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# PROBLEMS IN PARTITION THEORY AND RELATED TOPICS

**Final Technical Report**

By
E. M. WRIGHT

NOVEMBER 1967

EUROPEAN RESEARCH OFFICE
United States Army

Contract Number DAJA37 67 C 0020

University of Aberdeen
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## Abstract

The report falls into six sections together with five appendices. The work reported in each of the first five sections is described in detail in the corresponding appendix, which takes the form of a paper accepted for publication by a research periodical.

The first investigation studies rotatable partitions, i.e. those d-dimensional partitions whose representations are invariant under rotation of the axes of coordinates. If d is a power of a prime p, the number of irrotatable partitions is divisible by p. The number of rotatable partitions of small n is small and easily calculated. Consequences include a convenient check of the total number of partitions and a simple proof of the recently discovered fact that the long conjectured form of the generating function of solid partitions is mistaken.

In the second section I find a general identity involving $\xi$-functions. Particular cases of this have applications in partition theory.

The third section introduces a new combinatorial idea, the n-stack, gives generating functions for the number of n-stacks under certain restrictions and under no restrictions and finds asymptotic values for these numbers for large n.

The fourth appendix is a short, semi-expository paper correcting a statement by another author that a particular problem in partition theory is unsolved. To do this I give a new and simple derivation of the behaviour for large n of the number of partitions of n into just k parts. This has a picturesque interpretation in terms of railroad trucks.

The fifth section finds necessary and sufficient conditions that almost all graphs of a given kind on n unlabelled nodes shall be connected. This condition is closely related to that found earlier for the labelled case.

The sixth section reports preliminary investigations into the asymptotic expansions of, and relations between, the number of connected and disconnected graphs of a given kind on n labelled and on n unlabelled nodes when n is large. This work is at present far from complete but looks promising.

Table of contents

## The generating function for the number of solid partitions: rotatable partitions

1. A linear partition of a positive integer is a solution in positive integers of the equation

$$n = \sum_{h \geqslant 1} m_h \qquad (m_h \geqslant m_{h+1}).$$

If $p(n)$ is the number of such partitions, the generating function is easily seen to be

$$1 + \sum_{n=1}^{\infty} p(n) X^n = \prod_{k=1}^{\infty} (1-X^k)^{-1}$$

(see, for example, [12]). A plane partition is a solution of

$$n = \sum_{h,i \geqslant 0} m_{hi} \quad (m_{hi} \geqslant \max(m_{h+1,i}, m_{h,i+1})).$$

If $q(n)$ is the number of such partitions, then

$$1 + \sum_{n=1}^{\infty} q(n) X^n = \prod_{k=1}^{\infty} (1-X^k)^{-k}, \tag{1.1}$$

as was first proved by Macmahon [14]. His proof is lengthy and Chaundy [2] produced a simpler (but still fairly complicated) proof.

Let $r(n)$ be the number of solid partitions, i.e. solutions of

$$(1.2) \qquad n = \sum_{h,i,j} m_{hij}$$

$$m_{hij} \geq \max(m_{h+1,i,j}, m_{h,i+1,j}, m_{h,i,j+1}).$$

It has long been conjectured (but without great confidence) that

$$(1.3) \qquad 1 + \sum_{n=1}^{\infty} r(n)X^n = \prod_{k=1}^{\infty} (1-X^k)^{-k(k+1)/2}.$$

Just before the contract period opened Dr A.O.L. Atkin (then of the Atlas Computer Laboratory at Chilton, Berkshire, England and now of the University of Maryland) wrote to tell me that he and others had proved (1.3) false, essentially by evaluating the coefficients of the smaller powers of X on each side. The coefficients agreed up to $X^5$, but $r(6) = 140$ while the coefficient of $X^6$ in the product on the right-hand side of (1.3) is 141.

Atkin and his collaborators (I.G. Macdonald and J. Mackay) used a computer to calculate the number of d-dimensional partitions of fairly small n. The (d+1)-dimensional partitions into unit parts correspond one-to-one to the d-dimensional partitions into parts of any size. I picked out those partitions which, when expressed diogrammatically, are invariable under rotation of the axes of coordinates. I found that, if

$d = p^t$, where p is a prime, then the number of d-dimensional partitions of any n which are not rotatable is divisible by p. Hence the total number of d-dimensional partitions of n is congruent (mod p) to the number of rotatable partitions. For moderate sized n the rotatable partitions are very few in number and easily identified and counted. For $d = 3$ and $n = 6$ there are only two; for $d = 4$ and $n = 6$ and unit parts there are none.

My result enabled me to check (and find copying errors in) a table of numbers of partitions calculated on a computer by means of a programme written by Mackay. It also provides a very short proof that the product on the right-hand side of (1.3) is not the generating function for 3-dimensional partitions. It will be observed that $d = p^t$ or $d + 1 = p^t$ for every $d \leqslant 13$, so the result is useful for all dimensions up to 13. The result may yield information about the structure of the generating function for r(n).

My method and results (including two tables) are written up in Appendix 1 to this report; this has been accepted for publication in the Journal of the London Mathematical Society.

In [24] I found an expression for $\eta(a,b,c)$, the generating function for the number of solutions of (1.2) subject to the additional conditions that

$$m_{111} \leqslant a,\ m_{121} \leqslant b,\ m_{211} \leqslant c,\ m_{hi1} = 0 \qquad (h+i>3).$$

I did a little further work this year trying to determine $\eta(a,b,c,d)$, the generating function for the number of solutions of (1.2) subject to

$$m_{111} \leqslant a,\ m_{121} \leqslant b,\ m_{211} \leqslant c,\ m_{221} \leqslant d,\ m_{hi1} = 0$$
$$(h>2 \text{ or } i>2).$$

The calculations became very complicated. The function $\eta(a,b,c)$ had been expressed (and I was seeking to express $\eta(a,b,c,d)$) in terms of the function $\xi_a$, where

$$\xi_0 = 1, \quad \xi_a = \prod_{k=1}^{a}(1-X^k)^{-1}, \quad \xi_{-a} = 0 \qquad (a>0).$$

It seemed desirable to devote some study to the function $\xi_a$ and its properties and relations and this and its consequences I describe in sections 2,3 and 4. Otherwise I abandoned work on $\eta(a,b,c,d)$ for the present. While I am still sure that this is worth completing when time permits, it seemed good policy to pursue the more immediately rewarding lines which opened up.

The disproof of (1.3) is not, in this connection, a severe blow. The conjecture had long ceased to be a very plausible one. Again the correct form of the generating function for the number of plane partitions in (1.1) has been known for 50 years but this knowledge has so far been of no help whatever in finding a simple, transparent proof of the result.

2. We have

$$\lim_{X\to 1} (1-X)^a \xi_a = (a')^{-1} \quad (a>0)$$

and so

$$\lim_{X\to 1} \frac{\xi_a \xi_b}{\xi_{a+b}} = \frac{(a+b)!}{a!b!},$$

the binomial coefficient which is also the number of combinations of a+b things taken a at a time. There is thus a correspondence between certain identities involving $\xi$-functions and others involving binomial coefficients.

I managed to find the $\xi$-identities corresponding to certain well-known identities in binomial coefficients. Several turned out to be particular cases of the identity

$$(2.1) \qquad \sum_u X^{u(u-k)} \xi_u \xi_{u-k} \xi_{r-u} \xi_{s-u} \xi_{t-u} \xi_{r+s+t-u-k}^{-1}$$

$$= \frac{\xi_r \xi_{r-k} \xi_s \xi_{s-k} \xi_t \xi_{t-k}}{\xi_{s+t-k} \xi_{t+r-k} \xi_{r+s-k}}.$$

The proof of this and the deduction from it of several results for $\xi$ functions and for binomial coefficients (the last well-known) are given in Appendix 2, which has been accepted for publication in the American Mathematical Monthly.

There remains the question of when an identity involving binomial coefficients can be generalised to one in $\xi$-functions. (The converse question is of course trivial). Answers may well be either trivial or unobtainable, but it is just possible that there may be some more significant results obtainable.

An example of a well-known identity which does not generalise is

$$\binom{nk+t+n}{n} = t \sum_{i=0}^{n} \frac{1}{ik+i+t} \binom{ik+i+t}{i} \binom{(n-i)(k+1)}{n-i} .$$

Taking $n = 2$, I have proved that there are no indices $\lambda_1, \lambda_2, \lambda_3$ such that

$$\frac{\xi_2 \xi_{2k+t}}{\xi_{2k+t+2}} = (1-X^t) \sum_{i=0}^{2} \frac{X^{\lambda_i} \xi_i \xi_{ik+t} \xi_{2-i} \xi_{k(2-i)}}{\xi_{ik+i+t-1} \xi_{(2-i)(k+1)}} .$$

## Stacks

3. If we put $k = 1$ and let $s,t \to \infty$ in the identity (2.1) of section 2, we have

$$\sum_u x^{u(u-1)} \xi_u \xi_{u-1} \xi_{r-u} = \xi_r \xi_{r-1},$$

an identity which can be proved quite simply directly. Attempts to interpret this in combinatorial terms led to a study of structures which I call "stacks". An n-stack is an arrangement of n nodes in rows such that no row overlaps the one below it, thus:-

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I have found the generating functions for $s_r(b,t,n)$, the number of stacks with n nodes, r rows, t nodes in the top row and b in the base and for various similar enumerative functions. In particular,

$$\sum_n s(n)x^n = \sum_r x^r \xi_r \xi_{r-1} . \tag{3.1}$$

Most of this was fairly straightforward, but a more difficult problem was that of finding an asymptotic approximation for s(n) for large n, since the form of the generating function on the right of (3.1) does not lend itself to the usual "circle" method of Hardy and Ramanujan. However, first by manipulation of generating functions and then by an alternative direct "graphical" argument, I proved that

$$q(n) = s(n) + s(.,1,n+1), \tag{3.2}$$

where $s(.,1,n+1)$ is the number of $(n+1)$-stacks with a single node in the top row and $q(n)$ is defined by

$$1 + \sum_{n=1}^{\infty} q(n)X^n = \prod_{k=1}^{\infty} (1-X^k)^{-2}.$$

(This $q(n)$ is different from that of §1.) From (3.2) I could deduce that

$$q(n-1) \leqslant 2s(n) \leqslant q(n).$$

The asymptotic expansion of $q(n)$ is known ([13] and [21]) and so we can deduce that

$$s(n) = 8^{-1}(3^3n^5)^{-1/4} \exp[2\pi\sqrt{(n/3)}]\left\{1+O(n^{-\frac{1}{2}})\right\}.$$

All this work appears in Appendix 3, a paper which has been accepted for publication by the Quarterly Journal of Mathematics.

The reader's attention is drawn to the "Note added at proof stage" at the end of Appendix 3 (pp. 71,72). This refers to information received since the end of the contract period. It shows that, while another author studied structures equivalent to my unrestricted n-stacks in 19.1, his results overlap minimally with mine and his methods not at all. There seems to be some possibility of applications in statistical mechanics.

## Partitions into k parts

4. Let $p_k(n)$ be the number of partitions of n into just k parts and $q_k(n)$ the number into at most k parts. We have $q_k(n) = p_k(n+k)$ so that only one of these two functions need be studied. Again $x^k\xi_k$ and $\xi_k$ are the generating functions of $p_k(n)$ and $q_k(n)$. In a recent article [3] on arrangements, Collins described the problem of determining $p_k(n)$ and $q_k(n)$ as "an unsolved problem in partitions". This is, of course, nonsense, as the form of $p_k(n)$ has been studied at length by Sylvester [19], Glaisher [8] and several others [9,18] including me [20]. But these papers are fairly complicated and most text-books on enumeration describe how to determine $p_k(n)$ only for the first few values of k. It seemed worthwhile to write a short article (Appendix 4, accepted for publication by the Mathematical Gazette) showing how quite simple and elementary methods of partial fractions give the form of $p_k(n)$ as a semi-polynomial and also its asymptotic value for large n. This is perhaps as much exposition as research, but probably worth doing.

## Asymptotic consequences of a relation between generating functions; applications to graph theory

5. In the previous contract year I completed a study of the relation

$$(5.1) \qquad 1 + \sum_{n=1}^{\infty} G_n X^n = \exp\left(\sum_{n=1}^{\infty} g_n X^n\right) \qquad (g_n \geqslant 0)$$

(which is formal if the series diverge) and determined conditions that

$$(5.2) \qquad G_n \sim g_n$$

as $n \to \infty$. I showed that the case in which $g_n = 0$ for an infinite sequence of n is either trivial or can be ignored without loss of generality. Hence we take $g_n > 0$ for all $n > c$. (It follows that $G_n > 0$ for all $n > c$.) The chief result was that the necessary and sufficient condition for (5.2) is that

$$(5.3) \qquad \sum_{s=1}^{n-1} H_s H_{n-s} = o(H_n),$$

where the sequence $\{H_n\}$ is either $\{G_n\}$ or $\{g_n\}$. We can thus confine our investigation in any particular case to one only of the sequences $\{G_n\}$ and $\{g_n\}$, whichever we know most about. I also found sets of sufficient conditions on $H_n$ for (5.3) to hold. These results I published in [22] and [23], of which [22] is the more relevant here.

(5.1) occurs in the equivalent form

$$1 + \sum_{n=1}^{\infty} \frac{F_n}{n!} X^n = \exp\left(\sum_{n=1}^{\infty} \frac{f_n}{n!} X^n\right) \tag{5.4}$$

in the enumerative theory of graphs, where $f_n$ is the number of connected graphs on n labelled nodes which have a particular property and $F_n$ is the number of graphs on n labelled nodes each of whose connected components has this property. We have then found sufficient and necessary conditions that almost all such graphs should be connected.

When he read [22], Dr R.C. Read of the University of the West Indies wrote to me pointing out that my result (5.3) with $H_n = g_n$ meant that the number of those graphs on n labelled nodes which are disconnected is small compared with the number of those which are connected if and only if the number of those which have just two connected components is similarly small. It is interesting that my efforts to get useful conditions from an asymptotic and analytic point of view should have led to a result which had a simple interpretation in graph theory. In Read's interpretation the "only if" condition is obviously trivial and it is the "if" that is interesting. This distinction of depth showed, as one would expect, in my own analytic argument.

In the same letter Dr Read drew my attention to the relationship

$$1 + \sum_{n=1}^{\infty} T_n X^n = \prod_{k=1}^{\infty} (1-X^k)^{-t_n} \tag{5.5}$$

(again formal if series and product diverge) between $t_n$ the number of connected graphs on n unlabelled nodes with a particular property and $T_n$ the number of graphs on n unlabelled nodes each of whose connected components has the same property. He asked if I could establish the conditions under which

$$T_n \sim t_n \tag{5.6}$$

as $n \to \infty$. First I found that we could without loss of generality take $t_n > 0$ (and so $T_n > 0$) for all large enough n; then (after considerable work) that the necessary and sufficient condition for (5.6) is again (5.3) with the sequence $\{H_n\}$ either $\{T_n\}$ or $\{t_n\}$. This has the advantage that all the work on conditions for (5.3) in [22,23] applies without more ado. My proofs and results appear in Appendix 5 which has been accepted for publication in the Journal of the London Mathematical Society. The new result can be interpreted in terms of graphs exactly as Read interpreted my earlier one.

Read remarks that results about unlabelled graphs are usually more interesting than those about labelled graphs but also much more difficult to obtain. The truth of this appeared when I tried to find applications of my new result. In each of the sets of sufficient conditions for (5.3) which I developed in [22,23], there is always one which requires a reasonably steady rate of increase by $H_n$ (in one sense or another). It appears that, in the unlabelled graph case, it may sometimes be more troublesome to prove that this condition is satisfied (though it is very plausible) than that the apparently more stringent conditions are. Instead of taking this any further, however, at this stage, I turned my attention to a more general group of problems which I describe in the next section.

6. I have not worked out fully any of the results (which are in any case preliminary) described in this section. In particular, I have not constructed rigorous proofs nor written them out. One can never be absolutely sure that one is right until this has been done and this section should be read with this caution in mind. None the less the ideas seem interesting enough to be worth reporting.

Towards the end of the contract period the work described in §5 led me to consider the problem of relating $F_n, f_n, T_n$ and $t_n$ when n is large. As stated in §5, I have already found ([22] and Appendix 5 of this report) necessary and sufficient conditions that $F_n \sim f_n$ and that $T_n \sim t_n$, but I have now found that

$$T_n = t_n + T_1 t_{n-1} + T_2 t_{n-2} + \dots . \tag{6.1}$$

We might hope that this would give us an asymptotic expansion of $T_n$ for large n in terms of $t_n, t_{n-1}, \dots$, if the latter are in descending order of magnitude. This is only true if certain conditions are satisfied. The succeeding terms in the expansion do initially get smaller but this may not persist. The later terms are not all so simple in form and there are a large number of them. The point that the first term of an expansion is not an asymptotic approximation merely because the second term is of lower order, but only if the sum of all the other terms is of lower order, was made by Ford and Uhlenbeck [7], but in their examples the result was in fact true; their remark was

just a correct statement of what constitutes rigorous proof. In mine, I can produce a quite simple counter-example which shows that we must satisfy the more stringent condition or our result may in actual fact be false.

While (6.1) is interesting and attractively simple, it is useful to deduce from it that

$$t_n = T_n + \alpha_1 T_{n-1} + \alpha_2 T_{n-2} + \ldots , \tag{6.2}$$

since we are more likely to know $T_n$ than $t_n$. I seem able to find similar results for $F_n$ and $f_n$, viz.

$$F_n = f_n + \binom{n}{1} F_1 f_{n-1} + \binom{n}{2} F_2 f_{n-2} + \ldots$$

and so on.

We need next an asymptotic relationship between $F_n$ and $T_n$. This depends on a famous theorem due to Polya [16], developed and applied by Harary [10], de Bruijn (Chapter 5 of [1]) and others; in its general form it cannot be stated as simply as (6.1) and (6.2), but in special cases it can be used very effectively. The interest of all these relationships lies in the fact that $F_n$ is usually the easiest number to calculate while $f_n$, $T_n$ and $t_n$ are, increasingly in that order, the more interesting both theoretically and for applications.

If we take as a particular example the simple form of graph in which every pair of nodes is, or is not, joined by just one undirected edge (and there are no

slings), we have $F_n = 2^N$ and so $G_n = 2^N/n!$, where $N = n(n-1)/2$. It seems clear that our conditions are satisfied and so we get an asymptotic expansion for $f_n$. I used Polya's theorem to find an asymptotic expansion in the form

$$(6.3) \quad T_n = G_n + \psi_1(n)G_{n-1} + \psi_2(n)G_{n-2} + \dots ,$$

where $\psi_k(n)$ is a polynomial of degree k in n which can be readily calculated by my method for $k = 1,2,3,4,\dots$ . The result $n!T_n \sim 2^N$ was already known [7] and a few weeks ago a paper by Oberschelp [15] appeared which gave the result

$$n!T_n = 2^N\{1+\phi_1(n)2^{-n}+\phi_2(n)2^{-2n}+O(n^5 2^{-5n/2})\}$$

where $\phi_k(n)$ is a polynomial of degree 2k. (I can show that $\phi_k(n)$ is divisible by $\binom{n}{k+1}$. My own corresponding error term deduced from (6.3) would be $O(n^6 2^{-3n})$ and I have not yet discovered why Oberschelp gives the larger one). Using the expansion of $T_n$ given by (6.3) in (6.2), we have an asymptotic expansion of $t_n$.

If we restrict each graph to have just p edges, we may write $F_{np}, f_{np}, T_{np}, t_{np}$ for the numbers corresponding to $F_n, f_n, T_n, t_n$. It looks as if more complicated results of the same kind should be obtainable for $F_{np}, f_{np}, T_{np}, t_{np}$, but I have not gone far into this yet.

In particular, however, if we take the same example as above we have $F_{np} = \binom{N}{p}$ and it is known [7] that $f_{np} \sim F_{np}$ provided that $p > n \log n$, and that Polya has proved that

$$n!T_{np} \sim F_{np} , \tag{6.4}$$

provided $|p-\frac{1}{2}N| = O(n)$. From this it can be deduced that $n!t_{np} \sim F_{np}$. Oberschelp has proved (6.4) under the wider condition that $|p-\frac{1}{2}N| \leqslant Cn^{3/2}$ for suitable C. I seem to be able to do more, viz. to find an asymptotic approximation for the difference $n!T_{np} - F_{np}$ and that under the still wider condition that $|p-\frac{1}{2}N| = O(n^{2-\varepsilon})$ for any positive $\varepsilon$. But all this requires much more work.

The previous investigations which gave first approximations to the particular examples of $f_{np}$, $T_{np}$ and $t_{np}$ above were all undertaken because of the relevance of the results to statistical mechanics [4-7,17] and chemistry [16]. The theory of graphs, although heavily investigated as a fascinating branch of pure mathematics, has many applications [1]. We might expect that the asymptotic theory is likely to find its applications mainly in mathematical physics and expecially in statistical mechanics. Since physicists have shown an interest in the subject, it would seem useful to develop a coherent theory, as general and complete as possible. There appear also to be applications [11] to the theory of logical relations.

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## Appendix 1

# Rotatable Partitions

by E.M. Wright †

[To be published in the Journal of the London Mathematical Society]

1. In what follows all small latin letters denote non-negative rational integers or functions all of whose values are non-negative integers. By a d-dimensional b-restricted partition Y of n, where $d > 0$, $b > 0$, we understand a solution of the equation.

$$n = \sum_{x_1, x_2, \ldots, x_d \geq 0} y(x_1, x_2, \ldots, x_d), \tag{1}$$

where every $y \leqslant b$ and

$$y(x_1, x_2, \ldots, x_d) \geqslant y(x_1', x_2', \ldots, x_d'),$$

whenever $x_i \leqslant x_i'$ for all i. We may take $b = \infty$ when we

† The research reported herein has been sponsored by the European Research Office, United States Army.

shall call the partition unrestricted. The only other case of importance is that in which $b = 1$; such a partition we call a unit partition (more correctly, a partition into units). We write $q(d,b;n)$ for the number of d-dimensional b-restricted partitions of n. If we sum with respect to $x_{d+1}$, we see that

$$q(d+1,1;n) = q(d,\infty;n). \tag{2}$$

2. We require the following lemma.

Lemma. Let p be a prime number, $d = p^t$ and T be a transformation such that $T^d$ is the identity. If S is a finite set closed under T, then the number of members of S not invariant under T is divisible by p.

Corresponding to every member s of S, we construct the set $\sum(s)$, viz.

$$s, Ts, T^2s, \ldots, T^{c-1}s,$$

where c is the least positive integer such that $T^c = s$. Then $c \leqslant d$; let us write $d = uc + v$, where $0 \leqslant v < c$. We have $s = T^d s = T^v (T^c)^u s = T^v s$ and so $v = 0$, by the definition of c. Hence $c \mid d$.

Clearly $\sum(s) \subset S$. Again, if $s' \subset \sum(s)$, then $\sum(s') = \sum(s)$; hence any two sets $\sum(s_1)$ and $\sum(s_2)$ either coincide or are disjoint. We have then all the members of S arranged in disjoint sets $\sum(s_1)$, $\sum(s_2)$, ...

If s is invariant under T, then $c = 1$. For all other s we have $c > 1$ and so, if $d = p^t$, $p \mid c$. Hence all the s not invariant under T are arranged in disjoint sets and the number of members in each of these sets is a

multiple of p. Hence the total number of s not invariant under T is a multiple of p.

We now take the set S to be the set of d-dimensional b-restricted partitions Y of n, and T to be the transformation Y' = TY such that

$$y'(x_1,x_2,\ldots,x_d) = y(x_2,x_3,\ldots,x_d,x_1).$$

The conditions of the lemma are clearly satisfied. Let us call any partition Y which is invariant under T, i.e. one for which

$$y(x_1,x_2,\ldots,x_d) = y(x_2,x_3,\ldots,x_d,x_1)$$

for all sets $x_1,x_2,\ldots,x_d$, a rotatable partition and let q'(d,b;n) denote the number of d-dimensional b-restricted rotatable partitions of n. Then our lemma gives us at once the following theorem.

Theorem. If $d = p^t$, then

$$q(d,b;n) \equiv q'(d,b,n) \pmod{p}.$$

We observe that nothing like (2) is true for the q'.

3. It has long been conjectured that the generating function for $q(d,\infty;n)$, viz.

$$Q_d = Q_d(X) = 1 + \sum_{n=1}^{\infty} q(d,\infty;n)X^n$$

is equal to

$$R_d = R_d(X) = \prod_{k=1}^{\infty} (1 - X^k)^{-\binom{d+k-2}{k-1}} = 1 + r(d,n)X^n,$$

where $\binom{d+k-2}{k-1}$ takes the value 1 for $k = 1$ and otherwise denotes the usual binomial coefficient. For $d = 1$, this conjecture is true and its almost intuitive proof is due to Euler [4]. Macmahon [5] proved the conjecture true for $d = 2$, but neither his proof nor that of Chaundy [2] is at all simple. Attempts to produce a direct (i.e. combinatorial) proof for $d = 2$ have not got very far. Cheema and Gordon [3] found a combinatorial proof that

$$(1 - X)^{-1} \prod_{k=2}^{\infty} (1 - X^k)^{-2} = 1 + \sum_{n=1}^{\infty} q(2,2;n)X^n,$$

but it is not trivial and its further extension looks

difficult. Recently [1] the conjecture has been disproved for $d = 3$, essentially by showing that $q(d,\infty;6) \neq r(d,6)$. The authors of [1] do not give the details of the calculation (which they describe as "more tedious"), but its nature is clear. They have also used a computer to calculate $q(d,1;n)$ for $d \leqslant 8$ and a range of $n$.

The theorem of §2 enables me to give in §5 a very simple proof of the falsehood of the conjecture for $d = p$ or $d = p - 1$. This theorem might also provide a test for any other conjecture. When $d$ is a prime power, the theorem also provides a simple check of the accuracy of computed values of $q(d,1;n)$ and $q(d,\infty;n)$ for fairly small values of $n$.

It is interesting to learn that $R_3(X)$ is not the generating function of $q(3,\infty;n)$ and it would be of some interest to have a more plausible conjecture as to what is the correct generating function. But the case of $q(2,\infty;n)$ shows that it is unlikely that any such conjecture would help us greatly to prove what is the generating function. In that case, our knowledge of the generating function has not enabled us to produce a simple proof or a direct, enumerative proof.

In [6] I showed that the generating function for the number of solutions of (1) for d = 3 subject to

$y(0,0,0) \leqslant a,\ y(1,0,0) \leqslant b,\ y(0,1,0) \leqslant c \qquad (b<a,\ c<a)$

and

$$y(u,v,0) = 0 \qquad (u+v \geqslant 2)$$

is

$$\sum_{u=0}^{b} \sum_{v=0}^{c} \alpha(u,v)\, \zeta_{a+u+v}\, \zeta_{b-u}\, \zeta_{c-v} ,$$

where

$$\zeta_t = \prod_{s=1}^{t} (1 - X^s)^{-1}.$$

Here $\alpha(u,v)$ is a polynomial in X whose term of lowest degree in X is of degree $\frac{1}{2}(u-v)^2 + \frac{3}{2}(u+v) - 1$. The $\alpha(u,v)$ can in theory be calculated from (increasingly elaborate) recurrence relations. In particular, $\alpha(u,v) = \alpha(v,u)$ and

$\alpha(0,0,0) = 1,\ \alpha(0,1) = \alpha(1,0) = -X,\ \alpha(0,v) = 0 \quad (v \geqslant 2).$

I am investigating the next step, in which we allow y(1,1,0) to have positive values, but the work is not simple.

4. The values of $q'(d,1;n)$ for the smaller values of d and n can be readily calculated by enumerating the rotatable unit partitions. This is particularly easy when d is a prime, the most interesting case from our point of view.

The values of $q'(d,\infty;n)$ can be deduced from those for $q'(d,1;n')$ for $n' = 1,2,\ldots,n$, since a rotatable unbounded partition of n can be dissected into suitable unit partitions of n', where $n = \sum n'$.

The values are given in the tables. If $p > 3$, where p is a prime, we have

$$q'(p,1;1) = q'(p,1;p+1) = 1,\quad q'(p,1;2p+1) = \tfrac{1}{2}(p+1),$$

$$q'(p,1;n) = 0 \qquad (2\leqslant n\leqslant p,\ p+2\leqslant n\leqslant 2p,\ 2p+2\leqslant n\leqslant 3p),$$

$$q'(p,1;3p+1) = \tfrac{1}{8}(p^2-1) + 1.$$

Also

$$q'(p,\infty;n) = 1 \qquad (1\leqslant n\leqslant p),$$

$$q'(p,\infty;n) = 2 \qquad (p+1\leqslant n\leqslant 2p),$$

$$q'(p,\infty;2p+1) = \tfrac{1}{2}(p+5),$$

$$q'(p,\infty;n) = \tfrac{1}{2}(p+7) \qquad (2p+2\leqslant n\leqslant 3p),$$

$$q'(p,\infty;3p+1) = \tfrac{1}{8}(d+1)(d+3) + 3.$$

5. Here we compare $r(d,n)$ and $q(d,\infty;n) - q(d+1,\infty;n)$ for two classes of $d$. We take $p$ an odd prime. All congruences are to modulus $p$. We write $S_1(X) \sim S_2(X)$ to denote that

$$S_1(X) - S_2(X) = \sum_{n=0}^{\infty} a_n X^n \equiv \sum_{n=2p+1}^{\infty} a_n X^n .$$

First let us take $d = p$. From §4, we have

$$q(p,\infty;n) \equiv q'(p,\infty;n) = \begin{cases} 1 & (1 \leqslant n \leqslant p), \\ 2 & (p+1 \leqslant n \leqslant 2p). \end{cases}$$

Hence

$$Q_p = 1 + \sum_{n=1}^{\infty} q(p,\infty;n) \sim (1 - X)^{-1}(1 + X^{p+1}).$$

Again

$$\binom{p+k-2}{k-1} \equiv \begin{cases} 0 & (k \not\equiv 1) \\ 1 & (k \equiv 1) \end{cases}$$

and

$$(1 - X^t)^p \equiv 1 - X^{pt}.$$

Hence

$$R_p = \prod_{k=1}^{\infty} (1 - X^k)^{-\binom{p+k-2}{k-1}}$$

$$\sim (1 - X)^{-1}(1 - X^2)^{-p}(1 - X^{p+1})^{-1}$$

$$\sim (1 - X)^{-1}(1 - X^{2p})^{-1}(1 - X^{p+1})^{-1}$$

$$\sim (1 - X)^{-1}(1 + X^{2p})(1 + X^{p+1}) \sim Q_p + X^{2p}$$

and so

$$r(p,2p) \leqq q(p,\infty;2p).$$

Next let $d = p - 1$. We remark that $q(d,\infty;n) = q(p,1;n)$ and that

$$q(p,1;n) \equiv q'(p,1;n) = \begin{cases} 1 & (n=1,\ p+1), \\ 0 & (2 \leqslant n \leqslant p,\ p+2 \leqslant n \leqslant 2p), \end{cases}$$

so that

$$Q_d = Q_{p-1} \sim 1 + X + X^{p+1}.$$

We have

$$R_d = R_{p-1} = \prod_{k=1}^{\infty} (1 - X^k)^{-\binom{p+k-3}{k-1}}.$$

It is easily seen that

$$\binom{p+k-3}{k-1} = \begin{cases} 1 & (k = 1), \\ -1 & (k = 2), \\ 0 & (k \neq 1,2) \end{cases}$$

We have then

$$\begin{aligned} R_{p-1}(X) &\sim (1 - X)^{-1}(1 - X^2)^{-p+1}(1 - X^{p+1})^{-1}(1 - X^{p+2})^{-up+1} \\ &\sim (1 + X)(1 + X^{2p})(1 + X^{p+1})(1 - X^{p+2}) \\ &\sim (1 + X)(1 + X^{p+1} - X^{p+2} + X^{2p}) \\ &\sim 1 + X + X^{p+1} - X^{p+3} + X^{2p} \end{aligned}$$

and so

$$r(p-1,n) \neq q(p-1,\infty;n) \qquad (n = p+3, 2p).$$

Table of $q'(d,1;n)$

| $n \backslash d$ | 3 | 4 | 5 | 7 | 8 | 9 | 11 |
|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 4 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| 5 | 0 | 1 | 0 | 0 | 0 | 0 | 0 |
| 6 | 0 | 0 | 1 | 0 | 0 | 0 | 0 |
| 7 | 2 | 1 | 0 | 0 | 0 | 0 | 0 |
| 8 | 1 | 0 | 0 | 1 | 0 | 0 | 0 |
| 9 | 0 | 2 | 0 | 0 | 1 | 0 | 0 |
| 10 | 2 | 0 | 0 | 0 | 0 | 1 | 0 |
| 11 | 1 | 2 | 3 | 0 | 0 | 0 | 0 |
| 12 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| 13 | 4 | 2 | 0 | 0 | 1 | 0 | 0 |
| 14 | 3 | 0 | 0 | 0 | 0 | 0 | 0 |
| 15 | 0 | 4 | 0 | 4 | 0 | 0 | 0 |
| 16 | 5 | 1 | 4 | 0 | 0 | 0 | 0 |

Table of $q'(d,\infty;n)$

| n \ d | 3 | 4 | 5 | 7 | 8 | 9 | 11 |
|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 3 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 4 | 2 | 1 | 1 | 1 | 1 | 1 | 1 |
| 5 | 2 | 2 | 1 | 1 | 1 | 1 | 1 |
| 6 | 2 | 2 | 2 | 1 | 1 | 1 | 1 |
| 7 | 4 | 3 | 2 | 1 | 1 | 1 | 1 |
| 8 | 6 | 3 | 2 | 2 | 1 | 1 | 1 |
| 9 | 6 | 5 | 2 | 2 | 2 | 1 | 1 |
| 10 | 8 | 6 | 2 | 2 | 2 | 2 | 1 |
| 11 | 11 | 8 | 5 | 2 | 2 | 2 | 1 |
| 12 | 13 | 9 | 6 | 2 | 2 | 2 | 2 |
| 13 | 17 | 11 | 6 | 2 | 3 | 2 | 2 |
| 14 | 24 | 14 | 6 | 2 | 3 | 2 | 2 |
| 15 | 28 | 19 | 6 | 6 | 3 | 2 | 2 |
| 16 | 36 | 22 | 10 | 7 | 3 | 2 | 2 |

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## Appendix 2

### An identity and applications

by E.M. Wright†

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1. In what follows all small latin letters denote integers positive, negative or zero. X,Y are any complex numbers such that $|X| < 1$, $|Y| < 1$. We write

$$\xi_a = 0 \ (a<0), \ \xi_0 = 1, \ \xi_a = \prod_{u=1}^{a} (1-X^u)^{-1} \quad (a>0).$$

We use $\sum_u$ to denote summation over all u and $\sum_u'$ summation over all $u \leqslant r+s+t-k$. In every case all but a finite number of the terms vanish, so that the sum is a finite one. We shall prove

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Theorem:

$$(1)\quad \xi_{s+t-k}\xi_{t+r-k}\xi_{r+s-k}\sum_{u}\frac{X^{u(u-k)}\xi_u\xi_{u-k}\xi_{r-u}\xi_{s-u}\xi_{t-u}}{\xi_{r+s+t-k-u}}$$

$$= \xi_r\xi_{r-k}\xi_s\xi_{s-k}\xi_t\xi_{t-k}\ .$$

We first prove two lemmas.

Lemma 1: If $w > 0$, then

$$\xi_w\prod_{j=1}^{w}(1+YX^j) = \sum_x X^{x(x+1)/2}\xi_x\xi_{w-x}Y^x\ .$$

The result is trivial for $w = 1$. We can establish an induction with respect to w, provided we can show that

$$(1+YX^{w+1})\sum_x X^{x(x+1)/2}\xi_x\xi_{w-x}Y^x$$

$$= (1-X^{w+1})\sum_x X^{x(x+1)/2}\xi_x\xi_{w+1-x}Y^x\ .$$

The coefficient of $Y^x$ on the left-hand side is

$$X^{x(x+1)/2}\,\xi_x\xi_{w-x} + X^{w+1+x(x-1)/2}\,\xi_{x-1}\xi_{w+1-x}$$

$$= X^{x(x+1)/2}\,\xi_x\xi_{w+1-x}\,\{(1-X^{w+1-x}) + X^{w+1-x}(1-X^x)\}$$

$$= X^{x(x+1)/2}\,\xi_x\xi_{w+1-x}(1-X^{w+1}),$$

which is the coefficient of $Y^x$ on the right-hand side.

Lemma 2:

$$(2)\quad \xi_v\xi_w = \xi_{v+w}\sum_x (-1)^x\, X^{xv+x(x+1)/2}\,\xi_x\xi_{w-x}\ .$$

This is trivial if $w \leqslant 0$ or if $w > 0$, $v + w < 0$. We suppose then that $w > 0$ and $v + w \geqslant 0$. We put $Y = -X^v$ in Lemma 1 and use the result in (2). We find that we have to prove that

$$(3)\quad \xi_v = \xi_{v+w}\prod_{j=1}^{w}(1-X^{j+v}) \qquad (w > 0,\ v + w \geqslant 0).$$

If $v < 0$, the left-hand side vanishes; also $1 \leqslant -v \leqslant w$ and so one of the factors in the product on the right vanishes. If $v \geqslant 0$, (3) is immediate from the definition of $\xi_n$.

By Lemma 2; we have

$$\xi_{t-u}\xi_{r+s-k} = \xi_{r+s+t-k-u}\sum_z (-1)^z \chi^{z(t-u)+z(z+1)/2}\xi_z\xi_{r+s-k-z},$$

$$\xi_t\xi_{r-k} = \xi_{r+t-k}\sum_x (-1)^x \chi^{xt+x(x+1)/2}\xi_x\xi_{r-k-x},$$

$$\xi_{t-k}\xi_s = \xi_{s+t-k}\sum_y (-1)^y \chi^{y(t-k)+y(y+1)/2}\xi_y\xi_{s-y}.$$

If we use these in (1), we find that it is enough to prove that

$$\sum_u\sum_z (-1)^z \chi^{\alpha}\xi_u\xi_{u-k}\xi_{r-u}\xi_{s-u}\xi_z\xi_{r+s-k-z}$$

$$= \xi_r\xi_{s-k}\sum_x\sum_y (-1)^{x+y}\chi^{\beta}\xi_x\xi_{r-k-x}\xi_y\xi_{s-y},$$

where

$$\alpha = u(u-k)+z(t-u)+z(z+1)/2 = u(u-k-z)+zt+z(z+1)/2$$

and

$$\beta = xt+y(t-k)+x(x+1)/2+y(y+1)/2$$

$$= t(x+y)+y(y-k-x-y)+(x+y)(x+y+1)/2 .$$

Selecting those terms on the right for which $x+y=z$, we see that it is enough to prove that

$$(4)\quad \xi_z \xi_{r+s-z-k} \sum_u X^{u(u-k-z)} \xi_u \xi_{u-k} \xi_{r-u} \xi_{s-u}$$

$$= \xi_r \xi_{s-k} \sum_y X^{y(y-k-z)} \xi_y \xi_{z-y} \xi_{r-k-z+y} \xi_{s-y}$$

for every $z$.

Again, by Lemma 2,

$$\xi_u \xi_{r-u} = \xi_r \sum_v (-1)^v \, x^{vu+v(v+1)/2} \xi_v \xi_{r-u-v} \,,$$

$$\xi_{s-u} \xi_{u-k} = \xi_{s-k} \sum_w (-1)^w \, x^{w(s-u)+w(w+1)/2} \xi_w \xi_{u-k-w} \,,$$

$$\xi_y \xi_{z-y} = \xi_z \sum_v (-1)^v \, x^{vy+v(v+1)/2} \xi_v \xi_{z-y-v} \,,$$

$$\xi_{s-y} \xi_{r-z-k+y}$$

$$= \xi_{r+s-z-k} \sum_w (-1)^w \, x^{w(s-y)+w(w+1)/2} \xi_w \xi_{r-z-k+y-w} \,.$$

We substitute from these in (4). It is then enough to prove that the coefficient of

$$(-1)^{v+w} \, x^{wz+v(v+1)/2+w(w+1)/2} \xi_v \xi_w \xi_z \xi_{r+s-z-k} \xi_r \xi_{s-k}$$

on each side is the same, i.e. to prove that

$$\sum_u x^{u(u-k-z+v-w)} \xi_{r-u-v} \xi_{u-k-w}$$

$$= \sum_y x^{y(y-k-z+v-w)} \xi_{z-y-v} \xi_{r-z-k-w+y} \,.$$

If we put $y = k+z+w-v-u$ in the sum on the right-hand side, it becomes identical with that on the left-hand side. This completes the proof of our theorem.

2. If $a \to \infty$, then

$$\xi_a \to \xi_\infty = \prod_{u=1}^{\infty} (1-x^u)^{-1} ,$$

the generating function of the number of partitions of n into any number of parts. If we let $t \to \infty$ in (1) we have the identity

$$(5)\quad \xi_{r+s-k} \sum_u x^{u(u-k)} \xi_u \xi_{u-k} \xi_{r-u} \xi_{s-u} = \xi_r \xi_{r-k} \xi_s \xi_{s-k} .$$

If we let $s \to \infty$ in this, we have the further identity

$$(6)\quad \sum_u x^{u(u-k)} \xi_u \xi_{u-k} \xi_{r-u} = \xi_r \xi_{r-k} .$$

This last identity can be proved independently by induction with respect to r. It has an interpretation in terms of partition theory, but I have not yet found a direct combinatorial proof.

We remark that, if $a \geq 0$, $b \geq 0$, then

$$\lim_{X\to 1} \frac{\xi_a \xi_b}{\xi_{a+b}} = \frac{(a+b)!}{a!\,b!} ,$$

where $0! = 1$. Let us write $c = \min(0,k)$ and $d = \max(r,s,t)$. If $c \leqslant d$, we have

$$(7) \qquad \sum_{u=c}^{d} \frac{(r+s+t-u-k)!}{u!\,(u-k)!\,(r-u)!\,(s-u)!\,(t-u)!}$$

$$= \frac{(s+t-k)!\,(t+r-k)!\,(r+s-k)!}{r!\,(r-k)!\,s!\,(s-k)!\,t!\,(t-k)!}$$

if we let $X \to 1$ in (1). Similarly we can deduce from (5) that

$$\sum_{u=\max(0,k)}^{\min(r,s)} \frac{1}{u!\,(u-k)!\,(r-u)!\,(s-u)!} = \frac{(r+s-k)!}{r!\,(r-k)!\,s!\,(s-k)!} .$$

Recently Graham and Riordan [1] have shown that the solution of the recurrence relation

$$(8) \qquad \omega_{nm} = \sum_{v=0}^{m} \omega_{mv} \binom{n+v}{2m} \qquad (0 \leqslant m \leqslant n)$$

in terms of the undetermined $\omega_{vv}$ is

$$(9)\quad \omega_{nm} = \sum_{v=0}^{m} \frac{2v+1}{m+v+1} \binom{n+v}{m+v} \binom{n-1-v}{m-v} \omega_{vv} \quad (m<n).$$

An alternative proof to theirs would be to substitute from (9) in (8) and then seek to prove equal the coefficients of $\omega_{vv}$ on either side of the result. What is required readily reduces to the identity (7) with $k = 1$.

In the same way, the solution of the recurrence relation

$$\Omega_{nm} = \sum_{v=0}^{m} X^{(m-v)(m-v+1)} \xi_{2m} \xi_{n+v-2m} \Omega_{mv} / \xi_{n+v}$$

is

$$\Omega_{nm} = \sum_{v=0}^{m} \frac{(1-X^{2v+1}) \xi_{m+v+1} \xi_{n-m} \xi_{m-v} \xi_{n-1-m} \Omega_{vv}}{\xi_{n+v} \xi_{n-1-v}}$$

and the verification of this reduces to the identity (1) with $k = 1$.

Again let us put $r = s = p$ and $k = 0$ in (1) and let $X \to 1$. We have

$$\sum_{u=1}^{p} \binom{p}{u}^2 \binom{t + 2p - u}{2p} = \binom{t + p}{p}^2 ,$$

which is, of course, a well-known identity.

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## Appendix 3

## Stacks

by E.M. Wright†

[To be published in the Quarterly
Journal of Mathematics]

1. In what follows all small latin letters denote non-negative integers or functions all of whose values are such integers. X and Y are complex numbers such that $|X| < 1$, $|Y| < 1$ and sometimes $Y \to 1$. Under these conditions all the questions of convergence which arise are trivial and we ignore them.

An n-stack is a solution of the equation

$$n = \sum_{x,y \geqslant 1} z(x,y),$$

in which every z is 0 or 1, $z(1,1) = 1$,

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†The research reported herein has been sponsored by the European Research Office, United States Army.

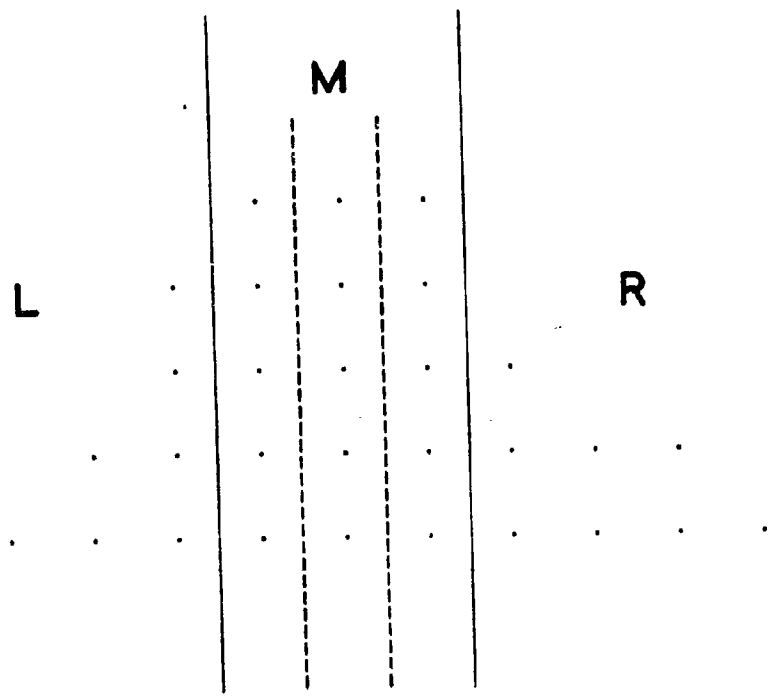
$$z(x,y_1) = 1 \Rightarrow z(x,y) = 1 \qquad (1 \leqslant y \leqslant y_1)$$

and

$$z(x_1,y) = z(x_2,y) = 1,\ x_1 < x_2 \Rightarrow z(x,y) = 1 \qquad (x_1 \leqslant x \leqslant x_2).$$

The stack may be represented by n nodes arranged at the points whose coordinates are x,y as in the figure (ignore the vertical lines, whether broken or unbroken). We write s(n) for the number of n-stacks and $s_r(n)$ for the number of n-stacks with just r rows. Our object here is to study $s_r(n)$ and s(n) and a number of related enumerative functions and to find their generating functions and certain relations between them. Finally we obtain asymptotic approximations to $s_r(n)$ and s(n).

We may dissect an n-stack with r rows by drawing a line parallel to the y-axis just to the left of the left-hand node in the top layer (the left-hand unbroken line in the figure). The stack is then dissected into two parts. If the left hand part contains $n_1$ nodes, it may be read as the graph of a


M
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partition of $n_1$ into not more than r-1 parts. We write $p_{r-1}(n_1)$ for the number of such partitions and take $p_{r-1}(0) = 1$, $p_{r-1}(-n) = 0$ $(n>0)$. The right hand part is the graph of a partition of $n_2$ into just r parts, where $n_2 = n-n_1 \geq r$; the number of such partitions is readily seen to be $p_r(n_2-r)$. (Remove one node from each part; there remains a partition of $n_2-r$ into at most r parts.) The generating function of $p_r(n)$ is well known to be

$$\xi_r = \xi_r(X) = \prod_{k=1}^{r}(1-X^k)^{-1} = \sum_{n=0}^{\infty} p_r(n)X^n$$

(see, for example, [1]). We write $\xi_0 = 1$ and $\xi_{-a} = 0$ $(a>0)$.

We have then

$$s_r(n) = \sum_{n_1+n_2=n} p_{r-1}(n_1)p_r(n_2-r)$$

and so the generating function of $s_r(n)$ is

$$(1)\qquad \sum_{n=r}^{\infty} s_r(n)X^n = \sum_{n=r}^{\infty} \sum_{n_1+n_2=n} p_{r-1}(n_1)p_r(n_2-r)X^n$$

$$= X^r \sum_{n_1=0}^{\infty} p_{r-1}(n_1)X^{n_1} \sum_{n_3=0}^{\infty} p_r(n_3)X^{n_3} = X^r \xi_r \xi_{r-1}.$$

Hence the generating function of s(n) is

$$(2)\qquad \sum_{n=1}^{\infty} s(n)X^n = \sum_{r=1}^{\infty} \sum_{n=r}^{\infty} s_r(n)X^n = \sum_{r=1}^{\infty} X^r \xi_r \xi_{r-1}.$$

2. We write $s_r(b,t,n)$ for the number of n-stacks with r rows, b nodes in the base or bottom row and t nodes in the top row. We write $s_r(.,t,n)$ for the number of n-stacks with r rows and t nodes in the top row; similarly $s_r(b,.,n)$ is the number with r rows and b nodes in the base. Again $s_r(.,\geqslant t,n)$ is the number with r rows and not less than t nodes in the top row. Finally, if the number of rows is unrestricted, we omit the suffix r. Thus

$$(3)\quad s_r(b,.,n) = \sum_{t\geqslant 1} s_r(b,t,n), s(b,.,n) = \sum_{r\geqslant 1} s_r(b,.,n)$$

and so on.

An n-stack with r rows, base b and top t may be dissected into three parts L, M, R as in the figure by drawing the two unbroken vertical lines, one just to the left of the left-hand node of the top layer and one to the right of the right-hand node of the top layer. (The broken lines should be ignored). The left hand part L is the graph of a partition of $n_1$ (say) into not more than $r-1$ parts of which the greatest is $b_1$ (say).

We write $p_{r-1}(b_1,n_1)$ for the number of such partitions; it is well known [3] that

$$P_{r-1}(X,Y) = 1 + \sum_{b_1=1}^{\infty} \sum_{n_1=1}^{\infty} p_{r-1}(b_1,n_1) Y^{b_1} X^{n_1} = \prod_{k=1}^{r-1} (1-YX^k)^{-1}.$$

The middle part M contains just rt nodes. The right hand part R is a partition of $n_2$ into not more than r-1 parts of which the greatest is $b_2$, where $n = n_1+n_2+rt$ and $b = b_1+b_2+t$. We have then

$$s_r(b,t,n) = \sum_{b_1+b_2=b-t} \sum_{n_1+n_2=n-rt} p_{r-1}(b_1,n_1) p_{r-1}(b_2,n_2)$$

and so

$$(4) \qquad S_r(t;X,Y) = \sum_{b,n} s_r(b,t,n) Y^b X^n$$

$$= X^{rt} Y^t [P_{r-1}(X,Y)]^2$$

$$= X^{rt} Y^t \prod_{k=1}^{r-1} (1-YX^k)^{-2}.$$

We write

$$S_r(X,Y) = \sum_{b,n\geqslant 1} s_r(b,.,n)Y^bX^n$$

and

$$S(X,Y) = \sum_{b,n\geqslant 1} s(b,.,n)Y^bX^n.$$

By (3) and (4)

$$S_r(X,Y) = \sum_{t\geqslant 1} S_r(t;X,Y)$$

$$= YX^r(1-YX^r)^{-1}\prod_{k=1}^{r-1}(1-YX^k)^{-2} \tag{5}$$

and

$$S(X,Y) = \sum_{r\geqslant 1} S_r(X,Y). \tag{6}$$

Since $s(n)$ is the number of n-stacks, with no restriction on base, top or number of rows, we have

$$s(n) = \sum_{b} s(b,.,n)$$

and so

$$\sum_{n} s(n)X^n = \sum_{b,n} s(b,.,n)X^n = S(X,1).$$

Using (5) and (6) we see that this is in agreement with (2).

3. By the definitions, we have

$$s_r(.,\geq t,n) = \sum_{u\geq t} s_r(.,u,n)$$

and

$$S_r(u;X,1) = \sum_{n=u}^{\infty} s_r(.,u,n)X^n$$

$$= X^{ru}\xi_{r-1}^2$$

by (4). Hence

$$\sum_{n=t}^{\infty} s_r(.,\geq t,n)X^n = \xi_{r-1}^2 \sum_{u=t}^{\infty} X^{ru} = X^{rt}\xi_{r-1}\xi_r$$

and

$$\sum_{n=t}^{\infty} s(.,\geq t,n)X^n = \sum_{r=1}^{\infty} X^{rt}\xi_{r-1}\xi_r = F(X,X^t), \tag{7}$$

where

$$F(X,Y) = \sum_{r=1}^{\infty} \xi_{r-1}\xi_r Y^r,$$

so that

$$F(X,X) = S(X,1) = \sum_{n=1}^{\infty} s(n)X^n. \tag{8}$$

4. Let us write

$$\omega_b = \omega_b(X) = \sum_{n=b}^{\infty} s(b,.,n)X^n,$$

so that

$$S(X,Y) = \sum_{b\geq 1} \omega_b Y^b. \tag{9}$$

By (5)

$$S_{r+1}(X,Y)(1-YX)^2 = S_r(X,YX)$$

and so, summing over r, we have

$$S(X,Y)(1-YX)^2 = YX(1-YX) + S(X,YX).$$

Substituting from (9) and equating the coefficients of $Y^b$, we have

$$\omega_1 = X\xi_1,\ \omega_2 = X^2(1+X)\xi_2,$$

$$\omega_b - 2X\omega_{b-1} + X^2\omega_{b-2} = X^b\omega_b \qquad (b\geq 3). \tag{10}$$

From this we can easily calculate that

$$\omega_3 = X^3\xi_3(1+X)^2,\ \omega_4 = X^4\xi_4(1+x)(1+2X+X^3),$$

$$\omega_5 = X^5\xi_5(1+X)(1+3X+2X^3+X^4+X^5).$$

Alternatively we may proceed as follows. We remark that $s(b,.,b) = 1$. If $n > b$, we may remove the base b from the stack. We are then left with a stack with base c (say) which may have occupied any one of $b-c+1$ positions in the original stack. Hence

$$s(b,.,n) = \sum_{c=1}^{b}(b-c+1)s(c,.,n-b) \qquad (n>b),$$

which leads at once to

$$\omega_b = X^b\{1 + \sum_{c=1}^{b}(b-c+1)\omega_c\} \qquad (b\geq 1).$$

From this (10) follows.

5. From (1) we can find an expression for $s_r(n)$ which gives us very easily the behaviour of $s_r(n)$ for fixed r and large n. The method is closely related to that of [4]. We note that

$$\xi_r^{-1}\xi_{r-1}^{-1} = \prod_{v=1}^{r}(1-X^v)\prod_{v=1}^{r-1}(1-X^v)$$

$$= \prod_{v=1}^{r}\prod_{\rho(v)}(1-\rho X)^{\lambda(r,v)},$$

where the last product is taken over all primitive v-th roots $\rho$ of unity and $\lambda(r,v) = [r/v]+[(r-1)/v]$. Hence by the elementary technique of partial fractions, we have

$$X^r\xi_r\xi_{r-1} = \sum_{v=1}^{r}\sum_{\rho(v)}\sum_{t=1}^{\lambda(r,v)} A(r,v,\rho,t)(1-\rho X)^{-t}$$

and so, by (1),

$$s_r(n) = \sum_{v=1}^{r} P(r,v,n)$$

where

$$P(r,v,n) = \sum_{\rho(v)} \sum_{t=1}^{\lambda(r,v)} A(r,v,\rho,t)\rho^n \frac{(n+t-1)!}{n!(t-1)!}$$

$$= \sum_{t=1}^{\lambda(r,v)} B(r,v,t,n)n^{t-1}$$

and

$$B(r,v,t,n) = \sum_{\rho(v)} C(r,v,\rho,t)\rho^n.$$

Hence $B(r,v,t,n)$ depends on the residue of n (mod v) but not otherwise on n. We say that $P(r,v,n)$ is a semi polynomial in n of degree $\lambda(r,v)-1$ and to modulus v.

We observe that $\lambda(r,1) = 2r-1$, $\lambda(r,2) = r-1$ and $\lambda(r,v) < 2r/3$ for $v > 2$. For large n and fixed r we have then

$$s_r(n) = P(r,1,n) + O(n^{r-2}),$$

where $P(r,1,n)$ is a polynomial of degree $2r-2$ in n.

Using a method similar to that of [4], we can evaluate the leading coefficients in $P(r,1,n)$ and we find that

$$s_r(n) = \frac{1}{r!(r-1)!} \sum_{t=1}^{[\frac{1}{2}r]} \Omega_r \frac{(n+R)^{2r-2t}}{(2r-2t)!} + O(n^{r-1}),$$

where $r>1$, $R = \frac{1}{2}r(r+2)$, $\Omega_1 = 1$ and

$$\Omega_2 = -\frac{r(2r^2+1)}{72}, \quad \Omega_3 = \frac{25r^2(2r^2+1)^2+6r(6r^4+10r^2-1)}{259200}.$$

6. We cannot deduce the asymptotic behaviour of $s(n)$ directly from a knowledge of its generating function, at least in the form given in (2). But the results of §3 enable us to find what we want. First we remark that

$$(1-Y)F(X,Y) = \sum_{r=1}^{\infty}(\xi_{r-1}\xi_r - \xi_{r-2}\xi_{r-1})Y^r$$

and so

$$\lim_{Y\to 1}(1-Y)F(X,Y) = \sum_{r=1}^{\infty}(\xi_{r-1}\xi_r - \xi_{r-2}\xi_{r-1}) \tag{11}$$

$$= \lim_{r\to\infty}\xi_{r-1}\xi_r = \xi_\infty^2,$$

where

$$\xi_\infty^2 = \prod_{k=1}^{\infty}(1-X^k)^{-2} = \sum_{n=0}^{\infty}q(n)X^n \tag{12}$$

(say).

Again we have

$$XYF(X,Y) = \sum_r \xi_{r-1}\xi_r XY^{r+1} = \sum_r \xi_{r-2}\xi_{r-1} XY^r$$

$$= \sum_r \xi_{r-1}\xi_r Y^r (X-X^r)(1-X^r)$$

$$= XF(X,Y)-(1+X)F(X,XY)+F(X,X^2Y)$$

and so

$$X(1-Y)F(X,Y)-(1+X)F(X,XY)+F(X,X^2Y) = 0.$$

Letting $Y \to 1$, we have

$$\sum_{n=0}^{\infty} q(n)X^{n+1}-(1+X)\sum_{n=1}^{\infty} s(n)X^n+\sum_{n=2}^{\infty} s(.,\geq 2,n)X^n = 0$$

by (7), (8) and (11). Hence

$$s(n+1)+s(n) = q(n)+s(.,\geq 2,n+1),$$

that is

$$(13)\quad q(n) = s(n)+s(n+1)-s(.,\geqslant 2,n+1) = s(n)+s(.,1,n+1).$$

Any (n+1)-stack with a top of one node can be converted into a unique n-stack by removing that one node. The reverse process is not unique. Hence

$$s(n) \leqslant s(.,1,n+1) \leqslant s(n+1),$$

the latter inequality being trivial. Hence, by (13),

$$2s(n) \leqslant q(n) \leqslant 2s(n+1).$$

From this, we have

$$(14)\qquad \tfrac{1}{2}q(n-1) \leqslant s(n) \leqslant \tfrac{1}{2}q(n).$$

If we write

$$\phi(n) = 8^{-1}(3^3n^5)^{-\frac{1}{4}}\exp\{2\pi\sqrt{(n/3)}\},$$

we have, as $n \to \infty$,

$$q(n) = 2\phi(n)[1 + O(n^{-\frac{1}{2}})]$$

by [2]. Clearly

$$\phi(n-1) = \phi(n)[1 + O(n^{-\frac{1}{2}})]$$

and so, by (14),

$$s(n) = \phi(n)[1 + O(n^{-\frac{1}{2}})]$$

as $n \to \infty$. This gives the asymptotic approximation to $s(n)$.

7 The essential relation (13) can be established by a more direct study of the n-stack. This has some interest in itself. We remark first that $\xi_\infty = \sum_n p(n)X^n$, where p(n) is the number of unrestricted partitions of n and p(0) = 1. Hence, by (12), we have

$$q(n) = \sum_{n_1+n_2=n} p(n_1)p(n_2).$$

If we take the graph of any partition of $n_1$ and the graph of any partition of $n_2$ and arrange them back-to-back we have an n-stack, where $n = n_1+n_2$. This is the reverse process to the dissection of a stack in §1. A little consideration shows that any n-stack with top t can be constructed thus in just t+1 ways though, of course, with different $n_1, n_2$. For, in the figure, we may separate the n-stack by any one of the vertical straight lines (broken or unbroken) into two partition graphs. Hence

$$q(n) = \sum_{t\geq 1} (t+1)s(.,t,n). \tag{15}$$

We consider again an n-stack with top t. We can place a single node above the top row in just t ways to form an (n+1)-stack with top 1. Nor can any one of these (n+1)-stacks be constructed in this way from any other n-stacks. Hence

$$s(.,1,n+1) = \sum_{t\geq 1} ts(.,t,n). \tag{16}$$

But trivially

$$s(n) = \sum_{t\geq 1} s(.,t,n) \tag{17}$$

and (13) follows at once from (15), (16) and (17).

Note added at proof stage. Dr A.O.L. Atkin, to whom I had communicated my results in this paper, drew my attention to [5], in which Auluck considers a combinatorial structure equivalent to the unrestricted n-stack. Auluck finds the generating function on the right-hand side of (2) and the asymptotic approximation $s(n) \sim \emptyset(n)$, but not my result about the order of the error. The other results of the present paper do not appear in [5] and the methods used in the two papers differ entirely. Auluck also shows that

$$\sum_{r=1}^{\infty} x^r \xi_r \xi_{r-1} = \xi_c^2 \sum_{n=1}^{\infty} (-1)^{n-1} x^{n(n+1)/2} ,$$

a result which Atkin had conjectured from results obtained on a computer and which I can prove very simply. Auluck also studies other combinatorial structures on which I have recently done some work. Again there is surprisingly little overlap, either in methods or results and I hope to publish further results later.

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## Appendix 4

### Number of arrangements

by E.M. Wright †

[To be published in the Mathematical Gazette]

In a recent article [1] in this Gazette, Collings discusses the number of arrangements of n railway trucks on k sidings under a variety of conditions. Let $q_k(n)$ be the number of ways of arranging n indistinguishable trucks on k indistinguishable sidings or, what is the same thing, the number of partitions of n into not more than k parts. Let $p_k(n)$ be the number of these arrangements which use all k sidings, that is, the number of partitions of n into exactly k parts. Collings describes the determination of $p_k(n)$ and $q_k(n)$ as an unsolved problem in partition theory; this is not quite correct.

It is convenient to write

$$q_k(n) = 0 \quad (n<0), \quad q_k(0) = 1, \quad q_0(n) = 0 \quad (n\geqslant 1). \qquad (1)$$

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It is easy to see that each arrangement of the n trucks on the k sidings is either (i) an arrangement which uses all k sidings or (ii) one which uses not more than k-1 sidings. Hence

$$q_k(n) = p_k(n) + q_{k-1}(n) \quad (n \geq 1). \tag{2}$$

Again, if we consider an arrangement of n trucks using all k sidings and remove a truck from each siding, we are left with n-k trucks on not more than k sidings; the reverse process holds also. Hence

$$p_k(n) = q_k(n-k). \tag{3}$$

This reduces the determination of $p_k(n)$ to that of $q_k(n)$.

Using (3) in (2), we have

$$q_k(n) = q_k(n-k) + q_{k-1}(n) \quad (k \geq 1,\ n \geq 1). \tag{4}$$

Now let us write

$$Q_k = Q_k(X) = \sum_{n=0}^{\infty} q_k(n) X^n \quad (k \geq 0, |X| < 1). \tag{5}$$

Multiplying (4) throughout by $X^n$ and summing over n, we have

$$Q_k = X^k Q_k + Q_{k-1},\ Q_k(1 - X^k) = Q_{k-1} \qquad (k \geqslant 1) \qquad (6)$$

and so

$$Q_k = \prod_{s=1}^{k} (1 - X^s)^{-1}, \qquad (7)$$

since $Q_0 = 1$. This result is due to Euler [2].

We ignored the question of the convergence of the series in (5). If we regard $Q_k$ as defined by (7), it is trivial that $Q_k$ can be expanded in a power series convergent for $|X| < 1$. Again this $Q_k$ clearly satisfies (6) and so the coefficients in the power series satisfy conditions corresponding to (1) and (4) and can be identified with the $q_k(n)$.

While in theory $q_k(n)$ can be calculated from (5) and (7), or indeed from (1) and (4), for any k and n, this process does not lead to a simple formula like those found by Collings [1] for the number of arrangements under other conditions.

The form of $q_k(n)$ has been studied by Sylvester [6], Glaisher [3], Rieger [5] and in [4] and [7]. Sylvester's and Glaisher's papers are lengthy and detailed. However

we can find out something about $q_k(n)$ by applying the well-known elementary technique of partial fractions to $Q_k$.

We must first find the linear factors of $Q_k^{-1}$. We have

$$1 - X^h = \prod(1 - \tau X),$$

where $\tau$ runs through all h-th roots of unity, i.e. all values of $\tau$ such that $\tau^h = 1$. Let $\rho$ be a primitive h-th root of unity, i.e. $\rho^h = 1$ and $\rho^t \neq 1$ for $1 \leq t < h$. Then $\rho$ is an hu-th root of unity for all positive integral u and so the factor $1 - \rho X$ occurs just $[k/h]$ times in $Q_k^{-1}$. Here $[F]$ denotes the greatest integer less than or equal to F. Hence

$$Q_k^{-1} = \prod_{s=1}^{k}(1 - X^s) = \prod_{h=1}^{k} \prod_{\rho(h)} (1 - \rho X)^{[k/h]},$$

where the last product is over all primitive h-th roots $\rho$ of unity.

If we split $Q_k$ into partial fractions in the usual way, we have

$$Q_k = \sum_{h=1}^{k} \sum_{\rho(h)} \sum_{t=1}^{[k/h]} A(k,h,\rho,t)(1 - \rho x)^{-t} \qquad (8)$$

and so

$$q_k(n) = \sum_{h=1}^{k} R(k,h,n), \qquad (9)$$

where

$$R(k,h,n) = \sum_{\rho(h)} \rho^n \sum_{t=1}^{[k/h]} A(k,h,\rho,t) \frac{(n+t-1)!}{n!(t-1)!}$$

$$= \sum_{\rho(h)} \rho^n \sum_{v=1}^{[k/h]} B(k,h,\rho,v) n^{v-1}$$

$$= \sum_{v=1}^{[k/h]} C(k,h,v,n) n^{v-1}$$

and

$$C(k,h,v,n) = \sum_{\rho(h)} B(k,h,v,n) \rho^n .$$

Since $\rho^h = 1$, we see that $C(k,h,v,n+h) = C(k,h,v,n)$ and so $C(k,h,v,n)$ depends on the residue of $n$ (mod $h$) but not otherwise on $n$. Hence, for large $n$, $C(k,h,v,n)$ is bounded and $R(k,h,n) = O(n^{[k/h]-1})$. Hence from (9) we can deduce that

$$q_k(n) = R(k,1,n) + O(n^{[\frac{1}{2}k]-1}). \tag{10}$$

If $h = 1$, the sole value of $\rho$ is 1 and so

$$R(k,1,n) = \sum_{t=1}^{k} A(k,1,1,t) \frac{(n+t-1)!}{n!(t-1)!}$$

$$= A(k,1,1,k) \frac{n^{k-1}}{(k-1)!} + O(n^{k-2}).$$

If we multiply (8) throughout by $(1 - X)^k$ and let $X \to 1$, we have

$$A(k,1,1,k) = \lim_{X \to 1} (1 - X)^k Q_k(X) = 1/(k!).$$

Hence, for large $n$,

$$q_k(n) = \frac{n^{k-1}}{k!(k-1)!} + O(n^{k-2}). \tag{11}$$

Thus by quite elementary reasoning we have learnt quite a lot about $q_k(n)$, especially (9), (10) and (11). The last result can be improved substantially with enough labour; in fact (see [7])

$$q_k(n) = \frac{(n+K)^{k-1}}{k!(k-1)!} - \frac{(k+1)(2k+1)(n+K)^{k-3}}{144(k-1)!(k-2)!} + O(n^{k-5})$$

if $k \geqslant 7$ and $K = \frac{1}{2}k(k+1)$. For smaller values of k the exact form of $q_k(n)$ can be readily determined by partial fractions. Detailed results are given in [4].

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## Appendix 5

## A relationship between two sequences III

by E.M. Wright†

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1. In an earlier paper [2] we studied two sequences $\{g_n\}$, $\{G_n\}$ whose relationship was defined by

$$1 + \sum_{n=1}^{\infty} G_n X^n = \exp\left( \sum_{n=1}^{\infty} g_n X^n \right) \tag{1.1}$$

(to be interpreted formally if the series diverge for all non-zero X) or by the equivalent

$$nG_n = ng_n + \sum_{s=1}^{n-1} sg_s G_{n-s} \ . \tag{1.2}$$

(1.1) and (1.2) holds good when $n!g_n$ is the number of

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connected graphs on n labelled nodes with a particular property and $n!G_n$ is the number of graphs on n labelled nodes each of whose connected components has that property. See, for example, Gilbert [1].

In [2] I took $g_n \geqslant 0$ and studied conditions under which

$$G_n \sim g_n \tag{1.3}$$

as $n \to \infty$. We found that, without loss of generality, we could assume that $g_n > 0$ for all sufficiently large n. The divergence of the power series in (1.1) for all non-zero X was a necessary condition for (1.3). An example showed that, even when combined with a condition of reasonably steady increase of $g_n$, this was not sufficient. A necessary and sufficient condition for (1.3) was that

$$\sum_{s=1}^{n-1} H_s H_{n-s} = o(H_n) \tag{1.4}$$

as $n \to \infty$, where the sequence $\{H_n\}$ is either $\{G_n\}$ or $\{g_n\}$. The advantage of (1.4) is that we need information

about only one of the sequences $\{G_n\}$, $\{g_n\}$. In [2] I found two sets of sufficient conditions on the rate of growth of $H_n$ for (1.4) to be true. In [3] I studied a further necessary condition (given that $H_n$ satisfies a condition of fairly smooth growth) and showed that it was not sufficient. The gap between these necessary and sufficient conditions was now not great.

When Dr R.C. Read read [2] he wrote to ask me whether I could find conditions that

$$T_n \sim t_n \tag{1.5}$$

as $n \to \infty$, where

$$1 + \sum_{n=1}^{\infty} T_n x^n = \prod_{k=1}^{\infty} (1 - x^k)^{-t_k} \tag{1.6}$$

(again interpreted formally if the series and product diverge). Here $t_n$ is the number of connected graphs on n unlabelled nodes with a particular property and $T_n$ is the number of graphs on n unlabelled nodes, each of whose connected components has that property. I show here that

the necessary and sufficient conditions on $\{T_n\}$ and $\{t_n\}$ are precisely the same as those on $\{G_n\}$ and $\{g_n\}$. This means that all the results about (1.4) in [2,3] apply to the new problem.

Dr Read pointed out that, if we take $H_n = g_n$, (1.4) has the following meaning. The number of graphs on n labelled nodes which are disconnected is small compared with the number which are connected if, and only if, the number of them which have just two connected components is similarly small. My result here extends this statement to the graphs on unlabelled nodes.

2. We suppose $t_n \geq 0$ for all $n$. Differentiating (1.5) logarithmically, we have

$$\frac{\sum nT_n X^n}{1+\sum T_n X^n} = \sum \frac{nt_n X^n}{1-X^n}$$

$$= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} nt_n X^{mn} = \sum_{s=1}^{\infty} sv_s X^s,$$

where

$$sv_s = \sum_{m|s} mt_m. \tag{2.1}$$

Hence

$$\sum_{n=1}^{\infty} nT_n X^n = \sum_{s=1}^{\infty} sv_s X^s \left(1+\sum_{h=1}^{\infty} T_h X^h\right)$$

and so

$$nT_n = nv_n + \sum_{s=1}^{n-1} sv_s T_{n-s}. \tag{2.2}$$

This relation (2.2) between $\{T_n\}$ and $\{v_n\}$ is the same as

(1.2) between $(G_n)$ and $(g_n)$. We use this fact in what follows. It follows from (2.1) and (2.2) that

$$t_n \leqslant v_n \leqslant T_n. \tag{2.3}$$

We prove first the following theorem

Theorem 1. If $T_n = 0$ for an infinite sequence of $n$, then $t_n = 0$ except when $n$ is a multiple of some $d > 1$.

Let us suppose the condition satisfied. Then, since (2.2) is equivalent to (1.2), Theorem 1 of [2] tells us that there is a $d > 1$ such that $v_n = 0$ if $d \nmid n$. Since $t_m \geqslant 0$ for all $m$, it follows from (2.1) that $t_n = 0$ if $d \nmid n$. This is our theorem.

If the condition of Theorem 1 is satisfied, but $T_n$ does not vanish for all $n$, let us take $d$ to have its largest possible value. (1.6) becomes

$$1+\sum_{m=1}^{\infty} T_{dm}X^{dm} = \prod_{m=1}^{\infty}\left(1-X^{md}\right)^{-t_{md}},$$

that is

$$1+\sum_{m=1}^{\infty} T_{dm} z^m = \prod_{m=1}^{\infty} \left(1-z^m\right)^{-t_{md}} .$$

There cannot now be an infinity of zero $T_{dm}$, since otherwise we could apply Theorem 1 again and, in an obvious way, find a contradiction to the hypothesis that d has its largest possible value. Hence there is no loss of generality if we suppose $T_n > 0$ for all sufficiently large n. We shall therefore suppose that

$$t_n > 0 \qquad (n \geqslant c) \tag{2.4}$$

since otherwise $T_n \sim t_n$ is clearly impossible.

In what follows we write K for a positive number, not always the same at each occurrence, independent of n. We use $K_1, K_2$ for fixed numbers of the same kind.

3. We prove next our main theorem.

Theorem 2. The necessary and sufficient condition that $T_n \sim t_n$ is that

$$\sum_{s=1}^{n-1} H_s H_{n-s} = o(H_n)$$

where the sequence $\{H_n\}$ is either $\{T_n\}$ or $\{t_n\}$.

We have to prove that

$$(\mathcal{A}) \qquad T_n \sim t_n$$

is equivalent to

$$(\mathcal{B}) \qquad \sum_{s=1}^{n-1} T_s T_{n-s} = o(T_n)$$

and also to

$$(\mathcal{b}) \qquad \sum_{s=1}^{n-1} t_s t_{n-s} = o(t_n).$$

In the course of the argument we shall consider also the logical standing of the statements

$(\mathfrak{D})$ $$T_n \sim v_n ,$$

$(\mathfrak{E})$ $$v_n \sim t_n ,$$

$(\mathfrak{F})$ $$\sum_{s=1}^{n-1} v_s v_{n-s} = o(v_n) .$$

By (2.3), we see that

$$\mathfrak{A} \longleftrightarrow \mathfrak{D} + \mathfrak{E} .$$

Again the relationship (2.2) between $\{T_n\}$ and $\{v_n\}$ is the same as (1.2) between $\{G_n\}$ and $\{g_n\}$. Hence

$$\mathfrak{B} \longleftrightarrow \mathfrak{D} \longleftrightarrow \mathfrak{F}$$

by the result of [2] quoted in §1 of this paper. We shall prove below that

(3.1) $$\mathfrak{F} \longrightarrow \mathfrak{E} ,$$

(3.2) $$\mathfrak{F} \longleftrightarrow \mathfrak{L} .$$

Using these, we have

$$\mathcal{A} \to \mathcal{D} \to \mathcal{B},\mathcal{B} \to \mathcal{D}+\mathcal{J} \to \mathcal{D}+\mathcal{C} \to \mathcal{A}.$$

$$\mathcal{A} \to \mathcal{D} \to \mathcal{J} \to \mathcal{L},\mathcal{L} \to \mathcal{J} \to \mathcal{D}+\mathcal{C} \to \mathcal{A}.$$

Hence

$$\mathcal{B} \leftrightarrow \mathcal{A} \leftrightarrow \mathcal{L}$$

and this is Theorem 2.

It remains to prove (3.1) and (3.2). We start by assuming $\mathcal{J}$. Let b be the least integer such that $v_b > 0$. By (2.3) and (2.4), $v_n > 0$ when $n \geq c$. Then, for $n > b + c$, we have

$$v_{n-b} > 0,\ v_b v_{n-b} = o(v_n),\ v_n/v_{n-b} > \infty$$

as $n \to \infty$. Hence $v_n \to \infty$. Again

$$\min_{s \geq c} v_s > K_1.$$

Hence, if $n > 2c$, we have by (2.1) and $\mathcal{J}$

$$v_n - t_n \leqslant \sum_{m \leqslant n/2} t_m \leqslant \sum_{m \leqslant n/2} v_m \leqslant K_1^{-1} \sum_{m \leqslant n/2} v_m v_{n-m} = o(v_n),$$

which is $\mathcal{t}$. By this and (2.3)

$$\sum_{s=1}^{n-1} t_s t_{n-s} \leqslant \sum_{s=1}^{n-1} v_s v_{n-s} = o(v_n) = o(t_n),$$

which is $\mathcal{b}$. Hence we have

$$\mathcal{F} \Rightarrow \mathcal{b} + \mathcal{t}.$$

It remains only to prove that $\mathcal{b} \Rightarrow \mathcal{F}$. Assume $\mathcal{b}$ true. We deduce $\mathcal{t}$ from $\mathcal{b}$ just as we deduced $\mathcal{t}$ from $\mathcal{F}$ above, so that we have $v_n \sim t_n$. Hence

(3.3) $$t_n \leqslant v_n \leqslant K t_n \qquad (n \geqslant c).$$

We now write

$$\sum_{s=1}^{n-1} v_s v_{n-s} = \sum_{s=1}^{c-1} + \sum_{s=c}^{n-c} + \sum_{s=n-c+1}^{n-1} = J_1 + J_2 + J_3$$

(say). By $\mathcal{b}$ and (3.3), we have

$$J_2 = \sum_{s=c}^{n-c} v_s v_{n-s} \leqslant K \sum_{s=c}^{n-c} t_s t_{n-s} = o(t_n).$$

If $n \geqslant 2c$, then by (2.1) and (3.3)

$$J_3 = J_1 = \sum_{s<c} v_s v_{n-s} < K \sum_{s<c} v_s t_{n-s}$$

$$= K \sum_{s<c} \sum_{m|s} t_m t_{n-s} = K \sum_{mk<c} t_m t_{n-mk}.$$

Now let $1 \leqslant m \leqslant mk < c$. We write $K_2$ for the least non-zero value of $t_m$, unless every $t_m = 0$, in which case we write $K_2 = 1$. By $\mathcal{L}$,

$$t_m t_{n-mk} = o(t_{n-mk+m}), \quad t_m^k t_{n-mk} = o(t_n),$$

$$t_m t_{n-mk} = K_2^{1-k}\, o(t_n).$$

We have then

$$J_3 = J_1 \leqslant K \sum_{mk<c} t_m t_{n-mk} \leqslant o(t_n) \sum_{mk<c} K_2^{1-k} = o(t_n).$$

Hence

$$\sum_{s=1}^{n-1} v_s v_{n-s} = J_1 + J_2 + J_3 = o(t_n) = o(v_n)$$

and this is $\mathcal{H}$.

4. Theorem 3. A necessary condition for (1.5) is that either side of (1.6) diverges for all non-zero X.

In this paragraph the words "for all non-zero X" are understood after every use of the word "divergence". By Theorem 2 of [2] the divergence of $\sum G_n X^n$ and of $\sum g_n X^n$ is a necessary condition for (1.3). But (1.3) is equivalent to (1.4) with $\{G_n\}$ or $\{g_n\}$ for $\{H_n\}$. Hence (1.4) implies the divergence of $\sum H_n X^n$ and so, by Theorem 2 of this paper, (1.5) implies the divergence of $\sum T_n X^n$ and of $\sum t_n X^n$. Finally the divergence of $\sum t_n X^n$ implies the divergence of the product on the right hand side of (1.6).

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13 ABSTRACT

The first investigation studies rotatable partitions, i.e. those d-dimensional partitions whose representations are invariant under rotation of the axes of coordinates. If d is a power of a prime p, the number of irrotatable partitions is divisible by p. The number of rotatable partitions of small n is small and easily calculated. Consequences include a convenient check of the total number of partitions and a simple proof of the recently discovered fact that the long conjectured form of the generating function of solid partitions is mistaken. In the second section is found a general identity involving $\zeta$-functions. Particular cases of this have applications in partition theory. The third section introduces a new combinatorial idea, the n-stack, gives generating functions for the number of n-stacks under certain restrictions and under no restrictions and finds asymptotic values for these numbers for large n. The fourth section is a short, semi-expository paper correcting a statement by another author that a particular problem in partition theory is unsolved. To do this, a new and simple derivation of the behaviour for large n of the number of partitions of n into just k parts is given. The fifth section finds necessary and sufficient conditions that almost all graphs of a given kind on n unlabelled nodes shall be connected. This condition is closely related to that found earlier for the labelled case. The sixth section reports preliminary investigations into the asymptotic expansions of, and relations between, the number of connected and disconnected graphs of a given kind on n labelled and on n unlabelled nodes when n is large.

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