A NOTE ON SIMPLE DIPOLAR STRESSES

by

A. E. Green

P. M. Naghdi

Report No. AM-67-15
Contract Nonr-222(69), Project NR-064 436
Division of Applied Mechanics
December 1967

COLLEGE OF ENGINEERING
UNIVERSITY OF CALIFORNIA, Berkeley

22

OFFICE OF NAVAL RESEARCH
Contract Nonr-222(69)
Project NR-064-436

Report No. AM-67-15

A Note on Simple Dipolar Stresses

by

A. E. Green and P. M. Naghdi

December 1967

Division of Applied Mechanics
University of California
Berkeley, California

A Note on Simple Dipolar Stresses

by

A. E. Green[*] and P. M. Naghdi[+]

Abstract: In the theory of simple force and stress dipoles (within the framework of the simple multipolar theory of Green and Rivlin) the anti-symmetric part of dipolar stresses make no contribution to the equations of motion or to the energy equation and are not determined by a free energy function. Here, the significance of these stresses (in a complete theory) is further discussed with reference to a simple example of torsion of a circular cylinder of an elastic material.

---

[*] University of Newcastle upon Tyne

[+] University of California, Berkeley

1. Introduction.

Green and Rivlin [1] have presented a general theory of a continuum in which stress multipoles of various order were introduced using conventional kinematics. We refer to this as simple multipolar theory and it is valid for any material and includes temperature effects. Further use of this theory has been made by Bleustein and Green [2]. Toupin [3], among other considerations, has developed a theory of couple stresses for isothermal elasticity which he calls materials of grade 2. A special case of the theory of Green and Rivlin for dipolar elastic materials can be shown to be equivalent to the theory of Toupin for materials of grade 2, apart from the presence in the theory of Green and Rivlin of a set of additional stresses which are not determined by the energy function. This difference is, at first sight, unimportant since these stresses make no contribution to the equations of motion, nor to the rate of work terms in the energy equation over any closed surface, and they do not contribute to the resultant force and couple over any closed smooth surface. Green and Rivlin [1] and later Bleustein and Green [2] made use of these stresses in order to find correct surface conditions associated with the basic differential equations of the theory.

One purpose of the present note is to display more clearly the relation between the theory of Green and Rivlin and that of Toupin for elastic materials. Moreover, a straightforward application of Toupin's theory, or that part of the theory of Green and Rivlin [1] which is equivalent to that of Toupin [3], to the problem of simple torsion of a right circular cylinder using linearized elasticity shows that care must be exercised in evaluating the couple by a direct method. This is discussed from a number of points of view in Sec. 4.

2. Basic equations.

We use a fixed system of rectangular Cartesian axes and denote the coordinates of a typical point of the continuum at time t by x_i, where Latin indices take the values 1,2,3. The corresponding reference state is denoted by X_i. We quote the basic equations from Green and Rivlin [1] but use the notation of Bleustein and Green [2].

The conservation of mass equation is

$$\dot{\rho} + \rho v_{k,k} = 0 \quad , \tag{2.1}$$

where ρ is density, v_k velocity and a comma denotes partial differentiation with respect to x_k holding t fixed. Also

$$\dot{\rho} = \frac{\partial\rho}{\partial t} + v_k \frac{\partial\rho}{\partial x_k} \quad . \tag{2.2}$$

The equations of motion for the non-symmetric monopolar stress tensor σ_{ij} are

$$\sigma_{ji,j} + \rho f_i = \rho \dot{v}_i \quad , \tag{2.3}$$

where f_i are monopolar body forces. The monopolar tractions t_i, acting across a surface with outward unit normal n_k, are

$$t_i = n_k \sigma_{ki} \quad , \tag{2.4}$$

The moment equations are

$$\tau_{ij} = \sigma_{ij} + \Sigma_{kij,k} + \rho(F_{ij} - \Gamma_{ij}) = \tau_{ji} \quad . \tag{2.5}$$

In (2.5), F_{ij} are dipolar body forces per unit mass and $\rho\Gamma_{ij}$ are dipolar inertia terms. If the dipolar tractions associated with a surface whose outward unit normal is n_k are T_{ij}, then for an elastic continuum

$$T_{ij} = n_k \Sigma_{kij} \quad . \tag{2.6}$$

Also, for any continuum

$$h = n_k q_k \quad , \tag{2.7}$$

where q_k is the heat flux vector and h the flux of heat across a surface with unit normal n_k .

When the continuum is elastic, we have

$$\Sigma_{(kj)i} = \rho \frac{\partial x_i}{\partial X_r} \frac{\partial x_i}{\partial X_s} \frac{\partial x_k}{\partial X_t} \frac{\partial A}{\partial E_{rst}} \quad , \tag{2.8}$$

$$\tau_{ji} = \rho \left(\frac{\partial x_i}{\partial X_r} \frac{\partial^2 x_j}{\partial X_s \partial X_t} + \frac{\partial x_j}{\partial X_r} \frac{\partial^2 x_i}{\partial X_s \partial X_t} \right) \frac{\partial A}{\partial E_{rst}} + 2\rho \frac{\partial x_i}{\partial X_r} \frac{\partial x_j}{\partial X_s} \frac{\partial A}{\partial E_{rs}} \quad , \tag{2.9}$$

where the Helmholtz free energy function A is specified by

$$A = A(E_{rs}, E_{rst}, T) \quad , \tag{2.10}$$

T is the temperature and

$$E_{rs} = \frac{\partial x_i}{\partial X_r}\frac{\partial x_i}{\partial X_s} \quad , \quad E_{rst} = \frac{\partial x_i}{\partial X_r}\frac{\partial^2 x_i}{\partial X_s \partial X_t} \quad . \tag{2.11}$$

In evaluating (2.8) and (2.9) A is written as a function of $\frac{1}{2}(E_{rs} + E_{sr})$ and $\frac{1}{2}(E_{rst} + E_{rts})$ to avoid ambiguity. Also

$$\Sigma_{kji} = \Sigma_{(kj)i} + \Sigma_{[kj]i} \tag{2.12}$$

with the usual notation for symmetry and anti-symmetry in the indices k , j . The anti-symmetric part of the dipolar stress $\Sigma_{[kj]i}$ is not determined by the Helmholtz function A . These stresses make no contribution to the equations of motion and no contribution to the rate of work equation over any closed surface in the continuum. Previously, Green and Rivlin [1] and Bleustein and Green [2] took the view that these stresses could be taken to be zero at points inside the body but that they played an important role in considering surface conditions.

3. Surface conditions.

Surface conditions were discussed by Green and Rivlin [1] and Bleustein and Green [2]. Here we develop the discussion a little further in order to show the relation of the present theory to that of Toupin [3]. The rate of work of surface forces at time t over any arbitrary closed surface A is

$$R = \iint_A (t_i v_i + T_{ji} v_{i,j}) dA \quad . \tag{3.1}$$

Now

$$v_{i,j} = D_j v_i + n_j Dv_i \quad , \tag{3.2}$$

where

$$D = n_k \frac{\partial}{\partial x_k} \quad , \quad D_j = (\delta_{jk} - n_j n_k) \frac{\partial}{\partial x_k} \quad . \tag{3.3}$$

Provided the quantities involved are single-valued and sufficiently smooth on a smooth closed surface A, we can show that

$$\iint_A [D_j (T_{ji} v_i) + 2Hn_j T_{ji} v_i] dA = 0 \quad , \tag{3.4}$$

$$\iint_A [D_j T_{ji} + 2Hn_j T_{ji}] dA = 0 \quad , \tag{3.5}$$

$$\epsilon_{kij} \iint_A [x_i (D_m T_{mj} + 2Hn_m T_{mj}) + T_{ij} - n_i n_m T_{mj}] dA = 0 \quad , \tag{3.6}$$

where H is the mean curvature at a point on A and ϵ_{kij} is the permutation symbol. Using (3.4), the rate of work expression (3.1) becomes

$$R = \iint_A (t_i^* v_i + t_i^{**} Dv_i)dA \quad , \tag{3.7}$$

where

$$t_i^* = t_i - D_j T_{ji} - 2Hn_j T_{ji} \quad , \tag{3.8}$$

$$t_i^{**} = n_j T_{ji} \quad . \tag{3.9}$$

Moreover, in view of (3.5) and (3.6),

$$\iint_A t_i \, dA = \iint_A t_i^* \, dA \quad , \tag{3.10}$$

$$\epsilon_{kij} \iint_A (x_i t_j + T_{ij})dA = \epsilon_{kij} \iint_A (x_i t_j^* + n_i t_j^{**})dA \quad . \tag{3.11}$$

Following the procedure of Bleustein and Green [2], we can show that t_i^* and t_i^{**} can be expressed in terms of the dipolar stresses $\Sigma_{(kj)i}$ and they do not depend on the stresses $\Sigma_{[kj]i}$. Thus

$$\begin{aligned} t_i^* = n_j \tau_{ji} - \rho n_j (F_{ji} - \Gamma_{ji}) - n_j n_k D\Sigma_{(jk)i} \\ - 2n_j D_k \Sigma_{(jk)i} - (D_k n_j) \Sigma_{(kj)i} \\ - 2Hn_j n_k \Sigma_{(jk)i} \end{aligned} \tag{3.12}$$

and

$$t_i^{**} = n_j n_k \Sigma_{(kj)i} \quad . \tag{3.13}$$

When surface forces are prescribed on any surface A , then t_i^* and t_i^{**} are also prescribed and are the appropriate conditions to be adjoined to the basic differential equations. The conditions (3.12) and (3.13) are equivalent to those used by Green and Rivlin [1] and Bleustein and Green [2] since, if t_i^{**} is prescribed, the corresponding term (the last) in (3.12) can be eliminated from t_i^* before the use of the stress boundary condition. The quantities t_i^* , t_i^{**} in (3.8) and (3.9) can be obtained without discussing the integral (3.1) simply by eliminating $\Sigma_{[kj]i}$ from surface conditions in which the twelve quantities t_i , T_{ij} are specified. They are therefore pointwise conditions.

Toupin [3] using a Hamiltonian principle applicable to perfectly elastic media at constant temperature, introduces (in Sec. 10 of his paper) surface forces by an integral which corresponds to (3.7). His equations for the equilibrium of an elastic solid can be shown to be equivalent to equations (2.3), (2.4), (2.5), (2.6), (2.8), (2.9), (3.8) and (3.9) above when inertia and body force terms are zero and when the stresses $\Sigma_{[kj]i}$ are put equal to zero. These stress components should not be confused with the anti-symmetric components $\Sigma_{k[ji]}$ which correspond to Toupin's stress couple. The stresses $\Sigma_{(kj)i}$ correspond to Toupin's <u>hyperstress tensor</u> which is defined in terms of derivatives of the work function (A here) and is necessarily symmetric in k , j . The stress $\Sigma_{[kj]i}$ plays no part in Toupin's theory because his starting point is equivalent to using (3.7) instead of (3.1).

The foregoing results are valid when all the functions involved are sufficiently smooth and the boundary surface is a closed smooth surface A. When there are discontinuities across an edge in the direction of the tangent planes to the surface, additional considerations are required. We now indicate briefly the necessary modifications.

Suppose C is a simple closed curve on a closed surface A dividing A into two parts A_1 and A_2, and suppose there are discontinuities in T_{ji} or in the tangent plane to A across C. For definiteness, let $\underset{\sim}{n}$ (with components n_i) stand for the outward unit normal vector to the surface, $\underset{\sim}{s}$ (with components s_i) for the unit tangent vector to C and $\underset{\sim}{m} = \underset{\sim}{s} \times \underset{\sim}{n}$ (with components $m_i = \epsilon_{ipq} s_p n_q$) for the unit normal vector to C. Further let $\underset{\sim}{n}^{(1)}$ and $\underset{\sim}{n}^{(2)}$ refer to the outward unit normals to A_1 and A_2; and let $\underset{\sim}{s}^{(1)}$, $\underset{\sim}{m}^{(1)}$ and $\underset{\sim}{s}^{(2)}$, $\underset{\sim}{m}^{(2)}$ stand for the values of $\underset{\sim}{s}$, $\underset{\sim}{m}$ when C is part of A_1 and when C is part of A_2, respectively. Then, instead of (3.4) to (3.6), we have

$$\iint_A [D_j(T_{ji} v_i) + 2Hn_j T_{ji} v_i]\, dA = \int_C \llbracket m_j T_{ji} v_i \rrbracket\, dC \tag{3.4a}$$

$$\iint_A [D_j T_{ji} + 2Hn_j T_{ji}]\, dA = \int_C \llbracket m_j T_{ji} \rrbracket\, dC \quad , \tag{3.5a}$$

$$\epsilon_{kij} \iint_A [x_i (D_m T_{mj} + 2Hn_m T_{mj}) + T_{ij} - n_i n_m T_{mj}]\, dA$$

$$= \epsilon_{kij} \int_C \llbracket x_i m_p T_{pj} \rrbracket\, dC \quad . \tag{3.6a}$$

In (3.4a) to (3.6a), the surface integrals are taken over the entire surface $A = A_1 + A_2$, the circuit line integrals are taken along the direction of the unit tangent to C with $s_i^{(2)} = - s_i^{(1)}$ (keeping the area in question to the left), and

$$[\![f]\!] = f_1 + f_2 \quad ,$$

where f is any function which takes different values f_1 and f_2 on either side of C in the surfaces A_1 and A_2, respectively.

Also, in view of (3.5a) and (3.6a), it follows that

$$\iint_A t_i \, dA = \iint_A t_i^* \, dA + \int_C [\![m_j T_{ji}]\!] \, dC \quad , \tag{3.10a}$$

$$\epsilon_{kij} \iint_A (x_i t_j + T_{ij}) \, dA$$

$$= \epsilon_{kij} \iint_A (x_i t_j^* + n_i t_j^{**}) \, dA + \epsilon_{kij} \int_C [\![x_i m_p T_{pj}]\!] \, dC \ . \tag{3.11a}$$

We have already seen that t_i^*, t_i^{**} do not involve the anti-symmetric stresses $\Sigma_{[kj]i}$. It can also be shown that the integrands in the line integrals in (3.10a) and (3.11a) are independent of $\Sigma_{[kj]i}$ provided these stresses are continuous functions at C.

4. Torsion of a circular cylinder.

In this section, we quote the results for the special case of linearized elasticity. All stresses are now referred to points in the initial reference body and are measured per unit surface area in this body. For convenience, we use x_k now to denote points in the initial body. Then, with ρ now denoting the initial density,

$$\rho A = \frac{1}{2} \lambda\, e_{rr}\, e_{ss} + \mu\, e_{rs}\, e_{rs}$$

$$+ a_1\, e_{i(ik)}\, e_{j(jk)} + a_2\, e_{i(ik)}\, e_{k(jj)}$$

$$+ a_3\, e_{k(ii)}\, e_{k(jj)} + a_4\, e_{i(jk)}\, e_{i(jk)}$$

$$+ a_5\, e_{i(jk)}\, e_{k(ji)} \tag{4.1}$$

for a solid which is initially isotropic with a center of symmetry, where λ , μ , $a_1, \ldots, a_5$ are constants. Also

$$e_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i}) \quad ,$$

$$e_{i(jk)} = u_{i,jk} \quad , \tag{4.2}$$

where u_i is the displacement vector. It follows that

$$\tau_{ij} = \lambda\, e_{rr}\, \delta_{ij} + 2\, \mu\, e_{ij} \quad , \tag{4.3}$$

$$\Sigma_{(kj)i} = a_1 [e_{r(rk)}\, \delta_{ji} + e_{r(rj)}\, \delta_{ki}]$$

$$+ \frac{1}{2} a_2 [e_{k(rr)}\, \delta_{ij} + 2e_{r(ri)}\, \delta_{jk} + e_{j(rr)}\, \delta_{ik}]$$

$$+ 2\, a_3\, e_{i(rr)}\, \delta_{jk} + 2\, a_4\, e_{i(jk)}$$

$$+ a_5 [e_{k(ji)} + e_{j(ki)}] \quad . \tag{4.4}$$

For equilibrium under zero body forces

$$\sigma_{ji,j} = 0 \quad ,$$

$$\tau_{ij} = \sigma_{ij} + \Sigma_{kij,k} \quad , \qquad (4.5)$$

or

$$\tau_{ji,j} - \Sigma_{(kj)i,kj} = 0 \quad . \qquad (4.6)$$

We now examine the pure torsion of a right circular cylinder $|x_3| \leqq \ell$, $x_1^2 + x_2^2 = r^2 \leqq a^2$, and assume that the displacements are of the form

$$u_1 = - \tau x_2 x_3 \quad , \quad u_2 = \tau x_1 x_3 \quad , \quad u_3 = 0 \quad , \qquad (4.7)$$

where τ is a constant. The only non-vanishing strains are

$$e_{13} = e_{31} = - \frac{1}{2} \tau x_2 \quad , \quad e_{23} = e_{32} = \frac{1}{2} \tau x_1 \quad ,$$

$$e_{1(23)} = - e_{2(13)} = - \tau \quad . \qquad (4.8)$$

From (4.3) and (4.4), we find that the only non-vanishing stresses are

$$\tau_{13} = \tau_{31} = - \mu \tau x_2 \quad , \quad \tau_{23} = \tau_{32} = \mu \tau x_1 \quad ,$$

$$\Sigma_{(32)1} = - \Sigma_{(31)2} = - \eta \tau \quad , \qquad (4.9)$$

where

$$\eta = 2\, a_4 - a_5 \quad . \tag{4.10}$$

The equations of equilibrium (4.6) are satisfied.

On the surface $r = a$ (in fact, on any surface r = constant) it can be verified at once from (3.12), (3.13) and (4.9) that

$$t_i^* = 0 \quad , \quad t_i^{**} = 0 \quad . \tag{4.11}$$

Moreover, over any surface x_3 = constant,

$$\begin{aligned} t_1^* &= \tau_{31} - 2\, \Sigma_{(32)1,2} \quad , \\ t_2^* &= \tau_{32} - 2\, \Sigma_{(31)2,1} \quad , \\ t_3^* &= 0 \quad , \quad t_i^{**} = 0 \quad . \end{aligned} \tag{4.12}$$

In order to evaluate the couple acting on the cylinder, in view of (4.11) for $r = a$, we use the right hand side of (3.11a) with the surface integral evaluated over any surface x_3 = constant and the line integral evaluated along the curve x_3 = constant, $r = a$. Thus

$$\begin{aligned} M = {} & \epsilon_{3ij} \iint_A (x_i\, t_j^* + n_i\, t_j^{**})\, dx_1\, dx_2 \\ & + \epsilon_{3ij} \int_c [\![x_i\, m_p\, T_{pj}]\!]\, dc \quad , \end{aligned} \tag{4.13}$$

the surface integral being evaluated over the plane area x_3 = constant and the line integral along the curve x_3 = constant, $r = a$. The anti-symmetric stresses $\Sigma_{[kj]i}$ make no contribution to either integral in (4.13) and we have

$$M = \iint_A [x_1 \tau_{32} - x_2 \tau_{31} - 2(x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2}) \Sigma_{(31)2}] \, dx_1 \, dx_2$$

$$+ 2 a^2 \int_0^{2\pi} \Sigma_{(31)2} \, d\theta$$

$$= \int_0^{2\pi} \int_0^a [\mu \tau r^2 + 4 \Sigma_{(31)2}] \, r dr d\theta = \frac{1}{2} \pi \mu \tau a^4 + 4 \pi \eta \tau a^2 , \qquad (4.14)$$

where (4.9), (4.11) and (4.12) have been used. It must be emphasized that the line integral in (4.13) contributes an important part to M and incorrect results would be obtained if this term is omitted. This term does not appear to be explicitly mentioned in Toupin's theory.*

On the other hand, from (4.1) and (4.8), we have

$$\rho A = \frac{1}{2} \mu \tau^2 r^2 + 2 \eta \tau^2 , \qquad (4.15)$$

from which

$$\iint \rho A \, dx_1 \, dx_2 = 2\pi \tau^2 a^2 \left(\frac{\mu a^2}{8} + \eta \right) . \qquad (4.16)$$

This must equal $\frac{1}{2} M \tau$ so that

$$M = \frac{1}{2} \pi \mu \tau a^4 + 4 \pi \eta \tau a^2 , \qquad (4.17)$$

in agreement with (4.14).

---

[*] Line integral terms were mentioned by Toupin in his paper on the old couple stress theory [4].

We now show how the same result can be recovered using the complete theory of Green and Rivlin. As in [1], we suppose that $\Sigma_{[kj]i}$ have zero values at points in the interior of the body but not on its boundary surface. Then, over the surface $r = a$ of the cylinder we can have

$$t_i = 0 \quad , \quad T_{ji} = 0 \quad , \tag{4.18}$$

provided

$$\begin{aligned} \Sigma_{[13]2} &= - \Sigma_{(13)2} = - \eta \tau \quad , \\ \Sigma_{[23]1} &= - \Sigma_{(23)1} = \eta \tau \quad , \end{aligned} \tag{4.19}$$

the remaining components of $\Sigma_{[kj]i}$ being zero.[+] Over any plane $x_3 =$ constant we have

$$\begin{aligned} t_1 &= \sigma_{31} = \tau_{31} - \Sigma_{k31,k} \quad , \\ t_2 &= \sigma_{32} = \tau_{32} - \Sigma_{k32,k} \quad , \\ t_3 &= 0 \quad , \\ T_{12} &= \Sigma_{312} \quad , \\ T_{21} &= \Sigma_{321} \quad , \end{aligned} \tag{4.20}$$

[+] See the remark after Eq. (4.20).

which involve only the tangential derivatives of $\Sigma_{[kj]i}$ and the remaining T_{ij} being zero. We remark here that the conditions $(4.18)_2$ can be satisfied over $r = a$ by the more general relations of the form

$$\begin{aligned} &\Sigma_{[13]2} = -\Sigma_{(13)2} + \alpha_1 n_2 \ , \quad \Sigma_{[23]2} = -\alpha_1 n_1 \ , \\ &\Sigma_{[13]1} = -\alpha_2 n_2 \ , \quad \Sigma_{[23]1} = -\Sigma_{(23)1} + \alpha_2 n_1 \ , \\ &\Sigma_{[13]3} = \alpha_3 n_2 \ , \quad \Sigma_{[23]3} = -\alpha_3 n_1 \ . \end{aligned} \tag{4.21}$$

However, the arbitrary coefficients α_i $(i=1,2,3)$ in (4.21) may be set equal to zero without loss in generality and in what follows we adopt (4.19) instead of (4.21).**

Using the left-hand side of (3.11), the couple M is now

** It may be noted that we are seeking a solution which would yield zero stresses on $r = a$ and a couple over $x_3 =$ constant and, as will become evident shortly, in obtaining this solution we only utilize the continuity of $\Sigma_{[kj]i}$ on $r = a$, $x_3 =$ const. If the forms (4.21) are retained instead of (4.19), we can then show that the additional terms give rise to zero resultant forces calculated from the left-hand side of (3.10) with $i = 1,2$ and zero couples obtained from the left-hand side of (3.11) with $k = 1,2$. Moreover, the additional terms in (4.21) will not contribute to the couple calculated from the left-hand side of (3.11) with $k = 3$.

$$M = \iint (x_1 t_2 - x_2 t_1 + T_{12} - T_{21}) \, dx_1 \, dx_2$$

$$= M_1 + M_2 \quad , \tag{4.22}$$

where

$$M_1 = \iint \Big\{ x_1 [\tau_{32} - \Sigma_{(13)2,1}] - x_2 [\tau_{31} - \Sigma_{(23)1,2}]$$

$$+ \Sigma_{(31)2} - \Sigma_{(32)1} \Big\} \, dx_1 \, dx_2 \quad , \tag{4.23}$$

$$M_2 = \iint \Big\{ - x_1 \Sigma_{[\alpha 3]2,\alpha} + x_2 \Sigma_{[\alpha 3]1,\alpha} + \Sigma_{[31]2} - \Sigma_{[32]1} \Big\} \, dx_1 \, dx_2 , (\alpha = 1,2). \tag{4.24}$$

The right hand-side of (4.23) is independent of $\Sigma_{[kj]i}$ and (4.24) which involves $\Sigma_{[kj]i}$ and its tangential derivatives may be expressed in the form

$$M_2 = \iint \Big\{ \frac{\partial}{\partial x_1} [x_1 \Sigma_{[31]2} + x_2 \Sigma_{[13]1}]$$

$$- \frac{\partial}{\partial x_2} [x_1 \Sigma_{[23]2} + x_2 \Sigma_{[32]1}] \Big\} \, dx_1 \, dx_2$$

$$= \int \Big\{ x_1 \Sigma_{[31]2} + x_2 \Sigma_{[13]1} \Big\} dx_2 + \Big\{ x_1 \Sigma_{[23]2} + x_2 \Sigma_{[32]1} \Big\} dx_1 \quad , \tag{4.25}$$

where the line integral is evaluated along $r = a$, $x_3 = \text{const.}$ With the help of (4.9), (4.23) yields

$$M_1 = \frac{1}{2} \pi \mu \tau a^4 + 2 \pi \eta \tau a^2 \quad . \tag{4.26}$$

Assuming that $\Sigma_{[kj]i}$ are continuous on $r = a$, $x_3 =$ constant, we also obtain

$$M_2 = \int \eta \tau (x_1 \, dx_2 - x_2 \, dx_1) = 2 \pi \eta \tau a^2 \quad , \tag{4.27}$$

since $(4.25)_2$ involves only the values of $\Sigma_{[kj]i}$ on $r = a$. Hence by $(4.22)_2$, we have

$$M_1 + M_2 = \frac{1}{2} \pi \mu \tau a^4 + 4 \pi \eta \tau a^2 \tag{4.28}$$

which is the required result and which is also in agreement with (4.14) and (4.17). In contrast to (4.13), the formula (4.22) for M contains no contribution from a line integral involving discontinuities.

Acknowledgement. The work of one of us (P.M.N.) was supported by the U. S. Office of Naval Research under Contract Nonr-222(69) with the University of California, Berkeley.

References

[1] A. E. Green and R. S. Rivlin, Arch. Rational Mech. Anal. 16, 325-353 (1964).

[2] J. L. Bleustein and A. E. Green, Int. J. Engng. Sci. 5, 323-240 (1967).

[3] R. A. Toupin, Arch. Rational Mech. Anal. 17, 85-112 (1964).

[4] R. A. Toupin, Arch. Rational Mech. Anal. 11, 385-414 (1962).

Unclassified
Security Classification

DOCUMENT CONTROL DATA - R&D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY *(Corporate author)* Division of Applied Mechanics University of California Berkeley, California	**2a. REPORT SECURITY CLASSIFICATION** Unclassified **2b. GROUP**

3. REPORT TITLE

A Note on Simple Dipolar Stresses

4. DESCRIPTIVE NOTES *(Type of report and inclusive dates)*

Technical Report

5. AUTHOR(S) *(Last name, first name, initial)*

Green, A. E.

Naghdi, P. M.

6. REPORT DATE December 1967	**7a. TOTAL NO. OF PAGES** 18	**7b. NO. OF REFS** 4

8a. CONTRACT OR GRANT NO. Nonr-222(69) **b. PROJECT NO.** NR-064-436 **c.** **d.**	**9a. ORIGINATOR'S REPORT NUMBER(S)** AM-67-15 **9b. OTHER REPORT NO(S)** *(Any other numbers that may be assigned this report)*

10. AVAILABILITY/LIMITATION NOTICES

11. SUPPLEMENTARY NOTES	**12. SPONSORING MILITARY ACTIVITY** Office of Naval Research Structural Mechanics Branch Washington, D. C.

13. ABSTRACT

In the theory of simple force and stress dipoles (within the framework of the simple multipolar theory of Green and Rivlin) the anti-symmetric part of dipolar stresses make no contribution to the equations of motion or to the energy equation and are not determined by a free energy function. Here, the significance of these stresses (in a complete theory) is further discussed with reference to a simple example of torsion of a circular cylinder of an elastic material.

DD FORM 1 JAN 64 1473

Unclassified
Security Classification

14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Simple dipolar stress						
Boundary conditions						
Elastic						
Torsion						

INSTRUCTIONS

1. ORIGINATING ACTIVITY: Enter the name and address of the contractor, subcontractor, grantee, Department of Defense activity or other organization *(corporate author)* issuing the report.

2*a*. REPORT SECURITY CLASSIFICATION: Enter the overall security classification of the report. Indicate whether "Restricted Data" is included. Marking is to be in accordance with appropriate security regulations.

2*b*. GROUP: Automatic downgrading is specified in DoD Directive 5200.10 and Armed Forces Industrial Manual. Enter the group number. Also, when applicable, show that optional markings have been used for Group 3 and Group 4 as authorized.

3. REPORT TITLE: Enter the complete report title in all capital letters. Titles in all cases should be unclassified. If a meaningful title cannot be selected without classification, show title classification in all capitals in parenthesis immediately following the title.

4. DESCRIPTIVE NOTES: If appropriate, enter the type of report, e.g., interim, progress, summary, annual, or final. Give the inclusive dates when a specific reporting period is covered.

5. AUTHOR(S): Enter the name(s) of author(s) as shown on or in the report. Enter last name, first name, middle initial. If military, show rank and branch of service. The name of the principal author is an absolute minimum requirement.

6. REPORT DATE: Enter the date of the report as day, month, year; or month, year. If more than one date appears on the report, use date of publication.

7*a*. TOTAL NUMBER OF PAGES: The total page count should follow normal pagination procedures, i.e., enter the number of pages containing information.

7*b*. NUMBER OF REFERENCES: Enter the total number of references cited in the report.

8*a*. CONTRACT OR GRANT NUMBER: If appropriate, enter the applicable number of the contract or grant under which the report was written.

8*b*, 8*c*, & 8*d*. PROJECT NUMBER: Enter the appropriate military department identification, such as project number, subproject number, system numbers, task number, etc.

9*a*. ORIGINATOR'S REPORT NUMBER(S): Enter the official report number by which the document will be identified and controlled by the originating activity. This number must be unique to this report.

9*b*. OTHER REPORT NUMBER(S): If the report has been assigned any other report numbers *(either by the originator or by the sponsor)*, also enter this number(s).

10. AVAILABILITY/LIMITATION NOTICES: Enter any limitations on further dissemination of the report, other than those imposed by security classification, using standard statements such as:

(1) "Qualified requesters may obtain copies of this report from DDC."

(2) "Foreign announcement and dissemination of this report by DDC is not authorized."

(3) "U. S. Government agencies may obtain copies of this report directly from DDC. Other qualified DDC users shall request through
______________________________."

(4) "U. S. military agencies may obtain copies of this report directly from DDC. Other qualified users shall request through
______________________________."

(5) "All distribution of this report is controlled. Qualified DDC users shall request through
______________________________."

If the report has been furnished to the Office of Technical Services, Department of Commerce, for sale to the public, indicate this fact and enter the price, if known.

11. SUPPLEMENTARY NOTES: Use for additional explanatory notes.

12. SPONSORING MILITARY ACTIVITY: Enter the name of the departmental project office or laboratory sponsoring *(paying for)* the research and development. Include address.

13. ABSTRACT: Enter an abstract giving a brief and factual summary of the document indicative of the report, even though it may also appear elsewhere in the body of the technical report. If additional space is required, a continuation sheet shall be attached.

It is highly desirable that the abstract of classified reports be unclassified. Each paragraph of the abstract shall end with an indication of the military security classification of the information in the paragraph, represented as *(TS)*, *(S)*, *(C)*, or *(U)*.

There is no limitation on the length of the abstract. However, the suggested length is from 150 to 225 words.

14. KEY WORDS: Key words are technically meaningful terms or short phrases that characterize a report and may be used as index entries for cataloging the report. Key words must be selected so that no security classification is required. Identifiers, such as equipment model designation, trade name, military project code name, geographic location, may be used as key words but will be followed by an indication of technical context. The assignment of links, roles, and weights is optional.