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SELECTION PROCEDURES FOR RESTRICTED FAMILIES OF PROBABILITY DISTRIBUTIONS

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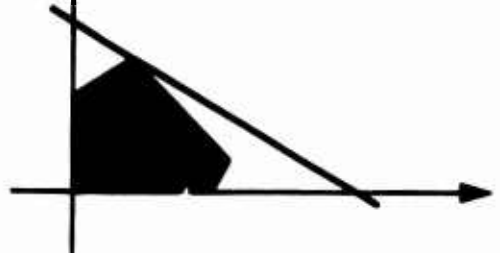
Richard E. Barlow and Shanti S. Gupta

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FAMILIES OF PROBABILITY DISTRIBUTIONS

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#### ABSTRACT

This paper is primarily concerned with selecting a subset of  $k$  populations such that the probability is at least  $P^*$  that the selected subset includes the population with the largest (smallest) quantile of a given order  $\alpha$  ( $0 < \alpha < 1$ ). In particular a procedure is proposed and studied which is valid for any family of distributions with increasing failure rate on the average (IFRA). It is compared, asymptotically, with a distribution-free procedure proposed by Rizvi and Sobel.

SELECTION PROCEDURES FOR RESTRICTED  
FAMILIES OF PROBABILITY DISTRIBUTIONS<sup>†</sup>

by

Richard E. Barlow and Shanti S. Gupta

1. INTRODUCTION AND SUMMARY

Let  $\Pi_1, \Pi_2, \dots, \Pi_k$  be  $k$  populations. The random variable  $X_i$  associated with  $\Pi_i$  has a continuous distribution  $F_i, i = 1, 2, \dots, k$ . We are primarily interested in selecting a subset such that the probability is at least  $P^*$  that the selected subset includes the population with the largest (smallest) quantile of a given order  $\alpha (0 < \alpha < 1)$ . We assume each  $F_i$  has a unique  $\alpha$ -quantile,  $\xi_{\alpha, i}$ . Let  $F_{[i]}(x) = F_{[i]}$  denote the cumulative distribution function of the population with the  $i$ th smallest  $\alpha$ -quantile. In the following, we consider families of distributions ordered in a certain sense with respect to a specified continuous distribution  $G$  and propose and study a selection procedure which is different from the nonparametric procedure of Rizvi and Sobel (1967). We assume

(a)  $F_{[i]}(x) \geq F_{[k]}(x) \quad i = 1, 2, \dots, k \text{ and all } x,$

(b)  $\exists$  a continuous distribution  $G \ni F_{[i]} \preceq G, \forall i = 1, 2, \dots, k,$

where  $\preceq$  denotes a partial ordering relation on the space of distributions.

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A relation  $\prec$  on the space of distribution is a *partial ordering* if

$$F \prec F \quad \forall \text{ distributions } F$$

$$F \prec G, G \prec H \text{ implies } F \prec H.$$

Note that  $F \prec G$  and  $G \prec F$  do not necessarily imply  $F \equiv G$ .

Various special cases in addition to stochastic ordering are:

$$F \prec_{*} G \text{ iff } F(0) = G(0) = 0 \text{ and } \frac{G^{-1}F(x)}{x} \text{ is} \quad (i)$$

nondecreasing in  $x \geq 0$  on the support of  $F$ .

$$F \prec_{c} G \text{ iff } G^{-1}F(x) \text{ is convex on the support of } F. \quad (ii)$$

$$F \prec_{r} G \text{ iff } F(0) = G(0) = \frac{1}{2} \text{ and } \frac{G^{-1}F(x)}{x} \text{ is} \quad (iii)$$

increasing (decreasing) for  $x$  positive (negative) on the support of  $F$ .

$$F \prec_{s} G \text{ iff } F(0) = G(0) = \frac{1}{2} \text{ and } G^{-1}F \text{ is concave-convex} \quad (iv)$$

about the origin, on the support of  $F$ .

If  $G(x) = 1 - e^{-x}$  for  $x \geq 0$ , then (i) defines the class of IFRA distributions studied by Birnbaum, Esary and Marshall (1966) while (ii) defines the class of IFR distributions studied by Barlow, Marshall and Proschan (1963). For any distribution  $G$ ,  $F \prec_{*} G$  iff  $F(x)$  crosses  $G(\theta x)$  at most once and from below if at all as a function of  $x$  for all  $\theta > 0$ . If  $G(x) = 1 - \exp(-x^{\lambda})$  for  $x \geq 0$  and  $\lambda > 0$ , then  $F \prec_{*} G$  implies that  $F$  is "sharper" than the family of Weibull distributions with shape parameter  $\lambda$ . Implications of

orderings defined by (iii) were studied by Lawrence (1966). Van Zwet (1964) studied orderings defined by both (ii) and (iv). Clearly  $\prec_c$  ordering implies  $\prec_s$  ordering and  $\prec_s$  ordering implies  $\prec_r$  ordering.

If  $\underline{X}_i = (X_{i1}, X_{i2}, \dots, X_{in})$  is the observed sample from the  $i$ th population, then we restrict ourselves to the class of statistics  $T_i = T(\underline{X}_i)$  that preserve both ordering relations (a) and (b), i.e.,

$$P_{F_{[1]}} \{T(\underline{X}) \leq x\} \geq P_{F_{[k]}} \{T(\underline{X}) \leq x\} \quad (a')$$

for all  $x$  and  $i = 1, 2, \dots, k$ .

$$F_{T(\underline{X}_i)} \prec G_{T(\underline{Y})} \quad (b')$$

$i = 1, 2, \dots, k$  where  $\underline{Y} = (Y_1, Y_2, \dots, Y_n)$  is a random sample from  $G$ .

In Section 2 of this paper, we propose and study procedures  $R$  ( $R'$ ) for selecting the population with the largest (smallest)  $\alpha$ -quantile for distributions which are  $\prec_s$  ordered with respect to a specified distribution  $G$ . The infimum of the probability of a correct selection is obtained in Theorem 2.1 and asymptotic evaluation is given in Theorem 2.2. Section 3 deals with quantile selection procedures for the class of IFRA distributions. In Section 4, we study the efficiency of procedure  $R$  with respect to a procedure studied by Rizvi and Sobel (1967) under scale type slippage configurations. Asymptotic relative efficiency of  $R$  with respect to a selection procedure for the gamma populations proposed by Gupta (1963) is also investigated. Section 5 deals with selection procedures for the median for distributions that are  $\prec_r$  ordered with respect to a specified  $G$ . In Section 6 we propose a selection procedure with respect to the sample means for distributions that are  $\prec_c$  ordered with respect to  $G(x) = 1 - e^{-x}$ . Application to the selection of gamma populations is also given in Section 6.

2. QUANTILE SELECTION RULES FOR DISTRIBUTIONS  $\prec$  ORDERED WITH RESPECT TO G  
\*

We are given a sample of size  $n$  from each of the  $k$  populations  $\Pi_{i,1} = 1, 2, \dots, k$ . The distributions  $F_{[i]}$  and the specified distribution  $G$  satisfy the assumptions (a) and (b) of Section 1. The distributions  $F_i$  are, otherwise, unspecified. Of course, the correct pairing of the unordered and ordered  $F_i$ 's is not known. We denote the  $k$ -tuples  $(F_1, F_2, \dots, F_k)$  by  $\Omega$ . Let  $T_{j,i}$  denote the  $j$ th order statistic from  $F_i$  where  $j \leq na < j+1$ . Clearly,  $T_{j,i} \xrightarrow[\text{a.s.}]{} \xi_{\alpha,i}$ , the  $\alpha$ -quantile as  $n \rightarrow \infty$  and  $\frac{1}{n} \rightarrow \alpha$ . The rule we propose for selecting the population with the largest  $\alpha$ -quantile is:

R : Select population  $\Pi_i$  iff

$$T_{j,i} \geq c \max_{1 \leq r \leq k} T_{j,r}, \quad j \leq na < n+1, \quad (2.1)$$

where  $c = c(k, P^*, n, j)$  is some number between 0 and 1 which is determined so as to satisfy the probability requirement

$$\inf_{\Omega} P\{CS|R\} = P^*, \quad (2.2)$$

where CS stands for a correct selection, i.e., the selection of any subset which contains the population  $\Pi_{[k]}$  with distribution  $F_{[k]}$ . Before discussing the main theorem concerned with the evaluation of  $P\{CS|R\}$ , we present a known result for order statistics. Let  $H_{j,i}(x)$  be the cdf of the  $j$ th order statistic from  $F_{[i]}$  and let  $G_j(x)$  be the cdf of the  $j$ th order statistic from  $G$ . Let us define

$$B_{j,n}(x) = \frac{n!}{(j-1)!(n-j)!} \int_0^x u^{j-1} (1-u)^{n-j} du$$

so that

$$H_{j,i}(x) = B_{j,n}(F_{[i]}(x)) \equiv B_{j,n}F_{[i]}(x) . \quad (2.3)$$

Since

$$G_j^{-1} H_{j,i}(x) = [B_j G]^{-1} B_j F_{[i]}(x) = G^{-1} F_{[i]}(x) . \quad (2.4)$$

we see that order statistics preserve each of the partial ordering relations

(i) - (iv). For additional applications of (2.4) see van Zwet (1964).

We now state and prove a theorem which enables us to compute the constant  $c$  which defines the procedure  $R$ .

Theorem 2.1:

If  $F_{[i]}(0) = G(0) = 0$ ,  $F_{[i]}(x) \geq F_{[k]}(x)$ ,  $x \geq 0$ ,  $i = 1, 2, \dots, k$  and  $F_{[k]} \prec_{\star} G$ , then

$$\inf_{\Omega} P\{CS|R\} = \int_0^{\infty} [G_j(\frac{x}{c})]^{k-1} dG_j(x) . \quad (2.5)$$

Proof:

Note that

$$\begin{aligned} P\{CS|R\} &= \int_0^{\infty} \left[ \prod_{i=1}^{k-1} H_{j,i}(x/c) \right] dH_{j,k}(x) \\ &\geq \int_0^{\infty} [H_{j,k}(x/c)]^{k-1} dH_{j,k}(x) . \end{aligned}$$

We wish to bound the right-hand side. Let  $X_{j,r}$  ( $r = 1, 2, \dots, k$ ) be i.i.d. with c.d.f.  $H_{j,k}(x)$ . (Note that  $X_{j,r} \equiv T_{j,r}$  when  $F_{[1]} \equiv F_{[k]}$ ,  $V_1$ .) Let  $\phi(x) = G_j^{-1} H_{j,k}(x) = G_j^{-1} F_{[k]}(x)$  so that  $\phi(x)/x$  is nondecreasing in  $x \geq 0$ .

Then

$$\frac{\phi(X_{j,r})}{X_{j,r}} \leq \frac{\phi(\max_{1 \leq r \leq k} X_{j,r})}{\max_{1 \leq r \leq k} X_{j,r}} = \frac{\max_{1 \leq r \leq k} \phi(X_{j,r})}{\max_{1 \leq r \leq k} X_{j,r}} \quad (2.6)$$

so that

$$\frac{\phi(X_{j,r})}{\max_{1 \leq r \leq k} \phi(X_{j,r})} \leq \frac{X_{j,r}}{\max_{1 \leq r \leq k} X_{j,r}} \quad (2.7)$$

Since  $Y_{j,r} = \phi(X_{j,r})$  has distribution  $G_j$  for  $r = 1, 2, \dots, k$ , we have

$$\begin{aligned} P(\text{CS}|\text{R}) &\geq P_{H_{j,k}} \left\{ \frac{X_{j,r}}{\max_{1 \leq r \leq k} X_{j,r}} \geq c \right\} \\ &\geq P_{G_j} \left\{ \frac{Y_{j,r}}{\max_{1 \leq r \leq k} Y_{j,r}} \geq c \right\} \\ &= \int_0^{\infty} [G_j(\frac{x}{c})]^{k-1} dG_j(x) \end{aligned} \quad (2.8)$$

provided  $c$  is between 0 and 1. ||

Remark 1:

The constant  $c = c(k, P^*, n, j)$  which defines the selection procedure  $R$  is determined by

$$\int_0^{\infty} [G_j(\frac{x}{c})]^{k-1} dG_j(x) = P^*, \quad (\frac{1}{k} < P^* < 1). \quad (2.9)$$

These constants are tabulated for  $G(x) = 1 - e^{-x}$  for selected values of  $n, k, j$  and  $P^*$  in the first set of tables in the companion paper by Barlow, Gupta and Panchapakesan (1967). Clearly,  $c$  is independent of scale.

Remark 2:

If  $G_j(x) = 1 - e^{-(\frac{x}{\theta})^\lambda}$ , for  $x \geq 0$  and  $\theta, \lambda > 0$ , then for  $j = 1$ , the values of  $c$  are independent of  $n$ . This can be seen from the fact that the distribution of the smallest order statistic involves  $n$  only as a scale parameter and that the selection procedure (2.1) is scale invariant.

Remark 3:

It should be pointed out that Theorem 2.1 requires only  $F_{[k]} \prec_{\star} G$ ; however, to apply the procedure  $R$ , we assume that  $F_{[1]} \prec_{\star} G, V_1$ .

Now we discuss the asymptotic evaluation of the probability of a correct selection. We state and prove the following theorem.

Theorem 2.2:

If  $F_{[k]}(x) \prec_{\star} G$ ,  $F_{[k]}(G)$  has a differentiable density  $f_{[k]}(g)$  in a neighborhood of the  $\alpha$ -quantile  $\xi_{\alpha}(\eta_{\alpha})$  and  $f_{[k]}(\xi_{\alpha}) \neq 0$  ( $g(\eta_{\alpha}) \neq 0$ ), then in our previous notation (see Theorem 2.1)

$$\begin{aligned} P_{H_{j,k}} \left\{ \frac{X_{1,k}}{\max_{1 \leq r \leq k} X_{j,r}} \geq c \right\} &\approx \int_{-\infty}^{\infty} \phi^{k-1} \left( \frac{x}{c} + \frac{(1-c)\xi_{\alpha} f_k(\xi_{\alpha}) \sqrt{n}}{c \sqrt{\alpha} \bar{\alpha}} \right) d\phi(x) \\ &\geq \int_{-\infty}^{\infty} \phi^{k-1} \left( \frac{x}{c} + \frac{(1-c)\eta_{\alpha} g(\eta_{\alpha}) \sqrt{n}}{c \sqrt{\alpha} \bar{\alpha}} \right) d\phi(x) \end{aligned} \quad (2.10)$$

as  $n \rightarrow \infty$ , where  $\bar{\alpha} = 1 - \alpha$  and  $\phi(\cdot)$  is the c.d.f. of the standard normal variate.

Proof:

$$\begin{aligned}
 & P \left\{ X_{j,k} \geq c \max_{1 \leq r \leq k-1} X_{j,r} \right\} \\
 &= P \left\{ \frac{(X_{j,k} - \xi_\alpha) f_{[k]}(\xi_\alpha) \sqrt{n}}{\sqrt{\alpha \bar{\alpha}}} \geq c \frac{\max_{1 \leq r \leq k-1} (X_{j,r} - \xi_\alpha) + \frac{c-1}{c} \xi_\alpha}{\sqrt{\alpha \bar{\alpha}} / (\sqrt{n} f_{[k]}(\xi_\alpha))} \right\} \quad (2.11) \\
 &\approx \int_{-\infty}^{\infty} \phi^{k-1} \left( \frac{x}{c} + \frac{(1-c) \xi_\alpha f_{[k]}(\xi_\alpha) \sqrt{n}}{c \sqrt{\alpha \bar{\alpha}}} \right) d\phi(x),
 \end{aligned}$$

since 
$$X_{j,k} \sim N \left( \xi_\alpha, \frac{\alpha \bar{\alpha}}{n f_{[k]}^2(\xi_\alpha)} \right).$$

To prove the second part of (2.10), note that  $F_{[k]} \prec_* G$  implies  $G^{-1} F_{[k]}(x) - x$  changes sign at most once and from  $-$  to  $+$ , if at all. Since  $F_{[k]} \prec_* G$  is invariant under scale changes, we can assume  $\eta_\alpha = \xi_\alpha$  so that  $F(\xi_\alpha) \equiv G(\xi_\alpha)$ . Either  $F = G$  in a neighborhood of  $\xi_\alpha$  or the slope of the tangent line to  $F$  at  $\xi_\alpha$  is greater than the slope of the tangent line to  $G$  at  $\xi_\alpha$ . In either case  $f_{[k]}(\xi_\alpha) \geq g(\xi_\alpha)$  and in general  $\xi_\alpha f_{[k]}(\xi_\alpha) \geq \eta_\alpha g(\eta_\alpha)$ .

Remark 4:

Setting

$$\int_{-\infty}^{\infty} \phi^{k-1} \left( \frac{x}{c} + \frac{(1-c) \eta_\alpha g(\eta_\alpha) \sqrt{n}}{c \sqrt{\alpha \bar{\alpha}}} \right) d\phi(x) = P^*, \quad (2.12)$$

we see that

$$c(k, P^*, n, j) \approx 1 - \frac{\gamma(k, P^*, \alpha)}{\sqrt{n}} \quad (2.13)$$

as  $n \rightarrow \infty$ .

For  $k = 2$ , and  $g(x) = e^{-x}$ , we see

$$c(2, P^*, n, j) = 1 - \frac{2^{1/2}C}{n^{1/2}} + \frac{C}{n} - \frac{3}{2^{3/2}} \frac{C^3}{n^{3/2}} + o\left(\frac{1}{n^2}\right) \quad (2.14)$$

where

$$C = \frac{\phi^{-1}(P^*)\sqrt{\alpha/\bar{\alpha}}}{(1-\alpha)[- \log(1-\alpha)]}$$

#### Subset Selection Rule for Smallest $\alpha$ -quantile

The rule for selecting the population with the smallest  $\alpha$ -quantile is

$R'$ : Select population  $\pi_1$  iff

$$dT_{j,i} \leq \min_{1 \leq i \leq k} T_{j,i}, \quad j \leq n < j+1 \quad (2.15)$$

where  $0 < d = d(k, P^*, n, j) < 1$  is determined so as to satisfy the basic probability requirement. If  $F_{[i]}(x) \leq F_{[1]}(x)$ ,  $i = 1, 2, \dots, k$  and all  $x \geq 0$  and  $F_{[1]} \prec G$ , then the constant  $d$  is given by the equation

$$\int_0^{\infty} [\bar{G}_j(xd)]^{k-1} dG_j(x) = P^* \quad (2.16)$$

where  $\bar{G}(x) = 1 - G(x)$ . In a manner similar to the proof of Theorem 2.1 we can show that

$$P\{CS|R'\} \geq \int_0^{\infty} [\bar{G}_j(xd)]^{k-1} dG_j(x)$$

The values of  $d$  are tabulated for selected values of  $k$ ,  $P^*$ ,  $n$  and  $j$  in the companion paper by Barlow, Gupta and Panchapakesan (1967).

The rules  $R$  and  $R'$  select nonempty subsets. The size of the selected subset is a random variable which takes values  $1, 2, \dots, k$ . The expected size of the selected subset is a common measure of the efficiency of the procedure (Gupta (1963b)). However, it is difficult in our more general framework to set meaningful bounds on the expected size without further assumptions. If we assume, in addition, that there exists  $G^+$  such that  $G^+ < F_{[1]}^*$  for all  $i$ , then we can obtain an upper bound on the probability of including the "worst" population in the selected subset. We consider this in more detail later for IFRA distributions.

If we assume that  $F_{[i]}(x)$  is stochastically increasing with respect to  $i$ , then

$$P\{\text{select } \Pi_{[i]} | R\} \geq P\{\text{select } \Pi_{[j]} | R\} \text{ if } i \geq j. \quad (2.17)$$

The proof is similar to the one given in Gupta (1967). A result similar to (2.17) is true for  $R_1$ .

### 3. QUANTILE SELECTION PROCEDURES FOR THE CLASS OF IFRA DISTRIBUTIONS

If  $F \prec_* G$  where  $G(x) = 1 - e^{-x}$  for  $x \geq 0$ , then  $F$  is an IFRA distribution. The problem of selecting the best one of several IFRA populations has been considered by J. K. Patel (1967). He was interested in selecting that population with the smallest failure rate at a prescribed time  $T$ . His decision rule depends only on the number of observed failures in  $[0, T]$  for each population and not on the times at which failure occurred. In this respect our procedure will utilize more information, though of course we are selecting with respect to quantiles rather than failure rate.

We show how to obtain  $c_\lambda$  values for the Weibull distribution with shape parameter  $\lambda > 0$ . We remark that the class of distributions  $F$ , such that

$F \prec_* G_\lambda$  where  $G_\lambda(x) = 1 - e^{-\frac{x^\lambda}{\theta}}$  for  $x \geq 0$  and  $\theta, \lambda > 0$  is the smallest

class of continuous distributions containing the Weibull class of distributions with shape parameter  $\lambda$  which is closed under the formation of coherent structures and limits in distribution.<sup>†</sup> To select populations  $\prec_*$  ordered with respect to  $G_\lambda$ , choose  $c$  corresponding to  $n, k, j$  and  $P^*$  based on an exponential assumption and set  $c_\lambda = (c)^{1/\lambda}$ . To see this, let  $Y_{j,i}$  denote the  $j$ -th order statistic from population  $i$  (all populations having the exponential distribution). Then  $Y_{j,i}^{1/\lambda}$  is distributed as the  $j$ -th order statistic from  $G_\lambda$  and

$$\begin{aligned}
 P_{G_\lambda} \{CS|R\} &= P \left\{ \frac{Y_{1,1}^{1/\lambda}}{\max_{1 \leq i \leq k} Y_{j,i}^{1/\lambda}} \geq c_\lambda \right\} \\
 &= P \left\{ \frac{Y_{j,1}}{\max_{1 \leq i \leq k} Y_{j,i}} \geq (c_\lambda)^\lambda \right\}
 \end{aligned}
 \tag{3.1}$$

<sup>†</sup> Private communication with James Esary.

or  $c = (c_\lambda)^\lambda$ .

If, in addition to the assumptions of Section 2 (see Theorem 2.1), we assume that a)  $F_{[1]}(x) \geq F_{[i]}(x)$  for all  $x \geq 0$ ,  $i = 1, 2, \dots, k$  and, b)  $G_\lambda \prec_* F_{[1]}$  for all  $i = 1, 2, \dots, k, \lambda > 1$  then we can obtain an upper bound on the probability of selecting the "worst" population, i.e.,

$$P\{\text{Selecting } \Pi_{[1]} | R\} \leq \int_0^\infty [G_j(\frac{x}{c})]^{k-1} dG_j(x) \quad (3.2)$$

where  $c$  is chosen so that

$$P\{CS|R\} \geq \int_0^\infty [G_j(\frac{x}{c})]^{k-1} dG_j(x) = P^* .$$

Clearly, the upper bound is an increasing function of  $\lambda$  for  $\lambda \geq 1$ .

Now we describe the relative performance for small sample size ( $n = 15$ ) for procedures  $R$  and  $R_1$  (the Rizvi-Sobel procedure - see (4.8)) using Monte Carlo techniques. For this purpose we chose gamma and Weibull distributions with densities

$$\text{gamma, } \frac{e^{-x/\theta_1}}{\theta_1 \Gamma(r)} \left(\frac{x}{\theta_1}\right)^{r-1}, \quad i = 1, 2$$

$$\text{Weibull, } e^{-(x/\theta_1)^r} \frac{r}{\theta_1} \left(\frac{x}{\theta_1}\right)^{r-1}, \quad i = 1, 2 .$$

Based on 5,000 simulations, we computed  $P\{CS|R\}$ ,  $E(S|R)$  and  $P\{CS|R_1\}$ ,  $E(S|R_1)$ . ( $E(S|R)$  is the expected size of the selected subset using procedure  $R$ .) These values are given below:

Monte Carlo Comparisons<sup>†</sup> of R and R<sub>1</sub>

P\* = .90, k = 2, n = 15

|                       | Gamma<br>r = 1<br>$\theta_1 = 1, \theta_2 = 2$ | Gamma<br>r = 5<br>$\theta_1 = 2, \theta_2 = 3$ | Weibull<br>r = 2<br>$\theta_1 = 1, \theta_2 = 2$ |
|-----------------------|--|--|--|
| P{CS R}               | .993   | 1.000  | .985   |
| E(S R)                | 1.47   | 1.88   | 1.85   |
| P{CS R <sub>1</sub> } | .997   | 1.000  | .939   |
| E(S R <sub>1</sub> )  | 1.64   | 1.96   | 1.76   |

<sup>†</sup> Computations performed by David Stanford of the Operations Research Center, University of California, Berkeley.

#### 4. EFFICIENCY OF PROCEDURE R UNDER SLIPPAGE CONFIGURATIONS

We consider slippage configurations  $F_{[i]}(x) = F\left(\frac{x}{\delta}\right)$ ,  $i = 1, 2, \dots, k-1$  and  $F_{[k]}(x) = F(x)$ ,  $0 < \delta < 1$ . We obtain asymptotic expressions for the probability of a correct selection and the expected size of the selected subset for procedure R and for two other procedures.

Using our previous notation

$$\begin{aligned} P\{CS|R\} &= P\left\{T_{j,k} \geq c \max_{1 \leq r \leq k} T_{j,r}\right\} \\ &= P_{H_{j,k}}\left\{X_{j,k} \geq c \max_{1 \leq r \leq k} \delta X_{j,r}\right\} \\ &\geq P_{G_j}\left\{Y_{j,k} \geq c\delta \max_{1 \leq r \leq k} Y_{j,r}\right\} \end{aligned} \quad (4.1)$$

where  $Y_{j,r}$ ,  $r = 1, 2, \dots, k$  are i.i.d. with the cdf  $G_j(y)$ . From (2.11), we obtain

$$P\{CS|R\} \approx \int_{-\infty}^{\infty} \phi^{k-1} \left( \frac{x}{c\delta} + \frac{(1-c\delta)\xi_\alpha f(\xi_\alpha)\sqrt{n}}{c\delta\sqrt{\alpha}\bar{\alpha}} \right) d\phi(x). \quad (4.2)$$

Note that the probability of a correct selection is a monotone decreasing function of  $\delta$ . For the slippage configuration

$$E(S|R) = P\{CS|R\} + (k-1) P\left\{T_{j,1} \geq c \max_{i \neq 1} T_{j,i}\right\} \quad (4.3)$$

$$\begin{aligned} &P\left\{T_{j,1} \geq c \max_{i \neq 1} T_{j,i}\right\} \\ &= P\left\{T_{j,1}^{-\delta\xi_\alpha} \geq c \max_{2 \leq i \leq k-1} (T_{j,i}^{-\delta\xi_\alpha}), T_{j,k}^{-\xi_\alpha + \xi_\alpha - \delta\xi_\alpha} + c\delta\xi_\alpha - \delta\xi_\alpha\right\} \quad (4.4) \\ &\approx \int_{-\infty}^{\infty} \phi\left(\frac{\delta x}{c} - \frac{f(\xi_\alpha)\xi_\alpha(1-\frac{\delta}{c})\sqrt{n}}{\sqrt{\alpha}\bar{\alpha}}\right) \phi^{k-2}\left(\frac{x}{c} - \frac{\xi_\alpha f(\xi_\alpha)(1-\frac{1}{c})\sqrt{n}}{\sqrt{\alpha}\bar{\alpha}}\right) d\phi(x). \end{aligned}$$

Setting  $k = 2$ , we have

$$E(S|R) - P\{CS|R\} \approx \Phi \left( - \frac{f(\xi_\alpha) \xi_\alpha \left(1 - \frac{\delta}{c}\right) \sqrt{n}}{\sqrt{\alpha} \bar{\alpha} \sqrt{1 + \left(\frac{\delta}{c}\right)^2}} \right). \quad (4.5)$$

Setting the right-hand side of (4.5) equal to  $\epsilon$ ,  $-f(\xi_\alpha) \xi_\alpha \left(1 - \frac{\delta}{c_n}\right) \sqrt{n} = \sqrt{\alpha} \bar{\alpha} \sqrt{1 + \left(\frac{\delta}{c_n}\right)^2} \Phi^{-1}(\epsilon)$ , where we have put  $c = c_n$ . Now using  $c_n \approx 1 - \frac{C}{\sqrt{n}}$

(from (2.14))

$$-f(\xi_\alpha) \xi_\alpha \left(1 - \delta \frac{\sqrt{2C\delta}}{\sqrt{n}}\right) \sqrt{n} = \Phi^{-1}(\epsilon) \sqrt{\alpha} \bar{\alpha} \sqrt{1 + \delta^2 \left(1 + \frac{2\sqrt{2C}}{\sqrt{n}} + \frac{2C^2}{n}\right)} \quad (4.6)$$

from which, keeping terms of order  $\sqrt{n}$ , we obtain

$$n_R(\epsilon) \approx \frac{[-\sqrt{\alpha} \bar{\alpha} \Phi^{-1}(\epsilon) (1 + \delta^2)^{1/2} + \sqrt{2C\delta} \xi_\alpha f(\xi_\alpha)]^2}{\xi_\alpha^2 f^2(\xi_\alpha) (1 - \delta)^2}. \quad (4.7)$$

#### Comparison with Rizvi - Sobel Procedure

Rizvi and Sobel (1967) propose and investigate a distribution-free procedure,  $R_1$ , for the quantile selection problem.

$R_1$ : Select population  $\Pi_1$  iff

$$T_{j,i} \geq \max_{1 \leq r \leq k} T_{j-a,r} \quad (4.8)$$

where  $a$  is the smallest integer with  $1 \leq a \leq j - 1$  for which

$$\inf_{\Omega} P\{CS|R_1\} \geq P^* \quad (4.9)$$

is satisfied.

A disadvantage of this procedure is that for any given  $\alpha$  and  $k$  a value

of  $a \leq j - 1$  may not exist for some pairs  $(n, P^*)$ . However if  $P^*$  is chosen not greater than some function  $P_1(n, \alpha, k)$  where  $1/k < P_1 < 1$ , then a value of  $a \leq j - 1$  does exist that satisfies (4.9). Rizvi and Sobel compare the efficiency of this procedure relative to several competing procedures under translation configurations.

We discuss the asymptotic probability of a correct selection using their procedure under the slippage configuration.

$$\begin{aligned}
 P\{CS|R_1\} &= P\left\{T_{j,k} \geq \max_{1 \leq r \leq k-1} T_{j-a,r}\right\} \\
 &= P\left\{\frac{(T_{j,k} - \xi_\alpha)\sqrt{n}}{\sqrt{\alpha\bar{\alpha}}} f(\xi_\alpha) \geq \max_{1 \leq r \leq k-1} \left(\frac{T_{j-a,r} - \delta\xi_\alpha + (\delta-1)\xi_\alpha}{\delta\sqrt{\alpha\bar{\alpha}}}\right) \delta\sqrt{n} f(\xi_\alpha)\right\} \\
 &\approx \int_{-\infty}^{\infty} \phi^{k-1} \left(\frac{x}{\delta} + \frac{\gamma}{\sqrt{\alpha\bar{\alpha}}} - \frac{(1-\delta)\xi_\alpha\sqrt{n}}{\delta\sqrt{\alpha\bar{\alpha}}} f(\xi_\alpha)\right) d\phi(x).
 \end{aligned} \tag{4.10}$$

The derivation above uses the fact that

$$\frac{T_{j-a,r} - \delta\xi_\alpha}{\sqrt{\alpha\bar{\alpha}}} \sqrt{n} \frac{1}{\delta} f(\xi_\alpha) \xrightarrow{\text{law}} N\left(\frac{-\gamma}{\sqrt{\alpha\bar{\alpha}}}, 1\right)$$

where  $\gamma/\sqrt{n} = a/(n+1)$ . (See Lemma 2 of Rizvi and Sobel (1967).)

Similarly

$$E(S|R_1) - P\{CS|R_1\} \approx$$

$$\int_{-\infty}^{\infty} \phi^{k-2} \left(\delta x + \frac{\gamma}{\sqrt{\alpha\bar{\alpha}}} - \frac{(1-\delta)\xi_\alpha\sqrt{n}}{\sqrt{\alpha\bar{\alpha}}} f(\xi_\alpha)\right) \phi^{k-2} \left(x + \frac{\gamma}{\sqrt{\alpha\bar{\alpha}}}\right) d\phi(x). \tag{4.11}$$

Setting  $k = 2$ , we have from (4.11)

$$E(S|R_1) - P\{CS|R_1\} \approx \phi\left(\frac{\gamma(1-\delta)\xi_\alpha + f(\xi_\alpha)\sqrt{n}}{\sqrt{\alpha\bar{\alpha}}\sqrt{1+\delta^2}}\right). \quad (4.12)$$

Equating the right-hand side of (4.12) to  $\epsilon$ , we obtain

$$n_{R_1}(\epsilon) \approx \frac{[\sqrt{\alpha\bar{\alpha}}\sqrt{1+\delta^2}\phi^{-1}(\epsilon) - \gamma]^2}{(1-\delta)^2[\xi_\alpha f(\xi_\alpha)]^2}. \quad (4.13)$$

For the slippage configuration above we define the asymptotic relative efficiency  $ARE(R, R_1; \delta)$  of  $R$  relative to  $R_1$  to be the limit as  $\epsilon \rightarrow 0$  of the ratio of  $n_{R_1}(\epsilon)$  to  $n_R(\epsilon)$ .

$$\begin{aligned} ARE(R, R_1; \delta) &= \lim_{\epsilon \rightarrow 0} \frac{n_{R_1}(\epsilon)}{n_R(\epsilon)} \\ &= \lim_{\epsilon \rightarrow 0} \frac{[\sqrt{\alpha\bar{\alpha}}\sqrt{1+\delta^2}\phi^{-1}(\epsilon) - \gamma]^2}{\left[\sqrt{\alpha\bar{\alpha}}\phi^{-1}(\epsilon)\sqrt{1+\delta^2} + \sqrt{2\delta C}\xi_\alpha f(\xi_\alpha)\right]^2} \\ &= 1. \end{aligned} \quad (4.14)$$

Using the fact that  $F < G$  implies

$$\xi_\alpha f(\xi_\alpha) \geq \eta_\alpha g(\eta_\alpha), \quad \text{we see that}$$

$$P_F\{CS|R_1\} \geq P_G\{CS|R_1\} \quad (4.15)$$

and

$$E_F(S|R_1) - P_F\{CS|R_1\} \leq E_G(S|R_1) - P_G\{CS|R_1\} \quad (4.16)$$

where both (4.15) and (4.16) are asymptotically true as  $n \rightarrow \infty$  for the slippage configuration.

#### Comparison with Gupta Procedure

Gupta (1963) gave a selection procedure for gamma populations with densities

$$\frac{1}{\Gamma(r)\theta_1^r} \exp\left(-\frac{x}{\theta_1}\right) \left(\frac{x}{\theta_1}\right)^{r-1}, \quad x > 0, \theta_1 > 0, \quad i = 1, 2, \dots, k.$$

This procedure,  $R_2$ , based on the means of sample size  $n$  from each of the  $k$  populations is:

$R_2$ : Select the population corresponding to the observed mean  $\bar{x}_i$  iff

$$\bar{x}_i \geq b \max_{1 \leq j \leq k} \bar{x}_j \quad (4.17)$$

where  $b$  is the largest constant ( $0 < b \leq 1$ ) chosen so that  $P\{CS|R_2\} \geq P^*$ .

Letting  $v = 2nr$ , it is shown that  $\log_e b \approx -d\sqrt{\frac{2}{v-1}}$  where  $d$  is independent of  $n$  and satisfies

$$\int_{-\infty}^{\infty} \phi^{k-1} (x+d) d\phi(x) = P^* \quad (4.18)$$

Assume that the ranked  $\theta_i$ 's have the slippage configuration  $\theta_{[i]} = \delta\theta_{[k]}$ ,

$0 < \delta < 1$ ,  $i = 1, 2, \dots, k-1$ . Then

$$E(S|R_2) - P\{CS|R_2\} \approx (k-1) \int_{-\infty}^{\infty} \phi^{k-2} \left(x - \frac{\log b}{\sqrt{2/(v-1)}}\right) \phi \left(x - \frac{(\log b/\delta)}{\sqrt{2/(v-1)}}\right) d\phi(x) \quad (4.19)$$

so that for  $k = 2$

$$E(S|R_2) - P(CS|R_2) \approx \phi\left(\frac{-\log(b/\delta)}{\sqrt{\frac{2}{v-1}}\sqrt{2}}\right) \quad (4.20)$$

Setting the right-hand side of (4.20) equal to  $\epsilon$  and solving for  $n = n_{R_2}(\epsilon)$

$$n_{R_2}(\epsilon) \approx \frac{[2\phi^{-1}(\epsilon) - \sqrt{2}]^2}{2r(\log \delta)^2} \quad (4.21)$$

$$A R E(R, R_2; \delta) \approx \lim_{\epsilon \rightarrow 0} \frac{n_{R_2}(\epsilon)}{n_R(\epsilon)} \quad (4.22)$$

$$= \frac{2(1-\delta)^2 [\xi_\alpha f(\xi_\alpha)]^2}{r(\log \delta)^2 \alpha \bar{\alpha}(1+\delta^2)}$$

$$\text{for } r \geq 1, \quad A R E(R, R_2; \delta) \geq \frac{2(1-\alpha)^2 (1-\alpha)^2 [-\log(1-\alpha)]^2}{r(\log \delta)^2 \alpha \bar{\alpha}(1+\delta^2)}$$

$$\text{Also } \lim_{\delta \uparrow 1} A R E(R, R_2; \delta) = \frac{[\xi_\alpha f(\xi_\alpha)]^2}{r \alpha \bar{\alpha}} \quad (4.23)$$

$$\text{so that } \lim_{\delta \uparrow 1} A R E(R, R_2; \delta) \geq \frac{(1-\alpha)^2 (-\log(1-\alpha))^2}{\alpha(1-\alpha)}, \quad (\text{letting } r = 1).$$

$$\text{For } \alpha = \frac{1}{2} \text{ and } r = 1, \lim_{\delta \uparrow 1} A R E(R, R_2; \delta) = [\log 2]^2 = .493.$$

5. SELECTION WITH RESPECT TO THE MEDIAN FOR DISTRIBUTIONS  $r$ -ORDERED WITH RESPECT TO A SPECIFIED DISTRIBUTION  $G$

We consider selection procedures with respect to the median for distributions  $F_i$  which have lighter tails than a specified distribution  $G$ . We say that  $F_i$  has lighter tails than  $G$  if  $F_i$  centered at its median,  $\Delta_i$ , is  $\leftarrow_r$ -ordered with respect to  $G$  ( $G(0) = \frac{1}{2}$ ) and  $\frac{d}{dx} F_i(x + \Delta_i)|_{x=0} \geq \frac{d}{dx} G(x)|_{x=0}$ . Here we are following an ordering proposed by Doksum (1967). (See van Zwet (1964), Lawrence (1966).)

We wish to select a subset of the  $k$  populations containing the population with the largest median  $\Delta_{[k]}$ . The selection rule, we propose, is in terms of the sample medians. We use the same notation as in Section 2. The rule,  $R_3$ , is:

$R_3$  : Select  $\pi_i$  iff

$$T_{j,i} \geq \max_{1 \leq r \leq k} T_{j,r} - D, \quad j \leq \frac{n}{2} < j + 1 \quad (5.1)$$

and  $D$  is chosen to satisfy

$$\inf_{\Omega_1} P\{CS|R_3\} = P^* \quad (5.2)$$

where  $\Omega_1$  is set of all  $k$ -tuples  $(F_1, F_2, \dots, F_k)$  satisfying assumptions given above.

Now, we state and prove a theorem related to the infimum of probability of a correct selection when rule  $R_3$  is used. Let  $F_{[i]}(x)$  denote the distributions with median  $\Delta_{[i]}$ ,  $i = 1, 2, \dots, k$ .

Theorem 5.1:

If  $F_{[i]}(x) \geq F_{[k]}(x)$ , for all  $x$ ,  $G(0) = \frac{1}{2}$  and  $G^{-1}F_{[k]}(x + \Delta_{[k]})$  is nondecreasing (nonincreasing) in  $x \geq 0$  ( $x \leq 0$ ) and

$$\frac{d}{dx} F_{[k]}(x + \Delta_{[k]}) \Big|_{x=0} \geq \frac{d}{dx} G(x) \Big|_{x=0} > 0,$$

then

$$\inf_{\Omega_1} P\{CS|R_3\} = \int_{-\infty}^{\infty} G_j^{k-1}(t+D) dG_j(t)$$

where  $G_j$  is defined as before.

Proof:

By stochastic ordering of the order statistics, we have

$$P\{CS|R_3\} \geq \int_{-\infty}^{\infty} H_{j,k}^{k-1}(t+D) dH_{j,k}(x) \quad (5.3)$$

$$= P\{X_{j,k} \geq \max_{1 \leq r \leq k-1} X_{j,r} - D\}$$

$$= P\{X_{j,k} - \Delta_{[k]} \geq \max_{1 \leq r \leq k-1} (X_{j,r} - \Delta_{[k]}) - D\}$$

where  $X_{j,1}, X_{j,2}, \dots, X_{j,k}$  are i.i.d.r.v. with distribution  $H_{j,k}(x) \equiv B_{j,n} F_{[k]}(x)$ .

Let  $\phi(x) = G^{-1} F_{[k]}(x + \Delta_{[k]}) = G_j^{-1} M(x)$  where  $M \equiv B_{j,n} F_{[k]}(x + \Delta_{[k]})$  is the distribution of  $X_{j,r} - \Delta_{[k]}$ . Note that  $\phi(X_{j,r} - \Delta_{[k]}) = Y_{j,r}$  has distribution  $G_j$ . Now  $\frac{\phi(x)}{x} \uparrow$  in  $x \geq 0$ ,  $\frac{\phi(x)}{x} \downarrow$  in  $x \leq 0$  and  $\phi'(0) > 1$  imply

$$\frac{\phi\left(\max_{1 \leq r \leq k-1} (X_{j,r} - \Delta_{[k]})\right) - \phi(X_{j,k} - \Delta_{[k]})}{\max_{1 \leq r \leq k-1} (X_{j,r} - \Delta_{[k]}) - (X_{j,k} - \Delta_{[k]})} \geq 1. \quad (5.4)$$

To see (5.4) we need only verify that

$$\frac{\phi(y) - \phi(x)}{y - x} \geq 1 \text{ for } y \geq x.$$

Note that  $\phi'(0) = \frac{f_{[k]}(\Delta_{[k]})}{g(0)} \geq 1$  where  $f_{[k]}(g)$  is the density of  $F_{[k]}(G)$ .

Case 1:

$$(0 < x < y) \frac{\phi(x)}{x} \uparrow \text{ in } x \geq 0 \text{ and } \phi'(0) \geq 1$$

imply

$$\frac{\phi(y) - \phi(x)}{y - x} - \frac{\phi(x)}{x} \geq 1.$$

Case 2:

$(x < y < 0)$  follows from  $\frac{\phi(x)}{x} \uparrow$  in  $x \leq 0$  and  $\phi'(0) \geq 1$ .

Case 3:

$(x < 0 < y)$ . In this case  $\phi(x) \leq x$  and  $\phi(y) \geq y$

imply

$$\frac{\phi(y) - \phi(x)}{y - x} \geq 1.$$

We have, by (5.4)

$$\max_{1 \leq r \leq k-1} Y_{j,r} - Y_{j,k} \geq \max_{1 \leq r \leq k-1} X_{j,r} - X_{j,k} \quad (5.5)$$

which implies

$$P\left\{Y_{j,k} \geq \max_{1 \leq r \leq k-1} Y_{j,r} - D\right\} \leq P\left\{X_{j,k} \geq \max_{1 \leq r \leq k-1} X_{j,r} - D\right\} \quad (5.6)$$

which proves the result.

Note that we need not have restricted attention only to medians.

## 6. SELECTION WITH RESPECT TO THE MEANS

Let  $\mu_i$  be the mean of the distribution  $F(x; \mu_i)$ ,  $i = 1, 2, \dots, k$  and assume

$$(a) \quad F(x; \mu_{[i]}) \geq F(x; \mu_{[k]}) \quad \text{for } i = 1, 2, \dots, k-1 \text{ and all } x.$$

$$(b) \quad F(x; \mu_{[i]}) \underset{c}{<} G(x) = 1 - e^{-x} \quad \text{for } i = 1, 2, \dots, k.$$

Note that by assumption (b) we are confining attention to the so-called IFR class of distributions. It will also be convenient to assume  $F(0; \mu_{[i]}) = 0$  for all  $i$ .

Let  $\bar{x}_i = \sum_{j=1}^n x_{ij}/n$ , where  $x_{ij}$  is the  $j$ -th observation in a random sample of size  $n$  from  $\Pi_i$ . Let  $K_i(x) = K(x; \mu_i)$  be the distribution of  $\bar{x}_i$ . Then if  $K_{[i]}(x) = K(x; \mu_{[i]})$

$$K_{[i]}(x) \geq K_{[k]}(x) \quad \text{for } i = 1, 2, \dots, k-1 \text{ and all } x. \quad (6.1)$$

$$K_{[i]} \underset{c}{<} G \quad \text{for } i = 1, 2, \dots, k. \quad (6.2)$$

(6.1) is an immediate consequence of (a) while (5.2) follows from (b) and the closure of IFR distributions under convolution (see Barlow, Marshall and Proschan (1963)).

If we are interested in selecting a subset containing the population with the largest  $\mu_{[k]}$ , we use the rule

$R_4$  : Select population  $\Pi_i$  iff

$$\bar{x}_i \geq c' \max_{1 \leq l \leq k} \bar{x}_l.$$

It follows that

Theorem 6.1:

$$P\{CS|R_4\} \geq \int_0^{\infty} [G(\frac{x}{c'})]^{k-1} dG(x)$$

where  $G(x) = 1 - e^{-x}$ .

The proof is the same as for Theorem 2.1. The disadvantage is that the right-hand side of the inequality is independent of  $n$ . However, by restricting the class of distributions to the gamma family we can obtain a lower bound which depends on  $n$ .

#### Application to the Selection of gamma Populations

Let us consider  $k$  populations with densities

$$\lambda_i^\alpha x^{\alpha-1} e^{-\lambda_i x} / \Gamma(\alpha), \quad x \geq 0, \quad \lambda_i > 0, \quad i = 1, 2, \dots, k.$$

Assume that  $\alpha \geq 1$ , but otherwise unknown. This implies that the distributions are IFR. We are interested in selecting the population with the largest (smallest) value,  $\lambda_{[k]}$  ( $\lambda_{[1]}$ ), based on an independent sample of size  $n$  from each of the  $k$  populations. Note that  $\mu_{[i]} = \alpha/\lambda_{[i]}$  for  $i = 1, 2, \dots, k$ . The subset selection rule based on the sample means,  $\bar{x}_i$ ,  $i = 1, 2, \dots, k$ , is  $R_4$  as before.

Let  $G^{(\alpha)}$  denote a gamma distribution with parameter  $\alpha$ . Since  $\sum_{j=1}^n x_{ij}$  is distributed as a gamma random variable with distribution,  $G^{(n\alpha)}$  it follows from a result of van Zwet (1964) that

$$G^{(n\alpha)} \prec_c G^{(n)}$$

when  $\alpha \geq 1$ . It follows that in this case

$$P\{CS|R_4\} > \int_0^{\infty} [G^{(n)}(\frac{x}{c'})]^{k-1} dG^{(n)}(x) . \quad (6.3)$$

The constant  $c'$  is determined by

$$\int_0^{\infty} [G^{(n)}(\frac{x}{c'})]^{k-1} dG^{(n)}(x) = P^* . \quad (6.4)$$

The values of  $c'$  are tabulated in Gupta (1963). It should be pointed out that for selecting the population with the smallest  $\lambda$ , the rule can be modified to:

$R_5$  : Select population  $\Pi_1$  iff

$$\bar{x}_1 \leq d \min_{1 \leq i \leq k} \bar{x}_i$$

where  $d$  is determined by

$$\int_0^{\infty} [1 - G^{(n)}(\frac{x}{d})]^{k-1} dG^{(n)}(x) = P^* .$$

The values of  $d$  are tabulated in Gupta and Sobel (1962).

The shape parameter  $\alpha \geq 1$  need not be the same for all populations. It is only necessary that the distribution of the population,  $\Pi_{[k]}$ , with the largest mean be stochastically larger than the others.

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