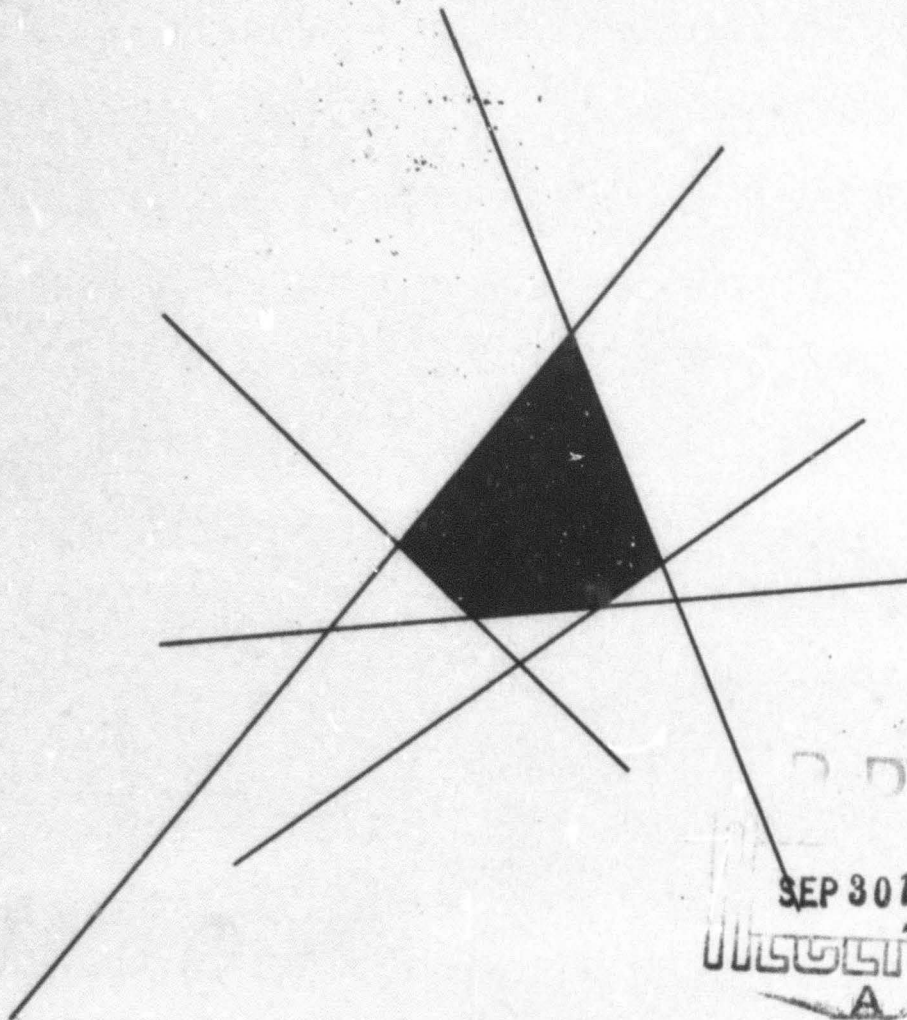


# SOME RECENT DEVELOPMENTS IN RELIABILITY THEORY

by  
Richard E. Barlow

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by

Richard E. Barlow  
Department of Industrial Engineering  
and Operations Research  
University of California, Berkeley

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#### SUMMARY

This is a survey of problems and recent developments in some selected areas of nonparametric reliability theory. The paper is divided into three parts: Models for Life Distributions, Tests of Hypotheses, and Estimation Procedures.

# SOME RECENT DEVELOPMENTS IN RELIABILITY THEORY

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## 1. MODELS FOR LIFE DISTRIBUTIONS

Reliability theory is largely concerned with questions about coherent structures, i.e., structures which can be represented as combinations of various series and parallel networks (allowing the possibility of component replication). Various attempts have been made to delimit the class of life distributions of interest for such structures. Extreme value theory provides an answer, in part, to the question: Given a coherent structure with a finite number of components and some procedure to increase the number of components without bounds, what are the possible limiting distributions for the structure lifetime for given component lifetime distributions? A series structure has a lifetime corresponding to the minimum of its component lifetimes and the limiting procedure adds one component at a time to the structure. The only possible relevant limiting distributions for this case are the Weibull and double exponential distributions [12]. If we add k-out-of-n structures to the structures of interest, we obtain the normal and lognormal distributions and distributions of the form

$$\begin{aligned} G_{\alpha,k}(x) &= 0 && \text{for } x \leq 0 \\ &= \frac{1}{(k-1)!} \int_0^{x^\alpha} e^{-y} y^{k-1} dy && \text{for } x > 0, \alpha > 0 \\ \Lambda_k(x) &= \frac{1}{(k-1)!} \int_0^{e^x} e^{-y} y^{k-1} dy && -\infty < x < \infty \end{aligned}$$

[8], [27]. Results for more general models have been obtained by Harris [14].

There is a vast literature concerned with the estimation of parameters for distributions in these classes. For a recent comprehensive survey of results for the Weibull distribution, see Mann [18].

We might call the preceding approach "the limiting distribution approach." A more recent approach to delimit the class of life distributions of interest has been to start with the exponential distribution and to ask the question: What is the smallest family of life distributions containing the exponential distribution which is closed under the formation of coherent structures and limits in distribution? This question was recently answered by Birnbaum, Esary and Marshall [5]. The class of distributions in question is precisely the class with increasing failure rate on the average (IFRA distributions), i.e.,  $\int_0^t r(x)dx/t$  is nondecreasing in  $t \geq 0$  where  $r(t)$  is the failure rate function. Research on estimation procedures for IFRA distributions and tests of hypotheses concerning such distributions is still underway [3], [9], [10].

A great deal of attention has been focused on models for life distributions based on properties of the failure rate function. For example, we could consider the class of distributions having failure rate  $r(t)$  for which  $r(t)/\psi(t)$  is increasing (or convex increasing) for specified  $\psi(t) > 0$ . If  $\psi(t)$  is constant, we have the class of IFR (for increasing failure rate) distributions. The case of more general  $\psi(t)$  has been investigated by Saunders [26] in connection with models for fatigue failure.

Some multivariate models have been proposed to describe the joint lifetimes of components in a coherent structure. The multivariate exponential distribution studied by Marshall and Olkin [19], [20] generalizes the univariate exponential distribution in a natural way. Harris [15] has proposed a class of multivariate distributions with IFR marginals which satisfy additional reasonable restrictions.

Esary, Proschan and Walkup [11] study a new concept of positive dependency describing the joint lifetimes of components in a coherent structure. They say that random variables  $T_1, T_2, \dots, T_n$  are *associated* if

$$\text{Cov}[f(\underline{T}), g(\underline{T})] \geq 0$$

for all nondecreasing functions  $f$  and  $g$  where  $\underline{T} = (T_1, T_2, \dots, T_n)$ . Random variables jointly distributed according to the multivariate exponential distribution, for example, are associated. This concept is partly motivated by the idea that working components tend to reinforce the contribution of each other to system performance. Stated in another way, a failed component may, if anything, stress the remaining working components so as to cause them to perform poorly. Also, components subject to a common environment tend to be associated. They obtain lower bounds on system reliability in terms of component marginal reliabilities when components have associated lifetimes. Lehmann [17] considers related ideas of positive dependency for the bivariate case and examines tests of independence versus positive dependency.

## 2. TESTS OF HYPOTHESES

Considerable attention has been focused on the problem of testing the validity of the IFRA and IFR models [1], [3], [24]. The problem usually posed is that of testing

$$\begin{aligned} H_0 &: F \text{ exponential} \\ \text{versus} & \\ H_1 &: F \text{ IFRA (or IFR)}. \end{aligned} \tag{1}$$

If  $G(x) = 1 - e^{-\lambda x}$  for  $x \geq 0$  and  $\lambda > 0$ , then  $G^{-1}F(x)/x$  is nondecreasing in  $x \geq 0$  if and only if  $F$  is IFRA. This observation suggests that we consider the following partial ordering on the space of life distributions; namely,  $F_1 \prec_* F_2$  if  $F_2^{-1}F_1(x)/x$  is nondecreasing in  $x \geq 0$ . Note that  $F_1 \prec_* F_2 \prec_* G$  suggests that a test for (1) should have greater power at  $F_1$  than at  $F_2$  no matter what significance level we choose for our test. Marshall, Walkup and Wets [22] have characterized the class of differentiable test statistics which produce tests having monotone power with respect to  $\prec_*$  ordering. These are just the tests based on functions  $h(x_1, x_2, \dots, x_n)$  having the properties:

(a)  $h$  is homogeneous;

$$(b) \sum_{i=0}^j x_{n-i} \frac{\partial h(x_1, \dots, x_n)}{\partial x_{n-i}} \geq 0 \text{ for } j = 0, 1, \dots, n-2$$

and all  $0 \leq x_1 \leq x_2 \leq \dots \leq x_n$ . The test associated with  $h$  would reject exponentiality if  $h(X_1, X_2, \dots, X_n) \geq c$  where  $c$  is some suitable critical number and  $0 \leq X_1 \leq X_2 \leq \dots \leq X_n$  are order statistics from  $F$ . For example,

$$h(X_1, \dots, X_n) = \sum_{i=1}^n (X_i - \bar{X})^2 / (\bar{X})^2$$

where  $\bar{X} = \sum_{i=1}^n X_i / n$  is such a function.

Let  $n(u)$  equal the number of items exposed to risk at time  $u$  and  
 $T(X_1) = \int_0^{X_1} n(u) du$ . Tests based on

$$h(X_1, \dots, X_n) = \sum_{i=1}^{n-1} \Delta_i T(X_i) / T(X_n) \quad (2)$$

for  $\Delta_i \geq 0$  are unbiased and have monotone power with respect to  $<^*$  ordering. These tests reject exponentiality in favor of IFRA for large values of  $h(X_1, X_2, \dots, X_n)$ . This statistic has a natural analogue for the case of incomplete data [2]. The corresponding test remains unbiased but we can no longer prove that it has monotone power with respect to  $<^*$  ordering.

Tests based on (2) have been extensively studied by Bickel and Doksum [3]. They call such tests studentized linear spacings tests and consider the following "pencils" of alternative densities to the exponential:

$$f_{\theta}^{(1)}(x) = [1 + \theta(1 - e^{-x})] \exp\{-[x + \theta(x + e^{-x} - 1)]\}$$

$$f_{\theta}^{(2)}(x) = (1 + \theta x) \exp\{-(x + \frac{1}{2}\theta x^2)\} \quad (\text{Linear Failure Rate})$$

$$f_{\theta}^{(3)}(x) = (1 + \theta)x^{\theta} \exp\{-x^{(1+\theta)}\} \quad (\text{Weibull})$$

$$f_e^{(4)}(x) = [x^{\theta} e^{-x}] / \Gamma(1 + \theta) \quad (\text{Gamma})$$

For each density  $x \geq 0$ ,  $\theta \geq 0$ , and the null hypothesis is obtained for  $\theta = 0$ . The asymptotically best studentized linear spacings test against  $f_{\theta}^{(1)}$  corresponds to weights  $\Delta_i = 1$  ( $i = 1, 2, \dots, n-1$ ). This statistic is also called the *total time on test* statistic. The asymptotic Pitman efficiency of this test compared to the asymptotically most powerful test for this reduced problem is only  $\frac{1}{4}$ . However, there is a great deal of evidence supporting the "robustness" of this test. The asymptotically best studentized linear spacings test for  $f_{\theta}^{(3)}$  --the Weibull density--seems to be a robust competitor to the total time on test statistic. They also consider the corresponding asymptotically best linear rank tests which are

asymptotically equivalent to the best linear spacings tests. However, the Monte Carlo power of the asymptotically best linear spacings tests is much superior for small sample sizes.

The Monte Carlo power of the total time on test statistic is computed in [1] against the likelihood ratio test for truncated exponentiality versus IFR distributions. Again, the total time on test statistic seems to be decidedly superior.

In [28], van Soest studied an omnibus Cramer-Von Mises-Smirnov type of statistic. The statistic is

$$C_n = n \int_0^{\infty} \{F_n(x) - \hat{F}(x)\}^2 d\hat{F}(x)$$

$$= 1/12n + \sum_{j=1}^n \left\{ \hat{F}(x_j) - \frac{2j-1}{n} \right\}^2$$

where  $F_n$  is the empirical distribution and  $\hat{F}$  is the maximum likelihood estimate of  $F$  under  $H_0$ . The null hypothesis is rejected for large values of  $C_n$ . He computes the Monte Carlo power of this test and compares it with a two sided test based on the total time on test statistic. The power curves below compare this test with the one sided test based on the total time on test statistic (2).

Doksum [10] has recently investigated the two sample problem for IFRA distributions. Let  $X_1, X_2, \dots, X_n$  and  $Y_1, Y_2, \dots, Y_n$  be two independent random samples from populations with continuous IFRA distributions  $F(\cdot)$  and  $F(\cdot/\Delta)$ , respectively, and let  $s_1, \dots, s_n$  denote the ranks of the  $Y$ 's in the combined sample. For testing  $H_0 : \Delta \leq 1$  versus  $H_1 : \Delta > 1$ , it is shown that the error probabilities of each monotone rank test  $\phi$  are bounded by the error probabilities for exponential alternatives, i.e., if  $G(t) = 1 - \exp(-t)$ ,  $t \geq 0$ , then

$$P[\text{reject } H_0 \mid F(\cdot), F(\cdot/\Delta)] \leq P[\text{reject } H_0 \mid G(\cdot), G(\cdot/\Delta)]$$



for  $\Delta \leq 1$ , and

$$P[\text{accept } H_0 \mid F(\cdot), F(\cdot/\Delta)] \leq P[\text{accept } H_0 \mid G(\cdot), G(\cdot/\Delta)]$$

for  $\Delta > 1$ .

The Savage test which rejects for small values of

$$S_N = \sum_{i=1}^n J_0(s_i) \quad \text{where} \quad J_0(k) = \sum_{j=N+1-k}^N \frac{1}{j} \quad (N = m+n)$$

is locally minimax for IFRA scale alternatives within the class of rank tests. For the two sample problem with indifference region, i.e.,  $H_0 : \Delta \leq 1$  versus  $H_1 : \Delta \geq \Delta_1 > 1$ , the Lehmann test which rejects for small values of

$$L_N = \sum_{j=1}^N \log \left[ j - \frac{(\Delta_1 - 1)}{\Delta_1} s'_j \right]$$

where  $s'_j$  is the number of  $Y$ 's greater than or equal to the  $(N+1-j)$ th order statistic in the combined sample, is minimax at  $\Delta = \Delta_1$  within the class of rank tests for IFRA scale alternatives.

In an earlier paper [9], Doksum considers the asymptotic efficiency of the best test for exponential models relative to the Savage,  $S_N$ , test. For the exponential distribution, the uniformly most powerful level  $\alpha$  test  $\phi_N^*$  of  $H_0 : \Delta = 1$  against  $H_1 : \Delta > 1$  rejects when

$$T = m^{-1} \sum_{i=1}^m X_i / n^{-1} \sum_{i=1}^n Y_i > F_{2m, 2n}(\alpha)$$

where  $F_{2m, 2n}(\alpha)$  is obtained from the tables of the  $F$  distribution with  $2m$  and  $2n$  degrees of freedom. Doksum considers the modified test  $\hat{\phi}$  so as to have an asymptotic level  $\alpha$  test when the distribution is IFRA and not exponential. The

test  $\hat{\phi}$  rejects  $H_0$  for large values of

$$N^{1/2}(T - 1)/(\hat{\sigma}/\hat{\mu})$$

where

$$\hat{\mu} = N^{-1} \left( \sum_{i=1}^n X_i + \sum_{i=1}^n Y_i \right)$$

and

$$(\hat{\sigma})^2 = N^{-1} \left( \sum_{i=1}^n X_i^2 + \sum_{i=1}^n Y_i^2 \right) - (\hat{\mu})^2 .$$

The efficiency of the Savage test with respect to  $\hat{\phi}$  goes from 1 to  $\infty$  as the shape parameter,  $b$ , of the Weibull distribution,  $G(t) = 1 - \exp[-\lambda t^b]$ , goes from 1 to  $\infty$  (or from 1 to 0). He also treats the case of censored sampling.

### 3. ESTIMATION PROCEDURES

The failure rate function is perhaps the most useful characterization of a life distribution. Parametric and nonparametric methods for estimating the failure rate are discussed in detail by Grenander [13]. He also characterizes the maximum likelihood estimate (MLE) of the failure rate function under the IFR assumption. The MLE can be easily computed even for very incomplete data (withdrawals may be allowed for example). If a total of  $n$  items are exposed to risk, failures are observed at times

$$Z_1 \leq Z_2 \leq \dots \leq Z_k \quad (k \leq n)$$

and  $n(u)$  is the number of items exposed to risk at time  $u$ , then the MLE estimate for the failure rate,  $r(t)$ , can be expressed as a step function, where

$$\hat{r}_n(t) = \begin{cases} 0 & 0 \leq t < Z_1 \\ \hat{r}_n(Z_i) & Z_i \leq t \leq Z_{i+1} \\ \infty & t \geq Z_k \end{cases}$$

and

$$\hat{r}_n(Z_i) = \text{Max}_{s \leq i} \text{Min}_{t \geq i+1} \left( \frac{t-s}{\sum_{j=s}^{t-1} \int_{Z_j}^{Z_{j+1}} n(u) du} \right).$$

Marshall and Proschan [21] proved that  $\hat{r}_n(t)$  is strongly consistent in the complete sample case. Since the life distribution is determined by the failure rate, we can also determine the MLE for the life distribution under the IFR assumption. Monte Carlo investigations, however, indicate that it is badly biased in the tails. For samples of size 100 or so, the empirical distribution appears to be a better estimate of the life distribution in the tails.

Rao [25] has characterized the limiting distribution of  $\hat{r}_n(t)$  assuming  $r(t)$  increasing and  $r'(t) > 0$ . He shows that

$$\mathcal{L} \left[ n^{1/3} \left\{ \frac{r'(t)r^2(t)}{2f(t)} \right\}^{-1/3} \left\{ \hat{r}_n(t) - r(t) \right\} \right] \rightarrow H(x)$$

where  $H(x)$  is a distribution whose density is determined implicitly as a solution to the heat equation. This result enables us to make asymptotic efficiency comparisons with other nonparametric estimators of the failure rate.

Parzen [23] and Weiss and Wolfowitz [30] investigate window estimators of the density  $f(t)$ , i.e.,

$$\bar{f}_n(t) = N/2n\epsilon_n$$

where  $N$  is the number of observations out of a sample of size  $n$  in  $(t - \epsilon_n, t + \epsilon_n)$ . If  $\epsilon_n = n^{-\alpha}$  and we assume only that  $f'(t)$  exists, then Weiss and Wolfowitz show that  $\alpha = 1/3$  provides an asymptotically efficient estimate in a certain sense. A natural nonparametric estimate of the hazard rate is then

$$\bar{r}_n(t) = \frac{\bar{f}_n(t)}{\int_t^{\infty} \bar{f}_n(x) dx}$$

Watson and Leadbetter [29] show that

$$\mathcal{L} \left[ [1 - F(t)][2n\epsilon_n/f(t)]^{1/2} [\bar{r}_n(t) - r(t)] \right] \rightarrow \Phi(x)$$

where  $\Phi(x)$  is the  $N(0,1)$  distribution. Following Hodges and Lehmann [4], we

define the asymptotic efficiency of  $\bar{r}_n(t)$  relative to  $\hat{r}_n(t)$  as

$$e(\bar{r}_n(t), \hat{r}_n(t)) = \frac{\left[ \frac{r'(t)r^2(t)}{2f(t)} \right]^{2/3} \sigma_0^2}{\left[ \frac{f(t)}{2} \right] \left[ \frac{1}{1-F(t)} \right]^2} \quad (3)$$

where  $\sigma_0$  satisfies

$$\inf_{\sigma} d[\phi(x), H(x/\sigma)] = d[\phi(x), H(x/\sigma_0)]$$

and

$$d[\phi(x), H(x)] = \sup_{-\infty < x < \infty} |\phi(x) - H(x)|.$$

In this computation, we have let  $\epsilon_n = n^{-1/3}$  for the window estimator. It seems clear from (3) that the MLE estimator will do best when  $r'(t) = 0$  and the window estimator will be better when  $r'(t)$  is very large, or in other words, when the failure rate is increasing very rapidly.

Since  $\bar{r}_n(t)$  is not necessarily increasing, a more acceptable competitor to  $\hat{r}_n(t)$  under the IFR assumption would be

$$\hat{\bar{r}}_n(t_i) = \text{Max}_{s \leq i} \text{Min}_{w \geq i} \left\{ \frac{\sum_{j=s}^w \bar{r}_n(t_j)}{w - s + 1} \right\}$$

which is nondecreasing in  $t_i$ . ( $0 \leq t_1 \leq t_2 \leq \dots \leq t_n \leq \dots$  are midpoints of intervals over which the window estimator is constant.) The results of Brunk [7] show that  $\hat{\bar{r}}_n(t)$  will inherit the consistency property of  $\bar{r}_n(t)$  and will improve on  $\bar{r}_n(t)$  in a least squares sense.

Similar nonparametric MLE estimates for U-shaped failure rate functions have been considered by Bray, Crawford and Proschan [6]. There is a very useful general smoothing technique behind all of these estimates. Brunk [7] has observed that  $\hat{r}_n(Z_i)$  is the "isotonic regression" on the index  $i$  of the sequence

$$r_n^*(Z_i) = 1 / \int_{Z_i}^{Z_{i+1}} n(u) du$$

with respect to a suitable measure, namely,

$$\mu\{i\} = \int_{Z_i}^{Z_{i+1}} n(u) du$$

in this case. The isotonic regression  $\hat{r}_n(Z_i)$  has a useful least squares property; namely,

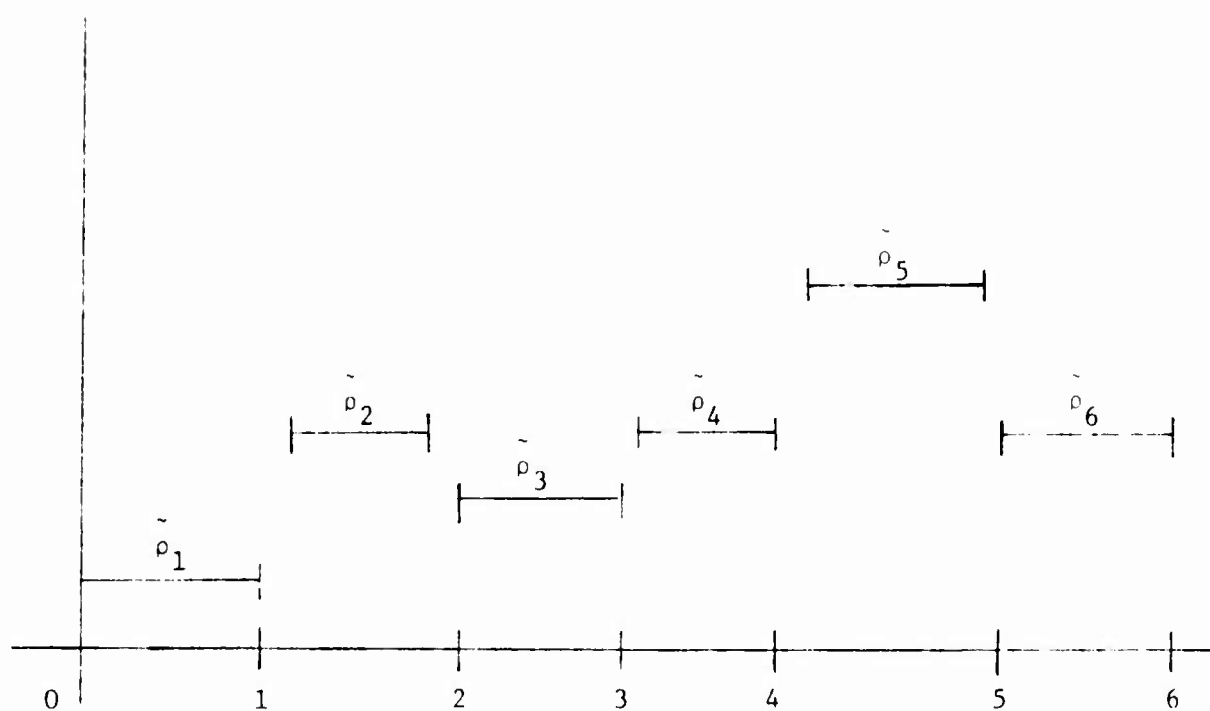
$$\begin{aligned} & \sum_{i=1}^{n-1} \left[ r_n^*(Z_i) - r(Z_i) \right]^2 \mu\{i\} \\ & \geq \sum_{i=1}^{n-1} \left[ r_n^*(Z_i) - \hat{r}_n(Z_i) \right]^2 \mu\{i\} + \sum_{i=1}^{n-1} \left[ \hat{r}_n(Z_i) - r(Z_i) \right]^2 \mu\{i\} \end{aligned}$$

holds for every increasing function  $r(Z_i)$ .

To illustrate another use of the isotonic regression, consider the following estimation problem. Suppose that items are inspected at random times and that failure is observed only through inspection. Our failure data, on each item which fails, consists of intervals  $(t_i, t_{i+1})$  where it is only known that the item survived to time  $t_i$  and failed sometime in the interval  $(t_i, t_{i+1})$ . Harris, Meier and Tukey [16] study the MLE estimate of the failure rate under these conditions. The approach is nonparametric in that it is assumed only that the

failure rate is constant (but unknown) over specified time intervals. The fact that the authors seek rates rather than probabilities for intervals produces certain differences in their treatment of observations extending over parts of intervals as compared to the "actuarial" treatment. The isotonic regression technique can be applied to the Harris, Meier, Tukey estimate, for example, if we assume that the failure rate first decreases and then increases.

Let  $\hat{\rho}_i$  denote the Harris-Meier-Tukey MLE of the failure rate in interval  $i$ . Let  $(0,1)$  denote the first interval;  $(1,2)$  the second interval, etc. Suppose the MLE looks as follows:



To obtain an estimate, say  $\hat{\rho}$ , which decreases and then increases and which is closest to  $\hat{\rho}$  in a least squares sense, proceed as follows:

- (1) Choose in turn,  $i_0 = 0, 1, 2, \dots, n$  for possible change points of the failure rate.
- (2) Suppose  $i_0 = 0$ . Then

$$\hat{\rho}_{1,0} = \underset{v \geq 1}{\text{Min}} \frac{\sum_{j=1}^v \tilde{\rho}_j}{v+1-1}$$

and

$$\hat{\rho}_{i,0} = \underset{v \geq i}{\text{Min}} \underset{u \leq i}{\text{Max}} \frac{\sum_{j=u}^v \tilde{\rho}_j}{v+1-u}.$$

(3) If  $0 < i_0 < n$ , then

$$(3a) \quad \hat{\rho}_{i,i_0} = \underset{i_0-1 \geq v \geq i}{\text{Max}} \underset{u \leq i}{\text{Min}} \frac{\sum_{j=u}^v \tilde{\rho}_j}{v+1-u} \quad \text{for } 1 \leq i \leq i_0 - 1$$

and

$$(3b) \quad \hat{\rho}_i = \underset{v \geq i}{\text{Min}} \underset{i_0 \leq u \leq j}{\text{Max}} \frac{\sum_{j=u}^v \tilde{\rho}_j}{v+1-u} \quad \text{for } i_0 \leq i \leq n.$$

(4) For each choice  $i_0$ , compute

$$\sum_{i=1}^n \left( \tilde{\rho}_i - \hat{\rho}_{i,i_0} \right)^2.$$

(5) Take as your estimate that  $\hat{\rho}_{i,i_0}$  corresponding to the change point which minimizes the sum of squares.

It can be shown that the estimate  $\hat{\rho}_{i,i_0}$  so obtained is closest to the Harris-Meier-Tukey estimate  $\tilde{\rho}_i$  in the sense that it minimizes the sum of squared errors relative to all competing estimates which first decrease and then increase.

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