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Use of the Weibull Distribution in Bayesian Decision Theory

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FOREWORD

The reliability of a complex system frequently enters into important decisions. The decision-making process may be difficult when the lifetime distribution of the particular system is known; it becomes more difficult when the lifetime distribution is unknown. Such may be the case when a new type of equipment is placed in operation on a large scale. Provisioning for replacements should be based on the lifetime distribution of the equipment, but a reliable estimate of this distribution is usually not available until a great deal of failure information has been obtained. In such a situation it is the aim of the Bayesian approach to enable all available data to be incorporated into the decision-making process.

This paper first discusses some characteristics that are desirable in the class of lifetime distributions to be examined by Bayesian methods and shows that the Weibull distribution possesses some of these characteristics. The remainder of the paper treats the case in which the shape parameter of the Weibull distribution is known and presents prior, posterior, and preposterior Bayesian analyses.

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CONTENTS

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ABSTRACT

The Weibull distribution is useful in analyzing the probabilistic lifetimes of many electrical components and complex systems. It is attractive for Bayesian decision-making because its right-hand cumulative function is of an exponential form which allows all life-test data to be easily incorporated into the decision-making process. Unfortunately no natural conjugate prior distribution exists if both the shape and scale parameters of the Weibull distribution are assumed to be unknown. If the shape parameter is assumed known, however, Bayesian analysis becomes little more difficult than for the exponential distribution, a special case of the Weibull. Prior, posterior, and preposterior analyses are given for the case of known shape parameter. In connection with preposterior analysis several sampling plans are discussed. The paper concludes with an analysis of a problem in optimal sampling.

1. INTRODUCTION

Many decisions involve the reliability of electrical components and complex systems. The decision-making process is often difficult when the lifetime distribution of the particular component or system is known; it becomes more difficult when the lifetime distribution is unknown. Such is the case, for example, when a new type of equipment is placed in operation on a large scale. Provisioning decisions should be based on the lifetime distribution of the equipment, but a reliable estimate of this distribution is not usually available until a considerable amount of failure information has been obtained. This paper first briefly discusses the benefits that may be obtained by applying the Bayesian approach to decision problems involving the reliability of systems. Section 2 then discusses some desiderata of the class of lifetime distributions to be examined by Bayesian methods and shows that the Weibull distribution possesses some of these desirable characteristics. Section 3 presents a Bayesian analysis of the Weibull distribution when the shape parameter is assumed known. This assumption is carried through the remainder of the paper. Experimentation and preposterior analysis are discussed in Sec 4, and Sec 5 illustrates the approach with an optimal sampling problem.

In the Bayesian approach we assume that the probability density function $f(x)$ of the useful lifetime $\tilde{x}$ (a tilde indicates a random variable) of a system is a member of a class of density functions indexed by a parameter vector θ. Thus we write $f(x|\theta)$. We assume that the value of θ which applies to the particular system under study is unknown and that the decision-maker treats $\tilde{\theta}$ as a random variable and expresses his judgment about $\tilde{\theta}$ in the form of a probability distribution on $\tilde{\theta}$. We write the probability density function of $\tilde{\theta}$ as $f(\theta|\mathcal{E})$, $\mathcal{E}$ indicating the experience of the decision-maker. By using the density functions $f(x|\theta)$ and $f(\theta|\mathcal{E})$ and his utilities for consequences associated with future events involving x and $\tilde{\theta}$, the decision-maker can choose among various courses of action. If some evidence z (failure data) is accumulated, the decision-maker revises his probability distribution on $\tilde{\theta}$ by Bayes' theorem:

$$f(\theta|z, \mathcal{E}) = \frac{\ell(z|\theta, \mathcal{E}) f(\theta|\mathcal{E})}{\int \ell(z|\theta, \mathcal{E}) f(\theta|\mathcal{E}) d\theta} . \tag{1}$$

Here $\ell(z|\theta, \mathcal{E})$ is the likelihood of the evidence z conditional on θ and $\mathcal{E}$.

The decision-maker will now choose among various courses of action by employing the revised or posterior density function $f(\theta|z, \mathcal{E})$. Indeed, whenever a decision must be made, the total evidence z accumulated up until that time can be incorporated into the decision-making process via the new density function $f(\theta|z, \mathcal{E})$. This is one of the main benefits of the Bayesian approach.

The notion of a sequence of decisions raises the possibility of deciding to delay a decision until more evidence is available. But just how much evidence should be collected before a terminal decision is made? The decision-maker can decide this by using the density functions $f(x|\theta)$ and $f(\theta|\mathcal{E})$ and his utilities for consequences associated with future events involving $\tilde{x}$, $\tilde{\theta}$, and the evidence $\tilde{z}$ (which is, a priori, a random variable). This is another major benefit of the Bayesian approach to decision-making.

The mathematical foundations of Bayesian decision theory are set forth in Raiffa and Schlaifer.[1] For further discussion of the Bayesian approach to decision problems involving the reliability of systems the reader is referred to Briggs,[2] Howard,[3] and Martel.[4]

2. DESIDERATA FOR BAYESIAN ANALYSIS OF RELIABILITY PROBLEMS

We first list some desirable properties of the density functions $f(x|\theta)$.

(1) $f(x|\theta)$ must be the density function of a nonnegative random variable.

(2) Depending on θ, $f(x|\theta)$ must assume a fairly wide range of shapes.

(3) The likelihood $\ell(z|\theta, \mathcal{E})$ should have a relatively simple form.

The first property is clearly necessary. The second is desirable if the analysis is to cover a wide range of reliability problems. For example, the exponential distribution given by $f(x|\theta) = e^{-\theta x}$ assumes that the failure rate does not change with time, a somewhat restrictive assumption.

At any time the total evidence z will consist of the information that r systems have failed after operating for times $x_1, \ldots, x_r$ and that $n - r$ systems have operated for times $x_{r+1}, \ldots, x_n$ without failing. We assume that systems fail independently. If we define

$$G(x|\theta) = \int_x^\infty f(y|\theta)dy ,$$

the likelihood of all this evidence is

$$\ell(z|\theta,\mathcal{E}) = \left[\prod_{i=1}^{r} f(x_i|\theta)\right]\left[\prod_{i=r+1}^{n} G(x_i|\theta)\right] .$$

Unless this likelihood has a relatively simple mathematical form the computations required by Bayes' theorem become very complicated. For example, the gamma-1 distribution with density function

$$f(x|r,y) = y^r x^{r-1} e^{-yx}/\Gamma(r),$$

where $\tilde{\theta} = (\tilde{r}, \tilde{y})$, does not yield a closed-form expression for $G(x|r, y)$ unless r is an integer, and even then $G(x|r, y)$ is a sum of r terms.

A density function which has the three properties given above is that of the Weibull distribution (see Qureishi[5] for references):

$$\begin{aligned} f_w(x|\lambda,\alpha) &= \lambda\alpha x^{\alpha-1}e^{-\lambda x^\alpha}, \\ &\qquad\qquad 0 \le x < \infty, \\ &\qquad\qquad 0 < \lambda, \alpha < \infty, \\ G_w(x|\lambda,\alpha) &= e^{-\lambda x^\alpha}. \end{aligned} \tag{2}$$

The parameters α and λ are shape and scale parameters, respectively. The likelihood $\ell(z|\theta, \mathcal{E})$ becomes in this case

$$\ell(z|\lambda, \alpha) = \alpha^r \lambda^r (x_1 \ldots x_r)^{\alpha-1} \exp\left[-\lambda \sum_{i=1}^{n} x_i^{\alpha}\right]. \tag{3}$$

We have just discussed some properties that are desirable in the density function $f(x|\theta)$. There are also some properties that are desirable in the prior density function $f(\theta | \mathcal{E})$. These are discussed by Raiffa and Schlaifer,[1] and we list three of them here.

(4) It should be fairly easy to ascertain the posterior density that results from a given prior density and given evidence z.

(5) $f(\theta | \mathcal{E})$ should be a member of a closed family f^* of density functions so that the posterior density $f(\theta | z, \mathcal{E})$ is also a member of f^*.

(6) The expectations of some simple utility functions with respect to any member of f^* should be expressible in convenient form.

Unfortunately it does not seem possible to find a family f^* that possesses these properties when the parameters λ and α of the Weibull distribution are both assumed to be unknown. This is basically because sufficient statistics of fixed dimensionality (see Raiffa and Schlaifer[1]) for the evidence z do not exist in this case. Appendix A briefly examines a joint density function for $(\tilde{\lambda}, \tilde{\alpha})$ that has property 4 and has property 5 in a weak sense but does not possess property 6.

It thus appears that the Weibull distribution with both parameters unknown is not amenable to useful Bayesian analysis.

3. BAYESIAN ANALYSIS OF THE WEIBULL DISTRIBUTION WITH KNOWN SHAPE PARAMETER

In the remainder of this paper we shall assume that $f(x|\theta)$ is given by the Weibull density function (expression 2) and that the value of the shape parameter α is known. In this case only the scale parameter $\tilde{\lambda}$ need be treated as a random variable. The following have been offered as justification for this assumption:

(1) For some items, such as vacuum tubes (see Kao[6]), an appropriate value of α may be known from previous test evidence.

(2) The same assumption is made in standard Weibull life-testing procedures (see Ref 7).

(3) Previous Bayesian analysis of the exponential distribution (see Raiffa and Schlaifer,[1] Briggs,[2] and Martel[4]) in effect makes the assumption $\alpha = 1$. It is no less valid to assume a value of α other than 1 in an appropriate situation.

Natural Conjugate Distribution

We need to find a family f^* of density functions that possesses properties 4, 5, and 6 of Sec 2; one such family is that of natural conjugates to the sample likelihood (expression 3). With α known, a kernel of the likelihood, i.e., a factor which depends on λ (see Raiffa and Schlaifer,[1] p 30), is

$$\lambda^r e^{-\lambda y}. \tag{4}$$

where

$$y = \sum_{i=1}^{n} t_i^{c} \, . \tag{5}$$

When $\tilde{\lambda}$ is a random variable, the natural conjugate of expression 4, obtained by treating expression 4 as a kernel of a density function for $\tilde{\lambda}$ (see Raiffa and Schlaifer,[7] p 47), is the gamma-1 density function.

$$f_{\gamma 1}(\lambda|r, y) = \frac{y^r \lambda^{r-1} e^{-\lambda y}}{\Gamma(r)} \qquad \begin{matrix} 0 \le \lambda < \infty, \\ 0 < r, y < \infty . \end{matrix} \tag{6}$$

Wilson[8] has previously used expression 6, in a different context, as a prior density function when the scale parameter λ of a Weibull distribution is unknown.

Prior and Posterior Analysis

Suppose the decision-maker must choose an act a from a set A of possible acts and his terminal utility (terminal in the sense that no experiment is to be performed) for an act a and particular value λ is $u_t(a, \lambda)$. He will choose an act a' such that

$$E'_{\lambda} u_t(a', \tilde{\lambda}) = \max_a E'_{\lambda} u_t(a, \tilde{\lambda}) .$$

The notation E' indicates that the expectation is taken with respect to the gamma-1 prior distribution on $\tilde{\lambda}$.

If a gamma-1 prior distribution with parameters r' and y' is assigned to $\tilde{\lambda}$ and an experiment e yields an outcome z (evidence z) with statistics r and y, the posterior distribution on $\tilde{\lambda}$ is gamma-1 with parameters r'' and y'', where

$$r'' = r' + r, \; y'' = y' + y . \tag{7}$$

The decision-maker will now choose an act a'' such that

$$E''_{\lambda|z} u_t(a'', \tilde{\lambda}) = \max_a E''_{\lambda|z} u_t(a, \tilde{\lambda}) .$$

Here $E''_{\lambda|z}$ denotes expectation with respect to the posterior distribution on $\tilde{\lambda}$, which is conditional on z.

4. EXPERIMENTATION AND PREPOSTERIOR ANALYSIS

When the decision-maker is contemplating a specific experiment he must consider the net value of that experiment. He will generally have a utility function $u(e, z, a, \lambda)$ defined for each combination of experiment e, outcome z, subsequent action a, and particular value λ. Before experiment e is performed the outcome $\tilde{z}$ is a random variable, and so the overall (expected) utility of experiment e is

$$u^*(e) = E_{z|e} \max_a E''_{\lambda|z} u(e, \tilde{z}, a, \tilde{\lambda}) . \tag{8}$$

Here $E_{z|e}$ indicates expectation with respect to the distribution of $\tilde{z}$. When a number of experiments are being considered, the final step in preposterior (before an experiment) analysis is the finding of an optimal experiment e^*, i.e., one such that

$$u^*(e^*) = \max_e u^*(e).$$

It is clear from Eq 8 that the distribution of $\tilde{z}$ is important for the successful completion of preposterior analysis. In the present case the statistic $\tilde{z}$ is the pair $(\tilde{r}, \tilde{y})$, where $\tilde{r}$ is the number of failures observed and $\tilde{y}$ (defined by Eq 5) is the sum of the αth powers of the operating times.

The next subsection briefly discusses a particular type of sampling plan (experiment) and gives the distributions of $\tilde{r}$ and $\tilde{y}$. We also consider the time $\tilde{t}$ required to complete the experiment. It is assumed that the prior distribution on $\tilde{\lambda}$ is gamma-1 with parameters r' and y'. Two other sampling plans are then mentioned in the subsection, "Other Sampling Plans."

Test n Items Simultaneously, without Replacement, until r Failures Occur

In this sampling plan n items are placed on life test at the same time and the experiment is terminated when r of them have failed. In the special case in which $r = n$, the failure times of all n are observed. The statistic r is predetermined and the distribution of $\tilde{y}$ is desired. It can be shown (see RAC-TP-215[9]) that this distribution is inverted-beta-2:

$$f(y|r', y'; r) = f_{i\beta 2}(y|r, r', y') = \frac{1}{B(r, r')} \frac{y^{r-1}(y')^{r'}}{(y+y')^{r+r'}}. \tag{9}$$

Note that this distribution does not depend on n. The time $\tilde{t}$ to complete the experiment (i.e., the time of the rth failure) does depend on n; the density function for $\tilde{t}$ is very complicated, but the pth moment is given by (see RAC-TP-215[9]):

$$E(\tilde{t}^p|r', y'; r, n) = E(\tilde{t}^p|1; r, n)(y')^{p/\alpha}\Gamma(r' - p/\alpha)/\Gamma(r'). \tag{10}$$

Here $E(\tilde{t}^p|1; r, n)$ is the expected value of the pth power of the rth order statistic in a sample of n items. Each of these items has a Weibull lifetime distribution with $\lambda = 1$ and the same value of α used in the Bayesian analysis. The first moment, $E(\tilde{t}|1; r, n)$, is tabulated by Harter[10] for $\alpha = 0.5(0.5)4(1)8$, $r = 1(1)n$, $n = 1(1)40$. The pth moment may be obtained by replacing α by α/p.

Other Sampling Plans

There are a number of other sampling plans that may be considered in practice and are therefore worthy of further investigation. Two of them are as follows:

(1) Test items sequentially until r failures occur. As soon as one item fails, another one is placed on life test. The statistic r is fixed and $\tilde{y}$ and $\tilde{t}$ are random variables. This sampling plan is discussed in RAC-TP-215.[9]

(2) Test n components simultaneously, without replacement, and stop at time $\tilde{t} = \min(\tilde{x}_m, T)$, where $\tilde{x}_m$ is the time of the mth failure and T is fixed in advance. In this case $\tilde{r}$, $\tilde{y}$, and $\tilde{t}$ are all random variables.

5. A PROBLEM IN OPTIMAL SAMPLING

To illustrate the previous developments we present an analysis (including prior, posterior, and preposterior analyses) of a hypothetical problem. The sampling plan is the same as the one considered previously, namely testing n items simultaneously without replacement, until r failures occur. It is desired that the optimal values of r and n be found.

Problem Statement

A manufacturer must supply a traveling-wave tube for use in a government satellite. He can either use one of type T1 (act a_1) whose lifetime distribution is known or use one of type T2 (act a_2) which has just been developed and whose lifetime distribution is unknown. He is willing to assume that the lifetime distribution of type T2 is Weibull with a particular value of the shape parameter α. He assigns a gamma-1 prior distribution with parameters r' and y' to the unknown scale parameter $\tilde{\lambda}$.

The manufacturer's terminal utility function $u_t(a, \lambda)$ is derived as follows. If the tube used in the satellite functions longer than time t_o, benefits with utility k accrue to the manufacturer. If the tube fails before time t_o, his utility is zero. His overall utility is thus k[Prob(tube life exceeds t_o)], and therefore

$$u_t(a_2, \lambda) = ke^{-\lambda t_o^{\alpha}} = k\rho = u_t(a_2, \rho),$$

where

$$\rho = e^{-\lambda t_o^{\alpha}} .$$

If $\tilde{\lambda}$ is a random variable, so is $\tilde{\rho}$. Since the lifetime distribution of type T1 is known we have $u_t(a_1, \lambda) = u_t(a_1, \rho) = K$, a constant.

Prior and Posterior Analysis

It is convenient in this problem to work with the random variable $\tilde{\rho}$. We assume that $K < k$ so that there exists a breakeven value of ρ, called ρ_b, such that $u_t(a_1, \rho_b) = u_t(a_2, \rho_b)$; thus $\rho_b = K/k$. Then a_1 is preferred for $\rho < \rho_b$, and a_2 is preferred for $\rho > \rho_b$.

Since $\tilde{\rho}$ is a random variable, if terminal action is to be taken on the basis of prior information the manufacturer should choose an act a' such that

$$E'_{\rho} u_t(a', \tilde{\rho}) = \max_a E'_{\rho} u_t(a, \tilde{\rho}) .$$

But $E'_{\rho} u_t(a_1, \tilde{\rho}) = K$ and $E'_{\rho} u_t(a_2, \tilde{\rho}) = k\bar{\rho}'$, where $\bar{\rho}' = [y'/(y' + t_o^{\alpha})]^{r'}$ is the mean of the prior distribution on $\tilde{\rho}$. Thus the optimal act under the prior distribution is

$$a' = \begin{cases} a_1 \text{ if } \bar{\rho}' \le \rho_b , \\ a_2 \text{ if } \bar{\rho}' > \rho_b . \end{cases}$$

If an experiment (life-test sample) on tubes of type T2 yields an outcome with statistics r and y, the posterior distribution on $\tilde{\lambda}$ will be gamma-1 with parameters $r'' = r' + r$ and $y'' = y' + y$. The optimal act under the posterior distribution will then be

$$a'' = \begin{cases} a_1 & \text{if } \bar{\rho}'' \le \rho_b , \\ a_2 & \text{if } \bar{\rho}'' \ge \rho_b , \end{cases}$$

where $\bar{\rho}'' = [y''/(y'' + t_o^{\alpha})]^{r''}$ is the mean of the posterior distribution on $\tilde{\rho}$.

Preposterior Analysis

In considering the value of sampling we assume that sampling and terminal utilities are additive, so that $u(e, z, a, \rho) = u_t(a, \rho) - c_s(e, z)$, where $c_s(e, z)$ is the cost of experiment e and outcome z. Equation 8 then becomes

$$\begin{aligned} u^*(e) &= E_{z|e} \max_a E''_{\rho|z} u_t(a, \tilde{\rho}) - E_{z|e} c_s(e, \tilde{z}) \\ &= u_t^*(e) - c_s^*(e) . \end{aligned}$$

We have $E''_{\rho|z} u_t(a, \tilde{\rho}) = u_t(a, \tilde{\bar{\rho}}'')$, where a tilde is placed over $\bar{\rho}''$ to indicate that prior to the experiment it is a random variable. We next note that

$$\max_a u_t(a, \bar{\rho}'') = \begin{cases} K & \text{if } \bar{\rho}'' \le \rho_b , \\ k\bar{\rho}'' & \text{if } \bar{\rho}'' \ge \rho_b . \end{cases}$$

We shall consider the sampling plan treated in the subsection "Test n Items Simultaneously, without Replacement, until r Failures Occur." The statistic $\tilde{y}$ is a random variable with inverted-beta-2 distribution given by Eq 9. Since $\bar{\rho}'' = [y''/(y'' + t_o^{\alpha})]^{r''}$, we obtain

$$u_t^*(e) = \int_0^{y_b} K f_{i\beta 2}(y|r, r', y')dy + \int_{y_b}^{\infty} k\left[(y+y')/(y+y'+t_o^{\alpha})\right]^{r''} f_{i\beta 2}(y|r, r', y')dy. \qquad (11)$$

where y_b is defined by

$$\rho_b = \left[(y_b + y')/(y_b + y' + t_o^{\alpha})\right]^{r''} .$$

Now we turn to the expected cost of sampling $c_s^*(e)$. We shall assume that $c_s(e, \tilde{z}) = c_1 n + c_2 \tilde{t} + c_3 \tilde{t}^2$, where n is the number of tubes placed on life test, $\tilde{t}$ is the duration of the test, and the c_i are positive constants. We then have

$$c_s^*(e) = c_1 n + c_2 E(\tilde{t}|r', y'; r, n) + c_3 E(\tilde{t}^2|r', y'; r, n),$$

where $E(\tilde{t}^2 | r', y'; r, n)$ is given by Eq 10.

The final step in preposterior analysis is to determine the optimal sampling plan, i.e., we want to choose r and n to maximize $u^*(e) = u_t^*(e) - c_s^*(e)$. It may happen, of course, that $u^*(e)$ is less than the utility of immediate terminal action $E'_{\rho} u_t(a', \tilde{\rho})$ for all combinations of r and n; in this case no experiment should be performed.

Determination of the Optimal Sampling Plan

We can use dynamic programming to find the pair (r^*, n^*) which maximizes $u^*(e)$. Let $u^*(r, n)$ be the (expected) utility of the sampling plan $e = (r, n)$.

Define

$$u^*(r) \quad \max_n u^*(r, n).$$

Then

$$u^*(r^*, n^*) \quad \max_r u^*(r).$$

The computations required to obtain $u^*(r)$ are relatively minor because only $c_s^*(c)$ depends upon n; $u_i^*(c)$ depends only upon r. Determination of $u^*(r)$ therefore amounts to minimizing $c_s^*(r, n)$ with respect to n and subtracting the result from $u_i^*(r)$ (we write $u_i^*(c) = u_i^*(r)$ when c specifies r failures). Recall that $c_s^*(r, n) = c_1 n + c_2 E(\widehat{t} \mid r', y'; r, n) + c_2 E(\widehat{t^2} \mid r', y'; r, n)$. It can be verified from the values in Harter[10] that $E(\widehat{t^p} \mid r', y'; r, n)$ is a convex function of n for fixed r. It then follows that $c_s^*(r, n)$ is a convex function of n for fixed r and so can be minimized over integer values of n fairly readily.

Calculation of $u_i^*(r)$ from Eq 11 involves determination of values of the incomplete beta-function. Depending on the computational facilities available, this determination might be done only for selected values of r. Then $u^*(r)$ could be computed for these values of r, and the results graphed as a function of r. The graph would give a good estimate of r^* and this estimate could be checked by computing $u^*(r)$ for appropriate values of r.

Appendix A

EXAMINATION OF A PRIOR DENSITY FUNCTION WITH TWO PARAMETERS UNKNOWN

Here we shall briefly examine a prior density function for the case in which the parameters λ and α of the Weibull distribution are both assumed to be unknown. Although this density function possesses property 4 of Sec 2, it does not really have property 5 and definitely does not possess property 6.

We take the joint density function on $(\tilde{\lambda}, \tilde{\alpha})$ in the form of the likelihood, expression 3:

$$f(\lambda, \alpha | r', d_1, \ldots, d_{n'}) = \lambda^{r'} \alpha^{r'} (d_1 \ldots d_{r'})^{\alpha - 1} \exp\left[-\lambda \sum_{i=1}^{n'} d_i^{\alpha}\right] \Big/ k(r', d_1, \ldots, d_{n'}) , \quad (12)$$

where $0 \le \lambda, \alpha < \infty$; $n' \ge r' \ge 1$; $d_i > 0$; and

$$k(r', d_1, \ldots, d_{n'}) = \int_0^{\infty} \int_0^{\infty} \lambda^{r'} \alpha^{r'} (d_1 \ldots d_{r'})^{\alpha - 1} \exp\left[-\lambda \sum_{i=1}^{n'} d_i^{\alpha}\right] d\lambda\, d\alpha$$

$$= r'! \int_0^{\infty} \alpha^{r'} (d_1 \ldots d_{r'})^{\alpha - 1} \left(\sum_{i=1}^{n'} d_i^{\alpha}\right)^{-r'-1} d\alpha \quad (13)$$

Equation 12 defines a proper density function if some d_i is greater than 1, since this suffices to show that $k(r', d_1, \ldots, d_{n'})$ is finite.

Now if the prior density function on $(\tilde{\lambda}, \tilde{\alpha})$ is $f(\lambda, \alpha | r', d_1, \ldots, d_{n'})$ and the likelihood of the evidence z is given by expression 3, then Bayes' theorem yields

$$f(\lambda, \alpha | r' + r, d_1, \ldots, d_{r'}, x_1, \ldots, x_r, d_{r'+1}, \ldots, d_{n'}, x_{r+1}, \ldots, x_n) \quad (14)$$

for the posterior density function. Thus the prior density function defined by Eq 12 possesses property 4. Although the prior and posterior density functions have similar forms, property 5 is not present because the two density functions do not have the same number of parameters. Property 6 is lacking for the simple reason that the constant $k(r', d_1, \ldots, d_{n'})$ cannot be evaluated in closed form.

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