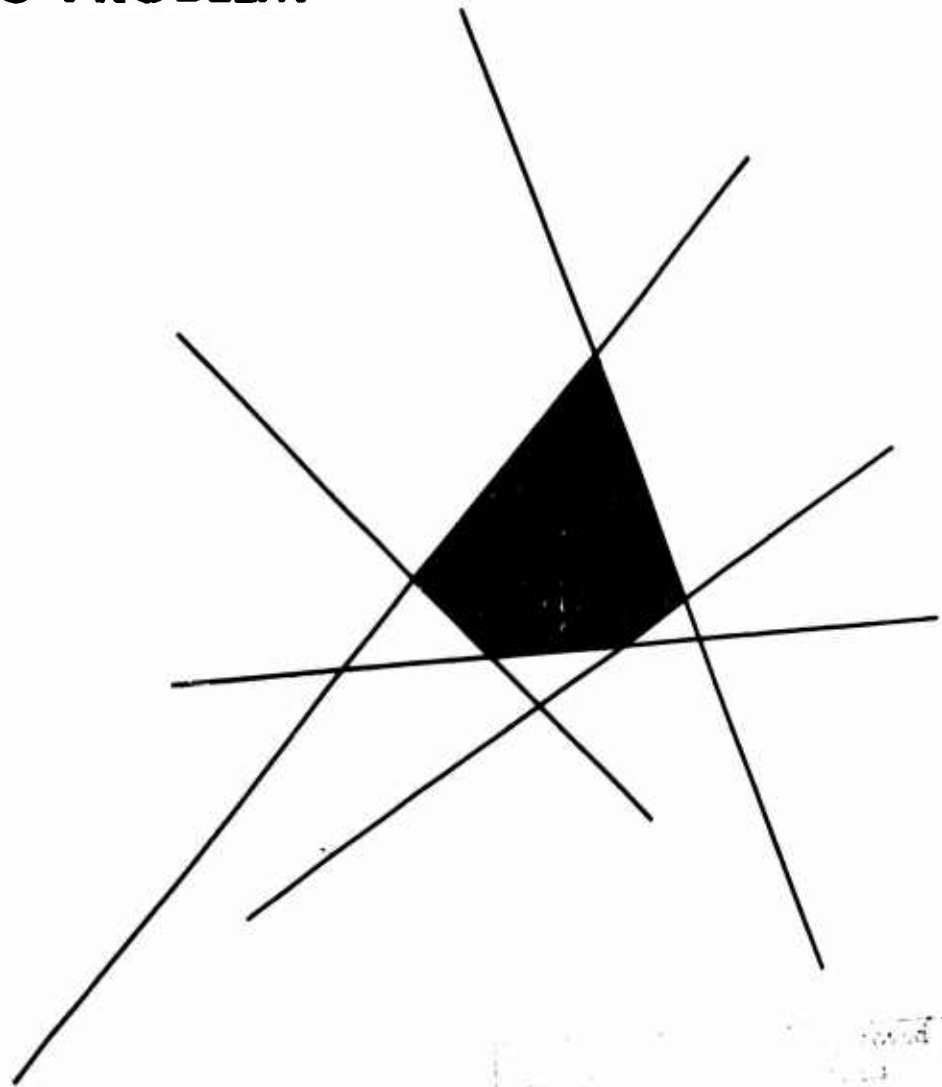


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ON THE NUMBER OF SOLUTIONS TO THE COMPLEMENTARY QUADRATIC PROGRAMMING PROBLEM

by
Katta G. Murty

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ABSTRACT

The relationship between the number of solutions to the complementarity problem, the right-hand constant vector q and the matrix M is explored. The main results proved in this work are summarized below.

The number of solutions to the complementarity problem is finite for all $q \in R^n$ if and only if all the principal subdeterminants of M are nonzero. The necessary and sufficient condition for this solution to be unique for each $q \in R^n$ is that all principal subdeterminants of M are strictly positive. When $M \geq 0$, there is at least one complementary feasible solution for each $q \in R^n$ if and only if all the diagonal elements of M are strictly positive; and, in this case, the number of these solutions is an odd number whenever q is nondegenerate. If all principal subdeterminants of M are nonzero, then the number of complementary feasible solutions has the same parity (odd or even) for all $q \in R^n$ which are nondegenerate. Also, if the number of complementary feasible solutions is a constant for each $q \in R^n$, then that constant is equal to one and M is a P-matrix.

Most of the proofs are based on mathematical induction. Counterexamples are given to show that the theorems fail if any of the hypotheses are not satisfied.

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ON THE NUMBER OF SOLUTIONS TO THE COMPLEMENTARY
QUADRATIC PROGRAMMING PROBLEM

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1. INTRODUCTION:

1.1 The complementary quadratic programming problem is that of finding column vectors $w = (w_i) \in R^n$ and $z = (z_i) \in R^n$ satisfying

$$\begin{aligned} w &= Mz + q \\ w \geq 0, z \geq 0, w^T z &= 0 \end{aligned} \tag{1}$$

where $M = (m_{ij})$ is a given $n \times n$ square matrix and $q = (q_i)$ is a given $n \times 1$ column vector and w^T denotes the transpose of w . R^n is the n -dimensional real Euclidean space.

1.2 Because w, z are nonnegative, the constraint

$$w^T z = \sum_{i=1}^n w_i z_i = 0 \Rightarrow w_i z_i = 0 \quad \text{for each } i = 1, \dots, n.$$

Thus if one of the variables in the pair w_i, z_i is positive, the other should be zero. Hence the constraint $w^T z = 0$ will be referred to as the *complementarity condition* and the problem is sometimes known as the *complementarity problem of order n* .

1.3 Consider the quadratic programming problem

$$\begin{aligned} &\text{Minimize } w^T z \\ &\text{Subject to } w - Mz = q \\ &w \geq 0, z \geq 0. \end{aligned}$$

If (1) has any solution $(w; z)$ then that solution also solves the

quadratic programming problem. Conversely if the minimum value for the objective function in the quadratic programming problem is zero, then any optimal solution to it also solves (1).

Thus solving (1) is equivalent to finding out whether the minimum objective value in the above quadratic program is zero or strictly positive. Hence the problem (1) is known as the *complementary quadratic programming problem*.

1.4 Cottle and Dantzig [1] and Lemke [7], [8] have shown that all the problems in linear programming, convex quadratic programming and also the problem of finding a Nash equilibrium point of a bimatrix game, can be posed in the form of (1). For other applications of (1) see Scarf [13]. Lemke and Howson [7], [9] have developed a simple algorithm for solving (1) which is based on pivot steps.

Lemke [7], Cottle and Dantzig [1] have shown that (1) has a solution if all the principal determinants of M are positive or if M is a nonnegative matrix with positive elements in the principal diagonal. Lemke [7] has also given sufficient conditions on M and q under which the number of solutions to (1) is finite.

In this paper our main interest is to examine the relationship of the number of solutions to (1) to the properties of the given matrices M and q . The motivation for this problem was provided by Gale when he asked me to try and prove or construct counterexamples to the following conjectures:

- (a) M is a P-matrix if and only if the complementarity problem has a unique solution for each $q \in \mathbb{R}^n$.
- (b) $M \geq 0$, is a Q-matrix if and only if $m_{ii} > 0$ for all $i = 1, \dots, n$.

(c) If M is a Q -matrix, the complementarity problem has an odd number of solutions whenever q is nondegenerate with respect to M .

The result of the investigation is the present work.

2. NOTATION AND PRELIMINARIES:

2.1 If A is any matrix, A^T denotes its transpose. $A_{.i}$ denotes the i -th row vector of A and $A_{.j}$ denotes the j -th column vector of A . I denotes the unit matrix.

2.2 A square matrix M is called a *P-matrix* if all its principal subdeterminants are strictly positive. The square matrix M is called *nondegenerate* if every matrix A obtained by taking $A_{.j}$ to be either $M_{.j}$ or $I_{.j}$ for each $j = 1, \dots, n$ is nonsingular. An equivalent definition is that M is a nondegenerate matrix if and only if all its principal subdeterminants are nonzero. M is called a *Q-matrix* if problem (1) has a solution for all $q \in R^n$.

2.3 Let A be any finite set of column vectors in R^n . The convex cone generated by the column vectors in A is denoted by $\text{Pos}\{A\}$. Thus $x \in \text{Pos}\{A\}$ if and only if x can be expressed as a nonnegative linear combination of the column vectors in A .

2.4 Suppose $L(q) \subset R^{2n}$ is the linear manifold determined by the linear equality constraints

$$\begin{aligned} w - Mz &= q \\ w \in R^n, z \in R^n \end{aligned} \tag{2}$$

without any nonnegativity constraints. The vector $\begin{pmatrix} w \\ z \end{pmatrix} \in L(q)$ if and only if it satisfies (2). Any $\begin{pmatrix} w \\ z \end{pmatrix} \in L(q)$ is called a *solution*. For convenience we will write down the vector $\begin{pmatrix} w \\ z \end{pmatrix} \in R^{2n}$ as $(w; z)$.

2.5 The convex polyhedron $K(q) \subset L(q)$ is the set of all *feasible solutions* $(w; z)$ which satisfy

$$\begin{aligned}
 w - Mz &= q \\
 w \geq 0, z \geq 0
 \end{aligned}
 \tag{3}$$

2.6 A *basic feasible solution* is a feasible solution $(w; z) \in K(q)$ such that the column vectors in (3) of the variables w_j and z_j which are strictly positive, are linearly independent. Every basic feasible solution is an extreme point of the convex polyhedron $K(q)$ and vice versa.

2.7 A *complementary feasible solution* is a feasible solution $(w; z) \in K(q)$ which satisfies the complementarity condition $w^T z = 0$. A complementary feasible solution is a solution to (1) and vice versa.

2.8 For each $i = 1, \dots, n$ the variables w_i, z_i constitute a *complementary pair* and each of the variables in the pair is the complement of the other. In the system (1) the column vector $I_{.j}$ is associated with the variable w_j and $-M_{.j}$ is associated with z_j . Thus the pair $(I_{.j}, -M_{.j})$ are the j -th *complementary pair* of column vectors in (1).

2.9 A *complementary set of column vectors* is a set of column vectors $\{A_{.j}, j = 1, \dots, n\}$ such that $A_{.j}$ is either $I_{.j}$ or $-M_{.j}$ for each $j = 1, \dots, n$. Thus any set of column vectors containing exactly one vector from each complementary pair of vectors is a complementary set of column vectors. The corresponding set of variables is called a complementary set of variables. Hence there are 2^n complementary sets of column vectors.

2.10 Each solution to (1) represents q as a nonnegative linear combination of some complementary set of column vectors.

Conversely if $\{A_{.j}\}$ is a complementary set of column vectors and if

$$q \in \text{Pos} \{A_{.j}, j = 1, \dots, n\}$$

$$\text{i.e., } q = \sum_{j=1}^n \beta_j A_{.j} \text{ where } \beta_j \geq 0 \text{ for each } j$$

then a solution to (1) is obtained by setting the variables associated with the column $A_{.j}$ equal to β_j for $j = 1, \dots, n$ respectively and all the other variables in $(w; z)$ not in this complementary set equal to zero.

The pos cone generated by any complementary set of column vectors is known as a *complementary cone*. Thus there are 2^n complementary cones and the union of all these cones is the set of all q for which (1) has a solution.

2.11 Any set of variables $\{y_1, \dots, y_{i-1}, y_{i+1}, \dots, y_n\}$ where y_r is either w_r or z_r for each r , is known as a *subcomplementary set* of variables. The column vectors associated with a subcomplementary set of variables constitute a subcomplementary set of column vectors. The complementary pair of variables (w_i, z_i) is the *left out complementary pair* of variables in the subcomplementary set $\{y_1, \dots, y_{i-1}, y_{i+1}, \dots, y_n\}$.

2.12 An *almost complementary feasible solution* is a feasible solution $(w; z) \in K(q)$ such that

$$\begin{aligned} w^T z &= w_i z_i && \text{for some } i \\ \text{i.e., } w_j z_j &= 0 && \text{for all } j \neq i, \text{ for some } i. \end{aligned}$$

2.13 The set $C_i(q)$ is the *almost complementary set* defined by

$$\begin{aligned} C_i(q) &= \{(w; z) : (w; z) \in K(q), w^T z = w_i z_i \\ &\text{i.e., } w_j z_j = 0 \text{ for } j \neq i\}. \end{aligned}$$

where i is any integer from 1 to n .

2.14 If $x \in \mathbb{R}^n$, $x \neq 0$ then the ray generated by x is

$$\text{Pos } \{x\} = \{y: y = \lambda x \text{ for some } \lambda \geq 0\} .$$

2.15 If $x^1, x^2 \in \mathbb{R}^n$, $x^1 \neq 0$ then the set

$$\{y: y = x^2 + \lambda x^1 \text{ for some } \lambda \geq 0\}$$

is the *half-line* through x^2 parallel to the ray generated by x^1 .

2.16 The column vector q is said to be *nondegenerate with respect to M* if and only if for all $(w; z) \in L(q)$, at most n of the $2n$ variables $\{w_j, z_j\}$ are zero. Equivalently q is nondegenerate with respect to M if it does not lie in any subspace generated by $(n-1)$ or less column vectors of $(I \begin{smallmatrix} \vdots \\ -M \end{smallmatrix})$. Thus the set of all q which are not nondegenerate with respect to M belong to a finite number of subspaces of \mathbb{R}^n .

2.17 Two basic feasible solutions $(w^1; z^1)$ and $(w^2; z^2)$ are said to be *adjacent extreme points* of $K(q)$ if every convex combination of $(w^1; z^1)$ and $(w^2; z^2)$ has a unique representation as a convex combination of extreme points of $K(q)$. The line segments joining any pair of adjacent extreme points of $K(q)$ is called an *edge* of $K(q)$.

2.18 If $K(q)$ is nonempty and unbounded, any basic feasible solution of

$$w - Mz = 0$$

$$\sum_{i=1}^n w_i + \sum_{i=1}^n z_i = 1$$

$$w, z \geq 0$$

is known as an *extreme homogeneous solution* of (3). Any half-line through

a basic feasible solution in $K(q)$ parallel through the ray generated by an extreme homogeneous solution of (3), lies in $K(q)$. Such a half-line is called an *unbounded edge* (or *extreme half-line*) of $K(q)$ if every point on the half line has a unique representation as the sum of a convex combination of basic feasible solutions of $K(q)$ and a nonnegative linear combination of extreme homogeneous solutions of (3).

2.19 Consider the set of equality constraints (2) again

$$w = Mz + q \quad (2)$$

The i -th constraint in this system is

$$w_i = M_{i.}z + q_i \quad (2i)$$

A *principal pivot* in the position (i,i) in (2) consists of the following steps:

- (i) Solve equation (2i) for the variable z_i in terms of $z_1, \dots, z_{i-1}, w_i, z_{i+1}, \dots, z_n$ and replace the i -th equation in (2) by this equation expressing z_i in terms of $z_1, \dots, z_{i-1}, w_i, z_{i+1}, \dots, z_n$.
- (ii) Substitute the expression obtained for z_i in (i) in each of the other equations in (2).

Thus a principal pivot in position (i,i) in (2) can only be performed if $m_{ii} \neq 0$. The result of this principal pivot is to exchange the variables (w_i, z_i) and we get a transformed system of equations which has the same form as (2), but the left-hand set of variables in it differ from the left-hand set in (2) in one component (the i -th). However, the set of the complimentary pairs of variables remains unchanged as a result of a principal pivot.

2.20 If a series of principal pivots are performed on the system (2), then it will be transformed into the system

$$u = \tilde{M}v + \tilde{q} \quad (4)$$

where each pair (u_i, v_i) is a permutation of the complementary pair of variables (w_i, z_i) . A complementary feasible solution to (4) is a solution to the system

$$\begin{aligned} u &= \tilde{M}v + \tilde{q} \\ u &\geq 0, v \geq 0, u^T v = 0. \end{aligned} \quad (5)$$

2.21 We notice that there is a one to one correspondence between solutions to (1) and solutions to (5). For example, suppose (5) is obtained from (1) by making only one principal pivot in which w_1, z_1 are exchanged, say. Then

$$\begin{aligned} \hat{w}, \hat{z} \text{ solves (1)} &\Leftrightarrow \hat{u} = (\hat{z}_1, \hat{w}_2, \dots, \hat{w}_n), \\ \hat{v} = (\hat{w}_1, \hat{z}_2, \dots, \hat{z}_n) &\text{ solves (5).} \end{aligned}$$

In general since u, v in (5) are such that (u_i, v_i) is a permutation of the variables (w_i, z_i) , we can construct a solution $(\hat{u}; \hat{v})$ to (5) corresponding to each solution $(\hat{w}; \hat{z})$ to (1) by taking the same permutation, and vice versa.

Thus the *number* of solutions to (1) is invariant under principal pivots.

2.22 Let N be a principal submatrix of M of order s , obtained by striking off from M all the rows excepting the i_1, \dots, i_s -th rows and all but the i_1, \dots, i_s -th columns. Let

$$\omega = (w_{i_1}, \dots, w_{i_s})^T, \xi = (z_{i_1}, \dots, z_{i_s})^T$$

and $Q = (q_{i_1}, \dots, q_{i_s})^T$. Then

$$\begin{aligned} \omega &= N\xi + Q \\ \omega &\geq 0, \xi \geq 0, \omega^T \xi = 0 \end{aligned} \tag{6}$$

is known as a *principal subproblem* of (1) in the variables $(\omega; \xi)$.

2.23 Suppose $(\hat{w}; \hat{z})$ is a complementary feasible solution to (1) such that

$$\hat{z}_i = 0 \text{ for all } i \neq i_1 \text{ or } i_2, \dots, \text{ or } i_s.$$

Let $\hat{\omega} = (\hat{w}_{i_1}, \dots, \hat{w}_{i_s})^T$ and $\hat{\xi} = (\hat{z}_{i_1}, \dots, \hat{z}_{i_s})^T$. Then from the definition of the principal subproblem (6) we see that $(\hat{\omega}; \hat{\xi})$ solves (6).

2.24 If r is any integer, its *parity* is said to be *odd* if r is an odd integer or *even* if r is an even integer. When considering a set of integers, it is said to be of *constant parity* if all the numbers in the set have the same parity.

2.25 A set of cones in R^n whose union is R^n is said to form a *partition of R^n* if each cone in the set has a nonempty interior and the intersection of the interiors of any two cones in the set is empty.

3. FINITENESS OF THE NUMBER OF COMPLEMENTARY FEASIBLE SOLUTIONS:

3.1 Lemke [7] has shown that the number of complementary feasible solutions is finite whenever q is nondegenerate with respect to M . Here we determine the necessary and sufficient conditions under which the number of solutions to (1) is finite for each $q \in R^n$.

3.2 Theorem:

The number of complementary feasible solutions is finite for all $q \in R^n$ if and only if M is nondegenerate.

Proof:

Suppose there exists a $q \in R^n$ such that (1) has an infinite number of distinct solutions. Each solution to (1) represents q as a nonnegative linear combination of some complementary set of column vectors. There are only 2^n distinct complementary sets of column vectors. Thus if (1) has an infinite number of distinct solutions, there must exist a complementary set of column vectors $\{A_{.j}, j = 1, \dots, n\}$ such that

$$\sum_{j=1}^n A_{.j} y_j = q \quad (7)$$

$$y_j \geq 0 \quad \text{for each } j = 1, \dots, n$$

has an infinite number of distinct solutions. (7) is a square system of n equations in n nonnegative variables. If (7) has an infinite number of solutions, then the set of column vectors $\{A_{.j}, j = 1, \dots, n\}$ must be linearly dependent. Since $\{A_{.j}\}$ is a complementary set of column vectors, this implies by the definition in 2.2 that M is not nondegenerate.

To prove the converse, suppose M is not nondegenerate.

Case 1: Suppose one of the column vectors of M , say $M_{.1}$, is zero. Then let $q = (0, 1, 1, \dots, 1)^T$. Then $(w; z) = (0, 1, 1, \dots, 1; \alpha, 0, 0, \dots, 0)$ is a complementary feasible solution for any $\alpha \geq 0$. Thus there are an infinite number of distinct complementary feasible solutions when $q = (0, 1, 1, \dots, 1)^T$ in this case.

Case 2: Suppose $M_{.1} \neq 0$. Since M is not nondegenerate, there exists a complementary set of columns, say $\{A_{.j}, j = 1, \dots, n\}$ which is linearly dependent. So there exists $\alpha = (\alpha_1, \dots, \alpha_n)^T \neq 0$ such that

$$\sum_{j=1}^n A_{.j} \alpha_j = 0.$$

Also $A_{.1}$ is either $I_{.1}$ or $-M_{.1}$ and hence in this case $A_{.1} \neq 0$.

If $\sum_{j=1}^n A_{.j} = 0$, let $q = A_{.1} \neq 0$. Then every $(w; z)$ obtained by setting the variable associated with $A_{.1}$ equal to $1 + \alpha$, the variable associated with $A_{.j}$ equal to α for $j \neq 1$, and all other variables in $(w; z)$ equal to zero is a complementary feasible solution for any $\alpha \geq 0$. Hence there are an infinite number of distinct complementary feasible solutions when $q = A_{.1} \neq 0$ in this case.

If $\sum_{j=1}^n A_{.j} \neq 0$, let $q = \sum_{j=1}^n A_{.j}$. Let

$$\theta = \min_{\substack{j \text{ such that} \\ \alpha_j < 0}} \left(\frac{1}{-\alpha_j} \right)$$

= $+\infty$ if there does not exist any $\alpha_j < 0$.

So $\theta > 0$. Every $(w; z)$ obtained by setting the variable associated with $A_{.j}$ equal to $1 + \lambda \alpha_j$ for $j = 1, \dots, n$ and all the other variables in

$(w; z)$ equal to zero, is a complementary feasible solution for any λ such that $0 \leq \lambda \leq \theta$. Hence there are an infinite number of distinct complementary feasible solutions when $q = \sum_{j=1}^n A_{.j} \neq 0$ in this case.

Hence if M is not nondegenerate there exists a $q \neq 0$ for which (1) has an infinite number of distinct solutions. This completes the proof of Theorem 3.2.

3.3 Corollary:

If M is not nondegenerate there exists a $q \neq 0$ for which there are infinite number of distinct complementary feasible solutions.

4. UNIQUENESS OF THE COMPLEMENTARY FEASIBLE SOLUTION:

4.1 We will now examine the question of when (1) has a unique solution for each $q \in R^n$ while M is fixed. Lemke [7], Cottle and Dantzig [1] have shown that (1) has a solution if M is a P-matrix. They have also shown that the solution is unique if M is positive definite. We will extend these results.

4.2 Theorem:

If M is a P-matrix then (1) has a unique solution for each $q \in R^n$.

Proof:

Proof is by induction on n . If $n = 1$, then $M = (m_{11})$ and M is a P-matrix $\Rightarrow m_{11} > 0$. In this case $q = (q_1)$. If $q_1 \geq 0$ then $w = (w_1) = (q_1)$ and $z = (z_1) = (0)$ is the only solution to (1). If $q_1 < 0$, then $w = (w_1) = 0$ and $z = (z_1) = (-q_1)$ is the only solution to (1). So the theorem is verified for $n = 1$.

Suppose the theorem is true for all complementarity problems of order $1, 2, \dots, n-1$. We will now prove that it also holds for problems of order n .

In Theorem 6 of [1] Cottle and Dantzig have proved that (1) has at least one solution when M is a P-matrix. Parsons has proved the same by using induction on n (Theorem 6.1 of [11]). Thus we need only prove the uniqueness of the solution. Suppose a solution to (1) is $(\hat{w}; \hat{z})$ where

$$\hat{z}_{j_1} > 0, \dots, \hat{z}_{j_r} > 0$$

and

(8)

$$\hat{z}_j = 0 \text{ for all } j \neq j_1 \text{ or } \dots \text{ or } j_r .$$

Make a principal pivot exchanging z_{j_1} and w_{j_1} which is possible because $m'_{j_1 j_1} > 0$ by the P-property of M . The result of this principal pivot is another system of the same form as (1), in which the matrix M' is again a P-matrix by Tucker's theorem [15].

In the new system make a principal pivot exchanging z_{j_2} and w_{j_2} which is possible because $m'_{j_2 j_2} > 0$ by the P-property of M' . Continue making principal pivots until all the original z_{j_1}, \dots, z_{j_r} are exchanged with w_{j_1}, \dots, w_{j_r} respectively. This is possible because after each principal pivot we get a new system in which the matrix is again a P-matrix by Tucker's theorem [15] and hence has strictly positive elements along the principal diagonal.

Suppose the system obtained at the end is

$$\begin{aligned} u &= \tilde{M}v + \tilde{q} \\ u \geq 0, v \geq 0, u^T v &= 0 \end{aligned} \tag{9}$$

By the manner in which (9) was obtained from (1) and since $(\hat{w}; \hat{z})$ is a solution to (1) we note using (8) and 2.21 that (9) has a solution in which $v = 0$. So in that solution $u = \tilde{q}$ and since u must be nonnegative, $\tilde{q} \geq 0$. Thus by a series of principal pivots we have transformed (1) into (9) in which \tilde{M} is a P-matrix (by Tucker's theorem [15]) and $\tilde{q} \geq 0$.

Since \bar{M} is a P-matrix all its principal submatrices are P-matrices. Hence by the induction hypothesis all principal subproblems (defined in 2.22) of (9) have unique solutions. (9) has one solution $u = \bar{q}$, $v = 0$. If (9) has another distinct solution $(u^*; v^*)$ in which for some j , $v_j^* = 0$, then the principal subproblem of (9) in the variables $(u_1, \dots, u_{j-1}, u_{j+1}, \dots, u_n; v_1, \dots, v_{j-1}, v_{j+1}, \dots, v_n)$ has two distinct solutions, namely

$$u_i = \bar{q}_i \text{ for each } i \neq j \text{ and } v_i = 0 \text{ for each } i \neq j$$

and

$$u_i = u_i^* \text{ for each } i \neq j \text{ and } v_i = v_i^* \text{ for each } i \neq j$$

and these two are distinct because $(u^*; v^*)$ is distinct from $(\bar{q}; 0)$.

This contradicts the induction hypothesis.

Hence if (9) has another solution $(u^*; v^*)$ distinct from $(\bar{v}; 0)$ then we must have $v^* > 0$. By the complementarity constraint $v^* > 0 \Rightarrow u^* = 0$. Therefore

$$\begin{aligned} \bar{M}v^* &= -\bar{q} \leq 0 \\ v^* &> 0. \end{aligned} \tag{10}$$

This is a contradiction because by the theorem of Gale and Nikaido (Theorem 1 of [6]) since \bar{M} is a P-matrix.

$$\left. \begin{aligned} \bar{M}v &\leq 0 \\ v &\geq 0 \end{aligned} \right\} \Rightarrow v = 0.$$

Hence (9) has a unique solution. By 2.21 we conclude that the solution to (1) is also unique.

Hence by induction, Theorem 4.2 is true for all n .

It is interesting to note that the converse of Theorem 4.2 is also true. This is proved next.

4.3 Theorem:

If (1) has at most one solution for each $q \in R^n$, then M is a P-matrix.

Proof:

Proof is by induction on n . Suppose $n = 1$. Then $M = (m_{11})$ and $q = (q_1)$.

If $m_{11} = 0$, then if $q = (q_1) = 0$, (1) has several solutions, namely

$$w = (w_1) = 0, z = (z_1) \text{ for any } z_1 \geq 0.$$

If $m_{11} < 0$, then if $q = (q_1) > 0$, (1) has two distinct solutions, namely,

$$w = (q_1), z = 0$$

and

$$w = 0, z = \left(\frac{q_1}{|m_{11}|} \right).$$

Thus if $n = 1$, by the hypothesis of Theorem 2 $m_{11} \not\leq 0$. So we must have $m_{11} > 0$ and hence $M = (m_{11})$ must be a P-matrix, which proves the theorem for the case $n = 1$. Suppose the theorem holds for all problems of order $(n - 1)$ or less. We will now show that it also holds for problems of order n .

If (1) has at most one solution for each $q \in \mathbb{R}^n$, then every principal subproblem of (1), has at most one solution for any of its right-hand side constant vector Q . To prove this consider the subproblem in the variables $\omega = (\omega_2, \dots, \omega_n)^T$, $\xi = (\xi_2, \dots, \xi_n)^T$. If there exists a $Q = (q_2, \dots, q_n)^T$ for which this principal subproblem has two distinct solutions, namely, $(\bar{\omega}; \bar{\xi})$ and $(\hat{\omega}; \hat{\xi})$, choose q_1 to satisfy

$$q_1 > \text{Max} \left\{ \left| \sum_{j=2}^n \bar{\xi}_j m_{1j} \right|, \left| \sum_{j=2}^n \hat{\xi}_j m_{1j} \right| \right\},$$

and let

$$\bar{\omega}_1 = q_1 + \sum_{j=2}^n \bar{\xi}_j m_{1j}, \quad \bar{z}_1 = 0$$

$$\hat{\omega}_1 = q_1 + \sum_{j=2}^n \hat{\xi}_j m_{1j}, \quad \hat{z}_1 = 0.$$

If $q = \begin{pmatrix} q_1 \\ -Q \end{pmatrix}$, then (1) has two distinct solutions, namely

$$\bar{w} = \begin{pmatrix} \bar{\omega}_1 \\ \bar{\omega} \end{pmatrix}, \quad \bar{z} = \begin{pmatrix} \bar{z}_1 \\ \bar{\xi} \end{pmatrix}$$

and

$$\hat{w} = \begin{pmatrix} \hat{\omega}_1 \\ \hat{\omega} \end{pmatrix}, \quad \hat{z} = \begin{pmatrix} \hat{z}_1 \\ \hat{\xi} \end{pmatrix}$$

contradicting the hypothesis.

A similar proof holds for all principal subproblems of (1) of order $n - 1$, and by repeating the argument we see that all principal subproblems of (1) have at most one solution for all right-hand side constant vectors. Hence by the induction hypothesis all principal submatrices of M of order $(n - 1)$ or less are P -matrices. In particular $m_{11} > 0$ for all

$i = 1, \dots, n$.

The matrix M itself should be nonsingular, as otherwise it will not be nondegenerate, in which case there exists a q for which (1) has an infinite number of solutions by Theorem 3.2. So M^{-1} exists. Left multiplying by M^{-1} problem (1) can be written as

$$\begin{aligned} z - M^{-1}w &= Q \\ z \geq 0, w \geq 0, z^T w &= 0 \end{aligned} \tag{11}$$

where $Q = -M^{-1}q$.

If (1) has at most one solution for each $q \in \mathbb{R}^n$, then (11) has at most one solution for each $Q \in \mathbb{R}^n$. Hence by the arguments used previously, all principal subdeterminants of M^{-1} of order $(n-1)$ or less are strictly positive. Let α be the value of the principal subdeterminant of M^{-1} obtained by striking off the first row and column from M^{-1} . Then

$$\alpha = \frac{m_{11}}{\text{determinant of } M} \tag{12}$$

But $\alpha > 0$, $m_{11} > 0$. So by (12) the determinant of M is positive. Also every principal submatrix of M is a P -matrix. Hence M itself is a P -matrix. Hence by induction Theorem 4.3 holds for all n .

4.4 Corollary:

If (1) has at most one solution for each $q \in \mathbb{R}^n$ (M being fixed) then it has exactly one solution for any $q \in \mathbb{R}^n$.

4.5 Corollary:

(1) has a unique solution for each $q \in \mathbb{R}^n$ if and only if M is a P -matrix.

4.6 Corollary:

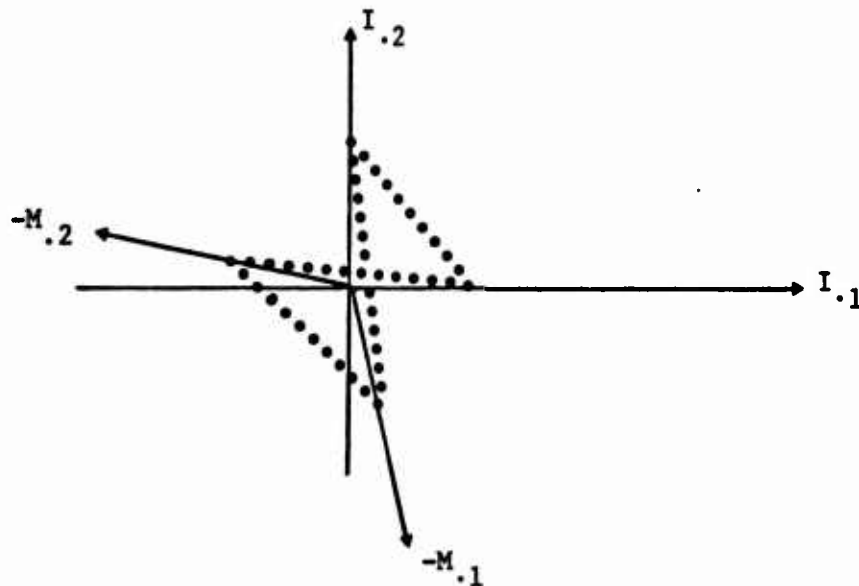
If (1) has a unique solution for each $q \in \mathbb{R}^n$, then any principal subproblem of (1) has a unique solution for each of its right-hand side constant vector.

4.7 Note: It is not possible to generalize Corollary 4.6 by dropping the condition of "uniqueness" of the solution to (1). As an example, let $M = \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix}$. Then the problem is to solve

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} - \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \quad (13)$$

$$\underline{w} \geq 0, \underline{z} \geq 0, \underline{w}^T \underline{z} = 0$$

In the diagram below, we indicate each complementary cone by drawing a dotted line segment running across its generators.



We see that the union of the complementary cones is the whole space, \mathbb{R}^2 . So by 2.10, (13) has a solution for all $q \in \mathbb{R}^2$. So M is a Q -matrix. Now let us examine the principal subproblem of (13) in w_2, z_2 . It is

$$\begin{aligned}
 w_2 - (-1)z_2 &= q_2 \\
 w_2 \geq 0, z_2 \geq 0, w_2 z_2 &= 0
 \end{aligned}
 \tag{14}$$

(14) has no solution if $q_2 < 0$. Also we verify that M violates the hypothesis of Corollary 4.6 because it has three distinct solutions for every q in the positive orthant.

4.8 Thus the fact that M is a Q -matrix does not imply that its principal submatrices are Q -matrices.

4.9 Note: After Theorems 4.2, 4.3 were conjectured and proved as described here, a theorem by Smelson, Thrall and Wesler [12] on the partition of R^n by convex cones which is equivalent to Corollary 4.5 has come to our notice. Their proof is geometric and not based on the mathematical programming approach.

The property of "uniqueness" of the solution to (1) also affects the nature of the solution. This is discussed below.

4.10 Theorem:

Suppose M has the property that (1) has a unique solution for each $q \in R^n$. Keep q_2, \dots, q_n fixed but let q_1 vary. Let $z_1(q_1)$ be the value of z_1 in the solution to (1) as a function of q_1 . Then $z_1(q_1)$ is monotonic decreasing in q_1 and it is strictly monotonic decreasing in the region in which it is positive.

Proof:

Proof by contradiction. Let $Q = (q_2, \dots, q_n)^T$ which is held fixed. Pick any value for q_1 and let $\beta > 0$ be arbitrary. Let $(\hat{w}; \hat{z})$ be the solution to (1) when $q = \begin{pmatrix} q_1 \\ -\beta Q \end{pmatrix}$, $(\bar{w}; \bar{z})$ be the solution to (1) when

$q = \left(\frac{q_1 + \beta}{Q} \right)$. Then $z_1(q_1) = \hat{z}_1$ and $z_1(q_1 + \beta) = \tilde{z}_1$. Let

$$\hat{w} = (\hat{w}_2, \dots, \hat{w}_n)^T, \quad \hat{\xi} = (\hat{z}_2, \dots, \hat{z}_n)^T$$

$$\tilde{w} = (\tilde{w}_2, \dots, \tilde{w}_n)^T, \quad \tilde{\xi} = (\tilde{z}_2, \dots, \tilde{z}_n)^T$$

and

$$m_{\cdot 1} = (m_{21}, \dots, m_{n1})^T.$$

If $\tilde{z}_1 > 0$ we wish to show that $\tilde{z}_1 < \hat{z}_1$. Suppose not, then $\tilde{z}_1 \geq \hat{z}_1 > 0$. By complementarity $\tilde{w}_1 = \hat{w}_1 = 0$. Then, if

$$q = \left(\frac{q_1 + \beta + m_{11}\hat{z}_1}{Q + m_{\cdot 1}\hat{z}_1} \right),$$

(1) has two solutions, namely

$$w = \left(\frac{\beta}{\hat{z}_1} \right), \quad z = \left(\frac{0}{\hat{\xi}} \right)$$

and

$$w = \left(\frac{0}{\tilde{z}_1} \right), \quad z = \left(\frac{\tilde{z}_1 - \hat{z}_1}{\tilde{\xi}} \right)$$

contradicting the uniqueness of the solution to (1) for each $q \in \mathbb{R}^n$. So if $z_1(q_1) > 0$ then $z_1(q_1 + \beta) < z_1(q_1)$ for any $\beta > 0$.

It remains to be shown that if $z_1(q_1) = 0$ then $z_1(q_1 + \beta) = 0$ for all $\beta > 0$. Suppose not, then $z_1(q_1) = \hat{z}_1 = 0$ and $z_1(q_1 + \beta) = \tilde{z}_1 > 0$.

Then if $q = \left(\frac{q_1 + \beta}{Q} \right)$, (1) has two solutions, namely

$$\tilde{w}; \tilde{z} \text{ and } w = \left(\frac{\hat{w}_1 + \beta}{\hat{z}_1} \right), \quad z = \left(\frac{0}{\hat{\xi}} \right) = \hat{z}$$

which is again a contradiction.

4.11 We now show that if M is a Q-matrix and (1) has a unique solution when q is any element of the set

$\{I_{.1}, I_{.2}, \dots, I_{.n}; -M_{.1}, \dots, -M_{.n}\}$, then M is a P-matrix.

4.12 Theorem:

Let

$[z_j]$ = union of all complementary cones which contain $-M_{.j}$ as a generator

$[w_j]$ = union of all complementary cones which contain $I_{.j}$ as a generator.

If $I_{.j} \notin [z_j]$ and $-M_{.j} \notin [w_j]$ for each $j = 1$ to n and M is a Q-matrix, then M is a P-matrix.

Proof:

4.13 Let N be the principal submatrix of M of order $(n-1)$, obtained by striking off the first row and column of M . We will now show that N is also a Q-matrix.

Suppose not. Let $\omega = (w_2, \dots, w_n)^T$ and $\xi = (z_2, \dots, z_n)^T$. Consider the principal subproblem in (ω, ξ) , which is to solve

$$\begin{aligned} \omega - N\xi &= Q \\ \omega \geq 0, \xi \geq 0, \omega^T \xi &= 0. \end{aligned} \tag{15}$$

If N is not a Q-matrix there exists a $\bar{Q} \in R^{n-1}$ such that when $Q = \bar{Q}$, (15) has no solution.

Let $\bar{q} = \begin{pmatrix} -\alpha \\ \bar{Q} \end{pmatrix}$. If (1) has a solution (\bar{w}, \bar{z}) when $q = \bar{q}$, then $\bar{z}_1 > 0$ in it, as otherwise $(\bar{w}, \bar{\xi})$ would be a solution to (15). Thus every

point on the line

$$\left\{ \bar{q} : \bar{q} = \begin{pmatrix} -\alpha \\ \bar{Q} \end{pmatrix}, \alpha \text{ real} \right\}$$

corresponds to only complementary feasible solutions in which $z_1 > 0$. Since there are only a finite number of complementary cones and each one is convex, there must exist an α_0 such that the half-line

$$\left\{ q : q = \begin{pmatrix} -\alpha_0 \\ \bar{Q} \end{pmatrix} + \theta I_{.1}, \theta \geq 0 \right\}$$

lies entirely in a complementary cone. By the above argument, in every complementary feasible solution corresponding to any point on this half-line we must have $z_1 > 0$. This implies that this half-line lies in a complementary cone for which $-M_{.1}$ is a generator. This implies that $\text{Pos}\{I_{.1}\}$ also lies in the same complementary cone, i.e., $I_{.1} \in [z_1]$, which is a contradiction. So N must be a Q -matrix. By a similar argument we conclude that all principal submatrices of M of order $(n-1)$ must be Q -matrices.

4.14 Let N be the principal submatrix of M of order $(n-2)$ obtained by striking off the first two rows and columns from M . We will now show that N must be a Q -matrix also.

Suppose not. Then there exists a $\bar{Q} \in \mathbb{R}^{n-2}$ such that the subproblem in $(w_3, \dots, w_n; z_3, \dots, z_n)$ has no complementary feasible solution when its right hand constant vector is \bar{Q} .

Let

$$\bar{q} = \begin{pmatrix} \lambda \\ \mu \\ \bar{Q} \end{pmatrix}. \quad (16)$$

Then for any λ, μ real, (1) has no solutions in which both z_1 and z_2

are equal to zero, when $q = \bar{q}$ by 2.23.

4.15 Fix $\lambda = \lambda_1, \mu = \mu_1$ where $\lambda_1 > 0$ and $\mu_1 > 0$ and consider the line

$$\left\{ q : q = \begin{pmatrix} \alpha \lambda_1 \\ \alpha \mu_1 \\ -\bar{q} \end{pmatrix}, \alpha \text{ real} \right\}.$$

Points on this line have complementary feasible solutions in which both z_1 and z_2 cannot be zero together. Since the number of complementary cones is finite and each is convex, there must exist an α_0 such that the entire half-line

$$\left\{ q : q = \begin{pmatrix} \alpha \lambda_1 \\ \alpha \mu_1 \\ -\bar{q} \end{pmatrix}, \alpha > \alpha_0 \right\} \quad (17)$$

is in a complementary cone. Suppose this half-line is in the complementary cone $\text{Pos}\{I_{.1}, -M_{.2}, A_{.3}, \dots, A_{.n}\}$. Then

$$\lambda_1 I_{.1} + \mu_1 I_{.2} \in \text{Pos}\{I_{.1}, -M_{.2}, A_{.3}, \dots, A_{.n}\}.$$

Suppose

$$\lambda_1 I_{.1} + \mu_1 I_{.2} = \alpha_1 I_{.1} + \alpha_2 (-M_{.2}) + \sum_{j=3}^n \alpha_j A_{.j}$$

where $\alpha_1, \alpha_2, \dots, \alpha_n \geq 0$.

If $\alpha_1 \geq \lambda_1$, then if we put $(\lambda, \mu) = (0, \mu_1)$, \bar{q} of (16) will have a complementary feasible solution in which both $z_1 = z_2 = 0$, which is a contradiction to 4.14.

If $\alpha_1 < \lambda_1$, then $(\lambda_1 - \alpha_1)I_{.1} + \mu_1 I_{.2}$ lies in the intersection of $\text{Pos}\{-M_{.2}, A_{.3}, \dots, A_{.n}\}$ with $\text{Pos}\{I_{.1}, I_{.2}\}$. We note that $\text{Pos}\{I_{.1}, I_{.2}\}$

cannot entirely lie in $\text{Pos}\{-M_{.2}, A_{.3}, \dots, A_{.n}\}$ because then $I_{.1} \in [z_1]$ and $I_{.2} \in [z_2]$ contradicting the hypothesis. So $\text{Pos}\{I_{.1}, I_{.2}\}$ and $\text{Pos}\{-M_{.2}, A_{.3}, \dots, A_{.n}\}$ intersect in a half-line and when $(\lambda, \mu) \geq 0$ unless λ, μ are such that

$$\mu = \mu_1$$

$$\lambda \geq \lambda_1 - \alpha_1$$

we have

$$\lambda I_{.1} + \mu I_{.2} \notin \text{Pos}\{I_{.1}, -M_{.2}, A_{.3}, \dots, A_{.n}\}.$$

Similarly we see that for $(\lambda, \mu) \geq 0$ if $\lambda I_{.1} + \mu I_{.2}$ is contained in a complementary cone $\text{Pos}\{-M_{.1}, I_{.2}, B_{.3}, \dots, B_{.n}\}$, then $\lambda I_{.1} + \mu I_{.2}$ must lie on some half-line in $\text{Pos}\{I_{.1}, I_{.2}\}$.

Hence when $(\lambda_1, \mu_1) \geq 0$, unless $(\lambda_1 I_{.1} + \mu_1 I_{.2})$ lies in the union of a finite number of half-lines in $\text{Pos}\{I_{.1}, I_{.2}\}$, the half-line in (17) can only be contained in a complementary cone for which both $-M_{.1}$ and $-M_{.2}$ are generators. This implies that all the points in $\text{Pos}\{I_{.1}, I_{.2}\}$ excepting those lying on a finite number of half-lines belong to the union of all complementary cones containing both $-M_{.1}$ and $-M_{.2}$ as generators. But this union is a closed cone and if it contains all points of $\text{Pos}\{I_{.1}, I_{.2}\}$ excepting those lying on a finite number of half-lines, then it contains all of $\text{Pos}\{I_{.1}, I_{.2}\}$. This implies that $I_{.1}$ lies in some complementary cone which has both $-M_{.1}$ and $-M_{.2}$ as generators, which is a contradiction to the hypothesis.

So N must be a Q-matrix. By a similar argument we can show that every principal submatrix of M is a Q-matrix. Hence all the elements in the principal diagonal of M must be strictly positive.

From the hypothesis of the theorem we see that every matrix \bar{M} , obtained from M by performing a series of principal pivots (as in 2.20) has the property that all its diagonal elements are strictly positive.

By Tucker's theorem [15] (see also Lemma 6.1 in [11]) this implies that M is a P-matrix.

4.16 Note: It may be possible to use Theorem 4.12 to develop an efficient algorithm for testing whether a given real square matrix M is a P-matrix or not.

5. ON THE Q-NATURE OF NONNEGATIVE MATRICES

5.1 Suppose the square matrix M is nonnegative, i.e., $m_{ij} \geq 0$ for each i and j . This case is of particular interest because the problem of finding a Nash equilibrium point of a bimatrix game can be formulated as a problem of the form (1) in which $M \geq 0$. See [1]. It is of interest to know when such a matrix is a Q-matrix. The following theorem discusses this question.

5.2 Theorem:

Let $M \geq 0$. M is a Q-matrix if and only if $m_{ii} > 0$ for each $i = 1, \dots, n$.

Proof of Sufficiency:

This has been proved by Cottle and Dantzig [1] in a corollary under their Theorem 5.

Proof of Necessity:

Proof by induction. Suppose $n = 1$. If $M = (m_{11}) = 0$ then for $q = (q_1) < 0$, (1) has no solution. So the theorem is verified for $n = 1$.

Suppose the conditions of the theorem are necessary for all problems of order $(n - 1)$ or less. We will show that they are then necessary for n also. Let $M \geq 0$ be a square matrix of order n , with at least one m_{ii} equal to zero. Without any loss of generality we can assume that $m_{11} = 0$.

Consider the principal subproblem of (1) in the variables

$$\omega = (w_1, \dots, w_{n-1})^T, \quad \xi = (z_1, \dots, z_{n-1})^T.$$

The matrix corresponding to this subproblem is nonnegative with a zero element ($m_{11} = 0$) in its diagonal. So by the induction hypothesis, there exists a $\bar{q} = (\bar{q}_1, \dots, \bar{q}_{n-1})^T$ such that this principal subproblem of (1) has no solution when its right-hand side constant vector is \bar{q} . Let $\bar{q}_n > 0$ and $\bar{q} = \begin{pmatrix} \bar{q} \\ \bar{q}_n \end{pmatrix}$. When $q = \bar{q}$, (1) has no solution because

(i) If (1) has a solution at all, say $(\bar{w}; \bar{z})$ then $\bar{z}_n \neq 0$, since $\bar{z}_n = 0$ implies that $(\bar{w}; \bar{z})$ solve the principal subproblem contradicting the manner in which \bar{q} was obtained.

(ii) So $\bar{z}_n > 0$. But by (1) $\bar{w}_n = M_n \bar{z} + \bar{q}_n > 0$ because $M_n \geq 0$, $\bar{z} \geq 0$, $\bar{q}_n > 0$.

So if (1) has any solution at all, both w_n, z_n must be positive in it which violates the complementarity constraint.

Then (1) has no solution when \bar{q} is obtained as above. So the theorem is true for n . By induction, it is true for all n .

5.3 Proof of Theorem 5.2 by Example:

Suppose $m_{11} = 0$. Then if

$$\bar{q} = (-1, 1, 1, \dots, 1)^T.$$

Then using the fact that $M \geq 0$ it is easily verified that (1) has no solution when $q = \bar{q}$. This example is given by Gale.

5.4 Corollary:

If $M \geq 0$ and M is a Q-matrix, then all principal submatrices of M are also Q-matrices.

6. ON THE CONSTANT PARITY OF THE NUMBER OF COMPLEMENTARY
FEASIBLE SOLUTIONS:

6.1 We will now examine how the number of solutions to (1) varies as q varies over R^n while M is fixed.

6.2 Theorem:

If M is nondegenerate, then the number of complementary feasible solutions has the same parity for all $q \in R^n$ which are nondegenerate with respect to M .

Proof:

In the proof of this theorem we will use some of the results proved by Lemke in [7].

6.3 Results [Lemke]:

If \tilde{q} is nondegenerate with respect to M

(i) Then (1) has a finite number of solutions when $q = \tilde{q}$.

(ii) For each $i = 1, \dots, n$, the almost complementary set $C_1(\tilde{q})$ is either empty or is the union of some edges (bounded or unbounded) of $K(\tilde{q})$.

(iii) The number of unbounded edges in $C_1(\tilde{q})$ differs from the number of solutions to (1) by an even number.

6.4 We will now prove that if M is nondegenerate and $\tilde{q} \in R^n$ then w_1 is unbounded on every unbounded edge of $C_1(\tilde{q})$. Suppose F is an unbounded edge of $K(\tilde{q})$ contained in $C_1(\tilde{q})$. Let

$$F = \{(w; z) : (w; z) = (w^1 + \theta w^2 ; z^1 + \theta z^2), \theta \geq 0\}$$

where $(w^1; z^1)$ is a basic feasible solution and $(w^2; z^2)$ is an extreme homogeneous solution of $K(\bar{q})$.

Along this unbounded edge F , $w_1 = w_1^1 + \theta w_1^2$, and it remains bounded for all $\theta \geq 0$ only if $w_1^2 = 0$. If $w_1^2 = 0$, then if we put $q = \bar{q} - w_1^1 I_{.1}$, (1) has an infinite number of solutions, namely

$$w = w^1 - w_1^1 I_{.1} + \theta w^2; \quad z = z^1 + \theta z^2 \quad \text{for all } \theta \geq 0$$

which is a contradiction to the hypothesis that M is nondegenerate by Theorem 3.2.

Thus, on every unbounded edge of $C_1(\bar{q})$, w_1 is unbounded.

6.5 We will now use the result obtained in 6.4 to show that if $\bar{q} \in R^n$ is nondegenerate with respect to M and α is any real number such that $\hat{q} = \bar{q} + \alpha I_{.1}$ is also nondegenerate with respect to M , then the number of unbounded edges in $C_1(\bar{q})$ and $C_1(\hat{q})$ are the same.

Pick any unbounded edge

$$F = \{(w; z) : (w; z) = (w^1 + \theta w^2; z^1 + \theta z^2), \theta \geq 0\} \subset C_1(\bar{q}).$$

By 6.4, $w_1^2 > 0$ and w_1 is unbounded on this edge. Let

$$v = \text{Max} \left[0, \frac{w_1^1 + \alpha}{-w_1^2} \right].$$

Then it is easily verified that

$$F^1 = \{(w; z) : (w; z) = (w^1 + \theta w^2 + \alpha I_{.1}; z^1 + \theta z^2), \theta \geq v\}$$

is an unbounded edge in $C_1(\hat{q})$.

Thus we have shown that there exists an unbounded edge F^1 in $C_1(\hat{q})$ corresponding to each unbounded edge F in $C_1(\tilde{q})$. Conversely by treating $\tilde{q} = \hat{q} + (-\alpha)I_{.1}$ we can establish a correspondence between unbounded edges in $C_1(\hat{q})$ and those of $C_1(\tilde{q})$. This establishes a 1 - 1 correspondence between the unbounded edges in $C_1(\tilde{q})$ and those in $C_1(\hat{q})$. Hence both $C_1(\tilde{q})$ and $C_1(\hat{q})$ must have the same number of unbounded edges.

6.6 Now to continue the proof of Theorem 6.2, let \tilde{q} and \bar{q} be any two column vectors in R^n both of which are nondegenerate with respect to M .

By 6.5 and (i), (iii) of 6.3 we conclude that the parity of the number of solutions to (1) does not change if we alter the vector q one component at a time so that it remains nondegenerate with respect to M both before and after the alteration.

It is always possible to alter \tilde{q} , by one component at a time, retaining the property of being nondegenerate with respect to M throughout, until it becomes equal to \bar{q} .

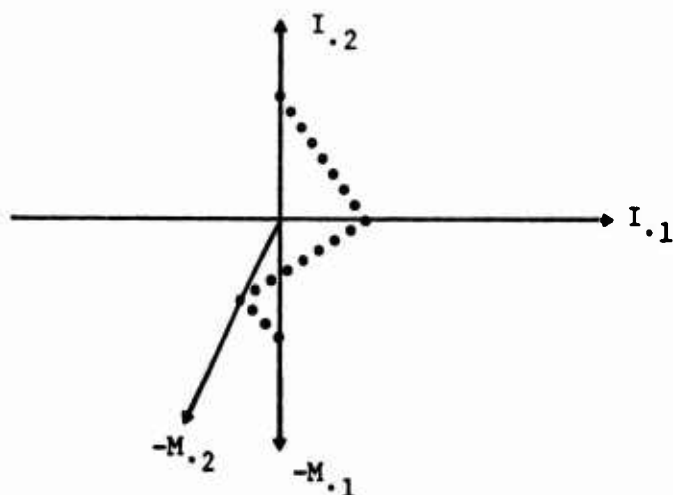
Hence the number of solutions to (1) has the same parity whether $q = \tilde{q}$ or \bar{q} . Hence the number of solutions to (1) has the same parity whenever q is nondegenerate with respect to M .

6.7 Note: The assumption that M is nondegenerate cannot be dropped from the hypothesis of Theorem 6.2, as can be seen from the example below.

Let $M = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}$. Then (1) is to solve

w_1	w_2	z_1	z_2	$= q$
1	0	0	-1	q_1
0	1	-1	-2	q_2

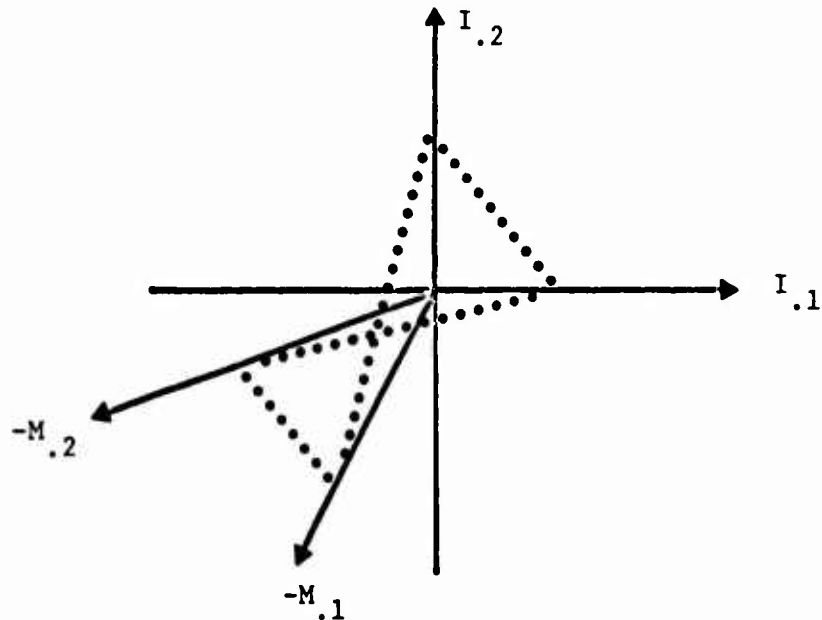
$$w_1, w_2, z_1, z_2 \geq 0, w_1 z_1 + w_2 z_2 = 0.$$



$q = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ leads to one solution to (1) and $q = \begin{pmatrix} -1 \\ -2 \end{pmatrix}$ leads to two solutions to (1) even though both these are nondegenerate with respect to M here.

Here M is not nondegenerate because the matrix $(I.2, -M.1)$ is singular. The argument in 6.4 fails.

6.8 Note: The assumption that q is nondegenerate with respect to M cannot be dropped from the hypothesis of Theorem 6.2 as can be seen from the example below. Let $M = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$.



When q is nondegenerate with respect to M the number of solutions to (1) is an odd number, but when $q = -M_{.2}$, (1) has exactly two distinct solutions.

6.9 Corollary:

If M is nondegenerate and not a Q -matrix then the number of solutions to (1) is an even number for all q which are nondegenerate with respect to M .

6.10 Proof:

By 2.10, the set of all q for which (1) has a solution is the union of the 2^n complementary cones. Each complementary cone is a closed set in R^n and hence their union (being a union of a finite number of closed sets) is itself closed. The set of all q for which (1) has no solution is the complement of this union, and hence is an open set. Because M is not a Q -matrix, this open set is nonempty. So the set of all q for which (1) has no solution is a nonempty open cone.

6.11 By 2.16, the set of all q which are not nondegenerate with respect to M is the union of a finite number of subspaces of R^n , each of which has dimension $\leq (n-1)$. Hence the set of all q which are not nondegenerate with respect to M has no interior.

6.12 By 6.10 and 6.11 we conclude that there must exist a q nondegenerate with respect to M , for which (1) has no solution, i.e., zero solutions. Now by applying Theorem 6.2 we conclude that the number of solutions to (1) has the same parity as zero, i.e., even parity, whenever q is nondegenerate with respect to M .

6.13 Note: Corollary 6.9 is not necessarily true if M is not nondegenerate as seen from Example 6.7.

6.14 Note: The converse of Corollary 6.9 is not necessarily true unless $M \geq 0$. This is discussed in 8.17.

7. ON PROBLEMS WITH A CONSTANT NUMBER OF COMPLEMENTARY FEASIBLE SOLUTIONS

7.1 Here we show that if the number of complementary feasible solutions is a constant for all nonzero $q \in R^n$ then that constant is equal to one and M is a P-matrix.

7.2 Theorem:

If the number of complementary feasible solutions is a constant for all $q \in R^n$, $q \neq 0$, then M is a P-matrix and that constant is equal to one.

Proof:

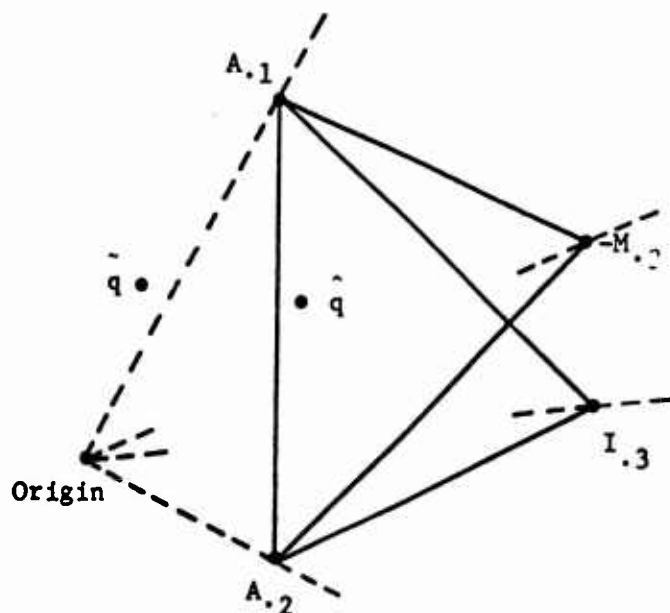
7.3 Whatever M may be, (1) always has at least one solution for every $q \geq 0$ (the solution is $w = q$; $z = 0$). If M is not a Q-matrix, there exists a $q \neq 0$ for which (1) has no solution at all. Hence, if M is not a Q-matrix then the number of solutions to (1) cannot be a constant for all $q \neq 0$. So under the hypothesis of Theorem 7.2, M must be a Q-matrix.

7.4 The number of complementary feasible solutions is finite whenever q is nondegenerate with respect to M . By Corollary 3.3, if M is not nondegenerate, there exists a $q \neq 0$ for which (1) has an infinite number of solutions.

Since the number of solutions to (1) is a constant for any $q \neq 0$, M must therefore be a nondegenerate matrix. Hence, all the principal submatrices of M are also nondegenerate. Also, every subcomplementary set of column vectors is linearly independent and every complementary cone has a nonempty interior.

7.5 Let $\{A_{.1}, \dots, A_{.n-1}\}$ be any subcomplementary set of column vectors. We will now show that the hyperplane generated by this subcomplementary set of column vectors strictly separates the points representing the left out complementary pair of column vectors $I_{.n}$ and $-M_{.n}$.

Suppose not. Then the interiors of the complementary cones $\text{Pos}\{A_{.1}, \dots, A_{.n-1}, I_{.n}\}$ and $\text{Pos}\{A_{.1}, \dots, A_{.n-1}, -M_{.n}\}$ have a nonempty intersection.



Then we pick a pair of points \hat{q} , \tilde{q} , both nondegenerate with respect to M , sufficiently close together, such that \hat{q} is in the intersection of $\text{Pos}\{A_{.1}, \dots, A_{.n-1}, I_{.n}\}$ and $\text{Pos}\{A_{.1}, \dots, A_{.n-1}, -M_{.n}\}$ and \tilde{q} lies outside both these cones and is strictly separated from \hat{q} by the hyperplane through $\text{Pos}\{A_{.1}, \dots, A_{.n-1}\}$. We can choose

$$\hat{q} = \sum_{j=1}^{n-1} \lambda_j A_{.j} + \alpha I_{.n}$$

where $\lambda_1, \dots, \lambda_{n-1}, \alpha$ are all > 0 , α sufficiently small $\lambda_1, \dots, \lambda_n, \alpha$ are such that \hat{q} is nondegenerate with respect to M and $\sum_{j=1}^{n-1} \lambda_j A_{.j}$ does not lie in any subspace generated by $(n-1)$ or less column vectors of $(I, -M)$ excepting $\text{Pos}\{A_{.1}, \dots, A_{.n-1}\}$. By the nondegeneracy of \hat{q} , $\hat{q} + \lambda_n I_{.n}$ is also nondegenerate with respect to M for all but a finite number of values of λ_n . So we could pick an $\alpha_0 > 0$ sufficiently small such that

$$q = \sum_{j=1}^{n-1} \lambda_j A_{.j} + \beta I_{.n}$$

is nondegenerate with respect to M for all β satisfying $\beta \neq 0$, $-\alpha_0 < \beta < \alpha_0$. So if we had chosen our original α so small that $0 < \alpha < \alpha_0$, and \hat{q} is in the interior of both $\text{Pos}\{A_{.1}, \dots, A_{.n-1}, I_{.n}\}$ and $\text{Pos}\{A_{.1}, \dots, A_{.n-1}, -M_{.n}\}$, then

$$\tilde{q} = \sum_{j=1}^{n-1} \lambda_j A_{.j} - \alpha I_{.n}$$

is outside both these complementary cones and \tilde{q} does not lie in any complementary cone in which \hat{q} does not lie. Thus, the number of solutions to (1) when $q = \tilde{q}$ is strictly less (by two) than the number when $q = \hat{q}$, leading to a contradiction.

By a similar argument, we verify that the hyperplane through any subcomplementary set of column vectors strictly separates the points representing the left out complementary pair of column vectors.

7.6 We will now show that the principal subproblem of (1) in the variables $(w_1, \dots, w_{n-1}; z_1, \dots, z_{n-1})$ satisfies a similar separation property. The column vector corresponding to w_j in this subproblem is the j -th column vector of the unit matrix of order $(n-1)$, which we denote by \downarrow_j , and the column vector corresponding to z_j in this subproblem is $-(m_{1j}, m_{2j}, \dots, m_{n-1,j})^T$ which we denote by $-m_j$. We note that the column vectors in the subproblem are obtained by deleting the last component from the column vectors in the original problem.

Let $\{a_{.1}, \dots, a_{.i-1}, a_{.i+1}, \dots, a_{.n-1}\}$ be any subcomplementary set of column vectors in the subproblem. We want to show that the hyperplane in R^{n-1} through these column vectors strictly separates $\downarrow_{.i}$ and $-m_{.i}$.

Let $A_{.r}$ be the column vector corresponding to $a_{.r}, r=1, \dots, i-1, i+1, \dots, n-1$, in the original problem. Then $\{A_{.1}, \dots, A_{.i-1}, A_{.i+1}, \dots, A_{.n-1}, I_{.n}\}$ is a subcomplementary set in the original problem. By 7.5 the hyperplane in R^n through these column vectors strictly separates $I_{.i}$ and $-M_{.i}$. Suppose this hyperplane is

$$DX = 0$$

where $D = (d_1, \dots, d_n)$ and $X \in R^n$. Then

$$\begin{aligned} DA_{.1} &= 0 \\ &\vdots \\ DA_{.i-1} &= 0 \\ DI_{.i} &> 0 \quad \text{and} \quad D(-M_{.i}) < 0 \\ DA_{.i+1} &= 0 \\ &\vdots \\ DA_{.n-1} &= 0 \\ DI_{.n} &= 0. \end{aligned}$$

Now $DI_{\cdot n} = d_n = 0$. Let $d = (d_1, \dots, d_{n-1})$. Because $d_n = 0$, the above equations imply that

$$\begin{aligned} da_{\cdot 1} &= 0 \\ &\cdot \\ &\cdot \\ &\cdot \\ da_{\cdot i-1} &= 0 \\ d\downarrow_{\cdot i} &> 0 \text{ and } d(-m_{\cdot i}) < 0 \\ da_{\cdot i+1} &= 0 \\ &\cdot \\ &\cdot \\ &\cdot \\ da_{\cdot n-1} &= 0. \end{aligned}$$

Let $x = (x_1, \dots, x_{n-1}) \in R^{n-1}$. Thus $dx = 0$ is the hyperplane in R^{n-1} through the subcomplementary set $\{a_{\cdot 1}, \dots, a_{\cdot i-1}, a_{\cdot i+1}, \dots, a_{\cdot n-1}\}$ of the subproblem and it strictly separates $\downarrow_{\cdot i}$ and $-m_{\cdot i}$.

Hence the subproblem also satisfies a similar separation property. By a similar argument we can verify that every principal subproblem of (1) of order $(n-1)$ satisfies the separation property.

7.7 Induction Hypothesis:

For any complementarity problem of order $r \leq n-1$, with column vectors $(I, -N)$, if N is nondegenerate and if the hyperplane through every subcomplementary set of column vectors strictly separates the points representing the left out complementary pair of column vectors in the problem, then N is a P-matrix.

7.8 The induction hypothesis is easily verified for the case $r = 1$. By nondegeneracy $N = (m_{11}) \neq 0$. Since $r = 1$, the subcomplementary set is the null set and hence the hyperplane through the subcomplementary set is the singleton consisting of the origin itself. Since this separates

the points on R^1 representing 1 and $-m_{11}$, we should have $-m_{11} < 0$. . .
 So $N = (m_{11}) > 0$ and hence is a P-matrix in this case.

7.9 Hence by the induction hypothesis and 7.6 every principal submatrix of M of order $n-1$ is a P-matrix. Hence all principal subdeterminants of M of order $\leq n-1$ are strictly positive.

Since M is nondegenerate by 7.4, determinant of $M \neq 0$. So M^{-1} exists. So the constraints (1) can be written as

$$\left. \begin{array}{l} z - M^{-1}w = Q \\ z \geq 0, w \geq 0, z^T w = 0 \end{array} \right\} \quad (19)$$

where

$$Q = -M^{-1}q.$$

If (1) has a constant number of solutions for every $q \in R^n$, $q \neq 0$, then (19) has a constant number of solutions for each $Q \in R^n$, $Q \neq 0$.

Hence by the arguments used previously all principal subdeterminants of M^{-1} of order $(n-1)$ or less are strictly positive. Let α be the value of the principal subdeterminant of M^{-1} obtained by striking off the first row and column from M^{-1} . Then

$$\alpha = \frac{m_{11}}{\text{determinant of } M}. \quad (20)$$

But $\alpha > 0$, $m_{11} > 0$. So by (20) the determinant of M is also strictly positive. So all principal subdeterminants of M are strictly positive. Hence M is a P-matrix.

So the induction hypothesis 7.7 holds when $r = n$ also. It has been verified for $r = 1$ in 7.8. So by induction it holds for all n .

Hence by 7.5, Theorem 7.2 is true for all n .

8. THE ODD NUMBER THEOREM FOR NONNEGATIVE Q-MATRICES:

8.1 Here we show that if M is a nonnegative Q -matrix, then the number of complementary feasible solutions is an odd number whenever q is nondegenerate with respect to M . This result may not hold if M is not nonnegative.

8.2 Theorem:

If $M \geq 0$ and is a Q -matrix, then the number of complementary feasible solutions is an odd number for any q nondegenerate with respect to M .

Proof:

Proof is by induction on n .

8.3 If $n = 1$, then $M = (m_{11})$ and M is a Q -matrix if and only if $m_{11} > 0$ by Theorem 5.2. Here $q = (q_1)$ and for each $q \in R^1$ there is exactly one complementary feasible solution. Hence Theorem 8.2 is true when $n = 1$.

8.4 Induction Hypothesis:

Suppose Theorem 8.2 is true for all complementarity problems of order $(n-1)$ or less. We will now show that this implies that Theorem 8.2 also holds for problems of order n .

8.5 By Corollary 5.4 all principal submatrices of M are also Q -matrices. Consider the principal subproblem in $(w_2, \dots, w_n; z_2, \dots, z_n)$ with the right hand constants $= Q$. If \bar{Q} is nondegenerate in the subproblem, then it has an odd number of complementary feasible solutions when $Q = \bar{Q}$, by the induction hypothesis 8.4.

Let $\bar{q} = \begin{pmatrix} q_1 \\ -\frac{q_1}{Q} \end{pmatrix}$ where $q_1 > 0$, be nondegenerate with respect to M .

Since $q_1 > 0$ and $M \geq 0$, the variable z_1 must be equal to zero in any solution to (1) when $q = \bar{q}$. Thus if $(\tilde{w}; \tilde{z})$ is a solution to (1) when $q = \bar{q}$, $\tilde{z}_1 = 0$ and hence $(\tilde{w}_2, \dots, \tilde{w}_n; \tilde{z}_2, \dots, \tilde{z}_n)$ is a complementary feasible solution to the subproblem when $Q = \bar{Q}$.

Also if $(w_2^*, \dots, w_n^*; z_2^*, \dots, z_n^*)$ is any complementary feasible solution to the subproblem when $Q = \bar{Q}$, define

$$w_1^* = q_1 + \sum_{j=2}^n m_{1j} z_j^* > 0$$

and then $(w_1^*, w_2^*, \dots, w_n^*; 0, z_2^*, \dots, z_n^*)$ is a complementary feasible solution to the original problem when $q = \bar{q}$.

Thus every complementary feasible solution of the original problem leads to a complementary feasible solution to the subproblem and vice versa. Hence both problems must have the same number of complementary feasible solutions. Hence when $q = \bar{q}$, (1) has an odd number of solutions.

By a similar argument we conclude that the original problem has an odd number of complementary feasible solutions whenever q is nondegenerate with respect to M and at least one component in the vector q is positive.

It only remains to be shown that the same result holds even when $q < 0$.

8.6 We will now show that on every unbounded edge of $K(q)$ lying in the almost complementary set $C_1(q)$, both the variables w_1 and z_1 tend to $+\infty$, while z_2, \dots, z_n remain finite.

From 2.5, $K(q)$ is the set of all $(w; z)$ satisfying

$$\begin{aligned} w_i &= M_{i1} z + q_i, & i &= 1, \dots, n \\ z &\geq 0, w &\geq 0 \end{aligned} \tag{21}$$

Since M is a Q -matrix, $m_{ii} > 0$ for all $i = 1, \dots, n$ by Theorem 5.2.

Consider any unbounded edge of $K(q)$. If all the variables z_1, \dots, z_n remain finite on this edge, then by (21) all the variables w_1, \dots, w_n also remain finite and hence the edge cannot be an unbounded edge. Hence, on every unbounded edge in $K(q)$ at least one of the variables z_1, \dots, z_n must tend to $+\infty$. If z_i tends to $+\infty$ on this edge, then from (21) and the facts that $M \geq 0$, $m_{ii} > 0$ and q_i is finite and fixed, w_i must also tend to $+\infty$ along this edge. Hence if any unbounded edge of $K(q)$ lies in the almost complementary set $C_1(q)$, then the variables z_2, \dots, z_n should all remain bounded on that edge. Hence z_1 must tend to $+\infty$ on that edge and consequently w_1 also tends to $+\infty$ on that edge.

Thus on every unbounded edge in $C_1(q)$, the variable $w_1 \rightarrow +\infty$.

8.7 Suppose \tilde{q} is nondegenerate with respect to M . Then there exists an $\alpha_0 > 0$ such that for all $\alpha > \alpha_0$, the point $\tilde{q} - \alpha I_{.1}$ is nondegenerate with respect to M . Hence the entire half-line

$$\{q : q = \tilde{q} - \alpha I_{.1}, \alpha > \alpha_0\} \quad (22)$$

lies in the interior of a set of complementary cones. We now show that the number of complementary cones in which this half-line lies is precisely the number of unbounded edges in $C_1(\tilde{q})$. Let

$$F = \{(w; z) : (w; z) = (w^1 + \theta w^2; z^1 + \theta z^2), \theta \geq 0\}$$

be an unbounded edge in $C_1(\tilde{q})$. Then

$$\begin{aligned} (w_i^1 + \theta w_i^2)(z_i^1 + \theta z_i^2) &= 0 && \text{for all } i \neq 1 \\ &&& \text{for all } \theta \geq 0 \end{aligned}$$

and $w_1^2 > 0$ by 8.6. Hence

$$(w; z) = (0, w_2^1 + \theta w_2^2, \dots, w_n^1 + \theta w_n^2; z_1^1 + \theta z_1^2, \dots, z_n^1 + \theta z_n^2)$$

is a complementary feasible solution for

$$q = \tilde{q} - (w_1^1 + \theta w_1^2) I_{.1} \quad \text{for all } \theta \geq 0$$

and since $w_1^2 > 0$, as θ varies from 0 to ∞

$$\left\{ q : q = \tilde{q} - (w_1^1 + \theta w_1^2) I_{.1}, \theta \geq 0 \right\}$$

is eventually the same half-line as in (22).

Also for any $\bar{\alpha} > \alpha_0$, $\tilde{q} - \bar{\alpha} I_{.1}$ cannot lie in any complementary cone which has $\text{Pos}\{I_{.1}\}$ as a generator. For, if it does, there exists a subcomplementary set of columns $\{B_{.2}, \dots, B_{.n}\}$ such that

$$\tilde{q} - \bar{\alpha} I_{.1} = \lambda_1 I_{.1} + \sum_{i=2}^n \lambda_i B_{.i}$$

for some $\lambda_1, \dots, \lambda_n \geq 0$. Then $\tilde{q} - (\bar{\alpha} + \lambda_1) I_{.1}$ lies in the subspace through the subcomplementary set $\{B_{.2}, \dots, B_{.n}\}$ contradicting the assumption that $\tilde{q} - \alpha I_{.1}$ is nondegenerate with respect to M for all $\alpha > \alpha_0$.

Hence if the half-line in (22) lies in some complementary cone, say, $\text{Pos}\{A_{.1}, A_{.2}, \dots, A_{.n}\}$ then $A_{.1}, \dots, A_{.n}$ must be linearly independent and $A_{.1} = -M_{.1}$. Then we can express this half-line as

$$\left\{ q : q = \sum_{i=1}^n \beta_i^1 A_{.i} + (\alpha - \alpha_0) \sum_{i=1}^n \beta_i^2 A_{.i}, \alpha > \alpha_0 \right\}$$

for some $\beta^1, \beta^2 \geq 0$. Thus

$$\bar{q} = \alpha I_{.1} + \sum_{i=1}^n \beta_i^1 A_{.i} + (\alpha - \alpha_0) \sum_{i=1}^n \beta_i^2 A_{.i} \quad (23)$$

for any $\alpha > \alpha_0$.

Suppose $(w^r; z^r)$ is obtained by setting $w_1^1 = \alpha_0$, $w_1^2 = 1$, and the variable associated with the column vector $A_{.i}$ equal to β_i^r , $i = 1, \dots, n$ and all the other variables in $(w; z)$ equal to zero, for $r = 1, 2$. Then (22) implies that

$$F = \{(w; z) : (w; z) = (w^1 + \theta w^2; z^1 + \theta z^2), \theta \geq 0\}$$

is an unbounded edge in $C_1(\bar{q})$.

Thus every unbounded edge in $C_1(\bar{q})$ gives rise to a complementary cone in the interior of which the half-line in (22) lies and vice versa. Hence the number of unbounded edges in $C_1(\bar{q})$ is equal to the number of complementary feasible solutions for $\bar{q} - \alpha I_{.1}$ where α is a sufficiently large number.

8.8 Thus for any q_1 such that $\hat{q} = (q_1, \bar{q}_2, \dots, \bar{q}_n)^T$ is nondegenerate with respect to M , the number of unbounded edges in $C_1(\hat{q})$ is a constant. This number is equal to the number of complementary cones in which the half-line (22) eventually lies as α_0 is made large.

8.9 By the nondegeneracy of \bar{q}_1 we know that there exists a β_0 such that for all $\beta > \beta_0$, $(\beta, \bar{q}_2, \dots, \bar{q}_n)^T$ is nondegenerate with respect to M . Hence we can always pick a $q_1^* > 0$ such that $q^* = (q_1^*, \bar{q}_2, \dots, \bar{q}_n)^T$ is nondegenerate with respect to M . Since $q_1^* > 0$, the number of complementary feasible solutions when $q = q^*$ is an odd number. Therefore by 6.3(iii) the number of unbounded edges in $C_1(q^*)$ is an odd number and hence by 8.8 the number of unbounded edges in $C_1(\bar{q})$ is an odd number. By 6.3(iii), the number of complementary feasible solutions when $q = \bar{q}$ is therefore an odd number.

Hence under the induction hypothesis, Theorem 8.2 holds for the original problem of order n . By 8.3 and by induction, Theorem 8.2 is true for all n .

8.10 Corollary:

If M is a Q -matrix and if there exists a complementary set of column vectors $\{A_{.1}, \dots, A_{.n}\}$ which is linearly independent, such that each of the remaining vectors $B_{.1}, \dots, B_{.n}$ among the column vectors of $(I, -M)$ satisfies

$$B_{.j} \in \text{Pos}\{-A_{.1}, \dots, -A_{.n}\} \quad \text{for all } j = 1, \dots, n$$

then the number of complementary feasible solutions is an odd number for all q which are nondegenerate with respect to M .

Proof:

Transform the column vectors $A_{.1}, \dots, A_{.n}$ into the column vectors of the unit matrix by making the necessary principal pivots. Then Corollary 8.10 follows from Theorem 8.2 and 2.21.

8.11 In the special case when $n = 2$, the restriction that $M \geq 0$ can be removed from the hypothesis of Theorem 8.2. This is discussed below.

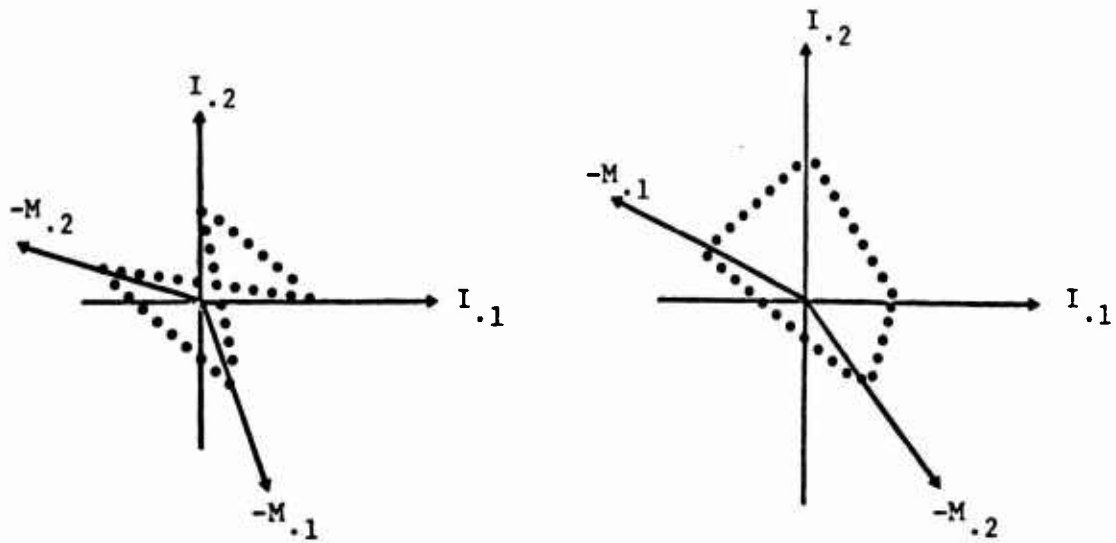
8.12 Theorem:

If $n = 2$ and M is a Q -matrix then the number of complementary feasible solutions is an odd number whenever q is nondegenerate with respect to M .

Proof:

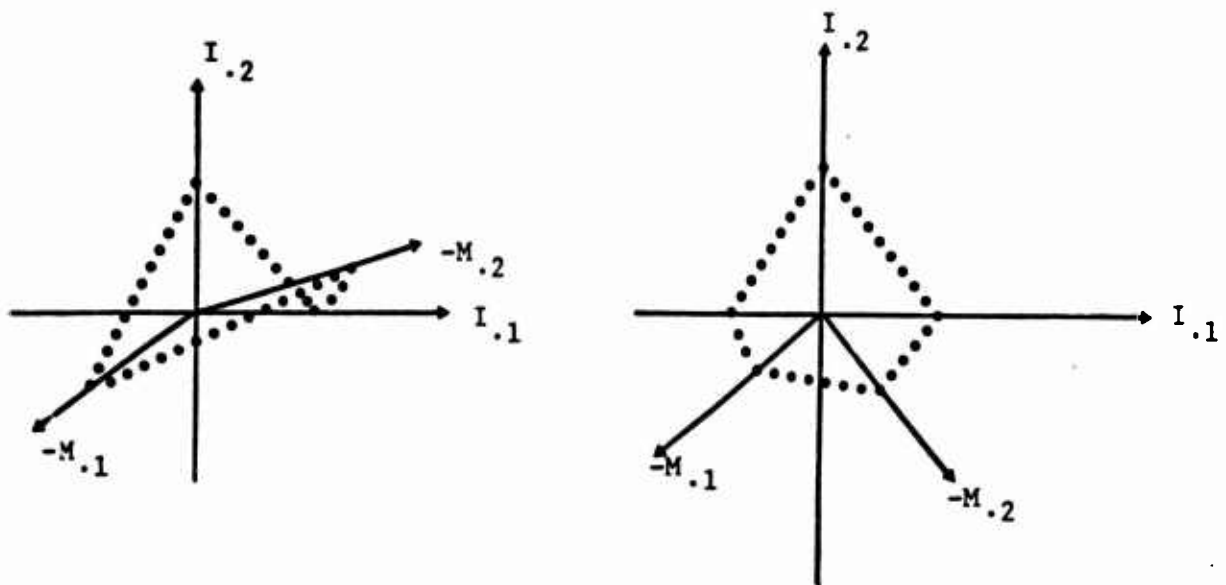
8.13 Case 1: If $\text{Pos}\{-M_{.1}, -M_{.2}\}$ is a subset of the nonpositive orthant of R^2 , Theorem 8.12 follows from Theorem 8.2.

8.14 Case 2: If $\text{Pos}\{-I_{.1}, -I_{.2}\} \subset \text{Pos}\{-M_{.1}, -M_{.2}\}$ the hypothesis that M is a Q -matrix implies that $-M_{.1}$ and $-M_{.2}$ are contained one each in $\text{Pos}\{I_{.1}, -I_{.2}\}$ and $\text{Pos}\{-I_{.1}, I_{.2}\}$ respectively.



We verify that in this case the number of complementary feasible solutions is 1 or 3 for every q nondegenerate with respect to M .

8.15 Case 3: Since M is a Q-matrix, the only other possibility is that exactly one of $-M_{.1}$ or $-M_{.2}$ is contained in the interior of $\text{Pos}\{-I_{.1}, -I_{.2}\}$. Suppose it is $-M_{.1}$. Then the hypothesis that M is a Q-matrix implies that either $-M_{.2} \in \text{Pos}\{I_{.1}, M_{.1}\}$ or $-M_{.2} \in \text{Pos}\{I_{.1}, -I_{.2}\}$.



In either case we verify that the number of complementary feasible solutions is either 1 or 3 for all q nondegenerate with respect to M .

8.16 Corollary:

If $n = 2$, there exists a q nondegenerate with respect to M , for which the number of complementary feasible solutions is at most one.

8.17 Note: When $n \geq 3$, Theorem 8.2 is not necessarily true if $M \not\geq 0$, and Corollary 8.16 may not be true.

As an example consider

$$M = \begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}.$$

It can be shown that this is a Q-matrix by verifying that the union of all the 8 complementary cones is R^3 . Also M is a nondegenerate matrix. We verify that $\tilde{q} = (1,1,1)^T$ is nondegenerate with respect to M . When $q = \tilde{q}$ there are four distinct complementary feasible solutions, because \tilde{q} lies in each of the complementary cones $\text{Pos}\{I_{.1}, I_{.2}, I_{.3}\}$, $\text{Pos}\{-M_{.1}, I_{.2}, I_{.3}\}$, $\text{Pos}\{I_{.1}, -M_{.2}, I_{.3}\}$ and $\text{Pos}\{I_{.1}, I_{.2}, -M_{.3}\}$ and in none of the others.

By Theorem 6.2, the number of complementary feasible solutions is an even number for all q nondegenerate with respect to M , and since M is a Q-matrix, this number must be ≥ 2 .

This shows that the converse of Corollary 6.9 is not necessarily true unless $M \geq 0$.

8.18 Note: When $n \geq 3$, the number of complementary feasible solutions can be ≥ 2 for all $q \in R^n$. The example in 8.17 shows this. Thus when $n \geq 3$ the complementary cover can span the whole space more than twice around.

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