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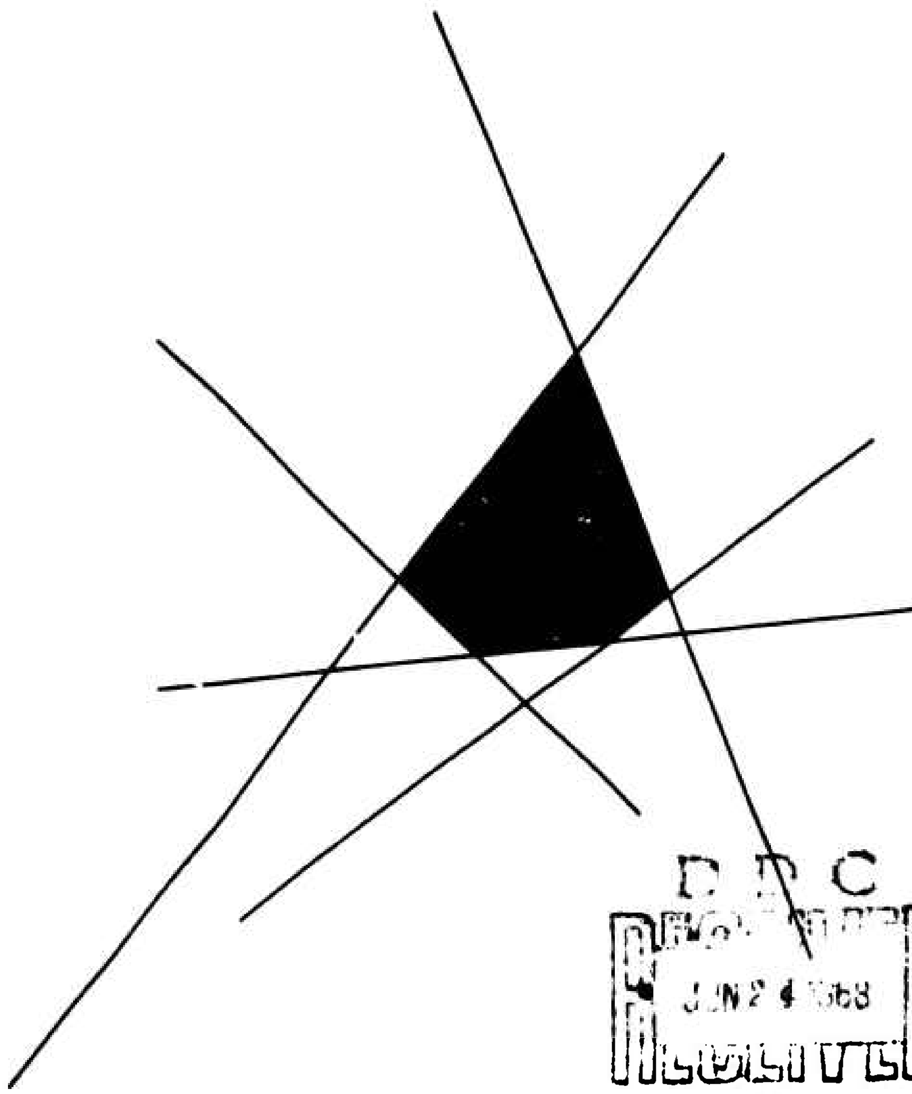
ON EXISTENCE OF WEAKLY MAXIMAL PROGRAMS IN VON NEUMANN GROWTH MODELS

by

WILLIAM A. BROCK



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Dr. Gale suggested investigating whether the optimal stationary program was optimal in the class of programs starting from it. This turned out not to be true but the investigation of it lead to the present work.

ABSTRACT

We consider, with D. Gale, an n-sector economy defined by a convex, compact set T consisting of input-output pairs $(x,y) \in R^{2m}$ where R^{2m} is real 2m-dimensional space. On T is defined a concave, continuous real function u that associates *utility* to each input-output pair. T and u are called the *technology* and the *utility function* of the economy. At each $(x,y) \in T$ we assume u has *bounded steepness*, i.e.,

$$\sup_{(x',y')\in T} \left| u(x',y') - u(x,y) \right| / \|(x',y') - (x,y)\| < \infty$$

where $\|\cdot\|$ is any norm on R^{2m}. A *program with initial stocks* y_o is a sequence $\left\{(x_n,y_n)\right\}_{n=1}^{\infty}$ of points of T such that $x_n \leqq y_{n-1}$, $n = 1,2, \ldots$ An *optimal stationary program* (OSP) is a pair $(\bar{x},\bar{y}) \in T$ such that $(\bar{x},\bar{y})$ solves: max $u(x,y)$ over $\{(x,y) \in T \mid x \leqq y\}$ where $\leqq$ is componentwise ordering on R^m. The initial stock y_o is *sufficient* if there is a program $\left\{(x_n,y_n)\right\}_{n=1}^{\infty}$ from y_o such that $y_N > 0$ for some N. The technology T is *productive* if there is $(x,y) \in T$ such that $x < y$ (i.e., $x_i < y_i$, $i = 1,2, \ldots, m$). We say *inaction* is possible if $(0,0) \in T$. A program $\{(x'_n,y'_n)\}$ from y_o is *optimal (weakly maximal)* if

$$\underline{\lim}_{N\to\infty} \sum_{n=1}^{N} (u(x_n',y_n') - u(x_n,y_n)) \geqq$$

$$0\left(\overline{\lim}_{N\to\infty} \sum_{n=1}^{N} (u(x_n',y_n') - u(x_n,y_n)) \geqq 0\right)$$

for all programs $\{(x_n,y_n)\}$ starting from y_o. The utility function u is *strictly concave* at $(\bar{x},\bar{y}) \in T$ if $u(\lambda(\bar{x},\bar{y}) + (1-\lambda)(x,y)) > \lambda u(\bar{x},\bar{y}) + (1-\lambda)u(x,y)$ for all $(x,y) \neq (\bar{x},\bar{y})$, $(x,y) \in T$, $0 < \lambda < 1$.

An easy proof is given of Gale's result: If y_o is sufficient, T productive, inaction is possible, and u is strictly concave at $(\bar{x},\bar{y})$ then there is an optimal program from y_o. The main result is: If $(\bar{x},\bar{y})$ is the unique OSP, y_o is sufficient, T is productive, and inaction is possible then there is a weakly maximal program from y_o. An example is given where the hypotheses of the main result are satisfied but the OSP does not outgrow every program starting from it.

The importance of the main result lies in removing the objectionable hypothesis of strict concavity on the utility function.

ON EXISTENCE OF WEAKLY MAXIMAL PROGRAMS
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In this paper we shall consider general models of economic growth, formulate an "optimality" criterion, and show that there exist "optimal programs" under weak, economically meaningful assumptions. We now turn to development of the model following Gale [1].

Let T be a compact, convex set in real $2m$ space R^{2m} ($R^{2m} \equiv \{(x,y) \mid x$ is a real m-vector and y is a real m-vector$\}$) . Let u be a concave, continuous numerical function defined on T . T represents the possible input-output combinations and u attaches utility to each. T,u are called the *technology* and the *utility function* respectively. Assume further that u has† bounded steepness over T , i.e., $\sup_{(x,y)\epsilon T} \sigma(x,y) < \infty$ where

$$\sigma(x,y) \equiv \sup_{(x',y')\epsilon T} |u(x',y') - u(x,y)| / ||(x',y') - (x,y)||$$

for each $(x,y)\epsilon T$, where $||\cdot||$ is a norm on R^{2m} . Bounded steepness at (x,y) , i.e., $\sigma(x,y) < \infty$ means that no infinitesimal movement from (x,y) can produce a large change in utility.

Let $0 \leq y_o \epsilon R^m$. A *program with initial stocks* y_o is a sequence $\{(x_n,y_n)\}_{n=1}^{\infty}$ of points $(x_n,y_n)\epsilon T$ such that $x_n \leq y_{n-1}$, $n = 1, 2, \ldots$ With each program there is associated a utility sequence $\{u(x_n,y_n)\}_{n=1}^{\infty}$.

The problem of optimal economic growth is: given initial stocks y_o , find a "reasonable" partial ordering of utility sequences generated by the economy and show there exist maximal elements with respect to this ordering.

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†It is an easy exercise to show $\forall (x,y)\epsilon T$, $\sigma(x,y) < \infty$ implies $u(x,y)$ has bounded steepness over T .

We say that $\{(x_n,y_n)\}$ *overtakes (strictly overtakes)* $\{(x'_n,y'_n)\}$ if

$$\varliminf_{N\to\infty}\left(\sum_{n=1}^{N} u(x_n,y_n) - u(x'_n,y'_n)\right) \geqq 0 \ , \ (> 0) \tag{1}$$

$\{(x_n,y_n)\}$ is *optimal* if it overtakes every other program starting from y_o. $\{(x_n,y_n)\}$ is *maximal (weakly maximal)* if no other program starting from y_o overtakes (strictly overtakes) it. Overtaking is clearly a partial ordering on the set of programs starting from y_o and a maximal program is a maximal element with respect to this ordering. We now introduce the notion of optimal stationary program (OSP) to facilitate construction of weakly maximal programs.

Definition:

$(\bar{x},\bar{y})\epsilon T$ is an *OSP* if $(\bar{x},\bar{y})$ solves:

$$\max \{u(x,y) \mid x \leqq y \ , \ (x,y)\epsilon T\} .$$

Let us assume,

Assumption 1:

T is *productive* and *inaction* is possible, i.e., there $(x,y)\epsilon T$ such that $x < y$, and $(0,0)\epsilon T$, respectively.

Using productivity of T we may prove

Lemma 1

If $(\bar{x},\bar{y})$ is an OSP and T is productive then there is $p \geqq 0$, $p \in R^m$ such that $(\bar{x},\bar{y})$, p solve:

$$\begin{array}{ll} \text{Maximize} & u(x,y) + p\cdot(y - x) \\ \text{Subject to} & (x,y)\epsilon T \end{array}$$

furthermore

$$p \cdot (\bar{y} - \bar{x}) = 0 .$$

Proof:

This is an application of the Kuhn-Tucker Theorem to the problem of maximizing $u(x,y)$ subject to the constraint $x \leqq y$ where, by productivity of T, the constraint can be satisfied strictly and this is sufficient (see [3]) for the validity of the Kuhn-Tucker Theorem.

Let us now consider an important example, i.e., the "Von Neumann" economy:

$$u(x,y) \equiv \max \{c.v \mid Av = x , Bv = y\} , T \equiv \{Av,Bv) \mid v \in K\}$$

where A,B are $m \times k$ nonnegative goods matrices and $K \subseteqq R^k$ is a compact polyhedron (e.g., labor limitations) constraining activity levels v and $c \cdot v = \sum_{i=1}^{k} c_i v_i$ is the "value" or utility obtained from operating the k activities at intensity v.

We want to find reasonable assumptions so that this economy has maximal or weakly maximal programs. In [1] Gale shows that strict concavity of u at $(\bar{x},\bar{y})$ implies existence of an optimal program from y_o. But the utility function of the Von Neumann economy is strictly concave at no $(x,y) \epsilon T$.

If one is satisfied with a weakly maximal program then it is sufficient to assume that the OSP, $(\bar{x},\bar{y})$, is unique and to make some mild assumptions on T, y_o. We shall now work toward the proof of this result.

Assumption 2:

y_o is *sufficient*, i.e., there exists a program $\{(x_n,y_n)\}_{n=1}^{\infty}$ from y_o such that† $y_n > 0$ for some n.

†Given a vector $x \epsilon R^m$, $x > 0$ $(x \geqq 0)$ means $x_i > 0$ $(x_i \geqq 0)$ $i = 1, 2, \ldots, m$.

We now define an important class of programs.

Definition:

A program is *good* if there exists a real number G such that

$$G \leq \sum_{n=1}^{N} u(x_n,y_n) - u(\bar{x},\bar{y}) ,$$

for all positive integers N .

For the sake of completeness we state and give Gale's proof of the existence of good programs.

Lemma 2 (Gale)

If y_o , T satisfy Assumptions 1,2, then there is a good program from y_o .

Proof:

Since y_o is sufficient we can, after finite time, produce a bundle $y_n > 0$. Since goodness does not depend on what happens in a finite amount of time we may as well assume $y_o > 0$. Using $(0,0)\epsilon T$ and convexity of T choose a productive pair $(x_p,y_p)\epsilon T$ such that $x_p < y_o$. Choose $0 \leq \lambda < 1$ such that $x_1 \equiv (1 - \lambda)\,\bar{x} + \lambda x_p \leq y_o$, $y_1 \equiv (1 - \lambda)\,\bar{y} + \lambda y_p$. Define the sequence $\{(x_n,y_n)\}$ by

$$(x_n,y_n) = (1 - \lambda^n)(\bar{x},\bar{y}) + \lambda^n(x_p,y_p) , n = 1, 2, \ldots$$

Using the recursion:

$$(x_n,y_n) = (1 - \lambda)(\bar{x},\bar{y}) + \lambda(x_{n-1},y_{n-1}) , n = 2, 3, \ldots$$

it is easy to check that $\{(x_n,y_n)\}$ is a program from y_o.

Using bounded steepness we get, for all N,

$$\sum_{n=1}^{N} |u(\bar{x},\bar{y}) - u(x_n,y_n)| \leqq \sum_{n=1}^{N} \sigma(x_n,y_n) \; ||(\bar{x},\bar{y}) - (x_n,y_n)||$$

$$\leqq \sum_{n=1}^{N} \sup_{(x,y)\epsilon T} \sigma(x,y) \; \lambda^n \; ||(x_p - \bar{x} \, , \, y_p - \bar{y})||$$

$$< \infty \text{ , since } \sup_{(x,y)\epsilon T} \sigma(x,y) < \infty \text{ , } 0 \leqq \lambda < 1 \text{ .}$$

It follows that $\{(x_n,y_n)\}$ is good.

It turns out that

<u>Lemma 3</u> (Gale)

A program is good or its associated series converges to $-\infty$.

<u>Proof</u>:

From this point on we set $u(\bar{x},\bar{y}) = 0$ to ease the writing. If a program is not good then for all real numbers G there is M depending on G such that

$$\sum_{n=1}^{M} u(x_n,y_n) \leqq G \text{ .}$$

By Lemma 1 and compactness of T

$$\sum_{n=M+1}^{N} u(x_n,y_n) \leqq \sum_{n=M+1}^{N} p \cdot (x_n - y_n) \leqq p \cdot (x_{M+1} - y_N) < B$$

where B is a bound independent of M and N. Hence

$$\sum_{n=1}^{N} u(x_n,y_n) \leq G - B \qquad \text{for } N \geq M .$$

It follows that

$$\sum_{n=1}^{N} u(x_n,y_n) \to -\infty , \; N \to \infty .$$

"Turnpike" properties appear to be everpresent in the theory of optimal economic growth. We prove an "average turnpike" property for good programs.

Lemma 4

If $\{(x_n,y_n)\}$ is good then $u(\bar{x}_n,\bar{y}_n) \to u(\bar{x},\bar{y})$ where $(\bar{x}_n,\bar{y}_n) \equiv \frac{1}{n} \sum_{i=1}^{n} (x_i,y_i) \varepsilon T$. In particular, if $(\bar{x},\bar{y})$ is the unique OSP then $(\bar{x}_n,\bar{y}_n) \to (\bar{x},\bar{y})$.

Proof:

By goodness of $\{(x_n,y_n)\}$ and concavity of u , there is G such that for each N

$$\frac{G}{N} \leq \frac{1}{N} \sum_{n=1}^{N} u(x_n,y_n) \leq u(\bar{x}_N,\bar{y}_N) .$$

Let (x,y) be any cluster point of the sequence $\{(\bar{x}_n,\bar{y}_n)\}$. Then $0 \leq u(x,y)$, $x \leq y$, and $(x,y) \varepsilon T$ because

$$(\bar{x}_n,\bar{y}_n) = \frac{1}{n} \sum_{i=1}^{n} (x_i,y_i) \varepsilon T , \; n = 1, 2, \ldots$$

and T is closed. Hence (x,y) is an OSP and

$$u(\bar{x}_n,\bar{y}_n) \to u(\bar{x},\bar{y}) \equiv 0 \ , \ n \to \infty \ .$$

In particular if $(\bar{x},\bar{y})$ is unique, $(x,y) = (\bar{x},\bar{y})$, thus

$$(\bar{x}_n,\bar{y}_n) \to (\bar{x},\bar{y}) \ , \ n \to \infty \ .$$

After proving one more Lemma we may prove the main result of this paper.

Lemma 5

If y_o, T satisfy Assumptions 1,2 then there is a nonnegative sequence $\{\delta_n\}_{n=1}^{\infty}$ associated with $\{u(x_n,y_n)\}$ such that

$$\sum_{n=1}^{N} u(x_n,y_n) = p \cdot (y_o - y_N) - \sum_{n=1}^{N} \delta_n \ , \ N = 1, 2, \ldots \tag{1}$$

Furthermore, a program $\{(x'_n,y'_n)\}$ may be found so that its associated series $\sum_{n=1}^{\infty} \delta'_n$ is minimal in the class of programs starting from y_o .

Proof:

By Lemma 1 there is $\beta_n \geqq 0$ such that

$$u(x_n,y_n) = p \cdot (x_n - y_n) - \beta_n \ , \ n = 1, 2, \ldots \tag{2}$$

Summing (2) to N and setting $\delta_n = -p \cdot (x_n - y_{n-1}) + \beta_n \geqq 0$ gives (1). For $\{(x_n,y_n)\}$ good there exist G, B such that

$$G \leqq \sum_{n=1}^{N} u(x_n,y_n) = p \cdot (y_o - y_N) - \sum_{n=1}^{N} \delta_n \leqq B - \sum_{n=1}^{N} \delta_n \ , \ N = 1, 2, \ldots$$

Thus

$$\sum_{n=1}^{N} \delta_n \leq B - G \;, \; N = 1, 2, \ldots$$

so that $\sum_{n=1}^{\infty} \delta_n < \infty$ for good programs. Our task is to find $\{(x_n', y_n')\}$ starting from y_o such that $\alpha \equiv \inf\left\{ \sum_{n=1}^{\infty} \delta_n : \{\delta_n\} \text{ is associated with } \{(x_n, y_n)\} \text{ starting from } y_o \right\} = \sum_{n=1}^{\infty} \delta_n'$. By existence of good programs (Lemma 2) we have $\alpha < \infty$.

Choose a "minimizing" sequence of programs $\left\{\left(x_n^N, y_n^N\right)\right\}$ such that

$$\sum_{n=1}^{\infty} \delta_n^N \leq \alpha + \frac{1}{N} \;, \; N = 1, 2, \ldots$$

By compactness of T, use the Cantor diagonal process to select a subsequence $\{N'\} \subseteq \{N\}$ such that for each n

$$\left(x_n^{N'}, y_n^{N'}\right) \rightarrow (x_n', y_n') \;, \; N' \rightarrow \infty \;.$$

It is easy to check that $\{(x_n', y_n')\}$ is a program. By compactness of T and continuity of u, $\{\delta_n^{N'}\}_{N'}$ is bounded for each n. Hence we may use the Cantor diagonal process again to choose a subsequence $\{N''\} \subseteq \{N'\}$ so that

$$\delta_n^{N''} \rightarrow \delta_n' \;, \; n = 1, 2, \ldots$$

It is clear from continuity of u and definition of δ_n that $\{\delta_n'\}$ corresponds to the program $\{(x_n', y_n')\}$. Hence

$$\sum_{n=1}^{\infty} \delta_n' \geq \alpha \;.$$

Suppose $\beta \equiv \sum_{n=1}^{\infty} \delta_n' > \alpha$. Let r_1, r_2 be chosen so that $\alpha < r_1 < r_2 < \beta$. Choose N_o so that $N \geq N_o$ implies $\sum_{n=1}^{N} \delta_n' \geq r_2$. Choose M_o (depending on N_o)

so that $M \in \{N''\}$, $M \geqq M_o$ implies $\sum_{n=1}^{N_o} \delta_n^M \geqq r_1$. But $\alpha + \frac{1}{M} \geqq \sum_{n=1}^{\infty} \delta_n^M \geqq \sum_{n=1}^{N_o} \delta_n^M$. Hence there are infinitely many $M \in \{N''\}$ such that $\alpha + \frac{1}{M} \geqq r_1 > \alpha$. This is a contradiction. Thus

$$\alpha = \sum_{n=1}^{\infty} \delta'_n \ .$$

We now prove our main result

<u>Theorem 1</u>

If there exists a unique OSP and T, y_o satisfy Assumptions 1,2 then there is a weakly maximal program from y_o . If $y_o = \bar{y}$ then a weakly maximal program is $(\bar{x},\bar{y})$.

<u>Proof</u>:

Let $\{(x'_n,y'_n)\}$ be a program with minimal $\sum_{n=1}^{\infty} \delta'_n$. From (1) of Lemma 5 we get

$$\sum_{n=1}^{N} u(x_n,y_n) - u(x'_n,y'_n) = p \cdot (y'_N - y_N) + \sum_{n=1}^{N} \delta'_n - \sum_{n=1}^{N} \delta_n \ , \ N = 1, 2, \ldots \qquad (1)$$

for any other program $\{(x_n,y_n)\}$. Assume $\{(x_n,y_n)\}$ strictly outgrows $\{(x'_n,y'_n)\}$. Using (1) above there are M , $\lambda > 0$ such that

$$\lambda \leqq p \cdot (y'_N - y_N) \ , \ N \geqq M \qquad (2)$$

Since we need only compare good programs (Lemma 2) we may use Lemma 4 (the average turnpike Lemma) to obtain

$$\lambda \leqq \underline{\lim}_{N\to\infty} p \cdot (\bar{y}'_N - \bar{y}_N) = 0$$

which is a contradiction. If $\bar{y} = y_0$ then $\sum_{n=1}^{\infty} \delta'_n = 0$ if $\{\delta'_n\}$ is associated with $\{(\bar{x},\bar{y})\}$. Hence $(\bar{x},\bar{y})$ is weakly maximal (note that, if $y_0 = \bar{y}$, the assumption that y_0 is sufficient is not needed for the weak maximality proof).

One might ask if it is possible to strengthen Theorem 1 by showing existence of a strongly optimal program, i.e.,

$$\underline{\lim_{N\to\infty}} \sum_{n=1}^{N} u(x_n,y_n) - u(x'_n,y'_n) > 0 \qquad \text{for all } \{(x'_n,y'_n)\} .$$

We produce a counterexample where the OSP is unique, but there is no strongly optimal program from $\bar{y}$.

We consider a one good, two activity Von Neumann economy. Let $v_1 \geqq 0$, $v_2 \geqq 0$ be activity levels of A_1, A_2 where A_1 doubles capital but produces no utility, A_2 halves capital and produces one unit of utility. Let the labor constraint be $v_1 + v_2 \leqq 1$. Then $A = (1,2)$, $B = (2,1)$. Solving the problem:

$$\text{Maximize} \quad c \cdot v = \max 0 \cdot v_1 + v_2$$

$$\text{Subject to} \quad v_1 + 2v_2 \leqq 2v_1 + v_2 \; ; \; v_1 + v_2 \leqq 1 \; , \; v_1 \geqq 0 \; , \; v_2 \geqq 0$$

gives us the unique OSP

$$\bar{v}_1 = \bar{v}_2 = \frac{1}{2} \, , \; \bar{x} \equiv \bar{v}_1 + 2\bar{v}_2 = \frac{3}{2} \, , \; \bar{y} \equiv 2\bar{v}_1 + \bar{v}_2 = \frac{3}{2} \, , \; \bar{u} = \bar{v}_2 = \frac{1}{2} \, .$$

Starting from $\bar{y}$ we have the program: (to see that the following is a program check that $Av_n \leqq Bv_{n-1}$, $n = 1, 2, \ldots$ where $v_0 = \bar{v}$)

$$\left(v_1^1, v_2^1\right) = \left(\frac{1}{6}, \frac{2}{3}\right) ; \ u_1 = \frac{2}{3}$$

$$\left(v_1^n, v_2^n\right) = (1,0) \ , \ n \ \text{even} \ ; \ u_n = 0$$

$$\left(v_1^n, v_2^n\right) = (0,1) \ , \ n \ \text{odd} \ ; \ u_n = 1 \ , \ n > 1$$

Note that

$$\overline{\lim} \sum_{n=1}^{N} (u_n - \bar{u}) > 0$$

so that $(\bar{x},\bar{y})$ does not outgrow $\{(x_n,y_n)\}$. However, by Theorem 1 $(\bar{x},\bar{y})$ is weakly maximal. Hence there can be no strongly optimal program from $(\bar{x},\bar{y})$. To get optimal programs it is sufficient to assume strict concavity of v at $(\bar{x},\bar{y})$.

The strong result of Gale may be obtained easily by the method of proof employed in this paper.

<u>Theorem 2</u> (Gale)

If u is[†] strictly concave at $(\bar{x},\bar{y})$ and y_o, T satisfy Assumptions 1,2 then there is a program $\{(x'_n,y'_n)\}$ from y_o such that for each $\{(x_n,y_n)\}$ starting from y_o

$$\underline{\lim}_{N\to\infty} \sum_{n=1}^{N} u(x'_n,y'_n) - u(x_n,y_n) \geq 0 \tag{1}$$

<u>Proof</u>:

Select $\{(x'_n,y'_n)\}$ with minimal $\sum_{n=1}^{\infty} \delta'_n$. It is easy to check that $\{(x'_n,y'_n)\}$ is good. By Lemma 3 we need only compare good programs. Consider

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[†]A function f on a set X is strictly concave at $\bar{x} \in X$ if for each $x \in X$, $0 < \lambda < 1$, $f(\lambda\bar{x} + (1 - \lambda)x) > \lambda f(\bar{x}) + (1 - \lambda)f(x)$.

a good program $\{(x_n,y_n)\}$. We show $(x_n,y_n) \to (\bar{x},\bar{y})$, $n \to \infty$. Since

$$\sum_{n=1}^{N} u(x_n,y_n) = \sum_{n=1}^{N} p \cdot (x_n - y_n) - \sum_{n=1}^{N} \beta_n \leqq p \cdot (y_o - y_N) - \sum_{n=1}^{N} \beta_n .$$

where $0 \leqq \beta_n = p.(x_n - y_n) - u(x_n,y_n)$, there is G (since $\{(x_n,y_n)\}$ is good) such that

$$\sum_{n=1}^{N} \beta_n \leqq p \cdot (y_o - y_N) - G \leqq B - G , \qquad N = 1,2, \ldots$$

Therefore $\beta_n \to 0$. Now let $(\tilde{x},\tilde{y})$ be any cluster point of $\{(x_n,y_n)\}$. Since $\beta_n \to 0$, therefore $u(\tilde{x},\tilde{y}) + p \cdot (\tilde{y} - \tilde{x}) = 0$. But strict concavity of u at $(\bar{x},\bar{y})$ implies $(\bar{x},\bar{y})$ is the unique maximizer over T of the nonpositive function, $u(x,y) + p \cdot (y - x)$. Thus $(\tilde{x},\tilde{y}) = (\bar{x},\bar{y})$ and $(x_n,y_n) \to (\bar{x},\bar{y})$ as $n \to \infty$. Now, as in Theorem 1, let $\{(x'_n,y'_n)\}$ be a program with minimum $\sum_{n=1}^{\infty} \delta'_n$. Since $y'_N \to \bar{y}$, $y_N \to \bar{y}$ for good programs it is now clear from (1) of Theorem 1 that

$$\underline{\lim}_{N\to\infty} \sum_{n=1}^{N} u(x'_n,y'_n) - u(x_n,y_n) \geqq 0 .$$

One might expect weakly maximal programs to exist in almost any model. Gale's [1] "cake eating" example with u strictly concave has no weakly maximal program. McKenzie [2] constructed an example of a Von Neumann model with linear utility that has no weakly maximal programs. Both of these examples have multiple OSP's. We doubt that it is possible to replace "weakly maximal" in Theorem 1 by "maximal".

BIBLIOGRAPHY

[1] Gale, D., "On Optimal Development in a Multisector Economy," Review of Economic Studies, Vol. 34, pp. 1-18, (1967).

[2] Private Communication of L. McKenzie to D. Gale.

[3] Karlin, S., MATHEMATICAL METHODS AND THEORY IN GAMES, PROGRAMMING AND ECONOMICS, Vol. I, Addison-Wesley, Reading, Massachusetts, pp. 200-201, (1959).

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