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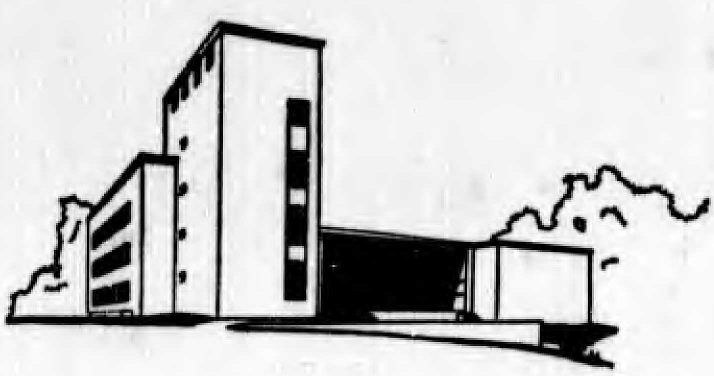

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DUALITY IN DISCRETE PROGRAMMING

III. NONLINEAR OBJECTIVE FUNCTION AND CONSTRAINTS

by

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Abstract

The results of "Duality in Discrete Programming" [1], [2] are extended in this paper (which, however, is self-contained) to the case of dual programs with nonlinear objective function and nonlinear constraints. The symmetric dual nonlinear programs studied by Dantzig, Eisenberg and Cottle [3], as well as the dual nonlinear programs formulated by Wolfe [4], Mangasarian [5] and Huard [6] are generalized by allowing some of the variables to be constrained to belong to arbitrary sets of real numbers, and dropping the requirement that the objective function and the constraints be convex (concave) in these variables. The basic properties established for the above problems in [3], [4], [5], [6], are shown to carry over, with some qualification, to their generalized counterparts which encompass, among others, various types of mixed-integer nonlinear programs.

DUALITY IN DISCRETE PROGRAMMING
III. NONLINEAR OBJECTIVE FUNCTION AND CONSTRAINTS

by Egon Balas

1. Introduction

In [1] we have studied a pair of dual problems, in which the min-max (max-min) of a linear function is to be found over a domain defined by linear inequalities, some of the variables being constrained to belong to arbitrary sets of real numbers. Mixed-integer, and hence all-integer (linear) programs are special cases of these problems. The above duality is involutory (symmetric); i.e. the dual of the dual is the primal. Subject to a qualification, the primal problem was shown to have an optimal solution if and only if the dual has one, and in this case the values of their respective objective functions are equal. A (partial) complementary slackness property holds for these dual programs.

In [2] the above results were extended to the case of quadratic objective functions with negative semi-definite quadratic forms.

The present paper extends the results of [1] to the case of nonlinear objective functions and nonlinear constraints. The symmetric dual nonlinear programs studied by Dantzig, Eisenberg and Cottle [3], as well as the dual nonlinear programs formulated by Wolfe [4], Mangasarian [5], and Huard [6], are generalized by allowing for some of the variables to be constrained to belong to arbitrary sets of real numbers, and dropping the requirement that the objective function and the constraints be concave (convex) in these variables. The basic properties established for the above problems are shown to carry over, with some qualification, to their generalized counterparts which encompass, among others, various cases of mixed-integer nonlinear programs.

2. Symmetric Dual Nonlinear Programs with Arbitrary Constraints

Let

$x \in R^n$, $u \in R^m$ (rows or columns, according to context)

$\{1,\ldots,n\} = N \quad , \quad \{1,\ldots,m\} = M$

$\{1,\ldots,n_1\} = N_1 \subset N$, i.e. $0 \leq n_1 \leq n$

$\{1,\ldots,m_1\} = M_1 \subset M$, i.e. $0 \leq m_1 \leq m$

$x = (x^1, x^2), \quad u = (u^1, u^2)$

$x^1 = (x_1, \ldots, x_{n_1}), \; x^2 = (x_{n_1+1}, \ldots, x_n)$

$u^1 = (u_1, \ldots, u_{m_1}), \; u^2 = (u_{m_1+1}, \ldots u_m)$

Consider a function $K(x,u)$ differentiable in x^2 and u^2, and denote

$$\nabla_{x^2} K(x,u) = \left(\frac{\partial K(x,u)}{\partial x_{n_1+1}}, \ldots, \frac{\partial K(x,u)}{\partial x_n} \right)$$

$$\nabla_{u^2} K(x,u) = \left(\frac{\partial K(x,u)}{\partial u_{m_1+1}}, \ldots, \frac{\partial K(x,u)}{\partial u_m} \right)$$

Let P be the problem of finding $x \in R^n$, $u \in R^m$, and

$$\min_{u^1} \max_{x,u^2} \; K(x,u) - u^2 \nabla_{u^2} K(x,u)$$

(P) subject to $\nabla_{u^2} K(x,u) \geq 0$

$$x^1 \in X^1, u^1 \in U^1$$

$$x^2, u^2 \geq 0$$

where $X^1 \subset R^n$ and $U^1 \subset R^m$ are arbitrary sets.

We define the dual of P as the problem D of finding $x \in R^n$, $u \in R^m$ and

$$\max_{x^1} \min_{x^2,u} K(x,u) - x^2 \nabla_{x^2} K(x,u)$$

(D) subject to $\nabla_{x^2} K(x,u) \leq 0$

$$x^1 \epsilon X^1, u^1 \epsilon U^1$$

$$x^2, u^2 \geq 0.$$

The duality defined in this way is obviou ly involutory (symmetric), i.e. the dual of D is P.

This pair of dual problems is a generalization of the symmetric dual nonlinear programs studied by Dantzig, Eisenberg and Cottle [3]. At this stage, the generalization consists in the fact that we allow for some of the variables to be constrained to belong to arbitrary sets.

In [1] we have stated a separation-lemma which will now be slightly generalized.

Definition 1. A scalar function $f(r,s)$ of vectors $r\epsilon R^m$, $s\epsilon R^n$, is said, for our purposes, to be separable with respect to r, if there exists a function $f_1(r)$, independent of s, and a function $f_2(s)$, independent of r, such that

$$f(r,s) \equiv f_1(r) + f_2(s).$$

Note that separability with respect to a vector r, as defined here, does not imply separability in each component of r (as the more familiar concept of separability used in the literature does).

Let $\Omega_r \subset R^m$ and $\Omega_s \subset R^n$ be two arbitrary sets of vectors $r\epsilon\Omega_r$, $s\epsilon\Omega_s$. We say that Ω_r is independent of Ω_s, if Ω_r is not defined in terms of elements $s\epsilon\Omega_s$ or of functions of $s\epsilon\Omega_s$.

Definition 2. A set $\Omega = \Omega_r \times \Omega_s$ of vectors $(r,s)\epsilon R^m \times R^n$ is said to be separable with respect to $r\epsilon R^m$, if Ω_r and Ω_s are independent of each other.

Thus if Ω is defined by some set of constraints $F(r,s) \leq 0$, where F is a vector function with arguments $r\epsilon R^m$, $s\epsilon R^n$, then the separability of Ω with respect to $r\epsilon R^m$ is the same thing as the componentwise separability of the vector function $F(r,s)$ with respect to r, as defined in [1] (i.e., the property that each component F_i of F can be written either as $F_i(r)$, or as $F_i(s)$).

Let $f(r,s,t)$ be a scalar function of $r\epsilon R^m$, $s\epsilon R^n$, $t\epsilon R^p$ and let $\Omega = \Omega_r \times \Omega_s \times \Omega_t$ be some arbitrary set such that $\Omega_r \subset R^m$, $\Omega_s \subset R^n$, $\Omega_t \subset R^p$.

Then we have

<u>Lemma 1 (Separation)</u>. If $f(r,s,t)$ and Ω are separable with respect to $r\epsilon R^m$ or $s\epsilon R^n$, then

$$\inf_s \sup_{r,t} \{f(r,s,t) \mid (r,s,t)\epsilon\Omega\} = \sup_r \inf_s \left\{\sup_t \{f(r,s,t) \mid (r,s,t)\epsilon\Omega\}\right\}.$$

<u>Proof</u>. Suppose $f(r,s,t)$ and Ω are separable with respect to $r\epsilon R^m$, i.e. $f(r,s,t) \equiv f_1(r) + f_2(s,t)$, and Ω_r and $\Omega_{st} = \Omega_s \times \Omega_t$ are independent of each other. Then both sides of the equality in the Lemma become

$$\sup_r \{f_1(r) \mid r\epsilon\Omega_r\} + \inf_s \sup_t \{f_2(s,t) \mid (s,t)\epsilon\Omega_{st})\} \cdot$$

If, on the other hand, separability holds with respect to $s\epsilon R^n$, then both sides of the equality become

$$\inf_s \{f_1(s) \mid s\epsilon\Omega_s\} + \sup_{r,t} \{f_2(r,t) \mid (r,t)\epsilon\Omega_{rt})\}.$$

This completes the proof of Lemma 1.

<u>Remark</u>. If in the above lemma we replace "$(r,s,t)\epsilon\Omega$" by "$G(r,s,t) \leq 0$" (G being a vector function), and the assumption that "Ω is separable with respect to $r\epsilon R^m$ or $s\epsilon R^n$," by the assumption that "$G(r,s,t)$ is componentwise separable (as defined in [1]) with respect to r or s," we obtain the separation Lemma of [1].

Denote

$$Z = \{(x,u) \mid \nabla_{u^2} K(x,u) \geq 0,\ x^1 \epsilon X^1, u^1 \epsilon U^1;\ x^2, u^2 \geq 0\}$$

$$W = \{(x,u) \mid \nabla_{x^2} K(x,u) \leq 0,\ x^1 \epsilon X^1, u^1 \epsilon U^1;\ x^2, u^2 \geq 0\}$$

$$f = K(x,u) - u^2 \nabla_{u^2} K(x,u)$$

$$g = K(x,u) - x^2 \nabla_{x^2} K(x,u)$$

Further, for any given $\hat{x}$, $\hat{u}$, let $\nabla_{x^2} K(\hat{x},\hat{u})$ and $\nabla_{u^2} K(\hat{x},\hat{u})$ stand for $\nabla_{x^2} K(x,u)$ and $\nabla_{u^2} K(x,u)$ respectively, evaluated at $(\hat{x},\hat{u})$. Finally, let $\nabla_{u^2 u^2} K(\hat{x},\hat{u}) = \nabla_{u^2}[\nabla_{u^2} K(\hat{x},\hat{u})]$ be the matrix of second partial derivatives of $K(x,u)$ in the components of u^2, evaluated at $(\hat{x},\hat{u})$.

The formulation of P and D allows for the possibility that $M = M_1$ and/or $N = N_1$, i.e. that x has no x^2 - components and/or u has no u^2 - components. If neither x^2 nor u^2 exists, then the relevant parts of Theorem 1 below can be obtained immediately from the Separation Lemma. Hence we consider this case solved, and assume that either $M \neq M_1$ or $N \neq N_1$.

Furthermore, for the sake of brevity, the following theorem is stated for the case when both $M \neq M_1$ and $N \neq N_1$. However, it will subsequently be shown that if x^2 (or u^2) does not exist, Theorem 1 can be made to hold by simply dropping the assumptions and statements concerning x^2 (or u^2).

Theorem 1. Assume that

1. $K(x,u)$ is concave in x^2 for each x^1,u, and convex in u^2 for each x,u^1.
2. $K(x,u)$ is twice differentiable in u^2 and x^2.
3. $K(x,u)$ and $X^1 \times U^1$ are separable (as defined above) with respect to u^1 or x^1.

Under assumptions 1, 2, 3, if P has an optimal solution $(\bar{x},\bar{u})$ such that $\nabla_{u^2u^2}K(\bar{x},\bar{u})$ is positive definite, then the following statements hold:

I. Joint Optimum: $(\bar{x},\bar{y})$ is an optimal solution to D, with

$$\min_{u^1} \max_{x,u^2} \{f \mid (x,u)\epsilon Z\} = \max_{x^1} \min_{x^2,u} \{g \mid (x,u)\epsilon W\}$$

II. Complementary Slackness:

$$\bar{x}^2 \nabla_{x^2}K(\bar{x},\bar{u}) = 0, \qquad \bar{u}^2 \nabla_{u^2}K(\bar{x},\bar{u}) = 0$$

III. Saddle-Point: $K(x,u)$ has a saddle-point at $(\bar{x},\bar{u})$:

$$K(x,\bar{u}) \leq K(\bar{x},\bar{u}) \leq K(\bar{x},u)$$

for all x such that $(x,\bar{u})\epsilon Z$, $\nabla_{u^2}K(x,\bar{u}) = \nabla_{u^2}K(\bar{x},\bar{u})$, and all u such that $(\bar{x},u)\epsilon W$, $\nabla_{x^2}K(\bar{x},u) = \nabla_{x^2}K(\bar{x},\bar{u})$.

Before proving this theorem, it seems useful to compare it with theorem 2 of Dantzig, Eisenberg and Cottle [3]. Assumptions 1 and 2 above, as well as the one that $\nabla_{u^2u^2}K(\bar{x},\bar{u})$ is positive definite, are also present in theorem 2 of [3]. Hence, while the problem considered in [3] is generalized here by allowing for some of the variables x_i and/or u_i to be constrained to belong to arbitrary sets of real numbers, and by dropping the requirement for $K(x,u)$ to be concave (convex in these variables, the only additional assumption which we need in order to extend the results of [3] to this case, is 3. In connection with assumption 3, we recall that separability with respect to the vector u^1 does not imply separability in each component u_i of u^1. Furthermore, we do not assume separability with respect to both x^1 and u^1. Thus, if $M_1 = \emptyset$ (M_1 being the index set for the variables u^1) then $K(x,u)$ is separable with respect to u^1; so this is sufficient for assumption 3 to hold. The results of [3] are thus extended without any further assumption to mixed-integer nonlinear programming, by taking X^1 to be the set of vectors $x^1 \epsilon R^{n_1}$ with nonnegative integer components, and $M_1 = \emptyset$.

Of course, in view of the symmetry of our dual problems, P and D can be interchanged in Theorem 1.

Proof of the Theorem. Denote

$$z = \min_{u^1} \max_{x,u^2} \{f \mid (x,u) \epsilon Z\}$$

$$w = \max_{x^1} \min_{x^2,u} \{g \mid (x,u) \epsilon W\}$$

Let us assume that $K(x,u)$ and $X^1 \times U^1$ are separable with respect to u^1. (A completely analogous reasoning holds if separability is assumed with respect to x^1). Then z can be written

$$(1) \quad z = \min_{u^1 \epsilon U^1} \max_{\substack{x^1 \epsilon X^1 \\ x^2,u^2 \geq 0}} \{K^1(u^1) + K^2(x,u^2) - u^2 \nabla_{u^2} K^2(x,u^2) \mid \nabla_{u^2} K^2(x,u^2) \geq 0\}$$

where

$$K(x,u) = K^1(u^1) + K^2(x,u^2)$$

According to Lemma 1, $\min_{u^1}$ and $\max_{x^1}$ can now be interchanged in (1), so that

$$(2) \quad z = \max_{x^1 \epsilon X^1} \min_{u^1 \epsilon U^1} \{K^1(u^1) + f_2(x^1)\}$$

where

$$(3) \quad f_2(x^1) = \max_{x^2,u^2 \geq 0} \{K^2(x,u^2) - u^2 \nabla_{u^2} K^2(x,u^2) \mid \nabla_{u^2} K^2(x,u^2) \geq 0\}$$

On the other hand, w can be written as

$$(4) \quad w = \max_{x^1 \epsilon X^1} \min_{\substack{u^1 \epsilon U^1 \\ x^2,u^2 \geq 0}} \{K^1(u^1) + K^2(x,u^2) - x^2 \nabla_{x^2} K^2(x,u^2) \mid \nabla_{x^2} K^2(x,u^2) \leq 0\}$$

or

$$(5) \quad w = \max_{x^1 \epsilon X^1} \min_{u^1 \epsilon U^1} \{K^1(u^1) + g_2(x^1)\}$$

where

(6) $g_2(x^1) = \min_{x^2,u^2\geq 0} \{K^2(x,u^2) - x^2\nabla_{x^2}K^2(x,u^2) \mid \nabla_{x^2}K^2(x,u^2)\leq 0\}$.

For any given x^1 and u^1, (3) and (6) are a pair of symmetric dual nonlinear programs of the type studied in [3]. Hence in view of assumptions 1 and 2, if for some $x^1 = \hat{x}^1$, $u^1 = \hat{u}^1$, (3) has an optimal solution $(\hat{x}^2,\hat{u}^2)$, such that $\nabla_{u^2u^2}K(\hat{x},\hat{u})$ is positive definite, then ([3], theorem 2) $(\hat{x}^2,\hat{u}^2)$ is also an optimal solution to (6), and $f_2(\hat{x}^1) = g_2(\hat{x}^1)$.

But since $(\bar{x},\bar{u})$ is an optimal solution to P, it follows that, for $x^1 = \bar{x}^1$, $u^1 = \bar{u}^1$, the vector $(\bar{x}^2,\bar{u}^2)$ is an optimal solution to (3); for, if it is not, then $(\bar{x},\bar{u}^2)$ does not maximize f for $u^1 = \bar{u}^1$. Hence, if $\nabla_{u^2u^2}K(\bar{x},\bar{u})$ is positive definite, then $(\bar{x}^2,\bar{u}^2)$ is also optimal for (6), and

(7) $f_2(\bar{x}^1) = g_2(\bar{x}^1)$.

Then statement I of our theorem follows from (2), (5) and (7).

Statement II follows from the fact that for $x^1 = \bar{x}^1$, $u^1 = \bar{u}^1$, the vector $(\bar{x}^2,\bar{u}^2)$ is an optimal solution to (3) and (6) ([3], theorem 1).

To prove statement III, suppose it is false. Then there exists $(\hat{x},\bar{u}) \in Z$ such that

(8) $K(\hat{x},\bar{u}) > K(\bar{x},\bar{u})$ and $\nabla_{u^2}K(\hat{x},\bar{u}) = \nabla_{u^2}K(\bar{x},\bar{u})$, or there exists $(\bar{x},\hat{u}) \in W$ such that

(9) $K(\bar{x},\hat{u}) < K(\bar{x},\bar{u})$ and $\nabla_{x^2}K(\bar{x},\hat{u}) = \nabla_{x^2}K(\bar{x},\bar{u})$.

But (8) implies that $\bar{u}^1$ does not minimize (with respect to u^1)

$$\max_{x,u^2} \{f \mid (x,u) \in Z\},$$

and (9) implies that $\bar{x}^1$ does not maximize (with respect to x^1)

$$\min_{x^2,u} \{g \mid (x,u) \in W\}.$$

In the first case $(\bar{x},\bar{u})$ is not an optimal solution to P, which contradicts the assumption in the theorem. In the second case, $(\bar{x},\bar{u})$ is not optimal for D, which contradicts statement I. This proves statement III.

In stating Theorem 1, we have assumed that $M \neq M_1$, and $N \neq N_1$.

Corollary 1. If $N = N_1$, delete from Theorem 1 the assumptions on concavity and differentiability in x^2, as well as the statement that $\bar{x}^2\nabla_{x^2}K(\bar{x},\bar{u}) = 0$. If $M = M_1$, delete the assumptions on convexity and differentiability in u^2, the requirement that $\nabla_{u^2u^2}K(\bar{x},\bar{u})$ be positive definite, and the statement that $\bar{u}_2 \nabla_{u^2}K(\bar{x},\bar{u}) = 0$. In both cases, statements I, III, and the remaining part of II, hold.

Proof. Suppose $M = M_1$. Then (3) and (6) become

(3') $$f_2(x^1) = \max_{x^2\geq 0} \{K^2(x^1,x^2)\}$$

(6') $$g_2(x^1) = \min_{x^2\geq 0} \{K^2(x^1,x^2) - x^2\nabla_{x^2}K^2(x^1,x^2) \mid \nabla_{x^2}K^2(x^1,x^2)\leq 0\}$$

Since K^2 is concave in x^2 for any x^1, (3') can be regarded as the convex programming problem

(3'') $$\min_{x^2} \; -K^2(x^1,x^2) \quad \text{s.t.} \quad x^2\geq 0$$

whose dual [4] is

(6'') $$\max_{x^2} \; -K^2(x^1,x^2)-tx^2 \quad \text{s.t.} \quad -\nabla_{x^2}K^2(x^1,x^2) = t\,\nabla_{x^2}x^2, \quad t \geq 0,$$

where $t\in R^{n-n_1}$, and $\nabla_{x^2}x^2$ is the matrix $[\nabla_{x^2}x_{n_1+1},\ldots,\nabla_{x^2}x_n] = I$

But since $\nabla_{x^2} x^2$ is the $(n-n_1)$-th order identity matrix, the right-hand side of the equality is t, and (6") is the same as (6), except that x^2 is unconstrained.

According to our assumption, $\bar{x}^2$ is an optimal solution to (3") for $x^1 = \bar{x}^1$, $u^1 = \bar{u}^1$. Then ([4], Theorem 2), there exists t such that $(\bar{x}^2, t)$ is an optimal solution to (6"), i.e., $\bar{x}^2$ is an optimal solution to (6), and $f_2(\bar{x}^1) = g_2(\bar{x}^1)$. Then statements I, II and III follow in the same way as in the proof of the theorem.

If, on the other hand, $N = N_1$, (3) and (6) become a pair of dual nonlinear programs in u^2, of the same type as above, but (3) is now the dual problem. The assumption that $\bar{u}^2$ is an optimal solution to (3) for $x^1 = \bar{x}^1$, $u^1 = \bar{u}^1$, and that $\nabla_{u^2 u^2} K(\bar{x}, \bar{u})$ is positive definite, is sufficient ([5], p,301; [6], p. 58) to ensure that $\bar{u}^2$ is also optimal for (6) — now the primal problem — and that $f_2(\bar{x}^1) = g_2(\bar{x}^1)$. Statements I, II and III follow then as before.

3. Special Cases

In this section we examine some of the special cases to which P and D reduce when K(x,u) is specified in certain ways. For most of these cases the statements I, II, III hold under weaker assumptions than in Theorem 1.

3.1. Dual linear programs with added arbitrary constraints.

This case subsumes linear integer and mixed-integer programming. If we let

(10) $$K(x,u) = cx + ub - uAx$$

where c, b and A are of appropriate dimensions, we obtain the pair of dual programs

$$\min_{u^1} \max_{x} \; cx + u^1y^1$$

(P)
$$\text{s.t.} \quad Ax + y = b$$
$$x^1 \epsilon X^1, \; u^1 \epsilon U^1$$
$$x^2, y^2 \geq 0$$

$$\max_{x^1} \min_{u} \; ub - v^1x^1$$

(D)
$$\text{s.t.} \; uA - v = c$$
$$u^1 \epsilon U^1, \; x^1 \epsilon X^1$$
$$u^2, v^2 \geq 0$$

<u>Corollary 2</u>. Let $K(x,u) = cx + ub - uAx$. Then, under assumption 3 of Theorem 1, if P has an optimal solution $(\bar{x},\bar{u}^1)$, there exists $\bar{u}^2$ such that statements I, II, III of Theorem 1 hold for $(\bar{x},\bar{u})$.

<u>Proof</u>. Going through the proof of Theorem 1, we find that in this case for any given x^1,u^1, (3) and (6) become a pair of dual linear programs. Hence the results of Theorem 1 can be derived without any assumptions other than 3.

Assumption 3 of Theorem 1 can itself be relaxed if, writing $-uAx = -u^1A^{11}x^1 - u^1A^{12}x^2 - u^2A^{21}x^1 - u^2A^{22}x^2$, we drop the term $-u^1A^{11}x^1$ from (10), by setting

(11) $K(x,u) = cx + ub - uAx + u^1A^{11}x^1.$

In this case the objective functions of P and D become

$$\min_{u^1} \max_{x} \; cx + u^1y^1 + u^1A^{11}x^1$$

and

$$\max_{x^1} \min_{u} \; ub - v^1x^1 + u^1A^{11}x^1.$$

respectively. This is the pair of dual programs studied in [1], where in addition X^1 and U^1 were assumed to be the sets of all m_1-vectors and n_1-vectors respectively, with nonnegative integer components. Then the assumption that K(x,u) is separable with respect to x^1 or u^1 can be replaced by the weaker assumption that y^2 (or v^2) is componentwise separable with respect to x^1 (to u^1) ([1], Theorem 2). To show that this assumption is weaker, consider the case when $A^{22} = 0$, but $u^1A^{12}x^2 + u^2A^{21}x^1$ is not separable with respect to either x^1 or u^1. Then both y^2 and v^2 are componentwise separable, but K(x,u) is not separable. On the other hand, it is easy to see that separability of K(x,u) implies componentwise separability of y^2 or v^2.

3.2. Symmetric dual quadratic programs with added arbitrary constraints. This case subsumes integer and mixed-integer quadratic programming. If we set

(12) $K(x,u) = cx + ub - uAx + \frac{1}{2}(xCx-uEu)$

where c,b and A are as before, while C and E are symmetric negative semi-definite matrices, our pair of dual programs becomes

$$\text{(P)}\quad \begin{array}{l} \min\limits_{u^1} \max\limits_{x,u^2} \; cx + \frac{1}{2}xCx + \frac{1}{2}uEu + u^1y^1 \\ \text{s.t.} \quad Ax + Eu + y = b \\ \qquad x^1 \epsilon X^1,\ u^1 \epsilon U^1 \\ \qquad x^2, u^2, y^2 \geq 0 \end{array}$$

$$\text{(D)}\quad \begin{array}{l} \max\limits_{x^1} \min\limits_{x^2,u} \; ub - \frac{1}{2}uEu - \frac{1}{2}xCx - v^1x^1 \\ \qquad uA - xC - v = c \\ \qquad u^1 \epsilon U^1,\ x^1 \epsilon X^1 \\ \qquad u^2, x^2, v^2 \geq 0 \end{array}$$

This pair of problems generalizes the symmetric dual quadratic programs studied by Cottle [7].

For this case, Theorem 1 becomes

Corollary 3. Let $K(x,u)$ be defined by (12). Then, under assumption 3 of Theorem 1, if P has an optimal solution $(\bar{x},\bar{u}^1)$, there exists $\hat{u}^2$ such that $(\bar{x},\hat{u})$, where $\bar{u} = (\bar{u}^1,\hat{u}^2)$, is also an optimal solution to P, with $E^{22}\bar{u}^2 = E^{22}\hat{u}^2$, and statements I, II, III hold for $(\bar{x},\hat{u})$.

Proof. Same as for Theorem 1, except that (3) and (6) now become, for any x^1,u^1, a pair of symmetric dual quadratic programs in x^2,u^2, to which we apply the results of [7] (Corollary 1 to theorem 2).

Remark. Corollary 3 also holds for P', and D' obtained from the problems P and D of this section, by dropping the constraints $u^2 \geq 0$ from P and $x^2 \geq 0$ from D.

In this case too, as in the previous subsection, assumption 3 of Theorem 1 can also be relaxed by setting, instead of (12),

(13) $K(x,u) = cx + ub + \frac{1}{2}(xCx - uEu) + u^1A^{11}x^1.$

Then the objective functions of P and D become

$$\min_{u^1} \max_{x,u^2} cx + \frac{1}{2}xCx + \frac{1}{2} uEu + u^1y^1 + u^1A^{11}x^1$$

and

$$\max_{x^1} \min_{x^2,u} ub - \frac{1}{2}xCx - \frac{1}{2} uEu - v^1x^1 + u^1A^{11}x^1$$

respectively, and we obtain the pair of dual programs considered in [2]. The assumption that $K(x,u)$ is separable with respect to x^1 (or u^1) can then be replaced by the assumption — which is easily seen to be weaker — that $C^{12}=0$ and y^2 is componentwise separable with respect to x^1 (or $E^{21} = 0$ and v^2 is componentwise separable with respect to u^1)([2], Theorem 2).

Since the dual quadratic programs considered by Dorn [8] are special cases of the above ones, their generalized counterparts obtained by adding the constraints $x^1 \epsilon X^1$, $u^1 \epsilon U^1$ are also covered by Corollary 2.

3.3. Other dual nonlinear programs

If we set

(14) $K(x,u) = f(x) - uF(x)$

where $f(x)$ is a scalar function and $F(x)$ an m-component vector function, and if we write $F(x) = \left[F^1(x), F^2(x)\right]$, where $F^1(x) = \left[F_1(x),\ldots,F_{m_1}(x)\right]$, $0 \leq m_1 \leq m$, and $F^2(x) = \left[F_{m_1+1}(x),\ldots,F_m(x)\right]$, then P and D become

$$
(P) \quad \begin{array}{l} \min\limits_{u^1} \; \max\limits_{x} \; f(x) - u^1F^1(x) \\ \text{s.t.} \qquad F^2(x) \leq 0 \\ \qquad x^1 \epsilon X^1, u^1 \epsilon U^1 \\ \qquad x^2 \geq 0 \end{array}
$$

$$
(D) \quad \begin{array}{l} \max\limits_{x^1} \; \min\limits_{x^2,u} \; f(x) - uF(x) - x^2\left[\nabla_{x^2} f(x) - u \nabla_{x^2} F(x)\right] \\ \text{s.t.} \qquad \nabla_{x^2} f(x) - u \nabla_{x^2} F(x) \leq 0 \\ \qquad x^1 \epsilon X^1, u^1 \epsilon U^1 \\ \qquad x^2, u^2 \geq 0 \end{array}
$$

where $\nabla_{x^2} F(x)$ is the $m \times (n-n_1)$ matrix whose m rows are $\nabla_{x^2} F_1(x),\ldots, \nabla_{x^2} F_m(x)$.

This generalizes the dual nonlinear programs considered by Wolfe [4], Mangasarian [5], and Huard [6].

Assumptions 1, 2 and 3 of Theorem 1 are now to be maintained, but the requirement for $\nabla_{u^2 u^2} K(\bar{x},\bar{u})$ to be positive definite, is replaced by the weaker assumption that, if $(\bar{x},\bar{u})$ is an optimal solution to P, then

4. $F^2(\bar{x}^1,x^2) \leq 0$ satisfies the Kuhn-Tucker constraint qualification at $\bar{x}^2$; i.e., at $\bar{x}^2$ any vector differential dx^2 such that

$$F_i(\bar{x}) = 0 \implies \nabla_{x^2} F_i(\bar{x}) dx^2 \geq 0$$

is tangent to an arc contained in the constraint set $F^2(\bar{x}^1,x^2) \leq 0$ ([9], p.483).

Denoting, as before, by z and w respectively the objective function-value of the optimum for P and D, we have

Corollary 4. Under assumptions 1, 2, 3 of Theorem 1, if P has an optimal solution $(\bar{x},\bar{u}^1)$ such that at $\bar{x}^2$ the set $F(\bar{x}^1,x^2) \leq 0$ satisfies the constraint qualification 4, there exists $\bar{u}^2$ such that $(\bar{x},\bar{u}) = (\bar{x},\bar{u}^1,\bar{u}^2)$ is an optimal solution to D.

Conversely, under the same assumptions 1, 2, 3, if D has an optimal solution $(\hat{x},\hat{u})$ such that the matrix $\nabla_{x^2}\left[\nabla_{x^2} f(\hat{x}) - \hat{u}\, \nabla_{x^2} F(\hat{x})\right]$ is negative definite, then $(\hat{x},\hat{u}^1)$ is an optimal solution to P.

In both cases, statements I, II and III of Theorem 1 hold.

Proof. The second part follows directly from Theorem 1. As to the first part, the proof is along the same lines as for Theorem 1. Problems (3) and (6) are in this case the dual nonlinear programs studied by Wolfe [4], except that the primal variable x^2 is taken in [4] to be unconstrained, and correspondingly the dual constraints are equalities rather than inequalities, which accounts for the fact that the term $-x^2\left[\nabla_{x^2} f(x) - u\, \nabla_{x^2} F(x)\right]$ does not appear in the objective function of Wolfe's dual. However, the results of [4] preserve their validity for the case when x^2 is constrained to be nonnegative and the dual is modified accordingly (cf. [5]).

As suggested by this, the results of Corollary 4 preserve their validity, under slightly modified assumptions, if the constraints $x^2 \geq 0$ are dropped. In this case, with K(x,u) still defined by (14), P and D become

$$\text{(P)}\quad \begin{aligned} &\min_{u^1}\ \max_{x}\ f(x) - u^1F^1(x) \\ &\quad \text{s.t.}\ \ F^2(x) \leq 0 \\ &\qquad x^1 \epsilon X^1,\ u^1 \epsilon U^1 \end{aligned}$$

$$\text{(D)}\quad \begin{aligned} &\max_{x^1}\ \min_{x^2,u}\ f(x) - uF(x) \\ &\quad \text{s.t.}\ \ \nabla_{x^2} f(x) - u\nabla_{x^2} F(x) = 0 \\ &\qquad x^1 \epsilon X^1,\ u^1 \epsilon U^1 \end{aligned}$$

Further, the requirement in Corollary 4 for $\nabla_{x^2}\left[\nabla_{x^2} f(x) - u\nabla_{x^2} F(x)\right]$ to be negative definite can be replaced by the weaker requirements formulated by Huard [6]. In this case Corollary 4 becomes

<u>Corollary 4a</u>. First part as in Corollary 4. Second part: Conversely, under the same assumptions 1, 2, 3, if D has an optimal solution $(\hat{x},\hat{u})$ such that at $(\hat{x},\hat{u})$ the set

$$\nabla_{x^2} f(\hat{x}^1,x^2) - \hat{u}\nabla_{x^2} F(\hat{x}^1,x^2) = 0$$

satisfies the Kuhn-Tucker constraint qualification and the matrix $\nabla_{x^2}\left[\nabla_{x^2} f(\hat{x}) - \hat{u}\nabla_{x^2} F(\hat{x})\right]$ is nonsingular, then $(\hat{x},\hat{u}^1)$ is an optimal solution to P, and statements I, II, [illegible] of Theorem 1 hold.

As an example, let us consider the case when K(x,u) is given by (14), but $M_1 = \emptyset$ (i.e., u has no u^1-component) and

$$X^1 = \left\{x^1 \epsilon R^{m_1} \,\middle|\, x_j \text{ integer} \geq 0,\ j = 1,\ldots,m_1\right\}.$$

Then P is the mixed-integer nonlinear program

$$\text{(P)}\qquad \begin{aligned} &\max\ f(x)\\ &\text{s.t.}\quad F(x)\leq 0\\ &\qquad x_j\geq 0,\ j\epsilon N\\ &x_j \text{ integer },\ j\epsilon N_1\end{aligned}$$

The dual of such a program is, according to the above theory,

$$\text{(D)}\qquad \begin{aligned} &\max_{x^1}\ \min_{x^2,u}\ f(x) - uF(x) - x^2\left[\nabla_{x^2} f(x) - u\nabla_{x^2} F(x)\right]\\ &\text{s.t.}\qquad \nabla_{x^2} f(x) - u\nabla_{x^2} F(x)\leq 0\\ &\qquad x,u\geq 0\\ &\qquad x_j \text{ integer},\ j\epsilon N_1\end{aligned}$$

If the constraint $x^2\geq 0$ is deleted from P, then the gradient-constraints of D become equalities, and the corresponding term in the objective function can be dropped.

Since $M_1 = \emptyset$ implies that $K(x,u)$ is separable with respect to u^1, Corollary 4 (or 4a, according to whether $x^2\geq 0$ is maintained or not) holds for this pair of dual programs without assumption 3. That is, the results of Wolfe [4]-Mangasarian [5]-Huard [6] are extended to mixed-integer concave programming without any other assumption than those in [4], [5] and/or [6].

Finally, let us consider the case when

(16) $\quad K(x,u) = f(x^1) + cx^2 + g(u^1) + u^2[b-F(x^1) - Ax^2]$

where f and g are arbitrary scalar functions, $F(x)\in R^{m-m_1}$ is an arbitrary vector function of $x\in R^n$, c and b are $(n-n_1)$-and $(m-m_1)$-vectors respectively, while A is an $(m-m_1)\times(n-n_1)$-matrix. Then P and D become

$$\text{(P)}\quad \begin{array}{lll} \min\limits_{u^1} & \max\limits_{x} & f(x^1) + cx^2 + g(u^1) \\ & \text{s.t.} & F(x^1) + Ax^2 \leq b \\ & & x^1 \epsilon X^1,\ u^1 \epsilon U^1 \\ & & x^2 \geq 0 \end{array}$$

$$\text{(D)}\quad \begin{array}{lll} \max\limits_{x^1} & \min\limits_{u} & f(x^1) + g(u^1) + u^2 b - u^2 F(x^1) \\ & \text{s.t.} & u^2 A \geq c \\ & & x^1 \epsilon X^1,\ u^1 \epsilon U^1 \\ & & u^2 \geq 0 \end{array}$$

Assumptions 1, 2 and 3 are now automatically satisfied. Moreover, if $A \neq 0$, for any given x^1, u^1, the primal constraint set (linear in x^2) as well as the dual constraint set (linear in u^2) satisfies the Kuhn-Tucker constraint qualification. Hence, for this case Theorem 1 reduces to

Corollary 5. If P has an optimal solution $(\bar{x}, \bar{u}^1)$, then there exists $\bar{u}^2$ such that $(\bar{x}, \bar{u})$, where $\bar{u} = (\bar{u}^1, \bar{u}^2)$, is an optimal solution to D. Conversely, if D has an optimal solution $(\bar{x}, \bar{u})$, then $(\bar{x}, \bar{u}^1)$ is an optimal solution to P. In both cases, statements I, II, III of Theorem 1 are true.

This generalizes the partitioning theorem of Benders [10].

The above discussion has been conducted in terms of "finite" programming, i.e. our problems were cast in terms of finite dimensional vector spaces. However, the results of Charnes, Cooper and Kortanek [11] on duality in semi-infinite programming can be generalized in a similar way for the case when some of the variables are constrained to belong to arbitrary sets.

Finally, it should be mentioned that the duality theory for discrete programming proposed in [1] and extended in this paper is also related to the work of Stoer [12], as well as to the perturbational duality theory for generalized convex programs developed by Rockafellar [13], [14].

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13. ABSTRACT

The results of "Duality in Discrete Programming" [1], [2], are extended in this paper (which, however, is self-contained) to the case of dual programs with nonlinear objective function and nonlinear constraints. The symmetric dual nonlinear programs studied by Dantzig, Eisenberg and Cottle [3], as well as the dual nonlinear programs formulated by Wolfe [4], Mangasarian [5] and Huard [6] are generalized by allowing some of the variables to be constrained to belong to arbitrary sets of real numbers, and dropping the requirement that the objective function and the constraints be convex (concave) in these variables. The basic properties established for the above problems in [3], [4], [5], [6], are shown to carry over, with some qualification, to their generalized counterparts which encompass, among others, various types of mixed-integer nonlinear programs.

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