

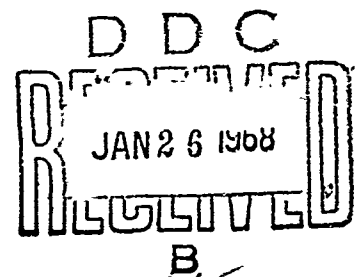
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EDDY CURRENT TORQUES AND MOTION DECAY ON ROTATING SHELLS

NOVEMBER 1967

J. F. A. Ormsby

Prepared for
DIRECTORATE OF PLANNING AND TECHNOLOGY
DEVELOPMENT ENGINEERING DIVISION
ELECTRONIC SYSTEMS DIVISION
AIR FORCE SYSTEMS COMMAND
UNITED STATES AIR FORCE
L. G. Hanscom Field, Bedford, Massachusetts



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Project 8051
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Bedford, Massachusetts
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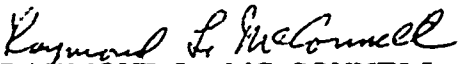
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FOREWORD

This report was prepared by The MITRE Corporation, Bedford, Massachusetts under Contract AF 19(628)-5165. The work was directed by the Development Engineering Division under the Directorate of Planning and Technology, Electronic Systems Division, Air Force Systems Command, Laurence G. Hanscom Field, Bedford, Massachusetts.

REVIEW AND APPROVAL

Publication of this technical report does not constitute Air Force approval of the report's findings or conclusions. It is published only for the exchange and stimulation of ideas.


RAYMOND L. MC CONNELL, ESLE
ESD Project Officer

ABSTRACT

This report describes a study conducted to estimate the torques acting on electrically conducting shells rotating in a magnetic field. The basic electromagnetic expressions leading to an evaluation of torque are given using various approaches with comparisons for the spherical shell. Application of these techniques to cylindrical shells with flat and hemisphere shell ends is described. The torques both slow and alter the direction of the rotating motion, and the resultant decay time constants can be in the order of days. Factors affecting this decay time and influencing shell design are examined.

ACKNOWLEDGEMENT

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SECTION I

INTRODUCTION

HISTORICAL SUMMARY

After the discovery of electromagnetic induction by M. Faraday and J. Henry (1841) the slowing down effect, which conductors experience when rotating in a magnetic field, was studied by H. Hertz (1896).^[1]* Shortly after, additional analytic treatment was described by R. Gans (1903).^[2, 3] These results for spherical shells also appear in the relatively modern text by Smythe.^[4]

Effort during the first half of this century concentrated on generator and motor design. While studies in electromagnetism, terrestrial magnetism, and mechanics appeared in various texts, a rekindling of interest in the basic questions concerning the interaction of conducting bodies moving in magnetic fields awaited the arrival of the space age around 1955. Then a number of investigations, for example, Rosenstock, Vinti, and Zunov^[5, 6, 7] considered the problem in terms of a spinning satellite moving in an earth orbit. Yet until the present, very little was done analytically on other than a few regularly shaped bodies such as spheres, cylinders, and prolate spheroids.^[8, 9, 10]

A good deal of effort has been given to using observed satellite data to arrive empirically at decay and motion perturbation estimates.^[11, 12] In this regard, a rather comprehensive treatment recently appeared by Yu.^[13]

* Numbers in brackets refer to References cited at the end of the report.

Finally, a relatively large effort also has been given to utilizing the interaction of the magnetosphere with on-board magnetic sources in order to effect attitude control.^[14, 15] In dealing with objects in orbit, the interaction of the magnetosphere need not only be with the induced eddy currents. It may also react with existing current or magnetic sources on the object as well as with the magnetic field induced. There may also be reactions of ambient electric fields with induced electric fields.

THE SCOPE OF THE PROBLEM

When a conductor and a magnetic field undergo relative motion*, an electromotive force is developed in the conductor. If closed conducting paths are present through which a change of magnetic flux occurs, then eddy currents flow in the conductor. The further interaction of this current flow with the magnetic field produces torques which in general both dissipate the rotational energy and cause an orientation change in the rotation axis. In addition, the currents if sufficiently strong produce a magnetic field which can appreciably perturb the background magnetic field.

The analytic effort of interest here is that concerned with deriving expressions for the resulting torques, and also with the decay time constants which result, rather than the over-all motion problem to account for all magnetic torque components and the resulting motion about the center of mass as the object moves in orbit.

It will also be our concern here to deal with modified cylinder shapes and to discuss the design problem for realizing desired motion lifetimes.

*This includes the case of a stationary conductor and a magnetic field varying in time-space.

SECTION II

ANALYTICAL APPROACHES

BASIC ANALYTICAL MODEL

To provide a background for the methods of calculating eddy current torques, we begin this section by describing the basic electromagnetic and mechanics equations appropriate to establishing mathematical models. The following symbols apply to the equations given in the report:

Electromagnetics

$\mathcal{E}$ = electromotive force
 $\vec{E}$ = electric intensity vector
 $\vec{H}$ = magnetic intensity vector
 $\vec{D}$ = electric flux vector
 $\vec{B}$ = magnetic flux vector
 Φ = flux linkage
 ρ = charge density
 $\vec{i}$ = current density vector
 σ = conductivity
 ϵ = emissivity
 μ = magnetic permeability
 c = speed of light

Mechanics - Geometry

M = mass
 I = moment of inertia
 $\vec{\omega}$ = angular velocity vector
 $\vec{V}$ = velocity vector
 $\vec{r}$ = distance from center of mass
 $\vec{f}$ = force vector
 $\vec{T}$ = torque vector
 $\vec{l}$ = path length on object

Using a Gaussian system of units, the applicable basic equations are:

$$\nabla \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t} \quad (1)$$

$$\vec{i} = \sigma \left[\vec{E} + \frac{\mu}{c} (\vec{V} \times \vec{H}) \right] + \rho \vec{V} \quad (2)$$

$$\nabla \times \bar{H} = \frac{4\pi}{c} \bar{i} + \frac{1}{c} \frac{\partial \bar{D}}{\partial t} \quad (3)$$

$$\bar{V} = \bar{\omega} \times \bar{r} \quad (4)$$

Equations (1) to (3) apply whether measured in an inertial frame at rest with the laboratory, or fixed with respect to an inertial observer instantaneously at rest with respect to a point in the moving object. For an inertial frame at rest for some point on the object, the following constitutive equations apply:

$$\bar{D}' = \epsilon \bar{E}' \quad (5)$$

$$\bar{B}' = \mu \bar{H}' \quad (6)$$

Using unprimed quantities to apply for a fixed frame and taking $\mu = 1$, $\epsilon \cong 1$ (for a metal) and $(V/c)^2 \ll 1$ we get, inside the metal, for our purposes [6]

$$\bar{D} \cong \bar{E} \quad (7)$$

$$\bar{B} \cong \bar{H} \quad (8)$$

Also for $(V/c)^2 \ll 1$

$$\bar{i} = \bar{i}' + \rho \bar{V} \quad (9)$$

$$\bar{E}' = \bar{E} + (\bar{V} \times \bar{H}) \mu/c \quad (10)$$

In addition, the contribution of the convection currents $\bar{\rho} \bar{V}$ in regard to producing torque can be neglected. Their elimination in this respect can be justified both due to the relatively small contribution to $\bar{i}$ and also a possible cancellation for torque due to geometrical symmetry. The only effect, which is very small, is a turning torque. [6] However, in general we take

$$\bar{\mathbf{i}} = \bar{\mathbf{i}}' + \rho \bar{\mathbf{V}} \quad (11)$$

$$\bar{\mathbf{i}}' = \sigma \left[\bar{\mathbf{E}} + \frac{\mu}{c} (\bar{\mathbf{V}} \times \bar{\mathbf{H}}) \right] \quad (12)$$

In respect to the discussion in the introduction, it is interesting to recall the two mechanisms for inducing an electromotive force $\mathcal{E}$ about a conducting path S . Using $\mu = 1$ we have

$$\begin{aligned} \mathcal{E} &= \int_S \bar{\mathbf{E}}' \cdot d\bar{\ell} = \int_S \left[\bar{\mathbf{E}} + \left(\frac{\bar{\mathbf{V}} \times \bar{\mathbf{H}}}{c} \right) \right] \cdot d\bar{\ell} \\ &= - \frac{1}{c} \left[\left(\frac{\partial \Phi}{\partial t} \right) \bar{\mathbf{V}} = 0 + \frac{\partial \Phi}{\partial t} \bar{\mathbf{B}} = \text{constant} \right] \\ &= - \frac{1}{c} \frac{d\Phi}{dt} \end{aligned} \quad (13)$$

where $d\Phi/dt$ is the rate of change of flux linking path S .

We are concerned here with $(\partial\Phi/\partial t) \bar{\mathbf{B}} = \text{constant}$ due to motion as well as $(\partial\Phi/\partial t) \bar{\mathbf{V}} = 0$ if $\partial\mathbf{H}/\partial t \neq 0$, that is, if the background magnetic field is disturbed.

If the background magnetic field is fixed and the eddy currents do not appreciatively alter it, then we may take $\partial\bar{\mathbf{H}}/\partial t = 0$ in Equation (1). This allows $\nabla \times \bar{\mathbf{E}} = 0$ which results in a convenient approach for the spherical shell described under "The Spherical Shell."

The angular motion $\bar{\omega}$ can be resolved as

$$\bar{\omega} = \bar{\omega}_{||} + \bar{\omega}_{\perp} \quad (14)$$

where

$\bar{\omega}_{||}$ is parallel to $\bar{H}$

$\bar{\omega}_{\perp}$ is normal to $\bar{H}$

The effect of $\bar{\omega}_{||}$ is to cause charge separation toward the conductor boundary, that is radially from the rotation axis, resulting in an electric field $\bar{E}$ which balances out the originally induced electric field caused by the motion. In other words with $\partial \bar{H} / \partial t = 0$ and taking $\bar{V}_{||} = \bar{\omega}_{||} \times \bar{r}$ then

$$\bar{E}' = \bar{E} + \frac{\mu}{c} \left(\bar{V}_{||} \times \bar{H} \right) = 0 \quad (15)$$

Using Equation (12) then gives

$$\bar{i}' = 0 \quad (16)$$

With respect to the fixed frame the zero torque result also obtains as can be seen from the force expression using Equation (11) for $\bar{i}$ as follows

$$\begin{aligned} \bar{f} &= \rho \bar{E} + \bar{i} \times \mu \bar{H} = \rho \bar{E} + [\rho \bar{V} + \sigma (\bar{E} + \bar{V} \times \mu \bar{H})] \times \mu \bar{H} \\ &= \rho (\bar{E} + \bar{V} \times \mu \bar{H}) + \sigma (\bar{E} + \bar{V} \times \mu \bar{H}) = 0 \end{aligned} \quad (17)$$

from Equation (15).

This is equivalent to saying that under $\bar{\omega}_{||}$ action, no closed $\bar{i}$ loops occur. Here $\nabla \cdot \bar{i} \neq 0$ and amounts to $\partial \bar{D} / \partial t \neq 0$. The effect is referred to as electrostatic shielding. For $\bar{\omega}_{\perp}$ the charge separation dissipates rapidly along closed current loops. This in essence amounts to assuming $\partial \bar{D} / \partial t = 0$, since using

$$\nabla \cdot \bar{D} = 4 \pi \rho \quad (18)$$

implies no charge build-up. More exactly, from Equations (3) and (18) with $\bar{D}'$, we have

$$\nabla \cdot \bar{i}' = -\frac{1}{4\pi} \frac{\partial \nabla \cdot \bar{D}'}{\partial t} = -\frac{\partial \rho}{\partial t} \quad (19)$$

The rates $\partial \rho' / \partial t$ are electromagnetic and so much more rapid than the motion rate ω . Thus, taking for the transient $\bar{i}' = \sigma \bar{E}'$, Equation (18) allows simply

$$\frac{\partial \rho'}{\partial t} = \frac{4\pi \sigma \rho'}{\epsilon} \quad (20)$$

that is

$$\rho' = \rho'_0 e^{-t/\tau}, \quad \tau = \epsilon/4\pi\sigma \quad (21)$$

For example, τ for aluminum $\cong 0.2 \times 10^{-18}$ sec. Thus, for a conductor such as aluminum, the time constant τ is extremely short.

Hence, any tendency for charge build-up is rapidly dissipated and in the steady state ($\omega \sim$ constant) causes no opposing electric field as in Equation (16). Notice from Equation (1) that when $\partial \bar{H} / \partial t \neq 0$, an electric field can be developed even though no charge build-up occurs. Thus we may summarize by stating that an electric field is produced to oppose that caused by the motion ($\bar{V} \times \bar{H}/c$) if

- (1) $\bar{i}$ causes $\partial \bar{H} / \partial t \neq 0$ (i. e., changes in the total magnetic field).
- (2) $\bar{i}$ causes a build-up of charge.

In the case of only $\bar{\omega}_{||}$ present, case (2), as we have noted, results in $\bar{i}' = 0$ and so no torque is produced.* In such a case the opposing electric field is generated by the charge build-up.

Considering Equations (3) and (11) leads to

$$\nabla \times \bar{H} = \frac{4\pi\sigma}{c} \left(\bar{E} + \frac{\bar{V} \times \bar{H}}{c} \right) + \frac{\partial \bar{D}}{c \partial t} + \frac{4\pi}{c} \rho \bar{V} \quad (22)$$

We see that $\partial \bar{D} / \partial t = 0$ requires

$$\frac{4\pi\sigma}{c} \bar{E} \gg \frac{\epsilon}{c} \frac{\partial \bar{E}}{\partial t} \quad (23)$$

or for harmonic time variation

$$\sigma / f \gg \epsilon$$

a condition satisfied for good conductors and frequencies up to 10^{10} mc.

In this discussion the condition is valid and results in the so-called quasi-stationary case with no displacement current.

Taking the case of Equation (22) we have

$$\nabla^2 \bar{H} = \frac{4\pi\sigma\mu}{c^2} \frac{\partial \bar{H}}{\partial t} - \frac{4\pi\sigma\mu}{c^2} \left(\nabla \times (\bar{V} \times \bar{H}) \right) \quad (24)$$

We consider $\bar{E} = \bar{E}_1 + \bar{E}_2$ such that we associate

$$\nabla \times \left[\bar{E}_1 + \left(\frac{\bar{V} \times \bar{H}}{c} \right) \right] = 0 \quad (25)$$

*Unless $\bar{\omega}_{||}$ is strictly constant, current transients occur. The torques developed from the resulting field disturbances are negligible, however, with respect to the torque due to the steady state $\bar{\omega}_{\perp}$ motion.

and

$$\frac{c}{4\pi} \nabla^2 \bar{H} = -\sigma \mu \nabla \times \bar{E}_2 \quad (26)$$

For example, an aluminum object of characteristic length $\ell \approx 125$ cm and with $\bar{V}$ on its surface between 0 and about 12 cm/sec (corresponding to $\omega \approx 2\pi/30$ rad/sec), has $\nabla \times (\bar{V} \times \bar{H}/c) \sim VH/c\ell$ and $c/4\pi\sigma\mu \nabla^2 \bar{H} \sim cH/4\pi\sigma\mu\ell$ with equal order effects on $\partial\bar{H}/\partial t$. Then $\bar{E}_1$ and $\bar{E}_2$ should be comparable where we have associated $\bar{E}_1$ with the motion $\bar{V} \times \bar{H}/c$ and $\bar{E}_2$ with $\nabla^2 \bar{H}$, that is the effect of the presence of the body in disturbing the background magnetic field.

We shall conclude this discussion by developing the basic torque expressions.*

From Equations (4) and (11),

$$\bar{i} = \sigma \left[\bar{E} + \frac{\mu}{c} (\bar{\omega} \times \bar{r}) \times \bar{H} \right] \quad (27)$$

and

$$d\bar{T} = \bar{r} \times d\bar{f} = \bar{r} \times \left[\frac{\mu}{c} (\bar{i} \times \bar{H}) dV \right] \quad (28)$$

where the differential torque $d\bar{T}$ is developed from the force occurring on a differential volume. Then with $\mu = 1$

*By neglecting the $\rho \bar{V}$ term in $\bar{i}$ we are in essence taking $\bar{E}$ to be effectively $\bar{E}_2$ in $\bar{E} = \bar{E}_1 + \bar{E}_2$ where $\bar{E}_1$ is due to the component of $\bar{\omega}$ parallel to $\bar{H}$, i.e., the component due to charge build-up.

$$\bar{T} = \int_{Vol} d\bar{T} = \frac{1}{c} \int_{Vol} \bar{r} \times \left\{ \sigma \left[\left(\frac{\bar{\omega} \times \bar{r}}{c} \right) \times \bar{H} + \bar{E} \right] \times \bar{H} \right\} dV \quad (29)$$

As noted previously, for $\bar{\omega} = \bar{\omega}_{||}$, $\bar{T} = 0$ (no flux linkage). Then expanding $(\bar{\omega} \times \bar{r}) \times \bar{H}$ and using $\bar{\omega}_{||} \times \bar{H} = 0$, we get

$$\frac{\sigma}{c} \int_{Vol} \bar{r} \times \left\{ \left[\frac{(\bar{\omega} \cdot \bar{H}) \bar{r}}{c} + \bar{E} \right] \times \bar{H} \right\} dV = 0 \quad (30)$$

Since $\bar{r}$ can have arbitrary orientation with respect to $\bar{H}$,*

$$\bar{E} = - \left(\frac{\bar{\omega} \cdot \bar{H}}{c} \right) \bar{r} ; \text{ for } \bar{\omega} = \bar{\omega}_{||} \quad (31)$$

which is an alternate form of Equation (15).

Let

$$\bar{H} = \bar{H}_{\perp} + \bar{H}_{||} \quad (32)$$

where

$\bar{H}_{||}$ is the component of $\bar{H}$ parallel to $\bar{\omega}$

$\bar{H}_{\perp}$ is the component of $\bar{H}$ normal to $\bar{\omega}$

Similarly let

$$\bar{E} = \bar{E}_{\perp} + \bar{E}_{||} \quad (33)$$

* It is to be recalled (see Equation (17)) that the $\bar{E}$ of Equation (31) is due to charge build-up. We have neglected the $\rho \bar{V}$ term in the torque expression here since it does not effectively contribute.

where

$$\bar{E}_{||} = - \left(\frac{\bar{\omega} \cdot \bar{H}}{c} \right) \bar{r} = - \frac{\bar{\omega} |\bar{H}_{||}| |\bar{r}|}{c} \quad (34)$$

from Equation (31) using $||$ to stand for magnitude and let

$$\bar{E}_{\perp} = \bar{E} - \bar{E}_{||} \quad (35)$$

We note if $\bar{\omega} = \bar{\omega}_{\perp}$, then $(\bar{\omega} \cdot \bar{H}) = 0$ so that $\bar{E} = \bar{E}_{\perp}$

Then,

$$\bar{T} = \frac{\sigma}{c} \int_{Vol} \bar{r} \times \left\{ \left[\bar{E}_{\perp} - \frac{(\bar{H} \cdot \bar{r}) \bar{\omega}}{c} \right] \times \bar{H} \right\} dV \quad (36)$$

$$= \frac{\sigma}{c} \int_{Vol} \bar{r} \times \left[\bar{E}_{\perp} \times \bar{H} \right] dV - \frac{\sigma}{c^2} \int_{Vol} \bar{r} \times \left[(\bar{H} \cdot \bar{r}) \bar{\omega} \times \bar{H} \right] dV$$

We can write Equation (36) as

$$\bar{T} = \frac{\sigma}{c} \int_{Vol} \bar{r} \times (\bar{E}_{\perp} \times \bar{H}) dV + 2 \bar{T}_0 \quad (37)$$

where

$$\bar{T}_0 = \frac{\sigma}{2c^2} (\bar{\omega} \times \bar{H}) \times \int_{Vol} (\bar{H} \cdot \bar{r}) \bar{r} dV \quad (38)$$

is an expression for the complete torque on a sphere shape if $\nabla \times \bar{E} = 0$

is assumed as well as $\partial \bar{D} / \partial t = 0$.

The Spherical Shell

We begin the discussion using an approximate but convenient method based on Vinti.^[6] This method also gives insight into torques for other shapes discussed in Section III.

The assumption made is that $\nabla \times \bar{\mathbf{E}} = 0$. This amounts to saying that the magnetic field is not appreciably perturbed by the rotating conducting sphere shell. We define the inner radius of the shell by a and the outer radius by b , and call the thickness $h = b - a$. As noted previously the torque is given by

$$\bar{T} = \int_{\text{Vol}} \left[\bar{\mathbf{r}} \times (\bar{\mathbf{i}} \times \bar{\mathbf{H}}) \right] \frac{dV}{c} \quad (39)$$

*with $\nabla \times \bar{\mathbf{E}} = 0$ (i.e. $\partial \bar{\mathbf{H}} / \partial t = 0$) then

$$\nabla \times \bar{\mathbf{i}} = \frac{\sigma}{c} \nabla \times (\bar{\mathbf{V}} \times \bar{\mathbf{H}}) \quad (40)$$

For $\bar{\mathbf{H}}$ constant and uniform and $\bar{\omega}$ constant, we find expanding the curl of the cross product,

$$\nabla \times (\bar{\mathbf{V}} \times \bar{\mathbf{H}}) = -\bar{\mathbf{H}} \times \bar{\omega} \quad (41)$$

Then

$$\nabla \times \bar{\mathbf{i}} = \frac{\sigma}{c} (\bar{\omega} \times \bar{\mathbf{H}}) \quad (42)$$

or

$$\bar{\mathbf{i}} = \frac{\sigma}{2c} (\bar{\omega} \times \bar{\mathbf{H}}) \times \bar{\mathbf{r}} + \nabla \phi \quad (43)$$

*The implication is that $\bar{\mathbf{E}} = \nabla \Psi$ with Ψ a scalar function, since $\nabla \times (\nabla \Psi) = 0$.

We note that $\nabla \cdot \bar{\mathbf{i}} = 0$ using $\partial \bar{D}/\partial t = 0$, and with Equation (43) and $\nabla \cdot [(\bar{\omega} \times \bar{\mathbf{H}}) \times \bar{\mathbf{r}}] = 0$, then

$$\nabla^2 \phi = 0 \quad (44)$$

that is, ϕ satisfies Laplace's equation.

Considering the surface boundary conditions due to continuity of the normal $\bar{\mathbf{i}}$ component

$\frac{\partial \phi}{\partial \eta} = \nabla \phi \big|_{\text{surface}} = 0$ and the geometrical radial symmetry applicable with the sphere, there results $\nabla \phi = 0$ or

$$\bar{\mathbf{i}} = \frac{\sigma}{2c} (\bar{\omega} \times \bar{\mathbf{H}}) \times \bar{\mathbf{r}} \quad (45)$$

We then obtain from Equation (39) by expanding the triple product and simplifying results in a single term,

$$\bar{\mathbf{T}} = \bar{\mathbf{T}}_0 = \frac{\sigma}{2c} (\bar{\omega} \times \bar{\mathbf{H}}) \times \left[\bar{\mathbf{H}} \cdot \int_{\text{Vol}} \bar{\mathbf{r}} \bar{\mathbf{r}} dV \right] \quad (46)$$

Now for shell sphere taking, * $\bar{\mathbf{r}} = x_1 \hat{\mathbf{i}}_1 + x_2 \hat{\mathbf{i}}_2 + x_3 \hat{\mathbf{i}}_3$

* The $\hat{\mathbf{i}}_i$; $i = 1, 2, 3$ are unit vectors in the 3 coordinate directions.

Equations (47) come from $\int_{\text{Vol}} x_i x_j dV = 0$, $i \neq j$ and

$$\int_{\text{Vol}} x_i^2 dV = \frac{1}{3} \int_a^b r^2 dV = \frac{4\pi}{15} (b^5 - a^5); i = 1, 2, 3.$$

$$\begin{aligned} \int_{\text{Vol}} \vec{r} \vec{r} dV &= \int \left(\sum_{i, j=1}^3 x_i x_j \hat{i}_i \hat{i}_j \right) dV \\ &= \frac{4\pi}{15} (b^5 - a^5) (\hat{i}_1 \hat{i}_1 + \hat{i}_2 \hat{i}_2 + \hat{i}_3 \hat{i}_3) \end{aligned} \quad (47)$$

With $\vec{H} = \bar{H}_1 \hat{i}_1 + \bar{H}_2 \hat{i}_2 + \bar{H}_3 \hat{i}_3$,

$$\vec{H} \cdot \int_{\text{Vol}} \vec{r} \vec{r} dV = \frac{4\pi}{15} (b^5 - a^5) \vec{H}$$

Thus

$$\vec{T}_0 = \left[\frac{2\pi}{15} \frac{\sigma}{c} (b^5 - a^5) (\vec{\omega} \times \vec{H}) \right] \times \vec{H} = \vec{M} \times \vec{H} \quad (48)$$

where we may take

M = magnetic moment of the sphere shell.

We define the constant factor

$$\begin{aligned} k &= \frac{2\pi\sigma}{15c} (b^5 - a^5) = \frac{I\sigma}{4\rho c^2} \\ &\cong \frac{2\pi\sigma a^4 h}{3c^2} \end{aligned} \quad (49)$$

where I is the moment of inertia of sphere shell, ρ = uniform mass density and* for $h/a \ll 1$,

* In general for $h/a \ll 1$, $b^n - a^n \cong a^{n-1} (nh)$ or $b^{n-1} (nh)$.

$$\begin{aligned} (b^5 - a^5) &\cong 5a^4 h \\ &\cong 5b^4 h \end{aligned} \quad (50)$$

Thus

$$\bar{T}_o = k \left[(\bar{\omega} \times \bar{H}) \times \bar{H} \right] \quad (51)$$

We may resolve $\bar{T}_o$ into components parallel and orthogonal to $\bar{\omega}$ denoted respectively as $\bar{T}_{oD}$ and $\bar{T}_{oT}$ (D for drag, T for turning).

Then

$$\bar{T}_o = \bar{T}_{oD} + \bar{T}_{oT} = \left(\bar{T}_o \cdot \hat{i}_\omega \right) \hat{i}_\omega + \hat{i}_\omega \times \left(\bar{T}_o \times \hat{i}_\omega \right) \quad (52)$$

Substituting from Equation (51) and expanding the triple cross product

$$\bar{T}_{oD} = -k \left[|\bar{H}|^2 - \left(\hat{i}_\omega \cdot \bar{H} \right)^2 \right] \bar{\omega} = -k |\bar{H}_\perp|^2 \bar{\omega} \quad (53)$$

and

$$\bar{T}_{oT} = k \omega \left(\hat{i}_\omega \cdot \bar{H} \right) \left[\hat{i}_\omega \times \left(\bar{H} \times \hat{i}_\omega \right) \right] = k \omega |\bar{H}_\parallel| \bar{H}_\perp \quad (54)$$

with Equation (49) it is to be noted that Equation (51) becomes

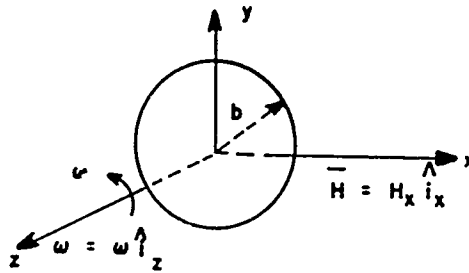
$$\bar{T}_{oD} \cong \frac{2\pi |\bar{H}_\perp|^2 \sigma (b^5 - a^5)}{15c^2} \bar{\omega} \quad (55)$$

Equation (55) is now described using an alternate approach which is less restrictive and applicable to other object shapes.

Let us consider the following geometry for the solid sphere of radius b .

Let $\{x, y, z\}$ be a fixed-in-space coordinate system.

$\{\xi, \eta, \zeta\}$ be a fixed-in-the-body coordinate system.



We now reformulate the model of a spinning shell in a fixed field to a shell in a field which changes in time to correspond to the relative motion, since torque depends only on relative motion. We note also that even though $\bar{H}$ is taken normal to $\bar{\omega}$ in the diagram so as to get only a damping torque, the model allows a general orientation between $\bar{H}$ and $\bar{\omega}$ as in the previous calculation for torque. For the motion assumed we can take $Z = z$. Then

$$\begin{aligned}\bar{H}_{\xi} &= (H_x \cos \omega t) \hat{i}_{\xi} \\ \bar{H}_{\eta} &= -(H_x \sin \omega t) \hat{i}_{\eta} \\ \bar{H}_z &= 0\end{aligned}\tag{56}$$

so that $\bar{H} = \bar{H}_{\xi} + \bar{H}_{\eta}$

In complex form these become

$$\begin{aligned}\tilde{H}_\xi &= H_x e^{-i\omega t}, & H_\xi &= \text{Re } \tilde{H}_\xi \\ \tilde{H}_\eta &= -iH_x e^{-i\omega t}, & H_\eta &= \text{Re } \tilde{H}_\eta \\ \tilde{H}_z &= 0\end{aligned}\tag{57}$$

The method now uses the formulation of the magnetic moment based on Landau and Tifshitz. [16]

Let M = total magnetic moment acquired by a conductor in a magnetic field. In general, the torque on the conductor in the magnetic field H is then

$$\bar{T} = \bar{M} \times \bar{H}\tag{58}$$

which compares to Equation (48) for the sphere. We take

$$\bar{M} = \sum_{i=1}^3 M_i \hat{f}_i, \quad \bar{H} = \sum_{j=1}^3 H_j \hat{f}_j\tag{59}$$

Then M_i can be written as

$$M_i = V\alpha_{ik}(\omega) H_k\tag{60}$$

where

- V = volume of the conductor
- ω = angular rate of conductor with respect to field direction

$V\alpha_{ik}(\omega)$ = magnetic polarizability, a tensor, where α_{ik} is symmetric and depends on the body shape and orientation of the external field but not on the volume.

For the sphere (solid) α is a scalar* $\alpha \delta_{ik}$ and we have

$$\bar{M} = V \alpha \bar{H} \quad (61)$$

with $\alpha = \alpha_1 + i\alpha_2$. For the sphere, the same α applies whether the $\bar{H}$ field is $\bar{H}_\perp$ or $\bar{H}_\parallel$ so that

$$\begin{aligned} M_\xi &= V H_x (\alpha_1 \cos \omega t + \alpha_2 \sin \omega t) \\ M_\eta &= V H_x (-\alpha_1 \sin \omega t + \alpha_2 \cos \omega t) \\ M_z &= 0 \end{aligned} \quad (62)$$

Using

$$\begin{aligned} M_x &= M_\xi \cos \omega t - M_\eta \sin \omega t \\ M_y &= M_\xi \sin \omega t + M_\eta \cos \omega t \end{aligned} \quad (63)$$

* δ_{ik} is the Kronecker delta. For the sphere the polarizability per unit

volume is $\alpha = \alpha_1 + i\alpha_2 = -\frac{3}{8\pi} \left[1 - \frac{3}{b^2 k^2} + \frac{3}{bk} \cot bk \right]$ where

$k = \frac{1+i}{\delta}$ and $\delta = \frac{1}{(2\pi\sigma\omega)^{1/2}}$ is the skin depth. It is to be noted also that since α is complex, a retardation exists between $\bar{M}$ and $\bar{H}$ and so non-colinearity and non-zero torque.

We get

$$M_x = V H_x \alpha_1$$

$$M_y = V H_x \alpha_2$$

$$M_z = 0 \quad (64)$$

Then with $\bar{H} = \bar{H}_x \hat{i}_x$ and Equation (61),

$$T_x = 0$$

$$T_y = 0$$

$$T_z = -M_y H_x = -V H_x^2 \alpha_2 \quad (65)$$

If the skin depth* $\delta \gg h$ then $\alpha_2 = \frac{b^2 \sigma \omega}{10c}$ so that with $V = \frac{4}{3} \pi b^3$,

$$T_z = - \frac{4\pi b^3 H_x^2 b^2 \sigma \omega \hat{i}_z}{30c^2} = \frac{2\pi H_x^2 \sigma b^5 \omega}{15c^2} \quad (66)$$

To approximate the thin spherical shell when $h/a \ll 1$, we subtract the result for $\bar{T}_z$ at a , the inner radius, from the result $\bar{T}_z$ at b , the outer radius. This gives for the spherical shell from Equation (66),

$$T_z = - \frac{2\pi}{15c^2} H_x^2 \sigma \left(b^5 - a^5 \right) \bar{\omega} \quad (67)$$

*For a shell $\delta \gg h$ (the shell thickness).

Equation (67) agrees exactly with Equation (49) since in our case

$$\bar{H} = \bar{H}_{\perp} = H_{\hat{x}} \hat{x}.$$

The form given by Equation (67) thus agrees with Equation (55) which was derived neglecting the effect of $\partial H / \partial t \neq 0$, if present.

For a spherical shell thick enough and with the $\bar{\omega} \perp \bar{H}$ geometry, the assumption $\partial \bar{H} / \partial t = 0$ is reasonable. The formula for a very thin shell given in Smyth^[4] includes the $\partial H / \partial t$ effect and restricts $\bar{H}$ normal to $\bar{\omega}$. Converting this result to our terminology gives,

$$T = - \frac{C_1 \omega}{1 + C_2 \omega^2} \quad (68)$$

where

$$C_1 = (2/3) \eta \mu^2 H^2 \sigma b^4 h$$

$$C_2 = \frac{\mu^2 b^2 \sigma^2 h^2}{9}$$

For us $\mu = 1$ and taking $\partial \bar{H} / \partial t = 0$ is to take $C_2 = 0$. Using

$b^4 \cong \frac{b^5 - a^5}{5h}$ then gives $C_1 = \frac{2\pi H^2 \sigma (b^5 - a^5)}{15}$. Applying this to the Gaussian system of units then produces agreement with Equations (55) and (67).

We close this section by noting that a recent investigation by Halverson and Cohen^[8] has included the generalization of considering both rotation with $\bar{\omega}$ and $\bar{B}$ arbitrarily related as in Equations (52) or (59) as well as $\partial \bar{H} / \neq 0$ as in Equation (68).

The model of using the assumed equivalence of a time-varying field and fixed conductor for the moving conductor in a fixed field is applied.

Using the vector potential $\bar{A}$ (see References (4) and (8)) where

$$\nabla \times \bar{A} = \bar{B} \quad (69)$$

then

$$\nabla^2 \bar{A} = \frac{4\pi\mu\sigma}{c} \frac{d\bar{A}}{dt} \quad (70)$$

and

$$\bar{I} = - \frac{c\sigma}{4\pi} \frac{d\bar{A}}{dt} \quad (71)$$

We note Equation (70) also applies to $\bar{E}$, $\bar{H}$ and $\bar{i}$.

The torque given by Equation (29) or (39) can be written in the equivalent form

$$\bar{T} = - \frac{\sigma}{c} \int_{\text{Vol}} \bar{r} \times \left[\frac{\partial \bar{A}}{\partial t} \times (\bar{v} \times \bar{A}) \right] dV \quad (72)$$

Solving Equation (70) with the appropriate boundary conditions for the sphere shell results in a general formula with rather complicated coefficients. [8] For a good conductor or with ω low [with $q \rightarrow 0$, see Reference (8)] or a good dielectric or high ω ($q \rightarrow \infty$), the form of the coefficients simplify, with $\bar{\omega} = \omega \hat{k}$ and $\chi(\bar{\omega}, \bar{B}) = \alpha$, to give for $q < 1$ ($q \rightarrow 0$)

$$\begin{aligned} \bar{T} = \frac{3\pi b^3 B^2}{\mu} \left\{ \left[-\frac{q^2}{90} (1 - \rho^5) \hat{i} + \frac{q^4}{108} \left(\frac{2}{35} - \frac{\rho^5}{5} + \frac{\rho^7}{7} \right) \hat{j} \right] \sin 2\alpha \right. \\ \left. + \left[\frac{q^2}{90} (1 - \rho^5) \hat{k} \right] 2 \sin^2 \alpha \right\} \end{aligned} \quad (73)$$

for $q > 100$ ($q \rightarrow \infty$)

$$T = \frac{3\pi b^3 B^2}{\mu} \left\{ \left[-\left(\frac{1}{q} - \frac{2}{2}\right) \hat{i} + \left(\frac{1}{3} - \frac{1}{q}\right) \hat{j} \right] \sin 2\alpha + \left[\left(\frac{1}{q} - \frac{2}{2}\right) \hat{k} \right] 2 \sin^2 \alpha \right\} \quad (74)$$

where

$$q = b (2P)^{1/2}; \quad P = \frac{\mu\omega}{\tau}, \quad \tau = \text{resistance/unit length}$$

$$\rho = b/a$$

For a thin shell $\rho \rightarrow 1$, so that taking $h = b - a$ with surface resistivity $\zeta = \tau/h$ remaining finite, then with q arbitrary,

$$\bar{T} = \frac{\tau\omega b^4 B^2}{9\zeta^2 + \mu^2 \omega^2 b^2} \left[\left(-3\zeta \hat{i} + \mu\omega b \hat{j} \right) \sin 2\alpha + \zeta \hat{k} \left(2 \sin^2 \alpha \right) \right] \quad (75)$$

For $\bar{B}_1 \bar{\omega}$ (i.e., $\alpha = \pi/2$)

$$\bar{T} = \frac{6\pi B^2 \zeta \bar{\omega} b^4}{9\zeta^2 + \mu^2 \omega^2 b^2} = \frac{6\pi B^2 \sigma \bar{\omega} b^4 h}{9 + \mu^2 b^2 \sigma^2 h^2} \quad (76)$$

Equation (67b) is the form given by Equation (68) with $\zeta = 1/\sigma h$.

The $\hat{i}$, $\hat{j}$, and $\hat{k}$ components of $\bar{T}$ in Equation (73), (74) or (75) are $-C_1 \sin 2\alpha$, $C_2 \sin 2\alpha$, and $2 C_1 \sin^2 \alpha$ respectively. These forms also apply, aside from the specific values for C_1 and C_2 , for the general case with no restrictions on q or h .

We see, as before, that

- (1) $\bar{T} = 0$ if $\alpha = 0$ (i.e., $\bar{\omega}$ and $\bar{B}$ are parallel)
- (2) The $\hat{i}$ and $\hat{j}$ components of $\bar{T}$ cause $\bar{\omega}$ to precess about $\bar{B}$.

The situation in (2) is consistent with the simplified form of $\bar{T}_{oT}$ in the $\bar{H}_\perp$ direction given in Equation (54). Except for perfect conductors, the vector $\bar{\omega}$ then precesses in the steady state until it becomes, for fixed $\bar{B}$, colinear with $\bar{B}$. As $\bar{\omega}$ approaches $\bar{B}$, $\bar{\omega}_\perp$ energy is transferred to $\bar{\omega}_\parallel$ which is dropping due to damping of $\bar{\omega}$ such that $\bar{\omega}_\parallel$ remains constant for constant $\bar{B}$.

SECTION III

APPLICATION TO CYLINDER-TYPE SHELLS

NON-SPHERICAL SHAPES

As noted the torque, in general, can be given as

$$\bar{T} = \frac{1}{c} \int_{\text{Vol}} \bar{r} \times \left\{ \sigma \left[\frac{(\bar{\omega} \times \bar{r}) \times \bar{H}}{c} + \bar{E} \right] \times \bar{H} \right\} dV \quad (77)$$

where the effect of convection currents can usually be neglected. Expanding the triple product then gives

$$\begin{aligned} \bar{T} = \frac{\sigma}{c} \int_{\text{Vol}} \bar{r} \times \left[\left(\bar{E} + \bar{r} \frac{(\bar{\omega} \cdot \bar{H})}{c} \right) \times \bar{H} \right] dV - \frac{\sigma}{c} \int_{\text{Vol}} \bar{r} \times \left[\bar{\omega} \frac{(\bar{H} \cdot \bar{r})}{c} \right] \\ \times \bar{H} \right] dV \end{aligned} \quad (78)$$

Using Equations (33) and (34) then allows for $\bar{T}$, in general, as

$$\bar{T} = \frac{\sigma}{c} \int_{\text{Vol}} \bar{r} \times \left(\bar{E}_{\perp} \times \bar{H} \right) dV + 2 \bar{T}_0 \quad (79)$$

where

$$\bar{T}_0 = \frac{\sigma}{2c} (\bar{\omega} \times \bar{H}) \times \left(\bar{H} \cdot \int_{\text{Vol}} \bar{r} \bar{r} dV \right) \quad (80)$$

In the case of a sphere shell $\bar{T} = \bar{T}_0$ which implies

$$\frac{\sigma}{c} \int_{\text{Vol}} \bar{r} \times \left(\bar{E}_{\perp} \times \bar{H} \right) dV = - \bar{T}_0 .$$

To say this in general would be to imply that $\nabla\phi = 0$ in

$$\bar{\mathbf{I}} = \frac{\sigma}{2c} (\bar{\omega} \times \bar{\mathbf{H}}) \times \bar{\mathbf{r}} + \nabla\phi \quad (81)$$

which is not true for non-spherical body geometries.

Thus, approaches using equations such as Equation (70) which take account of the field and currents internal to non-spherical geometries must be used. For cylinder shapes $\bar{\mathbf{T}}$ values can result which do not differ greatly from $\bar{\mathbf{T}}_0$. In addition, the dyadic form $\int \bar{\mathbf{r}} \bar{\mathbf{r}} dV$ in $\bar{\mathbf{T}}_0$ is useful for comparing different cylinder end configurations. (See Equation (83) and section entitled "Hemisphere End Cylinder".)

We also note that if conditions (1) and (2) (see page 9) do not hold, then as also noted in Reference (17), we may drop $\bar{\mathbf{E}}$ and take

$$\bar{\mathbf{I}} = \frac{\sigma}{c} (\bar{\omega} \times \bar{\mathbf{r}}) \times \bar{\mathbf{H}} \quad (82)$$

To do this would give for $\bar{\omega}_\perp \bar{\mathbf{H}}$,

$$\bar{\mathbf{T}} = 2 \bar{\mathbf{T}}_0 \quad (83)$$

which in the sphere case, at least, is in error by a factor of 2.

Thus, although $\nabla \times \bar{\mathbf{E}} = 0$ is assumed from $\partial\bar{\mathbf{H}}/\partial t = 0$ to get $\bar{\mathbf{T}} = \bar{\mathbf{T}}_0$ for the sphere, charge movement etc. does not allow dropping of the $\bar{\mathbf{E}}$ itself.

Finally when $\bar{\omega} = \bar{\omega}_{||}$ then $\bar{\mathbf{T}}_0 = 0$. With $\bar{\mathbf{T}} = 0$ in general for such orientation, then we might use the approximation

$$\bar{\mathbf{T}} = K \bar{\mathbf{T}}_0 \quad (84)$$

since $\bar{\mathbf{E}}_{||}$ and $\bar{\mathbf{E}}_\perp$ are not in general zero.

RIGHT CIRCULAR CYLINDERS

Initially in order to simplify the field equations, a spin axis $\bar{\omega}$ normal to $\bar{H}$ is assumed. This allows comparison with more complicated analysis. Here the assumptions of both zero displacement current $\partial\bar{D}/\partial t = 0$ and nonperturbed field $\partial\bar{H}/\partial t = \nabla \times \bar{E} = 0$ are made.

Considering the field induced in a current path moving with the body we write from Equation (10)

$$\nabla \times \bar{E}' = \mu \frac{\partial \bar{H}}{\partial t} + \nabla \times \left[\frac{\mu}{c} (\bar{V} \times \bar{H}) \right] \quad (85)$$

With $\nabla \times \bar{E} = \mu \partial \bar{H} / \partial t = 0$, $\bar{V}_\perp$ the component of $\bar{V}$ normal to $\bar{H}$, $\bar{l}$ the cylinder length at direction $\hat{l}$, and neglecting end effects, there results

$$\begin{aligned} \bar{E}' &= \frac{\mu H}{c} \bar{V}_\perp \hat{l} \\ &= \frac{\mu H}{c} r \omega \sin \alpha \hat{l} \end{aligned} \quad (86)$$

where $\alpha = \angle (\bar{V}, \bar{H})$.

For a path length l , the voltage $\mathcal{E}$ becomes*

$$\mathcal{E} = \frac{\mu H}{c} l r \omega \sin \alpha \quad (87)$$

* An approach similar to the one outlined here can be based on using
 V = Voltage = $d\phi/dt$; ϕ = flux linkage.
 P = power dissipated = $-V^2/R$; R = resistance of induced current paths.
 T = torque = $1/\omega dV/dt = 1/\omega P$.

Such an approach can be used to develop torque values for tumbling cylinders.^[18] However, such methods, although simple to construct, can be in error (e. g., page 30) since they do not account completely for the induced current paths and boundary conditions as do methods based on Maxwell field equations.

By calculating the path resistance next, a value of $\bar{i}$ and then $\bar{T}$ is obtained using

$$\bar{T} = \frac{\sigma}{c} \int \bar{r} \times (\bar{i} \times \bar{H}) dV$$

With a and b the inside and outside radii respectively for a cylinder of length ℓ , Hooper^[9] using this approach gives, with conducting end plates (in MKS units),

$$\bar{T} = - \frac{\pi \mu^2 H^2 \ell^2 \omega \sigma (b^4 - a^4)}{4 (\ell + a + b)} \quad (88)$$

or

$$T \cong - \frac{\pi \mu^2 H^2 \ell^2 \omega \sigma b^3 h}{(\ell + a + b)} \quad (89)$$

using $h = b - a$ and the footnote on page 14, and without end plates,

$$\bar{T} = - \frac{\pi \mu^2 H^2 \ell^2 \omega \sigma (b^4 - a^4)}{4 [\ell + \pi/4 (a + b)]} \quad (90)$$

or as in (89)

$$T \cong - \frac{\pi \mu^2 H^2 \ell^2 \omega \sigma b^3 h}{\ell + \pi/4 (a + b)} \quad (91)$$

Similar calculations on the sphere shell give

$$\bar{T} = - \frac{32}{75} \mu^2 H^2 \omega \sigma (b^5 - a^5) \quad (92)$$

which agrees very closely with Equation (67).

However, Smythe's analysis (Reference 4) accounting for $\partial \bar{H}/\partial t \neq 0$ gives for $\bar{\omega}$, the spinning cylinder shell without end plates, $\bar{H} \perp \bar{\omega}$, and h small

$$\bar{T} = - \frac{4 \pi \omega \sigma h b^3 B^2 \ell}{4 + \omega^2 \mu^2 b^2 h^2 \sigma^2} \quad (93)$$

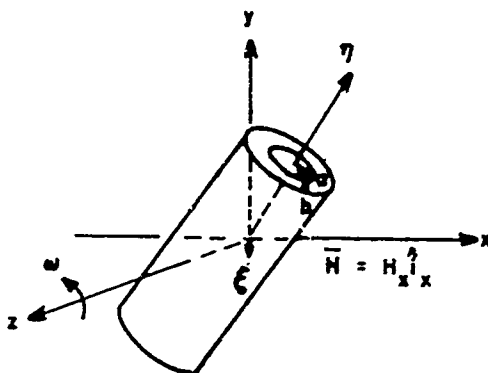
Neglecting $\partial \bar{H} / \partial t$ reduces Equation (93) to (in MKS units)

$$\bar{T} = - \pi \omega \sigma h b^3 B^2 \ell \quad (94)$$

which agrees with (91) if the term π/a ($a + b$) is neglected in the denominator. Since (94) agrees with a third calculation given below, it appears that the calculation of the resistance path in Reference 9 may be in error.

We now proceed to a calculation similar to that used on the sphere starting with Equation (56). This will be done for a tumbling (as well as spinning) cylinder. In both cases $\bar{\omega} \perp \bar{H}$ still applies.

We view the geometry as shown



For a solid cylinder the magnetic moment per unit length is

$$\bar{M} = V \alpha \bar{H}$$

where the tensor α_{ik} reduces to the scalar $\alpha \delta_{ik}$

$$\alpha = \alpha_1 + i\alpha_2$$

Equations (56) through (60) still apply. Now, however, we deal with two values of α , say α' and α'' , where α' applies when $\bar{H}$ is orthogonal to the cylinder axis and α'' when $\bar{H}$ is parallel to this axis.

Following as before, we now obtain

$$\begin{aligned} M_{\xi} &= V \operatorname{Re} (\alpha' H_{\xi}) = V \operatorname{Re} \left[(\alpha_1' + i \alpha_2') (H_x e^{-i\omega t}) \right] \\ &= V H_x (\alpha_1' \cos \omega t + \alpha_2' \sin \omega t) \end{aligned} \quad (95)$$

$$\begin{aligned} M_{\eta} &= V \operatorname{Re} (\alpha'' H_{\eta}) = V \operatorname{Re} \left[(\alpha_1'' + i \alpha_2'') (-i H_x e^{-i\omega t}) \right] \\ &= V H_x (-\alpha_1'' \sin \omega t + \alpha_2'' \cos \omega t) \\ &= \frac{V H_x}{2} (-\alpha_1' \sin \omega t + \alpha_2' \cos \omega t) \end{aligned} \quad (96)$$

$$M_z = 0 \quad (H_z = 0) \quad (97)$$

using, in the M_{η} equation, the fact that $\alpha'' = \alpha'/2$ (see Reference 16).

With Equation (63) we have

$$M_x = V H_x \left[\alpha_1' \left(1 - \frac{\sin^2 \omega t}{2} \right) + \alpha_2' \frac{\sin \omega t \cos \omega t}{2} \right] \quad (98)$$

$$M_y = V H_x \left[\alpha_1' \frac{\sin \omega t \cos \omega t}{2} + \alpha_2' \left(1 - \cos^2 \frac{\omega t}{2} \right) \right] \quad (99)$$

$$M_z = 0 \quad (100)$$

Now the torque per unit length gives

$$\begin{aligned}\bar{T} = \bar{M} \times \bar{H} &= \left(M_y H_z - M_z H_y \right) \hat{i}_x + \left(M_z H_x - M_x H_z \right) \hat{i}_y \\ &+ \left(M_x H_y - M_y H_x \right) \hat{i}_z \\ &= T_x \hat{i}_x + T_y \hat{i}_y + T_z \hat{i}_z\end{aligned}\quad (101)$$

so that

$$T_x = 0; \quad \left(M_z = H_z = 0 \right) \quad (102)$$

$$T_y = 0; \quad \left(M_z = H_z = 0 \right) \quad (103)$$

$$T_z = - M_y H_x; \quad \left(H_y = 0 \right) \quad (104)$$

where

$$\bar{H} = H_x \hat{i}_x$$

Thus using*

$$\alpha_1' = \frac{\tau b^4 \sigma^2 \omega^2}{6c^4}, \quad \alpha_2' = \frac{b^2 \sigma \omega}{4c^2}$$

*

The exact value has $\alpha' = -\frac{1}{2\pi} \left(1 - \frac{2J_1(ka)}{kb J_0(kb)} \right); \quad k = \frac{1+i}{\delta}$.

When the skin depth $\delta \gg b$ or for the shell, then the approximate values for α_1' , α_2' used above apply.

$$\begin{aligned}
T_z &= - M_y H_x = - V H_x^2 \left[- \frac{\pi b^4 \sigma \omega^2}{6c^4} \frac{\sin \omega t \cos \omega t}{2} \right. \\
&\quad \left. + \frac{b^2 \sigma \omega}{4c^2} \left(1 - \frac{\cos^2 \omega t}{2} \right) \right] \\
&= - \left(\frac{V H_x^2 b^2 \sigma \omega}{4c^2} \right) \left[- \frac{\pi b^2 \sigma \omega}{3c^2} \sin \omega t \cos \omega t + \left(1 - \frac{\cos^2 \omega t}{2} \right) \right] \\
&= - \left(\frac{\pi H_x^2 b^4 \sigma \omega}{4c^2} \right) \left[3/4 - \frac{\cos 2\omega t}{4} - \frac{\pi b^2 \sigma \omega}{6c^2} \sin 2\omega t \right] \quad (105)
\end{aligned}$$

Considering the average torque per period gives for the bracket contents,

$$\frac{1}{2\pi} \int_0^{2\pi} \left[\quad \right] d\theta = 3/4; \quad \theta = \omega t$$

so that

$$\text{Avg } \bar{T}_z = - \frac{3\pi H^2 b^4 \sigma \omega}{16c^2} \quad (106)$$

Doing the same for a cylinder of radius a and subtracting* for the shell of thickness $h = (b - a) \ll a$ gives the average torque for length ℓ (Tumbling)

$$\text{Avg } \bar{T}_z = - \frac{3\pi H^2 \sigma \omega \ell (b^4 - a^4)}{16c^2} = - \frac{3\pi H^2 \sigma \omega b^3 \ell}{4c^2} \quad (107)$$

*Since $\nabla \phi \neq 0$ for the cylinder, as it was for the sphere, the result now is an approximation.

Doing a similar calculation for a spinning cylinder shell where only α_1 applies gives

$$M_x = V H_x \alpha_1 \quad (108)$$

$$M_y = V H_x \alpha_2 \quad (109)$$

$$M_z = 0 \quad (110)$$

$$T_z = - M_y H_x \quad (111)$$

The steady state nature of the torque is revealed by Equations (108) through (111) and gives per unit length.

(Spinning)

$$\bar{T}_z = - M_y H_x = - \pi b^2 \ell H_x \left(\frac{b^2 \sigma \omega}{4c^2} \right) \quad (112)$$

Accounting for a thin shell of thickness $h = (b - a)$ leads after subtraction to the torque for a cylinder of length ℓ ,

$$\begin{aligned} \bar{T} &= - \frac{\pi \omega \sigma (b^4 - a^4) H^2 \ell}{4c^2} \\ &\approx - \frac{\pi \omega \sigma b^3 H^2 \ell}{c^2} \end{aligned} \quad (113)$$

Equation (113) agrees exactly with (94) based on Smythe^[4] where in (113) $\mu = 1$ is assumed*

Only α_2' appears in the average torque expression of Equation (107) for a tumbling cylinder shell and in the torque expression of Equation (113) for a spinning cylinder shell. We now examine for a possible case of interest the effect of using the exact value of α' as given in the footnote on page 30 as opposed to the approximate value (for $\delta \gg b$) also given on the bottom of page 30 and used in Equations (107) and (113).

For a solid aluminum cylinder with $\omega = 2\pi/100$ and of radius $b = 0.625m$

$$\alpha_2' \text{ (approx)} = \frac{b^2 \sigma \omega}{4c} = \frac{0.136}{2\pi}$$

$$\alpha_2' \text{ (exact)} = \frac{0.128}{2\pi}$$

Similarly for a solid cylinder of radius $a = 0.615m$,

$$\alpha_2' \text{ (approx)} = \frac{0.132}{2\pi}$$

$$\alpha_2' \text{ (exact)} = \frac{0.127}{2\pi}$$

* The possibility of agreement appears plausible in the spinning case where with $\bar{\omega}_1 \bar{H}$ a steady state condition might result so that the effect on the currents from $\nabla \phi$ cancels motion effects because of symmetry. The tumbling case affords less possibility for steady state conditions.

This amounts to a reduction of torque in each case by 4 percent, so that using α_2' (exact) in the shell cases gives

revised Avg $\bar{T}$ for tumbling cylinder shell = 0.96 (Avg $\bar{T}$ of Equation (107))

revised $\bar{T}$ for spinning cylinder shell = 0.96 ($\bar{T}$ of Equation (113)).

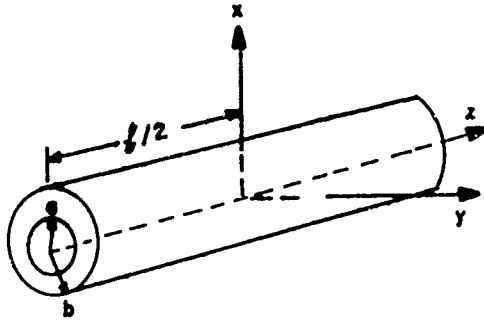
Finally with the shell thin $h \ll a$, the Bessel function form for α' provides an approximation using the subtraction approach employed above. The agreement in certain situations as we have noted, with torque values calculated in alternate ways, e.g. using Smythe values, supports this approach, in at least a limited application.

HEMISPHERE END CYLINDER

Rather than open or flat ends it may be useful for radar applications, for example, to have rounded ends. To provide some insight on the effect of such rounding, we choose to consider cylinder shells capped with hemisphere shells. To relate the relative contribution of such ends to the torque we turn to the dyadic calculation in the torque formulation expressed by $\bar{T}_0$ (see Section III).

a) Open End Cylinder Shell C_0

$$\int_{\text{Vol}} \bar{r} \bar{r} dV = \int \left[\sum_{k, \ell=1}^3 x_k x_\ell \hat{i}_k \hat{k}_\ell \right] dV ; \quad \begin{aligned} x_1 &= x \\ x_2 &= y \\ x_3 &= z \end{aligned} \quad (114)$$



$$r = (x^2 + y^2)^{1/2}$$

Now

$$\begin{aligned} \int_{\text{Vol}} x^2 dV &= \int_{\text{Vol}} y^2 dV = \int_{-l/2}^{l/2} \int_a^b \int_0^{2\pi} (r \cos \theta)^2 r d\theta dr dz \\ &= \frac{l\pi}{4} [b^4 - a^4] = A_0 \end{aligned} \quad (115)$$

$$\int_{\text{Vol}} z^2 dV = \int_{-l/2}^{l/2} \int_a^b \int_0^{2\pi} z^2 r d\theta dr dz = \frac{\pi}{12} (b^2 - a^2) l^3 = B_0 \quad (116)$$

$$\int_{\text{Vol}} x_i x_j dV = 0; \text{ all } i \neq j \quad (117)$$

Thus

$$\left(\int_{\text{Vol}} \bar{r} \bar{r} dV \right) C_0 = A_0 (\hat{i}_x \hat{i}_x + \hat{i}_y \hat{i}_y) + B_0 \hat{i}_z \hat{i}_z \quad (118)$$

and

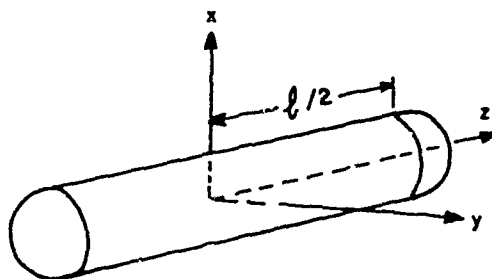
$$\bar{H} \cdot \left(\int_{\text{Vol}} \bar{r} \bar{r} dV \right) C_o = A_o \left(\bar{H}_x + \bar{H}_y \right) + B_o \bar{H}_z \quad (119)$$

Note with respect to the direction $\hat{i}_z$ we may call $\bar{H}_{||} = \bar{H} \cdot \hat{i}_z$ so that

$$A_o \left(\bar{H}_x + \bar{H}_y \right) + B_o \bar{H}_z = A_o \bar{H}_{\perp} + B_o \bar{H}_{||} \quad (120)$$

b) Hemisphere Capped Cylinder Shell C_1

We now apply Equation (114) to the object shown below



Now

$$\int_{\text{Vol}} x_i x_j dV = 0; i \neq j \quad (121)$$

$$\begin{aligned} \int_{\text{Vol}} x^2 dV &= \int_{\text{Vol}} y^2 dV = \frac{1}{3} \int_{\text{Vol}} \rho^2 (4\pi \rho^2) d\rho = \frac{4\pi}{15} (b^5 - a^5) = A_1 \\ &= \int_{\text{Vol}} \tilde{z}^2 dV \end{aligned} \quad (122)$$

where

$$\rho = (x^2 + y^2 + \tilde{z}^2)^{1/2} ; \tilde{z} = z \pm \ell/2$$

$$\int_{\text{Vol}} z^2 dV = \sum_{\text{each hemisphere}} \left(\int_{\text{Vol}} [\text{sgn}(z) \ell/2 + \tilde{z}]^2 dV \right)$$

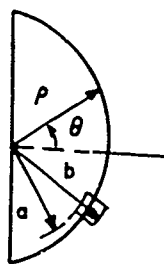
$$= \frac{\ell^2}{4} \text{Vol (sphere)} + \frac{4\pi}{15} (b^5 - a^5)$$

$$+ 2\ell \int_{\text{Hemisphere}} \tilde{z} dV$$

$$= \frac{4\pi}{3} \left\{ (b^3 - a^3) \frac{\ell^2}{4} + \frac{3\ell}{8} (b^4 - a^4) + \left(\frac{b^5 - a^5}{5} \right) \right\} = B_1 \quad (123)$$

In Equation (123) use has been made of

$$\int_{\text{Hemisphere}} \tilde{z} dV = 2 \int_{\text{Quadrant Shell}} \rho \cos \theta dV$$



$$= 2\pi \int_0^{\pi/2} \int_a^b \rho^3 \cos \theta \sin \theta d\theta d\rho$$

length of dV
slice for quadrant,
 $\pi \rho \sin \theta$
 $dV = \pi \rho \sin \theta dA$
 $= (\pi \rho \sin \theta) \rho d\theta d\rho$

$$= 2\pi \int_0^1 u du \int_a^b \rho^3 d\rho ; u = \sin \theta = \pi/4 (b^4 - a^4)$$

Thus

$$\bar{H} \cdot \left(\int_{\text{Vol}} \bar{r} \bar{r} dV \right)_{C_1} = A_1 (\bar{H}_x + \bar{H}_y) + B_1 \bar{H}_z \quad (124)$$

and as in Equation (120), we again may write,

$$A_1 (\bar{H}_x + \bar{H}_y) + B_1 \bar{H}_z = A_1 \bar{H}_\perp + B_1 \bar{H}_\parallel \quad (125)$$

Thus we may use (A_o, B_o) and (A_1, B_1) to compare the relative effect on torque of the hemisphere shell caps.

We choose a possible case of interest to compare $(\bar{T})_{C_o}$ and $(\bar{T})_{C_1}$.

For a tumbling cylinder C_o or C_1 , with $\bar{\omega}_\perp \bar{H}$, then over a rotation period with $\bar{H}$ fixed, Avg $\bar{T}$ is proportional to $A_o + B_o$ for C_o and proportional to $A_1 + B_1$ for C_1 since T_o is proportional to $A_k |H_\perp|^2 + B_k |H_\parallel|^2$; $k = 0, 1$ and $|H_\perp| = H \cos \omega t$, $|H_\parallel| = H \sin \omega t$.

CASE I

Considering in particular $\ell = 3m$, $b = 0.625m$ and $h = 0.01m$ we get

$$A_o + B_o = \frac{B_o}{4} + B_o = 1.25 B_o ; \text{ for } C_o$$

$$A_1 + B_1 = 1.25 A_o + 2.7 B_o \cong 3 B_o ; \text{ for } C_1 \quad (126)$$

This gives a factor increase of about 2.4 for C_1 over C_o .

CASE II

Again for $b = 0.625\text{m}$, $h = 0.01\text{m}$ and now $\ell = 2.4\text{m}$,

$$A_o + B_o = \frac{B_o}{3} + B_o = 1.33 B_o ; \text{ for } C_o$$

$$A_1 + B_1 = 1.3 A_o + 3.2 B_o \approx 3.6 B_o ; \text{ for } C_1 \quad (127)$$

This gives a factor increase of about 2.7 for C_1 over C_o . Of course the torque also is proportional directly to ℓ and the overall length of the C_1 cylinders is larger so that, although the effect of the ends being capped is high (factor of 2 to 3 increase in torque for the dimensions used), a rough comparison between cases 1 and 2 gives

$$\frac{T (\text{case 2})}{T (\text{case 1})} \approx \frac{2.7}{2.4} \frac{\ell_2}{\ell_1} \approx 0.9 \quad (128)$$

Thus, as the length of the cylinder section drops, the curved end caps contribute an increasing percentage to the torque. However, the reduction in cylinder section length affects the torque so as to result in about the same torque for two capped cylinders of the dimensions considered.

SECTION IV

DYNAMIC INTERACTIONS AND SHELL DESIGN

DECAY TIME FACTORS

As noted in the introduction, we are not concerned here directly with the time continuous motion of the body interacting with the eddy current torques which both oppose $\bar{\omega}$ (decay torque component) and alter the orientation of $\bar{\omega}$ (turning torque component).

We concentrate only on the decay effects and assume the $\bar{\omega}$ to $\bar{H}$ orientation fixed in order to assess in a conservative manner the time constants associated with decay. The basic dynamic equation relating decay torque $\bar{T}_D$ to acceleration $d\bar{\omega}/dt = \dot{\bar{\omega}}$ is

$$\bar{T}_D = I \dot{\bar{\omega}} \quad (129)$$

where I is the moment of inertia about the rotation axis in the $\bar{\omega}$ direction.

We shall consider with $\bar{\omega} \perp \bar{H}$ (i.e. $\bar{H} = \bar{H}_\perp$)

$$\bar{T}_{D_k} ; k = 1, 2, 3, 4$$

where

- $k = 1$: Spinning Spherical Shell
- $k = 2$: Spinning Cylindrical Shell (open-ended)
- $k = 3$: Tumbling Cylindrical Shell (open-ended)
- $k = 4$: Tumbling Cylindrical Shell with Hemisphere Shell Ends

For $k = 1$

From Equations (55) or (67) with $\partial \bar{H}/\partial t = 0$ assumed,

$$\bar{T}_{D_1} = - \frac{2\pi}{3c^2} H^2 \sigma b^4 h \bar{\omega} \quad (130)$$

or more exactly with $\partial \bar{H}/\partial t = \nabla \times \bar{E} \neq 0$, from Equation (68) or (76)
(i. e., at higher initial ω values)

$$\bar{T}_{D_1} = - \frac{2\pi H^2 \sigma b^4 h \bar{\omega}}{3c^2 \left[1 + \frac{b^2 \sigma^2 h^2 \omega^2}{9} \right]} \quad (131)$$

Equation (131) leads to

$$I \frac{d\omega}{dt} = - \frac{c_1 \omega}{1 + c_2 \omega^2} \quad (132)$$

where

$$c_1 = \frac{2\pi H^2 \sigma b^4 h}{3c^2}, \quad c_2 = b^2 \sigma^2 h^2 / 9.$$

Then

$$\left(\frac{I}{\omega} + I c_2 \omega \right) d\omega = - c_1 dt \quad (133)$$

Taking $\omega(t = 0) = \omega_0$, we get

$$\ln \left(\omega / \omega_0 \right) + c_2 \left(\frac{\omega^2 - \omega_0^2}{2} \right) = - \frac{c_1 t}{I} \quad (134)$$

or

$$\left(\omega/\omega_0\right) e^{k_1} \left[\left(\omega/\omega_0\right)^2 - 1 \right] = e^{-k_2 t} \quad (135)$$

with

$$k_1 = \frac{c_2 \omega_0^2}{2}, \quad k_2 = \frac{c_1}{I}$$

For $k_1 \ll 1$ (i. e., for large ω_0)

$$\omega/\omega_0 \cong e^{-k_2 t} \quad (136)$$

So that the exponential decay time constant becomes

$$\bar{\tau}_1 = 1/k_2 = \frac{I}{c_1} = \frac{I 3c^2}{2\pi H^2 \sigma b^4 h} \quad (137)$$

The same result comes directly from Equation (130) (the $\partial \bar{H}/\partial t = 0$ torque calculation).

Thus it is seen that τ_1 is proportional to I and inversely proportional to H^2 and h . This will hold as well for all τ_k . For example with aluminum of spherical shell thickness $h = .001m$.

$I = 3.4kg.m^2$, $b = 0.29m$, $H = 0.3$ oersted*, (see reference (7)).

$$\tau_1 \cong 70 \text{ days.}$$

*We take this to be representative value of earth magnetic intensity at about 300 nautical miles.

For k = 2

From Equations (94) or (113)

$$\bar{T}_{D_2} = - \frac{\pi H^2 \sigma b^3 \ell h \bar{\omega}}{c^2} \quad (138)$$

or more exactly from Equation (93)

$$\bar{T}_{D_2} = - \frac{\pi H^2 \sigma b^3 \ell h \bar{\omega}}{c^2 \left[1 + \frac{b^2 \sigma^2 h^2 \omega^2}{4} \right]} \quad (139)$$

Aside from new values for c_1 and c_2 , Equations (138) and (139) correspond exactly in form to Equations (130) and (131) respectively.

Then from Equation (137)

$$\tau_2 = \frac{I c^2}{\pi H^2 \sigma b^3 \ell h} \quad (140)$$

For $\ell = 3m$, $H = 0.3$ oersted, $b = 0.625m$, $h = 0.01m$ (1 cm) and aluminum together with an imposed value $I = 75 \text{ kg} \cdot m^2$ (this value cannot be attained with the dimensions given using only aluminum, see following section).

$$\tau_2 \cong 1.2 \text{ days}$$

We may note in passing that for fixed H , and the same material

$$\frac{\tau_2}{\tau_1} = \frac{2}{3} \times \frac{1}{\ell} \times \left(\frac{I_2}{I_1} \right) \left(\frac{b_2^4}{b_1^3} \right) \left(\frac{h_2}{h_1} \right) \quad (141)$$

For k = 3

From Equation (107) we obtain as an average time constant

$$\tau_3 = \frac{I 16c^2}{3\pi H^2 \sigma (b^4 - a^4) \ell} = \frac{I 4c^2}{3\pi H^2 \sigma b^3 \ell h} \quad (142)$$

For the same physical constants and dimensions as used in k = 2 we get

$$\tau_3 \cong 1.6 \text{ days .}$$

For k = 4

From Equations (107) and (129) we may write

$$I \frac{d\omega}{\omega} = - c_1 dt \quad (143)$$

Then

$$\tau_4 = \left(\frac{I}{c_1} \right)_{k=4} \cong \left(\frac{I_4}{I_3} \right) \left(\frac{(c_1)^{k=3}}{(c_1)^{k=4}} \right) \tau_3 \quad (144)$$

From Equations (84), (126), and (127) we write

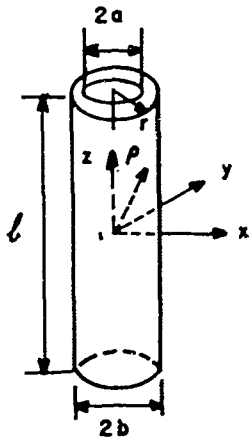
$$\left(\bar{T}_D \right)_{k=4} \cong K \left(\bar{T}_D \right)_{k=3} \quad (145)$$

where, for the dimensions of interest,

$$K = \frac{(c_1)_{k=4}}{(c_1)_{k=3}} \cong 2.5 \quad (146)$$

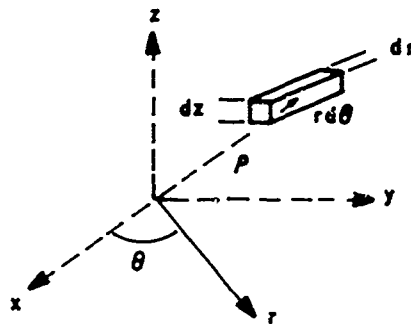
For tumbling we consider I as follows for $k = 3$

$$I = I_{xx} = \int_{\text{Vol}} z^2 dm + \int_{\text{Vol}} \gamma^2 dm$$



$$= \gamma \int_{-l/2}^{l/2} \int_a^b \int_0^{2\pi} \left[z^2 + (r \cos \theta)^2 \right] r d\theta dr dz \quad (147)$$

where γ is the mass density and with volume element shown by



Thus, using Equations (115) and (116)

$$\begin{aligned} \left(I_{xx} \right)_{k=3} &= I_3 = \gamma \left[\frac{\pi l^3}{r} (b^2 - a^2) + \frac{\pi l}{4} [b^4 - a^4] \right] \\ &= \gamma [A_o + B_o] \end{aligned} \quad (148)$$

In a similar way but following the type of calculation used in (122) and (123), we obtain for $k = 4$

$$\begin{aligned} \left(\frac{I_{xx}}{I_3} \right)_{k=4} &= I_4 = \gamma \left\{ \left[\tau (b^3 - a^3) \frac{\ell^2}{3} + \frac{\tau \ell}{2} (b^4 - a^4) \right. \right. \\ &\quad \left. \left. + \frac{8\pi}{15} (b^5 - a^5) \right] + \frac{\tau \ell^3}{12} (b^2 - a^2) \right. \\ &\quad \left. + \frac{\tau \ell}{4} (b^4 - a^4) \right\} \\ &= \gamma \left[A_1 + B_1 \right] \end{aligned} \quad (149)$$

Thus

$$\left(\frac{I_4}{I_3} \right) \left(\frac{\left(\frac{c_1}{I_3} \right)_{k=3}}{\left(\frac{c_1}{I_3} \right)_{k=4}} \right) = k \cdot \frac{1}{k} = 1 \quad (150)$$

so that

$$\tau_4 \cong \tau_3 \quad (151)$$

Therefore for the dimensions used in $k = 3$

$$\tau_4 = 1.6 \text{ days.}$$

INERTIA, WEIGHT AND ORBIT LIFE

For purposes of shell design with orbiting about the earth, we are interested not only in eddy current decay but a host of other effects. Here we will concentrate additionally only on orbit lifetime as determined by

aerodynamic drag. * Also the weight of the object is of interest as well as its size. For radar applications a conducting object is assumed and so eddy current decays will also be considered in the design.

The model considered will be as before with the cylinder shape tumbling so that the angular velocity is normal to H , but now with the orbiting motion experiencing the effects of drag. The choice of inertia and weight is coupled into both the eddy current and orbit drag considerations.

For the purposes at hand we take

$$\tau_k = \frac{K_k I_k}{h H^2} ; k = 3 \text{ or } 4 \quad (152)$$

where

K_k is a constant depending on body geometry

I_k is the tumbling moment of inertia

h is the shell thickness

Now, as noted previously, if I is not arbitrarily specified

$$K_3 I_3 = K_4 I_4 \quad (153)$$

so that only one set of eddy current calculations need be done for the case $k = 3$ or $k = 4$ cylinders whenever I is not imposed.

Of course for thin shells, I can be shown proportional to h . For example, from Equation (149)

*For aerodynamic drag effects on motion about the center of mass, see Reference (19).

$$I_4 = \gamma \tau h \left[\frac{\ell^3 b}{6} + \ell^2 b^2 + 3\ell b^2 + \frac{8b^3}{3} \right] \quad (154)$$

for $h \ll a$. Thus τ is independent of h (with I not imposed).

A useful value of τ for orbit applications might be about 1.7 years since this would result in 95 percent reduction in ω in about 5 years.

If we assume that orthogonality of $\bar{H}$ and $\bar{\omega}$ is unlikely, and take H reduced to $\leq 0.8H$ (H^2 to $\leq 0.64 H^2$) and if we impose a fixed value of $I = 75 \text{ kg-m}^2$, then for $\ell = 3\text{m}$, $b = 0.625\text{m}$, $H = 0.3$ oersted, $h = 0.01\text{m}$, as for τ_3 , we get with $k = 2.4$

$$\tau_4 \geq \frac{1}{k(.64)} \tau_3 \cong 1.05 \text{ days}$$

say, for example

$$\tau_4 = 1.2 \text{ days.}$$

The thickness then required to give $\tau_4 = 1.7$ years with all other factors remaining the same, becomes

$$h_4 \cong 0.0018 \text{ cm} \cong 0.75 \text{ mils.}$$

Now for most operational radar applications, the radar skin depth

$$\delta = \frac{c}{(2\pi\sigma\omega_{\text{radar}})^{1/2}} \geq 0.004 \text{ cm} = \delta_{\text{min}},$$

so that

$$h_4 = 4.5 \delta_{\text{min}}$$

where we desire

$$h \geq 3 \delta_{\text{min.}}$$

Considering that the all-aluminum shell determines the tumbling inertia itself (i. e., I is not imposed) then for $h \ll b$

$$\tau_3 = \tau_4 = \text{constant} \cong 1.8 \text{ weeks} \quad (155)$$

for $\bar{H} = 0.3$ oersted, $\ell = 3\text{m}$, $b = 0.625\text{m}$.

If the length had been decreased, then for example τ_4 would decrease slightly to about 1.3 weeks since the torque would remain about the same (see page 40) while the inertia would decrease. For the length-to-radius ratios considered here, the decrease in torque is less than that in inertia so that τ decreases.

In order to realize

$$\tau_3 = \tau_4 = 1.7 \text{ years}$$

we reduce the torque by taking small $h \cong 0.002 \text{ cm}$ and increase I by forming a non-conducting backing (say plastic) inside the conducting skin.

Using calculations similar to those for describing I , we find for the mass, M

$$M_3 = \gamma \pi (b^2 - a^2) \ell$$

$$M_4 = \gamma \tau \left[(b^2 - a^2) \ell + \frac{4}{3} (b^3 - a^3) \right] \quad (156)$$

Then the weight, W , in pounds with M in kg. is

$$W = 2.2 M \quad (157)$$

For thin shells

$$M = K_m h \quad (158)$$

with K_m constant for a fixed shape and size.

To illustrate a design, we find using a plastic with specific gravity
 $= 1.5$, and h (aluminum) $= 0.002$ cm,

$$h \text{ (plastic)} = 1/2 \text{ cm: } I \quad (\text{alum} + \text{plastic}) = 154 \text{ kg-m}^2$$

$$W \quad (\text{alum} + \text{plastic}) = 189 \text{ lbs.}$$

$$\tau \cong 3.5 \text{ years}$$

$$h \text{ (plastic)} = 1/4 \text{ cm: } I \quad (\text{alum} + \text{plastic}) = 76 \text{ kg-m}^2$$

$$W \quad (\text{alum} + \text{plastic}) = 96 \text{ lbs.}$$

$$\tau \cong 1.7 \text{ years}$$

Thus by striving to minimize torque by decreasing the conducting thickness while increasing the inertia with a plastic backing (without causing a large weight increase) reasonable eddy current time constants for rotating shells can be achieved.

It may also be mentioned that aerodynamic considerations, [19]
essentially independent of altitude, affecting the rotating motion require

$$\frac{I \text{ (tumble)}}{I \text{ (spin)}} > 4.5 \text{ in order that the stable mode be tumbling.}$$

With the spin inertia calculated as

$$I_3 \text{ (spin)} = \gamma \pi (b^4 - a^4) \frac{\ell}{2} \quad (159)$$

$$I_4 \text{ (spin)} = \gamma \left[\pi (b^4 - a^4) \frac{\ell}{2} + \frac{8\pi}{15} (b^5 - a^5) \right] \quad (160)$$

the above design yields

$$\frac{I \text{ (tumble)}}{I \text{ (spin)}} \approx 5 > 4.5$$

Since we have been varying weight, inertia and surface areas, we have been changing the orbit lifetime due to aerodynamic drag.

Omitting the details of the computation and considering eccentric orbits at 300 nautical miles perigee we find, using decay data,^[20] for $\epsilon = 0$ (circular orbit)

$$W = 200^{\#} \text{ gives orbit life } \cong 189 \text{ days}$$

$$W = 300^{\#} \text{ gives orbit life } \cong 280 \text{ days}$$

and for $\epsilon = 0.05$ (apogee = 694 nautical miles)

$$W = 200^{\#} \text{ gives orbit life } \cong 1658 \text{ days}$$

$$W = 300^{\#} \text{ gives orbit life } \cong 2430 \text{ days}$$

At a circular orbit of 450 nautical miles

$$W = 200^{\#} \text{ gives orbit life } \cong 2160 \text{ days}$$

$$W = 300^{\#} \text{ gives orbit life } \cong 3705 \text{ days}$$

In these calculations, $b = 0.625 \text{ m}$, $\ell = 3 \text{ m}$, an average inflight projected area for the hemisphere capped cylinder of 32.6 ft^2 and a $C_D = 2.2$ were used.

It turns out that, at the 450-nautical mile orbit, an orbit lifetime about twice as long as the eddy current time constant lifetime is achieved.

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13. ABSTRACT This report describes a study conducted to estimate the torques acting on electrically conducting shells rotating in a magnetic field. The basic electromagnetic expressions leading to an evaluation of torque are given using various approaches with comparisons for the spherical shell. Application of these techniques to cylindrical shells with flat and hemisphere shell ends is described. The torques both slow and alter the direction of the rotating motion, and the resultant decay time constants can be in the order of days. Factors affecting this decay time and influencing shell design are examined.		

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