THE CITY COLLEGE
of the
CITY UNIVERSITY OF NEW YORK

THE DYNAMIC RESPONSE OF FINITE, ELASTIC CYLINDERS ACCORDING TO VARIOUS SHELL THEORIES

VOLUME I

Selig Fisher
Sherwood B. Menkes

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FOREWORD

Under a Picatinny Arsenal contact administered through the Army Research Office (DA-31-124-ARO-D-419) the City College has been conducting research on the problem of large deformations of ICBM structures. The orientation is from the standpoint of the lethality analyst, and the intent has been to use whatever is presently available in the literature to provide rapid and reasonably good prediction tools.

In the course of this research, it has appeared that an explicit attack on certain phases of the elastic shell problem was in order. In a sense, then, this work is tangential to the main line of inquiry. When the effort is completed, it should be possible to pinpoint with reasonable precision the points of initial failure to shells subjected to dynamic loading.

The cooperation and assistance of Mr. Donald Miller and Mr. Murray Weinstein of PICATINNY ARSENAL-Engineering Science Laboratory is much appreciated. We should also like to acknowledge the helpful comments offered by Dr. Bernard Koplik, of Brooklyn Polytechnic Institute, and by Dr. Yi Yuan Yu, of the General Electric Company.

ABSTRACT

The forced response of finite, linearly elastic cylinders with prescribed edge conditions has not been sufficiently studied.

Various shell theories have been proposed to examine this problem. The simpler bending theories (called herein classical theories), more amenable to engineering approximation, having been examined by other criteria, may not actually be appropriate to all dynamic problems. Only limited use has been made of these classical theories for dynamic problems, and then only with very specialized edge conditions. More inclusive theories, which include transverse shear deformation and rotary inertia (usually termed refined or SR theories), though developed, have not been used to analyze dynamic problems of this nature.

We propose to compare the results of two shell theories, one classical and one SR theory, when they are used to analyze the forced dynamic deformation of an elastic cylinder with free and clamped edges. An essential feature of the analysis is a reliance on Hamilton's Variational Principle as the underlying, dominant governing physical law. Beginning with Hamilton's Principle, we have formulated two mutually consistent sets of descriptive equations and boundary conditions, as well as the required conditions of orthogonality. The equilibrium portions of these equations are shown to be identical with particular shell theories, previously developed by Yu, in part by different means.

After the solutions are obtained, they will be compared, and the differences noted. It is expected that we will be able to clearly delineate the limits of applicability of the classical theory.

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NOMENCLATURE
(PARTIAL)

The following list is an inventory of most of the symbols and letters used in this report. We have not made a serious attempt to restrict this nomenclature to unique or individual interpretation. Certain algebraic letters have historically been used so often to represent particular concepts that it would seriously detract from the presentation were other letters or symbols to be used. In addition, we have attempted to preserve much of the notation used in certain references, so as to make it easier to compare this material with that in the reference.

As a consequence, more than one meaning is often shown for a letter or symbol. The one which should be used will be clear in the context of the application. Not all symbols are listed; where the use of the character is restricted to only a few pages, and in a very narrow sense, it has not been included in this glossary of terms.

A_1	The determinant a_{rs} (see equation 76)
A_2	Frequency determinant (see equation 78)
C_{mN}	A constant, expressing amplitude of a solution u_k^n
C	Midsurface line (see Figure 4)
D	Linear operator on x
E	Modulus of elasticity
$\overline{F}$	Force
$F_k^n(x,\phi)$	Those terms in free vibration equations that are multiplied by ω_{mN}. (See 103). Traceable to K.E. variation.
$G_k(x,\phi)$	Distribution function for pressure pulse (Equation 142).
G	Lame's constant
G'	KG
H_k^n	Free vibration, deformation equilibrium equations
I	Hamilton's Integral
I_2'	Second invariant of deviatoric stress tensor
K_{mN-ps}	Stiffness coefficient (See equation E-24)

L	Length of cylinder
L	Lagrangian = T + U-W
M	Mass of entire cylinder
$M_{x,\phi}$	Moment resultants (equation 43)
M_{mN}	Generalized mass associated with mN^{th} mode (Eq. 129)
$N_{x,\phi}$	Stress resultants (equation 43)
N	Index of infinite number of eigenvalues of ω_n
$\overline{P}$	Pressure on surface of cylinder
P_o	Maximum value of $\overline{P}$, as scalar
Q^E	Generalized external force in the "direction" of $q(t)_{mN}$
Q_x, Q_ϕ	Shear stress resultant
R	Dimensionless parameter $= \frac{\sigma_r}{\sqrt{\rho/E\,(1-\nu^2)}\cdot 2h1/a}$
dS_k	Incremental distance along vector $\overline{e}_k$
S^n_{ik}	Components of boundary value determinant, free ends (see Eq. 94)
T	System kinetic energy
U	System strain energy
dV	Unit volume
V	Volume
W	System external work
Y_B	Location of center of gravity
a	mean radius of cylinder
a_{rs}	Free vibration coefficient matrix (see equation 75)
a_r	Coefficients of Fourier series (equation 168)
b^n_k	$(1+\frac{h}{a})h^n$
b_p	Coefficients of Fourier series (equation 166)
C^n_k	$-2\gamma\, h^* \frac{h}{(3a)}{}^{2n}_{n}$
$\overline{e}_k$	Body fixed triad
$f^n_k(x)$	Axial function in assumed solution
g	Coefficients in equation 79

h	Half-thickness of cylinder
h_k	Proportionality constant (see equation 15)
$h(\ )$	Unit step function
k	$1/3(h/a)^2$
m	Number of circumferential waves
$\bar{n}$	Vector normal to a surface $d\Sigma$ (Figure 2)
q	Generalized coordinate (t)
$\bar{r}$	Inertial vector location of typical particle
t	Time
t_1	Time terms involving rotary inertia
$\bar{u}$	Deformation vector
u_k^n	Midsurface deformation or rotation
x	Axial coordinate
z	Radial coordinate
Γ_{mi}	A constant amplitude factor (see equation 77)
K	$\kappa(1-\nu)/2$
Λ	Terms in free vibration equilibrium equations due to strain energy variation (See 103)
Ξ_k^n	Deformation equilibrium equations
$d\Sigma$	Surface element (see equation 13)
ϕ_k^n	Used in boundary condition equation (see equation 49)
X	Space dependent terms in Free Vibration equations (102)
ψ_k^n	Stress-motion equilibrium equations (see equation 44)
Ω_k^n	Used in boundary condition equations (see equation 50)
Ω_{mN}	Non-dimensional parameter (equation 185)
α_k	Parameter varied as coefficient of perturbation function
β_k	Parameter varied as coefficient of perturbation function

γ	Mass density
γ	Shear strain
δ	variational sign
δ_{ik}	Kronecker delta
ϵ	Strain
ϵ_{ik}	Strain tensor
ζ_p	Perturbation Function
ζ^n_{kmi}	Relative amplitudes of axial modes
η_k	Perturbation function
θ	Perturbation function (t) in generalized coordinate
κ	Mindlin's constant
λ_k	Body referenced coordinates (see equation 14)
λ_{ni}	A root of determinant A_2
ν	Poisson's ratio
ξ_p	Inertial axes
$\bar{\rho}$	Location vector to particle from center of gravity
σ	Stress (general notation)
σ_{ik}	Stress tensor
σ^n_{ik}	Stress or Moment Resultant (see equation 42)
σ_o	Yield stress in simple tension
σ_r	Stress intensity (see equation 176)
τ	Time of action of a pressure pulse
ϕ	Circumferential coordinate
ω	Natural frequency
ω_0	Lowest extensional frequency = $\sqrt{\frac{E}{\gamma a^2(1-\nu^2)}}$
∇	Del operator
$\cup$	Mathematical notation: "or"
$\bullet$	Notation for scalar multiplication
*	Arithmetic multiplier
$\{\ \}$	A column vector
$\mathcal{L}$	Lame's constant

I INTRODUCTION

1.1 Statement of the Problem

Many shell theories have been developed to predict the deformation response of thin cylinders to external stimuli. Shell theory is based upon the premise that deformation throughout the thickness of the shell is a function of the mid-surface motion (linear displacement, rotation, variation of rotation per inch, etc.), and the original distance of an element from the mid-surface. These mathematical models, when possible, have been compared with the assumed exact model, the three-dimensional theory of elasticity. Such comparisons have been limited due to the unavailability of exact(elasticity) solutions for other than extremely specialized loadings, constraints and response patterns, as for example, the axial shear vibrations of an infinite cylinder.

The classical shell theories are developed for a cylinder whose transverse shear deformation is considered negligible and whose rotary inertia is disregarded. Rotation of the mid-surface is seen then to be a dependent function of its linear displacement. The mathematical model to predict deformation response is therefore constructed from 3 independent variables, i.e., the orthogonal mid-surface displacements.

A more complete (non—classical) shell theory (SR) would account for transverse shear deformation and rotary inertia. Mid-surface variables are the linear displacements and rotation in the transverse planes. These develop a transverse strain approximately constant throughout the thickness. The five variables are independent.

In only one instance (wave propagation during free vibration of an infinite cylinder) has the dynamic characteristics of the models for both of these shell theories been compared with the exact three dimensional elastic solution. In that one case, the 5 variable shell theory can duplicate the elasticity solution while the classical (3 variable) shell theory is in substantial error in predicting the high frequency oscillations associated with short structural wavelengths.

Our intention is to discover whether or not this divergence of dynamic response characteristics between the two shell theories manifests itself in more typical engineering structures (finite cylinders, clamped and free-ended) subjected to familiar impulsive loadings. The comparison parameter will be the time history of the Hencky stress intensities developed on the cylinder's surface. A measurable difference in results for a particular shell geometry should indicate hesitation in the unlimited usage of the popular classical shell theories as models for many dynamic problems.

No dynamic response problem for a finite cylinder has been attempted using a non-classical theory. Nor has a free vibration problem, with the free and clamped boundary conditions, been attempted with a non-classical theory. Even with the classical theories, analytical dynamic response investigations are very limited. The deformation history for the finite - clamped or free cylinder under dynamic loading has not been investigated.

Shell equilibrium equations and required boundary conditions will be developed from Hamilton's Principle both for a classical theory and a shear deformation theory. The alterations necessary in both the equilibrium equations and the boundary requirements to transform one theory into its conjugate will be accomplished within Hamilton's integral and will therefore be consistent with the minimization principle.

The solution of the force-free equations is an infinite series of separable space (normal mode) and time (natural frequency) functions. Proof of the orthogonality of the normal mode functions will also be accomplished within the Hamilton Integral by a procedure shown to be applicable for any model, when the strain energy density of the shell is a quadratic function of the strains. This orthogonality is necessary to reduce Hamilton's Integral to a set of uncoupled Lagrange equations. The Lagrange equations determine the time functions necessary to construct the deformation response pattern during a forced vibration.

A truncation of the stress intensity series for both theories will rest upon upper limits of natural frequency magnitudes, to be selected on the basis of the precision desired in making the comparison.

1.2 History of Dynamic Analysis of Elastic Cylinders

Several authors have considered the free vibration characteristics of thin shells with various end conditions. In 1894, Rayleigh (1)* found the natural frequencies of a cylindrical shell with free edges, based upon an inextension of the mid-surface. Frequencies for an infinite cylinder were also computed for a pure inextensional and a pure extensional mode. Baron & Bleich (2), adding bending to Rayleigh's extensional energy, applied Rayleigh's Principle to the combined energy to construct a table of natural frequencies for an infinite cylinder. Arnold & Warburton (3) investigated the free vibrations of the simply supported cylinder. Later, (4) using the Rayleigh-Ritz procedure of assuming an axial mode shape, they developed the Lagrange equilibrium equations from Hamilton's Principle and computed an approximation to the natural mode frequencies of the clamped cylinder. In both cases, Arnold and Warburton used the classical strain expressions of Love. In 1955, Yu (5), also studied finite cylinders with clamped and simply supported edges, with Donnell's simplified equation.

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*Numbers in brackets ( ) refer to References

Yu employed an approximation which effectively restricted the minimum permissible wave length.

Recently Forsberg, (7), (8), presented a computer procedure to determine the exact mode shapes and frequencies for any edge conditions, using the classical theory of Flugge. The method was suggested by Flugge in 1934 but awaited the advent of the high speed computer. Warburton (9), used a variation of this procedure, with Flugge's equations, to present the edge condition neglected by Forsberg, but in our opinion used improper stresses at the boundary. Edge conditions do affect natural frequency, as shown by Weingarten (10).

Yu, (6), in 1958 included shear deformation and rotatory inertia (SR) in analyzing the five natural frequency branches for a simply-supported and an infinite cylinder. Other attempts to compute natural frequencies using a non-classical shell theory were made by Herrmann and Mirsky, (12), (13), for an infinite cylinder. The first was for an SR theory of five independent deformation coordinates and the latter for a six variable shell theory, including transverse normal stress (an SRN theory).

Weingarten (11), used a very much simplified classical theory (the Donnell equations with no longitudinal or circumferential inertia) and found that many classical theories (as well as many assumed mode shapes) lead to approximately the same values of natural frequencies.

The literature on the forced vibration problem of shell structures under transient loading is rather meager. Dynamic response of cylinders to an impulsive load has been investigated in a few cases of simplified geometry, using either a membrane theory or a classical bending theory. Payton (14) determined the dynamic membrane stress in an infinite cylinder for a number of radial pressure distributions. Humphreys and Winter (15) using Flugge's bending theory, solved the same problem and compared numerical results with Payton.

Sheng (16), Bushnell (17) and Yao (18) using, respectively, the classical theories of Donnell, Flugge, and again Donnell as equilibrium equations, have solved the response of a simply supported cylinder to a radial impulse. This was accomplished in a closed form solution, using the normal mode method. The complexity of the problem was greatly reduced, as the normal modes are well known as trigonometric functions for a simply-supported structure (19) (3).

Wah (2) considered the dynamic response of clamped-ended structures, but used Yu's approximation (see 5) after choosing the form of the axial normal mode. Therefore, he accepted, for the forcing function problem, an unproved (and incorrect) normal mode orthogonality.

A numerical solution by finite difference methods for an elastic (small deflection) cylinder by Johnson and Greif (21), was constructed from a classical shell theory (22), assuming first a normal mode solution. Results were indicated for various modal shapes for a clamped-free cylinder with a blast loading. Another numerical program (23) was developed for axisymmetric dynamic loading of an elastic cylinder, from a momentum distribution caused by traveling elastic waves.

Finite difference codes for large deformation, elastic-plastic response have been developed at MIT. An extensive account of the two-dimensional problem (axisymmetric loading) is presented by Witmer, et al, in reference (24). Pian extended the theoretical basis of the method to include the general three-dimensional shell (25). Recently Leech (26) offered the first 3 dimensional shell code, capable of handling large deflections and rotations.

A serious limitation to the numerical approach is in computer storage capability. This necessitates the use of a very coarse space mesh, which in turn restricts the precision of the result.

1.3 Various Shell Theories

Many shell theories have been proposed. The particular ones reviewed here have had considerable engineering application. The methods of developing these theories have been different, but they all rest on the premise that the mid-surface motion of the shell controls the behavior of the structure, and a particular motion is prescribed, either explicitly or implicitly.

Classical Theories

Usually, the dynamic equilibrium for a finite element as thick as the shell is considered. The force and moments acting upon this element are expressed first in terms of strain and change in curvature of the mid-surface and finally as functions of the mid-surface displacement. For a classical bending theory, where the normals to the median surface of the undeformed shell remain straight and normal to the median surface after deformation, the change in curvatures (rotations) of the mid-surface are directly dependent on the space-derivatives of the translation of the mid-surface. For a membrane theory, where deviations throughout the cross-section are disregarded, the effect of median-surface rotation, with respect to its contribution to this change of deformation throughout the cross-section, at least, is neglected.

Cylinder bending theories, developed following this finite element procedure, that have enjoyed wide usage, are those of Love (27), Timoshenko (28), Flugge (29) and Donnell (30). Love and Flugge make allowances for the trapezoidal shape of two faces of the element in computing the stress resultant. Timoshenko does not. Donnell's equations are derivable from those of Timoshenko or Flugge, by neglecting the first power of the thickness-mean radius ratio when compared with unity. The Donnell equations are applied most frequently as they can be easily manipulated to give an uncoupled equation in the transverse deformation coordinate and because they are amenable to approximate analytical solutions.

Other bending theories, perhaps more rigorously developed, have been compared by Budiansky and Sanders (31), and were justified for a variety of theoretical considerations. Two of the engineering theories above, Love's first approximation and Timoshenko's yield unsymmetrical equilibrium equations and are therefore in contradiction to the principle of conservation of energy (32). Their strain energy would not be invariant with a coordinate transformation (33). Implying the structure was not linearly elastic as assumed, Betti's law being violated (34). The major practical consequence of unsymmetric equilibrium equations is that a series solution of assumed normal mode functions would not be orthogonal and therefore not truly separable for analytical and computational purposes.

Non-Classical Shell Theories

Higher order (non-classical) shell theories have been developed which attempt to include the transverse shear strains and transverse normal stress. Such theories differ from the classical shell theories in that the material normal to the median surface of the undeformed shell no longer remains normal to the median surface after deformation. Shear deformation theories, including the effects of rotary inertia, are called SR theories, while the further inclusion of the normal stress creates what is termed an SRN theory

A different approach was necessary to create these theories as the geometry of the deformation is not as evident as in the classical bending theory. In a manner originally proposed by Cauchy and Poisson, and revived by Mindlin (35), the deformation of each particle throughout the thickness was described by a truncated power series for the transverse position coordinate of the shell;

$$u(x,\phi,z,t) = \sum_{n=0}^{N} z^n u_k^n(x,\phi,t)$$

where the superscript terms described the motions of the mid-surface. For an SR theory the circumferential and axial coordinates, u_1 and u_2 are linear functions of z. The radial deformation, u_3 is independent of z.

This is necessary to define $u_1^{(1)}$ and $u_2^{(1)}$, the rotations on the original normal to the median surface, as functions of the transverse strain.*

Consistent with the displacement of a particle, the three-dimensional stress-motion equilibrium of a particle is studied. The elastic constitutive equations at a point (with appropriate modifications for a transverse shear factor), coupled with exact strain-displacement relationships for curvilinear coordinates, transforms the particle equilibrium equations into functions of the mid-surface motion coordinates, $u_k^{(n)}$. An integration of these equilibrium equations over the thickness produces the shell eqiulibrium equations as a function of these coordinates. Herrmann and Mirsky (36) and Yu (6) developed their shell equilibrium equations in this manner. Herrmann and Mirsky determined their shear factor by studying the free, axi-symmetric motion of an infinite cylinder (36) by the three-dimensional theory of elasticity and comparing their shell theory to this.

Utilizing the true dynamic equilibrium equations at a point is the sufficient condition to satisfy the Virtual Work integral when there is a variation of the equilibrating displacements. Recognizing this, further work in developing shell equations was done, utilizing Hamilton's principle of minimizing the time integral of generalized Lagrange energy. The results should be exactly equivalent, as was checked by Yu (37) and Herrmann and Mirsky (12), as they redeveloped their shell theory equations, using the Hamilton method**. A great advantage of this procedure is that it readily and consistently permits the establishment of the appropriate boundary conditions. A further advantage, is that being developed from an energy basis, for a linear elastic system, the differential equations of motion (the equilibrium equations) possess the symmetry necessary for a series of orthogonal solutions.

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*As stated before, the classical bending theories use this method implicitly. When the transverse strain is set equal to zero, the dependency of the rotation $u_k^{(1)}$ on the translations of the mid-surface $u_k^{(o)}$ is found.

**This is a form of a Rayleigh-Ritz procedure. The actual deformations are approximated by Mindlin's series within the Hamilton integral and the mid-surface coordinates are minimized.

Klosner and Levine (38) developed two shear deformation theories starting with Donnell's and Flugge's bending theory by assuming changes in these bending strains consistent with transverse shear deformation and constant radial displacement. The strain energy was constructed and Hamilton's principle applied to develop the shell equilibrium equations.

The more complex SRN theories have also been developed using Hamilton's principle and the Mindlin power series deformation. Herrmann and Mirsky (13) proposed a linear power series for all three displacements. Naghdi, (43), applying the variational principle of Reissner (39), used a radial deformation that was a function of mid-surface coordinate terms up until the second power of z. The normal stress was not disregarded and the shear constants were obtained as a natural consequence of the displacement assumptions.

We will use Hamilton's principle to formulate our own shell equations of both the SR type and the classical bending type. These will prove to be the same as equations developed previously by Yu, by different means. This will incorporate a selection of the appropriate Mindlin deformation series (linear in the line-of-curvature coordinates and with only one transverse coordinate). The equilibrium equations in each case will be symmetric. This is a necessary requirement for solving a dynamic problem by the normal mode method. The boundary conditions, taken from within the Hamilton integral, will be consistent for each type of theory.

1.4 Comparison of Shell Theory Solutions - Dynamic & Static Response

The free vibration characteristics of infinite cylinders, analyzed by both the three dimensional theory of elasticity and various shell theories (classical and non-classical) have been investigated and compared.

Mindlin (45) investigated flexural wave propagation in an infinite plate and compared the known elasticity solution for the slowest moving wave for each structural wave length (first branch of the phase velocity spectrum) to solutions found from the membrane theory, a classical bending theory and a non-classical SR theory which used a shear factor, κ . All theories gave the same results as the membrane theory for long wave lengths but only the theory accounting for shear deformation could duplicate the elasticity solution (with an adjustment of κ as a function of Poission's ratio) for short wave lengths. If the first mode of thickness-shear vibration is compared, κ must have another value. The classical bending theory erroneously predicts wave propagation velocities for short structural wave lengths (high frequency terms).

Mindlin (46) did a similar analysis for axially symmetric flexural vibrations of a circular disk with free boundaries, comparing an SR with a classical theory where the rotary inertia is included. Natural frequencies for all mode shapes were compared. There was a significant difference in the high frequency range, (short structural wave length). The classical bending theory computes higher magnitudes for these frequencies due to the constraint imposed in suppressing shear deformation.

Herrmann and Mirsky (36) developed classical and SR shell equations (beginning with stress-equilibrium at a point and assuming a linear deformation pattern) and extended Mindlin's procedure of comparing first branch flexural wave propagation velocities, in an infinite cylinder with axisymmetric loading. Those computed from these shell theories were compared with the elastic solution. Mindlin's plate conclusions were verified for the cylinder. Interestingly, the effects of shell curvature were negligible for short structural wavelengths, and Mindlin's plate shear factor, κ could be used in the SR theory to produce the elasticity results. The bending theory prediction of wave velocities for these wave lengths were in error, while for long wavelengths the membrane theory was adequate.

Greenspon (40) compared the natural frequencies (associated with wave propagation) of an infinite shell computed from the membrane theory, various classical theories and non-classical SR theories. This was done for a range of structural wave lengths with more than one branch of the frequency spectrum studied. The upper branches contain higher frequencies associated with the same wave length (mode shape). The shear deformation theories were most successful in predicting the elasticity solution for the higher frequencies. Greenspon also found that, for this free vibration case, the deformations for thin shells according to the linear SR theory approached the theory of elasticity solution more successfully than did the classical bending theory. Even the SR deformations became less valid as the shell thickness increased.

Herrmann and Mirsky (12) duplicated the elasticity solution for the first mode frequency of an infinite cylinder, during thickness-shear vibration, with an SR theory, using Mindlin's plate shear factor. In their work (13) with an SRN shell theory the same results were obtained. (The shear factor, though, was slightly different from Mindlin's). In (13), where general vibration frequencies for six branches were obtained for each structural wavelength, and in another work (41), with their SRN theory, but with axi-symmetric vibrations, limitations of the shell theory were uncovered. The higher branches of the short structural wavelength frequencies (extremely high) were not predicted accurately for thick shells. This is partially accounted for by the non-universality use of the shear factor, the thickness-shear factor having been developed for a lower first mode frequency.

The forced dynamic response of infinite cylinders has been analyzed (at separate times) using the membrane theory, a classical bending theory, and the SR theory of Herrmann and Mirsky. Spillers (42) used the SR theory to describe the response of a semi-infinite cylinder to an axi-symmetric longitudinal impact at one end. He finds that at times shortly after the impact the deformation and stress could have been adequately described by the membrane theory. This must be due to high wave velocity of long structural wave lengths that are primarily associated with shell compression. At a later time, when the effects of bending and shear deformation (associated with the slower moving waves of short structural wavelengths) are felt, the membrane theory is invalid.

Spillers did not differentiate between the effects of bending and shear deformation, as we will. Humphrey and Winter (15) compared their work to Payton (14) to differentiate between the membrane and bending effects on an infinite cylinder in plane strain with radial impulsive loading. They found (29) their stresses, from Flugge's bending theory, diverged from the membrane solution only after some time.

The membrane-bending and membrane-shear deformation comparisons in an infinite cylinder during forced impulsive loading, therefore, do exhibit the characteristics found for free vibration wave propagation. As already noted, the free vibration analysis of an infinite cylinder also indicates an increase in exactness (towards the elasticity solution) of the SR theory over the classical bending theory. This difference in results between these theories has not been verified for a forced response, nor has the effect of end conditions in a finite cylinder in influencing this difference, if any, been investigated.* This is the subject of current research at CUNY. Any differences that will be uncovered will be assumed to be reinforcing the superiority of the SR theory.

This supposition of the superiority of the SR theory is by no means valid a priori if the criterion of comparison is the static response. Klosner and Levine (38) employing 5 shell theories (two classical, two SR, and the SRN theory of Naghdi) have investigated the displacement and stresses produced in an infinite cylinder, according to these theories, by axi-symmetric, periodically spaced radial loads. These results have been compared with the solution of the theory of elasticity. All shell theories predict stress and displacement erroneously in the vicinity of the applied loads. In predicting longitudinal stress and displacement the classical theory gave the best results. In predicting the circumferential stress, the classical theory gave better results than the shear deformation theory and was only inferior in predicting radial displacement. The authors attribute these seemingly surprising results to the particular static load used, which produces large circumferential stresses and constant longitudinal displacement throughout the shell thickness.

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*Bushnell (17) discovered marked differences in radial displacement and frequencies associated with this displacement when comparing a simply-supported and infinite shell response to radial impulse, using a classical theory.

The shell theories, though, assume a longitudinal displacement varying linearly throughout the thickness. The cross-section rotation coordinate, which accounts for this linear effect, is computed as overly large with the SR theory not suppressing the shear deformation. Iyengar and Yoganda (44) also found the classical shell theories acceptable in predicting deformations and stresses in a semi-infinite shell with a concentrated axi-symmetric line load at one end, especially for thin shells. The deformations and stresses in the vicinity of the concentrated load were in error. They did not investigate a shear-deformation theory in their paper.

1.5 Summary of Analysis

Forced Response of the Finite Elastic, Cylinder

The structural dynamic response of the free cylinder subjected to time dependent mechanical loading will be described by the resulting rigid body motion and the elastic deformation of the structure. The elastic deformation will be expressed in terms of modes of free vibration of the mathematical model chosen to represent the structure. The model will rest upon the choice of elastic parameters, permissible stresses and strains, and most importantly, the choice of displacement coordinates to define the shape of a deformed element within the structure.

Hamilton's Variation Principle

Hamilton's Integral, for an elastic cylinder in motion, is rearranged internally by choosing a set of displacement coordinates (Mindlin's power series, truncated,) [30] * leading to a particular shell theory. Utilizing Hamilton's principle to develop the ordinary differential equations associated with the rigid body motions, the partial differential equations associated with a finite element of the structure, and the natural form of required boundary

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* bracketed terms refer to subsequent equations

conditions, mode shapes and oscillation frequencies can be computed to construct a deformation pattern minimizing the Hamilton integral in the absence of external loadings (free vibration).

A variation of the propagation times for developing these mode shapes, is the method used to minimize the Integral formed when external loadings are applied. A set of generalized Lagrange equations are developed, as a consequence of this, which will define these required time functions. When the deformation pattern for this forced response is constructed, the stress pattern is, of course, determined.

Variation Technique - The Shell Equation

We will use the conventional variation to develop the three-dimensional, particle equilibrium equations within the time dependent Hamilton integral. In this method, all terms of the Lagrangian energy density are expressed as functions of the particle displacements. These displacements are ultimately varied during the variation of the entire integral. The applicability of this method for a continuous elastic structure was suggested by Love (27), Goldstein (47) and Wang (48). The actual variation technique (perturbation from a solution value) is described by Goldstein and also by Courant and Hilbert (49). Other variation forms have been suggested. Reisner (39)(50) varies displacement and stress independently. Yu has done considerable work (37)(51)(52), extending Washizu's method (53) of varying strain, displacement, and stresses independently, from the static to the dynamic case.

The SR shell equations are formed after making the power series approximation for the particle displacement, applying the variation to the mid-surface motion coordinates, and integrating over the cylinder thickness. The five shell equilibrium equations [53] - [57] as well as the five natural boundary conditions [59] - [60], are developed within Hamilton's integral, as functions of the five mid-surface coordinates, as a natural consequence of the variation principle. These are identical to Yu's (37) developed by his variational technique and those found when he used the three-dimensional stress-motion equations (6).

The transformation of Hamilton's integral when the rotary inertia is disregarded and the shear factor, K (inversely proportional to shear strain) approaches infinity, i.e.,

$$\operatorname{Lim} \left(\delta \int_{t_o}^{t_1} Ldt \Big|_{K \to \infty} = o \right)$$

results in a classical bending theory. The three equilibrium equations [69] - [71] and the four natural boundary conditions [68] are taken from within Hamilton's integral (see Appendix D).

Free Vibration Solution - Normal Modes, Natural Frequencies

As stated earlier, the only method of obtaining the exact mode shapes and natural frequencies, of a finite cylinder, with end conditions other than simply supported, was presented by Forsberg (7) in 1964. Many procedures are available to obtain approximations to the natural frequencies for a given modal pattern. These are either through approximations to the equilibrium equations (11) (5) or, Warburton's (3) (4) (9), assumed approximation to the mode shapes (the Rayleigh-Ritz procedure) which gives better results. Both Forsberg's and Warburton's work will be invaluable in solving the free vibration problem.

An assumed solution for the mid-surface deformation coordinates, in modal vibration, is made (54):

$$u_K^n = \left(\sum_{i=1}^{8,10} \zeta_{Ki}^n \Gamma_i e^{\lambda_i x} \right) \left(\cos m\phi \;\; U \;\; \sin m\phi \right) e^{i\omega t}$$

$K = 1,2,3$
$n = 0,1$

where the series of eight axial functions applies to the classical theory and the series of ten to the SR theory.

Substitution of these expressions into the homogeneous differential equations of equilibrium, after a Donnell-type manipulation, leads to a tenth (or eighth) order algebraic equation for λ_i as function of ω and equations involving the proportionality of $\zeta_{ki}^{(n)}$. With the boundary conditions specified, the problem is entirely determined. The detailed statement of these conditions leads directly to ten (eight) simultaneous algebraic equations for the ten (eight) unknown constants. These equations involve the roots of $\lambda_i(\omega)$. Since the boundary conditions will be homogeneous, the determinant [84] or [94] of these equations must be zero for a non-trival solution. A computer procedure solves the equilibrium equation and the boundary value determinant simultaneously, for a chosen circumferential wave pattern, m. A trial value of ω is chosen to determine λ_i, and $\zeta_{ki}^{(n)}$ for the determinant, which must go to zero, if the chosen ω was a true eigenvalue (natural frequency). If not, we iterate by trial increments to find the values of ω which will converge the determinant through zero.

The number of iterations required for convergence is greatly reduced if good initial estimates are available. The solutions developed by Arnold and Warburton (4)(9) will be excellent for this purpose, as found by Forsberg (8), for clamped-end structures, using Flugge's bending theory. The lower branch of the frequency spectrum will lend itself to this very well.* Higher frequency terms will be closer together and will not need an approximate initial estimate. Convergence time is reasonable, according to Forsberg. With Flugge's bending theory, 14 eigenvalues ω and their corresponding eigenvectors ($\zeta_{ki}^{(n)}, \Gamma_i, \lambda_i$) were computed in one minute.

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*Initially, the successive eigenvalues found will be from the lower branch spectrum for axial structural waves of increasing node number (long waves). They should be nearly the same for either shell theory, as the effects of shear deformation are only evident at higher frequencies, i.e., at shorter structural wavelengths or the higher branches of the frequency spectrum for longer waves.

Forced Vibrations - Lagrange Equation - Orthogonality of Normal Modes; Comparison of Shell Theories

Once the mode shapes and natural frequencies are determined, the forced vibration problem is solved, within Hamilton's Integral, by a variation of the time function [116] , [121] , [122]. This is the procedural form of the normal mode method, when Hamilton's principle is applied. The Lagrange equations can be taken from the Hamilton integral in uncoupled form, if the orthogonality of normal modes can be verified.

Kraus and Kalnins (55) verified the appropriateness of a normal mode dynamic solution for a bending theory shell with homogeneous boundary conditions. They need symmetric equilibrium equations to prove orthogonality by a particular Green identity. Tso (56) verified the orthogonality result, using the equilibrium equations and boundary conditions generated from Hamilton's principle, for a classical theory. His proof rests upon homogeneous boundary conditions (leading to a separable (space-time) solution series) and a linear elastic structure, where the strain energy is a symmetric quadratic function of the displacement variables and their space derivatives. Love (27), proves orthogonality for the normal modes of vibration for a structure whose strain energy is a quadratic function of the strain. For his proof, he utilizes the symmetry of the energy expression for this linear elastic system that makes Betti's law valid. We will extend a similar orthogonality proof to both the 3 variable and 5 variable (SR) case. The need for this orthogonality (to uncouple the Lagrange equations) will manifest itself within Hamilton's integral. We will prove the orthogonality, therefore, also within Hamilton's integral.

When the orthogonality proof is completed, the uncoupled Lagrange equations to determine the time coordinate as a function of an arbitrary force input are developed , [128] . The deformation pattern for an impulsive force input is examined, [146] and the resulting stress intensity on the cylinder's surface is computed, [180] [192] The results of utilizing the bending theory or the shear deformation theory will be compared for various edge conditions, shell geometry, and truncations of the deformation series. This will be the subject of a subsequent report. Previous dynamic solutions for a finite cylinder have only been attempted for simply-supported edges, where the normal mode functions are known. This work has been done with a classical bending theory. No attempt has been made previously to solve any dynamic problem for a finite cylinder by a non-classical (SR) theory. The comparison to a bending theory solution, therefore, has not been previously attempted.

1.6 Proposed Tasks (Subsequent Work)

All work described below must be done for the 5 variable (SR) theory and the 3 variable (classical theory equations). Equations referred to are for the 5 variable theory.

1. Perform a Donnell type manipulation of the free vibration equilibrium equations [53] - [57] to get uncoupled equilibrium equations [108]. The deformation series [100] is substituted in these equations to produce the eigenvalue equation [79] and the eigenvector equation [81].

2. Computer program for the evaluation of the five roots of λ_i^2 in the eigenvalue equation of a trial function, ω_T.

   a) Program for the evaluation of the lowest roots of the Warburton approximate frequency equation for estimates of trial ω_T.

   b) Calculation program for the eigenvector function $\Gamma_{ki}^{(n)}$.

3. Computer program for the evaluation of the boundary value determinant D (ω^T), ([84] or [94]) and iterative search procedure for convergence to zero.

4. Calculation program for the eigenvector function, Γ_i

5. Integration computer program for the amplitude factor, A_{mn}, at a given particular impulsive loading [146] . This includes the generalized mass and generalized force integration.

6. Truncated series summation program for the stress intensity [194] .

7. Comparison of stress intensities for various shell geometries, edge conditions, and solution series truncation, according to the classical or the SR theory Graph plotting and analysis.

SECTION II CO-ORDINATE SYSTEMS

The cylinder has length L, inside radius (a-h) and thickness 2h. It is subjected to an arbitrary pressure $\bar{P}$ on its surfaces. The usual practical case encountered would be that the pressure was exerted only on the outer cylindrical surface, but our analysis is not so restricted. We do require, however, that the body undergo only translation with respect to the inertial axes ζ_1, ζ_2, and ζ_3. (See Figure 1). With this limitation, only certain patterns of surface pressures are permitted.*

A point m in the material is located by means of the vector $\bar{\rho}$ from the center of gravity B. The body fixed triad $\bar{e}_1$, $\bar{e}_2$, $\bar{e}_3$ serves as a reference coordinate system both for the surface pressure $\bar{P}$ and for the subsequent deformation vector $\bar{u}$, locating the displaced particle at m'.

The symbol $\bar{e}_k$ is used to represent any one of the vectors $\bar{e}_1$, $\bar{e}_2$, $\bar{e}_3$, which are seen to retain their orientation with respect to the inertial set $\bar{\zeta}_p$.

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*Note: This particular limitation is not really essential to the analysis, but is chosen for the convenience it affords in presenting the problem clearly. The general desire is to prevent rigid body rotation, so the requirement is that there be no resultant external moment. If, for example, the loads are radially directed, rotation is automatically prevented about the ζ_3 axis. To avoid rotation about the ζ_2 axis they must be symmetric with respect to $\phi = 0$, and to prevent rotation about the ζ_1 axis, uniform along the longitudinal axis.

If there were a rigid body rotation, in the subsequent application of Hamilton's principle, terms deriving from the variation of strain energy would not be affected. The variation in kinetic energy and work would produce additional terms, identifiable as being associated with the rigid body rotation, but otherwise not particularly useful.

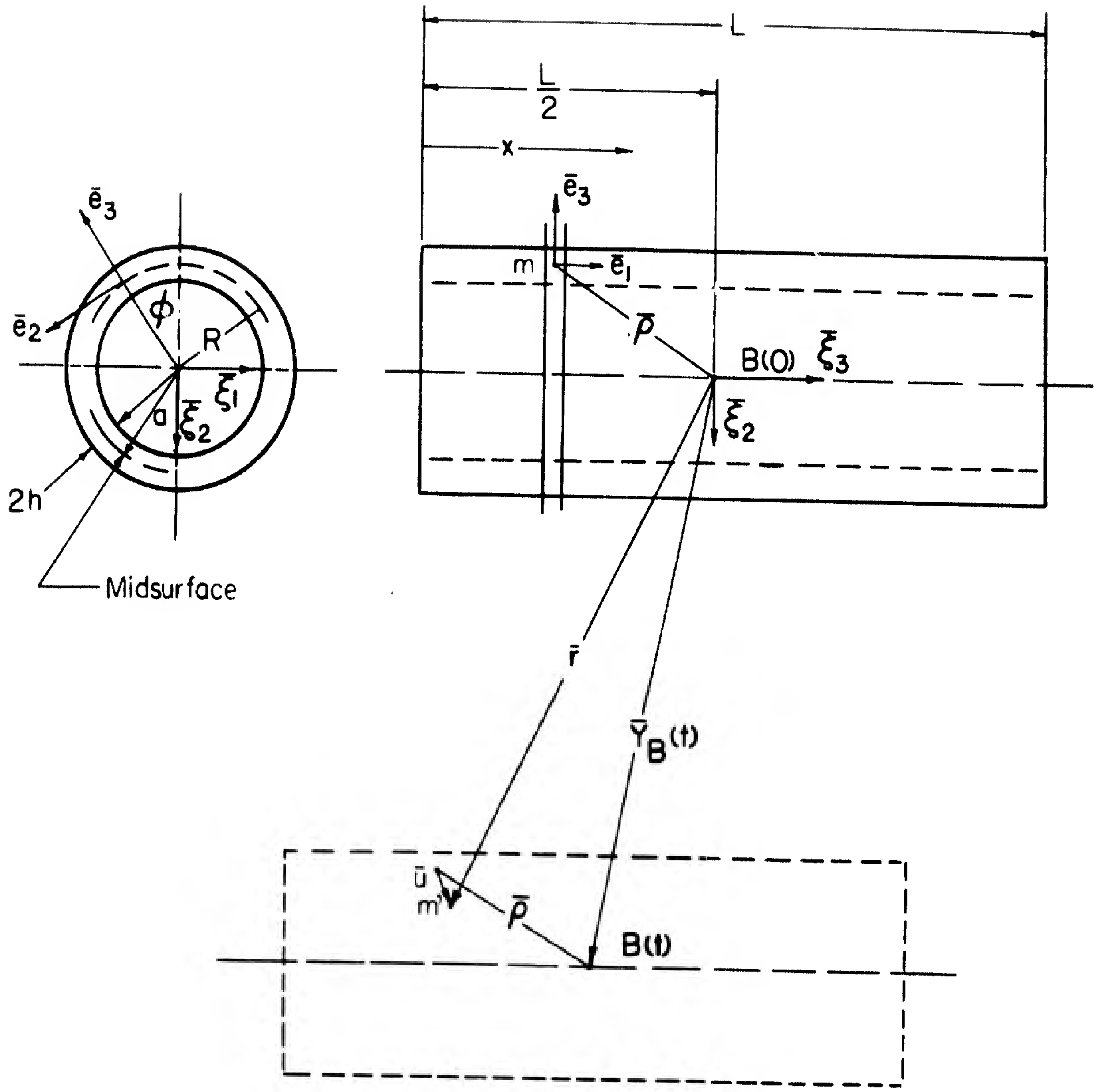

FIGURE 1: CO-ORDINATE SYSTEM

The point m may be visualized also as being located by the coordinates x, ϕ and z. x is taken as the distance from the left end, ϕ as positive counterclockwise from the vertical zero position, and z as the distance from the mid-surface, measured positive radially outward.

When the body has translated, the location of the center of gravity at time t_i is given by $Y_B(t)$. The location vector $\bar{\rho}$ is considered not to vary in time. The new position of m is given in inertial space by the vector $\bar{r}$. We may thus write:

$$(1) \qquad \bar{r}(x,\phi,z,t) = \bar{Y}_B(t) + \rho(x,\phi,z) + \bar{u}(x,\phi,z,t)$$

But we may express $\bar{\rho}$ as:

$$(2) \qquad \rho = R\bar{e}_3 + \left(\frac{L}{2} - x\right)(-\bar{I}_3)$$

or, more appropriately:

$$(3) \qquad \rho = (a+z)\bar{e}_3 + \left(\frac{L}{2} - x\right)(-\bar{I}_3)$$

so that equation (1) becomes

$$(4) \qquad \bar{r}(x,\phi,z,t) = \bar{Y}_B + (a+z)\bar{e}_3 - \left(\frac{L}{2} - x\right)\bar{I}_3 + \bar{u}$$

It should be noted that $\bar{e}_3$ is a function of ϕ. We now define the components of $\bar{Y}_B$ along the space-fixed axes as:

$$(5) \qquad Y_B(t)_p = \bar{Y}_B(t) \cdot \bar{I}_p$$

and along the body fixed axes as:

$$(6) \qquad Y_B(t)_K = \bar{Y}_B(t) \cdot \bar{e}_K$$

Equation (4) now may be written:

$$(7) \qquad \bar{r} = Y_B(t)_K \bar{e}_K + (a+z)\bar{e}_3 - \left(\frac{L}{2} - x\right)\bar{I}_3 + u_K \bar{e}_K$$

with the first and last terms understood to be in indicial notation.

Differentiating with respect to t, the second and third terms drop out and

(8) $$\dot{\bar{r}} = \frac{d}{dt}(Y_B)_K \bar{e}_K + \frac{\partial u_K}{\partial t} \bar{e}_K$$

Repeating the $\partial/\partial t$ operation, and employing the conventional dot notation:

(9) $$\ddot{\bar{r}} = (\ddot{Y}_B)_K \bar{e}_K + \ddot{u}_K \bar{e}_K$$

Expression (8) will be useful in formulating the system kinetic energy, and equation (9) in understanding the part played by the rigid body motion.

SECTION III DETERMINATION OF STRESS EQUATIONS OF MOTION AND STRESS BOUNDARY CONDITIONS

Hamilton's principle states that the motion of the system from time t_o to the time t_f is such that the line integral I is an extremum for the path of motion:

(10) $$I = \int_{t_o}^{t_f} (T - U + W)\, dt$$

The integral I is defined by equation (10), in which the kinetic energy T is:

(11) $$T = \int_V \frac{1}{2} \gamma \dot{\bar{r}} \cdot \dot{\bar{r}}\, dV$$

The system potential (strain)energy is:

(12) $$U = \int_V \frac{1}{2} \sigma_{ik} \epsilon_{ik}\, dV$$

and the term W is taken as:

(13) $$W = \int_{\Sigma} \bar{P} \cdot \bar{r} \, d\Sigma$$

In equations (11) - (13), we use

γ = density per unit mass

σ_{ik} = stress tensor

ϵ_{ik} = strain tensor

dV = unit volume

$d\Sigma$ = unit surface area

and $\bar{P}$, $\bar{r}$ are used as before. Some explanation of these terms is helpful.

The unit volume is defined as:

(14) $$dV = dS_1 dS_2 dS_3$$

where dS_k is the incremental distance along the body-fixed vector $\bar{e}_k$. The origin of the orthogonal body-fixed vector triad $\bar{e}_1$, $\bar{e}_2$, and $\bar{e}_3$ is formally located by the body referenced coordinates λ_1, λ_2, and λ_3. In the particular cylindrical coordinate system of Figure 1;

$\lambda_1 = x$ $\quad$ $dS_1 = 1 \quad dx$

$\lambda_2 = \phi$ $\quad$ $dS_2 = R d\phi$

$\lambda_3 = R = a+z$ $\quad$ $dS_3 = 1 \quad dz$

We will find it more convenient; however, to retain a more general symbolism:

(15) $$dS_K = h_K d\lambda_K$$

In the form of (15), we can use Hamilton's principle for any orthogonal curvilinear co-ordinate system.

The term $d\Sigma$ requires additional explanation. In equation (13) it is understood that the integration is taken over the entire surface of the body under study. In the actual case, which involves a cylinder, the surface occurs at either end of the cylinder, and on the inner and outer curved surfaces. At these boundaries, $d\Sigma$ has different forms:

Boundary	$d\Sigma$	
$\lambda_1 = x = 0$	$Rd\phi dz$	$= ds_2 ds_3$
$\lambda_1 = x = L$	$Rd\phi dz$	$= ds_2 ds_3$
$\lambda_3 = R = a+h$	$(a+h)\ d\phi dx$	$= ds_2]^+\ ds_1$
$\lambda_3 = R = a-h$	$(a-h)\ d\phi dx$	$= ds_2]^-\ ds_1$

The boundary is encountered in the first two cases, by traversing the body in the direction $\bar{e}_1$. In the latter two cases, the boundary is encountered by traversing in the direction $\bar{e}_3$. No boundary exists for a traversal in the direction $\bar{e}_2$.

In each of these cases, the surface $d\Sigma$ could be called $d\Sigma_1$ and $d\Sigma_3$ to reference the direction of traversal. Also, and fortuitously, the elemental surface at the actual boundary are normal to the direction of traversal.

In order to preserve the generality of the indicial notation, we do not assume that the surface $d\Sigma$ must lie in a plane normal to a body-fixed coordinate vector. Instead, we take the surface $d\Sigma$ as in figure 2, so oriented that $\bar{n}$ is normal to $d\Sigma$. The projection of $d\Sigma$ on any coordinate plane is found from:

(16) $$(d\Sigma)\, \bar{n}\cdot\bar{e}_k = ds_i\, ds_j \quad (i \neq j = k)$$

Furthermore, we assign a subscript to $d\Sigma$, e.g., $d\Sigma_k$ to reference the direction traversed in the body to encounter the surface. Thus (in this case) equation (13) should be understood to imply:

$$W = \int_{\Sigma} \bar{P} \cdot \bar{r} \, d\Sigma_1 + \int_{\Sigma} \bar{P} \cdot \bar{r} \, d\Sigma_3$$

The stress and strain tensors σ_{ik} and ϵ_{ik} follow the conventional notation that the first subscript indicates the plane on which the stress acts (normal to $\bar{e}_i$) and the second subscript the direction in which it acts.

Now, the extremum requirement on equation (10) is equivalent to:

(17) $$\delta I = 0$$

Setting L = T + U - W, equation (17) becomes:

(18) $$\delta \int_{t_0}^{t_f} L \, dt = 0$$

L is known to be a function of the displacement coordinates $Y_B(t)_p$ and u_k (x,ϕ,z,t) as well as of their appropriate space and time derivatives. (See Appendix A for details). Conventionally, u_k and Y_{Bp} are assumed in the form of a minimizing function plus a perturbation function:

(19) $$u_k(x,\phi,z,t,\alpha) = (u_k)_{MIN} + \alpha_k \eta_k(x,\phi,z,t) \; *$$

(20) $$Y_{Bp}(t,\beta) = (Y_{Bp})_{MIN} + \beta_p \zeta_p(t) \; *$$

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*See Reference 47

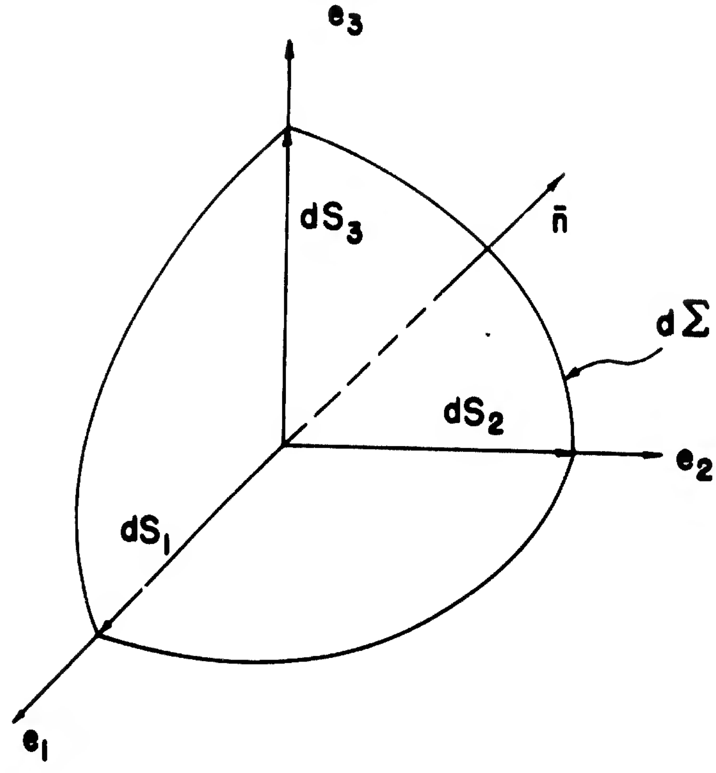

FIGURE 2: BOUNDARY SURFACE AREA $d\Sigma$

The terms η_k and ζ_p are the perturbation functions and α_k and β_p are the parameters to be varied. (Indicial notation is not implied here). The minimum forms $(u_k)_{MIN}$ and $(Y_B)_{MIN}$ are the forms of u_k and Y_{Bp} when α_k and β_p are set equal to zero. The perturbation functions η_k and ζ_p are required to be zero at t_o and t_f. The variation δI thus requires that:

(21) $$\delta I\ (\alpha_K, \beta_P) = \left.\frac{\partial I}{\partial \alpha_\ell}\right|_{\alpha_\ell = 0} d\alpha_\ell + \left.\frac{\partial I}{\partial \beta_\ell}\right|_{\beta_\ell = 0} d\beta_\ell \quad **$$

The application of equation (21) involves differentiation under the integral sign in (10) of the form:

(22) $$\left.\left(\frac{\partial I}{\partial \dot{u}_K}\right)\left(\frac{\partial \dot{u}_K}{\partial \alpha_K}\right)\right|_{\alpha_K = 0} d\alpha_K \quad \text{or} \quad \left.\left(\frac{\partial U}{\partial \sigma_{IK}}\right)\left(\frac{\partial \sigma_{IK}}{\partial \epsilon_{IK}}\right)\left(\frac{\partial \epsilon_{IK}}{\partial\left(\frac{\partial u_K}{\partial S_i}\right)}\right)\left(\frac{\partial\left(\frac{\partial u_K}{\partial S_i}\right)}{\partial \alpha_K}\right)\right|_{\alpha_K = 0} d\alpha_K$$

so that we must also use the notation:

(23) $$\delta u_K = \left.\frac{\partial u_K}{\partial \alpha_K}\right|_{\alpha_K = 0} d\alpha_K = \eta_K\, d\alpha_K$$

$$\delta \dot{u}_K = \left.\frac{\partial \dot{u}_K}{\partial \alpha_K}\right|_{\alpha_K = 0} d\alpha_K = \dot{\eta}_K\, d\alpha_K = \frac{\partial}{\partial t}(\delta u_K)$$

Expression (13) is an artificial form, chosen so that δW will be equal to the virtual work done by the surface tractions $\overline{P}$ during a virtual displacement. Equation (18) becomes:

(24) $$\delta I = \delta \int_t T dt - \delta \int_t U dt + \delta \int_t W dt = 0$$

**Indicial notation is implied

Substituting (11), (12) and (13) in (24):

(25)
$$\int_t\int_V\left[\delta\left(\tfrac{1}{2}\gamma\dot{\bar{r}}\cdot\dot{\bar{r}}\right)-\delta\left(\tfrac{1}{2}\sigma_{ik}\epsilon_{ik}\right)\right]dV\,dt+\int_t\int_\Sigma\bar{p}\cdot\delta\bar{r}\,d\Sigma\,dt=0$$

The actual variation of equation (25) is performed in Appendix A. The result is:

(26)
$$\delta\int_t L\,dt=\int_t\int_V\left[\frac{\partial\sigma_{ik}}{\partial s_i}+\sigma_{ik}\left(\bar{e}_t\cdot\frac{\partial\bar{e}_i}{\partial s_t}\right)+\sigma_{ti}\left(\bar{e}_k\cdot\frac{\partial\bar{e}_i}{\partial s_t}\right)-\gamma\left(\ddot{Y}_B(t)_k+\ddot{U}_k\right)\right]\delta U_k\,dV\,dt$$
$$+\int_t\int_{\Sigma_i}\left[P_k-\bar{n}\cdot\bar{e}_i\left\{(\sigma_{ik}\delta u_k)^{\lambda_i MAX}+(\sigma_{ik}\delta u_k)^{\lambda_i MIN}\right\}\right]d\Sigma_i\,dt$$
$$+\int_t\left[F(t)_p-M\ddot{Y}_B(t)_p\right]\delta Y_B(t)_p\,dt\qquad=0$$

---

* The equations are written to be most accessible for a free-body in space. For a constrained cylinder, the deformation (and strain energy) would referred to an inertial reference. As rigid body motion does not contribute to the strain energy of deformation, the Hamilton integral, as developed in Appendix A, would be acceptable for a constrained cylinder. The form of (26) would change;

In the first and second integral on the right hand side:

- δu_k would become δr_k
- $\ddot{u}_k+\ddot{Y}_B(t)_k$ would be $\ddot{r}_k$, as the c.g. acceleration is absorbed in the deformation coordinate

The third integral would be absorbed. It depends on the unspecified, (for the constrained case), motion of the c.g. of the body

The stress, σ_{ik}, would be developed as functions of the displacement coordinate, r_k, and its space derivatives.

In equation (26), the first integral provides the stress-motion equilibrium equations for a volumetric element. The second integral prescribes the boundary conditions and the third integral demonstrates the equilibrium equations for rigid body motion.

In the second integral, we interpret the subscript i as being associated with the traversal of the direction $\bar{e}_i$ to encounter a boundary surface element $d\Sigma_i$. The equation requires an evaluation of (σ_{ik} δ u_k) at the extreme values of the body referenced coordinate λ_i.

The stress-motion equilibrium equations (without the boundary conditions and rigid-body equations) may be separately obtained by setting the time integral of the virtual work equal to zero:

(27) $$\int_t W_{VIRT} = \int_t \int_V \bar{F}_{DYN} \cdot \delta \bar{r}\, dV dt = 0$$

In the conventional form, using τ as the stress tensor at a point, and ∇ as the del operator:

(28) $$\int_t W_{VIRT} = \int_t \int_V (\nabla \cdot \tilde{\tau} - \sigma \ddot{\bar{r}}) \cdot \delta \bar{r}\, dV dt = 0 \ *$$

---

*Note: This is shown as follows: $\nabla \cdot \tilde{\tau} = \bar{e}_k \frac{\partial(\;)}{\partial \xi_k} \cdot \sigma_{ij} \bar{e}_i \bar{e}_j$

in which σ_{ij} $\bar{e}_i\bar{e}_j$ is the stress dyad. Carrying out the differentiation:

$$\nabla \cdot \tilde{\tau} = \bar{e}_k \cdot \left[\frac{\partial \sigma_{ij}}{\partial \xi_k} \bar{e}_i \bar{e}_j + \sigma_{ij} \frac{\partial \bar{e}_i}{\partial \xi_k} \bar{e}_j + \sigma_{ij} \bar{e}_i \frac{\partial \bar{e}_j}{\partial \xi_k} \right]$$

continuing,

$$\nabla \cdot \tilde{\tau} = \bar{e}_k \cdot \bar{e}_i \frac{\partial \sigma_{ij}}{\partial \xi_k} \bar{e}_j + \left(\bar{e}_k \cdot \frac{\partial \bar{e}_i}{\partial \xi_k}\right) \sigma_{ij} \bar{e}_j + (\bar{e}_k \cdot \bar{e}_i) \sigma_{ij} \frac{\partial \bar{e}_j}{\partial \xi_k}$$

Using the Kronecker delta:

$$\nabla \cdot \tilde{\tau} = \delta_{ki} \frac{\partial \sigma_{ij}}{\partial \xi_k} \bar{e}_j + \left(\bar{e}_k \cdot \frac{\partial \bar{e}_i}{\partial \xi_k}\right) \sigma_{ij} \bar{e}_j + (\delta_{ki}) \sigma_{ij} \frac{\partial \bar{e}_j}{\partial \xi_k}$$

(Continuation of Note on preceding page)

or, contracting the first and third terms:

$$\nabla\cdot\tilde{\tau} = \frac{\partial\sigma_{Ki}}{\partial S_K}\bar{e}_j + \left(\bar{e}_K\cdot\frac{\partial\bar{e}_i}{\partial S_K}\right)\sigma_{ij}\,\bar{e}_j + \sigma_{Kj}\frac{\partial\bar{e}_j}{\partial S_K}$$

We now make use of the idemfactor $I = \delta_{tn}\bar{e}_t\bar{e}_n$ which has the property $I\cdot\bar{e}_1 = \bar{e}_1$

$$\sigma_{Kj}\,\delta t_m\,\bar{e}_t\bar{e}_m\cdot\frac{\partial\bar{e}_j}{\partial S_K} = \sigma_{Kj}\,\delta t_m\,\bar{e}_t\left(\bar{e}_m\cdot\frac{\partial\bar{e}_j}{\partial S_K}\right)$$

$$= \sigma_{Kj}\,\bar{e}_m\left(\bar{e}_n\cdot\frac{\partial\bar{e}_j}{\partial S_K}\right)$$

Replace m by t and interchange j and t to find

$$= \sigma_{Kj}\,\bar{e}_j\left(\bar{e}_j\cdot\frac{\partial\bar{e}_t}{\partial S_K}\right) = \sigma_{Kt}\,\bar{e}_j\cdot\frac{\partial\bar{e}_t}{\partial S_K}\,\bar{e}_j$$

Return to the expression for $\nabla\cdot\tilde{\tau}$ to find:

$$\nabla\cdot\tilde{\tau} = \frac{\partial\sigma_{Kj}}{\partial S_K} + \left(\bar{e}_K\cdot\frac{\partial\bar{e}_i}{\partial S_K}\right)\sigma_{ij}\bar{e}_j + \sigma_{Kt}\,\bar{e}_j\cdot\frac{\partial\bar{e}_t}{\partial S_K}\,\bar{e}_j$$

Replacing the index j by k, k by t and t by i:

$$\nabla\cdot\tilde{\tau} = \left[\frac{\partial\sigma_{tK}}{\partial S_t} + \bar{e}_t\cdot\frac{\partial\bar{e}_i}{\partial S_t}\sigma_{iK} + \sigma_{it}\bar{e}_K\cdot\frac{\partial\bar{e}_i}{\partial S_t}\right]\bar{e}_K$$

In the first expression replace t by i, and note that $\sigma_{ti} = \sigma_{it}$ to obtain

$$\nabla\cdot\tilde{\tau} = \left[\frac{\partial\sigma_{iK}}{\partial S_i} + \bar{e}_t\cdot\frac{\partial\bar{e}_i}{\partial S_t}\sigma_{iK} + \sigma_{ti}\bar{e}_K\cdot\frac{\partial\bar{e}_i}{\partial S_t}\right]\bar{e}_K = (\nabla\cdot\tilde{\tau})_K\,\bar{e}_K$$

The balance of the proof follows readily from the definition of $\bar{r}$.

The next step is to isolate the z dependence of the deformation coordinate $u_k(x,\phi,z,t)$, and to integrate equation (26) with respect to z over the thickness of the cylinder.

This will result in stress-motion equilibrium equations in terms of shell variables, i.e., as functions of x and ϕ, where the point x,ϕ is a location on the cylinder mid-surface. This is then a two-dimensional theory.

The first step is to assume that the deformation coordinate u_k can be expanded in the form:

(29) $$U_k(x,\phi,z,t) = \sum_{n=0}^{\infty} U_k^{(n)}(x,\phi,t)\, z^n \quad *$$

where the superscript (n) indicates a mid-surface property. Several prior investigators have found it appropriate to truncate this series as follows

(30) $$\begin{aligned} U_1(x,\phi,z,t) &= U_1^{o}(x,\phi,t) + z\,U_1^{1}(x,\phi,t) \\ U_2(x,\phi,z,t) &= U_2^{o}(x,\phi,t) + z\,U_2^{1}(x,\phi,t) \quad ** \\ U_3(x,\phi,z,t) &= U_3^{o}(x,\phi,t) \end{aligned}$$

---

*See Reference 35

**For a constrained cylinder, with inertially referenced deformation, the following condition holds;

$$u_k + Y_{B,k} = r_k$$
$$u_k^{o} + z u_k^{1} + Y_{B,k} = r_k^{o} + z r_k^{1}$$
$$(u_k^{o} + Y_{B,k}) + z u_k^{1} = r_k^{o} + z r_k^{1}$$

The deformation for constrained cylinder is a function of the inertially referenced, mid-surface motion, r_k^{n}.

In equations (30), the superscript o refers to a mid-surface deformation and the superscript 1 to a mid-surface rotation.

Since we require arbitrary variations in δu_k, it is necessary to develop equivalent expressions involving δu_k^o and δu_k^1. This is done as follows:

We have taken (see equation 19, repeated below for convenience):

(19) $$U_k(x,\phi,z,t,\alpha_k) = U_k(x,\phi,z,t,0)_{MIN} + \alpha_k \eta_k(x,\phi,z,t)$$

With the assumed z dependence:

(31) $$U_k(x,\phi,z,t,\alpha_k) = U_k^o(x,\phi,t,\alpha_k) + z\, U_k^1(x,\phi,t,\alpha_k)$$

Now, if the variations of u_k^o and u_k^1 are similarly defined:

(32) $$U_k^o(x,\phi,t,\alpha_k) = U_k^o(x,\phi,t,0)_{MIN} + \alpha_k \eta_k^o(x,\phi,t)$$

(33) $$U_k^1(x,\phi,t,\alpha_k) = U_k^1(x,\phi,t,0)_{MIN} + \alpha_k \eta_k^1(x,\phi,t)$$

We note the η_k^n (x,ϕ,t) are zero at t_o and t_f, but are arbitrary at other times.

Since any variation is still defined as:

(34) $$\left.\frac{\partial(\ )}{\partial \alpha_\ell}\right|_{\alpha_\ell=0} d\alpha_\ell + \left.\frac{\partial(\ )}{\partial \beta_\ell}\right|_{\beta_\ell=0} d\beta_\ell$$

We operate on (32) and (33) to yield:

(35) $$\delta U_k^o = \eta_k^o(x,\phi,t)\, d\alpha_k$$

$$\delta U_k^1 = \eta_k^1(x,\phi,t)\, d\alpha_k$$

Since:

(36) $$U_k(x,\phi,z,t,\alpha_k) = \left[U_k^0(x,\phi,t,0) + zU_k^1(x,\phi,t,0)\right] + \alpha_k\left[\eta_k^0(x,\phi,t) + z\eta_k^1(x,\phi,t)\right]$$

it becomes apparent that

(37) $$\delta U_k = \delta U_k^0 + z\,\delta U_k^1$$

We also require the volume element, which for cylindrical coordinates becomes:

(38) $$dV = (a+z)\,d\phi\,dx\,dt = \left(1+\frac{z}{a}\right)a\,d\phi\,dx\,dt$$

As has been explained, the boundary conditions may be related to several surfaces. We emply the table developed in section II to denote the proper forms of $d\Sigma_i$.

Using (37), (38) and $d\Sigma_i$ as needed, equation (26) becomes:

(39)

$$
\begin{aligned}
0 = & \int_t \int_x \int_\phi \left[\int_{-h}^{+h} \left\{ \frac{\partial \sigma_{iK}}{\partial S_K} + \sigma_{iK}\left(\bar{e}_t \cdot \frac{\partial \bar{e}_i}{\partial S_t}\right) + \sigma_{ti}\left(\bar{e}_K \cdot \frac{\partial \bar{e}_i}{\partial S_t}\right) \right. \right. \\
& \left. \left. - \gamma\left(\ddot{Y}_{BK} + \ddot{U}_K^o + z\ddot{U}_K^1\right) \right\} \left\{ \delta U_K^o + z\,\delta U_K^1 \right\} \left(1 + \frac{z}{a}\right) dz \right] a\, d\phi\, dx\, dt \\
& + \int_t \int_x \int_\phi \left(P_K - \sigma_{iK}\bar{e}_3 \cdot \bar{e}_i\right)_{z=h} \left(\delta U_K^o + hU_K^1\right)(a+h)\, d\phi\, dx\, dt \\
& + \int_t \int_x \int_\phi \left(P_K + \sigma_{iK}\bar{e}_3 \cdot \bar{e}_i\right)_{z=-h} \left(\delta U_K^o - hU_K^1\right)(a-h)\, d\phi\, dx\, dt \\
& + \int_t \int_\phi \left[\int_{-h}^{h} \left\{ P_K + \sigma_{iK}\bar{e}_1 \cdot \bar{e}_i \right\}_{x=0} \left\{ \delta U_K^o + z\,\delta U_K^1 \right\}_{x=0} \left(1 + \frac{z}{a}\right) dz \right] a\, d\phi\, dt \\
& + \int_t \int_\phi \left[\int_{-h}^{h} \left\{ P_K - \sigma_{iK}\bar{e}_1 \cdot \bar{e}_i \right\}_{x=L} \left\{ \delta U_K^o + z\,\delta U_K^1 \right\}_{x=L} \left(1 + \frac{z}{a}\right) dz \right] a\, d\phi\, dt \\
& + \int_t \left\{ F_p(t) - M\ddot{Y}_B(t)_p \right\} \delta Y_B(t)_p\, dt
\end{aligned}
$$

The detailed treatment of equation (39) is given in Appendix B. It is necessary, on a term by term basis, to assign the values 1, 2 and 3 to the index k, as well as to completely evaluate such terms as $\bar{e}_k \cdot \frac{\partial \bar{e}_i}{\partial S_t}$. For a better understanding of the form of the results, we consider equations (39) on a term by term basis.

The first part requires that:

(40)
$$0=\int_t\int_x\int_\phi\Big[\Big[\int_{-h}^{+h}\Big\{\frac{\partial\sigma_{ik}}{\partial S_i}+\sigma_{ik}\left(\bar{e}_t\cdot\frac{\partial\bar{e}_i}{\partial S_t}\right)+\sigma_{ti}\left(\bar{e}_k\cdot\frac{\partial\bar{e}_i}{\partial S_t}\right)$$
$$-\gamma(\ddot{Y}_{B_k}+\ddot{u}_k^0+z\ddot{u}_k^1)\Big\}(\delta u_k^0+z\,\delta u_k^1)\left(1+\frac{z}{a}\right)dz\Big]a\,d\phi\,dx\,dt$$

Manipulation of equation (40) results in the following equation assembled from terms developed in Appendix B:

(41)
$$\int_t\int_x\int_\phi\Big[\Big\{\int_{-h}^{h}\Big[\frac{\partial\sigma_{11}}{\partial S_1}+\frac{\partial\sigma_{21}}{\partial S_2}+\frac{\partial\sigma_{31}}{\partial S_3}+\frac{1}{a+z}\sigma_{31}-\gamma(\ddot{Y}_{B_1}+\ddot{U}_1^0+z\ddot{U}_1^1)\Big]\left(\frac{a+z}{a}\right)dz\Big\}\delta U_1^0$$
$$+\Big\{\int_{-h}^{h}\Big[\frac{\partial\sigma_{11}}{\partial S_1}+\frac{\partial\sigma_{21}}{\partial S_2}+\frac{\partial\sigma_{31}}{\partial S_3}+\frac{1}{a+z}\sigma_{31}-\gamma(\ddot{Y}_{B_1}+\ddot{U}_1^0+z\ddot{U}_1^1)\Big]z\left(1+\frac{z}{a}\right)dz\Big\}\delta U_1^1$$
$$+\Big\{\int_{-h}^{h}\Big[\frac{\partial\sigma_{12}}{\partial S_1}+\frac{\partial\sigma_{22}}{\partial S_2}+\frac{\partial\sigma_{32}}{\partial S_3}+\frac{\sigma_{32}}{a+z}+\frac{\sigma_{23}}{a+z}-\gamma(\ddot{Y}_{B_2}+\ddot{U}_2^0+z\ddot{U}_2^1)\Big]\left(1+\frac{z}{a}\right)dz\Big\}\delta U_2^0$$
$$+\Big\{\int_{-h}^{h}\Big[\frac{\partial\sigma_{12}}{\partial S_1}+\frac{\partial\sigma_{22}}{\partial S_2}+\frac{\partial\sigma_{32}}{\partial S_3}+\frac{\sigma_{32}}{a+z}+\frac{\sigma_{23}}{a+z}-\gamma(\ddot{Y}_{B_2}+\ddot{U}_2^0+z\ddot{U}_2^1)\Big]z\left(1+\frac{z}{a}\right)dz\Big\}\delta U_2^1$$
$$+\Big\{\int_{-h}^{h}\Big[\frac{\partial\sigma_{13}}{\partial S_1}+\frac{\partial\sigma_{23}}{\partial S_2}+\frac{\partial\sigma_{33}}{\partial S_3}+\frac{\sigma_{33}-\sigma_{22}}{a+z}-\gamma(\ddot{Y}_{B_3}+\ddot{U}_3^0)\Big]\left(1+\frac{z}{a}\right)dz\Big\}\delta U_3^0\Big]a\,d\phi\,dx\,dt$$
$$=0$$

Eꟹuation (41) is thus seen to contain in itself a series of five terms, which add up to zero because this is implicit in the independence of the terms in (39). In addition, each term in equation (41) is multiplied by an arbitrary integrating term, of the form δu_k^n. Therefore, each one of the terms, after integrating must equal zero. This result is obtained in Appendix B and summarized below.

(42)

$$\begin{aligned}\int_t\int_x\int_\phi \Big[& \Big\{ \frac{\partial\sigma_{11}^0}{\partial x} + \frac{1}{a}\frac{\partial\sigma_{21}^0}{\partial\phi} + P_1^+\left(1+\frac{h}{a}\right) - 2\gamma h\ddot{Y}_B(t)\cdot\bar{e}_1 - 2\gamma h\ddot{U}_1^0 - \frac{2\gamma h^3}{3a}\ddot{U}_1^1 \Big\}\delta U_1^0 \\ & + \Big\{ \frac{\partial\sigma_{11}^1}{\partial x} + \frac{1}{a}\frac{\partial\sigma_{21}^1}{\partial\phi} + P_1^+(h)\left(1+\frac{h}{a}\right) - \sigma_{13}^0 - \frac{2\gamma h^3}{3a}\ddot{Y}_B(t)\cdot\bar{e}_1 - \frac{2\gamma h^3}{3a}\ddot{U}_1^0 - \frac{2\gamma h^3}{3}\ddot{U}_1^1 \Big\}\delta U_1^1 \\ & + \Big\{ \frac{\partial\sigma_{12}^0}{\partial x} + \frac{1}{a}\frac{\partial\sigma_{22}^0}{\partial\phi} + \frac{\sigma_{33}^0}{a} + P_2^+\left(1+\frac{h}{a}\right) - 2\gamma h\ddot{Y}_B(t)\cdot\bar{e}_2 - 2\gamma h\ddot{U}_2^0 - \frac{2\gamma h^3}{3a}\ddot{U}_2^1 \Big\}\delta U_2^0 \\ & + \Big\{ \frac{\partial\sigma_{12}^1}{\partial x} + \frac{1}{a}\frac{\partial\sigma_{22}^1}{\partial\phi} + P_2^+\left(1+\frac{h}{a}\right)h - \sigma_{23}^0 - \frac{2\gamma h^3}{3a}\ddot{Y}_B(t)\cdot\bar{e}_2 - \frac{2\gamma h^3}{3a}\ddot{U}_2^0 - \frac{2\gamma h^3}{3}\ddot{U}_2^1 \Big\}\delta U_2^1 \\ & + \Big\{ \frac{\partial\sigma_{13}^0}{\partial x} + \frac{1}{a}\frac{\partial\sigma_{23}^0}{\partial\phi} - \frac{\sigma_{22}^0}{a} + P_3^+\left(1+\frac{h}{a}\right) - 2\gamma h\ddot{Y}_B(t)\cdot\bar{e}_3 - 2\gamma h\ddot{U}_3^0 \Big\}\delta U_3^0 \Big] a\,d\phi\,dx\,dt \\ & \equiv 0\end{aligned}$$

In equation (42), it is to be understood that there is no surface traction on the inside of the cylinder. The terms P_1^+, P_2^+ and P_3^+ represent the surface tractions on the outside of the cylinder. The meaning of such terms as σ_{11}^0, σ_{11}^1, etc. is shown on Figure 3. Generally speaking, the superscript o refers to a force resultant, and the superscript 1 to a mc.nent resultant. The first index designates the plane on which the resultant acts, and the second index the direction in which it acts. Stress resultants are to be taken as "per unit length" of surface on which they act.

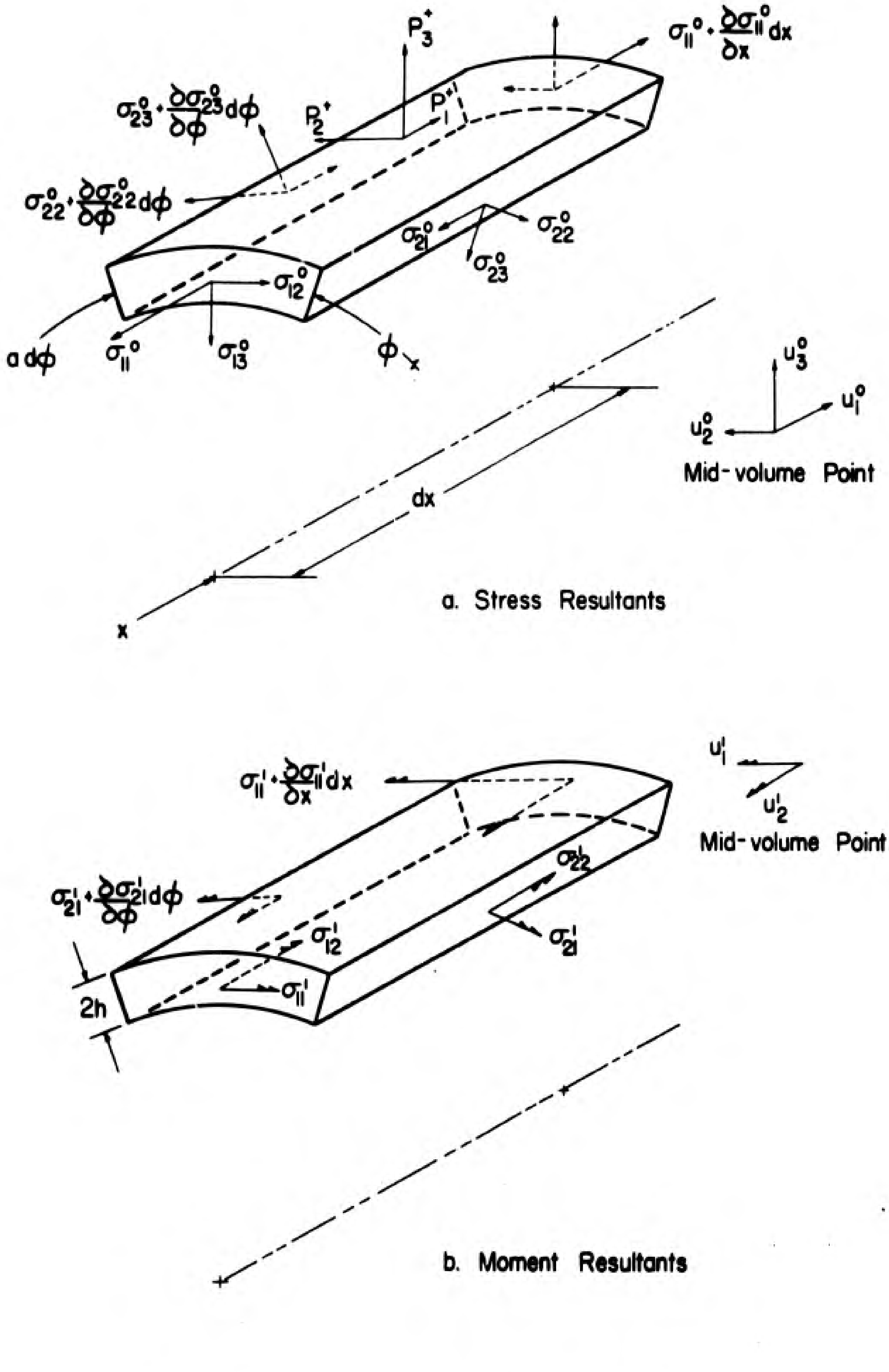

FIGURE 3: STRESS AND MOMENT RESULTANTS ON CYLINDRICAL ELEMENT

These equations may be more familiar if the notations for stress resultants N and Q, and moment resultants M, as discussed in Appendix B, are employed:

(43)

$$\int_t \int_x \int_\phi \Bigg[\left\{ \frac{\partial N_x}{\partial x} + \frac{1}{a}\frac{\partial N_{\phi x}}{\partial \phi} + P_1^+\left(1+\frac{h}{a}\right) - 2\gamma h\left(\ddot{Y}_B(t)\cdot\bar{e}_1 + \ddot{U}_1^o + \frac{h^2}{3a}\ddot{U}_1'\right)\right\}\delta U_1^o$$

$$+\left\{ \frac{\partial M_x}{\partial x} + \frac{1}{a}\frac{\partial M_{\phi x}}{\partial \phi} - Q_x + P_1^+\left(1+\frac{h}{a}\right)h - \frac{2\gamma h^3}{3}\left(\frac{\ddot{Y}_B(t)\cdot\bar{e}_1 + U_1^o}{a} + \ddot{U}_1'\right)\right\}\delta U_1'$$

$$+\left\{ \frac{\partial N_{x\phi}}{\partial x} + \frac{1}{a}\frac{\partial N_{\phi\phi}}{\partial \phi} + Q_\phi + P_2^+\left(1+\frac{h}{a}\right) - 2\gamma h\left(\ddot{Y}_B(t)\cdot\bar{e}_2 + \ddot{U}_2^o + \frac{h^2}{3a}\ddot{U}_2'\right)\right\}\delta U_2^o$$

$$+\left\{ \frac{\partial M_{x\phi}}{\partial x} + \frac{1}{a}\frac{\partial M_\phi}{\partial \phi} - Q_\phi + P_2^+\left(1+\frac{h}{a}\right)h - \frac{2\gamma h^3}{3}\left(\frac{\ddot{Y}_B(t)\cdot\bar{e}_2 + \ddot{U}_2^o}{a} + \ddot{U}_2'\right)\right\}\delta U_2'$$

$$\left\{ \frac{\partial Q_x}{\partial x} + \frac{1}{a}\frac{\partial Q_\phi}{\partial \phi} - \frac{N_{\phi\phi}}{a} + P_3^+\left(1+\frac{h}{a}\right) - 2\gamma h\left(\ddot{Y}_B(t)\cdot\bar{e}_3 + \ddot{U}_3^o\right)\right\}\delta U_3^o \Bigg] a\, d\phi\, dx\, dt$$

$$\equiv 0$$

In equations 42, 43, $\ddot{u}_k^o$ represent the second derivative (with respect to time) of the mid-surface deformation and $\ddot{u}_k^1$ represent the second derivative (with respect to time) of the mid-surface rotations.

We note that equation 42 (or, alternately, 43) in general forms a series

(44) $$\int_t \int_x \int_\phi \psi_k^n \, \delta U_k^n \, a\, d\phi\, dx\, dt \equiv 0$$

in which $k = 1,2,3$ and $n = 0,1$. (There is no term corresponding to δu_3^1.) Since, in general $\delta u_k^n \neq 0$, this requires that the individual terms $\psi_k^n \equiv 0$ identically. The set of five equations that results is known as the group of "equilibrium equations".

Returning to equation (39), the second, third, fourth and fifth integrals refer to the boundary conditions at the following boundaries:

At z = h	Outer Cylindrical Surface
At z = -h	Inner Cylindrical Surface
At x = 0	Left end
At x = L	Right end

The results are found in Appendix B, and the end products summarized below:

At z = h

(45)

$$\int_t \int_x \int_\phi \Big[(P_1^+ - \sigma_{31})_{z=h} (\delta U_1^\circ) + (P_1^+ - \sigma_{31})_{z=h} (\delta U_1' * h) + (P_2^+ - \sigma_{32})_{z=h} (\delta U_2^\circ) + (P_2^+ - \sigma_{32})_{z=h} (\delta U_2' * h) + (P_3^+ - \sigma_{33})_{z=h} (\delta U_3^\circ) \Big] (a+h)\, d\phi\, dx\, dt$$

At z = -h

(46)

$$\int_t \int_x \int_\phi \Big[(P_1^- + \sigma_{31})_{z=-h} (\delta U_1^\circ) + (P_1^- + \sigma_{31})_{z=-h} (\delta U_1' * h) + (P_2^- + \sigma_{32})_{z=-h} (\delta U_2^\circ) + (P_2^- + \sigma_{32})_{z=-h} (\delta U_2' * h) + (P_3^- + \sigma_{33})_{z=-h} (\delta U_3^\circ) \Big] (a-h)\, d\phi\, dx\, dt$$

At x = 0

(47)

$$\int_t \int_\phi \Big[\Big\{ \sigma_{11}^{\circ} + \int_{-h}^{h} P_1 \left(1+\frac{z}{a}\right) dz \Big\}_{x=0} (\delta U_1^{\circ})_{x=0}$$

$$+ \Big\{ \sigma_{11}^{1} + \int_{-h}^{h} P_1 \left(1+\frac{z}{a}\right) z\, dz \Big\}_{x=0} (\delta U_1^{1})_{x=0}$$

$$+ \Big\{ \sigma_{12}^{\circ} + \int_{-h}^{h} P_2 \left(1+\frac{z}{a}\right) dz \Big\}_{x=0} (\delta U_2^{\circ})_{x=0}$$

$$+ \Big\{ \sigma_{12}^{1} + \int_{-h}^{h} P_2 \left(1+\frac{z}{a}\right) z\, dz \Big\}_{x=0} (\delta U_2^{1})_{x=0}$$

$$+ \Big\{ \sigma_{13}^{\circ} + \int_{-h}^{h} P_3 \left(1+\frac{z}{a}\right) dz \Big\}_{x=0} (\delta U_3^{\circ})_{x=0} \Big] a\, d\phi\, dt = 0$$

At x = L

(48)

$$\int_t \int_\phi \Big[\Big\{ \sigma_{11}^{\circ} - \int_{-h}^{h} P_1 \left(1+\frac{z}{a}\right) dz \Big\}_{x=L} (\delta U_1^{\circ})_{x=L}$$

$$+ \Big\{ \sigma_{11}^{1} - \int_{-h}^{h} P_1 \left(1+\frac{z}{a}\right) z\, dz \Big\}_{x=L} (\delta U_1^{1})_{x=L}$$

$$+ \Big\{ \sigma_{12}^{\circ} - \int_{-h}^{h} P_2 \left(1+\frac{z}{a}\right) dz \Big\}_{x=L} (\delta U_2^{\circ})_{x=L}$$

$$+ \Big\{ \sigma_{12}^{1} - \int_{-h}^{h} P_2 \left(1+\frac{z}{a}\right) z\, dz \Big\}_{x=L} (\delta U_2^{1})_{x=L}$$

$$+ \Big\{ \sigma_{13}^{\circ} - \int_{-h}^{h} P_3 \left(1+\frac{z}{a}\right) dz \Big\}_{x=L} (\delta U_3^{\circ})_{x=L} \Big] a\, d\phi\, dt = 0$$

In the boundary condition equations (45 - 48) the quantities P_k refer to the external surface tractions at the boundary and the σ_{ik}^{n} to the stress resultants at the boundary. Equations (45) and (46) are of the form:

(49)

$$\int_t \int_x \int_\phi \Phi_k^n \, \delta U_k^n \, (a+h) \, d\phi \, dx \, dt = 0$$

while equations (47) and (48) are of the form:

(50)

$$\int_t \int_\phi \Omega_k^n \, \delta U_k^n \, a \, d\phi \, dt = 0$$

There are only two ways of satisfying equations (49) and (50). Either the $\delta u_k^n \neq 0$, in which case $\phi_k^n = 0$ and $\Omega_k^n = 0$, or alternately, the $\delta u_k^n = 0$. This prescription will laterbe considered in more detail.

Finally, the last term in equation (39), in this form, clearly demonstrates that Newton's second law must also hold for the entire body. The requirement is stated:

(51) $$F_p(t) - M\ddot{Y}_B(t)_p = 0$$

This fact, too, will be useful in our later development.

SECTION V
EQUATIONS IN TERMS OF
DEFORMATION COORDINATES

The next step is the transformation of the stress-motion equatins into a similar set involving deformation displacement coordinates. This work is done in Appendix C.

For the equilibrium equation 42, the result becomes:

(52) $$\int_t \int_x \int_\phi \Xi_k^n \, \delta U_k^n \, a \, d\phi \, dx \, dt \equiv 0$$

where the individual values $(\Xi)_k^n$ are, $(\delta u_k^n \neq 0)$;

(53)

$$\Xi_1^0 = \frac{2Eh}{1-\nu^2}\left\{ \frac{\partial^2 U_1^0}{\partial x^2} + \frac{1-\nu}{2a^2}\frac{\partial^2 U_1^0}{\partial \phi^2} + \frac{1+\nu}{2a}\frac{\partial^2 U_2^0}{\partial x \, \partial \phi} + \frac{\nu}{a}\frac{\partial U_3^0}{\partial x} + ka\left[\frac{\partial^2 U_1^1}{\partial x^2} - \frac{(1-\nu)}{2a^2}\frac{\partial^2 U_1^1}{\partial \phi^2}\right]\right\} + P_1^+\left(1+\frac{h}{a}\right)$$
$$-2\gamma h \ddot{Y}_B(t)\cdot \bar{e}_1 - 2\gamma h \frac{\partial^2 U_1^0}{\partial t^2} - \frac{2\gamma h^3}{3a}\frac{\partial^2 U_1^1}{\partial t_1^2} = 0$$

NOTE: In equations 53-60, replace u_k^n by r_k^n if dealing with an unconstrained cylinder.

(54)

$$\Xi_1^1 = \frac{2Eh^3}{3a(1-\nu^2)}\left\{\frac{\partial^2 U_1^0}{\partial x^2} - \frac{(1-\nu)}{2a^2}\frac{\partial^2 U_1^0}{\partial \phi^2} + a\frac{\partial^2 U_1^1}{\partial x^2} + \frac{1-\nu}{2a}\frac{\partial^2 U_1^1}{\partial \phi^2} + \frac{(1+\nu)}{2}\frac{\partial^2 U_2^1}{\partial x \partial \phi} - \frac{K}{ka}\left(U_1^1 + \frac{\partial U_3^0}{\partial x}\right)\right\} + P_1^+ h\left(1+\frac{h}{a}\right) - \frac{2\gamma h^3}{3a}\ddot{Y}_B(t)\cdot\bar{e}_1 - \frac{2\gamma h^3}{3}\left(\frac{\partial^2 U_1^0}{a\,\partial t_1^2} + \frac{\partial^2 U_1^1}{\partial t_1^2}\right) = 0$$

(55)

$$\Xi_2^0 = \frac{2Eh}{1-\nu^2}\left\{\frac{(1+\nu)}{2a}\frac{\partial^2 U_1^0}{\partial x \partial \phi} + \frac{(1-\nu)}{2}\frac{\partial^2 U_2^0}{\partial x^2} + \frac{1}{a^2}\frac{\partial^2 U_2^0}{\partial \phi^2} + \frac{1}{a^2}\frac{\partial U_3^0}{\partial \phi} + ka\left[\frac{(1-\nu)}{2}\frac{\partial^2 U_2^1}{\partial x^2} - \frac{1}{a^2}\frac{\partial^2 U_2^1}{\partial \phi^2}\right] - K\left[\frac{U_2^0}{a^2} - \frac{1}{a^2}\frac{\partial U_3^0}{\partial \phi} - \frac{U_2^1}{a}\right]\right\} + P_2^+\left(1+\frac{h}{a}\right) - 2\gamma h\,\ddot{Y}_B(t)\cdot\bar{e}_2 - \frac{2\gamma h^3}{3a}\frac{\partial^2 U_2^1}{\partial t_1^2} - 2\gamma h\frac{\partial^2 U_2^0}{\partial t_2^2} = 0$$

(56)

$$\Xi_2^1 = \frac{2Eh^3}{3a(1-\nu^2)}\left\{\frac{(1-\nu)}{2}\frac{\partial^2 U_2^0}{\partial x^2} - \frac{\partial^2 U_2^0}{a^2\,\partial \phi^2} - \frac{1}{a^2}\frac{\partial U_3^0}{\partial \phi} + \frac{(1+\nu)}{2}\frac{\partial^2 U_1^1}{\partial x \partial \phi} + \left(\frac{(1-\nu)}{2}\right)a\frac{\partial^2 U_2^1}{\partial x^2} + \frac{1}{a}\frac{\partial^2 U_2^1}{\partial \phi^2} + \frac{K}{k}\left(\frac{U_2^0}{a^2} - \frac{1}{a^2}\frac{\partial U_3^0}{\partial \phi} - \frac{U_2^1}{a}\right)\right\} + P_2^+\left(1+\frac{h}{a}\right)h - \frac{2\gamma h^3}{3a}\ddot{Y}_B(t)\cdot\bar{e}_2 - \frac{2\gamma h^3}{3}\left(\frac{\partial^2 U_2^0}{a\,\partial t_1^2} + \frac{\partial^2 U_2^1}{\partial t_1^2}\right) = 0$$

(57)

$$\Xi_3^0 = \frac{-2hE}{1-\nu^2}\left\{\frac{\nu}{a}\frac{\partial U_1^0}{\partial x} + \frac{1}{a^2}\frac{\partial U_2^0}{\partial \phi} + \frac{U_3^0}{a^2} - \frac{k}{a}\frac{\partial U_2^1}{\partial \phi} + K\left(\frac{1}{a^2}\frac{\partial U_2^0}{\partial \phi} - \nabla^2 U_3^0 - \frac{\partial U_1^1}{\partial x} - \frac{1}{a}\frac{\partial U_2^1}{\partial \phi}\right)\right\} + P_3^+\left(1+\frac{h}{a}\right) - 2h\gamma\frac{\partial^2 U_3^0}{\partial t^2} - 2h\gamma\,\ddot{Y}_B(t)\cdot\bar{e}_3 = 0$$

Equations (53-57) were obtained by Yu in 1958, except for the terms involving the external forces and the rigid body motion.

In equations (53-57),

(58)

$$k = \frac{1}{3}\left(\frac{h}{a}\right)^2$$

$$K = \frac{\kappa(1-\nu)}{2}$$

The significance of κ is explained in part in Appendix C and also in Appendix D. The notation t_1 is used to indicate those terms which are related to the inclusion of rotary inertia. This is further discussed in Section VI.

Boundary conditions (45) and (46) are in an appropriate form for use when the outer and inner cylindrical surfaces are free to move, and are not transformed.

Boundary conditions (47) and (48) transform as follows:

At x = 0

(59)

$$\int_t \int_\phi \Bigg[\left\{ \frac{2Eh}{1-\nu^2}\left(\frac{\partial U_1^0}{\partial x} + \frac{h^2}{3a}\frac{\partial U_1^1}{\partial x} + \frac{\nu}{a}\frac{\partial U_2^0}{\partial \phi} + \frac{\nu}{a}U_3^0\right) + \int_{-h}^{h} P_1\left(1+\frac{z}{a}\right)dz \right\}_{x=0} (\delta U_1^0)_{x=0}$$

$$+ \left\{ \frac{E}{1-\nu^2}\cdot\frac{2h^3}{3}\left(\frac{1}{a}\frac{\partial U_1^0}{\partial x} + \frac{\partial U_1^1}{\partial x} + \frac{\nu}{a}\frac{\partial U_2^1}{\partial \phi}\right) + \int_{-h}^{+h} P_1(z)\left(1+\frac{z}{a}\right)dz \right\}_{x=0} (\delta U_1^1)_{x=0}$$

$$+ \left\{ \frac{E}{1-\nu^2}\cdot\frac{1-\nu}{2}\cdot 2h\left(\frac{\partial U_2^0}{\partial x} + \frac{1}{a}\frac{\partial U_1^0}{\partial \phi} + \frac{h^2}{3a}\frac{\partial U_2^1}{\partial x}\right) + \int_{-h}^{h} P_2\left(1+\frac{z}{a}\right)dz \right\}_{x=0} (\delta U_2^0)_{x=0}$$

$$+ \left\{ \frac{2Eh^3}{3(1-\nu^2)}\cdot\frac{1-\nu}{2}\left(\frac{1}{a}\frac{\partial U_2^0}{\partial x} + \frac{\partial U_2^1}{\partial x} + \frac{1}{a}\frac{\partial U_1^1}{\partial \phi}\right) + \int_{-h}^{h} P_2 z\left(1+\frac{z}{a}\right)dz \right\}_{x=0} (\delta U_2^1)_{x=0}$$

$$+ \left\{ \frac{\kappa E}{1-\nu^2}\cdot\frac{1-\nu}{2}\cdot 2h\left(U_1^1 + \frac{\partial U_3^0}{\partial x}\right) + \int_{-h}^{h} P_3\left(1+\frac{z}{a}\right)dz \right\}_{x=0} (\delta U_3^0)_{x=0} \Bigg] a\, d\phi\, dt = 0$$

<u>At x = L</u>

(60)

$$
\begin{aligned}
\int_t \int_\phi \Bigg[& \left\{ \frac{2Eh}{1-\nu^2} \left(\frac{\partial U_1^0}{\partial x} + \frac{h^2}{3a} \frac{\partial U_1^1}{\partial x} + \frac{\nu}{a} \frac{\partial U_2^0}{\partial \phi} + \frac{\nu}{a} U_3^0 \right) - \int_{-h}^{h} P_1 \left(1+\frac{z}{a}\right) dz \right\}_{x=L} (\delta U_1^0)_{x=L} \\
& + \left\{ \frac{E}{1-\nu^2} \cdot \frac{2h^3}{3} \left(\frac{1}{a} \frac{\partial U_1^0}{\partial x} + \frac{\partial U_1^1}{\partial x} + \frac{\nu}{a} \frac{\partial U_2^1}{\partial \phi} \right) - \int_{-h}^{h} P_1 \left(1+\frac{z}{a}\right) z\, dz \right\}_{x=L} (\delta U_1^1)_{x=L} \\
& + \left\{ \frac{E}{1-\nu^2} \cdot \frac{1-\nu}{2} \cdot 2h \left(\frac{\partial U_2^0}{\partial x} + \frac{1}{a} \frac{\partial U_1^0}{\partial \phi} + \frac{h^2}{3a} \frac{\partial U_2^1}{\partial x} \right) - \int_{-h}^{h} P_2 \left(1+\frac{z}{a}\right) dz \right\}_{x=L} (\delta U_2^0)_{x=L} \\
& + \left\{ \frac{2Eh^3}{3(1-\nu^2)} \cdot \frac{1-\nu}{2} \left(\frac{1}{a} \frac{\partial U_2^0}{\partial x} + \frac{\partial U_2^1}{\partial x} + \frac{1}{a} \frac{\partial U_1^1}{\partial \phi} \right) - \int_{-h}^{h} P_2\, z \left(1+\frac{z}{a}\right) dz \right\}_{x=L} (\delta U_2^1)_{x=L} \\
& + \left\{ \frac{\kappa E}{1-\nu^2} \cdot \frac{1-\nu}{2} \cdot 2h \left(U_1^1 + \frac{\partial U_3^0}{\partial x} \right) - \int_{-h}^{h} P_3 \left(1+\frac{z}{a}\right) dz \right\}_{x=L} (\delta U_3^0)_{x=L} \Bigg] a\, d\phi\, dt = 0
\end{aligned}
$$

We retain this material for later use while we briefly consider its relationship to conventional treatments. This is discussed in Section VI.

SECTION VI
NEGLECT OF ROTARY INERTIA AND TRANSVERSE SHEAR DEFORMATION - THE "CLASSICAL" THEORY

Most frequently in classical engineering shell theory, the various descriptive equations are developed under the assumptions that there is no transverse shear, and that terms involving rotary inertia may be neglected. When this is done, the five differential equations reduce to a set of three, and the boundary conditions are also simplified. (Not all investigators have obtained the same set of three. This depends on the precise point at which these assumptions are made, as well as on the nature of other simplifying assumptions).

The usual next step is the transformation of the stress-motion equations into a similar set involving displacement (deformation) coordinates. From this point, some solutions have been found for the lower modes.* Prior studies have been directed generally at the free vibration problem. It has been found that slight differences in the original differential equations do not cause very much change in the natural frequencies found. It has also been demonstrated that for each set of mode parameters, three natural frequencies result from the set of three equilibrium equations. Similarly, one might expect five natural frequencies from the five equation set.

We are, however, interested in the response of the shell to impulsive loading, rather than in the free vibration problem. The response is formed from an infinite series of expressions for shell deformations, each term of which is at one of the shell natural frequencies.

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*NOTE: A mode is characterized by the size of an axial and a circumferential wave length. Ordinarily, only the lowest natural frequency for the mode is found.

Lacking information to the contrary, it is premature to assert that these assumptions may be made, and the problem simplified, without loss of accuracy. The actual response will depend on the nature of the load. Different forcing functions will certainly excite different combinations of the normal modes.

In order to assure complete consistency with the equilibrium equations and boundary conditions already obtained for five variables we essentially return to equation (39) et seq., obtained directly from Hamilton's principle. (The details are included in Appendix D.)

Neglect of rotary inertia affects only equation 42, in a relatively minor way. Terms in h^3 are dropped from each one. The result is the equation (D-7).

Neglect of transverse shear deformation is considerably more subtle. Essentially we require:

(61) $$\epsilon_{13} = 0 = \epsilon_{23}$$

This is the same as requiring that:

(62) $$U_1^{\circ} + \frac{\partial U_3^{\circ}}{\partial x} = 0 = U_2^{1} + \frac{1}{a}\frac{\partial U_3^{\circ}}{\partial \phi} - \frac{1}{a} U_2^{\circ}$$

This can be accomplished by letting Mindlin's Constant $K \to \infty$, <u>so long as $\sigma_{13}{}^{\circ}$ and $\sigma_{23}{}^{\circ}$ remain finite although undefined</u>.

The Hamilton Integral as $K \to \infty$, leads to the following stress-motion equilibrium equations which are (D-22, D-23) and (D-24), shown below as (63-65), ($\delta u_k{}^{\circ} \neq 0$);

(63)
$$\left\{ \frac{\partial \sigma_{11}^{\circ}}{\partial x} + \frac{1}{a}\frac{\partial \sigma_{21}^{\circ}}{\partial \phi} + P_1^{+}\left(1+\frac{h}{a}\right) - 2\gamma h \ddot{Y}_B(t)\cdot \bar{e}_1 - 2\gamma h \ddot{U}_1^{\circ} \right\} \delta U_1^{\circ} = 0$$

(64) $$\left\{ \frac{\partial \sigma_{12}^{\circ}}{\partial x} + \frac{1}{a}\frac{\partial \sigma_{22}^{\circ}}{\partial \phi} + P_2^{+}\left(1+\frac{h}{a}\right) - 2\gamma h \ddot{Y}_B(t)\cdot \bar{e}_2 - 2\gamma h \ddot{U}_2^{\circ} \right.$$
$$\left. + \frac{1}{a}\frac{\partial \sigma_{12}^{1}}{\partial x} + \frac{1}{a^2}\frac{\partial \sigma_{22}^{1}}{\partial \phi} + P_2^{+}\cdot\frac{h}{a}\cdot\left(1+\frac{h}{a}\right) \right\} \delta U_2^{\circ} = 0$$

(65)
$$\Big\{ P_3^+\left(1+\frac{h}{a}\right) - \frac{\sigma_{22}^{\circ}}{a} - 2\gamma h \ddot{Y}_B(t)\cdot E_3 - 2\gamma h \ddot{U}_3^{\circ} + \frac{\partial^2 \sigma_{11}'}{\partial x^2} + \frac{1}{a}\frac{\partial^2 \sigma_{21}'}{\partial x \partial \phi}$$
$$+ \frac{\partial P_1^+}{\partial x}\left(1+\frac{h}{a}\right)h + \frac{1}{a}\frac{\partial^2 \sigma_{12}'}{\partial x \partial \phi} + \frac{1}{a}\frac{\partial^2 \sigma_{22}'}{\partial \phi^2} + \frac{\partial P_2^+}{\partial \phi}\cdot\frac{h}{a}\cdot\left(1+\frac{h}{a}\right)\delta U_3^{\circ} = 0$$

Boundary condition equations (45) and (46) are unaffected by this assumption.

The complete set of boundary conditions (as $K \rightarrow \infty$) at x = 0 is given below for reference purposes:

At x = 0

(66)
$$\int_t \int_\phi \Bigg[\left\{ \sigma_{11}^{\circ} + \int_{-h}^{h} P_1\left(1+\frac{z}{a}\right)dz \right\}_{x=0} (\delta U_1^{\circ})_{x=0}$$
$$- \left\{ \sigma_{11}' + \int_{-h}^{h} P_1 z\left(1+\frac{z}{a}\right)dz \right\}_{x=0} \left(\delta \frac{\partial U_3^{\circ}}{\partial x}\right)_{x=0}$$
$$+ \left\{ \sigma_{12}^{\circ} + \int_{-h}^{h} P_2\left(1+\frac{z}{a}\right)dz + \frac{\sigma_{12}'}{a} + \frac{1}{a}\int_{-h}^{h} P_2 z\left(1+\frac{z}{a}\right)dz \right\}_{x=0} (\delta U_2^{\circ})_{x=0}$$
$$+ \Big\{ P_1^+\left(1+\frac{h}{a}\right)h + \frac{\partial \sigma_{11}'}{\partial x} + \frac{\partial \sigma_{21}'}{\partial x} + \frac{1}{a}\frac{\partial \sigma_{12}'}{\partial \phi} + \int_{-h}^{h} P_3\left(1+\frac{z}{a}\right)dz$$
$$+ \frac{1}{a}\frac{\partial}{\partial \phi}\int_{-h}^{h} P_2\left(1+\frac{z}{a}\right)z\,dz \Big\}_{x=0} (\delta U_3^{\circ})_{x=0} \Bigg] a\,d\phi\,dt = 0$$

For a clamped (fixed-ended) cylinder, at both x = 0, and x = L, the end conditions are prescribed and reduce to:

(67)
$$U_1^{\circ} = 0 = U_2^{\circ}$$
$$U_3^{\circ} = 0 = \frac{\partial U_3^{\circ}}{\partial x}$$

For a free-ended cylinder, $P_1 = P_2 = P_3 = 0$ and for arbitrary $\delta u_k{}^{\circ} \neq 0$, the boundary conditions at x = 0 and x = L become:

(68)
$$\sigma_{11}^{0} = N_x = 0$$

$$\sigma_{11}^{1} = M_x = 0$$

$$\sigma_{12}^{0} + \frac{1}{a}\sigma_{12}^{1} = N_{x\phi} + \frac{1}{a}M_{x\phi} = 0 \qquad \text{ERSATZ SHEAR}$$

$$0 = P_1^{+}\left(1+\frac{h}{a}\right)h + \frac{\partial \sigma_{11}^{1}}{\partial x} + \frac{\partial \sigma_{21}^{1}}{a\partial\phi} + \frac{1}{a}\frac{\partial \sigma_{12}^{1}}{\partial\phi} = P_1^{+}\left(1+\frac{h}{a}\right)h + \frac{\partial M_x}{\partial x} + \frac{\partial M_{\phi x}}{a\partial\phi} + \frac{1}{a}\frac{\partial M_{x\phi}}{\partial\phi}$$

ERSATZ TRANSVERSE SHEAR

where the first 3 terms of the last equation define $Q_x\big|_{x=0,L}$ with no rotary inertia considered.

In addition, we can obtain the appropriately consistent set of three equations in deformation displacement coordinates. A new set of stress resultant equations as $K \to \infty$ (see equations (D-30)(a-j) is formed by using equation (62). The set is then substituted in equations (63-65) to form: (assuming $\delta u_1^{0} \neq \delta u_2^{0} \neq \delta u_3^{0} \neq 0$).

(69)
$$+\frac{2Eh}{1-\nu^2}\left\{\frac{\partial^2 U_1^0}{\partial x^2} + \frac{(1-\nu)}{2a^2}\frac{\partial^2 U_1^0}{\partial\phi^2} + \frac{(1+\nu)}{2a}\frac{\partial^2 U_2^0}{\partial x\partial\phi} + \frac{\nu}{a}\frac{\partial U_3^0}{\partial x} + k\left(\frac{1-\nu}{2a}\frac{\partial^3 U_3^0}{\partial x\partial\phi^2} - a\frac{\partial^3 U_3^0}{\partial x^3}\right)\right\}$$
$$-\left\{2\gamma h \ddot{Y}_B(t)\cdot\bar{e}_1 + 2\gamma h \ddot{U}_1 - P_1^{+}\left(1+\frac{h}{a}\right)\right\} = 0$$

(70)
$$\frac{2Eh}{1-\nu^2}\left\{\frac{1+\nu}{2a}\cdot\frac{\partial^2 U_1^0}{\partial x\partial\phi} + \frac{1-\nu}{2}\cdot\frac{\partial^2 U_2^0}{\partial x^2} + \frac{1}{a^2}\cdot\frac{\partial^2 U_2^0}{\partial\phi^2} + \frac{1}{a^2}\frac{\partial U_3^0}{\partial\phi}\right.$$
$$\left. + k\left[\frac{3}{2}(1-\nu)\frac{\partial^2 U_2^0}{\partial x^2} - \frac{1}{a^2}\frac{\partial^2 U_2^0}{\partial\phi^2} - \frac{1}{a^2}\frac{\partial U_3^0}{\partial\phi} - \frac{(3-\nu)}{2}\cdot\frac{\partial^3 U_3^0}{\partial\phi\partial x^2}\right]\right\}$$
$$-\left\{2\gamma h \ddot{Y}_B(t)\cdot\bar{e}_2 + 2\gamma h \ddot{U}_2^0 - P_2^{+}\left(1+\frac{h}{a}\right) - P_2^{+}\cdot\left(\frac{h}{a}\right)\cdot\left(1+\frac{h}{a}\right)\right\} = 0$$

(71)
$$-\frac{2Eh}{1-\nu^2}\left\{\frac{1}{a^2}\frac{\partial U_2^0}{\partial\phi} + \frac{U_3^0}{a^2} + \frac{\nu}{a}\cdot\frac{\partial U_1^0}{\partial x} - ka^2\left[\frac{1}{a}\frac{\partial^3 U_1^0}{\partial x^3} - \frac{(1-\nu)}{2a^3}\frac{\partial^3 U_1^0}{\partial x\partial\phi^2} + \frac{(3-\nu)}{2a^2}\frac{\partial^3 U_2^0}{\partial\phi\partial x^2} + \frac{1}{a^4}\frac{\partial U_2^0}{\partial\phi}\right.\right.$$
$$\left.\left. - \frac{2}{a^4}\frac{\partial^2 U_3^0}{\partial\phi^2} - \nabla^4 U_3^0\right]\right\} - \left\{2\gamma h \ddot{Y}_B(t)\cdot\bar{e}_3 + 2\gamma h \ddot{U}_3^0 - P_3^{+}\left(1+\frac{h}{a}\right) - \frac{\partial P_1^{+}}{\partial x}\cdot h\left(1+\frac{h}{a}\right)\right.$$
$$\left. - \frac{\partial P_2^{+}}{\partial\phi}\cdot\frac{h}{a}\left(1+\frac{h}{a}\right)\right\} = 0$$

SECTION VII
THE FREE VIBRATION PROBLEM

We have obtained the shell equilibrium equations, expressed in terms of the mid-surface variables. (Equations 53-57)). The set may be written as:

(72)

$$\Xi_K^n\left(U_K^n, \frac{\partial U_K^n}{\partial S_L}, \frac{\partial^2 U_K^n}{\partial S_L^2}, \frac{\partial^2 U_K^n}{\partial S_i \partial S_j}, \frac{\partial^2 U_K^n}{\partial t^2}, \nu, a, \frac{h}{a}, E, P_K, \ddot{Y}_B(t)\right) = 0$$

where it is understood that

$$k = 1, 2, 3$$
$$n = 0, 1$$
$$S_1 = x$$
$$S_2 = ad\phi$$
$$S_3 = R$$

The solution to this general problem (which includes the surface tractions and the acceleration of the mass-center) is discussed in Section IX.

In the free vibration problem, the surface tractions $\overset{+}{P}_k$ and the acceleration $\ddot{Y}_B(t)$ are eliminated from the set Ξ. For clarity of notation, we write the set now as:

(73) $$H_K^n\left(U_K^n, \frac{\partial U_K^n}{\partial S_L}, \frac{\partial^2 U_K^n}{\partial S_L^2}, \frac{\partial^2 U_K^n}{\partial S_i \partial S_j}, \frac{\partial^2 U_K^n}{\partial t^2}, \nu, a, \frac{h}{a}, E\right) = 0$$

It should be emphasized that H_k^n will refer only to the free vibration problem, and that the only difference between Ξ_k^n and H_k^n is in the elimination of the P_k^+ and $\ddot{Y}_B(t)$ terms.

We assume that the solutions u_k^n may be expressed as an infinite series of terms, separable in the functions x, ϕ, t:

(74)

$$U_1^0 = \sum_{m=0}^{\infty} f_1^0(x)_m \cos m\phi \, C_m e^{i\omega_m t}$$

$$U_1^1 = \sum_{m=0}^{\infty} f_1^1(x)_m \cos m\phi \, C_m e^{i\omega_m t}$$

$$U_2^0 = \sum_{m=0}^{\infty} f_2^0(x)_m \sin m\phi \, C_m e^{i\omega_m t}$$

$$U_2^1 = \sum_{m=0}^{\infty} f_2^1(x)_m \sin m\phi \, C_m e^{i\omega_m t}$$

$$U_3^0 = \sum_{m=0}^{\infty} f_3^0(x)_m \cos m\phi \, C_m e^{i\omega_m t}$$

When these solutions are substituted in the set $\left\{H_k^n\right\}$ the result becomes:

(75)

$$H_1^0 = \sum_{m=0}^{\infty} \cos m\phi \, C_m e^{i\omega_m t}\left[a_{11} f_1^0 + a_{12} f_1^1 + a_{13} f_2^0 + a_{14} f_2^1 + a_{15} f_3^0\right] = 0$$

$$H_1^1 = \sum_{m=0}^{\infty} \cos m\phi \, C_m e^{i\omega_m t}\left[a_{21} f_1^0 + a_{22} f_1^1 + a_{23} f_2^0 + a_{24} f_2^1 + a_{25} f_3^0\right] = 0$$

$$H_2^0 = \sum_{m=0}^{\infty} \sin m\phi \, C_m e^{i\omega_m t}\left[a_{31} f_1^0 + a_{32} f_1^1 + a_{33} f_2^0 + a_{34} f_2^1 + a_{35} f_3^0\right] = 0$$

$$H_2^1 = \sum_{m=0}^{\infty} \sin m\phi \, C_m e^{i\omega_m t}\left[a_{41} f_1^0 + a_{42} f_1^1 + a_{43} f_2^0 + a_{44} f_2^1 + a_{45} f_3^0\right] = 0$$

$$H_3^0 = \sum_{m=0}^{\infty} \cos m\phi \, C_m e^{i\omega_m t}\left[a_{51} f_1^0 + a_{52} f_1^1 + a_{53} f_2^0 + a_{54} f_2^1 + a_{55} f_3^0\right] = 0$$

The trigonometric terms satisfy the necessary angular periodicity and the imaginary exponential time terms satisfy the periodicity requirements of free vibration.

The coefficients a_{rs} are functions of K, a, m, ω_n, ν , $\frac{h}{a}$, D and D^2, where D signifies differentiation with respect to x. Since these equations are valid for any value of ϕ or t, the bracketed terms must be equal to zero for any value of m.

Thus, equations (75) essentially form a matrix equation:

(76)

$$\begin{bmatrix} H_1^0 \\ H_1^1 \\ H_2^0 \\ H_2^1 \\ H_3^0 \end{bmatrix}_m = \begin{bmatrix} & & \\ & A_1 & \\ & & \end{bmatrix}_m \begin{bmatrix} f_1^0 \\ f_1^1 \\ f_2^0 \\ f_2^1 \\ f_3^0 \end{bmatrix}_m = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}_m$$

in which A_1 is a 5 x 5 determinant, as a function of γ, K, a, m, ω_n, ν , $\frac{h}{a}$, D and D^2. This can be solved if the typical solution is taken as:

(77)

$$f_k^n(x,m)_{mi} = \chi_{kmi}^n \Gamma_{mi} e^{\lambda_{mi} \bar{x}}$$

The subscripts i indicate that more than one such solution will be possible. The term Γ_{mi} refers to a factor of the solution common to all f_k^n for a particular m, i set of five. The terms χ_{kmi}^n then express the relative amplitudes for each f_k^n, for a particular m,i set.

Using this postulated solution form, the determinantal equation becomes:

(78)

$$\begin{bmatrix} \\ A_2(\omega_m, m, \frac{\lambda_{mi}}{a}, \frac{h}{a}) \\ \\ \end{bmatrix}_{mi} \begin{bmatrix} \mathcal{Z}_1^0 \\ \mathcal{Z}_1^1 \\ \mathcal{Z}_2^0 \\ \mathcal{Z}_2^1 \\ \mathcal{Z}_3^0 \end{bmatrix}_{mi} \Gamma_{mi} e^{\lambda_{mi} \frac{x}{a}} = 0$$

The matrix A_2 is of course also a function of γ, ν and K, but the form chosen highlights the variable factors in it.

To satisfy equation (78), the determinant of the matrix A_2 must equal zero for each value of i and m. Expansion of the determinant would result in an equation of the form:

(79) $$\Delta \cdot A_2 = \lambda_i^{10} + g_{8i}\lambda_i^8 + g_{6i}\lambda_i^6 + g_{4i}\lambda_i^4 + g_{2i}\lambda_i^2 + g_{0i} = 0$$

In equation (79), the coefficients g_{pi} are functions of h/a, ω_m and m. For specified values of h/a, ω_m and m, there are five values of λ_i^2, or ten values of λ_i, that will satisfy equation (79). There must be at least one real root λ_i^2, and this will be associated either with four other real roots, or with either one or two pairs of complex conjugate roots.

Assuming for the moment that the 10 values of λ_i are known, the index i in equation (77) is seen to take the values 1 - 10. Equation (77) thus yields:

(80) $$f_K^n(x,m)_m = \sum_{i=1}^{10} f_K^n(x,m)_{mi} = \sum_{i=1}^{10} \mathcal{Z}_{Kmi}^n \Gamma_{mi} e^{\lambda_{mi} \frac{x}{a}}$$

It is convenient to set $(\mathcal{L}_3^o)_{mi} = 1$, for normalizing purposes. Now, if λ_{mi} can be obtained as analytic functions of ω_m, it will be possible to obtain the relative amplitude factor:

(81) $$\mathcal{L}^n_{kmi}(\omega_m, \lambda_{mi}) = \frac{U^n_{kmi}}{U^o_{3mi}}$$

(This is done by substituting the λ_{mi} into four of the five equations (78).)

To reiterate, it is at this point possible to write equation (80) in the form:

(82)

$$f_1^o(x,m)_m = \sum_{i=1}^{10} \mathcal{L}^o_{1m1}\Gamma_{m1}e^{\lambda_{m1}\frac{x}{a}} + \mathcal{L}^o_{1m2}\Gamma_{m2}e^{\lambda_{m2}\frac{x}{a}} + \ldots + \mathcal{L}^o_{1m10}\Gamma_{m10}e^{\lambda_{m10}\frac{x}{a}}$$

$$f_1^1(x,m)_m = \sum_{i=1}^{10} \mathcal{L}^1_{1m1}\Gamma_{m1}e^{\lambda_{m1}\frac{x}{a}} + \mathcal{L}^1_{1m2}\Gamma_{m2}e^{\lambda_{m2}\frac{x}{a}} + \ldots + \mathcal{L}^1_{1m10}\Gamma_{m10}e^{\lambda_{m10}\frac{x}{a}}$$

$$f_2^o(x,m)_m = \sum_{i=1}^{10} \mathcal{L}^o_{2m1}\Gamma_{m1}e^{\lambda_{m1}\frac{x}{a}} + \mathcal{L}^o_{2m2}\Gamma_{m2}e^{\lambda_{m2}\frac{x}{a}} + \ldots + \mathcal{L}^o_{2m10}\Gamma_{m10}e^{\lambda_{m10}\frac{x}{a}}$$

$$f_2^1(x,m)_m = \sum_{i=1}^{10} \mathcal{L}^1_{2m1}\Gamma_{m1}e^{\lambda_{m1}\frac{x}{a}} + \mathcal{L}^1_{2m2}\Gamma_{m2}e^{\lambda_{m2}\frac{x}{a}} + \ldots + \mathcal{L}^1_{2m10}\Gamma_{m10}e^{\lambda_{m10}\frac{x}{a}}$$

$$f_3^o(x,m)_m = \sum_{i=1}^{10} 1\,\Gamma_{m1}e^{\lambda_{m1}\frac{x}{a}} + 1\,\Gamma_{m2}e^{\lambda_{m2}\frac{x}{a}} + \ldots + 1\,\Gamma_{m10}e^{\lambda_{m10}\frac{x}{a}}$$

Until the value of ω_m is found, it should be noted that the values of $\mathcal{L}^o_{1m1}$, $\mathcal{L}^o_{1m2}$, etc. are undetermined.

In order to proceed further, it is necessary to appreciate that the five functions (74 a-e) must satisfy the problem boundary conditions, as presented in equations (59) and (60).

BOUNDARY CONDITIONS FOR CLAMPED ENDS

For the purpose of continuing this discussion, we now note that for clamped ends equations (59) and (60) yield the following 10 homogeneous boundary conditions

(83)

At $x=0$		At $x=L$
$r_1^0=0$	*	$r_1^0=0$
$r_1'=0$		$r_1'=0$
$r_2^0=0$		$r_2^0=0$
$r_2'=0$		$r_2'=0$
$r_3^0=0$		$r_3^0=0$

When boundary conditions (83) are substituted in equations (74 a-e), each m dependent term will yield 10 equations. Since the boundary conditions do not depend on either ϕ or t, these essentially reduce to the 10 equations found by setting equations (82 a-e) equal to zero at x = 0 and x = L.

(84)

$$\begin{bmatrix} \zeta^0_{1m1} & \zeta^0_{1m2} & \zeta^0_{1m3} & \cdots & \zeta^0_{1m10} \\ \zeta'_{1m1} & \zeta'_{1m2} & \zeta'_{1m3} & \cdots & \zeta'_{1m10} \\ \zeta^0_{2m1} & \cdot & \cdot & \cdots & \cdot \\ \zeta'_{2m1} & \cdot & \cdot & \cdots & \cdot \\ | & \cdot & \cdot & \cdots & | \\ \zeta^0_{1m1} e^{\lambda_{m1}\frac{L}{a}} & \zeta^0_{1m2} e^{\lambda_{m2}\frac{L}{a}} & \cdot & \cdots & \zeta^0_{1m10} e^{\lambda_{m10}\frac{L}{a}} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 1e^{\lambda_{m1}\frac{L}{a}} & \cdot & \cdot & \cdots & 1e^{\lambda_{m10}\frac{L}{a}} \end{bmatrix}_m \begin{bmatrix} \Gamma_{m1} \\ \Gamma_{m2} \\ \Gamma_{m3} \\ \Gamma_{m4} \\ \Gamma_{m5} \\ \Gamma_{m6} \\ \Gamma_{m7} \\ \Gamma_{m8} \\ \Gamma_{m9} \\ \Gamma_{m10} \end{bmatrix}_m \equiv 0.$$

* See note on Page 43

Since the column $\Gamma_{m1} \ldots \Gamma_{m10}$ is arbitrary, the determinant of the coefficient matrix must = 0.

Recall that both equation (79) and the determinant of equation (84) are functions of ω for a particular value of m. Equation (79) is imagined to provide λ_{mi} as functions of ω_m, so that all that remains is to find the value or values of ω_m that satisfy equation (84).

The determinant of equation (84) is a transcendental equation which yields an infinite number of solutions. We employ the index N to denote these values. Thus N = 1,2,3 The natural frequency ω should thus be subscripted as ω_{mN}, and is seen (in this case) to be one of the eigenvalues of ω in the determinant of (84). Thus, to recapitulate:

- Each value of ω_{mN} (N=1,2,3 .... ∞) determines λ_{mN1} and $\mathcal{L}_{kmNi}$.
- Placing one value of ω_{mN} in 9 of 10 equations (84) produces Γ_{mNi}, the relative amplitudes of the axial functions, where Γ_{mN10} is taken as 1.
- Thus, equation (80) may be written:

$$f_K^n(x,m)_m = f_K^n(x,m,N)_{mN} = \sum_{i=1}^{10} \mathcal{L}_{KmNi} \Gamma_{mNi} e^{\lambda_{mNi} \frac{x}{a}} \tag{85}$$

There are an infinite number of solutions N = 1,2,3 .... ∞ for each m that will satisfy the boundary conditions (83) and the equilibrium equations (75). Each of these solutions may be conveniently expressed:

$$U_{KmN}^n = C_{mN} e^{i\omega_{mN} t} \cos m\phi \cup \sin m\phi \; f_K^n(x,m,N)_{mN} \tag{86}$$

The set of solutions correponding to the eigenvalue ω_{mN} is represented by the solution vector:

$$\left\{U_{KmN}^n\right\} = C_{mN} e^{i\omega_{mN} t} \cos m\phi \cup \sin m\phi \left\{f_K^n(x,m,N)_{mN}\right\} \tag{87}$$

Each of these solution vectors exist independently for each ω_{mN}. Thus the complete solution is represented by a doubly infinite series formed as a linear combination of the solution vectors:

$$\left\{U_K^n\right\} = \sum_{m=0}^{\infty} \sum_{N=1}^{\infty} \left\{U_{KmN}^n\right\} \tag{88}$$

or,

$$\left\{U_K^n\right\} = \sum_{m=0}^{\infty} \sum_{N=1}^{\infty} C_{mN} e^{i\omega_{mN} t} \cos m\phi \; U \sin m\phi \left\{f_K^n(x,m,N)_{mN}\right\} \tag{89}$$

Before proceeding further, let us also develop the boundary conditions for free ends.

BOUNDARY CONDITIONS FOR FREE ENDS:

For the free ended cylinder, the boundary conditions can be obtained from equations (59) and (60), by noting that the variables u_1^0, u_1^1, etc. are not prescribed, so that the several terms $\delta u_k^n \neq 0$. If we take no surface tractions on the end faces, equations (59) and (60) reduce to a system of 10 equations:

(90)

At $x=0$	At $x=L$
$\sigma_{11}^0 = 0$	$\sigma_{11}^0 = 0$
$\sigma_{11}^1 = 0$	$\sigma_{11}^1 = 0$
$\sigma_{12}^0 = 0$	$\sigma_{12}^0 = 0$
$\sigma_{12}^1 = 0$	$\sigma_{12}^1 = 0$
$\sigma_{13}^1 = 0$	$\sigma_{13}^0 = 0$

These are, of course, expressible in terms of deformation coordinates as follows:

(91)

a. $$\frac{2Eh}{1-\nu^2}\left(\frac{\partial U_1^0}{\partial x}+\frac{h^2}{3a}\frac{\partial U_1^1}{\partial x}+\frac{\nu}{a}\frac{\partial U_2^0}{\partial \phi}+\frac{\nu}{a}U_3^0\right)\Big|_{x=0,L}=0$$

b. $$\frac{2Eh}{1-\nu^2}\cdot\frac{h^2}{3}\left(\frac{1}{a}\frac{\partial U_1^0}{\partial x}+\frac{\partial U_1^1}{\partial x}+\frac{\nu}{a}\frac{\partial U_2^1}{\partial \phi}\right)\Big|_{x=0,L}=0$$

c. $$\frac{2Eh}{1-\nu^2}\cdot\frac{(1-\nu)}{2}\left(\frac{\partial U_2^0}{\partial x}+\frac{1}{a}\frac{\partial U_1^0}{\partial \phi}+\frac{h^2}{3a}\frac{\partial U_2^1}{\partial x}\right)\Big|_{x=0,L}=0$$

d. $$\frac{2Eh}{1-\nu^2}\cdot\frac{h^2}{3}\left(\frac{1}{a}\frac{\partial U_2^0}{\partial x}+\frac{\partial U_2^1}{\partial x}+\frac{1}{a}\frac{\partial U_1^1}{\partial \phi}\right)\Big|_{x=0,L}=0$$

e. $$\frac{2Eh}{1-\nu^2}\cdot\frac{\kappa(1-\nu)}{2}\left(U_1^1+\frac{\partial U_3^0}{\partial x}\right)\Big|_{x=0,L}=0$$

By way of example, consider (91a) in terms of equations (74) and (80). This becomes:

(92)

$$\begin{aligned}\frac{2Eh}{1-\nu^2}\sum_{m=0}^{\infty}\Bigg(&\sum_{i=1}^{10}\zeta_{1mi}^{0}\lambda_{mi}e^{\lambda_{mi}\frac{x}{a}}\Gamma_{mi}\cos m\phi\\&+\frac{h^2}{3a}\sum_{i=1}^{10}\zeta_{1mi}^{1}\lambda_{mi}e^{\lambda_{mi}\frac{x}{a}}\Gamma_{mi}\cos m\phi\\&+\frac{\nu}{a}m\sum_{i=1}^{10}\zeta_{2mi}^{0}\Gamma_{mi}e^{\lambda_{mi}\frac{x}{a}}\cos m\phi\\&+\frac{\nu}{a}\sum_{i=1}^{10}1\cdot\Gamma_{mi}e^{\lambda_{mi}\frac{x}{a}}\cos m\phi\Bigg)\Big|_{x=0,L}C_m e^{i\omega_m t}=0\end{aligned}$$

Factoring and simplifying:

(93)

$$\begin{aligned}\frac{2Eh}{1-\nu^2}\sum_{m=0}^{\infty}C_m e^{i\omega_m t}\cos m\phi\Big\langle\sum_{i=1}^{10}\Gamma_{mi}\Big[e^{\lambda_{mi}\frac{x}{a}}\Big(&\zeta_{1mi}\lambda_{mi}\\&+\frac{h^2}{3a}\zeta_{1mi}^{1}\lambda_{mi}+\frac{\nu}{a}\zeta_{2mi}^{0}+\frac{\nu}{a}\cdot 1\Big)\Big]\Big\rangle_{x=0,L}=0\end{aligned}$$

We designate the term in brackets as $S^n_{1k}(x)_m$ and in brackets as $S^n_{1k}(x)_{mi}$ where the subscript k and superscript n serve to assicate it with the corresponding stress resultant. The i subscript, of course, again serves to identify $S^n_{1k,mi}$ with one of the 10 roots λ_{mi}. Since a similar development can be made for each of the terms of equation (91) the result is again a 10 x 10 determinantal equation:

(94)

$$\begin{bmatrix} S^{0}_{11,m1} & S^{0}_{11,m2} & S^{0}_{11,m3} & \cdots\cdots & S^{0}_{11,m10} \\ S'_{11,m1} & S'_{11,m2} & & & \\ S^{0}_{12,m1} & & & & \\ S'_{12,m1} & & & & \\ S^{0}_{13,m1} & & & & \\ S^{0}_{11,m1} e^{\lambda_{m1} \ell} & & & & \\ & & & & \\ S^{0}_{13,m1} e^{\lambda_{m1} \ell} & & & & S^{0}_{13,m10} e^{\lambda_{m10} \ell} \end{bmatrix}_m \begin{bmatrix} \Gamma_{m1} \\ \Gamma_{m2} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \Gamma_{m10} \end{bmatrix}_m = 0$$

where all terms in the left matrix are analytic functions of ω_m, and the solution for ω_{mN} will proceed in an analagous manner as with clamped ends.

We now consider the forms of the equilibrium equations and the boundary conditions when the solution is taken in the form of equation (89), the doubly infinite series.

First, noting the form of equation (85), we write the solution set for a given m, N pair as:

(95)

$$(U_1^0)_{mN} = C_{mN} e^{i\omega_{mN}t} \cos m\phi \, f_1^0(x,m,N)_{mN}$$

$$(U_1^1)_{mN} = C_{mN} e^{i\omega_{mN}t} \cos m\phi \, f_1^1(x,m,N)_{mN}$$

$$(U_2^0)_{mN} = C_{mN} e^{i\omega_{mN}t} \sin m\phi \, f_2^0(x,m,N)_{mN}$$

$$(U_2^1)_{mN} = C_{mN} e^{i\omega_{mN}t} \sin m\phi \, f_2^1(x,m,N)_{mN}$$

$$(U_3^0)_{mN} = C_{mN} e^{i\omega_{mN}t} \cos m\phi \, f_3^0(x,m,N)_{mN}$$

Now, substitute the solution vector $\{u_k^n\}_{mN}$ into the set of equilibrium equations. In the process, we note from equations (75) that each one of the terms f_k^n is multiplied by a coefficient of the form a_{rs}. The coefficient is identified by row with a term H_k^n and by column with a term f_k^n. We therefore find it convenient to display the equilibrium equations in the form:

(96)

$$(H_1^0)_{mN} = C_{mN} e^{i\omega_{mN}t} \cos m\phi \left[a_{11}^{00} f_1^0 + a_{11}^{01} f_1^1 + a_{12}^{00} f_2^0 + a_{12}^{01} f_2^1 + a_{13}^{00} f_3^0\right]_{mN}$$

$$(H_1^1)_{mN} = C_{mN} e^{i\omega_{mN}t} \cos m\phi \left[a_{11}^{10} f_1^0 + a_{11}^{11} f_1^1 + a_{12}^{10} f_2^0 + a_{12}^{11} f_2^1 + a_{13}^{10} f_3^0\right]_{mN}$$

$$(H_2^0)_{mN} = C_{mN} e^{i\omega_{mN}t} \sin m\phi \left[a_{21}^{00} f_1^0 + a_{21}^{01} f_1^1 + a_{22}^{00} f_2^0 + a_{22}^{01} f_2^1 + a_{23}^{00} f_3^0\right]_{mN}$$

$$(H_2^1)_{mN} = C_{mN} e^{i\omega_{mN}t} \sin m\phi \left[a_{21}^{10} f_1^0 + a_{21}^{11} f_1^1 + a_{22}^{10} f_2^0 + a_{22}^{11} f_2^1 + a_{23}^{10} f_3^0\right]_{mN}$$

$$(H_3^0)_{mN} = C_{mN} e^{i\omega_{mN}t} \cos m\phi \left[a_{31}^{00} f_1^0 + a_{31}^{01} f_1^1 + a_{32}^{00} f_2^0 + a_{32}^{01} f_2^1 + a_{33}^{00} f_3^0\right]_{mN}$$

Equation set (96 a-e) may be more compactly displayed as the column vector:

(97)
$$\left\{H_k^n\right\}_{mN} = C_{mN} e^{i\omega_{mN} t} \sin m\phi \cup \cos m\phi \left\{a_{kr}^{nj} f_r^j(x)\right\}_{mN}$$

It should be carefully noted that the coefficients a_{kr}^{nj} are functions of m, D and ω^2, and that they are precisely the same as those displayed in equation (75). The only difference is that ω has been shown to be dependent not only on m but also on N.

Since these coefficients formed the matrix of equation (76), we may now write:

(98)
$$\left\{H_k^n\right\}_{mN} = 0$$

The notation cos $m\phi$ $\cup$ sin $m\phi$ may be incorporated into each term f_k^n by defining:

(99)
$$f_k^n(x,\phi)_{mN} = (\cos m\phi \cup \sin m\phi) f_k^n(x,m,N)_{mN}$$

Thus we write:

(100)
$$\left\{U_k^n\right\} = \sum_{m=0}^{\infty}\sum_{N=1}^{\infty} C_{mN} e^{i\omega_{mN} t}\left[f_k^n(x,\phi)\right]$$

By considering the development of equation (96), it is also evident that:

(101)
$$\left\{H_k^n\right\} = 0 = \sum_{m=0}^{\infty}\sum_{N=1}^{\infty} C_{mN} e^{i\omega_{mN} t}\left[a_{kr}^{nj} f_r^j(x,\phi)_{mN}\right]$$

It is convenient to write

(102) $$\left\{a^{nj}_{kr} f^{j}_{r}(x,\phi)_{mN}\right\} = \left\{\chi^{n}_{k}(x,\phi)_{mN}\right\}$$

and then to separate out the terms containing ω^2 in each function:

(103) $$\left\{\chi^{n}_{k}(x,\phi)_{mN}\right\} = \left\{\Lambda^{n}_{k}\left(\left\{f^{n}_{k}(x,\phi)_{mN}\right\}\right) + F^{n}_{k}(x,\phi)_{mN}\,\omega^{2}_{mN}\right\}$$

Equation (101) is now written:

(104) $$\left\{H^{n}_{k}\right\} = 0 = \sum_{m=0}^{\infty}\sum_{N=1}^{\infty} C_{mN}\, e^{i\omega_{mN}t}\left\{\Lambda^{n}_{k}\left(\left\{f^{n}_{k}(x,\phi)_{mN}\right\}\right) + F^{n}_{k}(x,\phi)_{mN}\,\omega^{2}_{mN}\right\}$$

In equations (103) and (104), the terms $\Lambda^{n}_{k,mN}$ are due to terms with just space derivatives, and stem from a variation of the strain energy. The terms $F^{n}_{k}(x,\phi)_{mN}\ \omega^{2}_{mN}$ come from terms with just time derivatives, and are due to the variation in kinetic energy.

(Each term, $\Lambda^{n}_{k,mN}$ and $\omega^{2}_{mN} F^{n}_{k,mN}$ is formed by using one of the mN series of $\{(u^{k}_{n})_{mN}\}$ in the equilibrium equations.)

Similarly, from equations (90-93), for any one of the series of $\{(u^{k}_{n})_{mN}\}$, the boundary conditions for free ends are typically noted as:

(105) $$\sigma^{n}_{1k}(x,\phi,t)_{mN}\Big|_{x=0,L} = 0 = S^{n}_{1k}(x,\phi)_{mN}\Big|_{x=0,L} C_{mN}\, e^{i\omega_{mN}t}$$

where $S^{n}_{1k}(x)_{mN}$ cos mϕ Usin mϕ = $S_{1k}(x,\phi)_{mN}$ (see, typically, equation (92).

It is thus also evident that we may define

$$(106)\qquad \sigma_{1K}^{n}(x,\phi,t)\Big|_{x=0,L} = 0 = \sum_{m=0}^{\infty}\sum_{N=1}^{\infty} S_{1K}^{n}(x,\phi)_{mN}\Big|_{x=0,L} C_{mN}\, e^{i\omega_{mN}t}$$

Later, consideration is given as to whether or not the series of equations (100), (104) and (106) actually do effect the required minimization of Hamilton's Integral for Free Vibration.

Now, we pause to note another important point. Equation (79) was displayed as an equation of the 10th order in λ_i. It could also have been shown as of the 10th order in ω_m or more precisely, of the fifth order in ω_m^2 Thus, having located any one set of λ_i values which, together with one value of ω , satisfy both equation (79) and (84) or (94) there must be four other values of ω that are compatible with each of the λ_{mi} set.

The interpretation of this situation is that a nodal pattern is established, consisting of m circumferential waves and r axial half waves. For each such nodal pattern there are five natural frequencies. (The r axial half waves are interpreted as having r+1 axial nodes).

Thus, as equation (84) or (94) is solved for the successive values of ω , corresponding to N = 1,2... ∞, we may expect that we could, with some effort, identify the particular five values of N corresponding to the same number of axial nodes. (This requires plotting the axial function, equation (85) for each N). Thus, the following notations are equally appropriate:

(107)

$$\omega_{mN} \rightleftharpoons \omega_{mrj}$$

$$\mathcal{L}_{KmNi}^{n} \rightleftharpoons \mathcal{L}_{Kmrji}^{n}$$

$$\lambda_{mNi} \rightleftharpoons \lambda_{mrji}$$

$$\Gamma_{mNi} \rightleftharpoons \Gamma_{mrji}$$

$$f_K^n(x,m,N)_{mN} \rightleftharpoons f_K^n(x,m,r,j)_{mrj} \rightleftharpoons \sum_{i=1}^{10} \Gamma_{mrji}\, \mathcal{L}_{Kmrji}^{n}\, e^{\frac{\lambda_{mrji}x}{a}}$$

We now proceed to consider the actual approach to be used in the numerical solution of the problem of free vibration.

NUMERICAL APPROACH

In reviewing the steps necessary to isolate the values of ω_m which satisfy the boundary value determinant, we note that it was necessary to develop the values of λ_{mi} as functions of ω, explicitly. Having these values, it was then necessary to use equations (78) to find ζ^n_{kmi} also as explicit functions of ω.

Since it does not appear feasible to obtain analytic expressions for λ_{mi}, an alternate approach is necessary. This was suggested by Forsberg (Reference 7). This method is outlined below:

- We assume a given shell $(\frac{L}{a}, \frac{h}{a}, \nu)$
- Use a trial value for ω
- For the shell, determine λ_{mi} from equation (79). Also find ζ_{kmi}.
- Using the appropriate boundary condition determinant (84 or 93), test whether the determinant = 0.
- Iterate on ω until the result converges, repeating the above process.

It will be noted that computations on ζ^n_{kmi} are in themselves also laborious. This effort can be somewhat reduced by applying a Donnell type manipulation to the equilibrium equation set (73). The method is alluded to in Reference 6.

The result of the Donnell manipulation will be to provide a different set of five equilibrium equations. One of the equations will be a function of u^o_3 alone, and the four other equations are each functions of u^o_3 and one other (unique) variable:

(108)
$$\Theta_1(U_3^0) = 0$$
$$\Theta_2(U_3^0, U_1^0) = 0$$
$$\Theta_3(U_3^0, U_2^0) = 0$$
$$\Theta_4(U_3^0, U_1^1) = 0$$
$$\Theta_5(U_3^0, U_2^1) = 0$$

Equation (108a) is essentially the same as equation (79). The other equations (108 a-3) provide algebraic expressions for ζ^n_{kmi}. The complete operational requirements are not given here, because they are very lengthy, and because they are available in Reference 57.

Asymptotic values are available for an infinite cylinder, to serve as appropriate starting points for the iterative process. These are:

Flexural Vibration of a Ring

(109)
$$\omega_1^2 = \frac{E}{\gamma a(1-\nu^2)} \cdot \frac{h^2}{3a^2} \cdot \frac{m^2(m^2-1)}{m^2+1}$$

Axial Shear Vibration

(110)
$$\omega_2^2 = \frac{E}{\gamma a(1-\nu^2)} \cdot \frac{(1-\nu)m^2}{2}$$

Extensional Vibration of a Ring

(111)
$$\omega_3^2 = \frac{E}{\gamma a(1-\nu^2)} \cdot (m^2+1)$$

Usually two of these frequencies are several orders of magnitude higher than the minimum frequency and thus the lowest one is the first to be tried.

From this starting frequency, a detailed, small increment in ω upwards should yield, for a given m, the eigenvalues corresponding to N = 1, 2, 3... ∞

POSSIBLE MEANS OF REDUCING THE ITERATIVE EFFORT

Previous investigators have been concerned with the analysis of the free vibration problem, rather than with the response problem. Thus, they have attempted to provide means by which the lowest frequency for a particular axial node pattern may be identified, as well as other frequencies also associated with that axial node pattern.

This method of designating the natural frequencies utilizies the r, j indices rather than what we have here termed N.

We have some reason to believe that the smaller eigenvalues corresponding to our index N may be correlated to the r, j index as follows:

N	r	j
1	1	1
2	2	1
3	3	1
4	4	1
.	.	.
.	.	.
.	.	.
N	1	2

It is not presently known at what point we will encounter the second frequency corresponding to r = 1, but the calculation should not be too difficult. Approximate solutions for the three frequency system for free ends are available in Reference 9, and for the clamped end case in Reference 4. At any rate, it is likely that for the lower eigenvalues, these approximations can serve as good starting points for iteration near N = 1, 2, 3, etc.

We should emphasize that for the response problem there is no need to identify the actual numbers of axial nodes, r+1. We will be concerned rather with a simply establishing a cutoff frequency as a truncation criterion for the deformation series, equation (100). This will be further considered in the next section.

SECTION VIII
THE FORCED VIBRATION PROBLEM

To reiterate, the free vibration problem, (in which both the surface tractions $\overline{F}$ and the acceleration $\overline{\ddot{Y}_B}$ of the center of gravity vanish) is solved by the equation set (100). This is repeated below for convenience, except that we now use the subscripts p, s instead of m,N. (The reason for this change is associated with a desire for clarity in the development of the orthogonality requirements later in the section):

(100) $$\{U_K^n\} = \sum_{p=0}^{\infty} \sum_{s=1}^{\infty} C_{ps} e^{i\omega_{ps} t} \{f_K^n(x,\phi)_{ps}\}$$

The equilibrium equations became equation (104):

(104) $$\{\Xi_K^n\} = \{H_K^n\} = \sum_{p=0}^{\infty} \sum_{s=1}^{\infty} C_{ps} e^{i\omega_{ps} t} \{\Lambda_K^n(\{f_K^n(x,\phi)_{ps}\})_{ps} + \omega_{ps}^2 F_K^n(x,\phi)_{ps}\} = 0$$

and the boundary conditions for the free ended cylinder became equation (106):

(106) $$\sigma_{1K}^n(x,\phi,t)\Big|_{x=0,L} = 0 = \sum_{p=0}^{\infty} \sum_{s=1}^{\infty} S_{1K}^n(x,\phi)_{ps}\Big|_{x=0,L} C_{ps} e^{i\omega_{ps} t}$$

Each p, s solution set independently and sequentially satisfy the equilibrium equations and the boundary conditions.

The original variation was presented in equation (39). Making use of the transformation to deformation coordinates (refer to equations (52-56), and of expressions (45-48), we recast the variation as follows:

(112)
$$0 = \delta I = \int_t \int_x \int_\phi \Xi_K^n \, \delta U_K^n \, a \, d\phi \, dt + \int_t \int_{\Sigma_3} \Phi_K \, \delta U_K \, d\Sigma_3 \, dt + \int_t \int_c \left(\left[P_K^n - \sigma_{iK}^n \bar{n} \cdot \bar{a}_i \right] \delta U_K^n \right)_c dC \, dt$$

In equation (112) $d\Sigma_3$ refers to the element of surface $(a \pm h)\, d\phi dx$, and C is the midsurface line. (See Figure 4). The term P_K^n is defined:

(113)
$$P_K^n\Big|_c = \int_{-h}^{h} P_K\Big|_c \left(1 + \frac{z}{a}\right) z^n dz \quad \text{or} \quad \int_{-h}^{h} P_K\Big|_c z^n dz$$

Of the three terms in equation (112), the first includes the set of terms Ξ_K^n (52-56). The second integral may be summarized:

(114)
$$\int_t \int_{\Sigma_3} \Phi_K \, \delta U_K \, d\Sigma_3 \, dt = \int_t \int_\phi \int_x \left(P_K(z) \mp \sigma_{3K} \right) \delta U_K (a + z) \Big|_{h,-h} d\phi \, dx \, dt$$

Since in both free and forced vibration, δu_K on the cylindrical surface is <u>not</u> defined, equation (114) serves here merely to emphasize that $P_K\big|_{h,-h} = \pm\sigma_{3K}\big|_{h,-h}$

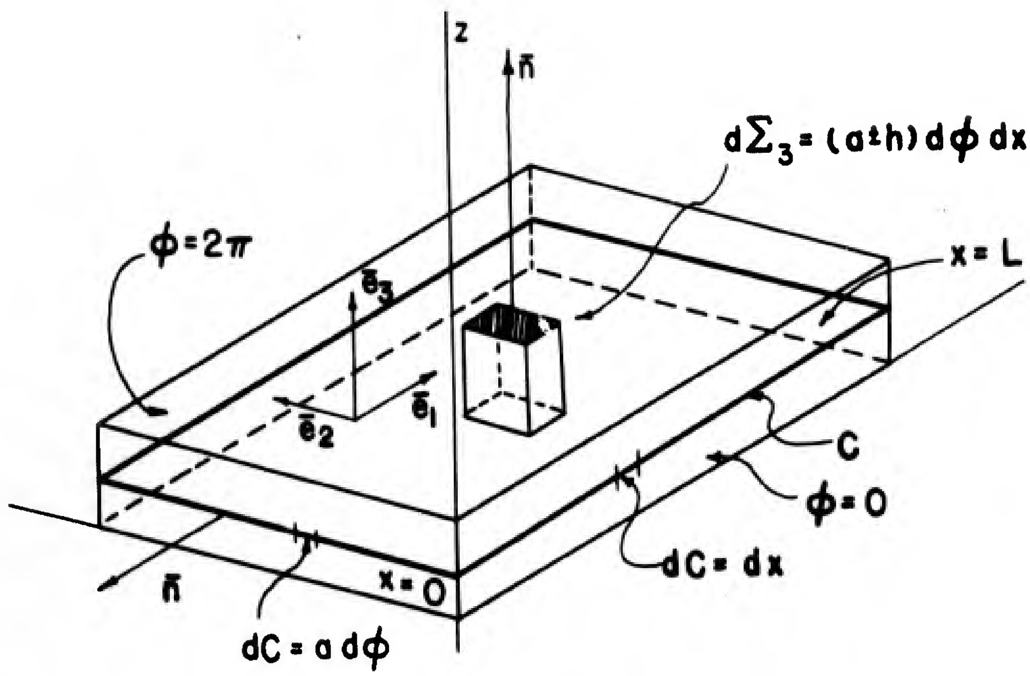

FIGURE 4: DEVELOPED
CYLINDER SECTION

Because of the equivalence of forces and deflections at $\phi = 0$ and $\phi = 2\pi$, the third integral reduces to:

(115) $$\int_t \int_\phi (P_K^n * \sigma_{1K}^n)(\delta U_K^n)\Big|_{x=0,L} a\, d\phi\, dt$$

in which

$$P_K^n\Big|_{x=0,L} = \int_{-h}^{h} P_K\Big|_{x=L,0} \left(1+\frac{z}{a}\right) z^n dz$$

To solve the forced vibration Hamilton Integral, a proposed deformation function must:

a. Satisfy the third (boundary condition) integral
b. Satisfy the first (equilibrium) integral
c. Satisfy the <u>limit</u> of the integral as $\overline{P}$ and $\ddot{Y}_B \rightarrow 0$ (i.e., the free vibration integral).

We propose a deformation function similar to equation (100), with the same function but a different time function:

(116) $$\{U_K^n\} = \sum_{p=0}^{\infty} \sum_{s=1}^{\infty} q(t)_{ps} \{f_K^n(x,\phi)_{ps}\}$$

and will also require that:

(117) $$\text{Limit}_{\overline{P},\ddot{Y} \rightarrow 0}\, q(t)_{ps} = C_{ps}\, e^{i\omega_{ps} t}$$

The proposal follows the classical concept of the so-called "normal mode" method of solution.

The obvious reason for such a choice is that the axial function already found for the free vibration solution satisfies the appropriate boundary conditions.

From an examination of equations (100) and (104), it is evident that were we to have used equation (116) for the free vibration problem, the set representing the five equations of equilibrium would have become:

$$(118) \qquad \left\{H_K^n\right\} = \sum_{p=0}^{\infty}\sum_{s=1}^{\infty}\left\{\Lambda_K^n(\{f_K^n(x,\phi)_{ps}\})q(t)_{ps} - F_K^n(x,\phi)_{ps}\ddot{q}(t)_{ps}\right\}$$

One of the equations would have read:

$$(119) \qquad H_K^n = \sum_{p=0}^{\infty}\sum_{s=1}^{\infty}\left[\Lambda_K^n(\{f_K^n(x,\phi)_{ps}\})q(t)_{ps} - F_K^n(x,\phi)_{ps}\ddot{q}(t)_{ps}\right]$$

The difference between the terms H_k^n and Ξ_k^n is in the existence of the terms dealingwith P_k^+ and $\ddot{Y}_B(t)_k$. Thus, if equation (116) is now substituted into (112) we obtain:

(120)

$$0 = \delta I = \int_t\int_x\int_\phi\left[\left(\sum_{p=0}^{\infty}\sum_{s=1}^{\infty}\left[\Lambda_K^n(\{f_K^n(x,\phi)_{ps}\})q(t)_{ps} - F_K^n(x,\phi)_{ps}\ddot{q}(t)_{ps}\right]\right)\right.$$

$$\left. + \left\langle P_K^+\left(1+\frac{h}{a}\right)h^n - \ddot{Y}_{B,K}\, 2\gamma h\cdot\frac{h^{2n}}{(3a)^n}\right\rangle\right]\left[\delta U_K^n\right] a\,d\phi\,dx\,dt$$

$$+ \int_t\int_\Sigma \Phi_K\,\delta U_K\,d\Sigma_3\,dt$$

$$+ \int_t\int_\phi P_K^n\,\delta U_K^n\Big|_{x=L,0} - \delta U_K^n\left(\sum_{p=0}^{\infty}\sum_{s=1}^{\infty} S_{1K,ps}^n(\{f_K^n(x,\phi)_{ps}\})q(t)_{ps}\right|_{x=0}^{x=L} a\,d\phi\,dt$$

We now recognize that the term δu_k^n also involves a doubly summed series. In order to distinguish the modal pairs in the δu_k^n term from those used elsewhere in (120), we call these by the indices m, N. The variation δu_k^n (with the space dependent function $f_k^n(x,\phi)_{mN}$ already defined) involves the variation of the time function $q(t)_{mN}$:

(121) $$\delta U_K^n = \sum_{m=0}^{\infty}\sum_{N=1}^{\infty} f_K^n(x,\phi)_{mN}\,\delta q(t)_{mN}$$

Now, we substitute equation (121) for δu_K^n into (120) to obtain:

(122)

$$0=\delta I = \int_t \sum_{m=0}^{\infty}\sum_{N=1}^{\infty} \delta q(t)_{mN}\Big[\int_x\int_\phi \Big[\Big(\sum_{p=0}^{\infty}\sum_{s=1}^{\infty}\big[\Lambda_K^n(\{f_K^n(x,\phi)_{ps}\})q(t)_{ps} - F_{K,ps}^n\ddot{q}(t)_{ps}\big]\Big)$$
$$+\Big\langle P_K^+(1+\tfrac{h}{a})h^n - \ddot{Y}_{\theta,K}2\gamma h + \frac{h^{2n}}{(3a)^n}\Big\rangle\Big] f_K^n(x,\phi)_{mN}\, a\,d\phi\,dx\Big]dt$$
$$+\int_t\int_{\Sigma_3} \Phi_K\,\delta U_K\,d\Sigma_3\,dt$$
$$+\int_t \sum_{m=0}^{\infty}\sum_{N=1}^{\infty} \delta q(t)_{mN}\Big[\int_0^{2\pi} P_K^n f_K^n(x,\phi)_{mN}\Big|_{x=L,0} a\,d\phi - \sum_p\sum_s q(t)_{ps}\int_0^{2\pi} S_{1K,ps}^n(\{f_K^n(x,\phi)_{ps}\})\, f_K^n(x,\phi)_{mN}\Big|_{z=0} a\,d\phi\Big]dt$$

In equation (122), the second integral is equal to zero for arbitrary δu_K^n. The third integral is also equal to zero for arbitrary $\delta q(t)_{mN}$;* as long as the deformation coordinates were chosen in the form of equation (116).

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*NOTE:

We take $$U_K^n(x,\phi,t,\alpha_K) = U_K^n(x,\phi,t,0) + \alpha_K \eta_K^n(x,\phi,t)$$
$$= \sum_{m=0}^{\infty}\sum_{N=1}^{\infty} f_K^n(x,\phi)_{mN}\big[q(t,0)_{mN} + \alpha_K \Theta_{mN}(t)\big]$$
$$= \sum_{m=0}^{\infty}\sum_{N=1}^{\infty} f_K^n(x,\phi)_{mN}\, q(t,0)_{mN} + \alpha_K \sum_{m=0}^{\infty}\sum_{N=1}^{\infty} f_K^n(x,\phi)_{mN}\,\Theta(t)_{mN}$$

therefore:

$$\delta U_K^n = \frac{\partial U_K^n}{\partial \alpha_K}\Big|_{\alpha_K=0} d\alpha_K = \sum_{m=0}^{\infty}\sum_{N=1}^{\infty} f_K^n(x,\phi)_{mN}\,\Theta(t)_{mN}\,d\alpha_K = \sum_{m=0}^{\infty}\sum_{N=1}^{\infty} f_K^n(x,\phi)_{mN}\,\delta q(t)_{mN}$$

Because

$$\delta q(t)_{mN} = \frac{\partial q(t)_{mN}}{\partial \alpha_K}\Big|_{\alpha_K=0} d\alpha_K = \Theta(t)_{mN}\,d\alpha_K$$

For free ended cylinders: (at x = 0 and x = L)

(123) $$P_K^n = 0 \quad \text{and} \quad S^n_{1K,ps}(\{f_K^n(x,\phi)_{ps}\}) = 0$$

For clamped end cylinders: (at x = 0 and x = L)

(124) $$r_{K,mN} = 0 \quad ; \quad f_K^n(x,\phi)_{mN} = 0$$

The third integral is thus demonstrated also to be zero.

Our assumed choice thus satisfies requirement a. Before examining the criteria involved in the satisfaction of requirement b (which will incorporate an orthogonality condition as a subsidiary requirement), we seek to demonstrate that requirement c is satisfied. This would necessitate using:

(117) $$\operatorname{Limit}_{\bar{p},\ddot{Y}_B \to 0} q(t)_{ps} = C_{ps} e^{i\omega_{ps} t}$$

to demonstrate that $\delta I|_{\bar{p},\ddot{Y}_B \to 0} = 0.$

We write:

(125)

$$\delta I|_{\bar{p},\ddot{Y}_B \to 0} = 0 = \int_t \sum_m \sum_N \delta(C_{mN} e^{i\omega_{mN} t}\Big[\sum_p \sum_s C_{ps} e^{i\omega_{ps} t} \int_x\int_\phi \langle \Lambda^n_{K,ps}(\{f^n_{K\,ps}\})$$

$$+ F^n_{K,ps}\omega^2_{ps}\rangle \cdot f_K^n(x,\phi)_{mN}\, a\, d\phi\, dx\Big] dt$$

$$+ \int_t \int_{\Sigma_3} \Phi_K\, \delta U_K\, d\Sigma_3\, dt$$

$$+ \int_t \sum_m \sum_N \delta(C_{mN} e^{i\omega_{mN} t})\Big[\int_0^{2\pi} \bar{P}^n_K f^n_{K,mN}\Big|_{x=L,0} a\, d\phi - \sum_p \sum_s C_{ps} e^{i\omega_{ps} t} \int_0^{2\pi} S^n_{1K,ps} f^n_{K,mN}\Big|_{x=0}^{x=L} a\, d\phi\Big] dt$$

Each component of this integral may readily be shown to be equal to zero (see equation (104)), for any arbitrary variation of $C_{mN}e^{i\omega_{mN}t}$.

We now return to equation (122) for an examination of the first integral. For the Hamilton condition to be satisfied, it must be equal to zero for any arbitrary variation $\delta q(t)_{mN}$. Therefore the term within the [ ] brackets must be brought to zero. This will require a minimization of each time coordinate, $q(t)_{mN}$, (see equation (128)). Specifically, this requirement (the equilibrium equation) is written:

(125)
$$0 = \int_x \int_\phi \sum_p^\infty \sum_s^\infty \Big[\big(\Lambda^n_{K,ps}(\{f^n_K(x,\phi)_{ps}\}) q(t)_{ps} - F^n_{K,ps}\ddot{q}(t)_{ps} \big) + \big(P^+_K (1+\tfrac{h}{a}) h^n - \ddot{Y}_{B,K} 2\gamma h \cdot \tfrac{h^{2n}}{(3a)^n} \big) \Big] \cdot f^n_K(x,\phi)_{mN}\, a\, d\phi\, dx$$

To equation (125), add and substract:

(126)
$$\int_x \int_\phi \sum_p \sum_s F^n_{K,ps}\,\omega^2_{ps}\, q(t)_{ps}\, f^n_K(x,\phi)_{mN}\, a\, d\phi\, dx$$

So that equation (125) becomes:

(125a)
$$\sum_p^\infty \sum_s^\infty q(t)_{ps} \int_x \int_\phi \big(\Lambda^n_{K,ps}(\{f^n_K(x,\phi)_{ps}\}) + F^n_{K,ps}\omega^2_{ps} \big) \cdot f^n_K(x,\phi)_{mN}\, a\, d\phi\, dx$$
$$- \sum_{p=0}^\infty \sum_{s=1}^\infty \big[\ddot{q}(t)_{ps} + \omega^2_{ps} q(t)_{ps} \big] \int_0^L \int_0^{2\pi} F^n_K(x,\phi)_{ps} f^n_K(x,\phi)_{mN}\, a\, d\phi\, dx$$
$$+ \int_0^L \int_0^{2\pi} \big(P^+_K (1+\tfrac{h}{a}) h^n - \ddot{Y}_{B,K} 2\gamma h \cdot \tfrac{h^{2n}}{(3a)^n} \big) \cdot f^n_K(x,\phi)_{mN}\, a\, d\phi\, dx \equiv 0$$

In equation (125a) the first term does equal zero, because it contains the factor H^n_K, ps = 0.

Now if the following orthogonality requirement is satisfied:

(127)
$$\int_x \int_\phi F^n_K(x,\phi)_{ps} f^n_K(x,\phi)_{mN}\, a\, d\phi\, dx \begin{cases} = 0 & m \neq p,\ s \neq N \\ \neq 0 & m = p,\ s = N \end{cases}$$

then the first integral of equation (125a) leads to the Lagrange equilibrium equations for principal (separable) time coordinates, as the coefficient of the arbitrary variation of these time coordinates:

(128)
$$\delta I = 0 = \int_t \sum_{mN}^{\infty\infty} \delta q(t)_{mN} \left[\overset{\prime}{Q}(t)_{mN} - M_{mN} \left(\ddot{q}(t)_{mN} + \omega_{mN}^2 \, q(t)_{mN} \right) \right] dt$$

where the generalized mass associated with the mNth normal mode is:

(129)
$$M_{mN} = \int_x \int_\phi F_K^n (x,\phi)_{mN} \, f_K^n (x,\phi)_{mN} \, a \, d\phi \, dx$$

and the generalized external force in the "direction" of $q(t)_{mN}$ associated with that mode is:

(130)
$$\overset{\prime}{Q}(t)_{mN} = \int_x \int_\phi \left(P_K^+ \left(1 + \frac{h}{a}\right) h^n - \ddot{Y}_{B,K} \, 2\gamma h \cdot \frac{h^{2n}}{(3a)^n} \right) \cdot f_K^n (x,\phi)_{mN} \, a \, d\phi \, dx$$

The proof of required orthogonality is shown in Appendix E.

This proof is interesting because it proceeds from Hamilton's integral directly, and does not require examination of the actual mode shapes.

The following pre-conditions are necessary:

- The system must be linearly elastic, with no dissipation of energy internally possible.
- The mode shapes must satisfy the equilibrium equations for free vibration.
- The mode shapes must satisfy the boundary conditions.

NOTE: In Appendix E, Betti's Law is seen to rest on Maxwell's Reciprocity theorem, $K_{mN-ps} = K_{ps-mN}$. Maxwell's theorem is verified below for an elastic system undergoing infinitesimal strain.

It then rests on Betti's Law, which is an implicit generalization of Maxwell's Law of Reciprocal relations*. Eventually, we demonstrate that since the natural frequencies for any two modal pairs are not equal, equation (127) must be satisfied.

NOTE: (Continuation of note)

A linear stress-strain relationship, $\sigma_{ij} = C_{ijkl}\epsilon_{kl}$ is formed for an elastic system whose strain energy density is a quadratic function of the strains, $U_{vol} = \frac{1}{2}C_{ij}\ \epsilon_{kl}\ \epsilon_{ij}$, (higher order strain products are disregarded with infinitesimal strain).

For a system in free vibration with homogeneous boundary conditions, the strain, which is composed of the separable deformation coordinates (100) and their space derivatives (A-2) is;

$$(1)\qquad \epsilon_{ij} = \sum_{mN}^{\infty\infty} h_{ij}(x,\phi,z)_{mN}\, q(t)_{mN}$$

$$(2)\qquad \epsilon_{kl} = \sum_{ps}^{\infty\infty} h_{kl}(x,\phi,z)_{ps}\, q(t)_{ps}$$

This develops a strain energy variation

$$(E\text{-}6)\qquad -\int_{t_0}^{t_1}\delta U dt = -\delta\int_t\int_V \sum_{mN}^{\infty\infty}\sum_{ps}^{\infty\infty}\tfrac{1}{2}C_{ijkl}h_{kl}(x,\phi,z)_{ps}\,q(t)_{ps}\,h_{ij}(x,\phi,z)_{mN}\,q(t)_{mN}\,dVdt$$

and in terms of the generalized force associated with the increase in strain energy;

$$(E\text{-}20)\qquad -\delta\int_{t_0}^{t_1}U dt = -\int_t \sum_{mN}^{\infty\infty}\left[\frac{\partial U}{\partial q(t)_{mN}}\right]\delta q_{mN}\,dt = +\int_t \sum_{mN}\left[\overset{(U)}{Q}(t)_{mN}\right]\delta q(t)_{mN}$$

$$= -\int_t\int_V \frac{\partial U_{vol}}{\partial \epsilon_{ij}}\,\delta\epsilon_{ij}\,dVdt = -\int_t\int_V \sigma_{ij}\sum_{mN}\frac{\partial\epsilon_{ij}}{\partial q(t)_{mN}}\,\delta q_{mN}\,dVdt$$

$$(3)\qquad -\int_{t_0}^{t_1}\delta U dt = -\int_{t_0}^{t_1}\sum_{mN}^{\infty\infty}\delta q(t)_{mN}\left[\sum_{ps}^{\infty\infty} q(t)_{ps}\int_V C_{ijkl}h_{kl,ps}h_{ij,mN}\,dV\right]dt$$

The term within the integral is the stiffness coefficient equivalent to E-24;

$$(4)\qquad K_{\underline{mN}-\underline{ps}} = \frac{\partial \overset{(U)}{Q}(t)_{\underline{mN}}}{\partial q(t)_{\underline{ps}}} = \frac{\partial\left(\frac{\partial U}{\partial q_{mN}}\right)}{\partial q(t)_{ps}} = -\int_V C_{ijkl}\,h_{ij,\underline{mN}}\,h_{kl,\underline{ps}}\,dV$$

NOTE: (Continuation of Note)

By the same token:

$$(5) \quad K_{\underline{ps}-\underline{mN}} = \frac{\partial Q(t)_{\underline{ps}}^{(U)}}{\partial q(t)_{\underline{mN}}} = -\frac{\partial\left(\frac{\partial U}{\partial q_{\underline{ps}}}\right)}{\partial q(t)_{\underline{mN}}} = -\int_V C_{ijk\ell}\, h_{ij,ps}\, h_{k\ell,\underline{mN}}\, dV = -\int_V C_{ijk\ell}\, h_{k\ell,\underline{ps}}\, h_{ij,\underline{mN}}\, dV$$

The latter equivalency is due to an interchange of the dummy variables in the summation.
Due to the symmetry of the elastic constants $C_{ijk\ell} = C_{k\ell ij}$;

$$(6) \quad K_{\underline{mN}-\underline{ps}} = K_{\underline{ps}-\underline{mN}} = -\int_V C_{ijk\ell}\, h_{ij,\underline{mN}}\, h_{k\ell,\underline{ps}}\, dV$$

This guarantees the proper usuage of Betti's Law for this system.

SECTION VIII

SOLUTION OF THE FORCED VIBRATION PROBLEM

With an assumed solution of the form:

(131) $$\left\{U^n_{k,mN}\right\} = \left\{f^n_k(x,\phi)_{mN}\right\} q(t)_{mN}$$

we have shown that the Lagrange equation (128) is to be satisfied. Equation (128) is repeated below for convenience:

(128) $$\ddot{q}(t)_{mN} + \omega^2_{mN} q(t)_{mN} = \frac{Q(t)_{mN}}{M_{mN}}$$

The definitions for the generalized mass M_{mN} and the generalized force $Q(t)_{mN}$ are given in equations (129) and (130).

We now consider a forcing function which is arbitrary when taken as a function of x and ϕ, but restricted to a step time function from o to τ and zero thereafter. Thus, the forcing function is written:

(132) $$\bar{P}^+(x,\phi,\tfrac{h}{2},t) = P^+_k \bar{e}_k = P_o\left[h(t) - h(t-\tau)\right]\bar{G}(x,\phi)$$

or, alternately:

(133) $$P^+_k \bar{e}_k = P_o\left[h(t) - h(t-\tau)\right] G_k(x,\phi)\bar{e}_k$$

We note also that one of the terms required in equation (130) is $\ddot{Y}_B(t)k$.

At this point, we should emphasize that although we have frequently referred to a "fixed-end" boundary, the constrained end structure equations are developed for the absolute (space-referenced) coordinate $\bar{r}$ and the c.g. motion is absorbed in this displacement coordinate. Therefore, any reference to the deformation coordinate $\bar{u}$, and the rigid body displacement, $\bar{Y}$, refers to the free cylinder in space. The acceleration of $\bar{Y}_B(t)$ with respect to the space fixed origin, is then found by computing the resultant force acting on the whole body, and dividing by the entire mass:

(134) $$\ddot{Y}_B(t) = \frac{\int_x\int_\phi (\bar{P}\cdot\bar{L}_p)\bar{L}_p(a+h)\,d\phi\,dx}{\gamma\int_x\int_\phi\int_z (a+z)\,dz\,d\phi\,dx}$$

From (134) and, using (133), we may write the components of $\bar{Y}_B(t)$ along the body-fixed triad $\bar{e}_k$:

(135) $$\ddot{Y}_B(t)_k = \ddot{\bar{Y}}_B(t)\cdot\bar{e}_k = \bar{L}_p\cdot e_k\left[\frac{P_0[h(t)-h(t-\tau)]\int\int\bar{\sigma}(x,\phi)\cdot\bar{L}_p(1+\frac{h}{a})\,a\,d\phi}{4h\pi a L\gamma}\right]$$

We now apply the Laplace transform operation to equation (128) (assuming zero initial conditions).

(136) $$q(s)_{mN} = \frac{Q(s)_{mN}}{(s^2+\omega_{mN}^2)M_{mN}}$$

We find it convenient to usethe following notation for generalized force:

(137) $$Q(t)_{mN} = \int_x\int_\phi (b_K^n P_K^+ + C_K^n \ddot{Y}_B(t)_K)\cdot f_K^n(x,\phi)_{mN}\,a\,d\phi\,dx$$

in which:

(138) $$b_K^n = (1+\frac{h}{a})h^n$$

(139) $$C_K^n = -2\gamma h + \frac{h^{2n}}{(3a)^n}$$

In order to transform $Q(t)_{mN}$, we consider the transform first of the component (133) and then the transform of (134). We pause to note that the era of interest is $t > \tau$, and that we intend to consider the limit, of pure impulse ($P_o\tau$ constant as $\tau \to 0$). The transform of (133) is:

$$P_K(s) = \mathcal{L} P_K(t) = \frac{P_o}{S} G_K(x,\phi)(1-e^{-s\tau}) \quad \text{For } t > \tau \tag{140}$$

Expanding the exponential term in (140):

$$P_K(s) = \frac{P_o G_K(x,\phi)}{S}\left[1-\left(1 - s\tau + \frac{s^2\tau^2}{2} - \frac{s^3\tau^3}{3} + \ldots\right)\right] \tag{141}$$

We now recognize that as $\tau \to 0$, the influence of the latter terms may be discounted. The expression becomes:

$$P_K(s)\Big|_{\tau \to 0} = P_o\tau G_K(x,\phi) \tag{142}$$

In the same way we can write, for $t > \tau$, as $\tau \to 0$ and $P_o\tau$ constant,

$$\ddot{Y}_B(s)_K = (\bar{\mathcal{L}}_p \cdot \bar{e}_K)\frac{P_o\tau}{4\pi h a L \gamma}\int_x\int_\phi\left[\bar{G}(x,\phi)\cdot\bar{\mathcal{L}}_p\right]\left(1+\frac{h}{a}\right)a\,d\phi\,dx \tag{143}$$

We now transform equation (137), making use of (142) and (143):

$$Q(s)_{mN} = P_o\tau\int_x\int_\phi\Big\langle\Big[b_K^n G_K(x,\phi) + \frac{C_K^n(\bar{\mathcal{L}}_p\cdot\bar{e}_K)}{4\pi h a L\gamma}\int_x\int_\phi \bar{G}(x,\phi)\cdot\bar{\mathcal{L}}_p\left(1+\frac{h}{a}\right)a\,d\phi\,dx\Big] + f_K^n(x,\phi)_{mN}\Big\rangle\, a\,d\phi\,dx \tag{144}$$

We substitute (144) into (136), and at the same time, take the inverse Laplace transform:

(145)

$$q(t)_{mN} = \frac{P_0 T}{\omega_{mN} M_{mN}} \left[\int_x \int_\phi \left[b_K^n G_K(x,\phi) + \frac{C_K^n(\bar{\mathcal{E}}_p \cdot \bar{e}_K)}{4\pi h a L \gamma} \int_x \int_\phi \bar{G}(x,\phi) \cdot \bar{\mathcal{E}}_p \left(1+\frac{h}{a}\right) a\, d\phi\, dx \right] f_{K,mN}^n\, a\, d\phi\, dx \right] \sin \omega_{mN} t$$

We substitute equation (145) in (131), and sum over all mN modes:

(146)

$$\left\{U_K^n\right\}_{t>T} = \sum_{mN}^{\infty} \left\{f_K^n(x,\phi)_{mN}\right\} \frac{P_0 T}{\omega_{mN} M_{mN}} \left[\int_x \int_\phi h^n G_K(x,\phi) f_K^n(x,\phi)_{mN} (a+h)\, d\phi\, dx \right.$$

$$\left. - \left(\int_x \int_\phi \frac{G_K(x,\phi)(\bar{e}_K \cdot \bar{\mathcal{E}}_p)\, a\, d\phi\, dx}{4\pi h a L \gamma} \right) \int_x \int_\phi (\bar{\mathcal{E}}_p \cdot \bar{e}_K) \frac{h^{2n}(2\gamma h)}{(3a)^n} f_K^n(x,\phi)_{mN} (a+h)\, d\phi\, dx \right] \sin \omega_{mN} t$$

Equation (146), it should be recalled, is the solution to the problem for the <u>limiting case</u> of the <u>forcing function in equation (132)</u>.

We seek to compare this solution with the one to be derived for the "initial velocity' problem:

(147) $$\left\{U_K^n\right\}_{t=0} = 0 \qquad \left\{\dot{U}_n^K\right\}_{t=0} \neq 0$$

For this we use the free vibration solution, equation (100), repeated below for convenience:

(100) $$\left\{U_K^n\right\} = \sum_m \sum_N \left\{f_K^n(x,\phi)_{mN}\right\} C_{mN} e^{i\omega_{mN} t}$$

We find it convenient to use the trigonometric equivalent (when adding the complex conjugate of the time term to (100):

(148)

$$\left\{U_K^n\right\} = \sum_m \sum_N \left\{f_K^n(x,\phi)_{mN}\right\} A_{mN} \sin \omega_{mN} t + B_{mN} \cos \omega_{mN} t$$

Since the initial displacement is taken equal to zero, the terms $B_{mN} = 0$. Equation (148) reduces to:

(149) $$\left\{U_K^n\right\} = \sum_m \sum_N \left\{f_K^n(x,\phi)_{mN}\right\} A_{mN} \sin \omega_{mN} t$$

Differentiate with respect to time, and substitute t = 0 to obtain:

(150) $$\left\{\dot{U}_K^n\right\}_{t=0} = \sum_m \sum_N \left\{f_K^n(x,\phi)_{mN}\right\} \omega_{mN} A_{mN}$$

We select <u>one</u> coefficient A_{mN} for one frequency $\omega_{\underline{mN}}$ for examination. Multiply each one of the five equations represented in 150 by the appropriate $F_K^n(x,\phi)_{mN}$, and integrate over the limits of x and ϕ:

(151) $$\int_x \int_\phi \left(\dot{U}_K^n\big|_{t=0}\right)\left(F_K^n(x,\phi)_{\underline{mN}}\, a\, d\phi\, dx = \int_x \int_\phi \sum_m \sum_N f_K^n(x,\phi)_{mN} F_K^n(x,\phi)_{\underline{mN}}\, \omega_{\underline{mN}} A_{\underline{mN}}\, a\, d\phi\, dx$$

Due to the orthogonality relationship (see Appendix E), only one term remains on the right hand side. Thus, for each nodal pair <u>mN</u>:

(152) $$A_{\underline{mN}} = \frac{\int_x \int_\phi (\dot{U}_K^n|_{t=0}) F_K^n(x,\phi)_{\underline{mN}}\, a\, d\phi\, dx}{\omega_{\underline{mN}} \int_x \int_\phi f_K^n(x,\phi)_{\underline{mN}} F_K^n(x,\phi)_{\underline{mN}}\, a\, d\phi\, dx}$$

Thus, equation (149) becomes:

(153) $$\left\{U_K^n\right\} = \sum_m \sum_N \left\{f_K^n(x,\phi)_{mN}\right\} \left[\frac{\int_x \int_\phi (\dot{U}_K^n|_{t=0}) F_K^n(x,\phi)_{mN}\, a\, d\phi\, dx}{\omega_{mN} \int_x \int_\phi f_K^n(x,\phi)_{mN} F_K^n(x,\phi)_{mN}\, a\, d\phi\, dx}\right] \sin \omega_{mN} t$$

If we state $\dot{u}_k^n\big|_{t=0}$, equation (153) may be used to present the solution to the initial velocity problem.

In order to specify the initial velocity distribution, we examine the equilibrium equations for forced vibration, (53-57).

We make the important assumption that at very short times, no deformation or strain terms exist. These equations then reduce to:

(154)

$$\text{a.}\quad P_1^+\left(1+\frac{h}{a}\right) - 2\gamma h\,\ddot{Y}_B(t)\cdot\bar{e}_1 - 2\gamma h\frac{\partial^2 U_1^0}{\partial t^2} - \frac{2\gamma h^3}{3a}\frac{\partial^2 U_1'}{\partial t^2} = 0$$

$$\text{b.}\quad P_1^+ h\left(1+\frac{h}{a}\right) - \frac{2\gamma h^3}{3a}\ddot{Y}_B(t)\cdot\bar{e}_1 - \frac{2\gamma h^3}{3a}\frac{\partial^2 U_1^0}{\partial t^2} - \frac{2\gamma h^3}{3}\frac{\partial^2 U_1'}{\partial t^2} = 0$$

$$\text{c.}\quad P_2^+\left(1+\frac{h}{a}\right) - 2\gamma h\,\ddot{Y}_B(t)\cdot\bar{e}_2 - 2\gamma h\frac{\partial^2 U_2^0}{\partial t^2} - \frac{2\gamma h^3}{3a}\frac{\partial^2 U_2'}{\partial t^2} = 0$$

$$\text{d.}\quad P_2^+(h)\left(1+\frac{h}{a}\right) - \frac{2\gamma h^3}{3a}\ddot{Y}_B(t)\cdot\bar{e}_2 - \frac{2\gamma h^3}{3a}\frac{\partial^2 U_2^0}{\partial t^2} - \frac{2\gamma h^3}{3}\frac{\partial^2 U_2'}{\partial t^2} = 0$$

$$\text{e.}\quad P_3^+\left(1+\frac{h}{a}\right) - 2\gamma h\,\ddot{Y}_B(t)\cdot\bar{e}_3 - 2\gamma h\frac{\partial^2 U_3^0}{\partial t^2} = 0$$

During the period $0<t<\tau$, P_k^+ is constant:

(155)
$$P_k^+ = P_0 G_k(x,\phi)$$

Since P_k^+ is constant, the acceleration $\ddot{Y}_{B,k}$ for $0<t<\tau$ is also constant, and by equation (135):

(156)
$$\ddot{Y}_{B,k}(0<t<\tau) = (\bar{\mathcal{E}}_p\cdot\bar{e}_k)\,\frac{P_0\int_L\int_\phi G_k(x,\phi)(\bar{e}_k\cdot\bar{\mathcal{E}}_p)\left(1+\frac{h}{a}\right)a\,d\phi\,dx}{4\pi h a L\gamma}$$

We now note that we can represent (154) in a shorthand (although not entirely satisfactory) manner:

(157)
$$P_k^+\left(1+\frac{h}{a}\right)h^n - 2\gamma h\frac{h^{2n}}{(3a)^n}\ddot{Y}_{B,k} - 2\gamma h\cdot\frac{h^{2n}}{3^n}\ddot{U}_k^{\,n} - \frac{2\gamma h^3}{3a}\ddot{U}_k^{\,n\pm1} = 0$$

(In the last term, the plus and minus signs **alternate** in a, b, c and d).

Since $\ddot{u}_k^{\,n}$ and $\ddot{u}_k^{\,n\pm 1}$ are constant during this interval, the velocity components are given by:

(158) $$\dot{U}_K^n(\tau) = \int_0^\tau \ddot{U}_K^n(t)\,dt = \ddot{U}_K^n \tau$$

We solve (157) for $u_k^{\,n}$ and multiply by τ :

(159)
$$\dot{U}_K^n(\tau) = \ddot{U}_K^n \tau = (P_K^+ \tau)\frac{3^n}{(2\gamma h)}\frac{(1+\frac{h}{a})h^n}{(h^{2n})} - \frac{3^n}{2\gamma h\,(h^{2n})}\cdot\frac{2\gamma h(h^{2n})}{(3a)^n}\ddot{Y}_{B,K}$$
$$-\frac{3^n}{2\gamma h}(h^{2n})\cdot\frac{2\gamma h^3}{3a}\cdot\ddot{U}_K^{n\pm 1}\tau$$

Substituting for $P_k^{\,+}$ and $\ddot{Y}_{B,k}$:

(160) $$\dot{U}_K^n(\tau) = \frac{1}{2\gamma h}\left[\frac{3}{h^2}\right]^n P_o \tau\, G_K(x,\phi)\left(1+\frac{h}{a}\right)h^n$$
$$-\frac{1}{2\gamma h}\left[\frac{3}{h^2}\right]^n \frac{P_o\tau(\bar{\mathcal{E}}_p\cdot\bar{e}_K)}{4\pi a L h\gamma}\int_x\int_\phi G_K(x,\phi)(\bar{e}_K\cdot\bar{\mathcal{E}}_p)\,2\gamma h\left[\frac{h^2}{3a}\right]^n\left(1+\frac{h}{a}\right)a\,d\phi\,dx$$
$$-\frac{1}{2\gamma h}\left[\frac{3}{h^2}\right]^n\left[\frac{2\gamma h^3}{3a}\dot{U}_K^{n\pm 1}(\tau)\right]$$

We now substitute equation (160) into equation (153) (noting that the demoninator of (153) can be expressed as M_{mN}), that we are considering the initial velocity as $\dot{u}_k^{\,n}(\tau)$ and that equation (153) will be valid for $t>\tau$

(161) $$\left\{U_K^n\right\}\Big|_{t>\tau} = \sum_m\sum_N\left\{f_K^n(x,\phi)_{mN}\right\}\frac{\sin\omega_{mN}t}{\omega_{mN}M_{mN}}\Bigg[P_o\tau\int_x\int_\phi\frac{h^n}{2\gamma h}\left[\frac{3}{h^2}\right]^n G_K(x,\phi)F_K^n(x,\phi)_{mN}(a+h)\,d\phi\,dx$$
$$-P_o\tau\int_x\int_\phi\frac{G_K(x,\phi)(\bar{e}_K\cdot\bar{\mathcal{E}}_p)\,a\,d\phi\,dx}{4\pi a L h\gamma}\int_x\int_\phi 2\gamma h\left[\frac{h^2}{3a}\right]^n\frac{(\bar{\mathcal{E}}_p\cdot\bar{e}_K)F_K^n(x,\phi)_{mN}(a+h)\,d\phi\,dx}{2\gamma h\left[\frac{h^2}{3}\right]^n}$$
$$-\int_x\int_\phi 2\gamma h\left[\frac{h^2}{3a}\right]^n\dot{U}_K^{n\pm 1}(\tau)\frac{F_K^n(x,\phi)_{mN}}{2\gamma h\left[\frac{h^2}{3}\right]^n}\,a\,d\phi\,dx\Bigg]$$

But we have used $F_K^{\,n}(x,\phi)_{mN}$ to refer to those portions of the solution multiplied by ω^2_{mN}. From equation (103) and (157):

(162) $$F^n_{K,mN} = 2\gamma h\left[\frac{h^2}{3}\right]^n f^n_{K,mN} + \frac{2\gamma h^3}{3a} f^{n\pm1}_{K,mN}$$

We substitute (162) into (161) to obtain:

(161a)
$$\left\{U^n_K\right\}_{t>\tau} = \sum_m \sum_N \{f^n_K(x,\phi)_{mN}\} \frac{\sin\omega_{mN}t}{\omega_{mN}M_{mN}} \Bigg[P_o\tau\int_x\int_\phi h^n G_K(x,\phi) f^n_K(x,\phi)_{mN}(a+h)\,d\phi\,dx$$
$$-P_o\tau\int_x\int_\phi \frac{G_K(x,\phi)(\bar{e}_K\cdot\bar{\mathcal{L}}_p)\,a\,d\phi\,dx}{4\pi a L\gamma}\int_x\int_\phi 2\gamma h\left[\frac{h^2}{3a}\right]^n(\bar{\mathcal{L}}_p\cdot\bar{e}_K) f^n_K(x,\phi)_{mN}(a+h)\,d\phi\,dx$$
$$+\frac{2\gamma h^3}{3a}\cdot\left[\frac{3}{h^2}\right]^n\frac{1}{2\gamma h}\int_x\int_\phi\Big\langle P_o\tau G_K(x,\phi)h^2\left(1+\frac{h}{a}\right) - P_o\tau\int_x\int_\phi \frac{G_K(x,\phi)(\bar{e}_K\cdot\bar{\mathcal{L}}_p)\,a\,d\phi\,dx}{4\pi a L h\gamma}\,\cdot$$
$$2\gamma h\left[\frac{h^2}{3a}\right]^n(\bar{\mathcal{L}}_p\cdot\bar{e}_K)\left(1+\frac{h}{a}\right) - \frac{2\gamma h^3}{3a}\dot{U}^{n\pm1}_K\Big\rangle f^{n\pm1}_K(x,\phi)_{mN}\,a\,d\phi\,dx$$
$$-\frac{2\gamma h^3}{3a}\int_x\int_\phi \dot{U}^{n\pm1}_K f^n_K(x,\phi)_{mN}\,a\,d\phi\,dx\Bigg]$$

The first two terms of (162) are indentically equal to (146). It remains to show that the latter four terms vanish.

Compare the section between ⟨ ⟩ brackets to equation (160), and it is then seen that the last four terms reduce to:

(163) $$+\frac{2\gamma h^3}{3a}\int_{x=0}^{L}\int_{\phi=0}^{2\pi}\left(\dot{U}^n_K(\tau) f^{n\pm1}_K(x,\phi)_{mN} - \dot{U}^{n\pm1}_K(\tau) f^n_K(x,\phi)_{mN}\right) a\,d\phi\,dx$$

In the integral the terms k, n, n±1 have the following values:

(164)

k=1	n=0 n=1	n±1=1 n±1=0
k=2	n=0 n=1	n±1=1 n±1=0
k=3	n=0	n±1 (not defined)

In the summation, the terms under the integral add to zero.

We conclude that equation (162) is identically equal to equation (146) in the limit, so that the forcing function problem and the initial velocity problem both yield the same answer.

We now briefly consider the characteristics of two typical forcing functions:

Cosine Frontal

The so-called "cosine frontal" loading is often used as a mathematical model for certain nuclear weapons effects on ICBM structures. In our nomenclature, the "cosine frontal" loading is specified:

(165) a. For $|\phi| < \frac{\pi}{2}$; $\bar{P}^{+} = -P_0\left[h(t) - h(t-\tau)\right]\cos\phi \; h(x)\bar{e}_3$

b. $\pi > |\phi| > \frac{\pi}{2}$; $\bar{P}^{+} = 0$

In expression (165a), h(t) and h(x) are unit step functions. τ represents the time duration of the pressure pulse. The load is thus seen to be directed radially inward, and symmetric in the xz plane with respect to ϕ=0, and uniform along the longitudinal axis.

The fact that the loading around the circumference exists only over the top half is mathematically cumbersome. We replace equations (165) by the equivalent expression involving a Fourier Series expansion in ϕ:

(166) $$P^{+} = -\bar{e}_3 P_0\left[h(t) - h(t-\tau)\right]\sum_{p=0}^{\infty} b_p \cos p\phi$$

In equation (166), the Fourier coefficients are:

(167)
$$b_0 = \frac{1}{\pi}$$
$$b_1 = 0.5$$
$$b_p = \frac{(-2)(-1)^{p/2}}{\pi(p^2-1)} \qquad p = 2,4,6,\ldots$$
$$b_p = 0 \qquad p = 1,3,5,\ldots$$

We see that this series, which includes all the even cosine waves, may not yield the most information, because the coefficients of the series diminish rapidly in magnitude.

A similar problem exists with respect to a Fourier series expansion in the x direction. The double series becomes:

(168)
$$P^{+} = -\bar{e}_3 P_0 \left[h(t) - h(t-\tau)\right] \sum b_p \cos p\phi \sum_{r=1,3,5} a_r \sin \frac{r\pi x}{L}$$

The coefficients a_r are found by:

(169)
$$a_r = \frac{4}{r\pi} \qquad r = 1,3,5$$

and are also noted as diminishing rapidly in magnitude.

<u>Impulsive Force at Center</u>

As an alternate to the cosine frontal load, we consider a single concentrated force acting radially inward at the position $x = \frac{L}{2}$, $\phi = o$, applied to the outer surface. We set the value of this force as:

(170)
$$F = (W_0)(a+h)(\Delta\phi)(\Delta x)$$

The force can then be expressed as a combination of unit step functions as follows:

(171)
$$\bar{P}^{+} = -\bar{e}_3 \frac{F}{(a+h)\Delta\phi\Delta x}\left[h\left(x-\left[\frac{L}{2}-\frac{\Delta x}{2}\right]\right) - h\left(x-\left[\frac{L}{2}+\frac{\Delta x}{2}\right]\right)\right] \cdot$$
$$\cdot\left[h(\phi) - h\left(\phi-\left[2n\pi+\frac{\Delta\phi}{2}\right]\right) - h\left(\phi-\left[2n\pi-\frac{\Delta\phi}{2}\right]\right)\right.$$
$$\left. - h(\phi-2n\pi)\right] \cdot \left[h(t) - h(t-\tau)\right]$$

When this is expanded in a Fourier series both in x and ϕ, the result becomes:

$$\bar{p}^{+} = -\bar{e}_3 \left[\frac{F}{2\pi L(a+h)}\right] \cdot \left[h(t) - h(t-\tau)\right] \cdot \sum_{p=0,1,2} b_p \cos p\phi \sum_{r=1,3,5} a_r \sin \frac{\pi r x}{L} \quad (172)$$

In expression (172), the Fourier coefficients are given by:

$$(173) \quad \begin{aligned} & b_p = 1 \qquad p = 0 \\ & b_p = 2 \qquad p = 1,2,3,4,5 \ldots \\ & a_r = 2 \sin \frac{r\pi}{2} \qquad r = 1,3,5 \ldots \end{aligned}$$

All the waves that are excited are thus represented as of the same magnitude. This is probably a better test function to use than the cosine frontal form because the transmissibility associated with a particular normal mode will be more easily identified.

While these forcing functions are suggested for initial analysis, consideration need not be so limited. We proceed to consideration of the parameters to be used to evaluate the response of the structure to a forcing function.

According to the Mises-Hencky theory, plastic flow is initiated when the second invariant (I'_2) of the deviatoric stress tensor at a point reaches a certain maximum value. The second invariant (I'_2) may be conveniently expressed in terms of the stress components σ_{ik} as follows:

$$I'_2(x,\phi,z,t) = \frac{1}{6}\left[(\sigma_{11}-\sigma_{22})^2 + (\sigma_{22}-\sigma_{33})^2 + (\sigma_{33}-\sigma_{11})^2 + 6(\sigma_{12}^2 + \sigma_{13}^2 + \sigma_{23}^2)\right] \quad (174)$$

The critical value for $I'_2(x,\phi,z,t)$ is determined by σ_0, the yield stress in simple (uniaxial) tension:

$$\frac{1}{3}\sigma_0^2 \geq I'_2(x,\phi,z,t) \quad (175)$$

A term called the "stress intensity", σ_r, has been defined as:

$$\sigma_r = \frac{1}{\sqrt{2}}\left[(\sigma_{11}-\sigma_{22})^2+(\sigma_{22}-\sigma_{33})^2+(\sigma_{33}-\sigma_{11})^2+6\left(\sigma_{12}^2+\sigma_{13}^2+\sigma_{23}^2\right)\right]^{\frac{1}{2}} \tag{176}$$

Thus, the Mises-Hencky yield criterion may be stated simply as:

$$\sigma_o \geq \sigma_r\,(x,\phi,z,t) \tag{177}$$

Clearly, then the parameter of interest is σ_r. Equation (177) determines whether or not plastic flow will be initiated, as well as when or where this will happen.

The deformation, stress and strain acting on planes normal to the cylinder surface (σ_{11}, σ_{22}, and σ_{12}) are maximum at either the inner or outer surfaces ($z=\pm h$). The stress on the surface (σ_{31}, σ_{32} and σ_{33}) is zero, once the initial impulsive pressure subsides.

Consequently, we assume that the maximum stress intensity occurs at the inner and outer surface:

$$\sigma_r\,(x,\phi,z,t)_{max} = \sigma_r\,(x,\phi,\pm h,t) \tag{178}$$

The stress intensity becomes:

$$\sigma_r\,(x,\phi,h,t) = \frac{1}{\sqrt{2}}\left[(\sigma_{11}-\sigma_{22})^2+\sigma_{22}^2+\sigma_{11}^2+6\sigma_{12}^2\right]^{\frac{1}{2}} \tag{179}$$

Equivalently,

$$\sigma_r\,(x,\phi,h,t) = \frac{1}{2}\left[\sigma_{11}^2+\sigma_{22}^2-2\sigma_{11}\sigma_{22}+3\sigma_{12}^2\right]^{\frac{1}{2}} \tag{180}$$

In expression,

(181) a. $$\sigma_{11} = \frac{E}{1-\nu^2}\left(\epsilon_{11} + \nu\epsilon_{22}\right)$$

b. $$\sigma_{22} = \frac{E}{1-\nu^2}\left(\epsilon_{22} + \nu\epsilon_{11}\right)$$

c. $$\sigma_{12} = \frac{E}{1-\nu^2} \cdot \frac{1-\nu}{2} \cdot \gamma_{12}$$

The strain-deformation relations have also been previously developed. The pertinent results are repeated below:

(182) a. $$\epsilon_{11}\big|_h = \frac{\partial U_1}{\partial x} = \frac{\partial U_1^0}{\partial x} + h\frac{\partial U_1'}{\partial x}$$

b. $$\epsilon_{22}\big|_h = \frac{1}{a+h}\left(\frac{\partial U_2}{\partial \phi} + U_3\right) = \frac{1}{a+h}\left[\frac{\partial U_2^0}{\partial \phi} + U_3^0 + h\frac{\partial U_2'}{\partial \phi}\right]$$

c. $$2\epsilon_{12}\big|_h = \gamma_{12}\big|_h = \frac{\partial U_2}{\partial x} + \frac{1}{a+h}\frac{\partial U_1}{\partial \phi} = \frac{\partial U_2^0}{\partial x} + h\frac{\partial U_2'}{\partial x} + \frac{1}{a+h}\frac{\partial U_1^0}{\partial \phi} + \frac{h}{a+h}\frac{\partial U_1'}{\partial \phi}$$

Substituting into the stress-strain expressions:

(183) a. $$\sigma_{11} = \frac{E}{1-\nu^2}\left[\frac{\partial U_1^0}{\partial x} + h\frac{\partial U_1'}{\partial x} + \frac{\nu}{a+h}\left(\frac{\partial U_2^0}{\partial \phi} + U_3^0 + h\frac{\partial U_2'}{\partial \phi}\right)\right]$$

b. $$\sigma_{22} = \frac{E}{1-\nu^2}\left[\frac{1}{a+h}\left(\frac{\partial U_2^0}{\partial \phi} + U_3^0 + h\frac{\partial U_2'}{\partial \phi}\right) + \nu\left(\frac{\partial U_1^0}{\partial x} + h\frac{\partial U_1'}{\partial x}\right)\right]$$

c. $$\sigma_{12} = \frac{E}{1-\nu^2}\left[\left(\frac{1-\nu}{2}\right)\left(\frac{\partial U_2^0}{\partial x} + h\frac{\partial U_2'}{\partial x} + \frac{1}{a+h}\frac{\partial U_1^0}{\partial \phi} + \frac{h}{a+h}\frac{\partial U_1'}{\partial \phi}\right)\right]$$

The solution to the free vibration problem, with the initial velocity specified, has been found, as equation (146), repeated below for convenience:

(146) $$\left\{U_K^n\right\} = \sum_m \sum_N \left\{f_K^n(x,\phi)_{mN}\right\} \frac{P_0 \tau}{\omega_{mN} M_{mN}} \sin \omega_{mN} t \left[\int_x \int_\phi h^n G_K(x,\phi) f_K^n(x,\phi)_{mN} (a+h)\, d\phi\, dx - \left(\int_x \int_\phi \frac{G_K(x,\phi)(\bar{e}_K \cdot \bar{\mathcal{L}}_p)\, a\, d\phi\, dx}{4\pi h a L \gamma}\right) \int_x \int_\phi (\bar{\mathcal{L}}_p \cdot \bar{e}_K) \frac{h^{2n}(2\gamma h)}{3a^n} f_K^n(x,\phi)_{mN} (a+h)\, d\phi\, dx\right]$$

We now introduce a normalizing factor for the natural frequency, ω_0, which is the lowest extensional frequency of a ring in plane strain:

(184) $$\omega_0 = \left[\frac{E}{\gamma a^2(1-\nu^2)}\right]^{\frac{1}{2}}$$

Correspondingly, we write:

(185) $$\omega_{mN} = \Omega_{mN}\,\omega_0$$

in which Ω_{mN} is a non-dimensional parameter.

In equation (146), we recall the definition of the generalized mass M_{mN}, equation (129):

(129) $$M_{mN} = \int_x\int_\phi F_K^n(x,\phi)_{mN}\, f_K^n(x,\phi)_{mN}\, a\,d\phi\,dx$$

We find it convenient to introduce the notation:

(186) $$M_{mN} = 2\pi h\, M'_{mN}$$

where we understand that the $(2\pi h)$ is factored out of the term $F^n_{K,mN}$ (see equation(162)). It will be seen that equation (146) can be written:

(187) $$\left\{U_K^n\right\} = \frac{P_0\tau}{2\pi h\omega_0}\sum_{mN}\left\{f_K^n(x,\phi)_{mN}\right\}\cdot\left[\frac{K'_{mN}}{\Omega_{mN}M'_{mN}}\right]\sin\omega_{mN}t$$

in which K' refers to the bracketed terms in equation (146). Both K' and M', for a particular mN pair, are defined.

We now note that, for any one equation $u_k{}^n$,

(188) $$\frac{\partial U_K^n}{\partial x} = \frac{P_0\tau}{\omega_0 2\pi h}\sum_{mN}\frac{\partial f^n_{K,mN}}{\partial x}\left[\frac{K'_{mN}}{\Omega_{mN}M'_{mN}}\right]\sin\omega_0\Omega_{mN}t$$

(189) $$\frac{\partial U_k^n}{a\partial\phi} = \frac{P_0\tau}{\omega_0 2\bar{s}h}\sum_{mN}\frac{\partial f^n_{k,mN}}{a\partial\phi}\left[\frac{K'_{mN}}{\Omega_{mN}M'_{mN}}\right]\sin\omega_0\Omega_{mN}t$$

The term $f_k^n(x,\phi)_{mN}$ is defined by equation (99) and (85). We see that, for each mN pair:

(190) $$\frac{\partial f_k^n}{\partial x} = \frac{1}{L}\cdot\frac{L}{a}\sum_{i=1}^{10}\mathcal{T}^n_{kmNi}\Gamma_{mNi}\lambda_{mNi}e^{\frac{\lambda_{mNi}x}{a}}(\cos m\phi \cup \sin m\phi)$$

(191) $$\frac{\partial f_k^n}{a\partial\phi} = \frac{1}{L}\cdot\frac{mL}{a}\sum_{i=1}^{10}\mathcal{T}^n_{kmNi}\Gamma_{mNi}e^{\frac{\lambda_{mNi}x}{a}}\frac{\partial(\cos m\phi \cup \sin m\phi)}{\partial(m\phi)}$$

Employing (190) and (191) in (188) and (189), the form of σ_{11}, σ_{22}, σ_{12} (refer to (183)) is seen to be typically:

(192)
$$\sigma_{ps} = \frac{E}{1-\nu^2}\cdot\frac{P_0\tau}{2\bar{s}h\omega_0 L}\left[\sum_{mN}\left[\frac{K'_{mN}}{\Omega_{mN}M'_{mN}}\right]\sin\omega_0\Omega_{mN}t \cdot \cos m\phi \cup \sin m\phi \,\times\right.$$
$$\left.\times\sum_{i=1}^{10}\Gamma_{mNi}e^{\lambda_{mNi}\frac{x}{a}}\mathcal{F}(\{\mathcal{T}^n_{kmNi}\},\lambda_{mNi},m,\frac{h}{a},\frac{L}{a},a)\right]_{ps}$$

In (179), the stress intensity σ_r is noted to involve the square root of the sum of several terms, each of which is a product $\sigma_{ps}\sigma_{qr}$. Thus:

(193) $$\sigma_r = \sqrt{\sum\sigma_{ps}\sigma_{qr}}$$

Examining (192), we now note the equivalent form:

(194) $$\sigma_r = \frac{E}{1-\nu^2}\cdot\frac{P_0\tau}{2\bar{s}h\omega_0 L}\sqrt{\sum\left[\sum_{mN}\quad\right]_{ps}\cdot\left[\sum_{mN}\quad\right]_{qr}}$$

where the terms in [ ] brackets are obtainable from (192). Now, utilize equation (184) to write:

$$\sigma_r = \left[\frac{1}{(1-\nu^2)E}\right]^{\frac{1}{2}} \frac{P_0 t}{2\frac{h}{a}\cdot\frac{L}{a}\cdot a}\left(\sqrt{\quad}\right) \tag{195}$$

The dimensionless parameter $R(x,\phi,z,t)$ becomes:

$$R\left(x,\phi,z,t,\frac{L}{a},\frac{h}{a},a\right) = \sqrt{\quad} = \frac{\sigma_r(x,\phi,z,t)}{\left[\frac{1}{(1-\nu^2)E}\right]^{1/2} 2\cdot\frac{h}{a}\cdot\frac{L}{a}\cdot a} \tag{196}$$

Equation (196) provides the basic tools necessary for parametric examination.

Given any particular cylinder, we would propose to plot R vs. time for various points that may be expected to display a critical section. We expect, actually, that the value of R for a free-free cylinder will be greatest at $x = \frac{L}{2}$, $\phi = 0$, and for a clamped cylinder, at $x = 0$, $\phi = 0$.

Assuming that it is possible to locate a critical section, then for a given cylinder the value of R found using the five variable theory (R_5) will be compared with the value of R found from the three variable theory.

It is hoped that the importance of this effect, perhaps measured as $(R_5-R_3)/R_5$, can be presented as a function of the significant parameters, including time of divergence and series truncation.

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APPENDIX A

VARIATION OF THE LAGRANGIAN DENSITY

The application of Hamilton's principle requires that:

(A-1) $$\int_t\int_V \delta\left(\tfrac{1}{2}\rho\dot{\bar{r}}\cdot\dot{\bar{r}}\right)dV\,dt - \int_t\int_V \delta\left(\tfrac{1}{2}\sigma_{ik}\epsilon_{ik}\right)dV\,dt + \int_t\int_\Sigma \bar{P}\cdot\delta\bar{r}\,d\Sigma\,dt = 0$$

We evaluate these, term by term, starting with the second term, which relates to the strain energy.

STRAIN ENERGY

The strain-displacement equations in curvilinear coordinates for a linear infinitesimal strain theory are represented by the tensor:

(A-2) $$\epsilon_{ik} = \frac{1}{2}\left[\frac{\partial u_i}{\partial S_k} + \frac{\partial u_k}{\partial S_i} + \left(\bar{e}_i\cdot\frac{\partial \bar{e}_t}{\partial S_k} + \bar{e}_k\cdot\frac{\partial \bar{e}_t}{\partial S_i}\right)u_t\right]$$

The constitutive equations of an isotropic material are:

(A-3) $$\sigma_{ik} = \mathcal{L}\epsilon_{tt}\delta_{ik} + 2G\epsilon_{ik}$$

in which ϵ_{tt} is the dilatation and $\mathcal{L}$, G are Lame's constants. For a homogeneous material, $\mathcal{L}$ and G do not vary. (δ_{ik} is the Kronecker delta, which vanishes for $i \neq k$, and is equal to 1 for $i = k$).

Both σ_{ik} and ϵ_{ik} are thus functions of the deformation coordinates u_k and their space derivatives. We have already indicated that u_k is taken as a function containing the parameter α_ℓ (refer to equation 19). Thus:

(A-4)

$$-\delta\left(\frac{1}{2}\sigma_{ik}e_{ik}\right) = -\frac{\partial}{\partial\alpha_\ell}\left(\frac{1}{2}\sigma_{ik}e_{ik}\right)d\alpha_\ell$$

Before proceeding, we shall demonstrate that:

(A-5) $$-\delta U = -\delta\left(\frac{1}{2}\sigma_{ik}e_{ik}\right) = -\sigma_{ik}\delta(e_{ik}) = -\frac{\partial U}{\partial e_{ik}}\delta(e_{ik})$$

where $\sigma_{ik} = \partial U/\partial e_{ik}$ for an elastic material with no dissipative properties. This may be done as follows. First, expand $-\delta(1/2\,\sigma_{ik}e_{ik})$

(A-6) $$-\delta\left(\frac{1}{2}\sigma_{ik}e_{ik}\right) = -\frac{\partial}{\partial\sigma_{ik}}(\quad)\delta\sigma_{ik} - \frac{\partial}{\partial e_{ik}}(\quad)\delta e_{ik}$$

(A-7) $$-\delta\left(\frac{1}{2}\sigma_{ik}e_{ik}\right) = -\frac{1}{2}e_{ik}\delta\sigma_{ik} - \frac{1}{2}\sigma_{ik}\delta e_{ik}$$

The expression for σ_{ik} is available from (A-3). We rewrite (A-3) below:

(A-3) $$\sigma_{ik} = \mathcal{L}e_{tt}\delta_{ik} + 2Ge_{ik}$$

Introducing an equivalent form,

(A-8) $$\sigma_{ik} = \mathcal{L}(e_{tj}\delta_{tj})\delta_{ik} + 2Ge_{ik}$$

Taking $\delta\sigma_{ik}$:

(A-9) $$\delta\sigma_{ik} = \frac{\partial\sigma_{ik}}{\partial e_{tj}}\delta(e_{tj}) + \frac{\partial\sigma_{ik}}{\partial e_{ik}}\delta(e_{ik})$$

This reduces to:

(A-10) $$\delta\sigma_{ik} = \mathcal{L}\delta_{tj}\delta_{ik}\delta(e_{tj}) + 2G\delta(e_{ik})$$

Now, substitute (A-10) in (A-7):

(A-11) $$-\delta\left(\tfrac{1}{2}\sigma_{ik}\epsilon_{ik}\right) = -\tfrac{1}{2}\sigma_{ik}\delta(\epsilon_{ik}) - \tfrac{1}{2}\epsilon_{ik}\left\{\mathcal{L}\delta_{tj}\delta_{ik}\delta(\epsilon_{tj}) + 2G\delta(\epsilon_{ik})\right\}$$

Factor out the ½ to obtain:

(A-12) $$-\delta\left(\tfrac{1}{2}\sigma_{ik}\epsilon_{ik}\right) = -\tfrac{1}{2}\left\{\sigma_{ik}\delta(\epsilon_{ik}) + \mathcal{L}\epsilon_{ik}\delta_{tj}\delta_{ik}\delta(\epsilon_{tj}) + 2G\epsilon_{ik}\delta(\epsilon_{ik})\right\}$$

In the second term, $\epsilon_{ik}\delta_{ik} \rightarrow \epsilon_{tt}$

(A-13) $$-\delta\left(\tfrac{1}{2}\sigma_{ik}\epsilon_{ik}\right) = -\tfrac{1}{2}\left\{\sigma_{ik}\delta(\epsilon_{ik}) + \mathcal{L}\epsilon_{tt}\delta_{tj}\delta(\epsilon_{tj}) + 2G\epsilon_{ik}\delta(\epsilon_{ik})\right\}$$

Now change the indices so that $\delta_{tj}\delta(\epsilon_{tj}) \rightarrow \delta_{ik}\delta(\epsilon_{ik})$

(A-14) $$-\delta\left(\tfrac{1}{2}\sigma_{ik}\epsilon_{ik}\right) = -\tfrac{1}{2}\left\{\sigma_{ik}\delta(\epsilon_{ik}) + \mathcal{L}\epsilon_{tt}\delta_{ik}\delta(\epsilon_{ik}) + 2G\epsilon_{ik}\delta(\epsilon_{ik})\right\}$$

Factor out $\delta(\epsilon_{ik})$

(A-15) $$-\delta\left(\tfrac{1}{2}\sigma_{ik}\epsilon_{ik}\right) = -\tfrac{1}{2}\left\{\sigma_{ik} + \mathcal{L}\epsilon_{tt}\delta_{ik} + 2G\epsilon_{ik}\right\}\delta(\epsilon_{ik})$$

Note that the last two terms are equal to σ_{ik} by eqnation (A-3), so that (A-15) reduces to:

(A-5) $$-\delta\left(\tfrac{1}{2}\sigma_{ik}\epsilon_{ik}\right) = -\sigma_{ik}\delta(\epsilon_{ik})$$

(This is what we set out to demonstrate).

To proceed with the actual variation of the strain energy term, use (A-2) and (A-5) in (A-1):

(A-16) $$-\int_t\int_V \delta\left(\tfrac{1}{2}\sigma_{ik}\epsilon_{ik}\right)dV\,dt = -\int_t\int_V \frac{\sigma_{ik}}{2}\,\delta\left\{\frac{\partial u_i}{\partial S_k} + \frac{\partial u_k}{\partial S_i} + \right.$$
$$\left. + \left(\bar{e}_i\cdot\frac{\partial\bar{e}_t}{\partial S_k} + \bar{e}_k\cdot\frac{\partial\bar{e}_t}{\partial S_i}\right)u_t\right\}dV\,dt$$

Consider the first two terms on the right side of (A-16):

(A-17) $$-\int_t\int_V \frac{\sigma_{ik}}{2}\delta\left\{\frac{\partial u_i}{\partial S_k}+\frac{\partial u_k}{\partial S_i}\right\} = -\int_t\int_V\left\{\frac{\sigma_{ik}}{2}\delta\frac{\partial u_i}{\partial S_k}+\frac{\sigma_{ik}}{2}\delta\frac{\partial u_k}{\partial S_i}\right\}dVdt$$

Since k and i now become dummy indices, interchange them in the first term:

(A-18) $$= -\int\int\left(\frac{\sigma_{ki}}{2}\delta\frac{\partial u_k}{\partial S_i}+\frac{\sigma_{ik}}{2}\delta\frac{\partial u_k}{\partial S_i}\right)dVdt$$

In general, because the stress tensor is symmetric,

(A-19) $$\sigma_{ki} = \sigma_{ik}$$

Using (A-19), and interchanging the operators δ and $\frac{\partial}{\partial S_i}$

(A-20) $$-\int_t\int_V \frac{\sigma_{ik}}{2}\delta\left\{\frac{\partial u_i}{\partial S_k}+\frac{\partial u_k}{\partial S_i}\right\} dVdt = -\int_t\int_V \sigma_{ik}\frac{\partial}{\partial S_i}\delta u_k\, dVdt$$

We now employ the definition of dV (equation (14) to yield:

(A-21) $$= -\int_t\int_{S_1}\int_{S_2}\int_{S_3} \sigma_{ik}\frac{\partial}{\partial S_i}\delta u_k\, dS_1 dS_2 dS_3\, dt$$

It is understood that i can take on in succession the values 1, 2, 3. Using equation (15), we rewrite (A-21) to:

(A-22) $$= -\int_t\int_{\lambda_1}\int_{\lambda_2}\int_{\lambda_3} \sigma_{ik}\frac{\partial}{\partial S_i}(\delta u_k)h_1 d\lambda_1\, h_2 d\lambda_2\, h_3 d\lambda_3\, dt$$

In general,

(A-23) $$\frac{\partial}{\partial S_i} = \frac{\partial}{\partial \lambda_i} \cdot \frac{\partial \lambda_i}{\partial S_i} = \frac{1}{h_i} \frac{\partial}{\partial \lambda_i}$$

The form of (A-22) when i = 1 is:

(A-24) $$= -\int_t \int_{\lambda_1} \int_{\lambda_2} \int_{\lambda_3} \sigma_{1K} \frac{\partial}{\partial \lambda_1} (\delta u_K) d\lambda_1 \; h_2 d\lambda_2 \, h_3 d\lambda_3 \, dt$$

Integrating (A-24) by parts with respect to λ_1

(A-25) $$= -\int_t \int_{\lambda_2} \int_{\lambda_3} (\sigma_{1K} h_2 h_3) \delta u_K \Big|_{\lambda_1(MIN)}^{\lambda_1(MAX)} d\lambda_2 \, d\lambda_3 \, dt$$

$$+ \int_t \int_{\lambda_1} \int_{\lambda_2} \int_{\lambda_3} \delta u_K \frac{\partial}{\partial \lambda_1} (\sigma_{1K} h_2 h_3) d\lambda_1 d\lambda_2 d\lambda_3 dt$$

Similarly, i can = 2 or 3, so that equation (A-25) can serve as the model for the complete expression of the result of (A-22). We can thus write that (A-22) becomes:

(A-26) $$= -\int_t \int_{\lambda_2} \int_{\lambda_3} (\sigma_{1K} \delta u_K h_2 h_3) \Big|_{\lambda_1(MIN)}^{\lambda_1(MAX)} d\lambda_2 \, d\lambda_3 \, dt$$

$$- \int_t \int_{\lambda_1} \int_{\lambda_3} (\sigma_{2K} \delta u_K h_1 h_3) \Big|_{\lambda_2(MIN)}^{\lambda_2(MAX)} d\lambda_1 d\lambda_3 \, dt$$

$$- \int_t \int_{\lambda_1} \int_{\lambda_2} (\sigma_{3K} \delta u_K h_1 h_2) \Big|_{\lambda_3(MIN)}^{\lambda_3(MAX)} d\lambda_1 d\lambda_2 dt$$

$$+ \int_t \int_{\lambda_1} \int_{\lambda_2} \int_{\lambda_3} \delta u_K \Big[\frac{\partial}{\partial \lambda_1} (\sigma_{1K} h_2 h_3) + \frac{\partial}{\partial \lambda_2} (\sigma_{2K} h_1 h_3)$$

$$+ \frac{\partial}{\partial \lambda_3} (\sigma_{3K} h_1 h_2) \Big] d\lambda_1 d\lambda_2 d\lambda_3 \, dt$$

In each of the first three terms, (σ_{ik} δu_k) be evaluated at the surface, at the extreme positions taken by the direction coordinates λ_i. Examine the first term and designate $d\Sigma_1$ to represent the surface element encountered. We may now use equation (16) to write, for the first term only:

(A-27) $$= -\int_t \int_{\Sigma_1} (\sigma_{1K} \delta u_K) \bar{n}\cdot\bar{e}_1 \Big|_{\lambda_1(MIN)}^{\lambda_1(MAX)} d\Sigma_1 \, dt$$

Expand (A-27) to find:

(A-28) $$= -\int_t \int_{\Sigma_1} (\sigma_{1K} \delta u_K) \bar{n}\cdot\bar{e}_1 \Big|^{\lambda_1(MAX)} d\Sigma_1 dt + \int_t \int_{\Sigma_1} (\sigma_{1K} \delta u_K) \bar{n}\cdot(-\bar{e}_1) \Big|^{\lambda_1(MIN)} d\Sigma_1 dt$$

To find:

(A-29) $$= -\int_t \int_{\Sigma_1} \bar{n}\cdot\bar{e}_1 \left\{ (\sigma_{1K} \delta u_K)^{\lambda_1(MAX)} + (\sigma_{1K} \delta u_K)^{\lambda_1(MIN)} \right\} d\Sigma_1 \, dt$$

The first three terms of (A-26) thus become

(A-30) $$-\int_t \int_{\Sigma_i} \bar{n}\cdot\bar{e}_i \left\{ (\sigma_{iK} \delta u_K)^{\lambda_i(MAX)} + (\sigma_{iK} \delta u_K)^{\lambda_i(MIN)} \right\} d\Sigma_i \, dt$$

Consider now the fourth term of (A-26) and letting ℓ, m, n represent the subscripts of λ, we summarize it as follows:

(A-31) $$= + \int_t \int_{\lambda_\ell} \int_{\lambda_m} \int_{\lambda_n} \frac{\partial}{\partial \lambda_\ell} (\sigma_{\ell K} h_m h_n) \delta u_K \, d\lambda_\ell \, d\lambda_m \, d\lambda_n \, dt$$

with $\ell \neq m \neq n$.

Expanding under the integral sign:

(A-32)

$$= +\int_t\int_{\lambda_\ell}\int_{\lambda_m}\int_{\lambda_n} \frac{\partial \sigma_{\ell k}}{\partial \lambda_\ell}(h_m h_n)\delta u_k d\lambda_\ell d\lambda_m d\lambda_n dt$$

$$+\int_t\int_{\lambda_\ell}\int_{\lambda_m}\int_{\lambda_n} \sigma_{\ell k}\frac{\partial}{\partial \lambda_\ell}(h_m h_n)\delta u_k d\lambda_\ell d\lambda_m d\lambda_n dt$$

Rewrite (A-32) as follows:

(A-33)

$$= +\int_t\int_{\lambda_\ell}\int_{\lambda_m}\int_{\lambda_n} \left(\frac{1}{h_\ell}\frac{\partial \sigma_{\ell k}}{\partial \lambda_\ell}\right)\delta u_k\, h_\ell d\lambda_\ell\, h_m d\lambda_m h_n d\lambda_n dt$$

$$+\int_t\int_{\lambda_\ell}\int_{\lambda_m}\int_{\lambda_n} \frac{\sigma_{\ell k}}{h_\ell h_m h_n}\frac{\partial}{\partial \lambda_\ell}(h_m h_n)\delta u_k\, h_\ell d\lambda_\ell\, h_m d\lambda_m h_n d\lambda_n dt$$

Simplifying,

(A-34) $$= +\int_t\int_V\left\{\frac{1}{h_\ell}\frac{\partial \sigma_{\ell k}}{\partial \lambda_\ell} + \sigma_{\ell k}\left[\frac{1}{h_\ell h_n}\frac{\partial h_n}{\partial \lambda_\ell} + \frac{1}{h_\ell h_m}\frac{\partial h_m}{\partial \lambda_\ell}\right]\right\}\delta U_k dVdt$$

Since we know that $\partial h_\ell/\partial\lambda$ = 0, we can add $1/h_\ell h_\ell\ \partial h_\ell/\partial\lambda_\ell$ to the second term without changing anything. Noting that $h_\ell\partial\lambda_\ell = \partial s_\ell$ we write:

(A-35) $$= +\int_t\int_V\left\{\frac{\partial \sigma_{\ell k}}{\partial s_\ell} + \sigma_{\ell k}\left[\frac{1}{h_\ell h_n}\frac{\partial h_n}{\partial \lambda_\ell} + \frac{1}{h_\ell h_m}\frac{\partial h_m}{\partial \lambda_\ell} + \frac{1}{h_\ell h_\ell}\frac{\partial h_\ell}{\partial \lambda_\ell}\right]\right\}\delta U_k dVdt$$

The second term can be replaced by a tensor:

(A-36) $$= +\int_t\int_V \left\{\frac{\partial\sigma_{\ell k}}{\partial s_\ell} + \sigma_{\ell k}\frac{\partial h_t}{\partial\lambda_\ell}\cdot\frac{1}{h_\ell h_t}\right\}\delta U_k\, dV dt$$

Replace the index ℓ by i to obtain:

(A-37) $$= +\int_t\int_V \left\{\frac{\partial\sigma_{ik}}{\partial s_i} + \sigma_{ik}\frac{\partial h_t}{\partial\lambda_i}\frac{1}{h_i h_t}\right\}\delta U_k\, dV dt$$

It may be shown that

(A-38) $$\bar{a}_t\cdot\frac{\partial\bar{a}_i}{\partial\lambda_t} = \frac{1}{h_i}\frac{\partial h_t}{\partial\lambda_i}$$

Multiply both sides of (A-38) by $1/h_t$

(A-39) $$\bar{a}_t\cdot\frac{1}{h_t}\frac{\partial\bar{a}_i}{\partial\lambda_t} = \frac{1}{h_i}\frac{1}{h_t}\frac{\partial h_t}{\partial\lambda_i} = \bar{a}_t\cdot\frac{\partial\bar{a}_i}{\partial s_t}$$

Thus, equation (37) becomes:

(A-40) $$= +\int_t\int_V \left\{\frac{\partial\sigma_{ik}}{\partial s_i} + \sigma_{ik}\left(\bar{a}_t\cdot\frac{\partial\bar{a}_i}{\partial s_t}\right)\right\}\delta u_k\, dV dt$$

We return now to the third and fourth terms of (A-16):

(A-41) $$= -\int_t\int_V \frac{\sigma_{ik}}{2}\,\delta\left\{\left(\bar{a}_i\cdot\frac{\partial\bar{a}_t}{\partial s_k} + \bar{a}_k\cdot\frac{\partial\bar{a}_t}{\partial s_i}\right)u_t\right\} dV dt$$

This equals:

(A-42) $$= -\int_t\int_V \left(\frac{\sigma_{ik}}{2}\bar{a}_i\cdot\frac{\partial\bar{a}_t}{\partial s_k}\delta u_t - \frac{\sigma_{ik}}{2}\bar{a}_k\cdot\frac{\partial\bar{a}_t}{\partial s_i}\delta u_t\right) dV dt$$

In the second term, interchange i and k, to yield

(A-43) $$= -\int_t\int_V \left(\frac{\sigma_{ik}}{2}\bar{e}_i \cdot \frac{\partial \bar{e}_t}{\partial S_k}\delta u_t + \frac{\sigma_{ki}}{2}\bar{e}_i \cdot \frac{\partial \bar{e}_t}{\partial S_k}\delta u_t\right) dV dt$$

Again, using $\sigma_{ik} = \sigma_{ki}$, this reduces to:

(A-44) $$-\int_t\int_V \left(\sigma_{ik}\bar{e}_i \cdot \frac{\partial \bar{e}_t}{\partial S_k}\delta u_t\right) dV dt$$

Interchange t and k to obtain:

(A-45) $$-\int_t\int_V \left(\sigma_{ti}\bar{e}_i \cdot \frac{\partial \bar{e}_k}{\partial S_t}\delta u_k\right) dV dt$$

Now, noting that:

(A-46) $$\bar{e}_i \cdot \frac{\partial \bar{e}_k}{\partial S_t} = -\bar{e}_k \cdot \frac{\partial \bar{e}_i}{\partial S_t}$$

Equation (A-45) becomes:

(A-47) $$+\int_t\int_V \left(\sigma_{ti}\bar{e}_k \cdot \frac{\partial \bar{e}_i}{\partial S_t}\right)\delta u_k \, dV dt$$

Thus, all the terms resulting from the variation in strain energy have been found. They are specifically contained in equations (A-30), (A-40) and (A-47).

The terms resulting from strain energy variation are summarized below:

(A-48) $$\int_t\int_V \left[\frac{\partial \sigma_{ik}}{\partial S_i} + \sigma_{ik}\left(\bar{e}_t \cdot \frac{\partial \bar{e}_i}{\partial S_t}\right) + \sigma_{ti}\left(\bar{e}_k \cdot \frac{\partial \bar{e}_i}{\partial S_t}\right)\delta u_k dV dt\right.$$

$$-\int_t\int_{\Sigma_i} \bar{n}\cdot\bar{e}_i\left\{(\sigma_{ik}\delta u_k)^{\lambda_i\,MAX} + (\sigma_{ik}\delta u_k)^{\lambda_i\,MIN}\right\} d\Sigma_i\, dt$$

We now proceed to the variation of kinetic energy.

KINETIC ENERGY

The variation in kinetic energy is expressed:

(A-49) $$\delta\int_t T\,dt = \int_t\int_V \delta\left(\tfrac{1}{2}\gamma\dot{\bar{r}}\cdot\dot{\bar{r}}\right) dV\,dt$$

But, from equation (8):

(A-50) $$\dot{\bar{r}} = \dot{\bar{Y}}_B + \dot{\bar{u}}$$

Substitute (A-50) in (A-49) to obtain:

(A-51) $$\delta(K.E.) = \int_t\int_V \frac{\gamma}{2}\,\delta\left\{\left[\dot{\bar{Y}}_B + \dot{\bar{u}}\right]\cdot\left[\dot{\bar{Y}}_B + \dot{\bar{u}}\right]\right\} dV\,dt$$

Expanding:

(A-52) $$\delta(K.E.) = \int_t\int_V \frac{\gamma}{2}\,\delta\left[\dot{\bar{Y}}_B\cdot\dot{\bar{Y}}_B + 2\dot{\bar{u}}\cdot\dot{\bar{Y}}_B + \dot{\bar{u}}\cdot\dot{\bar{u}}\right] dV\,dt$$

But:

(A-53) $$\dot{\bar{Y}}_B = \dot{Y}_B(t)_p\bar{\bar{L}}_p$$

$$\dot{\bar{u}} = \dot{u}_K\bar{e}_K$$

Also, from equations (19) and (20),

(A-54) $$U_K = (U_K)_{MIN} + \alpha_K \eta_K(x,\phi,z,t)$$

$$Y_B(t)_P = (Y_B(t)_P)_{MIN} + \beta_P \mathcal{L}_P(t)$$

Since $\delta(\frac{1}{2}\dot{\bar{r}}\cdot\dot{\bar{r}})$ ultimately depends on the two parameters α and β, we must take:

(A-55) $$\delta\left(\frac{\gamma}{2}\dot{\bar{r}}\cdot\dot{\bar{r}}\right) = \frac{\partial}{\partial\alpha_\ell}\left(\frac{\gamma}{2}\dot{\bar{r}}\cdot\dot{\bar{r}}\right)d\alpha_\ell + \frac{\partial}{\partial\beta_\ell}\left(\frac{\gamma}{2}\dot{\bar{r}}\cdot\dot{\bar{r}}\right)d\beta_\ell$$

Performing the operations indicated in (A-55):

(A-56) $$\delta\left(\frac{\gamma}{2}\dot{\bar{r}}\cdot\dot{\bar{r}}\right) = \frac{\gamma}{2}\dot{\bar{r}}\cdot\frac{\partial\dot{\bar{r}}}{\partial\alpha_\ell}d\alpha_\ell + \frac{\gamma}{2}\dot{\bar{r}}\cdot\frac{\partial\dot{\bar{r}}}{\partial\alpha_\ell}d\alpha_\ell + \gamma\dot{\bar{r}}\cdot\frac{\partial\dot{\bar{r}}}{\partial\beta_\ell}d\beta_\ell$$

or, noting that $\frac{\partial\dot{\bar{r}}}{\partial\alpha_K}d\alpha_K = \delta\dot{\bar{u}}$ and $\frac{\partial\dot{\bar{r}}}{\partial\beta_P}d\beta_P = \delta\dot{\bar{Y}}_B$

(A-57) $$\delta\left(\frac{\gamma}{2}\dot{\bar{r}}\cdot\dot{\bar{r}}\right) = \gamma\left[(\dot{\bar{Y}}_B + \dot{\bar{U}})\cdot\delta\dot{\bar{U}}\right] + \gamma\left[(\dot{\bar{Y}}_B + \dot{\bar{U}})\cdot\delta\dot{\bar{Y}}_B\right]$$

(A-58) $$\delta\left(\frac{\gamma}{2}\dot{\bar{r}}\cdot\dot{\bar{r}}\right) = \gamma\left[\dot{\bar{Y}}_B\cdot\delta\dot{\bar{U}} + \dot{\bar{U}}\cdot\delta\dot{\bar{Y}}_B + \dot{\bar{Y}}_B\cdot\delta\dot{\bar{Y}}_B + \dot{\bar{U}}\cdot\delta\dot{\bar{U}}\right]$$

(Equation (58) could have been derived directly from (A-52)). Note that $\delta\dot{\bar{Y}}_B = \delta\left(\frac{\partial\bar{Y}_B}{\partial t}\right)$ and that δ and $\partial/\partial t$ operators may be interchanged. Substitute (A-58) in (A-52):

(A-59) $$\delta(\text{K.E.}) = \int_t \dot{\bar{Y}}_B\cdot\delta\dot{\bar{Y}}_B\int_V \gamma\,dV\,dt + \int_t\left[\frac{\partial}{\partial t}\int_V \gamma\bar{U}\,dV\right]\cdot\delta\dot{\bar{Y}}_B\,dt$$
$$+\int_t\int_V \delta\dot{\bar{Y}}_B\cdot\frac{\partial}{\partial t}(\delta\bar{U})\,dV\,dt + \int_t\int_V \gamma\dot{\bar{U}}\cdot\frac{\partial}{\partial t}(\delta\bar{U})\,dV\,dt$$

The second term vanishes, because of the definition of center of gravity, and the remaining terms are integrated by parts. The result is:

(A-60)

$$\delta(K.E.) = M\dot{\bar{Y}}_B \cdot \delta\bar{Y}_B \Big|_{t_1}^{t_2} - \int_t M\ddot{\bar{Y}}_B \cdot \delta\bar{Y}_B dt$$
$$+ \int_V \gamma\dot{\bar{Y}}_B \cdot \delta\bar{U} \Big|_{t_1}^{t_2} - \int_t\int_V \gamma\ddot{\bar{Y}}_B \cdot \delta\bar{U} dV dt$$
$$+ \int_V \gamma\dot{\bar{U}} \cdot \delta\bar{U} \Big|_{t_1}^{t_2} - \int_t\int_V \gamma\ddot{\bar{U}} \cdot \delta\bar{U} dV dt$$

Since $\delta\bar{Y}_B$ and $\delta\bar{u}$ vanish at the time boundaries, equation (60) reduces to:

(A-61) $$\delta(K.E.) = -\int_t M\ddot{\bar{Y}}_B \cdot \delta\bar{Y}_B dt - \int_t\int_V \gamma[\ddot{\bar{Y}}_B + \ddot{\bar{U}}] \cdot \delta\bar{U} dV dt$$

M is used to represent body mass. We replace the vector dot products by the appropriate scalar tensor forms:

(A-62) $$\delta\int_t\int_V T\, dV dt = -\left\{\int_t M\ddot{Y}_B(t)_p \,\delta Y_B(t)_p dt + \int_t\int_V \gamma[\ddot{Y}_B(t)_K + \ddot{U}_K]\delta U_K dV dt\right\}$$

We now proceed to the variation in the work term:

(A-63) $$\delta\int_t W dt = \delta\int_t\int_\Sigma \bar{P}\cdot\bar{r}\, d\Sigma\, dt$$

This may be written:

(A-64) $$\delta\int_t W dt = \int_t\int_\Sigma \bar{P}\cdot\delta\bar{Y}_B(t) d\Sigma dt + \int_t\int_\Sigma \bar{P}\cdot\delta\bar{U} d\Sigma dt$$

Rearranging slightly:

(A-65)
$$\delta\int_t W dt = \int_t \left\{\int_\Sigma \bar{p}\, d\Sigma\right\} \cdot \delta\bar{Y}_B(t)\, dt + \int_t\int_\Sigma \bar{p}\cdot\delta\bar{u}\, d\Sigma\, dt$$

The term $\int_\Sigma \bar{p}\, d\Sigma$ may be replaced by $\bar{F}(t)$, the resultant force on the body. Using appropriate subscripts:

(A-66)
$$\delta\int_t W dt = \int_t F(t)_p\, \delta Y_B(t)_p\, dt + \int_t\int_\Sigma P_K\, \delta U_K\, d\Sigma\, dt$$

(It should be noted that the integration with respect to $d\Sigma$ must proceed over $d\Sigma_1$, $d\Sigma_2$, and $d\Sigma_3$, as previously discussed).

The final result is available from equations (A-48), (A-62) and (A-66), and is shown below:

(A-67)

$$\begin{aligned}\delta\int_t L\, dt = \int_t\int_V & \left\{\left[\frac{\partial\sigma_{iK}}{\partial S_i} + \sigma_{iK}\left(\bar{e}_t\cdot\frac{\partial\bar{e}_i}{\partial S_t}\right) + \sigma_{ti}\left(\bar{e}_K\cdot\frac{\partial\bar{e}_i}{\partial S_t}\right)\right]\right.\\ & \left. - \gamma\left[\ddot{Y}_B(t)_K + \ddot{U}_K\right]\right\}\delta U_K\, dV dt\\ & + \int_t\int_{\Sigma_i}\left[P_K - \bar{n}\cdot\bar{e}_i\left\{(\sigma_{iK}\delta U_K)^{\lambda_i MAX} + (\sigma_{iK}\delta U_K)^{\lambda_i MIN}\right\}\right] d\Sigma_i\, dt\\ & + \int_t\left[F(t)_p - M\ddot{Y}_B(t)_p\right]\delta Y_B(t)_p\, dt = 0\end{aligned}$$

The first integral provides the equilibrium equations, the second provides the boundary conditions, and the third shows the conditions for rigid body motion.

APPENDIX B

DERIVATION OF
SHELL EQUATIONS

As we examine equation (39) prior to integration, we recognize the need to find the values of such terms as $\partial\bar{e}_i/\partial s_t$ in cylindrical coordinates. It may be shown that:

(B-1) $$\frac{\partial \bar{e}_3}{\partial s_2} = \frac{1}{R}\bar{e}_2 \quad \text{and} \quad \frac{\partial \bar{e}_2}{\partial s_2} = -\frac{1}{R}\bar{e}_3$$

and that all other values of the indices yield 0. This, of course, considerably simplifies the problem.

We turn our attention first to the equilibrium equations themselves. Two forms may be isolated, one to do with δu_k^0 and the other with δu_k^1. In each case, the coefficients of these arbitrary variations must be identically zero:

(B-2) $$\int_{-h}^{+h}\Big[\frac{\partial\sigma_{ik}}{\partial s_i} + \sigma_{ik}\Big(\bar{e}_t\cdot\frac{\partial\bar{e}_i}{\partial s_t}\Big) + \sigma_{ti}\Big(\bar{e}_k\cdot\frac{\partial\bar{e}_i}{\partial s_t}\Big) - \gamma\big(\ddot{Y}_{B,K} + \ddot{U}_K^{(0)} + z\ddot{U}_K^{(1)}\big)\Big]\Big(1+\frac{z}{a}\Big)\delta U_K^{(0)}\,dz = 0$$

(B-3) $$\int_{-h}^{+h}\Big[\frac{\partial\sigma_{ik}}{\partial s_i} + \sigma_{ik}\Big(\bar{e}_t\cdot\frac{\partial\bar{e}_i}{\partial s_t}\Big) + \sigma_{ti}\Big(\bar{e}_k\cdot\frac{\partial\bar{e}_i}{\partial s_t}\Big) - \gamma\big(\ddot{Y}_{B,K} + \ddot{U}_K^{(0)} + z\ddot{U}_K^{(1)}\big)\Big](z)\Big(1+\frac{z}{a}\Big)\delta U_K^{(1)}\,dz = 0$$

As k assumes successively the values 1, 2, 3, there will be three equations from (B-2) and two from (B-3). <u>For k = 1</u>, equation (B-2) becomes:

(B-4) $$\int_{-h}^{+h}\left[\frac{\partial\sigma_{11}}{\partial S_1}+\frac{\partial\sigma_{21}}{\partial S_2}+\frac{\partial\sigma_{31}}{\partial S_3}+\left(\frac{1}{a+z}\right)\sigma_{31}-\rho(\ddot{Y}_{01}+\ddot{U}_1^{(0)}+z\ddot{U}_1^{(1)})\right]\left(\frac{a+z}{a}\right)dz=0$$

Using the curvilinear values for S_1, S_2, S_3, we write (B-4) as:

(B-5)

$$\frac{\partial}{\partial x}\int_{-h}^{h}\sigma_{11}\left(1+\frac{z}{a}\right)dz+\frac{1}{(a+z)}\frac{\partial}{\partial\varphi}\int_{-h}^{h}\sigma_{21}\left(1+\frac{z}{a}\right)dz$$
$$+\int_{-h}^{h}\frac{\partial\sigma_{31}}{\partial z}\left(1+\frac{z}{a}\right)dz+\frac{1}{a}\int_{-h}^{h}\sigma_{31}dz$$
$$-\rho\int_{-h}^{h}(\ddot{Y}_{01}+\ddot{U}_1^{(0)}+z\ddot{U}_1^{(1)})\left(1+\frac{z}{a}\right)dz=0$$

The third term is expanded and then integrated by parts to yield:

(B-6) $$\sigma_{31}\Big|_{-h}^{+h}+z\frac{\sigma_{31}}{a}\Big|_{-h}^{+h}-\int_{-h}^{h}\frac{\sigma_{31}}{a}dz$$

We confine our attention to the case of surface traction on the outer surface (^+h) only. Further, we define:

(B-7) $$\sigma_{11}^{(0)}=N_{xx}=\int_{-h}^{h}\sigma_{11}\left(1+\frac{z}{a}\right)dz$$

(B-8) $$\sigma_{21}^{(0)}=N_{\phi x}=\int_{-h}^{h}\sigma_{21}dz$$

Utilizing (B-6),(B-7) and (B-8), with the restriction cited for surface traction, equation (B-5) becomes:

(B-9) $$\frac{\partial \sigma_{11}^{(0)}}{\partial x} + \frac{1}{a}\frac{\partial \sigma_{21}^{(0)}}{\partial \phi} + P_1^{+}\left(1+\frac{h}{a}\right) - 2\delta h \ddot{Y}_B(t)\cdot \bar{e}_1 - 2\delta \ddot{U}_1^{(0)} h - \frac{2}{3a}\delta h^3 \ddot{U}_1^{(1)} = 0$$

We retain equation (B-9), and continue for k = 2 to explore what happens to (B-2):

(B-10) $$\int_{-h}^{h}\left[\frac{\partial \sigma_{12}}{\partial S_1} + \frac{\partial \sigma_{22}}{\partial S_2} + \frac{\partial \sigma_{32}}{\partial S_3} + \frac{\sigma_{32}}{a+z} + \frac{\sigma_{23}}{a+z} - \delta\left(\ddot{Y}_{B_2} + \ddot{U}_2^{(0)} + z\ddot{U}_2^{(1)}\right)\right]\left(1+\frac{z}{a}\right)\delta u_2^{0}\, dz = 0$$

Again, using the curvilinear values for S_1, S_2 and S_3:

(B-11)

$$\frac{\partial}{\partial x}\int_{-h}^{h}\sigma_{12}\left(1+\frac{z}{a}\right)dz + \frac{1}{a}\frac{\partial}{\partial \phi}\int_{-h}^{h}\sigma_{22}\, dz$$
$$+\int_{-h}^{h}\frac{\partial \sigma_{32}}{\partial z}\left(1+\frac{z}{a}\right)dz + \frac{2}{a}\int_{-h}^{h}\sigma_{32}\, dz$$
$$-\int_{-h}^{h}\delta\left(\ddot{Y}_{B2} + \ddot{U}_2^{(0)} + z\ddot{U}_2^{(1)}\right)\left(1+\frac{z}{a}\right)dz = 0$$

Now, defining:

(B-12)

$$\sigma_{12}^{(0)} = \int_{-h}^{h}\sigma_{12}\left(1+\frac{z}{a}\right)dz = N_{x\phi}$$
$$\sigma_{22}^{(0)} = \int_{-h}^{h}\sigma_{22}\, dz = N_{\phi\phi}$$
$$\sigma_{23}^{(0)} = \int_{-h}^{h}\sigma_{32}\, dz = Q_{\phi}$$

Using (B-12) and integrating the third term of (B-11) by parts, and confining our interest to outside surface traction only, equation (B-11) reduces to:

(B-13) $$\frac{\partial \sigma_{12}^{(0)}}{\partial x} + \frac{1}{a}\frac{\partial \sigma_{22}^{(0)}}{\partial \phi} + \frac{\sigma_{32}^{(0)}}{a} + P_2^+\left(1+\frac{h}{a}\right) - 2\gamma h \ddot{Y}_B(t)\cdot\bar{e}_2 - 2\gamma h \ddot{U}_2^{(0)} - \frac{2\gamma h^3}{3a}\ddot{U}_2^{(1)} = 0$$

We retain equation (B-13), and continue to examine (B-2) when k = 3:

(B-14) $$\int_{-h}^{h}\left[\frac{\partial \sigma_{13}}{\partial S_1} + \frac{\partial \sigma_{23}}{\partial S_2} + \frac{\partial \sigma_{33}}{\partial S_3} + \frac{\sigma_{33}}{(a+z)} - \frac{\sigma_{22}}{(a+z)} - \gamma\left(\ddot{Y}_{B3} + \ddot{U}_3^{(0)} + z\ddot{U}_3^{(1)}\right)\right]\left(1+\frac{z}{a}\right)dz = 0$$

Again, using the curvilinear values for S_1, S_2, and S_3:

(B-15)

$$\frac{\partial}{\partial x}\int_{-h}^{h}\sigma_{13}\left(1+\frac{z}{a}\right)dz + \frac{1}{a}\frac{\partial}{\partial \phi}\int_{-h}^{h}\sigma_{23}\,dz + \int_{-h}^{h}\frac{\partial \sigma_{33}}{\partial z}\left(1+\frac{z}{a}\right)dz$$
$$+\frac{1}{a}\int_{-h}^{h}\sigma_{33}\,dz - \frac{1}{a}\int_{-h}^{h}\sigma_{22}\,dz - \int_{-h}^{h}\gamma\left(\ddot{Y}_{B3} + \ddot{U}_3^{(0)} + z\ddot{U}_3^{(1)}\right)\left(1+\frac{z}{a}\right)dz = 0$$

The third term is integrated by parts. We define

(B-16) $$\sigma_{13}^{(0)} = Q_x = \int_{-h}^{h}\sigma_{13}\left(1+\frac{z}{a}\right)dz$$

Note that $\sigma_{23} = \sigma_{32}$, and finally obtain:

(B-17)

$$\frac{\partial \sigma_{13}^{(0)}}{\partial x} + \frac{1}{a}\frac{\partial \sigma_{23}^{(0)}}{\partial \phi} + P_3^+\left(1+\frac{h}{a}\right) - \frac{\sigma_{22}^{(0)}}{a}$$
$$-2\gamma h \ddot{Y}_B(t)\cdot\bar{e}_3 - 2\gamma h \ddot{U}_3^{(0)} - \frac{2\gamma h^3}{3a}\ddot{U}_3^{(1)} = 0$$

To summarize (B-9), (B-13) and (B-17) are the equations that derive from (B-2) for k = 1, 2 and 3 respectively. They represent equilibrium equations in the x, ϕ and z directions, and must be viewed as being multiplied by the arbitrary variations δu_1^0, δu_2^0 and δu_3^0.

We now return to (B-3) and evaluate it for $k = 1$ and $k = 2$. First, for $k = 1$, (B-3) becomes:

(B-18)

$$\int_{-h}^{h} \Big[\frac{\partial \sigma_{11}}{\partial S_1} + \frac{\partial \sigma_{21}}{\partial S_2} + \frac{\partial \sigma_{31}}{\partial S_3} + \frac{1}{(a+z)} \sigma_{31} - \gamma(\ddot{Y}_{B1} + \ddot{U}_1^{(0)} + z\ddot{U}_1^{(1)}) \Big] z \left(1 + \frac{z}{a}\right) dz = 0$$

Using the curvilinear coordinates for S_1, S_2, and S_3, we write:

(B-19)

$$\frac{\partial}{\partial x} \int_{-h}^{h} z\sigma_{11}\left(1+\frac{z}{a}\right)dz + \frac{1}{a}\frac{\partial}{\partial \phi}\int_{-h}^{h} z\sigma_{21}dz + \int_{-h}^{h} z\frac{\partial \sigma_{31}}{\partial z}\left(1+\frac{z}{a}\right)dz + \frac{1}{a}\int_{-h}^{h} z\sigma_{31}dz - \gamma\int_{-h}^{h}(\ddot{Y}_{B1} + \ddot{U}_1^{(0)} + z\ddot{U}_1^{(1)}) z\left(1+\frac{z}{a}\right)dz = 0$$

We define

(B-20)

$$\sigma_{11}^{(1)} = M_x = \int_{-h}^{h} z\sigma_{11}\left(1+\frac{z}{a}\right)dz$$

$$\sigma_{21}^{(1)} = M_{\phi x} = \int_{-h}^{h} z\sigma_{21}dz$$

and obtain:

(B-21)

$$\frac{\partial}{\partial x}\sigma_{11}^{(1)} + \frac{1}{a}\frac{\partial \sigma_{21}^{(1)}}{\partial \varphi} + P_1^{+}h\left(1+\frac{h}{a}\right) - \sigma_{13}^{(0)} - \frac{2\gamma h^3}{a}\ddot{Y}_B(t)\cdot \bar{e}_1 - \frac{2\gamma h^3}{a}\ddot{U}_1^{(0)} - 2\gamma h^3 \ddot{U}_1^{(1)} = 0$$

Last, we return to (B-2) and evaluate it for $k = 2$ to find:

(B-22)

$$\int_{-h}^{h}\Big[\frac{\partial\sigma_{12}}{\partial S_1}+\frac{\partial\sigma_{22}}{\partial S_2}+\frac{\partial\sigma_{32}}{\partial S_3}+\frac{\sigma_{32}}{a+z}+\frac{\sigma_{32}}{a+z}$$

$$-\gamma(\ddot{Y}_{B2}+\ddot{u}_2^{(0)}+z\ddot{u}_2^{(1)})\Big](z)(1+\frac{z}{a})dz\equiv 0$$

and this may be written as:

(B-23)

$$\frac{\partial}{\partial x}\int_{-h}^{h}z\sigma_{12}(1+\frac{z}{a})dz+\frac{1}{a}\frac{\partial}{\partial\varphi}\int_{-h}^{h}z\sigma_{22}\,dz$$

$$+\int_{-h}^{h}z\frac{\partial\sigma_{32}}{\partial z}(1+\frac{z}{a})dz+\frac{2}{a}\int_{-h}^{h}z\sigma_{32}\,dz$$

$$-\int_{-h}^{h}\gamma(\ddot{Y}_{B2}+\ddot{u}_2^{(0)}+z\ddot{u}_2^{(1)})(z)(1+\frac{z}{a})dz\quad\equiv 0$$

Define:

(B-24)

$$\sigma_{12}^{(1)}=M_{x\phi}=\int_{-h}^{h}z\sigma_{12}(1+\frac{z}{a})dz$$

$$\sigma_{22}^{(1)}=M_{\phi}=\int_{-h}^{h}z\sigma_{22}\,dz$$

Equation (B-22) becomes:

(B-25)

$$\frac{\partial\sigma_{12}^{(1)}}{\partial x}+\frac{1}{a}\frac{\partial\sigma_{22}^{(1)}}{\partial\varphi}+P_2^{+}h(1+\frac{h}{a})-\sigma_{32}^{(0)}$$

$$-\frac{2\gamma h^3}{a}\ddot{Y}_B(t)\cdot\bar{e}_2-\frac{2\gamma h^3}{a}\ddot{u}_2^{(0)}-2h^3\gamma\ddot{u}_2^{(1)}\equiv 0$$

The additional equations required are (B-21) and (B-25).*

Equation (39) may also be used to provide the boundary conditions. For each boundary, namely $x = 0$, $x = L$, $z = \pm h$, there are five equations, obtained by iterating through the arbitrary variations δu_k^o and δu_k^1. We consider them in the order in which they are given in equation (39).

In general, for clamped end cylinders the displacements will be prescribed at $x = 0$ and $x = L$. For free ends, the stress and moment resultants will be specified at $x = 0$ and $x = L$. In each case, the surface tractions will govern the boundary conditions at $z = \pm h$.

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*NOTE: In the several equations (B-9), (B-13), (B-17), (B-21) and (B-25), the substitutions:

$$P_1^+ = \sigma_{31})_h$$

$$P_2^+ = \sigma_{32})_h$$

$$P_3^+ = \sigma_{33})_h$$

have actually been made before a rigorous demonstration supports the notation. These follow logically from the boundary conditions equations (B-26). When $\delta u_k^n \neq o$, being not fixed at the free outside boundary, it is necessary that $P_1^+ = \sigma_{31}^+$, etc.

In order to preserve maximum generality, we will retain the multipliers δu_k^0 and δu_k^1 for each case.

At z = h

(B-26)

$$\int_t \int_x \int_\phi \Big[(P_1^+ - \sigma_{31})_{z=h} (a+h)\,\delta U_1^{0}$$
$$+ (P_1^+ - \sigma_{31})_{z=h} (a+h)h(\delta U_1^{(1)})$$
$$+ (P_2^+ - \sigma_{32})_{z=h} (a+h)\,\delta U_2^{(0)}$$
$$+ (P_2^+ - \sigma_{32})_{z=h} (a+h)h(\delta U_2^{(1)})$$
$$+ (P_3^+ - \sigma_{33})_{z=h} (a+h)\,\delta U_3^{(0)} \Big] dx\,d\phi\,dt = 0$$

At z = -h

(B-27)

$$\int_t \int_x \int_\phi \Big[(P_1^- + \sigma_{31})_{z=-h} (a-h)\,\delta U_1^{(0)}$$
$$+ (P_1^- + \sigma_{31})_{z=-h} (a-h)(-h)\,\delta U_1^{(1)}$$
$$+ (P_2^- + \sigma_{32})_{z=-h} (a-h)\,\delta U_2^{(0)}$$
$$+ (P_2^- + \sigma_{32})_{z=-h} (a-h)(-h)\,\delta U_2^{(1)}$$
$$+ (P_3^- + \sigma_{33})_{z=-h} (a-h)\,\delta U_3^{(0)} \Big] dx\,d\phi\,dt = 0$$

At x = 0

(B-28)
$$\int_t \int_\phi \Big\{ \Big[\sigma_{11}^{(0)} + \int_{-h}^{h} P_1 (1 + \tfrac{z}{a}) dz \Big]_{x=0} \delta U_1^{(0)} + \Big[\sigma_{11}^{(1)} + \int_{-h}^{h} P_1 z (1 + \tfrac{z}{a}) dz \Big]_{x=0} \delta U_1^{(1)}$$
$$+ \Big[\sigma_{12}^{(0)} + \int_{-h}^{h} P_2 (1 + \tfrac{z}{a}) dz \Big]_{x=0} \delta U_2^{(0)} + \Big[\sigma_{12}^{(1)} + \int_{-h}^{h} P_2 z (1 + \tfrac{z}{a}) dz \Big]_{x=0} \delta U_2^{(1)}$$
$$+ \Big[\sigma_{13}^{(0)} + \int_{-h}^{h} P_3 (1 + \tfrac{z}{a}) dz \Big]_{x=0} \delta U_3^{(0)} \Big\} a\,d\phi\,dt = 0$$

In expressions (B-28), P_1, P_2 and P_3 refer to values of surface traction at $x = 0$.

Finally,

<u>At x = L</u>

(B-29)

$$\int_t \int_\phi \Big\{ \Big[\sigma_{11}^{(0)} - \int_{-h}^{h} P_1 (1+\frac{z}{a}) dz\Big]_{x=L} \delta u_1^{(0)}$$

$$+ \Big[\sigma_{11}^{(1)} - \int_{-h}^{h} P_1(z)(1+\frac{z}{a}) dz\Big]_{x=L} \delta u_1^{(1)}$$

$$+ \Big[\sigma_{12}^{(0)} - \int_{-h}^{h} P_2 (1+\frac{z}{a}) dz\Big]_{x=L} \delta u_2^{(0)}$$

$$+ \Big[\sigma_{12}^{(1)} - \int_{-h}^{h} P_2(z)(1+\frac{z}{a}) dz\Big]_{x=L} \delta u_2^{(1)}$$

$$+ \Big[\sigma_{13}^{(0)} - \int_{-h}^{h} P_3 (1+\frac{z}{a}) dz\Big]_{x=L} \delta u_3^{(0)} \Big\} a\, d\phi\, dt = 0$$

Again, in these equations, P_1, P_2 and P_3 represent the surface tractions at the face $x = L$.

APPENDIX C

TRANSFORMATION OF EQUATIONS INTO DEFORMATION COORDINATES

Our task is to transform the set of equilibrium equations, (42), as well as the boundary condition equations (45-48), into ones that involve the deformation coordinates u_k^n.

The sequence of steps are planned as follows:

- Express the stresses in terms of the strains
- Write the strain-displacement relations
- Transform the strain-displacement relations into the deformation coordinates u_k^n of the mid-surface.
- Derive the stresses in terms of u_k^n
- Integrate over z to find the stress-resultants in terms of u_k^n
- Substitute these equations for the stress-resultants into equations (42) and (45-48).

The Stress-Strain Relations

To begin, we reiterate that the constitutive equations for a homogeneous, isotropic material are specified as in equation (A-3), repeated below for convenience:

(A-3) $$\sigma_{ik} = \mathcal{L}\,\epsilon_{tt}\delta_{ik} + 2G\epsilon_{ik}$$

When this is true, the individual stress-strain relations may be written:

(C-1) $$\sigma_{11} = \frac{E}{1-\nu^2}\left[\epsilon_{11} + \nu\epsilon_{22}\right] + \frac{\nu}{1-\nu}\,\sigma_{33}$$

(C-2) $$\sigma_{22} = \frac{E}{1-\nu^2}\left[\epsilon_{22} + \nu\epsilon_{11}\right] + \frac{\nu}{1-\nu}\,\sigma_{33}$$

(C-3) $$\sigma_{12} = 2G\epsilon_{12} = \frac{E}{1-\nu^2}(1-\nu)\epsilon_{12}$$

(C-4) $$\sigma_{13} = 2G\epsilon_{13} = \frac{E}{1-\nu^2}(1-\nu)\epsilon_{13}$$

(C-5) $$\sigma_{23} = 2G\epsilon_{23} = \frac{E}{1-\nu^2}(1-\nu)\epsilon_{23}$$

In order to prepare a path for the eventual neglect of transverse shear, we create what might be termed a pseudo-isotropic material by using $G' = \kappa G$ for the last two equations, (C-4). κ is known as Mindlin's constant, as he suggested this device in another connection. κ is a constant which may be chosen at a particular value for a particular purpose. In our case, at the appropriate time, κ will be permitted to approach ∞ in the limit.

We thus replace equations (C-4) and (C-5) with:

(C-6) $$\sigma_{13} = 2G'\epsilon_{13} = \frac{\kappa E}{1-\nu^2}(1-\nu)\epsilon_{13}$$

(C-7) $$\sigma_{23} = 2G'\epsilon_{23} = \frac{\kappa E}{1-\nu^2}(1-\nu)\epsilon_{23}$$

The Strain-Displacement Relations

In Appendix A, the strain-displacement equations for curvilinear coordinates in infinitesimal strain are shown as:

(A-2)
$$\epsilon_{ik}(x,\phi,z,t) = \frac{1}{2}\Big[\frac{\partial u_i}{\partial S_k} + \frac{\partial u_k}{\partial S_i} + \Big(\bar{e}_i \cdot \frac{\partial \bar{e}_t}{\partial S_k} + \bar{e}_k \cdot \frac{\partial \bar{e}_t}{\partial S_i}\Big) u_t\Big]$$

In equation (A-2) $\bar{e}_k$ is the body-referenced orthogonal triad shown in Figure 1. Each of the terms u_i, u_k and u_t are the deformation coordinates of the point. We reiterate that $\bar{e}_k$ is a function of ϕ, while u_i, etc. are dependent on x, ϕ, z and t.

The dS_k set has also been defined, and is repeated below for convenience: (Refer to equations (14) and (15)).

$$dS_1 = dx$$

$$dS_2 = (a)\ (1 + \frac{z}{a})\ d\phi = R d\phi$$

$$dS_3 = dz$$

Using this information, the strain-displacement relations for an arbitrary point become (see reference (27)).

(C-8) $$\epsilon_{11} = \frac{\partial u_1}{\partial x}$$

(C-9) $$\epsilon_{22} = \left(\frac{1}{a+z}\right)\left(\frac{\partial u_2}{\partial \phi} + u_3\right)$$

(C-10) $$\epsilon_{33} = \frac{\partial u_3}{\partial z}$$

(C-11) $$\epsilon_{12} = \frac{1}{2}\gamma_{12} = \frac{1}{2}\left[\frac{\partial u_2}{\partial x} + \frac{1}{a+z}\frac{\partial u_1}{\partial \phi}\right]$$

(C-12) $$\epsilon_{13} = \frac{1}{2}\gamma_{13} = \frac{1}{2}\left[\frac{\partial u_1}{\partial z} + \frac{\partial u_3}{\partial x}\right]$$

(C-13) $$\epsilon_{23} = \frac{1}{2}\gamma_{23} = \frac{1}{2}\left[\frac{\partial u_2}{\partial z} + \frac{1}{a+z}\frac{\partial u_3}{\partial \phi} - \frac{u_2}{a+z}\right]$$

We have also adopted the following truncated series expansion so as to be able to express the deformations of a point in terms of the deformations and rotations of a mid-surface:

(30) $$u_1 = u_1^{(0)} + z u_1^{(1)}$$

$$u_2 = u_2^{(0)} + z u_2^{(1)}$$

$$u_3 = u_3^{(0)}$$

In these equations, for clarity, we note once again that:

u_k = Deformation components of arbitrary point

u_k^o = Deformation components of the middle surface of the shell

u_k^1 = Changes in slope of normal to mid-surface in the $-\overline{e}_1$ and $\overline{e}_2$ directions.

Transformation of the Strain-Displacement Relations

Substitution of equations (30) into equations (C-8) through (C-13) yields:

a. $\epsilon_{11} = \frac{\partial u_1^o}{\partial x} + z\frac{\partial u_1^1}{\partial x}$

(C-14)

b. $\epsilon_{22} = \frac{1}{a+z}\left\{\frac{\partial u_2^o}{\partial \phi} + z\frac{\partial u_2^1}{\partial \phi} + u_3^o\right\}$

c. $\epsilon_{33} = \frac{\partial u_3^o}{\partial z} = 0$

d. $\epsilon_{12} = \frac{1}{2}\gamma_{12} = \frac{1}{2}\left[\frac{\partial u_2^o}{\partial x} + z\frac{\partial u_2^1}{\partial x} + \frac{1}{a+z}\frac{\partial u_1^o}{\partial \phi} + \frac{z}{a+z}\frac{\partial u_1^1}{\partial \phi}\right]$

e. $\epsilon_{13} = \frac{1}{2}\gamma_{13} = \frac{1}{2}\left[\frac{\partial u_1^o}{\partial z} + z\frac{\partial u_1^1}{\partial z} + u_1 + \frac{\partial u_3^o}{\partial x}\right]$

f. $\epsilon_{23} = \frac{1}{2}\gamma_{23} = \frac{1}{2}\left\{\frac{\partial u_2^o}{\partial z} + z\frac{\partial u_2^1}{\partial z} + u_2^1 + \frac{1}{a+z}\frac{\partial u_3^o}{\partial \phi} - \frac{1}{a+z}(u_2^o + zu_2^1)\right\}$

Equations (C-14e) and (C-14f) are further simplified by noting that $\frac{\partial u_1^o}{\partial z}$ and $\frac{\partial u_1^1}{\partial z}$ = 0, etc. These become:

e. $\epsilon_{13} = \frac{1}{2}\gamma_{13} = \frac{1}{2}\left\{u_1^1 + \frac{\partial u_3^o}{\partial x}\right\}$

f. $\epsilon_{23} = \frac{1}{2}\gamma_{23} = \frac{1}{2}\left\{\frac{a}{a+z}\cdot\left(u_2^1 + \frac{1}{a}\frac{\partial u_3^o}{\partial \phi} - \frac{1}{a}u_2^o\right)\right\}$

We note, in passing that ϵ_{13} is a function of x and ϕ alone, (constant throughout the thickness), and that ϵ_{23} is equal to $\frac{a}{a+z}$ multiplied by a function of x and ϕ.

Stresses in Terms of u_k^n

Equations (C-14) are now substituted into equations (C-1) through (C-5) to obtain the point stresses in terms of the mid-surface displacements u_k^n:

(C-15)

a. $$\sigma_{11} = \frac{E}{1-\nu^2}\left[\frac{\partial u_1^0}{\partial x} + z\frac{\partial u_1^1}{\partial x} + \frac{\nu}{a+z}\left\{\frac{\partial u_2^0}{\partial \phi} + z\frac{\partial u_2^1}{\partial \phi} + u_3^0\right\}\right] + \frac{\nu}{1-\nu}\sigma_{33}$$

b. $$\sigma_{22} = \frac{E}{1-\nu^2}\left[\frac{1}{a+z}\left\{\frac{\partial u_2^0}{\partial \phi} + z\frac{\partial u_2^1}{\partial \phi} + u_3^0\right\} + \nu\left\{\frac{\partial u_1^0}{\partial x} + z\frac{\partial u_1^1}{\partial x}\right\}\right] + \frac{\nu}{1-\nu}\sigma_{33}$$

c. $$\sigma_{12} = \frac{E}{1-\nu^2}\frac{(1-\nu)}{2}\left[\frac{\partial u_2^0}{\partial x} + z\frac{\partial u_2^1}{\partial x} + \frac{1}{a+z}\frac{\partial u_1^0}{\partial \phi} + \frac{z}{a+z}\frac{\partial u_1^1}{\partial \phi}\right]$$

d. $$\sigma_{13} = \frac{KE}{1-\nu^2}\frac{(1-\nu)}{2}\left[u_1^1 + \frac{\partial u_3^0}{\partial x}\right]$$

e. $$\sigma_{23} = \frac{KE}{1-\nu^2}\frac{(1-\nu)}{2}\left[\frac{a}{a+z}\cdot\left(u_2^1 + \frac{1}{a}\frac{\partial u_3^0}{\partial \phi} - \frac{1}{a}u_2^0\right)\right]$$

Integration Over z To Get Stress Resultants

The shell stress resultants are defined in Appendix B as follows, and are repeated below for convenience:

(C-16)

a. $$\sigma_{11}^{(0)} = N_x = \int_{-h}^{h} \sigma_{11}\left(1 + \frac{z}{a}\right) dz$$

b. $$\sigma_{12}^{(0)} = N_{x\phi} = \int_{-h}^{h} \sigma_{12}\left(1 + \frac{z}{a}\right) dz$$

c. $$\sigma_{13}^{(0)} = Q_x = \int_{-h}^{h} \sigma_{13}\left(1 + \frac{z}{a}\right) dz$$

d. $$\sigma_{21}^{(0)} = N_{\phi x} = \int_{-h}^{h} \sigma_{12}\, dz$$

e. $$\sigma_{22}^{(0)} = N_\phi = \int_{-h}^{h} \sigma_{22}\, dz$$

f. $$\sigma_{23}^{(0)} = Q_\phi = \int_{-h}^{h} \sigma_{23}\, dz$$

g. $$\sigma_{11}^{(1)} = M_x = \int_{-h}^{h} (z)\left(1+\frac{z}{a}\right)\sigma_{11}\, dz$$

h. $$\sigma_{12}^{(1)} = M_{x\phi} = \int_{-h}^{h} (z)\left(1+\frac{z}{a}\right)\sigma_{12}\, dz$$

i. $$\sigma_{22}^{(1)} = M_\phi = \int_{-h}^{h} z\,\sigma_{22}\, dz$$

j. $$\sigma_{21}^{(1)} = M_{\phi x} = \int_{-h}^{h} z\,\sigma_{21}\, dz$$

It is now necessary to compute each of the expressions (C-16), with the assistance of equations (C-15). This is done on a term by term basis:

a. $$\sigma_{11}^{(0)} = N_x = \frac{E}{1-\nu^2}\int_{-h}^{h}\left[1+\frac{z}{a}\right]\left[\frac{\partial u_1^{(0)}}{\partial x} + z\frac{\partial u_1^{(1)}}{\partial x} + \frac{\nu}{a+z}\left(\frac{\partial u_2^{(0)}}{\partial \phi} + z\frac{\partial u_2^{(1)}}{\partial \phi} + u_3^{(0)}\right) + \frac{(1+\nu)}{E}\sigma_{33}\right] dz$$

(C-17)

b. $$\sigma_{12}^{(0)} = N_{x\phi} = \frac{E}{1-\nu^2}\cdot\frac{1-\nu}{2}\int_{-h}^{h}\left[1+\frac{z}{a}\right]\left\{\left[\frac{\partial u_2^{(0)}}{\partial x} + z\frac{\partial u_2^{(1)}}{\partial x} + \frac{1}{a+z}\frac{\partial u_1^{(0)}}{\partial \phi} + \left(\frac{z}{a+z}\right)\frac{\partial u_1^{(1)}}{\partial \phi}\right]\right\} dz$$

c. $$\sigma_{13}^{(0)} = Q_x = \frac{KE(1-\nu)}{(1-\nu^2)(2)}\int_{-h}^{h}\left(1+\frac{z}{a}\right)\left(u_1^1 + \frac{\partial u_3^0}{\partial x}\right) dz$$

d. $$\sigma_{21}^{(0)} = N_{\phi x} = \frac{E(1-\nu)}{(1-\nu^2)(2)}\int_{-h}^{h}\left[\frac{\partial u_2^0}{\partial x} + z\frac{\partial u_2^1}{\partial \phi} + \frac{1}{a+z}\frac{\partial u_1^0}{\partial \phi} + \frac{z}{a+z}\frac{\partial u_1^1}{\partial \phi}\right] dz$$

e. $$\sigma_{22}^{0} = N_\phi = \frac{E}{1-\nu^2}\int_{-h}^{h}\left[\frac{1}{a+z}\left\{\frac{\partial u_2^0}{\partial \phi} + z\frac{\partial u_2^1}{\partial \phi} + u_3^0\right\} + \nu\left\{\frac{\partial u_1^0}{\partial x} + z\frac{\partial u_1^1}{\partial x}\right\} + \frac{(1+\nu)(\nu)}{E(1-\nu)}\sigma_{33}\right] dz$$

f. $$\sigma_{23}^{0} = Q_\phi = \frac{KE(1-\nu)}{(1-\nu^2)(2)}\int_{-h}^{h}\frac{a}{a+z}\left(u_2^1 + \frac{1}{a}\frac{\partial u_3^0}{\partial \phi} - \frac{1}{a}u_2^0\right) dz$$

g. $$\sigma_{11}^{1} = M_x = \frac{E}{1-\nu^2}\int_{-h}^{h}(z)\left(1+\frac{z}{a}\right)\left\{\frac{\partial u_1^0}{\partial x} + z\frac{\partial u_1^1}{\partial x} + \frac{\nu}{a+z}\left[\frac{\partial u_2^0}{\partial \phi} + z\frac{\partial u_2^1}{\partial \phi} + u_3^0\right] + \frac{\nu(1+\nu)}{E(1-\nu)}\sigma_{33}\right\} dz$$

h. $$\sigma_{12}' = M_{x\phi} = \frac{E(1-\nu)}{(1-\nu^2)(2)} \int_{-h}^{h} (z)\left(1+\frac{z}{a}\right)\left\{\frac{\partial u_2^0}{\partial x} + z\frac{\partial u_2^1}{\partial x} + \frac{1}{a+z}\frac{\partial u_1^0}{\partial \phi} + \frac{z}{a+z}\frac{\partial u_1^1}{\partial \phi}\right\} dz$$

i. $$\sigma_{22}' = M_{\phi} = \frac{E}{1-\nu^2} \int_{-h}^{h} (z)\left[\frac{1}{a+z}\left\{\frac{\partial u_2^0}{\partial \phi} + z\frac{\partial u_2^1}{\partial \phi} + u_3^0\right\} + \nu\left\{\frac{\partial u_1^0}{\partial x} + z\frac{\partial u_1^1}{\partial x}\right\} + \frac{(\nu)(1+\nu)}{E(1-\nu)}\sigma_{33}\right] dz$$

j. $$\sigma_{21}' = M_{\phi x} = \frac{E(1-\nu)}{(1-\nu^2)(2)} \int_{-h}^{h} z\left\{\frac{\partial u_2^0}{\partial x} + z\frac{\partial u_2^1}{\partial x} + \frac{1}{a+z}\frac{\partial u_1^0}{\partial \phi} + \frac{z}{a+z}\frac{\partial u_1^1}{\partial \phi}\right\} dz$$

The integration of equations (17a-j) is straightforward, although laborious. It is worth noting that after the integration, only terms containing odd powers of z survive.

We also pause to note the presence of the factor K in (17c) and (17f), which define the transverse shear resultants. These are the specific terms which will eventually be affected by the assumption of no transverse shear.

In the integration of equations (17), we shall make the following assumptions:

(C-18)

$$\int_{-h}^{h} \left(1+\frac{z}{a}\right)(\sigma_{33})\,dz = 0$$

$$\int_{-h}^{h} \sigma_{33}\,dz = 0$$

$$\int_{-h}^{h} (z)\left(1+\frac{z}{a}\right)\sigma_{33}\,dz = 0$$

$$\int_{-h}^{h} z\,\sigma_{33}\,dz = 0$$

Wherever, in the integration it occurs, we take

(C-19) $$\frac{1}{a+z} = \frac{1}{a} - \frac{z}{a^2} + \frac{z^2}{a^3} - \ldots$$

In the results of the integration, use is made of the "thin shell requirement" that h^2/a^2 may be neglected compared to 1.

Proceeding with the integration, we obtain:

(C-20) a. $$\sigma_{11}^0 = N_x = \frac{2Eh}{1-\nu^2}\left\{\frac{\partial u_1^0}{\partial x} + \frac{h^2}{3a}\frac{\partial u_1'}{\partial x} + \frac{\nu}{a}\frac{\partial u_2^0}{\partial \phi} + \frac{\nu}{a}u_3^0\right\}$$

b. $$\sigma_{12}^0 = N_{x\phi} = \frac{E}{1-\nu^2}\cdot\frac{1-\nu}{2}\cdot 2h\left\{\frac{\partial u_2^0}{\partial x} + \frac{1}{a}\frac{\partial u_1^0}{\partial \phi} + \frac{h^2}{3a}\frac{\partial u_2'}{\partial x}\right\}$$

c. $$\sigma_{13}^0 = Q_x = \frac{KE}{1-\nu^2}\cdot\frac{1-\nu}{2}\cdot 2h\left\{u_1' + \frac{\partial u_3^0}{\partial x}\right\}$$

d. $$\sigma_{21}^0 = N_{\phi x} = \frac{2Eh}{1-\nu^2}\cdot\frac{1-\nu}{2}\left\{\frac{\partial u_2^0}{\partial x} + \frac{1}{a}\frac{\partial u_1^0}{\partial \phi} - \frac{h^2}{3a^2}\frac{\partial u_1'}{\partial \phi}\right\}$$

e. $$\sigma_{22}^0 = N_\phi = \frac{2Eh}{1-\nu^2}\left\{\frac{1}{a}\frac{\partial u_2^0}{\partial \phi} + \frac{u_3^0}{a} + \nu\frac{\partial u_1^0}{\partial x} - \frac{h^2}{3a^2}\frac{\partial u_2'}{\partial \phi}\right\}$$

f. $$\sigma_{23}^0 = Q_\phi = \frac{2KEh}{1-\nu^2}\cdot\frac{1-\nu}{2}\left\{u_2' + \frac{1}{a}\frac{\partial u_3^0}{\partial \phi} - \frac{1}{a}u_2^0\right\}$$

g. $$\sigma_{11}' = M_x = \frac{E}{1-\nu^2}\cdot\frac{2h^3}{3}\left\{\frac{1}{a}\frac{\partial u_1^0}{\partial x} + \frac{\partial u_1'}{\partial x} + \frac{\nu}{a}\frac{\partial u_2'}{\partial \phi}\right\}$$

h. $$\sigma_{12}' = M_{x\phi} = \frac{2Eh^3}{3(1-\nu^2)}\cdot\frac{1-\nu}{2}\left\{\frac{1}{a}\frac{\partial u_2^0}{\partial x} + \frac{\partial u_2'}{\partial x} + \frac{1}{a}\frac{\partial u_1'}{\partial \phi}\right\}$$

i. $$\sigma_{22}' = M_\phi = \frac{2Eh^3}{3(1-\nu^2)}\left\{\frac{1}{a}\frac{\partial u_2'}{\partial \phi} - \frac{1}{a^2}\frac{\partial u_2^0}{\partial \phi} - \frac{1}{a^2}u_3^0 + \nu\frac{\partial u_1'}{\partial x}\right\}$$

j. $$\sigma_{21}' = M_{\phi x} = \frac{2Eh^3}{3(1-\nu^2)}\cdot\frac{1-\nu}{2}\left\{\frac{\partial u_2'}{\partial x} - \frac{1}{a^2}\frac{\partial u_1^0}{\partial \phi} + \frac{1}{a}\frac{\partial u_1'}{\partial \phi}\right\}$$

Equations (C-20) were obtained by Yu in 1958 (Reference 1). They will now be substituted in equations (42) and (45-48).

Substitution into Equation 42

Equation 42 becomes, after substitution:

(C-21)

$$\int_t\int_x\int_\phi \Bigg\{\Bigg[\frac{2Eh}{1-\nu^2}\Bigg\{\Bigg[\frac{\partial^2 u_1^0}{\partial x^2} + \frac{1-\nu}{2a^2}\frac{\partial^2 u_1^0}{\partial \phi^2} + \frac{1+\nu}{2a}\frac{\partial^2 u_2^0}{\partial x \partial \phi} + \frac{\nu}{a}\frac{\partial u_3^0}{\partial x}$$

$$+ka\Big[\frac{\partial^2 u_1^1}{\partial x^2} - \Big(\frac{1-\nu}{2a^2}\Big)\frac{\partial^2 u_1^1}{\partial \phi^2}\Big]\Bigg\} + P_1^+\Big(1+\frac{h}{a}\Big) - 2\gamma h \ddot{Y}_B(t)\cdot \bar{e}_1$$

$$-2\gamma h \frac{\partial^2 u_1^0}{\partial t^2} - \frac{2\gamma h^3}{3a}\frac{\partial^2 u_1^1}{\partial t_1^2}\Bigg]\delta u_1^0$$

$$+\Bigg[\frac{2Eh^3}{3a(1-\nu^2)}\Bigg\{\frac{\partial^2 u_1^0}{\partial x^2} - \frac{(1-\nu)}{2a^2}\frac{\partial^2 u_1^0}{\partial \phi^2} + a\frac{\partial^2 u_1^1}{\partial x^2} + \frac{(1-\nu)}{2a}\frac{\partial^2 u_1^1}{\partial \phi^2} + \frac{(1+\nu)}{2}\frac{\partial^2 u_2^1}{\partial x \partial \phi}$$

$$-\frac{K}{ka}\Big(u_1^1 + \frac{\partial u_3}{\partial x}\Big)\Bigg\} + P_1^+ h\Big(1+\frac{h}{a}\Big) - \frac{2\gamma h^3}{3a}\ddot{Y}_B(t)\cdot \bar{e}_1$$

$$-\frac{2\gamma h^3}{3}\Big[\frac{\partial^2 u_1^0}{a\partial t_1^2} + \frac{\partial^2 u_1^1}{\partial t_1^2}\Big]\Bigg]\delta u_1^1$$

$$+\Bigg[\frac{2Eh}{1-\nu^2}\Bigg\{\Big(\frac{1+\nu}{2a}\Big)\frac{\partial^2 u_1^0}{\partial x \partial \phi} + \frac{(1-\nu)}{2}\frac{\partial^2 u_2^0}{\partial x^2} + \frac{1}{a^2}\frac{\partial^2 u_2^0}{\partial \phi^2} + \frac{1}{a^2}\frac{\partial u_3^0}{\partial \phi}$$

$$+ka\Big[\frac{(1-\nu)}{2}\cdot\frac{\partial^2 u_2^1}{\partial^2 x} - \frac{1}{a^2}\frac{\partial^2 u_2^1}{\partial \phi^2}\Big] - K\Big[\frac{u_2^0}{a^2} - \frac{1}{a^2}\frac{\partial u_3^0}{\partial \phi} - \frac{u_2^1}{a}\Big]\Bigg\}$$

$$+P_2^+\Big(1+\frac{h}{a}\Big) - 2h\gamma\frac{\partial^2 u_2^0}{\partial t^2} - \frac{2\gamma h^3}{3a}\frac{\partial^2 u_2^1}{\partial t_1^2} - 2\gamma h \ddot{Y}_B(t)\cdot \bar{e}_2\Bigg]\delta u_2^0$$

$$+\Bigg[\frac{2Eh^3}{3a(1-\nu^2)}\Bigg\{\Big(\frac{1-\nu}{2}\Big)\frac{\partial^2 u_2^0}{\partial x^2} - \frac{\partial^2 u_2^0}{a^2\partial \phi^2} - \frac{1}{a}\frac{\partial u_3^0}{\partial \phi} + \Big(\frac{1+\nu}{2}\Big)\frac{\partial^2 u_1^1}{\partial x \partial \phi} + \frac{(1-\nu)}{2}a\frac{\partial^2 u_2^1}{\partial x^2}$$

$$+\frac{1}{a}\frac{\partial^2 u_2^1}{\partial\phi^2}+\frac{K}{k}\left(\frac{u_2^0}{a^2}-\frac{1}{a^2}\frac{\partial u_3^0}{\partial\phi}-\frac{u_2^1}{a}\right)\Big\}$$

$$+P_2^+ h\left(1+\frac{h}{a}\right)-\frac{2\gamma h^3}{3a}\ddot{Y}_B(t)\cdot\bar{e}_2-\frac{2\gamma h^3}{3}\left[\frac{\partial^2 u_2^0}{\partial t_1^2}\cdot\frac{1}{a}+\frac{\partial^2 u_2^1}{\partial t_1^2}\right]\Bigg]\delta u_2^1$$

$$+\Bigg[\frac{-h2E}{1-\nu^2}\Big\{\frac{\nu}{a}\frac{\partial u_1^0}{\partial x}+\frac{1}{a^2}\frac{\partial u_2^0}{\partial\phi}+\frac{u_3^0}{a^2}-\frac{k}{a}\frac{\partial u_2^1}{\partial\phi}+K\left(\frac{1}{a^2}\frac{\partial u_2^0}{\partial\phi}-\nabla^2 u_3^0-\frac{\partial u_1^1}{\partial x}-\frac{1}{a}\frac{\partial u_2^1}{\partial\phi}\right)\Big\}$$

$$+P_3^+\left(1+\frac{h}{a}\right)-2h\gamma\frac{\partial^2 u_3^0}{\partial t^2}-2\gamma h\ddot{Y}_B(t)\cdot\bar{e}_3\Bigg]\delta u_3^0\Bigg\}\,a\,dx\,d\phi\,dt \;=0$$

In equation (C-21), the following notations are used:

(C-22)

$$k=\frac{1}{3}\frac{h^2}{a^2}$$

$$K=\frac{k(1-\nu)}{2}$$

Also, the subscript 1 is used on t for time to identify those terms which are associated with rotary inertia.

Equation (C-21) was also obtained by Yu in Reference 6, although not by the Variational Principle.

<u>Substitution into Equations (45-48) (Boundary Conditions)</u>

In much the same manner, we now substitute equations (C-20) in equations (45-48), which become:

(C-23) Same as (45) (see text)

(C-24) Same as (46) (see text)

(C-25)

$$\int_t\int_\phi \Bigg\{\Bigg[\frac{2Eh}{1-\nu^2}\left(\frac{\partial u_1^0}{\partial x}+\frac{h^2}{3a}\frac{\partial u_1^1}{\partial x}+\frac{\nu}{a}\frac{\partial u_2^0}{\partial \phi}+\frac{\nu}{a}u_3^0\right)+\int_{-h}^{h}P_1\left(1+\frac{z}{a}\right)dz\Bigg]_{x=0}\delta u_1^0$$

$$+\Bigg[\frac{E}{1-\nu^2}\cdot\frac{2h^3}{3}\left\{\frac{1}{a}\frac{\partial u_1^0}{\partial x}+\frac{\partial u_1^1}{\partial x}+\frac{\nu}{a}\frac{\partial u_2^1}{\partial \phi}\right\}+\int_{-h}^{h}P_1(z)\left(1+\frac{z}{a}\right)dz\Bigg]_{x=0}\delta u_1^1$$

$$+\Bigg[\frac{E}{1-\nu^2}\cdot\frac{1-\nu}{2}\cdot 2h\left\{\frac{\partial u_2^0}{\partial x}+\frac{1}{a}\frac{\partial u_1^0}{\partial \phi}+\frac{h^2}{3a}\frac{\partial u_2^1}{\partial x}\right\}+\int_{-h}^{h}P_2\left(1+\frac{z}{a}\right)dz\Bigg]_{x=0}\delta u_2^0$$

$$+\Bigg[\frac{2Eh^3}{3(1-\nu^2)}\cdot\frac{1-\nu}{2}\left\{\frac{1}{a}\frac{\partial u_2^0}{\partial x}+\frac{\partial u_2^1}{\partial x}+\frac{1}{a}\frac{\partial u_1^1}{\partial \phi}\right\}+\int_{-h}^{h}P_2 z\left(1+\frac{z}{a}\right)dz\Bigg]_{x=0}\delta u_2^1$$

$$+\Bigg[\frac{KE}{1-\nu^2}\cdot\frac{1-\nu}{2}\cdot 2h\left\{u_1^1+\frac{\partial u_3^0}{\partial x}\right\}+\int_{-h}^{h}P_3\left(1+\frac{z}{a}\right)dz\Bigg]_{x=0}\delta u_3^0\Bigg\}\,a\,d\phi\,dt=0$$

(C-26)

$$\int_t\int_\phi \Bigg\{\Bigg[\frac{2Eh}{1-\nu^2}\left(\frac{\partial u_1^0}{\partial x}+\frac{h^2}{3a}\frac{\partial u_1^1}{\partial x}+\frac{\nu}{a}\frac{\partial u_2^0}{\partial \phi}+\frac{\nu}{a}u_3^0\right)-\int_{-h}^{h}P_1\left(1+\frac{z}{a}\right)dz\Bigg]_{x=L}\delta u_1^0$$

$$+\Bigg[\frac{E}{1-\nu^2}\cdot\frac{2h^3}{3}\left(\frac{1}{a}\frac{\partial u_1^0}{\partial x}+\frac{\partial u_1^1}{\partial x}+\frac{\nu}{a}\frac{\partial u_2^1}{\partial \phi}\right)-\int_{-h}^{h}P_1 z\left(1+\frac{z}{a}\right)dz\Bigg]_{x=L}\delta u_1^1$$

$$+\Bigg[\frac{E}{1-\nu^2}\cdot\frac{1-\nu}{2}\cdot 2h\left(\frac{\partial u_2^0}{\partial x}+\frac{1}{a}\frac{\partial u_1^0}{\partial \phi}+\frac{h^2}{3a}\frac{\partial u_2^1}{\partial x}\right)-\int_{-h}^{h}P_2\left(1+\frac{z}{a}\right)dz\Bigg]_{x=L}\delta u_2^0$$

$$+\Bigg[\frac{2Eh^3}{3(1-\nu^2)}\cdot\frac{1-\nu}{2}\left(\frac{1}{a}\frac{\partial u_2^0}{\partial x}+\frac{\partial u_2^1}{\partial x}+\frac{1}{a}\frac{\partial u_1^1}{\partial \phi}\right)-\int_{-h}^{h}P_2 z\left(1+\frac{z}{a}\right)dz\Bigg]_{x=L}\delta u_2^1$$

$$+\Bigg[\frac{KE}{1-\nu^2}\cdot\frac{1-\nu}{2}\cdot 2h\left[u_1^1+\frac{\partial u_3^0}{\partial x}\right]-\int_{-h}^{h}P_3\left(1+\frac{z}{a}\right)dz\Bigg]_{x=L}\delta u_3^0\Bigg\}\,a\,d\phi\,dt=0$$

APPENDIX D

EFFECT OF DISREGARDING ROTARY INERTIA AND TRANSVERSE SHEAR

The effect of disregarding rotary inertia in equations 42 is considered first. By rotary inertia we mean the inertia deriving from the kinetic energy of rotation of the plane normal to the mid-surface.

The total kinetic energy of the body has been given in Appendix A as:

(A-49) $$K.E. = \int_V \frac{\gamma}{2}(\dot{r}\cdot\dot{r})\,dV$$

Expanding by the definition of $\dot{\bar{r}}$,

(D-1) $$K.E. = \int_V \frac{\gamma}{2}\left\{\dot{\bar{Y}}_B\cdot\dot{\bar{Y}}_B + 2\dot{\bar{u}}\cdot\dot{\bar{Y}}_B + \dot{\bar{u}}\cdot\dot{\bar{u}}\right\}dV$$

We note that:

(D-2) $$\dot{\bar{u}} = (\dot{u}_K^{\circ} + z\dot{u}_K^{\prime})\bar{e}_K$$

and

(D-3) $$\dot{\bar{Y}}_B = \dot{\bar{Y}}_{B_K}\bar{e}_K$$

Using the volume coordinates dS_k, and expressions (D-2) and (D-2) in (D-1), we obtain:

(D-4)

$$K.E. = \int_x\int_\phi \sum_{k=1}^{3} \int_z \frac{\gamma}{2}\left\{\left[\dot{Y}_{B_K}\right]^2 + 2\left[\dot{Y}_{B_K}\right]\left[\dot{u}_K^o + z\dot{u}_K'\right] + \left[u_K^o + zu_K'\right]^2\right\} R\,d\phi\,dx\,dz$$

Expanding (D-4), and taking R = a+z,

(D-5)

$$K.E. = \int_x\int_\phi \sum_{k=1}^{3} \int_{-h}^{h} \frac{\gamma}{2}\left\{\left[\dot{Y}_{B_K}\right]^2 + 2\dot{Y}_{B_K}\dot{u}_K^o + 2z\dot{Y}_B\dot{u}_K' + \dot{u}_K^{o\,2} + 2\dot{u}_K^o z\dot{u}_K' + z^2\dot{u}_K'^{\,2}\right\}\left(1+\frac{z}{a}\right) dz\, a\,d\phi\,dx$$

We examine the results of the integration in z, to the extent that it affects the terms containing $\dot{u}_k^n$ and $\ddot{Y}_{B,k}$. Such terms are;

(D-6) $$R.I.\ terms = \frac{\gamma}{2}\int_x\int_\phi \ldots \frac{4}{3a}h^3 Y_{B_K}\dot{u}_K' + \ldots \frac{4\dot{u}_K^o\dot{u}_K'}{3a}h^3 + \ldots \frac{2h^3}{3}\dot{u}_K'^{\,2}$$

We may thus conclude that in equations (42), the R. I. terms are those that contain $\underline{h^3}$. These terms may be identified by calling the time t_1 rather than t. (This is done in the development of the system of five equations in deformation coordinates, equations (61-65)).

If these terms due to rotary inertia are eliminated from the stree-motion equations, (42), they become; ($\delta u_k^n \neq o$):

(D-7)

$$\int_t\int_x\int_\phi \Big\{\Big[\frac{\partial \sigma_{11}^{\circ}}{\partial x} + \frac{1}{a}\frac{\partial \sigma_{21}^{\circ}}{\partial \phi} + P_1^+(1+\frac{h}{a}) - 2\gamma h \ddot{Y}_B(t)\cdot \bar{e}_1 - 2\gamma \ddot{u}_1^{\circ} h\Big]\delta u_1^{\circ}$$

$$+\Big[\frac{\partial \sigma_{11}^{1}}{\partial x} + \frac{1}{a}\frac{\partial \sigma_{21}^{1}}{\partial \phi} + P_1^+(h)(1+\frac{h}{a}) - \sigma_{13}^{\circ}\Big]\delta u_1^{1}$$

$$+\Big[\frac{\partial \sigma_{12}^{\circ}}{\partial x} + \frac{1}{a}\frac{\partial \sigma_{22}^{\circ}}{\partial \phi} + \frac{\sigma_{23}^{\circ}}{a} + P_2^+(1+\frac{h}{a}) - 2\gamma h \ddot{Y}_B(t)\cdot \bar{e}_1 - 2\gamma h \ddot{u}_2^{\circ}\Big]\delta u_2^{\circ}$$

$$+\Big[\frac{\partial \sigma_{12}^{1}}{\partial x} + \frac{1}{a}\frac{\partial \sigma_{22}^{1}}{\partial \phi} + P_2^+ h(1+\frac{h}{a}) - \sigma_{23}^{\circ}\Big]\delta u_2^{1}$$

$$+\Big[\frac{\partial \sigma_{13}^{\circ}}{\partial x} + \frac{1}{a}\frac{\partial \sigma_{23}^{\circ}}{\partial \phi} + P_3^+(1+\frac{h}{a}) - \frac{\sigma_{22}^{\circ}}{a} - 2\gamma h \ddot{Y}_B(t)\cdot \bar{e}_3 - 2\gamma \ddot{u}_3^{\circ} h\Big]\delta u_3^{\circ}\Big\}\, a\,d\phi\,dx\,dt = 0$$

We now turn our attention to the neglect of transverse shear.

From Appendix C, equations (C-14):

(C-14) e. $\epsilon_{13} = \frac{1}{2}\gamma_{13} = \frac{1}{2}\left\{u_1^{1} + \frac{\partial u_3^{\circ}}{\partial x}\right\}$

f. $\epsilon_{23} = \frac{1}{2}\gamma_{23} = \frac{1}{2}\left\{\frac{a}{a+z}\left[u_2^{1} + \frac{1}{a}\frac{\partial u_3^{\circ}}{\partial \phi} - \frac{1}{a}u_2^{\circ}\right]\right\}$

and also:

(C-17) e. $\sigma_{13}^{\circ} = Q_x = \frac{KE(1-\nu)}{(1-\nu^2)2}\int_{-h}^{h}(1+\frac{z}{a})(u_1^{1} + \frac{\partial u_3^{\circ}}{\partial x})\,dz$

f. $\sigma_{23}^{\circ} = Q_\phi = \frac{KE(1-\nu)}{(1-\nu^2)(2)}\int_{-h}^{h}\frac{a}{a+z}(u_2^{1} + \frac{1}{a}\frac{\partial u_3^{\circ}}{\partial \phi} - \frac{1}{a}u_2^{\circ})\,dz$

We now combine equations (C-14e) and (C-17e), as well as (C-14f) and (C-17f), to yield:

(D-8)
$$\frac{1}{2}\left\{u_1^1 + \frac{\partial u_3^0}{\partial x}\right\} = \frac{\sigma_{13}^0}{\frac{KE(1-\nu)}{1-\nu^2}\int_{-h}^{h}\left(1+\frac{z}{a}\right)dz}$$

(D-9)
$$\frac{1}{2}\left\{u_2^1 + \frac{1}{a}\frac{\partial u_3^0}{\partial \phi} - \frac{1}{a}u_2^0\right\} = \frac{\sigma_{23}^0}{\frac{KE(1-\nu)}{1-\nu^2}\int_{-h}^{h}\left(\frac{a}{a+z}\right)dz}$$

In order to require that the transverse deformation go to zero:

(D-10)
$$\epsilon_{13} = 0$$
$$\epsilon_{23} = 0$$

This is the same as requiring:

(D-11)
$$u_2^1 + \frac{\partial u_3^0}{\partial x} = 0$$

(D-12)
$$u_2^1 + \frac{1}{a}\frac{\partial u_3^0}{\partial \phi} - \frac{1}{a}u_2^0 = 0$$

This can be accomplished by letting $K \to \infty$ in (D-8) and (D-9), so long as $\sigma_{13}^{\,0}$ and $\sigma_{23}^{\,0}$ remain finite, although undefined.

As $K \to \infty$, conclude, from (D-11) and (D-12), that:

(D-13)
$$u_1^1 = -\frac{\partial u_3^0}{\partial x}$$

(D-14)
$$u_2^1 = \frac{1}{a}u_2^0 - \frac{1}{a}\frac{\partial u_3^0}{\partial \phi}$$

We consider now the variations in u_1^1 and u_2^1:

(D-15) $$\delta u_1^1 = -\delta \frac{\partial u_3^0}{\partial x} = -\frac{\partial}{\partial x}\delta u_3^0$$

(D-16) $$\delta u_2^1 = \frac{1}{a}\delta u_2^0 - \frac{1}{a}\frac{\partial}{\partial \phi}\delta u_3^0$$

In equation (D-15), one must evaluate the integration with respect to x, and a similar operation is involved in (D-16) with respect to ϕ. We substitute (D-15) and (D-16) in (D-7);

(D-17)

$$\begin{aligned}
\int_t\int_x\int_\phi \Big\{ & \Big[\frac{\partial \sigma_{11}^0}{\partial x} + \frac{1}{a}\frac{\partial \sigma_{21}^0}{\partial \phi} + P_1^+\left(1+\frac{h}{a}\right) - 2\gamma h \ddot{Y}_B(t)\cdot\bar{e}_1 - 2\gamma \ddot{u}_1^0 h\Big]\delta u_1^0 \\
& + \Big[\frac{\partial \sigma_{11}^1}{\partial x} + \frac{1}{a}\frac{\partial \sigma_{21}^1}{\partial \phi} + P_1^+ h\left(1+\frac{h}{a}\right) - \sigma_{13}^0\Big]\Big[-\frac{\partial}{\partial x}\delta u_3^0\Big] \\
& + \Big[\frac{\partial \sigma_{12}^0}{\partial x} + \frac{1}{a}\frac{\partial \sigma_{22}^0}{\partial \phi} + \frac{\sigma_{23}^0}{a} + P_2^+\left(1+\frac{h}{a}\right) - 2\gamma h \ddot{Y}_B(t)\cdot\bar{e}_1 - 2\gamma h \ddot{u}_2\Big]\delta u_2^0 \\
& + \Big[\frac{\partial \sigma_{12}^1}{\partial x} + \frac{1}{a}\frac{\partial \sigma_{22}^1}{\partial \phi} + P_2^+ h\left(1+\frac{h}{a}\right) - \sigma_{23}^0\Big]\Big[\frac{1}{a}\delta u_2^0 - \frac{1}{a}\frac{\partial}{\partial \phi}\delta u_3^0\Big] \\
& + \Big[\frac{\partial \sigma_{13}^0}{\partial x} + \frac{1}{a}\frac{\partial \sigma_{23}^0}{\partial \phi} + P_3^+\left(1+\frac{h}{a}\right) - \frac{\sigma_{22}}{a} - 2\gamma h \ddot{Y}_B(t)\cdot\bar{e}_3 - 2\gamma \ddot{u}_3^0 h\Big]\delta u_3^0\Big\}\, a\, d\phi\, dx\, dt = 0
\end{aligned}$$

In equation (D-17), the terms may now be regrouped as coefficents of δu_1^0, δu_2^0 and δu_3^0 to form new equilibrium equations. The terms which are coefficients of $\frac{\partial}{\partial x}\delta u_3^0$ and $\frac{\partial}{\partial \phi}\delta u_3^0$ must be integrated with respect to x and ϕ respectively.

The term that combine to contribute initially to the equilibrium equations become:

(D-18)

$$\int_t\int_x\int_\phi \Big\{\Big[\frac{\partial\sigma_{11}^{o}}{\partial x}+\frac{1}{a}\frac{\partial\sigma_{21}^{o}}{\partial\phi}+P_1^{+}\Big(1+\frac{h}{a}\Big)-2\gamma h\ddot{Y}_B(t)\cdot\bar{e}_1-2\gamma\ddot{u}_1^{o}h\Big]\delta u_1^{o}$$

$$+\Big[\frac{\partial\sigma_{12}^{o}}{\partial x}+\frac{1}{a}\frac{\partial\sigma_{22}^{o}}{\partial\phi}+P_2^{+}\Big(1+\frac{h}{a}\Big)-2\gamma h\ddot{Y}_B(t)\cdot\bar{e}_1-2\gamma h\ddot{u}_2^{o}+\frac{1}{a}\frac{\partial\sigma_{12}^{1}}{\partial x}+\frac{1}{a^2}\frac{\partial\sigma_{22}^{1}}{\partial\phi}+\frac{P_2^{+}}{a}h\Big(1+\frac{h}{a}\Big)\Big]\delta u_2^{o}$$

$$+\Big[\frac{\partial\sigma_{13}^{o}}{\partial x}+\frac{1}{a}\frac{\partial\sigma_{23}^{o}}{\partial\phi}+P_3^{+}\Big(1+\frac{h}{a}\Big)-\frac{\sigma_{22}^{o}}{a}-2\gamma h\ddot{Y}_B(t)\cdot\bar{e}_3-2\gamma\ddot{u}_3^{o}h\Big]\delta u_3^{o}\Big\}\,a\,d\phi\,dx\,dt$$

We note, in the coefficient of δu_2^{o}, that $\frac{\sigma_{23}(o)}{a} - \frac{\sigma_{23}(o)}{a} = o$ has disappeared.

The additional terms are:

(D-19)

$$\int_t\int_x\int_\phi \Big\{\Big[\frac{\partial\sigma_{11}^{1}}{\partial x}+\frac{1}{a}\frac{\partial\sigma_{21}^{1}}{\partial\phi}+P_1^{+}h\Big(1+\frac{h}{a}\Big)-\sigma_{13}^{o}\Big]\Big(-\frac{\partial}{\partial x}\delta u_3^{o}\Big)$$

$$+\Big[\frac{\partial\sigma_{12}^{1}}{\partial x}+\frac{1}{a}\frac{\partial\sigma_{22}^{1}}{\partial\phi}+P_2^{+}h\Big(1+\frac{h}{a}\Big)-\sigma_{23}^{o}\Big]\Big(-\frac{1}{a}\frac{\partial}{\partial\phi}\delta u_3^{o}\Big)\Big\}\,a\,dx\,d\phi\,dt$$

Integrate term by term of equation (D-19), by parts:

(D-20) a. $$\int_x\Big(\frac{\partial\sigma_{11}^{1}}{\partial x}\Big)\Big(-\frac{\partial}{\partial x}\delta u_3^{o}\Big)dx = -\frac{\partial\sigma_{11}^{1}}{\partial x}\delta u_3^{o}\Big|_o^L+\int_x\frac{\partial^2\sigma_{11}^{1}}{\partial x^2}\delta u_3^{o}\,dx$$

b. $$\int_x\frac{1}{a}\frac{\partial\sigma_{21}^{1}}{\partial\phi}\Big(-\frac{\partial}{\partial x}\delta u_3^{o}\Big)dx = -\frac{\partial\sigma_{21}^{1}}{a\partial\phi}\delta u_3^{o}\Big|_o^L+\int_x\frac{1}{a}\frac{\partial^2\sigma_{21}^{1}}{\partial\phi\partial x}\delta u_3^{o}\,dx$$

c. $$\int_x P_1^{+}h\Big(1+\frac{h}{a}\Big)\Big(-\frac{\partial}{\partial x}\delta u_3^{o}\Big)dx = -P_1^{+}h\Big(1+\frac{h}{a}\Big)\delta u_3^{o}\Big|_o^L+\int_x\frac{\partial P_1^{+}}{\partial x}(h)\Big(1+\frac{h}{a}\Big)\delta u_3^{o}\,dx$$

d. $$\int_x{+}\,\sigma_{13}^{o}\frac{\partial}{\partial x}(\delta u_3^{o})dx = \sigma_{13}^{o}\delta u_3^{o}\Big|_o^L-\int_x\frac{\partial\sigma_{13}^{o}}{\partial x}\delta u_3^{o}\,dx$$

(D-21) a. $\int_\phi \frac{\partial\sigma_{12}^1}{\partial x}\left(-\frac{1}{a}\frac{\partial}{\partial\phi}\delta u_3^0\right)a\,d\phi = -\frac{\partial\sigma_{12}^1}{\partial x}\delta u_3^0\Big|_0^{2\pi} + \int_\phi \frac{1}{a}\frac{\partial^2\sigma_{12}^1}{\partial x\partial\phi}\delta u_3^0\,d\phi$

b. $\int_\phi \left(\frac{1}{a}\frac{\partial\sigma_{22}^1}{\partial\phi}\right)\left(-\frac{1}{a}\frac{\partial}{\partial\phi}\delta u_3^0\right)a\,d\phi = -\frac{1}{a}\frac{\partial\sigma_{22}^1}{\partial\phi}\delta u_3^0\Big|_0^{2\pi} + \int_\phi \frac{1}{a^2}\frac{\partial^2\sigma_{22}^1}{\partial\phi^2}\delta u_3^0\,a\,d\phi$

c. $\int_\phi P_2^+ h\left(1+\frac{h}{a}\right)\left(-\frac{1}{a}\frac{\partial}{\partial\phi}\delta u_3^0\right)a\,d\phi = -P_2^+ h\left(1+\frac{h}{a}\right)\delta u_3^0\Big|_0^{2\pi} + \int_\phi \frac{\partial P_2^+}{\partial\phi}\frac{h}{a}\left(1+\frac{h}{a}\right)\delta u_3^0\,a\,d\phi$

d. $\int_\phi + \frac{\sigma_{23}^0}{a}\frac{\partial}{\partial\phi}(\delta u_3^0)\,a\,d\phi = \sigma_{23}^0\,\delta u_3^0\Big|_0^{2\pi} - \int_\phi \frac{1}{a}\frac{\partial\sigma_{23}^0}{\partial\phi}\delta u_3^0\,a\,d\phi$

In equations (D-20 (a-d)) the left hand terms add to the boundary conditions and the right hand terms to the coefficient of δu_3^0. In equations (D-21 (a-d)) the left hand terms are identically zero, and the right hand terms contribute to the coefficient of δu_3^0. In the process, the terms $\frac{\partial\sigma_{13}^0}{\partial x}$ and $\frac{\partial\sigma_{23}^0}{a\partial\phi}$ vanish.

The final result, which involves a combination of (D-18), (D-20) and (D-21), form the equilibrium equation set: (Since δu_1^0, δu_2^0 and δu_3^0 are arbitrary):

(D-22) $$\left[\frac{\partial\sigma_{11}^0}{\partial x} + \frac{1}{a}\frac{\partial\sigma_{21}^0}{\partial\phi} + P_1^+\left(1+\frac{h}{a}\right) - 2\gamma h\ddot{Y}_B(t)\cdot\bar{e}_1 - 2\gamma h\ddot{u}_1^0\right]\delta u_1^0 = 0$$

(D-23) $$\left[\frac{\partial\sigma_{12}^0}{\partial x} + \frac{1}{a}\frac{\partial\sigma_{22}^0}{\partial\phi} + P_2^+\left(1+\frac{h}{a}\right) - 2\gamma h\ddot{Y}_B(t)\cdot\bar{e}_1 - 2\gamma h u_2^0 + \frac{1}{a}\frac{\partial\sigma_{12}^1}{\partial x} + \frac{1}{a^2}\frac{\partial\sigma_{22}^1}{\partial\phi} + P_2^+\frac{h}{a}\left(1+\frac{h}{a}\right)\right]\delta u_2^0 = 0$$

(D-24) $$\left[+P_3^+\left(1+\frac{h}{a}\right) - \frac{\sigma_{22}^0}{a} - 2\gamma h\ddot{Y}_B(t)\cdot\bar{e}_3 - 2\gamma h\ddot{u}_3^0 + \frac{\partial^2\sigma_{11}^1}{\partial x^2} + \frac{1}{a}\frac{\partial^2\sigma_{21}^1}{\partial\phi\,\partial x} + \frac{\partial P_1^+}{\partial x}(h)\left(1+\frac{h}{a}\right) + \frac{1}{a}\frac{\partial^2\sigma_{21}^1}{\partial x\partial\phi} + \frac{1}{a^2}\frac{\partial^2\sigma_{22}^1}{\partial\phi^2} + \frac{\partial P_2^+}{\partial\phi}\frac{h}{a}\left(1+\frac{h}{a}\right)\right]\delta u_3^0 = 0$$

Equations (D-22), (D-23) and (D-24) are the forms of the stress-motion equations when rotary inertia and transverse shear deformation are neglected. We note:

- Only three equations are necessary
- All terms containing σ_{13}^{o} and σ_{32}^{o} have vanished. (These corresponded to the shear stress resultants Qx and Qø).

The second condition is particularly important, because in our assumption $\kappa \rightarrow \infty$ (refer to equations (D-12) and (D-13)). Qx and Qø remained undefined.

Now, we proceed to the boundary conditions, shown both in Appendix B and as equations (45 - 48) in the text. Again, we must make use of (D-15), and (D-16) repeated below for convenience:

(D-15) $$\delta u_1^{\prime} = -\frac{\partial}{\partial x}\delta u_3^{o}$$

(D-16) $$\delta u_2^{\prime} = \frac{1}{a}\delta u_2^{o} - \frac{1}{a}\frac{\partial}{\partial \phi}\delta u_3^{o}$$

In the boundary conditions on z, at $\pm h$, equations (45) and (46) are not affected by the manipulations of this section.

First, we reconstitute the boundary conditions at x = 0, by using equation (47) and parts of (D-20).

At x = 0

(D-25) $$\int_t\int_\phi \Big\{\Big[\sigma_{11}^{o} + \int_{-h}^{h} P_1\left(1+\frac{z}{a}\right)dz\Big]\delta u_1 + \Big[\sigma_{11}^{\prime} + \int_{-h}^{h} P_1 z\left(1+\frac{z}{a}\right)dz\Big]\delta u_1^{\prime}$$
$$+\Big[\sigma_{12}^{o} + \int_{-h}^{h} P_2\left(1+\frac{z}{a}\right)dz\Big]\delta u_2^{o} + \Big[\sigma_{12}^{\prime} + \int_{-h}^{h} P_2 z\left(1+\frac{z}{a}\right)dz\Big]\delta u_2^{\prime}$$

$$+ \left[\sigma_{13}^{o} + \int_{-h}^{h} P_3\left(1+\frac{z}{a}\right)dz\right]\delta u_3^{o} + \left[\frac{\partial\sigma_{11}^{1}}{\partial x} + \frac{\partial\sigma_{21}^{1}}{a\partial\phi} + P_1^{+}h\left(1+\frac{h}{a}\right) - \sigma_{13}^{o}\right]\delta u_3^{o}\Big\} a\,d\phi\,dt = 0$$

First, note that the σ_{13}^{o} terms may be cancelled. Then, substitute (D-15) and (D-16), and regroup:

<u>At x = 0</u>

(D-26)

$$\int_t \int_\phi \Big\{ \left[\sigma_{11}^{o} + \int_{-h}^{h} P_1\left(1+\frac{z}{a}\right)dz\right]_{x=0} \delta u_1^{o}$$

$$-\left[\sigma_{11}^{1} + \int_{-h}^{h} P_1 z\left(1+\frac{z}{a}\right)dz\right]_{x=0} \frac{\partial\delta u_3^{o}}{\partial x}$$

$$+\left[\sigma_{12}^{o} + \int_{-h}^{h} P_2\left(1+\frac{z}{a}\right)dz + \frac{\sigma_{12}^{1}}{a} + \frac{1}{a}\int_{-h}^{h} P_2 z\left(1+\frac{z}{a}\right)dz\right]_{x=0} \delta u_2^{o}$$

$$+\left[\int_{-h}^{h} P_3\left(1+\frac{z}{a}\right)dz + P_1^{+}h\left(1+\frac{h}{a}\right) + \frac{\partial\sigma_{11}^{1}}{\partial x} + \frac{\partial\sigma_{21}^{1}}{\partial\phi}\right]_{x=0} \delta u_3^{o}$$

$$-\frac{1}{a}\left[\sigma_{12}^{1} + \int_{-h}^{h} P_2 z\left(1+\frac{z}{a}\right)dz\right]_{x=0} \frac{\partial}{\partial\phi}\,\delta u_3^{o}\Big\} a\,d\phi\,dt = 0$$

The last term may be integrated by parts with respect to ϕ, and the final result is:

At x = 0

(D-27)

$$\int_t \int_\phi \Big\{ \Big[\sigma_{11}^{\circ} + \int_{-h}^{h} P_1 (1+\tfrac{z}{a}) dz \Big]_{x=0} \delta u_1^{\circ}$$
$$- \Big[\sigma_{11}' + \int_{-h}^{h} P_1 z (1+\tfrac{z}{a}) dz \Big]_{x=0} \frac{\partial}{\partial x} (\delta u_3^{\circ})$$
$$+ \Big[\sigma_{12}^{\circ} + \int_{-h}^{h} P_2 (1+\tfrac{z}{a}) dz + \frac{\sigma_{12}'}{a} + \frac{1}{a} \int_{-h}^{h} P_2 z (1+\tfrac{z}{a}) dz \Big]_{x=0} \delta u_2^{\circ}$$
$$+ \Big[\int_{-h}^{h} P_3 (1+\tfrac{z}{a}) dz + P_1^{+} h (1+\tfrac{h}{a}) + \frac{\partial \sigma_{11}'}{\partial x} + \frac{\partial \sigma_{21}'}{\partial \phi}$$
$$+ \frac{1}{a} \frac{\partial \sigma_{12}'}{\partial \phi} + \frac{1}{a} \frac{\partial}{\partial \phi} \int_{-h}^{h} P_2 z (1+\tfrac{z}{a}) dz \Big]_{x=0} \delta u_3^{\circ} \Big\} a\, d\phi\, dt = 0$$

For a fixed-end c[illegible]nder, the end conditions must clearly be prescribed: (at both x = 0 and x = L)

(D-28)

$$u_1 = 0$$
$$u_2 = 0$$
$$u_3 = 0$$
$$\frac{\partial}{\partial x} u_3 = 0$$

For a free-ended cylinder, $P_1 = P_2 = P_3 = 0$, and for arbitrary δu_1^o, etc., the boundary conditions at $x = 0$ and also $x = L$ become:

(D-29)

$$\sigma_{11}^o = N_x = 0$$

$$\sigma_{11}^1 = M_x = 0$$

$$\sigma_{12}^o + \frac{1}{a}\sigma_{12}^1 = N_{x\phi} + \frac{1}{a}M_{x\phi} = 0 \qquad \text{ERSATZ SHEAR}$$

$$\frac{\partial \sigma_{11}^1}{\partial x} + \frac{\partial \sigma_{21}^1}{\partial \phi} + P_1^+(h)(1+\frac{h}{a}) + \frac{1}{a}\frac{\partial \sigma_{12}^1}{\partial \phi}$$

$$= \frac{\partial M_x}{\partial x} + \frac{\partial M_{\phi x}}{a\partial x} + P_1^+(h)(1+\frac{h}{a}) + \frac{1}{a}\frac{\partial M_{x\phi}}{\partial \phi} = 0 \qquad \text{ERSATZ TRANSVERSE SHEAR}$$

where the first 3 terms of the last equation define $Q_x\big|_{x=o,L}$, with no rotary inertia considered.

We may now obtain the corresponding expressions in terms of deformation coordinates. Starting with the equation set (C-20 a through j), we substitute for

$$u_1^1\Big|_{\kappa\to\infty} = -\frac{\partial u_3^o}{\partial x}$$

and

$$u_2^1\Big|_{\kappa\to\infty} = \frac{1}{a}u_2^o - \frac{1}{a}\frac{\partial u_3^o}{\partial \phi}$$

to obtain:

(D-30)

a. $$\sigma_{11}^o = N_x = \frac{2Eh}{1-\nu^2}\left[\frac{\partial u_1^o}{\partial x} + \frac{h^2}{3a}\left(-\frac{\partial^2 u_3^o}{\partial x^2}\right) + \frac{\nu}{a}\frac{\partial u_2^o}{\partial \phi} + \frac{\nu}{a}u_3^o\right]$$

b. $$\sigma_{12}^o = N_{x\phi} = \frac{E}{1-\nu^2}\cdot\frac{1-\nu}{2}\cdot 2h\left\{\left(1+\frac{h^2}{3a^2}\right)\frac{\partial u_2^o}{\partial x} + \frac{1}{a}\frac{\partial u_1^o}{\partial \phi} - \frac{h^2}{\nu a^2}\frac{\partial^2 u_3^o}{\partial \phi \partial x}\right\}$$

c. $$\sigma_{13}^o = Q_x = \frac{\kappa E}{1-\nu^2}\cdot\frac{1-\nu}{2}\cdot 2h\left[0\right] = \infty\cdot 0$$

d. $$\sigma_{21}^o = N_{\phi x} = \frac{2Eh}{1-\nu^2}\cdot\frac{1-\nu}{2}\left\{\frac{\partial u_2^o}{\partial x} + \frac{1}{a}\frac{\partial u_1^o}{\partial \phi} + \frac{h^2}{3a^2}\frac{\partial^2 u_3^o}{\partial \phi \partial x}\right\}$$

e. $$\sigma_{22}^o = N_\phi = \frac{2Eh}{1-\nu^2}\left[\left(\frac{1}{a} - \frac{h^2}{3a^3}\right)\frac{\partial u_2^o}{\partial \phi} + \frac{u_3^o}{a} + \frac{h^2}{3a^3}\frac{\partial^2 u_3^o}{\partial \phi^2} + \nu\frac{\partial u_1^o}{\partial x}\right]$$

f. $\sigma_{23}^{o} = Q_\phi = \frac{2KEh}{1-\nu^2}\cdot\frac{1-\nu}{2}\left\{0\right\} = \infty\cdot 0$

g. $\sigma_{11}^{1} = M_x = \frac{E}{1-\nu^2}\cdot\frac{2h^2}{3}\left\{\frac{1}{a}\frac{\partial u_1^o}{\partial x} - \frac{\partial^2 u_3^o}{\partial x^2} - \frac{\nu}{a^2}\frac{\partial^2 u_3^o}{\partial \phi^2} + \frac{\nu}{a^2}\frac{\partial u_2^o}{\partial \phi}\right\}$

h. $\sigma_{12}^{1} = M_{x\phi} = \frac{2Eh^3}{3(1-\nu^2)}\cdot\frac{1-\nu}{2}\left\{\frac{2}{a}\cdot\frac{\partial u_2^o}{\partial x} - \frac{2}{a}\frac{\partial^2 u_3^o}{\partial \phi\,\partial x}\right\}$

i. $\sigma_{22}^{1} = M_\phi = \frac{2Eh^3}{3(1-\nu^2)}\left\{\frac{1}{a}\frac{\partial u_2^o}{\partial \phi} - \frac{1}{a^2}\frac{\partial^2 u_3^o}{\partial \phi^2} - \frac{1}{a^2}\frac{\partial u_2^o}{\partial \phi} - \frac{1}{a^2}u_3^o - \nu\frac{\partial^2 u_3^o}{\partial x^2}\right\}$

j. $\sigma_{21}^{1} = M_{\phi x} = \frac{2Eh^3}{3(1-\nu^2)}\cdot\frac{1-\nu}{2}\left\{\frac{1}{a}\frac{\partial u_2^o}{\partial x} - \frac{2}{a}\frac{\partial^2 u_3^o}{\partial \phi\,\partial x} - \frac{1}{a^2}\frac{\partial u_1^o}{\partial \phi}\right\}$

We can now make use of these expressions for the stress resultants in terms of deformation coordinates by appropriate substitution in (D-22), (D-23) and (D-24). This becomes the Hamilton Integral as $K\to\infty$. The results are: (for δu_1^o, δu_2^o and δu_3^o all $\neq 0$)

(D-31)
$$\frac{2Eh}{1-\nu^2}\left[\frac{\partial^2 u_1^o}{\partial x^2} + \frac{(1-\nu)}{2a^2}\frac{\partial^2 u_1^o}{\partial \phi^2} + \frac{1+\nu}{2a}\frac{\partial^2 u_2^o}{\partial x\,\partial \phi} + \frac{\nu}{a}\frac{\partial u_3^o}{\partial x}\right.$$
$$\left. + k\left(\frac{1-\nu}{2a}\cdot\frac{\partial^3 u_3^o}{\partial \phi^2\,\partial x} - a\frac{\partial^3 u_3^o}{\partial x^3}\right)\right] - \left\{2\gamma h\ddot{Y}_B(t)\cdot\bar{e}_1 + 2\gamma h\ddot{u}_1^o - P_1^+\left(1-\frac{h}{a}\right)\right\} = 0$$

(D-32)
$$\frac{2Eh}{1-\nu^2}\left\{\frac{1+\nu}{2a}\frac{\partial^2 u_1^o}{\partial \phi\,\partial x} + \frac{1-\nu}{2}\frac{\partial^2 u_2^o}{\partial x^2} + \frac{1}{a^2}\frac{\partial^2 u_2^o}{\partial \phi^2} + \frac{1}{a^2}\frac{\partial u_3^o}{\partial \phi}\right.$$
$$\left. + k\left[\frac{3}{2}(1-\nu)\frac{\partial^2 u_2^o}{\partial x^2} - \frac{1}{a^2}\frac{\partial^2 u_2^o}{\partial \phi^2} - \frac{1}{a^2}\frac{\partial u_3^o}{\partial \phi} - \left(\frac{3-\nu}{2}\right)\frac{\partial^3 u_3^o}{\partial x^2\,\partial \phi}\right]\right\}$$
$$-\left\{2\gamma h\ddot{Y}_B(t)\cdot\bar{e}_1 + 2\gamma h\ddot{u}_2^o - P_2^+\left(1+\frac{h}{a}\right) - \frac{P_3^+}{a}h\left(1+\frac{h}{a}\right)\right\} = 0$$

(D-33)
$$-\frac{2Eh}{1-\nu^2}\left[\frac{1}{a^2}\frac{\partial u_2^o}{\partial \phi} + \frac{u_3^o}{a^2} + \frac{\nu}{a}\frac{\partial u_1^o}{\partial x} - ka^2\left\{\frac{1}{a}\frac{\partial^3 u_1^o}{\partial x^3} - \frac{(1-\nu)}{2a^3}\frac{\partial^3 u_1^o}{\partial \phi^2\,\partial x}\right.\right.$$
$$\left.\left. + \frac{(3-\nu)}{2a^2}\frac{\partial^3 u_2^o}{\partial x^2\,\partial \phi} + \frac{1}{a^4}\frac{\partial u_2^o}{\partial \phi} - \frac{2}{a^4}\frac{\partial^2 u_3^o}{\partial \phi^2} - \nabla^4 u_3^o\right\}\right]$$
$$-\left\{2\gamma h\ddot{Y}_B(t)\cdot\bar{e}_3 + 2\gamma h\ddot{u}_3^o - P_3^+\left(1+\frac{h}{a}\right) - \frac{\partial P_1^+}{\partial x}h\left(1+\frac{h}{a}\right) - \frac{\partial P_2^+}{\partial \phi}\frac{h}{a}\left(1+\frac{h}{a}\right)\right\} = 0$$

Equations (D-31) through (D-33) are essentially the same as those obtained by Yu in Reference 6, except of course for the terms containing the external load and the $Y_B(t)$ terms. They differ from the well-known Flugge equations only in terms with the coefficient k. (Yu used a force balance method to obtain his equations).

We contend that this system of three equations, derived directly from Hamilton's Principle, is inherently consistent with the basic (correct) system of five equations, and will use them instead of the Flugge equations.

APPENDIX E

DEMONSTRATION OF
ORTHOGONALITY OF THE
NORMAL MODES

In section VII, we indicated (equation 127) that it would be necessary to satisfy the following orthogonality requirement:

$$\text{(E-1)} \qquad \int_x \int_\phi F_K^n(x,\phi)_{ps} f_K^n(x,\phi)_{mN}\, a\, d\phi\, dx \begin{cases} =0 & m \neq p \\ =0 & s \neq N \\ \neq 0 & m=p,\ s=N \end{cases}$$

The term $F_k^n(x,\emptyset)_{ps}$ is identified in equation (104) as being associated with the variation in kinetic energy, and the $f_k^n(x,\emptyset)_{mN}$ are the normal modes in free vibration. (Refer to equations (80), (85) and (99)).

This condition (E-1) is essential to the solution of the forced vibration problem, in that it ensures that it is possible to develop the Lagrange equilibrium equations for prinicpal (separable) time coordinates.

This type of orthogonality requirement is commonly encountered whenever the solution to a forced vibration problem is attempted by the classical "normal mode method". The usual practice is to demonstrate that the necessary orthogonality conditions are met by manipulating the actual expressions for the normal modes. In this appendix, the verification is taken directly from Hamilton's equations.

There are three major assumptions in the proof:

- The system must be linearly elastic, without energy dissipation
- The normal mode functions must satisfy the equilibrium equations for free vibration

• The normal mode functions must satisfy the boundary conditions

(These will be cited as needed).

We begin by restating Hamilton's principle (Refer to equations (10), (17), and (18)):

(E-2) $$\delta\int_0^t (T-U+W)\,dt = 0$$

For the case of free vibration, there are no surface tractions $\overline{P}$, hence no term W, and no acceleration of the center of gravity $\ddot{Y}_B(t)$.*

Thus, for free vibration, Hamilton's principle reduces to:

(E-3) $$\delta\int_0^t (T-U)\,dt = 0$$

the kinetic energy variation is expressed (see Appendix A)

(E-4) $$\int_0^t \delta T\,dt = \int_0^t\int_V \delta\left(\tfrac{1}{2}\rho\dot{\overline{r}}\cdot\dot{\overline{r}}\right)\Big|_{\dot{Y}_B = \text{CONST (FREE CYL.)}} dV\,dt$$

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*NOTE: Here, we are of course glossing over certain subtleties that have been adequately discussed elsewhere in this analysis. For a free-ended cylinder $\overline{P} = 0$, while in a constrained end situation, no work is done by the constraint. Again, were this simply a question of free vibration, no use of the center of gravity as the origin of a bodyfixed coordinate system would even have been considered for a constrained end (i.e. $\overline{u} \rightarrow \overline{r}$).

Since we are now dealing with an inertia system ($\dot{Y}_B$ = constant), (E-4) reduces to:

(E-5) $$\int_0^t \delta T dt = \int^t \int_V \gamma \delta(\tfrac{1}{2}\dot{U}\cdot\dot{U}) dV dt = \int_t \int_V -\gamma \ddot{U}_K \delta U_K dV dt$$

In equation (E-5), u_k are deformation coordinates and indicial notation is of course implied.

We proceed to the variation of the strain-energy term. For a linearly elastic material, we showed in Appendix A (Equations A1-A15) that:

(E-6) $$-\int_0^t \delta \mathcal{U} dt = -\int_t \int_V \frac{\partial \mathcal{U}_{VOL}}{\partial \epsilon_{ij}} \delta\epsilon_{ij} dV dt = \int_t \int_V \sigma_{ij} \delta\epsilon_{ij} dV dt$$

In Appendix A (Refer to (A-20) and (A-47)), this expression was shown to be:

(E-7) $$-\int_0^t \delta U dt = -\int_t \int_V \sigma_{ik} \frac{\partial}{\partial S_i} \delta U_K dV dt + \int_t \int_V (\sigma_{ti} \bar{e}_K \cdot \frac{\partial \bar{e}_i}{\partial S_t}) \delta U_K dV dt$$

We have shown, in Appendix A, that the right hand side is equal to:

(E-8) $$-\int_0^t \delta U dt = -\int_t \int_{\Sigma_i} \bar{n}\cdot\bar{e}_i [(\sigma_{ik}\delta U_K)^{\lambda_i MAX} + (\sigma_{ik}\delta U_K)^{\lambda_i MIN}] d\Sigma_i dt$$
$$+ \int_t \int_V (\nabla\cdot\tau)_K \delta U_K dV dt$$

in which it was understood that the subscript i refers to the direction of traversal. During the subsequent manipulation of (E-8), we have utilized (see equations (30) and (37):

(E-9) $$u_K = u_K^{(0)} + z u_K^{(1)} \qquad \delta u_K = \delta u_K^{(0)} + z \delta u_K^{(1)}$$

and used a slightly different form to emphasize the boundary surface in the direction of z as contrasted with the boundary surface normal to the mid-surface.

Using all of the above information, we may write (E-3) as:

(E-10)
$$\delta\int_0^t (T-U)\,dt = \int_t\int_x\int_\phi \left[\int_{-h}^{h}\left[(\nabla\cdot\tau)_k\left(1+\frac{z}{a}\right)z^n - \gamma\left(z^0\ddot{U}_k^{(m)} + z^1\ddot{U}_k^{(m)}\right)z^n\right]dz\; a\,d\phi\, dx\, dt\right.$$
$$-\int_t\oint_c \left(\left[\sigma_{ik}^{(m)}\bar{n}\cdot\bar{e}_i\right]\delta U_k^{(m)}\right)_c dC\, dt$$
$$-\int_t\int_{\Sigma_3} \sigma_{3k}(\pm h)\,\bar{n}\cdot\bar{e}_3\,\delta U_k(\pm h)\, d\Sigma_3\, dt \quad = 0$$

For free vibration with homogeneous conditions the solutions have been developed (which of course satisfy Hamilton's integral (E-10) in the form:

(E-11)
$$\left\{U_k^n\right\} = \sum_p\sum_s\left\{f_k^n(x,\phi)_{ps}\right\}q(t)_{ps} = \sum_p\sum_s\left\{f_k^n(x,\phi)_{ps}\right\}C_{ps}\,e^{i\omega_{ps}t}$$

As has been explained (Refer to equation (121)) we want to distinguish the modal pairs in terms δu_k^n from those used elsewhere in the equations. We retain the choice adopted there, employing m, N subscripts for any terms involving δu_k^n. We thus write, for use in equation (E-10):

(E-12)
$$\delta U_k^n = \sum_m\sum_N f_k^n(x,\phi)_{mN}\,\delta q(t)_{mN}$$

(E-13)
$$\delta U_k(h) = \sum_m\sum_N\left[f_k^0(x,\phi)_{mN} + h f_k^1(x,\phi)_{mN}\right]\delta q(t)_{mN}$$

(E-14) $$\bar{n}\cdot\bar{e}_i\sigma^n_{ik} = \sum_p\sum_s \bar{n}\cdot\bar{e}_i S^n_{ik}(x,\phi)_{ps}q(t)_{ps}$$

(where $\bar{n}\cdot\bar{e}_1 \; S^n_{1k} \; (x,\phi)ps = 0$ for a free-ended cylinder

(E-15) $$\bar{n}\cdot\bar{e}_3\sigma_{3K}(h) = \sum_p\sum_s \bar{n}\cdot\bar{e}_3 S_{3K}(x,\phi,h)_{ps}\,q(t)_{ps}$$

and this equals zero for no traction.

(E-16) $$-\int_{-h}^{h}\gamma(z^0\ddot{U}^{(0)}_K + z^1 U^{(1)}_K)z^n(1+\frac{z}{a})dz = \sum_p\sum_s F^n_K(x,\phi)_{ps}\,\omega^2_{ps}\,q(t)_{ps}$$

(E-17) $$-\int_{-h}^{h}(\nabla\cdot\tau)_K(1+\frac{z}{a})z^n\,dz = \sum_p\sum_s \Lambda^n_K(\{f^n_K(x,\phi)_{ps}\})_{ps}\,q(t)_{ps}$$

If we now substitute equations (E-12) through (E-17) in (E-10), the result becomes:

(E-18)

$$0 = \int_t\sum_m\sum_N \delta q(t)_{mN}\Big[\sum_p\sum_s q(t)_{ps}\Big(\int_x\int_\phi(\Lambda^n_K(\{f^n_K(x,\phi)_{ps}\})_{ps} + F^n_K(x,\phi)_{ps}\omega^2_{ps})f^n_K(x,\phi)_{mN}\,a\,d\phi\,dx\Big)\Big]dt$$

$$-\int_t\sum_m\sum_N \delta q(t)_{mN}\Big[\sum_p\sum_s q(t)_{ps}\Big(\oint_C \bar{n}\cdot\bar{e}_i\,S^n_{ik}(x,\phi)_{ps}f^n_K(x,\phi)_{mN}\,dC\Big)\Big]dt$$

$$-\int_t\sum_m\sum_N \delta q(t)_{mN}\Big[\sum_p\sum_s q(t)_{ps}\Big(\int_{\Sigma_3}\bar{n}\cdot\bar{e}_3\,S_{3K}(x,\phi,\pm h)_{ps}(f^0_K(x,\phi)_{mN} + hf^1_K(x,\phi)_{mN})\,d\Sigma_3\Big)\Big]dt$$

Equation (18) can also be written in the form:

(E-19) $$0 = \int_t\sum_m\sum_N \delta q(t)_{mN}\Big[\overset{(1)}{Q}(t)_{mN} + \overset{(2)}{Q}(t)_{mN}\Big]dt = \int_t\sum_m\sum_N \delta q(t)_{mN}\big[Q(t)_{mN}\big]dt$$

in which:

$\overset{[U]}{Q}(t)_{mN}$ is the generalized force associated with the conservative work due to the increase in strain energy.

$\overset{[T]}{Q}(t)_{mN}$ is the generalized d'Alembert force associated with the work done by the variation of kinetic energy.

Thus:

(E-20) $$\int_t -(\delta U)\,dt = \int_t \Big(\sum_m \sum_N \delta q(t)_{mN} \overset{[U]}{Q}(t)_{mN}\Big)dt$$

(E-21) $$\int_t (\delta T)\,dt = \int_t \Big(\sum_m \sum_N \delta q(t)_{mN} \overset{[T]}{Q}(t)_{mN}\Big)dt$$

We write:

(E-22) $$\overset{[U]}{Q}(t)_{mN} = \sum_p \sum_s q(t)_{ps} K_{mN\text{-}ps} = \sum_p \sum_s \overset{[U]}{Q}(t)^{ps}_{mN}$$

(E-23) $$\overset{[T]}{Q}(t)_{mN} = \sum_p \sum_s q(t)_{ps} \int_x \int_\phi \omega^2_{ps} F^n_K(x,\phi)_{ps} f^n_K(x,\phi)_{mN}\, a\, d\phi\, dt = \sum_p \sum_s \overset{[T]}{Q}(t)^{ps}_{mN}$$

From equation (E-18), it is apparent that:

(E-24) $$K_{mN\text{-}ps} = \int_x \int_\phi \Lambda^n_K\left(\{f^n_K(x,\phi)_{ps}\}\right) f^n_K(x,\phi)_{mN}\, a\, d\phi\, dx$$
$$-\oint_c \bar{n}\cdot\bar{e}_i\, S^n_{iK}(x,\phi)_{ps}\Big|_c f^n_K(x,\phi)\Big|_c dC$$
$$-\int_{\Sigma_3} \bar{n}\cdot\bar{e}_3\, S_{3K}(x,\phi,\pm h)\Big|_{\Sigma_3}\left(f^o_K(x,\phi)_{mN} + h f^1_K(x,\phi)_{mN}\right)\Big|_{\Sigma_3} d\Sigma_3$$

Now for any function satisfying the boundary conditions, (either free or clamped ends) the latter two integrals vanish, and K_{mN-ps} reduces to:

(E-25) $$K_{mN-ps} = \int_x\int_\phi \Lambda_K^n(\{f_K^n(x,\phi)_{ps}\})_{ps} f_K^n(x,\phi)_{mN}\, a\,d\phi\, dx$$

We retain (E-25), and rewrite (E-19), substituting (E-22) and (E-23):

(E-26) $$0 = \int_t \sum_m \sum_N \delta q(t)_{mN} \Big[\sum_p \sum_s \{\overset{[U]}{Q}(t)^{ps}_{mN} + \overset{[T]}{Q}(t)^{ps}_{mN}\}\Big] dt$$

Since the term $\delta q(t)_{mN}$ is arbitrary, the interior brackets must equal zero, which leads to all and any:

(E-27) $$\overset{[U]}{Q}(t)^{ps}_{mN} + \overset{[T]}{Q}(t)^{ps}_{mN} = 0$$

Equation (E-27) holds for all terms in (E-26), and the ps sum should be considered as the coefficient of $\delta q(t)_{mN}$.

Now consider a particular term of (E-26), identified as including $\delta q(t)_{\underline{mN}}$. Associated with this term is the doubly infinite series:

(E-28) $$\sum_p \sum_s \{\overset{[U]}{Q}(t)^{ps}_{\underline{mN}} + \overset{[T]}{Q}(t)^{ps}_{\underline{mN}}\}$$

In (E-28), isolate one term, corresponding to <u>ps</u>. This coefficient of $\delta q(t)_{\underline{mN}}$ is thus:

(E-29) $$\overset{[U]}{Q}(t)^{\underline{ps}}_{\underline{mN}} + \overset{[T]}{Q}(t)^{\underline{ps}}_{\underline{mN}} = 0$$

Multiply this coefficient by $q(t)_{\underline{mN}}$ (this is still equal to zero).

(E-30) $$\overset{[U]}{Q}(t)^{\underline{ps}}_{\underline{mN}}\, q(t)_{\underline{mN}} + \overset{[T]}{Q}(t)^{\underline{ps}}_{\underline{mN}}\, q(t)_{\underline{mN}} = 0$$

In a parallel way, we can invert the values of <u>mN</u> and <u>ps</u>, and multiply the corresponding coefficient by $q(t)_{\underline{ps}}$ to obtain:

(E-31) $$\overset{[U]}{Q}(t)^{\underline{mN}}_{\underline{ps}}\, q(t)_{\underline{ps}} + \overset{[T]}{Q}(t)^{\underline{mN}}_{\underline{ps}}\, q(t)_{\underline{ps}} = 0$$

We now substract (E-31) from (E-30) to obtain:

(E-32) $$\left[\overset{[U]}{Q}(t)^{\underline{ps}}_{\underline{mN}}\, q(t)_{\underline{mN}} - \overset{[U]}{Q}(t)^{\underline{mN}}_{\underline{ps}}\, q(t)_{\underline{ps}}\right] + \left[\overset{[T]}{Q}(t)^{\underline{ps}}_{\underline{mN}}\, q(t)_{\underline{mN}} - \overset{[T]}{Q}(t)^{\underline{mN}}_{\underline{ps}}\, q(t)_{\underline{ps}}\right] = 0$$

<u>Now by Betti's Law the first bracket is equal to zero.*</u>

Also note, by reference to equations (100) and (104), (E-32) is exactly;

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*NOTE: Betti's Law applies to linear, elastic systems. In this case:

$$\overset{[U]}{Q}(t)^{\underline{ps}}_{\underline{mN}}\, q(t)_{\underline{mN}} - \overset{[U]}{Q}(t)^{\underline{mN}}_{\underline{ps}}\, q(t)_{\underline{ps}} = 0$$

which may be written:

$$\left(q(t)_{\underline{ps}} K_{\underline{mN}-\underline{ps}}\right) q(t)_{\underline{mN}} - \left(q(t)_{\underline{mN}} K_{\underline{ps}-\underline{mN}}\right) q(t)_{\underline{ps}} = 0$$

This is then seen to be a generalized statement of Maxwell's reciprocity theory.

$$K_{\underline{mN}-\underline{ps}} = K_{\underline{ps}-\underline{mN}}$$

(E-33) $$0 = \int_x^L \int_\phi^{2\pi} \left(H^n_{K,ps} u^n_{K,mN} - H^n_{K,mN} u^n_{K,ps} \right) a\, d\phi\, dx$$

which is the equivalent Sturm-Liouville[2] orthogonality scheme for a set of partial differential equations with homogeneous boundary conditions. This proof is attempted by a detailed evaluation of (E-33), utilizing the boundary conditions. What remains of (E-32) is therefore:

(E-34) $$\overset{[T]}{Q}(t)^{\underline{ps}}_{\underline{mN}}\, q(t)_{\underline{mN}} - \overset{[T]}{Q}(t)^{\underline{mN}}_{\underline{ps}}\, q(t)_{\underline{ps}} = 0$$

Using equation (E-23) this may be written as:

(E-35) $$q(t)_{\underline{mN}}\, q(t)_{\underline{ps}} \left(\omega^2_{\underline{ps}} - \omega^2_{\underline{mN}} \right) \int_x \int_\phi F^n_K(x,\phi)_{\underline{ps}}\, f^n_K(x,\phi)_{\underline{mN}}\, a\, d\phi\, dx = 0$$

because it can be shown that

(E-36) $$F^n_K(x,\phi)_{\underline{mN}}\, f^n_K(x,\phi)_{\underline{ps}} = F^n_K(x,\phi)_{\underline{ps}}\, f^n_K(x,\phi)_{\underline{mN}}$$

Since, in general, unless m,N = p,s, $\omega^2_{ps} \neq \omega^2_{mN}$, equation (E-35) demonstrates the desired orthogonality relation.

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13. ABSTRACT *The forced response of finite, linearly elastic cylinders with prescribed edge conditions has not been sufficiently studied.* Various shell theories have been proposed to examine this problem. The simpler bending theories (called herein classical theories), more amenable to engineering approximation, having been examined by other criteria, may not actually be appropriate to all dynamic problems. Only limited use has been made of these classical theories for dynamic problems, and then only with very specialized edge conditions. More inclusive theories, which include include transverse shear deformation and rotary inertia (usually termed refined or SR theories), though developed, have not been used to analyze dynamic problems of this nature. We propose to compare the results of two shell theories, one classical and one SR theory, when they are used to analyze the forced dynamic deformation of an elastic cylinder with free and clamped edges. An essential feature of the analysis is a reliance on Hamilton's Variational Principle as the underlying, dominant governing physical law. Beginning with Hamilton's Principle, we have formulated two mutually consistent sets of descriptive equations and boundary conditions, as well as the required conditions of orthogonality. The equilibrium portions of these equations are shown to be identical with particular shell theories, previously developed by Yu, in part of different means. *After the solutions are obtained, they will be compared, and the differences noted. It is expected that we will be able to clearly delineate the limits of applicability of the classical theory.*

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