lemma 4 to compute a $\{1\}$ - g. i. of

$$A = \begin{pmatrix} 1 & 2 & -1 & 0 \\ -1 & 0 & 1 & -1 \\ 2 & 1 & -3 & 1 \end{pmatrix}$$

by diagonalizing the $m \times (n + m)$ matrix

$$A^{(0)} = (A, I_m) :$$

$A^{(0)} =$	$\begin{array}{cccc ccc} \textcircled{1} & 2 & -1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & -1 & 0 & 1 & 0 \\ 2 & 1 & -3 & 1 & 0 & 0 & 1 \end{array}$	pivots are circled
$A^{(1)} =$	$\begin{array}{cccc ccc} 1 & 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & \textcircled{2} & 0 & -1 & 1 & 1 & 0 \\ 0 & -3 & -1 & 1 & -2 & 0 & 1 \end{array}$	
$A^{(2)} =$	$\begin{array}{cccc ccc} 1 & 0 & -1 & 1 & 0 & -5/2 & -1 \\ 0 & 1 & 0 & -1/2 & 1/2 & 1/2 & 0 \\ 0 & 0 & \textcircled{-1} & -1/2 & -1/2 & -3/2 & -1 \end{array}$	
$A^{(3)} =$	$\begin{array}{cccc ccc} 1 & 0 & 0 & 3/2 & 1/2 & -5/2 & -1 \\ 0 & 1 & 0 & -1/2 & 1/2 & 1/2 & 0 \\ 0 & 0 & 1 & 1/2 & 1/2 & -3/2 & -1 \end{array}$	

Assumption 3 is satisfied if the last matrix $A^{(m)}$ is of the form

$$A^{(m)} = ((I_m, \Delta) P \mid E)$$

Indeed from $A^{(3)}$ we read

$$(33) \quad \Delta = \begin{pmatrix} 3/2 \\ -1/2 \\ 1/2 \end{pmatrix}, \quad P = I, \quad E = \begin{pmatrix} 1/2 & -5/2 & -1 \\ 1/2 & 1/2 & 0 \\ 1/2 & -3/2 & -1 \end{pmatrix}$$

and by (14) it follows that $N(A)$ is spanned by the vector

$$(34) \quad N = \begin{pmatrix} -3/2 \\ 1/2 \\ -1/2 \\ 1 \end{pmatrix}$$

and that assumption 2 is satisfied for $c = \begin{pmatrix} 2 \\ -1 \\ -1 \\ 3 \end{pmatrix}$.

From (13) and (33) we get a $\{1\}$ - g. i. of A :

$$(35) \quad T = \begin{pmatrix} 1/2 & -5/2 & -1 \\ 1/2 & 1/2 & 0 \\ 1/2 & -3/2 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

for which

$$c^T T = (2, -1, -1, 3) \begin{pmatrix} 1/2 & -5/2 & -1 \\ 1/2 & 1/2 & 0 \\ 1/2 & -3/2 & -1 \\ 0 & 0 & 0 \end{pmatrix} = (0, -4, -1)$$

so that

$$(36) \quad I_- = \{2, 3\}, \quad I_+ = \emptyset, \quad I_0 = \{1\}.$$

The general optimal solution of (32) is from (24), (34), (35) and

(36):

$$(37) \quad x = \theta_1 \begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 0 \end{pmatrix} - 3 \begin{pmatrix} -5/2 \\ 1/2 \\ -3/2 \\ 0 \end{pmatrix} + 1 \begin{pmatrix} -1 \\ 0 \\ -1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -3/2 \\ 1/2 \\ -1/2 \\ 1 \end{pmatrix}$$

with $0 \leq \theta_1 \leq 1$

and λ arbitrary.

The optimal value of $(c, x) = 11$.

A different choice of pivots in $A^{(i)}$ ($i = 0, 1, 2$) could result in different matrices in (33) and (35), but the sets (36) and the manifold (37) are unchanged.

Discussion

Linear programs arising in concrete applications are usually feasible and possess bounded optimal solutions. Therefore assumptions 1 and 2 are not too restrictive.

Also any linear program with inequality constraints can be rewritten as our problem (LP), by setting the missing a_i and b_i as: $-M$ and $+M$ respectively, where $M > 0$ is a sufficiently large number. (If M appears in an optimal solution, then the problem has unbounded optimal solutions).

The remaining assumption 3 is a true restriction on the scope of our method. It is typically violated by linear programs of the form:

$$\begin{aligned} &\text{Maximize } (c, x) \\ &\text{subject to} \end{aligned}$$

$$(38) \quad \begin{aligned} Ax &\leq b \\ x &\geq 0 \end{aligned}$$

which are rewritten as our (LP) :

$$\text{Maximize } (c, x)$$

$$(39) \quad \begin{pmatrix} -Me \\ 0 \end{pmatrix} \leq \begin{pmatrix} A \\ I \end{pmatrix} x \leq \begin{pmatrix} b \\ Me \end{pmatrix}$$

where $M > 0$ is sufficiently large, and $e = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$.

There are several ways of applying our method to problems (LP) without assuming A to have full row rank. One possibility is to partition (LP) as follows:

$$\text{Max } (c, x)$$

$$(40) \quad \begin{aligned} a^1 &\leq A^1 x \leq b^1 \\ a^2 &\leq A^2 x \leq b^2 \end{aligned}$$

where A^1 is an $r \times n$ submatrix of the $m \times n$ matrix A and

$$(41) \quad \text{rank } A^1 = \text{rank } A = r.$$

From (41) it follows that

$$(42) \quad N(A) = N(A^1)$$

and by lemma 6, problem (40) has a bounded optimal solution iff

$$(43) \quad c \perp N(A^1) .$$

The subproblem

$$(44) \quad \begin{aligned} &\text{Max } (c, x) \\ &a^1 \leq A^1 x \leq b^1 \end{aligned}$$

thus satisfies our assumptions, and (24) can be used to obtain the manifold

$$(45) \quad x^1 + N(A)$$

of optimal solutions of (44).

Any vector in (45) which satisfies the remaining constraints

$$a^2 \leq A^2 x \leq b^2$$

of (40) is clearly an optimal solution of (40). In the absence of such a vector, an optimal solution of (40) can be found in a finite number of iterations, where at each iteration A^1 is changed by one row in an obvious manner. Our method may thus serve as a start for a dual simplex method.

Another possibility is to partition (LP) into k subproblems

$$\text{Max } (c, x^i)$$

$$(46. i) \quad a^i \leq A^i x^i \leq b^i \quad (i = 1, \dots, k)$$

where each subproblem satisfies our assumptions 1 and 3. Since assumption 2 is assumed for (LP), we solve (46. i) by (24) as if $c \perp N(A^i)$, $i = 1, \dots, k$. The resulting optimal solutions are generally different, but may be forced to coincide in a finite number of iterations of the decomposition method of Dantzig-Wolfe [9]. These results are contained in [15].

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