AD 567411

DECEMBER 1967
T-R-557

RESEARCH LIBRARY
AIR FORCE CAMBRIDGE RESEARCH LABORATORIES
L. G. HANSCOM FIELD, BEDFORD, MASSACHUSETTS

RESEARCH TRANSLATION

The Evolution of Stratus Clouds in the Surface Boundary Layer of the Atmosphere

IU. V. SHULEPOV

OFFICE OF AEROSPACE RESEARCH
United States Air Force

AMERICAN METEOROLOGICAL SOCIETY
45 BEACON STREET
BOSTON, MASSACHUSETTS
02108

THE EVOLUTION OF STRATUS CLOUDS IN THE SURFACE BOUNDARY LAYER OF THE ATMOSPHERE

Translation of

K evoliutsii sloistoi oblachnosti v prizemnom sloe atmosfery

by

Iu. V. Shulepov

Kiev. Ukrainskii Nauchno-Issledovatel'skii Gidrometeorologicheskii Institut, Trudy, No. 48: 13-20, 1965.

This translation was produced by the
American Meteorological Society under
Contract AF 19(628)-3880, through
the support and sponsorship of the

AIR FORCE CAMBRIDGE RESEARCH LABORATORIES
OFFICE OF AEROSPACE RESEARCH
L. G. HANSCOM FIELD
BEDFORD, MASSACHUSETTS
01731

1. The evolution of stratus clouds in the surface boundary layer of the atmosphere

2. Shulepov, Iu. V. K evoliutsii sloistoi oblachnosti v prizemnom sloe atmosfery, Kiev. Ukrainskii Nauchno-Issledovatel'skii Gidrometeorologicheskii Institut, Trudy, No. 48: 13-20, 1965 [in Russian].

3. 14 typewritten pages

4. Date of translation: June 1967

5. Translator: George E. Brady, Jr.

6. Produced for Air Force Cambridge Research Laboratories, Office of Aerospace Research, United States Air Force, L. G. Hanscom Field, Bedford, Massachusetts, by the American Meteorological Society, Contract number AF 19(628)-3880.

7. Unclassified and complete

THE EVOLUTION OF STRATUS CLOUDS IN THE SURFACE BOUNDARY LAYER OF THE ATMOSPHERE

by

Iu. V. Shulepov

The evolution of stratus clouds, whose base is in the surface boundary layer of the atmosphere, is investigated. Formulas are derived that describe the displacement of the base of stratus clouds during a brief time interval ($t \lesssim 10$ to 20 min).

Recently much attention has been paid to the theory of stratus cloud formation. The work in this field has primarily been directed to describing the cloud in a hydrodynamic approximation while neglecting its discrete structure.

The most systematic works of this type were the investigations of L. T. Matveiev, V. S. Kozharin, and Ie. M. ..eigel'son [3, 5].

The recent paper of K. V. Klubovich [2] is a step backwards compared to the aforementioned papers since, instead of consistently excluding terms that describe the condensation of water vapor and the evaporation of cloud droplets [3, 5], in [2] the author attempts to express these members only in terms of the vapor density, assuming that the mean radius and number of droplets is constant. But it can easily be seen that the liquid-water content can be expressed in terms of the mean radius of droplets and their number. Consequently, the assumption that the latter are constant leads to the constancy of the liquid-water content and, thus, it need not be defined.

However, even [3, 5] primarily considered the formation of stratiform clouds and the effect of various factors on them. Less attention was paid to the evolution of these clouds, i.e., to how their properties change with the passage of time under the influence of various external factors. Nonetheless this problem is of definite interest for predicting the base of the clouds.

In order to predict the behavior of the cloud base, one need not know the entire history of its origin and development.

But, once the state of the atmosphere and the characteristics of the cloud are known for a given moment of time, a full description of the process of evolution requires that one be able to predict the time-dependent changes not only of the state of the cloud but also of the state of the atmosphere (the level of turbulence, vertical and horizontal motions, etc.). At present this is a complex problem.

However, for practical applications one often needs to know the tendency of changes in the stratus clouds for brief periods of time during which the state of the atmosphere can be assumed to be constant (the coefficient of turbulent diffusion, and vertical and horizontal currents are not time dependent).

Let us pass to the examination of the behavior of stratus clouds whose base is within the limits of the surface boundary layer.

The system of equations that describes thermal and moisture transfer together with the processes of phase transition have the form (one should write the kinetic equation for the distribution function of cloud droplets by size [6] rather than the equation for ω.

But [6] shows that when the rate of descent of the cloud droplets is neglected, the kinetic equation reduces to the equation for ω):

$$\left.\begin{aligned}
&\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial z}-\frac{\partial}{\partial z}k(z)\frac{\partial w}{\partial z}=4\pi D(q-q_s(T))\,\bar{r}N_0,\\
&\frac{\partial q}{\partial t}+u\frac{\partial q}{\partial z}-\frac{\partial}{\partial z}k(z)\frac{\partial q}{\partial z}=-4\pi D(q-q_s(T))\,\bar{r}N_0,\\
&\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial z}-\frac{\partial}{\partial z}k(z)\frac{\partial T}{\partial z}=4\pi D\alpha(q-q_s(T))\,\bar{r}N_0,
\end{aligned}\right\} \quad (1)$$

where ω is the liquid-water content of the cloud; q is the concentration of the water vapor; T is the temperature in K^{o}; $q_s(T)$ is the concentration of saturated vapor at temperature T; D is the coefficient of water vapor diffusion; N_0 is the number of droplets per cm^3; k(z) is the coefficient of turbulent diffusion; u is the velocity of the ascending currents; $\alpha = L/c_p\rho_B$; L is the heat of the water vapor phase transition; c_p is the specific heat of the air at constant pressure; ρ_B is the air density; γ_a is the adiabatic lapse rate; z is the vertical coordinate with axis OZ directed upward; z = 0 is the ground level and $\bar{r}(z, t)$ is the mean radius of the cloud droplets (expressed on the basis of function of the cloud droplet size distribution) which is a function of z and t. [The actual form of $\bar{r}(z, t)$ is not given since we will use a method of integrating system (1) that consists of the elimination of terms with $\bar{r}(z, t)$.]

Let us use the procedure employed previously in [6] as a method for solving system (1). The essence of this method is that we are not interested in finding the functions ω, q, and T for each point. Rather, we want to derive the equation for the movement of the cloud boundary. Let us introduce two functions:

$$\left.\begin{aligned}
&S=w+q,\\
&\Phi=T-\alpha w.
\end{aligned}\right\} \quad (2)$$

For these functions the system of equations (1) dissociates into two independent equations:

$$\frac{\partial S}{\partial t} + u\frac{\partial S}{\partial z} - \frac{\partial}{\partial z}k(z)\frac{\partial S}{\partial z} = 0, \quad (3)$$

$$\frac{\partial \Phi}{\partial t} + u\frac{\partial \Phi}{\partial z} - \frac{\partial}{\partial z}k(z)\frac{\partial \Phi}{\partial z} = -\gamma_a u. \quad (4)$$

In this approximate method, none of the equations of system (1) is solved since with the condition that $q = q_s(T)$ and $w = 0$ at the boundary of the cloud simultaneously (the criteria of this approximation are indicated in [6]), the equation for the motion of the boundary can be derived on the basis of the functions S and Φ

$$S = q_s[\Phi(z, t)]. \quad (5)$$

Let us examine the selection of the manner in which k(z) and u are dependent on the vertical coordinate.

The function k(z) is selected for the surface boundary layer of the atmosphere in the form of the degree law [1]: $k(z) = k_\infty(z/z_\infty)^p$ when $0 \le z \le z_\infty$; when $z \gtrsim z_\infty$ $k(z) = k_\infty$, $0 \le p < 1$. There is no exact boundary $z = z_\infty$ that divides the surface boundary layer from the free atmosphere with $k(z) = k_\infty$, therefore the conclusions resulting from the solution to the problem are valid for z close to z_∞ only to the extent that the above indicated function accurately represents k(z) at z close to z_∞.

The velocity of ascending motions within the surface boundary layer is assumed to be zero.

The very question of the existence of vertical motions within the friction layer requires discussion. Since it is precisely in the friction layer that the transition takes place from the final velocities of motion to the zero velocities at the earth's surface, and since this layer is vertically very inhomogeneous, it is difficult to speak of the existence of directed motions. Beyond the surface boundary layer ($z > z_\infty$), the velocity of vertical motions is assumed to be constant ($u = \text{const}$). The above observation regarding the boundary $z = z_\infty$ is also appropriate here.

Now the determination of the functions S and Φ is separated into finding S_1, S_2, Φ_1, and Φ_2, as determined in the intervals

$$S = \begin{cases} S_1(z, t), & 0 \leqslant z \leqslant z_\infty; \\ S_2(z, t), & z_\infty \leqslant z < \infty; \end{cases}$$

$$\Phi = \begin{cases} \Phi_1, & 0 \leqslant z \leqslant z_\infty; \\ \Phi_2, & z_\infty \leqslant z < \infty \end{cases}$$

and satisfying the equations (from now on, when solving similar equations for S and Φ, we will refer only to Φ_1 and Φ_2, while S_1 and S_2 will be derived by substituting the appropriate boundary and initial conditions and the conversion of γ_2 into 0).

$$\frac{\partial \Phi_1}{\partial t} = \frac{\partial}{\partial z} z^p \frac{\partial \Phi_1}{\partial z}; \tag{6}$$

$$\frac{\partial \Phi_2}{\partial t} + \frac{u z_\infty}{k_\infty} \frac{\partial \Phi_2}{\partial z} = \frac{\partial^2 \Phi_2}{\partial z^2} - \frac{\gamma_a u z_\infty^2}{k_\infty}. \tag{7}$$

In equations (6) and (7) a transition was made to the dimensionless variables

$$z \to \frac{z}{z_\infty}, \quad t \to \frac{t}{t_0}, \quad t_0 = \frac{z_\infty^2}{k_\infty}.$$

Equations (6) and (7) as well as similar equations for S_1 and S_2 were solved with initial and boundary conditions that were derived from corresponding restrictions for T, w, and q

$$T = T(0, z), \; w = w(0, z), \; q = q(0, z) \text{ when } t = 0;$$
$$T = T(t, 0), \; w = w(t, 0), \; q = q(t, 0) \text{ when } z = 0.$$

Thus, in conformity with equation (2), for the functions S and Φ, we have:

$$\left.\begin{aligned} S(0,\ z) &= q\,(0,\ z) + w\,(0,\ z), \\ \Phi\,(0,\ z) &= T\,(0,\ z) - aw\,(0,\ z); \\ S(t,\ 0) &= q\,(t,\ 0) + w\,(t,\ 0), \\ \Phi\,(t,\ 0) &= T\,(t,\ 0) - aw\,(t,\ 0). \end{aligned}\right\} \tag{8}$$

The solution to equation (6) and the corresponding solution to $S_1(z,\ t)$ are found in appendix 1 by generalizing the method of potentials for the case where the coefficient of turbulence is dependent on the coordinates [formula (A.11)]. They are determined by the formulas:

$$\left.\begin{aligned} \Phi_1(z,\ t) &= \int_0^\infty dz'G\,(z,\ z',\ t)\,\Phi\,(z',\ 0) + \int_0^t d\tau G\,(z,\ 0,\ t-\tau)\,\mu_1(\tau) + \\ &\quad + \int_0^t d\tau G\,(z,\ 1,\ t-\tau)\,\mu_2(\tau), \\ G\,(z,\ z',\ t) &= \frac{(zz')^{\frac{1-p}{2}}}{(2-p)\,t}\exp\left(\frac{z'^{2-p} + z^{2-p}}{(2-p)^2 t}\right) I_{\frac{p-1}{2-p}}\left(\frac{2\,(z\,z')^{\frac{2-p}{2}}}{(2-p)^2 t}\right), \end{aligned}\right\} \tag{9}$$

where $I_\nu(x)$ is the Bessel function of the imaginary argument of subscript ν. It is well known that the solution to the heat conductivity equation (7) in the interval $1 \le z < \infty$ has the form

$$\Phi_2(z,\ t) = \left(\int_0^\infty \frac{\Psi\,(z',\ 0)}{2\sqrt{\pi t}}\, e^{-\frac{(z-z')^2}{4t} - \frac{uz_\infty}{2k_\infty} z'}\, dz' + \right.$$

$$\left. + \int_0^t \frac{\mu_3(\tau)\, e^{-\frac{(z-1)^2}{4(t-\tau)}}}{2\sqrt{\pi\,(t-\tau)}}\, d\tau\right) \exp\left[\frac{uz_\infty}{2k_\infty} z - \left(\frac{uz_\infty}{2k_\infty}\right)^2 t\right] - \frac{\gamma_a u z_\infty^2}{k_\infty}\, t. \tag{10}$$

The three unknown functions μ_1, μ_2, and μ_3, must be determined from the boundary condition $\Phi_1(z,\ t)/_{z\,=\,0} = \Phi(0,\ t)$ and also from the equality of Φ_1 and Φ_2 and their derivatives when there is a tendency of $z \to 1$ on the left in the first case and on the right in the second case.

In the form in which equations (9) and (10) appear, the function Φ satisfies equation (4) as well as the boundary conditions and the initial condition (8). This form can be used to numerically compute μ_1, μ_2, and μ_3.

The analytic expressions for μ_1, μ_2, and μ_3 can be found for brief time intervals ($t^{1/2} \ll 1$) (in this case μ_1 does not require definition as a value for the term of the exponential order of small magnitude), but in the equation for boundary motion (5) they produce a contribution that is negligible except when $(h - 1)^2 \ll 4t$, where h is the initial height of the cloud base. Whence it is seen that μ_2 and μ_3 need only be allowed for when the base of the cloud coincides with z_∞. But, since it is impossible to determine the latter exactly, for practical purposes another limiting case which does not require μ_2 and μ_3 to be determined, viz., $[(h - 1)^2 \gg 4t]$ is important.

Now, consistent with the general nature of the assumptions which we have admitted, let us assume that, during the time intervals examined, the cloud base is only able to move a distance that is much less than the height of the cloud (h)

$$z(t) = h + \chi(t), \quad h \gg \chi.$$

After substituting this expression for z(t) into (10) and (9), let us asymptotically compute the first integrals, assuming that in the general case the functions $\Phi_{1,2}(z, 0)$ [like $S_{1,2}(z, 0)$] may undergo discontinuity of the derivatives at the cloud boundary

$$\begin{aligned}\Phi_1(\chi, t) = {} & T(h, 0) + \frac{\chi}{2}\left[T'(h-0, 0) + T'(h+0, 0) - aw'(h+0, 0)\right] + \\ & + \frac{t^{1/2}}{\sqrt{\pi}}\left[T'(h+0, 0) - T'(h-0, 0) - aw'(h+0, 0)\right] + \\ & + \frac{t}{2}\left\{\frac{p}{h^{1-p}}\left[T'(h+0, 0) + T'(h-0, 0) - aw'(h+0, 0)\right] + \right. \\ & \left. + h^p\left[T''(h+0, 0) + T''(h-0, 0) - aw''(h+0, 0)\right] - \frac{(4-3p)p}{8h^{2-p}}T(h, 0)\right\} + \\ & + \frac{\chi^2}{2}\left\{T''(h+0, 0) + T''(h-0, 0) - aw''(h+0, 0) - \right. \\ & \left. - \frac{(4-p)p}{8h}\left[T''(h+0, 0) + T''(h-0, 0) - aw''(h+0, 0)\right]\right\} + \\ & + \chi t^{1/2}\left\{\frac{3p}{4h^{\frac{2-p}{2}}}\left[T'(h+0, 0) - T'(h-0, 0) - aw'(h+0, 0)\right] + \right. \\ & \left. + \frac{h^{p/2}}{\sqrt{\pi}}\left[T''(h+0, 0) - T''(h-0, 0) - aw''(h+0, 0)\right]\right\} - \\ & - \exp\left(-\frac{(h+\chi)^{2-p}}{(2-p)^2 t}\right)\frac{(2-p)t^{1/2}}{2h^{\frac{2-p}{2}}\sqrt{\pi}}\left(1 - \frac{2-p}{2}\frac{\chi}{h} + \ldots\right)T(h, 0) + \\ & + \frac{T'(h+0, 0) - T'(h-0, 0) - aw'(h+0, 0)}{2h^{p/2}} \cdot \frac{\chi}{t^{1/2}} + O, \qquad (11)\end{aligned}$$

where O are terms of the subsequent small orders of magnitude.

For the function Φ_2, as determined beyond the height of the surface boundary layer, the expansion of brief time intervals and small displacements by degrees $[\chi, (t)]$ has the form

$$\begin{aligned}\Phi_2(\chi, t) = {} & T(h, 0) + \frac{\chi}{2}\left[T'(h-0, 0) + T'(h+0, 0) - aw'(h+0, 0)\right] + \\ & + \frac{t^{1/2}}{\sqrt{\pi}}\left[T'(h+0, 0) - T'(h-0, 0) - aw'(h+0, 0)\right] + \\ & + \frac{t}{2}\left\{T''(h+0, 0) + T''(h-0, 0) - aw''(h+0, 0) - \frac{uz_\infty}{k_\infty}\left[T'(h+0, 0) + \right.\right. \\ & \left.\left. + T'(h-0, 0) - aw'(h+0, 0)\right]\right\} +\end{aligned}$$

$$+\frac{\chi^2}{2}\Big\{T''(h+0,0)-\alpha w''(h+0,0)-\frac{uz_\infty}{k_\infty}[T'(h+0,0)-\alpha w'(h+0,0)]+$$
$$+T''(h-0,0)+3\left(\frac{uz_\infty}{2k_\infty}\right)^2T(h)-\frac{uz_\infty}{k_\infty}T'(h-0,0)\Big\}+$$
$$+\frac{\chi t^{1/2}}{\sqrt{\pi}}\Big\{T''(h+0,0)-\alpha w''(h+0,0)-\frac{uz_\infty}{k_\infty}[T''(h+0,0)-\alpha w''(h+0,0)-$$
$$-T'(h+0,0)+\alpha w'(h+0,0)+T'(h-0,0)]\Big\}-\frac{\gamma_1 uz_\infty^2}{k_\infty}t-\frac{T(h,0)}{2h\sqrt{\pi}}\times$$
$$\times\exp\left(-\frac{(h+\chi)^2}{4t}\right)\left(1-\frac{\chi}{h}+\ldots\right)+O, \qquad (12)$$

where O are terms of the subsequent small orders of magnitude.

In expressions (11) and (12) the derivatives at the points h - 0 and h + 0 means that they are taken on the left and right, respectively, of the point z = h. An analysis of expressions (11) and (12) permits us to draw the conclusion that, during brief time intervals, i.e., when $t^{1/2} \ll 1$, $\chi/t^{1/2} \ll 1$, and $h \simeq 1$, Φ_1 and Φ_2 can be limited to the first three terms. The expressions for S_1 and S_2 are derived from Φ_1 and Φ_2, respectively, by the substitution of $T \rightarrow q$ and $-\alpha w \rightarrow w$.

The substitution of Φ_1, S_1, Φ_2, and S_2 in equation (5) makes it possible to determine the lower and upper displacements (χ_B and χ_N) of the boundary

$$\chi_N(t)=-\frac{1}{A}\Big\{t^{1/2}[q_s'(T)T'(h-0,0)-q'(h-0,0)+$$
$$+w'(h+0,0)(1+\alpha q_s'(T))]+\frac{q_s'(T)T-q_s(T)}{16h^{2-p}}(4-3p)pt\Big\}; \qquad (13)$$
$$\chi_B(t)=-\frac{1}{A}\{t^{1/2}[q_s'(T)T'(h-0,0)-q'(h-0,0)+$$
$$+w'(h+0,0)(1+\alpha q_s'(T))]-q_s'(T)\frac{\gamma_2 uz_\infty^2}{k_\infty}t, \qquad (14)$$

where

$$A=\frac{\sqrt{\pi}}{\cdot}[w'(h+0,0)(1+\alpha q_s'(T))-q_s'(T)T'(h-0,0)+q'(h-0,0)].$$

When computing (14) it was assumed that the velocity u does not exceed 1 cm/sec.

The time intervals during which expressions (13) and (14) hold are small: $t^{1/2} << |1 - h|$; in dimensional units, assuming that $z_\infty \simeq 100$ m, we have: $t^{1/2} << 10^{3/2}$, i.e., t is about 2 min. However, considering that it is important for us to know at least the qualitative pattern of the displacement, i.e., the sign of the displacement, the sign can approximately be assumed to hold up to several tens of minutes. The sign in equations (13) and (14) is obviously determined by the sign of the numerator, since the sign of the denominator in the majority of actual situations is positive.

It is interesting to note that, given the same initial stratifications of temperature, liquid-water content, and water vapor concentration, boundaries within the limits of the surface boundary layer and above it can have different signs for the displacement.

Appendix 1

In the appendix we determine in detail the solution to the equation in the form (6), i.e., the generalized heat conductivity equation in the interval $0 < z < \ell$ under the appropriate initial and boundary conditions.

Let the equation sought for have the form

$$\frac{\partial \varphi}{\partial t} = \frac{\partial}{\partial z} z^{p} \frac{\partial \varphi}{\partial z}. \qquad \text{(A.1)}$$

The initial and boundary conditions for the function φ are written in the form:

$$\varphi(z,\ t)\big|_{t=0}=\varphi(z,\ 0);\quad \varphi(z,\ t)\big|_{z=l}=\varphi(l,\ t);$$
$$\varphi(z,\ t)\big|_{z=0}=\varphi(0,\ t). \qquad (A.2)$$

Let us attempt to formulate a solution to (A.1) that is analogous to the method of heat potentials for the heat conductivity equation. In order to do this, one must find the Green's function of equation (A.1). The function of influence (The Green's function) can be derived from the solution to (A.1) with the initial condition (A.2) in infinite space.

In this case let us seek a solution in the form of the separation of variables

$$\varphi^*(z,\ t)=\int_0^\infty d\lambda\, v(\lambda,\ z)\, a(\lambda,\ z). \qquad (A.3)$$

The substitution of (A.3) into (A.1) produces the equation

$$\frac{1}{a}\ \frac{da}{dt}=\frac{1}{v}\ \frac{d}{dz}\,z^p\,\frac{dv}{dz}=-\lambda^2. \qquad (A.4)$$

The sign in front of λ^2 on the right-hand side of (A.4) is determined by the tendency of $\varphi^*(z,\ t)\to 0$ when $t\to\infty$. From (A.4) we find for $a(\lambda,\ t)$

$$a(\lambda,\ t)=c(\lambda)\,e^{-\lambda^2 t}, \qquad (A.5)$$

where $c(\lambda)$ is the function of the parameter λ.

For $v(\lambda,\ z)$ we have from (A.4)

$$\frac{d}{dz}\,z^p\,\frac{dv}{dz}+\lambda^2 v=0. \qquad (A.6)$$

This equation is a Bessel type equation, and its solution can be written in the form [4]

$$v(z, \lambda) = z^{\frac{1-p}{2}} I_{\frac{p-1}{2-p}}\left(\frac{2\lambda}{2-p} z^{\frac{2-p}{2}}\right), \tag{A. 7}$$

where $I_\nu(x)$ is the Bessel function of the subscript ν from the argument x.

The selection of the Bessel function as a solution to (A. 6) is explained by the fact that the solution to $\varphi^*(z, t)$ must be finite when $z = 0$. The sign in front of the subscript and argument will be understood from further discussion.

The substitution of (A. 7) and (A. 5) into (A. 4) produces the solution to $\varphi^*(z, t)$. The function $c(\lambda)$ is determined from the initial condition (A. 2). Assuming that $t = 0$ in (A. 4), we have

$$\varphi^*(z, 0) = \int_0^\infty c(\lambda) z^{\frac{1-p}{2}} I_{\frac{p-1}{2-p}}\left(\frac{2\lambda}{2-p} z^{\frac{2-p}{2}}\right) d\lambda.$$

Using the "orthogonality" of the Bessel functions in an infinite interval

$$\int_0^\infty I_k(s\rho) I_k(st)\, s\, ds = \frac{1}{t}\delta(\rho - t),$$

for $c(\lambda)$ we obtain the equations

$$c(\lambda) = \frac{2\lambda}{2-p}\int_0^\infty I_{\frac{p-1}{2-p}}\left(\frac{2\lambda}{2-p} z'^{\frac{2-p}{2}}\right) \varphi(z', 0)\, z'^{\frac{2-p}{2}} dz'. \tag{A. 8}$$

In the derivation we assumed that $\varphi(z', 0) = 0$ when $z' < 0$.

Substituting (A.8) into (A.5) and integrating over λ, we obtain for $\varphi^*(z, t)$ the equation

$$\varphi^*(z, t) =$$

$$= \frac{1}{(2-p)t} \int_0^\infty dz' (zz')^{\frac{1-p}{2}} \varphi(z', 0)\, e^{-\frac{z'^{2-p} + z^{2-p}}{(2-p)^2 t}} I_{\frac{p-1}{2-p}}\left(\frac{2(zz')^{\frac{2-p}{2}}}{(2-p)^2 t}\right), \quad \text{(A.9)}$$

where $I_\nu(x)$ is the Bessel function for the imaginary argument of the subscript ν.

The function G(z, z', t) appearing in (A.9) in addition to φ(z', 0) is called the Green's function for the operator of equation (A.1)

$$G(z, z', t) = \frac{(zz')^{\frac{1-p}{2}}}{(2-p)t} e^{-\frac{z^{2-p} + z'^{2-p}}{(2-p)^2 t}} I_{\frac{p-1}{2-p}}\left(\frac{2(zz')^{\frac{2-p}{2}}}{(2-p)^2 t}\right). \quad \text{(A.10)}$$

Now, just as in the case of the heat conductivity equation, one can solve equation (A.1) in the interval $0 < z < l$ with the initial and boundary conditions (A.2), in the form

$$\varphi(z, t) = \int_0^\infty dz' G(z, z', t)\, \varphi(z', 0) + \int_0^t d\tau G(z, 0, t-\tau)\, \mu_1(\tau) +$$

$$+ \int_0^t d\tau G(z, l, t-\tau)\, \mu_2(\tau). \quad \text{(A.11)}$$

In this form φ(z, t) satisfies equation (A.1) since the function G(z, z', t) satisfies this equation. Satisfaction of the initial condition (A.2) follows directly from the formulation of the solution to $\varphi^*(z, t)$ and the boundary conditions (A.2) are allowed for by the appropriate selection of μ_1 and μ_2.

References

1. Gandin, L. S., et. al. Osnovy dinamicheskoi meteorologii [Principles of dynamic meteorology]. Leningrad, Gidrometeoizdat, 1955.

2. Klubovich, K. V. Formation of clouds in a homogeneous air mass, Vestnik MGU, Fizika, astronomiia 1, 1963[in Russian].

3. Matveev, L. T. and V. S. Kozharin. The role of turbulent mixing in the formation of the structure of stratiform clouds, I, Izvestiia AN SSSR, Ser. geofiz. No. 11:1338-1353,1956[in Russian].

4. Ryzhik, I. M. and I. S. Gradshtein. Tablitsy integralov, summ, riadov i proizvedenii, 3rd ed. Moscow, Gostekhizdat, 1951[Engl. transl. of 4th ed.: Tables of Integrals, Series, and Products. New York, Academic, 1965].

5. Feigel'son, E. M. Formation and development of stratiform clouds, Izvestiia AN SSSR, Ser. geofiz. No. 3:408-417,1962[in Russian; Engl. transl. in corresponding number of Bulletin of the Academy of Sciences, USSR, Geophysics Ser.].

6. Shulepov, Iu. V. and M. V. Buikov. Dissipation of stratus clouds in a turbulent atmosphere, Trudy UkrNIGMI No. 48: 3-12,1965[in Russian; Engl. transl.: Am. Met. Soc. T-R-556].