THE EFFECT OF INITIAL DISPLACEMENTS ON PROBLEMS OF LARGE DEFLECTION AND STABILITY

P. V. MARÇAL

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by

P. V. Marçal[#]

Division of Engineering
BROWN UNIVERSITY
Providence, Rhode Island

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[#]Assistant Professor of Engineering, Brown University.

ABSTRACT

A finite element approach is used to obtain equations for large displacement and stability analysis via the principle of virtual work. The equations are then written in incremental form. It is found that the inclusion of the nonlinear strain displacement terms results in two additional 'initial stiffness' matrices. The first is the well known 'initial stress' matrix. An alternate general method is proposed for the derivation of the 'initial stress' matrix. The second matrix is a function of previous displacements and is referred to here as the 'initial displacement' matrix. It is obtained as a result of writing the nonlinear strain displacement relations in incremental form. The 'initial displacement' matrix was found to be of the same order as the 'initial stress' matrix but appears not to have been previously recognized in finite element analysis.

Results obtained for a simple truss and an arch problem agreed with results in the literature. Analysis without the 'initial displacement' matrix resulted in an overestimate of the buckling loads by factors of 1.8 and 2 for the truss and arch respectively. A linear eigenvalue analysis resulted in an overestimate of the buckling load by factors of 5 and 2 respectively.

It was concluded that 'initial displacement' stiffness matrices should be included in the finite element analysis of large deformation and instability problems.

NOTATION AND DEFINITIONS

$\{a\}$ = generalized coordinates, coefficients of displacement polynomials = $[\alpha]\{u\}$.

$[B]$ = differential operator that transforms generalized coordinates to increments of strains. $\Delta e_i = [B]\Delta\{a\}$.

$[D]$ = constitutive relation, strain to stress transformation matrix.

$\{e\}$ = strain vector.

$[f(u,x)]$ = displacement strain relation.

$[k^{(o)}]$ = stiffness matrix due to nonlinear strain-displacement relation.

$[k^{(1)}]$ = stiffness matrix due to initial stress.

$\{P\}$ = load vector.

$[T(u,x)]$ = local to global coordinate transformation matrix.

u,v = axial and normal displacements, respectively.

$\{u\}$ = displacement vector, referred to local coordinate system when not preceded by $[T(u,x)]$.

V = volume.

$\{x\}$ = position vector.

∂ = a virtual change.

Δ = incremental change.

$\{\sigma\}$ = stress vector.

(u,x) = function of displacement and geometry.

Subscripts

i = subscript denoting matrix row or column.

$o,1$ = subscript denoting initial and final geometry.

INTRODUCTION

Large displacement analysis by the finite element method was first proposed by Turner et al.[1] 'Initial stress' matrices were suggested to account for the effect of initial stress in truss and plane stress assemblies. Subsequent work on the derivation of the 'initial stress' matrices for other elements were reported by Argyris et al,[2] Gallagher et al[3] and Kapur and Hartz.[4] In a recent work, Martin[5] obtained the 'initial stress' matrix by considering the potential energy of the element. Nonlinear displacement-strain relations from the classical theory of elasticity were used.

This paper examines large displacement element behavior by means of the principle of virtual work. An alternate general method is obtained for finding the 'initial stress' matrix. All previous authors have neglected the contribution of initial displacements to the element stiffness matrices. This paper considers the effect of including the initial displacement terms and shows that it results in 'initial displacement' matrices which are of the same order as the 'initial stress' matrices.

Numerical results are then presented to justify the introduction of the new 'initial displacement' stiffness matrix.

PRINCIPLE OF VIRTUAL WORK

The equations for large deformation analysis are derived by the principle of virtual work. The general equations are considered without regard to subscription for elements and nodal points. The subdivision and expression of the formulation into finite elements and assembly into large master stiffness and other matrices follow the usual procedure.

In general the strain in the element may be described as a nonlinear function of the current geometry, previous displacements and the displacement

at the nodes $\{u\}$.

$$\{e\} = f(u,x) \tag{1}$$

The strain due to a virtual displacement $\partial\{u\}$ is

$$\partial\{e\} = [\frac{\partial f}{\partial u}(u,x)][T(u,x)]\partial\{u\} \tag{2}$$

where the transformation matrix $[T(u,x)]$ has been included to indicate transformation from local to global coordinate systems. The brackets (u,x) are used to indicate the dependence of the matrix on current nodal displacements and geometry respectively.

By the principle of virtual work

$$\int_V \partial\lfloor u\rfloor[T(u,x)]^T [\frac{\partial f}{\partial u}(u,x)]^T\{\sigma\}dV = \partial\lfloor u\rfloor\{P\} \tag{3}$$

so that

$$\int_V [T(u,x)]^T [\frac{\partial f}{\partial u}(u,x)]^T\{\sigma\}dV = \{P\} \tag{4}$$

where $\{P\}$ is the load vector at the nodes and should be understood to include the effect of uniform loading.

This is the equation describing the finite deformation behavior of a solid. The dependence of the various matrices on geometry and previous displacements are to be particularly noted.

ITERATIVE SOLUTIONS

Equation (4) has been solved by Bogner et al[6] by the use of energy search procedures. Turner et al[1] and subsequent work summarized by Zienkiewicz[7] however have sought to linearize Eq. (4) and account for the large displacement by iterative solutions for the final geometry. A symmetric stiffness

element based on final geometry was used. It can be seen that, for an elastic structure, this is equivalent to replacing the stress vector by

$$\{\sigma\} = [E] \; [\; \frac{\partial f}{\partial u} \, (u,x)]_1 [T(u,x)]_1 \{u\} \tag{5}$$

where subscript 1 denotes values at the end of the loading. It is the author's opinion that a better approximation would result from

$$\{\sigma\} = \frac{1}{2} [E] \; \left([\; \frac{\partial f}{\partial u} \, (o,x)]_o [T(o,x)]_o + [\; \frac{\partial f}{\partial u} \, (u,x)]_1 [T(u,x)]_1 \right) \{u\} \tag{6}$$

where subscript o denotes initial values.

Thus the linearized iterative solutions may be interpreted as application of integration methods where strain rates at the beginning and end of the integration are used. Equation (6) is a higher order formula and still more refined methods may be introduced which in the limit will tend to the incremental solution.

INCREMENTAL SOLUTIONS

Equation (4) is a nonlinear equation and must be solved by iteration. It is often more advantageous to solve a series of linear equations. This has given rise to the incremental piece-wise linear solutions.

To obtain the appropriate equations, Eq. (4) is subjected to a small perturbation and to first order

$$\int_V \Delta[T(u,x)]^T \; [\; \frac{\partial f}{\partial u} \, (u,x)]^T \{\sigma\} dV + \int_V [T(u,x)]^T \Delta[\; \frac{\partial f}{\partial u} \, (u,x)]^T \{\sigma\} dV$$

$$+ \int_V [T(u,x)]^T \; [\; \frac{\partial f}{\partial u} \, (u,x)]^T \; [D] \; [\; \frac{\partial f}{\partial u} \, (u,x)] \; [T(u,x)] \Delta\{u\} dV \;\; = \;\; \Delta\{P\} \tag{7}$$

where [D] is the linearized stress-strain relation for the increment.

The first term in Eq. (7) gives the so-called 'geometric stiffness' matrix,[2] the second term will be shown to be the 'initial stress' matrix[5] and the third term is the element stiffness matrix which also contains the 'initial displacement' matrix. Following previous works,[3,5] the geometric stiffness is neglected in the present study.

INITIAL STRESS MATRIX

We now show a method for obtaining the 'initial stress' matrix from the second term in Eq. (7). In this approach, separate matrices are first formed for each row of the initial stress vector $\{\sigma\}$.

Hence the matrix term due to the ith column of $[\frac{\partial f}{\partial u}]$ and ith row of $\{\sigma\}$ is given by

$$\int_V \sigma_i [T]^T \, \Delta\{\frac{\partial f}{\partial u}\}_i^T \, dV$$

now

$$\Delta\{\frac{\partial f}{\partial u}(u,x)\}_i^T = \frac{\partial\{\frac{\partial f}{\partial u}(u,x)\}_i^T}{\partial\{a(u,x)\}} \cdot \frac{\partial\{a(u,x)\}}{\partial\{u\}} \Delta\{u\} \qquad (8)$$

By definition of the differential operator $[B]$

$$\{\frac{\partial f}{\partial u}\}_i^T = [\alpha]^T \{B\}_i^T \qquad (9)$$

and also since

$$\frac{\partial\{a\}}{\partial\{u\}} = [\alpha] \qquad (10)$$

Eq. (8) becomes

$$\Delta\{\frac{\partial f}{\partial u}(u,x)\}_i^T = [\alpha]^T [\frac{\partial\{B\}_i^T}{\partial\{a\}}][\alpha] \Delta\{u\} \qquad (11)$$

and the initial stress matrix for stress i is given by

$$\int_V \sigma_i [T]^T \Delta\left\{\frac{\partial f}{\partial u}\right\}_i dV = \int_V \sigma_i [T]^T [\alpha]^T \left[\frac{\partial \{B\}_i^T}{\partial \{a\}}\right] [\alpha][T]\Delta\{u\}dV$$

$$= [k_i^{(1)}]\Delta\{u\} \tag{12}$$

The full 'initial stress' matrix may then be obtained by summing all the terms for all the stresses. It is noted that the 'initial stress' matrix is dependent on the current stress and the current geometry. This procedure yields the same 'initial stress' matrix as the procedures suggested in References 3-5. It is described here for its compactness and generality. For example, it may be used for any element with any degree of nonlinearity included in the strain displacement equations.

INITIAL DISPLACEMENT MATRIX

We consider the effect of initial displacements on the third term of Eq. (7). A second order strain displacement relation (usually of rotation) results in a first order current displacement term in the matrix $[\frac{\partial f}{\partial u}(u,x)]$. If [D] is essentially constant throughout the analysis, pre-multiplication by [D] results in terms of the order of the total stress. Hence the third term in Eq. (7) yields additional terms which are of the same order as the 'initial stress' matrix. These terms in the element stiffness are referred to here as the 'initial displacement' matrix in analogy with the 'initial stress' matrix. As far as the author is aware, the 'initial displacement' matrix has been ignored in all previous matrix finite element analysis. Its effect is, of course, implicitly included in the analysis of Bogner et al.[6]

Example

The initial stress matrix and the element stiffness matrix is derived for

the beam-column. The notation of Reference 5 will be adopted here with the same assumption of linear and cubic displacements in the axial and normal displacements u and v.

$$e = \frac{du}{dx} + \frac{1}{2}\left(\frac{dv}{dx}\right)^2 - y\frac{d^2y}{dx^2} \tag{13}$$

$$u = a_o + a_1x$$

$$v = b_o + b_1x + b_2x^2 + b_3x^3 \tag{14}$$

$$\{a\} = \begin{bmatrix} 1 & & & \\ -\frac{1}{L} & \frac{1}{L} & & \\ & & 1 & \\ & & & 1 \\ \frac{-3}{L^2} & \frac{3}{L^2} & \frac{-2}{L} & \frac{-1}{L} \\ \frac{2}{L^3} & \frac{-2}{L^3} & \frac{1}{L^2} & \frac{1}{L^2} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ v_1 \\ v_2 \\ \theta_1 \\ \theta_2 \end{Bmatrix} = [\alpha]\{u\} \tag{15}$$

An increment of strain is given by

$$\Delta e = \Delta\frac{du}{dx} + \frac{dv}{dx}\Delta\frac{dv}{dx} - y\Delta\frac{d^2v}{dx^2} \tag{16}$$

separating into membrane and bending components.

$$\Delta\{e\} = \begin{bmatrix} 0 & 1 & 0 & \dfrac{dv}{dx} & 2x\,\dfrac{dv}{dx} & 3x^2\,\dfrac{dv}{dx} \\[2ex] 0 & 0 & 0 & 0 & -2y & -6xy \end{bmatrix} \{a\} \qquad (17)$$

$$= [B]\,\Delta\{a\} = [B]\,[\alpha]\,\Delta\{u\}$$

Now only considering the membrane strain effect, i.e., only taking the first column of matrix $[B]$,

$$\frac{\partial\{B\}_m^T}{\partial\{a\}} = \underbrace{\left\{ \begin{array}{c} 0 \\[1ex] 1 \\[1ex] 0 \\[1ex] \dfrac{dv}{dx} \\[1.5ex] 2x\,\dfrac{dv}{dx} \\[1.5ex] 3x^2\,\dfrac{dv}{dx} \end{array} \right\}}_{\partial\{a\}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2x & 3x^2 \\ 0 & 0 & 0 & 2x & 4x^2 & 6x^3 \\ 0 & 0 & 0 & 3x^2 & 6x^3 & 9x^4 \end{bmatrix} \qquad (18)$$

By Eq. (12) the initial stress matrix becomes

$$[k_m^{(1)}] = \int_V \sigma_m [T]^T [\alpha]^T \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2x & 3x^2 \\ 0 & 0 & 0 & 2x & 4x^2 & 6x^3 \\ 0 & 0 & 0 & 3x^2 & 6x^3 & 9x^4 \end{bmatrix} [\alpha][T]\,dV \qquad (19)$$

This is identical with the expression derived by Martin.[5]

The third term gives rise to the element stiffness matrix $[k^{(o)}]$ for elastic behavior and neglecting second order terms, we have the element

stiffness matrix

$$[k] = \int_V [T][\alpha]^T E \begin{bmatrix} 0 & & & & & \\ 0 & 1 & & & & \\ 0 & 0 & 0 & & & \\ 0 & \dfrac{dv}{dx} & 0 & 0 & & \\ 0 & 2x\dfrac{dv}{dx} & 0 & 0 & 4y^2 & \\ 0 & 3x^2\dfrac{dv}{dx} & 0 & 0 & 12xy^2 & 36xy^2 \end{bmatrix} [\alpha][T]dV \qquad (20)$$

There are linear coefficients in displacements in the stiffness matrix $[k^{(o)}]$.
These linear coefficients form the 'initial displacement' matrix.

INSTABILITY

The instability that is sought exists as stationary points in the solution
of Eq. (4). It is only when the incremental equation (7) is used that we
obtain the eigenvalue problem.

From this standpoint, we discern several levels of approximation in the
literature. Martin[5] and Argyris[2] suggested a straightforward use of the
'initial stress' matrix by scaling as in the classical eigenvalue approach.
Gallagher et al[3] recognized the need for the correct evaluation of the stress
terms in the 'initial stress' matrix. This was achieved by iteration. An
improved level of approximation may be achieved by inclusion of the 'initial
displacement' matrix in an incremental large deformation analysis. Unless
the mode of instability is known, the master stiffness equations must be
examined for instability after each increment of load. That is to say
unsymmetric modes may not be obtained from a symmetric pre-buckling
displacement pattern in an incremental solution.

COMPUTER PROGRAM

A general purpose finite element computer program was modified to allow a power sweep for the lowest eigenvalue and eigenvector after each increment of load. The geometry was updated after each increment of load. The analysis was load controlled as opposed to deflection controlled. It was thought that such an approach would allow a search for asymmetric modes in cases with non-proportional loading. For this reason, no results were obtained for the post-buckled region. Stiffness matrices were developed for the truss and beam-column element. The arch was represented by a series of straight beam-column elements.

NUMERICAL EXAMPLES

Results were obtained for a simple truss and an arch example. Figure 1 shows the single-bar structure analyzed by Mallett and Berke.[8] The results from the present theory agree with that of Reference 8. The effect of neglecting the 'initial displacement' matrix is also shown. The linear instability load is included to show the effect of neglecting changes in geometry and stress.

Figure 2 shows an arch tested by Gjelsvik and Bodner.[9] The experimental and predicted buckling loads agree when the analysis is performed with 16 elements. Again, results obtained without the 'initial displacement' matrix demonstrate the need for this matrix.

DISCUSSION

The results for the arch show an unexpected variation with the number of elements. Two factors contribute to this. First of all the nature of the point load can be expected to cause rapid changes of displacements near its point of application. The second factor is that the curved arch is being

approximated by a straight beam column.

Calculations with 32 elements showed yet another effect of using the beam column approximation. This was that the equations of equilibrium showed poor convergence. The cause of this may best be explained by noting that the representation of the correct arch effect by a beam element depends on the axial component of the relative vertical displacements between the nodes. For the element adjacent to the mid-point of the arch, both the slope of the element and the relative vertical displacements are obtained as a result of small differences of large quantities. Hence, appreciable round-off errors can be expected in solutions with small elements. This was the reason for abandoning the analysis with 32 elements.

The choice of a linear axial displacement which resulted in a constant axial stress was not a good one. The second order strain terms resulted in a saw-toothed axial stress distribution along the arch. For the present, the saw-toothed stress distribution was removed by assuming that the stress state throughout each arch element was given by its centroidal value. Though the results are no longer 'consistent' within the principle of virtual work, this stop-gap element allows an investigation of the 'initial displacement' matrix.

Reference 8 only required 4 elements to obtain good results. It is thought that this is due to the element used there which always satisfied axial equilibrium. It may be that, with the use of an improved axial displacement function, the same results could be achieved here with fewer elements.

The results in the above examples demonstrate the need for including the 'initial displacement' matrix in a large deformation analysis. The structures tested, however, are noted for their sensitivity to changes in geometry. It may well be that, in general, initial displacements do not play such a large role.

NOTE ON LARGE DISPLACEMENT ANALYSIS BY ASSUMED MODES

The 'initial displacement' matrix has been obtained here and shown to be as important as the 'initial stress' matrix. This appears to be a new effect in the context of finite element analysis. However, it is important to note that, within the larger framework of the literature on large displacement analysis in continuum mechanics, the role of initial displacements is not new and has always been recognized. It is particularly interesting to regard the finite element method as a special case of the Ritz assumed modes technique in which the modes are written in terms of nodal point displacements. A large body of literature already exists on the use of the Ritz assumed modes technique for the solution of large displacement and instability problems. For example, in a summary of Koiter's work[10] dealing with post-buckling behavior, Budiansky[11] has presented equations which can be seen to include Eqs. (4) and (7) as special cases (from Eqs. (3) and (4) in Reference 11). In general, the equations[11] do not lead to such a simple and clear-cut matrix as the 'initial displacement' matrix found here. The difference between the present work and References 10 and 11 is that as a first approximation, the latter works neglect the effect of displacements prior to instability. This displacement has been shown to be important in the examples considered here.

CONCLUSIONS AND RECOMMENDATIONS

1. A compact and general method has been suggested for finding the 'initial stress' matrix for finite element analysis.

2. Initial displacements were shown to contribute an 'initial displacement' matrix to the stiffness matrix of an element in a large deformation analysis. This matrix was seen to be of the same order as that of the 'initial stress' matrix.

3. Numerical examples were presented which justify the introduction of
    the initial displacement matrix.

4. The effect of including the 'initial displacement' matrix should be
    investigated over a wider variety of elements and material behavior.

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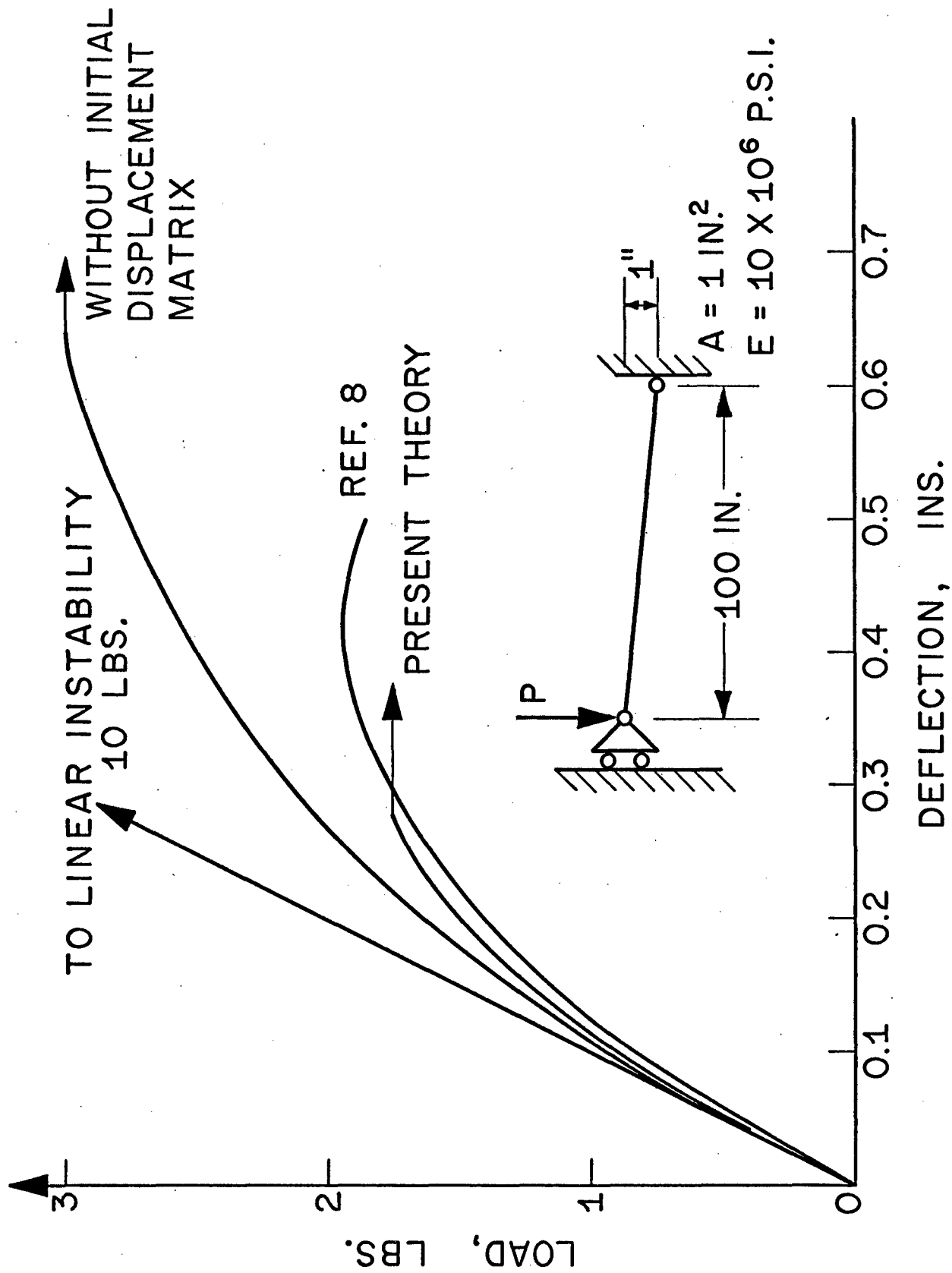Structures</u>, edited by G. Herrmann, Pergamon Press, 1967, pp. 83-106.
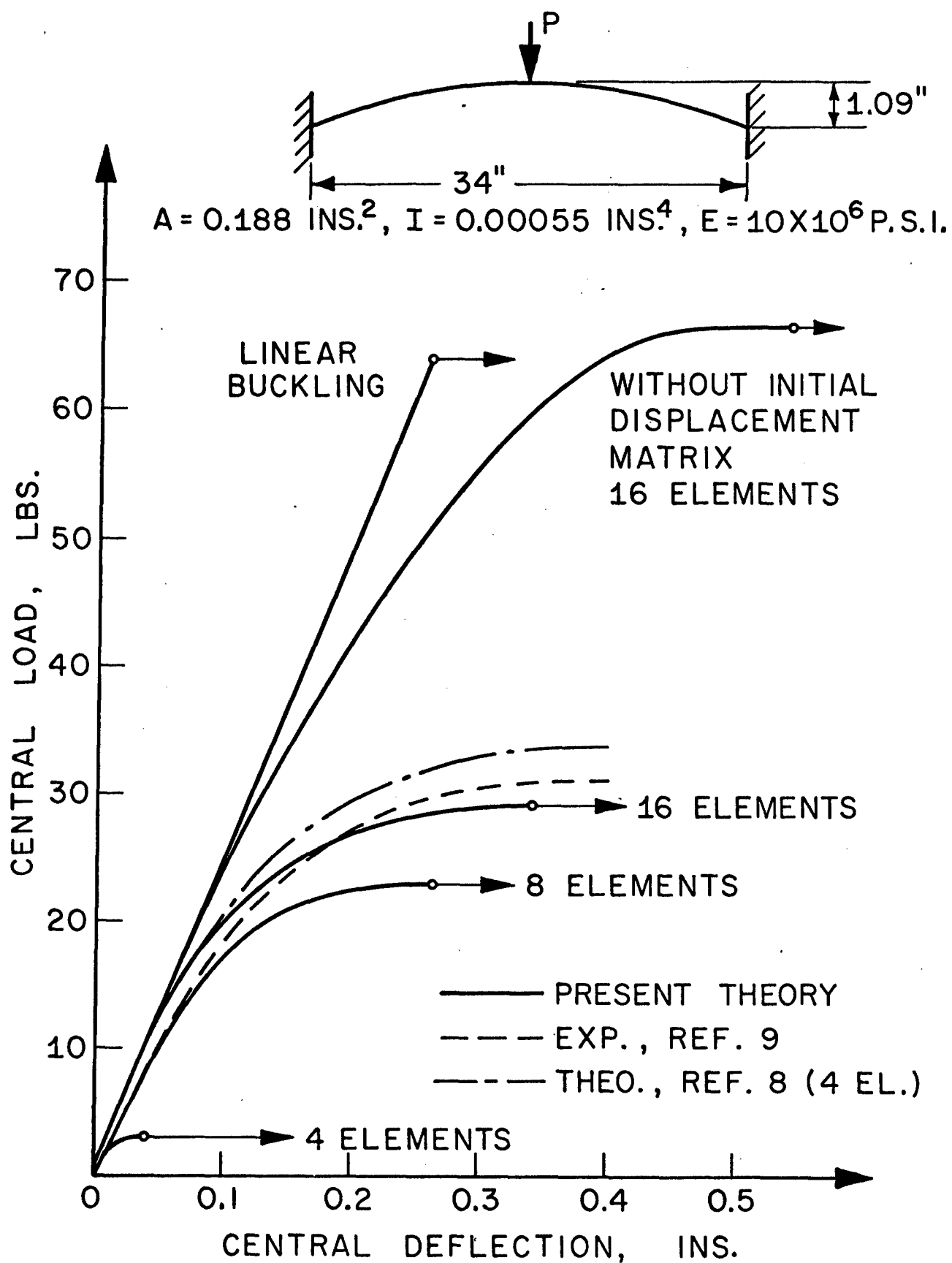
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FIGURE I

FIGURE 2