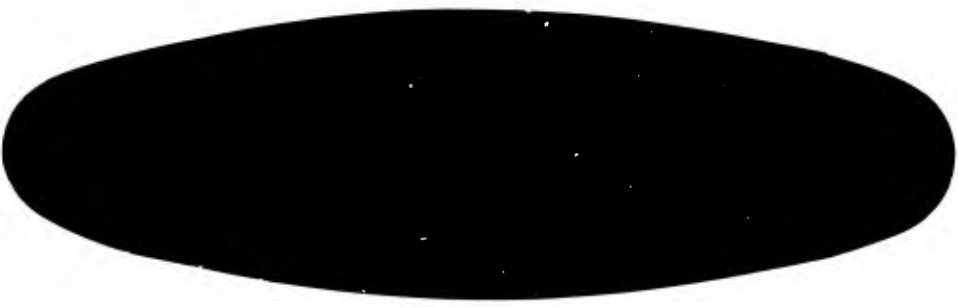


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TWO PAPERS ON ADAPTIVE INVENTORY CONTROL

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## TWO PAPERS ON ADAPTIVE INVENTORY CONTROL

### Summary

In this final report for the contract year we present two papers on adaptive inventory control. The first entitled

#### An Adaptive Inventory Model with Lead Time

employs process control techniques, developed by G. E. P. Box and G. Jenkins [1] to obtain an optimal order procedure for a multiperiod inventory problem. At the start of each of an infinite number of periods an order may be placed to adjust the inventory level in anticipation of future demands. The cost of ordering is a fixed cost  $C_1$  which is to be balanced against a loss proportional to the square of the on-hand inventory (or inventory shortage).

The paper first considers the problem in which there are a finite number of periods and in which the lead time between an order and its delivery is  $T$  units. The results obtained are then extended to the case in which the number of periods approaches infinity. The optimal order procedure is adaptive in the sense that it uses information on past demand to forecast future requirements.

The second paper

#### A Bayesian Approach to an Inventory Problem

based on the work of Jacob Marschak [2] considers a similar problem; orders to satisfy anticipated demand are to be placed at the start of each of a finite number of periods. If too many units are ordered each unit of surplus at the end of the period incurs a cost  $C_1$  while if too

few units are ordered each unit of shortage results in a cost  $C_2$ .

The optimal procedure is adaptive in that Bayesian procedures are used with the accumulated demand data to estimate the unknown parameters of the demand distribution.

## AN ADAPTIVE INVENTORY MODEL WITH LEAD TIME

### 1. Introduction

In this note we consider an inventory process at discrete time points  $0, 1, 2, \dots, p, p+1, \dots$ . Let  $d_p$  be the demand in the interval  $[p, p+1]$ , and let  $x_p^*$  be the amount of inventory ordered at time  $p$ . Then  $z_p = \sum_{i=0}^p d_i$  represents the cumulative demand at the end of period  $[p, p+1]$  and  $X_p^* = \sum_{i=0}^p x_i^*$  represents the cumulative orders by time  $p$ . We assume a leadtime of  $T$  units, that an order placed at time  $p$  will not arrive until time  $p+T$ . Let  $x_p(T) = x_p^*$  represent the amount of inventory which arrive at time  $p+T$  but which was ordered at time  $p$ .

Let

$$X_p(T) \equiv \sum_{i=0}^p x_i(T) .$$

$X_p(T)$  then represents the total inventory that has arrived by time  $p+T$ .

There is a fixed administrative cost  $C$  of placing an order and at each time  $p$  we loose an amount  $a(z_p - X_{p-T}(T))^2$ ,  $a > 0$ . The quantity  $|z_p - X_{p-T}(T)|$  is the magnitude of a shortage or a surplus the end of the  $p^{\text{th}}$  period. At each time  $p$  we must decide on one of two actions, to order or not to order. If we decide to order we must know the quantity to order. Since there is a leadtime of  $T$  units, our ordering policy of time  $p$  will not become effective until time  $p+T$ . Suppose that at time  $p$  we place an order for an amount  $x_p^*$ . In time period  $p+T$  we can therefore expect to loose an amount  $C + a(z_{p+T} - X_p(T))^2$ . If we place no order at time  $p$ , we will loose an amount  $a(z_{p+T} - X_{p-1}(T))^2$ . Hence if we decide to order at time  $p$  we should order an amount  $x_p^*$

such that

$$E(a(z_{p+T} - X_p(T))^2 | z_1, \dots, z_p) = E(a(z_{p+T} - (x_p^* + X_{p-1}(T)))^2 | z_1, \dots, z_p)$$

is a minimum.

If we decide not to order, the expected cost of not ordering should be less than the expected cost of ordering the optimal amount, that is

$$E(a(z_{p+T} - X_{p-1}(T))^2 | z_1, \dots, z_p) < E(a(z_{p+T} - X_p(T))^2 | z_1, \dots, z_p) .$$

## 2. Cumulative demand model

We assume that  $z_{p+1} = \gamma_0 S \alpha_p + \alpha_{p+1}$ , where  $S \alpha_p = \sum_{i=1}^p \alpha_i$  and the  $\alpha_p$  are independent normally distributed random variables with mean 0 and variance  $\sigma^2$ . Hence

$$z_{p+T} = \gamma_0 S \alpha_{p+T-1} + \alpha_{p+T} .$$

The minimum variance estimate of  $z_{p+T}$  at time  $p$  is given by

$$\hat{z}_p(T) = E(z_{p+T} | z_1, z_2, \dots, z_p) = \gamma_0 S \alpha_p .$$

Suppose that we have observed the  $z$  process up to time  $p$  and must decide whether or not to place a final order at time  $p$ . If we order an amount  $x_p^*$ , then we want to insure that  $E(a(z_p(T) - X_p(T))^2 | z_1, \dots, z_p)$  is a minimum. Since

$$X_p(T) = x_p^* + X_{p-1}(T)$$

we choose

$$x_p^* = \hat{z}_p(T) - X_{p-1}(T)$$

so that

$$X_p(T) = \hat{z}_p(T) .$$

Let

$L_1^0(\hat{z}_p(T))$  = the time expected loss if we do not place an order at time  $p$ ,

and

$L_1^1(\hat{z}_p(T))$  = the expected loss if an order is placed.\*

We have

$$L_1^0(\hat{z}_p(T)) = E(a(z_{p+T} - X_{p-1}(T))^2 | z_1, \dots, z_p)$$

and

$$z_{p+T} = \hat{z}_p(T) + \alpha_{p+T} + \gamma_0 \sum_{i=1}^{T-1} \alpha_{p+i} = \hat{z}_p(T) + \sigma C_T u$$

where  $C_T = \sqrt{1 + \gamma_0^2(T-1)}$  and  $u$  is a unit normal random variable independent of  $\hat{z}_p(T)$ . Therefore

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\* The losses are really a function of the variable  $|\hat{z}_p(T) - X_{p-1}(T)|$  although in keeping with Box's notation we have suppressed the dependence on the  $X$ 's.

$$L_1^0(\hat{z}_p(T)) = E[a(\hat{z}_p(T) + C_T \sigma u - X_{p-1}(T))^2 | z_1, \dots, z_p] \\ = a(\hat{z}_p(T) - X_{p-1}(T))^2 + a C_T^2 \sigma^2 .$$

Similarly

$$L_1^1(\hat{z}_p(T)) = E(a(z_{p+T} - \hat{z}_p(T))^2 | z_1, \dots, z_p) + C \\ = E(a(\sigma^2 C_T^2 u^2)) + C = a\sigma^2 C_T^2 + C .$$

Hence we order if and only if  $L_1^1(\hat{z}_p(T)) \leq L_1^0(\hat{z}_p(T))$  or equivalently if and only if  $|\hat{z}_p(T) - X_{p-1}(T)| \geq \sqrt{\frac{C}{a}} = \lambda_1$ . The loss for this case is

$$L_1(\hat{z}_p(T)) = \text{Min}\{L_1^0(\hat{z}_p(T)), L_1^1(\hat{z}_p(T))\} = \text{Min}\{a\sigma^2 C_T^2 + C, a\sigma^2 C_T^2 + a(\hat{z}_p(T) - X_{p-1}(T))^2\}$$

Now suppose that we can observe the  $z$  process from time  $p$  to time  $p+N-1$ . We must make  $N$  decisions, one at each of time points  $p, p+1, \dots, p+N-1$ . Let

$$L_N^0(\hat{z}_p(T)) = \text{the expected loss if we do not order at time } p$$

and

$$L_N^1(\hat{z}_p(T)) = \text{the expected loss if we order at time } p.$$

For the case of  $N=2$  we have

$$L_2^0(\hat{z}_p(T)) = a(\hat{z}_p(T) - X_{p-1}(T))^2 + aC_T^2 \sigma^2 \\ + \int \text{Min}\{a^2 C_T^2 \sigma^2 + C, a(\hat{z}_{p+1}(T) - X_{p-1}(T))^2 + aC_T^2 \sigma^2\} dF(\hat{z}_{p+1}(T) | z_1, \dots, z_p)$$

where  $F(\hat{z}_{p+1}(T)|z_1, \dots, z_p)$  represents the conditional distribution of  $\hat{z}_{p+1}(T)$  given  $z_1, \dots, z_p$ . But

$$\hat{z}_{p+1}(T) = \gamma_0 S \alpha_{p+1} = \gamma_0 \alpha_{p+1} + \hat{z}_p(T) = \gamma_0 \sigma u + \hat{z}_p(T)$$

where

$$u \sim N(0,1) .$$

So

$$L_2^0(z_p(T)) = aC_T^2 \sigma^2 + a(\hat{z}_p(T) - X_{p-1}(T))^2 + \int \text{Min}\{aC_T^2 \sigma^2 + C, a(\hat{z}_p(T) + \gamma_0 \sigma u - X_{p-1}(T))^2 + aC_T^2 \sigma^2\} \frac{e^{-u^2/2}}{\sqrt{2\pi}} du$$

Similarly

$$L_2^1(\hat{z}_p(T)) = aC_T^2 \sigma^2 + C + \int \text{Min}\{aC_T^2 \sigma^2 + C, a(\sigma \gamma_0 u)^2\} \frac{e^{-u^2/2}}{\sqrt{2\pi}} du .$$

Because of the symmetry of the normal distribution

$$L_2^1(\hat{z}_p(T)) \geq L_2^0(\hat{z}_p(T)) \text{ if and only if } |\hat{z}_p(T) - X_{p-1}(T)| \leq \lambda_2 .$$

In general

$$L_N^0(\hat{z}_p(T)) = aC_T^2 \sigma^2 + a(\hat{z}_p(T) - X_{p-1}(T))^2 + \int L_{N-1}(\hat{z}_p(T) + \gamma_0 \sigma u) \frac{e^{-u^2/2}}{\sqrt{2\pi}} du$$

and

$$L_N^1(\hat{z}_p(T)) = aC_T^2\sigma^2 + C + \int L_{N-1}(\hat{z}_p(T) + \gamma_0\sigma u) \frac{e^{-u^2/2}}{\sqrt{2\pi}} du .$$

The last terms of these expressions depend on  $X_p(T)$ , and, in general, are different because they depend on which action was taken at time  $p$ .

By symmetry it follows that  $L_N^0(\hat{z}_p(T)) \geq L_N^1(\hat{z}_p(T))$  if and only if  $|\hat{z}_p(T) - X_p(T)| = |\hat{z}_p(T) - X_{p-1}(T)| \geq \lambda_N$ . The  $\lambda_N$  form a decreasing sequence of numbers and hence  $\lambda = \lim \lambda_N$  exists.  $\lambda$  corresponds to the case where we observe the process at infinitely many time points.

Summarizing, our decision rule is the following: Order an amount

$$x_p^* = \hat{z}_p(T) - X_{p-1}(T) \quad \text{if} \quad |\hat{z}_p(T) - X_{p-1}(T)| \geq \lambda$$

and do not order if

$$|\hat{z}_p(T) - X_{p-1}(T)| < \lambda .$$

### 3. Estimation of $\lambda$

Suppose that at time  $p$  we order but do not order again until time  $p+\hat{n}$ . Let  $\hat{n}$  be called the run length. The average loss during a run may be written as

$$L_R = C + E(S_{\hat{n}})$$

where

$$S_{\hat{n}} = \sum_{i=0}^{\hat{n}-1} a(z_{p+T+1} - \hat{z}_p(T))^2 .$$

$S_{\hat{n}}$  represents the loss within a run and  $C$  is the cost of placing the one order. Hence the average loss per period may be written as

$$L = \frac{C}{E(\hat{n})} + \frac{E(S_{\hat{n}})}{E(\hat{n})} .$$

$$E(S_{\hat{n}}) = E(E(S_{\hat{n}}|\hat{n})) \quad \text{and} \quad E(S_{\hat{n}}|\hat{n}) = \sum_{i=0}^{\hat{n}-1} aE[(z_{p+T+1} - \hat{z}_p(T))^2 | z_1, z_2, \dots, z_{p+1}] .$$

Using the relation

$$z_{p+T+1} = \hat{z}_{p+1}(T) + \gamma_0 \sum_{j=1}^{T-1} \alpha_{p+j+1} + \alpha_{p+T+1}$$

and substituting into the previous expression yields,

$$E(S_{\hat{n}}|\hat{n}) = \sum_{i=0}^{\hat{n}-1} a(\hat{z}_{p+1}(T) - \hat{z}_p(T))^2 + \sigma^2 C_T^2 \hat{n} .$$

Hence the average loss per observation may be written as

$$L = \frac{C}{E(\hat{n})} + \sigma^2 C_T^2 + \frac{E\left(\sum_{i=1}^{\hat{n}-1} a(\hat{z}_{p+1}(T) - \hat{z}_p(T))^2\right)}{E(\hat{n})} \quad (1)$$

For any  $i$  we have  $(\hat{z}_{p+1}(T) - \hat{z}_p(T)) = \gamma_0 \sigma (u_{p+1} + u_{p+2} + \dots + u_{p+T})$  where the  $u$ 's are independent normal random variables with mean 0 and variance 1. It suffices to know the value of  $\lambda$  if  $\gamma_0 = \sigma = 1$  since  $|\hat{z}_{p+1}(T) - \hat{z}_p(T)| > \lambda$  if and only if  $|\sum_{i=1}^{\hat{n}} u_{p+i}| > \lambda/\gamma_0 \sigma = \Lambda$ . Hence if

we can find  $\Lambda$  we obtain  $\lambda$  by multiplication. So assume that  $\gamma_0 = \sigma = 1$ . The procedure for computing  $\Lambda$  is to pick a  $\Lambda$  compute the loss  $L(\Lambda)$  associated with it and use for the optimum  $\Lambda$  the number  $\Lambda_{\text{opt}}$  such that  $L(\Lambda_{\text{opt}}) = \inf_{\Lambda} L(\Lambda)$ .

Suppose that  $\Lambda$  is chosen. Compute quantities  $\hat{n}_1, \dots, \hat{n}_h$  by the following method. For each  $j = 1, 2, \dots, h$  draw from the unit normal distribution  $u_1, u_2, \dots, u_{\hat{n}_j}$  where  $|\sum_{i=1}^{\hat{n}_j} u_i| > \Lambda$ , and  $K < \hat{n}_j$  implies that  $|\sum_{i=1}^K u_i| \leq \Lambda$ . Hence each  $\hat{n}_j$  is a value of the run length. The estimate  $\hat{E}(\hat{n})$  of  $E(\hat{n})$  is given by  $\frac{1}{h} \sum_{j=1}^h \hat{n}_j$ . To estimate

$$E\left(\sum_{i=1}^{\hat{n}-1} a(\hat{z}_{p+1}(T) - \hat{z}_p(T))\right)^2$$

we proceed as follows.

For each  $\hat{n}_j$  and for each  $k < \hat{n}_j$  we draw the sample  $u_{1,k}^j, u_{2,k}^j, \dots, u_{k,k}^j$  from the unit normal distribution, and compute the quantities  $(\sum_{\ell=1}^k u_{\ell,k}^j)^2$  and  $\sum_{k=1}^{\hat{n}_j-1} a(\sum_{\ell=1}^k u_{\ell,k}^j)^2$ . It should be emphasized that for each  $k < \hat{n}_j$  a different sample is drawn from the unit normal distribution. The quantity

$$\sum_{k=1}^{\hat{n}_j-1} a\left(\sum_{\ell=1}^k u_{\ell,k}^j\right)^2$$

is equal to

$$\sum_{i=1}^{\hat{n}_j-1} a(\hat{z}_{p+1}(T) - \hat{z}_p(T))^2$$

by relation (1). Our estimate  $\hat{E}(S_{\hat{n}})$  of  $E(S_{\hat{n}})$  is given by

$$\frac{1}{h} \sum_{j=1}^h \left( \sum_{i=1}^{\hat{n}_j-1} a \left( \sum_{n=1}^i u_{n,i}^j \right)^2 \right) + c_T^2 \hat{E}(\hat{n}) .$$

Hence

$$L(\Lambda) = \frac{c}{\hat{E}(\hat{n})} + \frac{\hat{E}(S_{\hat{n}})}{\hat{E}(\hat{n})} .$$

We repeat this process until we find a  $\Lambda_{\text{opt}}$  which minimizes  $L(\Lambda)$ .

## A BAYESIAN APPROACH TO AN INVENTORY PROBLEM

### 1.0 Introduction

Suppose we are observing a finite discrete stochastic process  $X_t$ ,  $0 \leq t \leq T$  where  $X_0 = 0$  and  $X_t$  represents cumulative loss by the end of the period  $(t-1, t)$  for  $1 \leq t \leq T$ . In time period  $(t, t+1)$  there is a demand  $d_t$  and a stock level  $Y_t$ . The quantity  $|d_t - Y_t|$  represents the magnitude of either a shortage or a surplus which we measure in dollars. We assume that there is a cost  $C_1$  associated with over ordering and a cost  $C_2$  associated with not ordering enough. That is, if we have a surplus of one unit we pay  $C_1$ , and if we have a shortage of 1 unit we are penalized an amount  $C_2$ .

Assume that

$$X_t = X_{t-1} + C_1(Y_{t-1} - d_{t-1})^+ + C_2(Y_{t-1} - d_{t-1})^-$$

where

$$(Y_{t-1} - d_{t-1})^+ = \max(0, Y_{t-1} - d_{t-1}) \quad \text{and}$$

$$(Y_{t-1} - d_{t-1})^- = \max(0, d_{t-1} - Y_{t-1}) \quad .$$

The cumulative loss at time  $t$  is equal to the cumulative loss at time  $t-1$  plus a loss due to having a surplus or a shortage. The demands  $d_t$  are independent, identically distributed random variables and each assumes a finite number of values  $0, 1, 2, \dots, N$ . Let

$$\alpha_k = P[d_t = k], \quad k = 0, 1, \dots, N \quad .$$

We assume that the  $\alpha_k$  have a joint prior distribution  $P_0$ . Let

$$\bar{\alpha}_j = \int \alpha_j dP_0$$

and

$$\bar{\alpha}_j(d_0, \dots, d_t) = \int \alpha_j dP_1[d_0, d_1, \dots, d_t]$$

where  $P_1[d_0, d_1, \dots, d_t]$  is the posterior distribution of the  $\alpha$ 's given the demands  $d_0, d_1, \dots, d_t$ .

It is desired to choose an inventory policy  $(Y_0^*, Y_1^*, \dots, Y_{T-1}^*)$  such that the expected value of  $X_T$ , the cumulative loss at time  $T$  is a minimum. Each  $Y_t^*$  will represent the optimum stock level at the beginning of the time period  $(t, t+1)$ , time period  $t$ . The policy  $(Y_0^*, Y_1^*, \dots, Y_{T-1}^*)$  will be adaptive in the sense that the choice of  $Y_t^*$  will depend upon the previous demands  $d_0, d_1, d_2, \dots, d_{t-1}$ .

## 2.0 Example

Consider  $N$  identical systems which will become obsolete after  $T$  quarters. Assume that orders can be placed for spare parts quarterly up until the units are scrapped. For a given part which is used  $k$  times in each system we wish to know how many spare parts to order in each of the quarters. Since the systems are to be scrapped, it would be desirable to order in such a way as to minimize the total cumulative loss at time  $T$  associated with this particular part. Let  $X_T$  denote the accumulation of losses incurred in each quarter due to having a surplus or shortage of units. The model yields an ordering policy

$(W_0^*, W_1^*, \dots, W_{T-1}^*)$  for which  $X_T$  is a minimum.  $W_t^*$ , the number of units ordered at the start of the  $t^{\text{th}}$  quarter, depends upon previous demands through the a posteriori probability distribution on the parameters of the demand distribution.

### 3.0 Case 1: $T=2$ and Bernoulli Demand

Assume now that  $T=2$  and  $P[d_1=1] = \alpha$  and  $P[d_1=0] = 1-\alpha$  for  $i = 0, 1, 2$ . Let  $P_0$  be the prior distribution of the  $\alpha_i$ .  $P_0$  is assumed to be discrete and

$$p_k = P[\alpha=\alpha_k] \quad k = 1, 2, \dots, M .$$

In this case the posterior distribution of  $\alpha$  given  $d_0$  is

$$P_1[d_0] = \{P_1(d_0), \dots, P_M(d_0)\}$$

where, for  $j = 1, 2, \dots, M$

$$P_j(d_0) = \begin{cases} \alpha_j p_j / \sum_{k=1}^M \alpha_k p_k & \text{if } d_0 = 1 \\ (1-\alpha_j) p_j / \sum_{k=1}^M (1-\alpha_k) p_k & \text{if } d_0 = 0 \end{cases}$$

We first want to choose  $Y_1^*$  to minimize

$$\begin{aligned} E_{P_1[d_0]} E_{d_1} X_2 &= E_{P_1[d_0]} E_{d_1} [X_1 + C_1(Y_1 - d_1)^+ + C_2(Y_1 - d_1)^-] \\ &= X_1 + E_{P_1[d_0]} E_{d_1} [C_1(Y_1 - d_1)^+ + C_2(Y_1 - d_1)^-] , \end{aligned}$$

where  $E_{d_1}$  denotes an expectation taken with respect to  $d_1$  and  $E_{P_1[d_0]}$  denotes an expectation taken with respect to the posterior distribution of  $\alpha$  given  $d_0$ . Therefore

$$\begin{aligned} & E_{d_1} [C_1(Y_1-d_1)^+ + C_2(Y_1-d_1)^-] \\ &= \alpha[C_1(Y_1-1)^+ + C_2(Y_1-1)^- - C_1Y_1^+ - C_2Y_1^-] + C_1Y_1^+ + C_2Y_1^- \end{aligned}$$

and hence

$$\begin{aligned} & E_{P_1[d_0]} E_{d_1} [C_1(Y_1-d_1)^+ + C_2(Y_1-d_1)^-] \\ &= \bar{\alpha}(d_0)[C_1(Y_1-1)^+ + C_2(Y_1-1)^- - C_1Y_1^+ - C_2Y_1^-] + C_1Y_1^+ + C_2Y_1^- . \end{aligned}$$

The value  $Y_1^*$  which minimizes this expression is

$$0 \text{ if } \bar{\alpha}(d_0)c_2 < (1-\bar{\alpha}(d_0))c_1$$

and

$$1 \text{ if } \bar{\alpha}(d_0)c_2 > (1-\bar{\alpha}(d_0))c_1 .$$

Hence the loss will be

$$\text{Min}\{\bar{\alpha}(d_0)c_2, (1-\bar{\alpha}(d_0))c_1\} .$$

Therefore

$$\begin{aligned} \text{Min}_{Y_1} [E_{P_1[d_0]} E_{d_1} X_2] &= X_1 + \text{Min}\{\bar{\alpha}(d_0)c_2, (1-\bar{\alpha}(d_0))c_1\} \\ &= X_0 + C_1(Y_0-d_0)^+ + C_2(Y_0-d_0)^- + \text{Min}\{\bar{\alpha}(d_0)c_2, (1-\bar{\alpha}(d_0))c_1\} \end{aligned}$$

Now we wish to choose the value of  $Y_0^*$  to minimize the expression

$$\begin{aligned} & E_{P_0} E_{d_0} [C_1(Y_0 - d_0)^+ + C_2(Y_0 - d_0)^-] \\ &= \bar{\alpha}[C_1(Y_0 - 1)^+ + C_2(Y_0 - 1)^- - C_1Y_0^+ - C_2Y_0^-] + C_1Y_0^+ + C_2Y_0^- . \end{aligned}$$

The value of  $Y_0^*$  is

$$\begin{aligned} & 0 \quad \text{if } \bar{\alpha}C_2 < (1 - \bar{\alpha})C_1 \\ & 1 \quad \text{if } \bar{\alpha}C_2 > (1 - \bar{\alpha})C_1 . \end{aligned}$$

Hence, the optimal procedure is to choose

$$Y_0^* = \begin{cases} 0 & \bar{\alpha}C_2 < (1 - \bar{\alpha})C_1 \\ 1 & \bar{\alpha}C_2 > (1 - \bar{\alpha})C_1 \end{cases}$$

and

$$Y_1^* = \begin{cases} 0 & \bar{\alpha}(d_0)C_2 < (1 - \bar{\alpha}(d_0))C_1 \\ 1 & \bar{\alpha}(d_0)C_2 > (1 - \bar{\alpha}(d_0))C_1 . \end{cases}$$

The minimal expected loss is given by

$$\begin{aligned} & X_0 + \text{Min}\{\bar{\alpha}C_2, (1 - \bar{\alpha})C_1\} + \bar{\alpha} \text{Min}\{\bar{\alpha}(1)C_2, (1 - \bar{\alpha}(1))C_1\} + \\ & (1 - \bar{\alpha}) \text{Min}\{\bar{\alpha}(0)C_2, (1 - \bar{\alpha}(0))C_1\} . \end{aligned}$$

At each time point we order an amount  $W_t^*$  where  $t = 0, 1$ .  $W_t^*$  is determined by  $Y_{t-1}^*$  and  $d_{t-1}$  in the sense that we order an amount necessary to bring the stock level up to  $Y_t^*$ . It should be noted that this allows the possibility of negative orders which would be the case if the cost of holding inventory is too high. Clearly,

$$W_0^* = Y_0^* \quad \text{and} \quad W_1^* = Y_1^* - Y_0^* + d_0 \quad .$$

#### 4. Case 2: General Case

In general, let

$$X_t = X_{t-1} + C_1(Y_{t-1} - d_{t-1})^+ + C_2(Y_{t-1} - d_{t-1})^-$$

and let demand during each period be characterized by independent, identically distributed discrete random variables with probability function

$$P[d_t = k] = \alpha_k \quad k = 0, 1, 2, \dots, N$$

Let

$$g(k, d_0, d_1, \dots, d_{T-j}) = k[C_1 \sum_{i=0}^k \bar{\alpha}_i(d_0, \dots, d_{T-j}) - C_2 \sum_{i=k+1}^N \bar{\alpha}_i(d_0, \dots, d_{T-j})] + C_2 \sum_{i=0}^k i \bar{\alpha}_i(d_0, \dots, d_{T-j}) - C_1 \sum_{i=k+1}^N i \bar{\alpha}_i(d_0, d_1, \dots, d_{T-j}) \quad ,$$

and let

$$f(d_0, d_1, \dots, d_{T-j}) = \min_{0 \leq k \leq N} g(k, d_0, d_1, \dots, d_{T-j}) \quad .$$

We first choose  $Y_{T-1}$  in order to minimize

$$\begin{aligned}
 E_{P_1}[d_0, d_1, \dots, d_{T-2}] E_{d_{T-1}} X_T &= X_{T-1} + E_{P_1}[d_0, \dots, d_{T-2}] E_{d_{T-1}} (C_1(Y_{T-1} - d_{T-1})^+ \\
 &\quad + C_2(Y_{T-1} - d_{T-1})^-) \\
 &= X_{T-1} + \sum_{j=0}^N \bar{\alpha}_j(d_0, \dots, d_{T-2}) [C_1(Y_{T-1} - j)^+ + C_2(Y_{T-1} - j)^-] \\
 &= X_{T-1} + Y_{T-1} (C_1 \sum_{i=0}^k \bar{\alpha}_i(d_0, \dots, d_{T-2}) \\
 &\quad - C_2 \sum_{u=k+1}^N \bar{\alpha}_i(d_0, \dots, d_{T-2})) + C_2 \sum_{i=0}^k i \bar{\alpha}_i(d_0, \dots, d_{T-2}) \\
 &\quad - C_1 \sum_{i=k+1}^N i \bar{\alpha}_i(d_0, \dots, d_{T-2})
 \end{aligned}$$

$$\text{for } k \leq Y_{T-1} \leq k+1; \quad k = 0, \dots, N-1 .$$

This function is continuous and piecewise linear in the intervals  $(k, k+1)$ . The value of the function at the end point  $k$  of the interval  $(k, k+1)$  is given by

$$X_{T-1} + g(k, d_0, \dots, d_{T-2})$$

and the minimum value is given by

$$X_{T-1} + f(d_0, d_1, \dots, d_{T-2}) .$$

Then, choose

$$Y_{T-1}^* = k^*$$

where

$$g(k^*, d_0, \dots, d_{T-2}) = f(d_0, d_1, \dots, d_{T-2}) \quad .$$

Hence

$$\begin{aligned} \text{Min}_{Y_{T-1}} E_{P_1}[d_0, \dots, d_{T-2}] E_{d_{T-1}} X_T &= X_{T-1} + f(d_0, d_1, \dots, d_{T-2}) \\ &= X_{T-2} + C_1(Y_{T-2} - d_{T-2})^+ + C_2(Y_{T-2} - d_{T-2})^- + f(d_0, d_1, \dots, d_{T-2}) \quad . \end{aligned}$$

Now choose  $Y_{T-2}$  to minimize  $E_{P_1}[d_0, \dots, d_{T-3}] E_{d_{T-2}} X_{T-1}$ . This value is

$$Y_{T-2}^* = k^*$$

where

$$g(k^*, d_0, \dots, d_{T-3}) = f(d_0, \dots, d_{T-3})$$

and

$$\text{Min}_{Y_{T-2}} E_{P_1}[d_0, \dots, d_{T-3}] E_{d_{T-2}} X_{T-1} = X_{T-2} + f(d_0, \dots, d_{T-3})$$

Hence

$$\begin{aligned} \text{Min}_{Y_{T-2}} \{ E_{P_1}[d_0, \dots, d_{T-3}] E_{d_{T-2}} [\text{Min}_{Y_{T-1}} E_{P_1}[d_0, \dots, d_{T-2}] E_{d_{T-1}} X_T] \} \\ = X_{T-2} + f(d_0, d_1, \dots, d_{T-3}) + \sum_{j=0}^N \bar{\alpha}_j(d_0, \dots, d_{T-3}) f(d_0, \dots, d_{T-3}, j) \quad . \end{aligned}$$

We continue in this fashion each time choosing  $Y_j^* = k^*$  such that

$$g(k^*, d_0, \dots, d_{T-j-1}) = f(d_0, \dots, d_{T-j-1}) .$$

This holds for  $j = 1, 2, \dots, N$ . We pick  $Y_0^*$  to minimize

$$E_{P_0} E_{d_0} (X_0 + C_1(Y_0 - d_0)^+ + C_2(Y_0 - d_0)^-)$$

The required value of  $Y_0^*$  will be that  $k^*$  such that + quantity

$$k^*(C_1 \sum_{i=0}^{k^*} \bar{\alpha}_i - C_2 \sum_{i=k^*+1}^N \bar{\alpha}_i) + C_2 \sum_{i=0}^{k^*} i \bar{\alpha}_i - C_1 \sum_{i=k^*+1}^N i \bar{\alpha}_i$$

is a minimum.

We have now chosen a sequence  $Y_0^*, Y_1^*, Y_2^*, \dots, Y_{T-1}^*$  with the property that

$$E_{d_j} X_{j+1} = E_{d_j} [X_j + C_1(Y_j^* - d_j)^+ + C_2(Y_j^* - d_j)^-]$$

is a minimum and the parameters of the distribution of  $d_j$  are the posterior means of the  $\alpha_k$  given the demands  $d_0, \dots, d_{j-1}$ . In this sense the procedure uses the information provided by the previous demands.

Having determined the optimal stock levels  $(Y_0^*, Y_1^*, \dots, Y_{T-1}^*)$  we can determine our ordering policy  $(W_0^*, W_1^*, \dots, W_{T-1}^*)$  by the following recursive formula

$$W_0^* = Y_0^*$$

and

$$W_k^* = Y_k^* - Y_{k-1}^* + d_{k-1} \quad \text{for } k = 1, 2, \dots, T-1 .$$

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