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RM-5567-PR
JANUARY 1968

ON SOLUTIONS FOR n-PERSON GAMES

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PREFACE

This Memorandum reports two theoretical results in the mathematical theory of n-person cooperative games in characteristic function form. It represents a further extension of the discovery initially reported in RM-5518-PR, A Game With No Solution, and RM-5543-PR, The Proof That a Game May Not Have a Solution, that certain conjectures based on the von Neumann-Morgenstern theory of solutions for n-person games are false. Game theory is a continuing study sponsored by Project RAND.

SUMMARY

A solution concept for n-person cooperative games in characteristic function form was introduced by von Neumann and Morgenstern. This Memorandum reviews the definitions of an n-person game and then describes two particular games whose sets of solutions are rather restricted. The first is a five-person game which has a unique solution that is nonconvex. The second is an eight-person game which has no solution which possesses the symmetry of the characteristic function.

CONTENTS

ON SOLUTIONS FOR n-PERSON GAMES

1. INTRODUCTION

In 1944 von Neumann and Morgenstern [6] introduced a theory of solutions (stable sets) for n-person games in characteristic function form. Earlier results in solution theory led to various conjectures such as: that every game has at least one solution, that at least some of the solutions for a game can be characterized in an elementary manner, and that the union and intersection of all solutions for a game had certain properties. More recent developments, however, show that several of these conjectures about solutions are false [1, 2, 5] and that there are even games which do not have solutions [3, 4]. This Memorandum reviews the essential definitions for a game and then describes two particular games which illustrate some additional developments of this latter type.

Section 3 describes a five-person game which has a solution which is unique and nonconvex. An eight-person game with a unique and nonconvex solution has already been described in [2]. The present example is of interest because of the fewer number of players involved and because its core differs somewhat from those in the previous papers [1, 2, 5].

Section 4 describes an eight-person game which has solutions, but none of its solutions possesses the symmetry of the characteristic function. This result is not surprising in light of the counterexample on existence [3, 4]. In fact, it can be viewed as the "two-dimensional" analog to the "three-dimensional" aspects of this counterexample. However, the author arrived at the results in this Memorandum before that in [3, 4], and they are still of some interest on their own. After the results in [1, 2, 5] were known, L. S. Shapley suggested to the author that the derivation of a game without a symmetric solution may be the next step in arriving at a game with no solution.

2. DEFINITIONS

An *n-person game* is a pair (N, v) where $N = \{1, 2, \ldots, n\}$ is a set of n players and v is a characteristic function on 2^N, i.e., v assigns the real number $v(S)$ to each subset S of N and $v(\phi) = 0$ for the empty set ϕ. The set of *imputations* is

$$A = \{x: \sum_{i \in N} x_i = v(N) \text{ and } x_i \geq v(\{i\}) \text{ for all } i \in N\}$$

where $x = (x_1, x_2, \ldots, x_n)$ is a vector with real components.

If x and $y \in A$ and S is a nonempty subset of N, then y *dominates* x *via* S if

(1) $\qquad y_i > x_i \quad$ for all $i \in S$

and

(2) $\qquad \sum_{i \in S} y_i \leq v(S)$,

and this is denoted by $y \text{ dom}_S\, x$. If there exists an S such that $y \text{ dom}_S\, x$, then one says that y *dominates* x and denotes this by $y \text{ dom } x$. For any $y \in A$ and $Y \subset A$ define the following *dominions*: $\text{Dom}_S\, y = \{x \in A : y \text{ dom}_S\, x\}$, $\text{Dom } y = \{x \in A : y \text{ dom } x\}$, $\text{Dom}_S\, Y = \bigcup_{y \in Y} \text{Dom}_S\, y$, and $\text{Dom } Y = \bigcup_{y \in Y} \text{Dom } y$; and the *inverse dominions*: $\text{Dom}^{-1} y = \{z \in A : z \text{ dom } y\}$ and $\text{Dom}^{-1} Y = \bigcup_{y \in Y} \text{Dom}^{-1} y$.

To simplify the notation in (2) let

$$y(S) = \sum_{i \in S} y_i .$$

Also, expressions such as $v(\{1,3,5,7\})$ and $x(\{2,5,7\})$ will be shortened to $v(1357)$ and $x(257)$ respectively.

A subset K of A is a solution if

(3) $K \cap \text{Dom } K = \phi$

and

(4) $K \cup \text{Dom } K = A$.

If $K' \subset X \subset A$, then K' is a solution for X if

(3') $K' \cap \text{Dom } K' = \phi$

and

(4') $K' \cup \text{Dom } K' \supset X$.

The core of the game (N, v) is

$$C = \{x \epsilon A: x(S) \geq v(S) \text{ for all } S \subset N\}.$$

The core is a convex polyhedron (possibly empty), and for any solution K, $C \subset K$ and $K \cap \text{Dom } C = \phi$.

3. A GAME WITH A UNIQUE SOLUTION WHICH IS NONCONVEX

Consider the five-person game (N, v) where $N = \{1, 2, 3, 4, 5\}$ and v is given by:

$$v(N) = 3, \quad v(234) = v(345) = 2,$$

$$v(12) = v(45) = v(35) = v(34) = 1,$$

$$v(S) = 0 \text{ for all other } S \subset N.$$

For this game

$$A = \{x : x(N) = 3 \text{ and } x_i \geq 0 \text{ for all } i \in N\}.$$

In studying this game it is helpful to introduce the three-dimensional triangular wedge B which has the six vertices:
$c^0 = (0, 1, 1, 1, 0)$, $c^1 = (0, 1, 0, 1, 1)$, $c^2 = (0, 1, 1, 0, 1)$,
$c^3 = (1, 0, 1, 1, 0)$, $d^1 = (1, 0, 0, 1, 1)$, $d^2 = (1, 0, 1, 0, 1)$.
One can show that

$$B = \{x \in A : x(S) \geq v(S) \text{ for all } S \text{ except } \{2, 3, 4\}\}.$$

One can also prove that the core C is the convex hull of c^0, c^1, c^2, and c^3, and that

$$C = \{x \in B : x(234) \geq 2\}.$$

The unique solution for this game is

$$K = C \cup D_3 \cup D_4$$

where $D_3 = \{x \in B : x_3 = 1\} - C$ and $D_4 = \{x \in B : x_4 = 1\} - C$. This solution is pictured in Figure 1. To prove that K is the unique

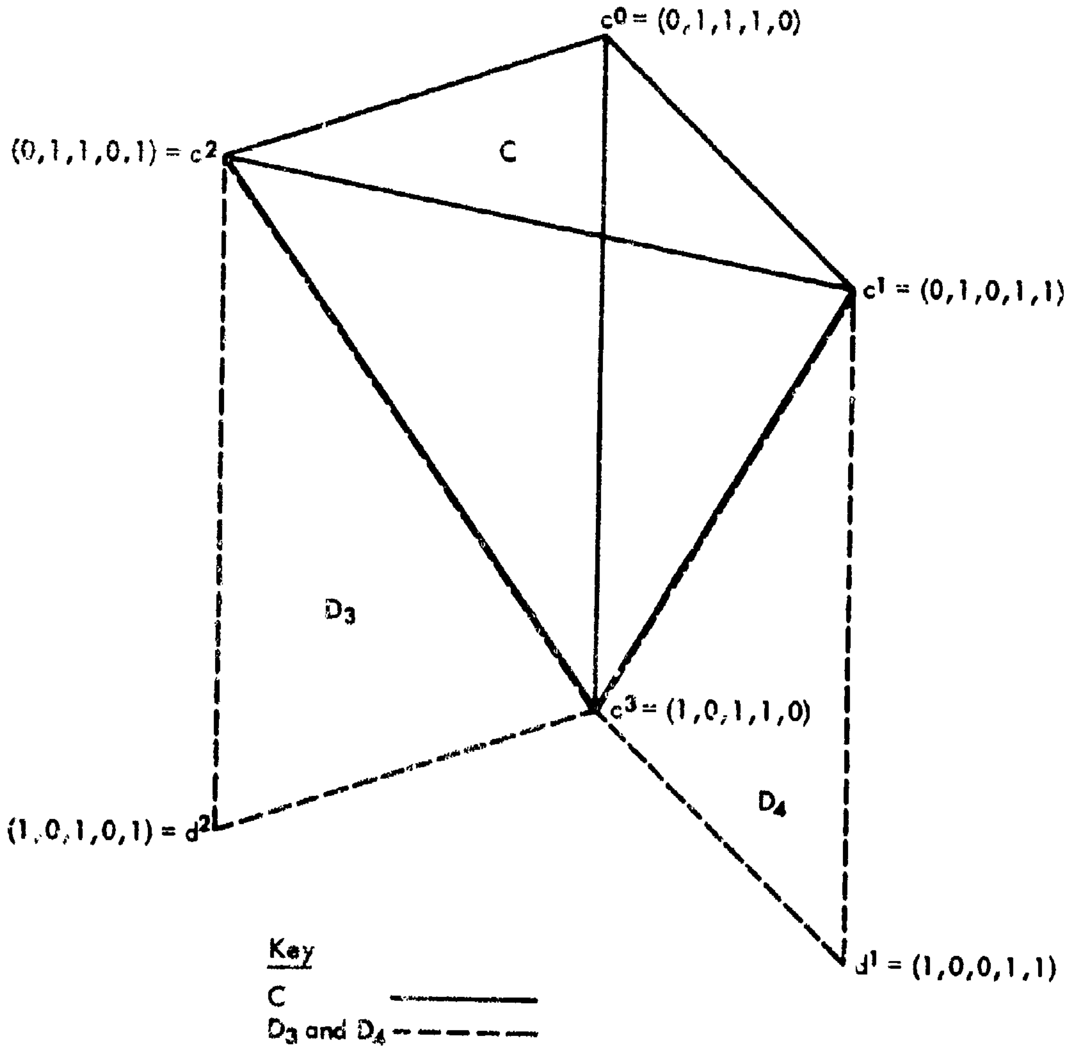


Fig. 1 — A unique solution which is nonconvex

solution it is sufficient to verify that Dom C $\supset$ A–B and to observe that K is precisely those elements in B which are maximal with respect to the relation "$\mathrm{dom}_{\{2,3,4\}}$." Therefore, this game has a unique solution which is clearly nonconvex.

4. A GAME WITH NO SYMMETRIC SOLUTION

Consider the eight-person game (N, v) where $N = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and v is given by:

$$v(N) = 4, \quad v(1357) = 3, \quad v(257) = v(457) = 1,$$

$$v(12) = v(34) = v(56) = v(78) = 1,$$

$$v(S) = 0 \text{ for all other } S \subset N .$$

This game is symmetric in the sense that one can interchange 1 with 3 and 2 with 4 and the characteristic function remains invariant. For this game:

$$A = \{x: x(N) = 4 \text{ and } x_i \geq 0 \text{ for all } i \in N\} .$$

It is helpful to introduce the four-dimensional hypercube

$$H = \{x \in A: x(12) = x(34) = x(56) = x(78) = 1\} .$$

One can prove that the core for this game is

$$C = \{x \in H: x(1357) \geq 3\}$$

and that C is the convex hull of the following five vertices of H: $c^0 = (1, 0, 1, 0, 1, 0, 1, 0)$, $c^2 = (0, 1, 1, 0, 1, 0, 1, 0)$, $c^4 = (1, 0, 0, 1, 1, 0, 1, 0)$, $c^6 = (1, 0, 1, 0, 0, 1, 1, 0)$, and $c^8 = (1, 0, 1, 0, 1, 0, 0, 1)$.

Define the following eleven regions in H:

$$F_i = \{x \in H: x_i = 1\} \qquad i = 1, 3, 5, 7$$

$$F = F_1 \cup F_3 \cup F_5 \cup F_7 - C$$

$$E_i = \{x \in F_i : x(i+1, 5, 7) < 1\} \qquad i = 1, 3$$

$$E = E_1 \cup E_3$$

$$G_1 = \{x \in E_1 : x(457) = 1\}$$

$$G_3 = \{x \in E_3 : x(257) = 1\}$$

$$G = G_1 \cup G_3.$$

The traces of these regions on some three-dimensional cubical traces of H are shown in Figure 2. The sets G_1 and G_3 are triangles and are illustrated in Figure 3. The regions A–H, $H-[C \cup (F-E) \cup E]$, C, F–E, and E form a partition of A.

One can use arguments like those in [4] to prove that

(5) $\quad \operatorname{Dom} C = [A-H] \cup [H-(C \cup F)]$

and thus any solution K for this game is contained in $C \cup F$. One can also check various cases to prove that

(6) $\quad (F-E) \cap \operatorname{Dom}(C \cup F) = \phi$

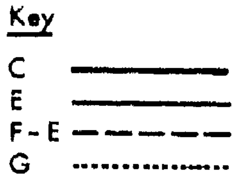
and

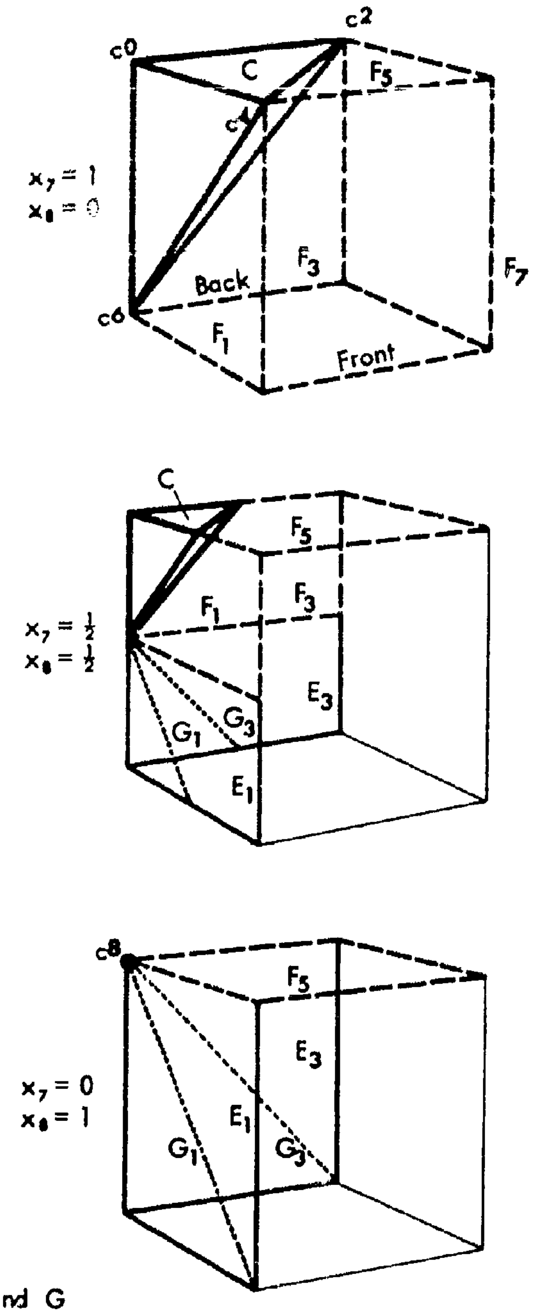
(7) $\quad E \cap \operatorname{Dom}[C \cup (F-E)] = \phi$.

Therefore, any solution is of the form

$$K = C \cup (F-E) \cup K'$$

where K' is a solution for E. The sets C, F–E, E, and G do exhibit the symmetry of the characteristic function.

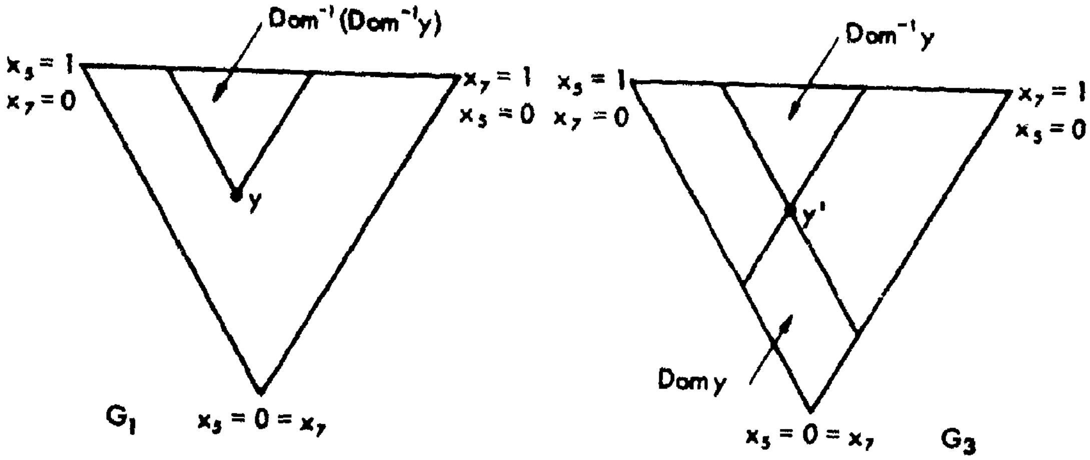




In each cube

Left face:	$x_1 = 1,\ x_2 = 0$
Right face:	$x_1 = 0,\ x_2 = 1$
Back face:	$x_3 = 1,\ x_4 = 0$
Front face:	$x_3 = 0,\ x_4 = 1$
Top face:	$x_5 = 1,\ x_6 = 0$
Bottom face:	$x_5 = 0,\ x_6 = 1$

Fig.2—Traces in H of C, E, F - E, and G



Note: The common top edge is in the core and not in G
In G_1: $x_2 = 0$, $x_4 > 0$
In G_3: $x_2 > 0$, $x_4 = 0$

Fig. 3 — The region G

It will follow from the following two lemmas that the problem of finding a solution K' for E is equivalent to finding a solution K'' for G.

LEMMA 1. For any solution K' for E

$K' \cap \{x \in E: x(257) < 1 \text{ and } x(457) < 1\} = \phi.$

PROOF. Assume that the LEMMA is false, and pick an x in this intersection. If $x \in E_1$ pick $y \in E_1$ so that $y(457) = 1$ and $y_i > x_i$ for $i = 4, 5, 7$. Then $y \notin K'$ since $y \operatorname{dom}_{\{4,5,7\}} x$. Thus there exists $z \in K'$ such that z dom y. One can then see that $z \operatorname{dom}_{\{2,5,7\}} y$; and clearly $z_2 > y_2 = 0 = x_2$. Therefore, $z \operatorname{dom}_{\{2,5,7\}} x$ and $x \notin K'$. A symmetrical argument shows that if $x \in E_3$ then x is not in this intersection.

LEMMA 2. Let $L(x, x')$ be the closed line segment joining x and x', and let K' be a solution for E. If $y \in G_1$ and $y' = (y_1, y_2, 0, 1, y_5, y_6, y_7, y_8)$, then $L(y, y') \cap K' \neq \phi$ implies that $L(y, y') \subset K'$. If $z \in G_3$ and $z' = (0, 1, z_3, z_4, z_5, z_6, z_7, z_8)$, then $L(z, z') \cap K' \neq \phi$ implies that $L(z, z') \subset K'$.

PROOF. Assume that $x \in L(y, y') - K'$. Then $x \in \operatorname{Dom} K'$, and by checking cases one can see that $x \in \operatorname{Dom}_{\{2,5,7\}} K'$. However, $x_i = y_i = y_i'$ when $i = 2, 5$, and 7, and thus $L(y, y') \subset \operatorname{Dom}_{\{2,5,7\}} K'$ or $L(y, y') \cap K' = \phi$. A similar proof works for the second part of the LEMMA.

One can now show that there is no solution K'' for G such that K'' has the symmetry of the characteristic function, i.e., if $y \epsilon K''$ then $y' = (y_3, y_4, y_1, y_2, y_5, y_6, y_7, y_8) \notin K''$. Clearly, $K'' \neq \phi$. Pick an arbitrary $y \epsilon K''$, and assume that $y \epsilon G_1$. Condition (3') implies that

(8) $$K'' \cap \mathrm{Dom}^{-1} y = \phi$$

where

$$G \cap \mathrm{Dom}^{-1} y = \{x \epsilon G_3 : x_i > y_i \text{ for } i = 5 \text{ and } 7\} .$$

Conditions (4') and (8) imply that

(9) $$K'' \cap \mathrm{Dom}^{-1}(\mathrm{Dom}^{-1} y) \neq \phi$$

where

$$G \cap \mathrm{Dom}^{-1}(\mathrm{Dom}^{-1} y) = \{z \epsilon G_1 : z_i > y_i \text{ for } i = 5 \text{ and } 7\} .$$

See Figure 2 for an illustration of these sets. If z is any imputation in the intersection in (9), then $z \epsilon K''$ and $z \text{ dom}_{\{4, 5, 7\}} y'$, because $z_4 > 0 = y_2 = y'_4$, $z_5 > y_5 = y'_5$, and $z_7 > y_7 = y'_7$. Therefore, $y \epsilon K''$ but the symmetrical point $y' \notin K''$. A symmetric argument holds if one assumes $y \epsilon G_3$. It follows that there is no symmetric solution K'' for G, and thus no symmetric solution K for this eight-person game.

This game does however have solutions. For example, G_1 and G_3 are solutions for G. There are also infinitely many other solutions for G, each of which contains imputations from both G_1

and G_3. The existence of these latter solutions was pointed out by L. S. Shapley. Any solution K'' for G can be extended to a solution K' for E by making use of the LEMMAS. The set

$$K = C \cup (F-E) \cup K'$$

will then be a solution for this game.

The classical theory of games assumed that the characteristic function is superadditive, i.e., $v(S_1 \cup S_2) \geq v(S_1) + v(S_2)$ whenever $S_1 \cap S_2 = \phi$. The two games in this paper can be transformed into superadditive games which have the same A, C, and solutions K.

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10. ABSTRACT A solution concept for n-person cooperative games in characteristic function form was introduced by von Neumann and Morgenstern in 1944. This study reviews the definitions of an n-person game and then describes two games whose sets of solutions are rather restricted. The first is a five-person game which has a unique solution that is nonconvex. The second is an eight-person game that has no solution possessing the symmetry of the characteristic function.	11. KEY WORDS Game theory Set theory