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INTERIOR BALLISTICS

by

M. E. Serevryakov

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SECTION SEVEN - NUMERICAL METHODS OF SOLUTION

USE OF NUMERICAL ANALYSIS IN INTERNAL BALLISTICS

Various variable quantities possessing definite physical significance usually take part in processes which occur in nature or are considered in technology. In this connection, numerical variations of one or more quantities are accompanied by or associated with variations of other variables. Thus there always exists a definite functional relation between the variable quantities under consideration. This functional relation may be expressed by means of three methods, such as tabulations, diagrams, and formulas. In the vast majority of processes, especially those encountered in technology, this relation is expressed by the aid of tables or diagrams obtained directly from experiment or from observation of the process, whereas the formulas appear only after subsequent analysis of the results obtained, and then only in the case of the simplest processes. It is thus apparent that the most natural means of expressing a functional relation between variable quantities representing in their totality the process under investigation is a tabulation; this is especially true when such a process is being utilized directly for technological purposes, in which case a formula or even a diagram will not serve the purpose, and only numerical values of the variable quantity considered to be of primary importance on the basis of practical considerations must be had.

Numerical analysis must thus serve as a means for studying and making practical utilization of the functional relations between the variable quantities involved.

Ordinary differential equations may be approximately integrated by means of any of the known methods of which there are a great many. These methods include the following:

1) Expansion in a Taylor's series in powers of the argument.

2) Integration by the method of successive approximations.

3) Expansion in a series in powers of small parameters entering into the equation.

4) Expansion in a series in powers of the initial values of the unknown function and its derivatives.

5) Method of successive approximations applied to equations for vibrational motion.

6) Methods of Euler, E.L. Bravin, and others.

7) Method of numerical integration.

The first six methods do not require the use of finite differences; all variants of the method of numerical integration are based on the use of such differences.

The principal variants of the method of numerical integration are discussed in the book by Academician A.N. Krylov, "Lectures on Approximate Computations". [4]

Fundamental information on the theory of finite differences and on the technical features of their application to the engineering of artillery may be found in the book by Professor G.V. Oppokov, "Numerical Analysis Applied to the Science of Artillery". [5]

Internal ballistics is an applied science possessing a perfectly definite technical content (the study of the motion of a projectile in the bore of a gun and of the laws of burning of powder) and a perfectly well-defined technical objective: the creation of means for plotting the curve of the speed of the projectile in the bore and for plotting the curve of the pressure of the powder gases in the bore as functions of the path of the projectile and of time.

These curves can be plotted after obtaining suitable tabulations giving the functional relation between the various variable quantities participating in the phenomenon of a shot. The necessary tabulations are obtained by analyzing primary experimental data and those formulas which, in their simplest form, express the relation existing between the initial variable quantities.

The use of numerical analysis constitutes the subject treated in this section of the book. The essence of numerical analysis, its specific features and its principal operations will also be considered here in the proper degree.

CHAPTER 1 - NUMERICAL INTEGRATION BY FINITE DIFFERENCES

(Written by Professor G.V. Oppokov)

1. APPLICATION OF NUMERICAL INTEGRATION TO THE DETERMINATION OF FUNCTIONS

1) Concept of Tabular Functions

That variable function whose numerical values can be chosen arbitrarily is usually designated as the argument or as the independent variable quantity. The remaining variable quantities taking part in the process under consideration are then designated as functions.

Let us assume that a certain independent variable quantity takes on a series of particular values:

$$x_0,\ x_1,\ x_2, \ldots,\ x_i, \ldots,\ x_n,$$

which are separated from one another by invariable equal intervals, so that:

$$x_1 - x_0 = x_2 - x_1 = \ldots = x_{i+1} - x_i = \ldots = x_n - x_{n-1} = h.$$

This interval is called the step of the argument and is designated by h.

In addition to the step h, the limits of the region x_0 and x_n must be stated in the form of finite numerical values.

Cases in which the step h is variable, or in which the variable x assumes infinitely large values in the region under consideration from x_0 to x_n, including the limits of the region themselves, are not considered at all.

Furthermore, let some other variable quantity y also assume a series of particular values:

$$y_0,\ y_1,\ y_2, \ldots,\ y_i, \ldots,\ y_n,$$

each of these particular values corresponding to one of the particular values of the argument x, so that it is always true that:

$$y_i = f(x_i),$$

where:

$$i = 0,\ 1,\ 2, \ldots,\ n - 1,\ n.$$

It follows that the functional relation between the variables y and x is established in the form of a tabulation.

An example of such a relation is presented in Table 4.

Table 4 - Pressure Curve as a Function of Time t.

$t \cdot 10^3$ sec	0	4	8	12	16	18	20	21	22	23	23.5	24	24.5	25
p $\frac{kg}{cm^2}$	21	23	26	32	48	63	88	105	128	175	223	274	330	394
$t \cdot 10^3$ sec	25.5	26	26.5	27.0	27.5	28	28.5	29	29.5	0	30.5	30.75	31	-
p $\frac{kg}{cm^2}$	466	546	636	739	857	990	1137	1312	1516	1743	1983	2097	2175	-

This table is the result of analysis of experimental data obtained by burning in a bomb a weighed sample of strip powder grade SP (1 x 18 x 40 mm) at $\Delta = 0.201$. The values of t were chosen when evaluating the experimental data, so that t is the argument; the values of pressure p were taken from the experimental curve for the chosen values of t. Consequently, the pressure of the powder gases is a function of the time t.

Use has been made in Table 4 of the so-called vertical notation, which we shall adopt henceforth. In this notation the particular values of each of the variable quantities are invariably placed in one row, while each column contains one particular value of each of these variables. This notation is found to be most convenient for the various operations to which the functions stated by the tabulation are subjected.

2) Finite Differences of Various Orders.

Once a function has been given in table form, it is not difficult to find the so-called finite differences of this function or, more simply, its table of differences, these differences being of various orders. Thus, for a portion of Table 4, we obtain the following table of differences.

Table 4-a.

$t \cdot 10^3$	23	23.5	24	24.5	25	25.5
p	175	223	274	330	3[illegible]1	466
Δp	48	51	56	64	72	-
$\Delta^2 p$	3	5	8	8	-	-
$\Delta^3 p$	2	3	0	-	-	-

Thus on the basis of the table of particular function values it is possible to compute the following differences:

$$\Delta p_0 = p_1 - p_0 = 223 - 175 = 48$$
$$\Delta p_1 = p_2 - p_1 = 274 - 223 = 51$$
$$\Delta p_2 = p_3 - p_2 = 330 - 274 = 56$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

These differences are called differences of the first order or, more briefly, as the first differences.

From the first differences, the following new differences can be easily found:

$$\Delta^2 p_0 = \Delta p_1 - \Delta p_0 = 51 - 48 = 3$$
$$\Delta^2 p_1 = \Delta p_2 - \Delta p_1 = 56 - 51 = 5$$
$$\Delta^2 p_2 = \Delta p_3 - \Delta p_2 = 64 - 56 = 8$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

These new differences are now called differences of the second order, or second differences.

From the second differences it is also possible to compute in a similar manner the third differences:

$$\Delta^3 p_0 = \Delta^2 p_1 - \Delta^2 p_0 = 5 - 3 = 2$$
$$\Delta^3 p_1 = \Delta^2 p_2 - \Delta^2 p_1 = 8 - 5 = 3$$
$$\Delta^3 p_2 = \Delta^2 p_3 - \Delta^2 p_2 = 8 - 8 = 0$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

followed by the fourth differences, etc. Generally, by definition, the k^{th} difference equals:

$$\Delta^k y_i = \Delta^{k-1} y_{i+1} - \Delta^{k-1} y_i .$$

Rule. In formulating a table of differences, the number at the left must be subtracted from the number at the right in the same row, and the result recorded in the next lower row under the number at the left.

It is useful to point out that in following this rule the differences with the same subscript (the function itself also being considered as a zero order difference) are automatically recorded in one and the same column.

Table 4-b.

x	x_0	x_1	x_2	x_3	x_4
y	y_0	y_1	y_2	y_3	y_4
Δy	Δy_0	Δy_1	Δy_2	Δy_3	-
$\Delta^2 y$	$\Delta^2 y_0$	$\Delta^2 y_1$	$\Delta^2 y_2$	-	-
$\Delta^3 y$	$\Delta^3 y_0$	$\Delta^3 y_1$	-	-	-

In the table of differences, in the column for $i = 0$, for example, are found differences such as:

$$p_0 = 175, \quad \Delta p_0 = 48,$$
$$\Delta^2 p_0 = 3, \quad \Delta^3 p_0 = 2$$

etc.

3) Certain Properties of Finite Differences.

1. All differences of any order can be expressed only in terms of particular functions of the function itself.

By definition of the first differences:

$$\Delta y_0 = y_1 - y_0, \quad \Delta y_1 = y_2 - y_1, \quad \Delta y_2 = y_3 - y_2.$$

Then, by definition of the second differences:

$$\Delta^2 y_0 = \Delta y_1 - \Delta y_0 = (y_2 - y_1) - (y_1 - y_0) = y_2 - 2y_1 + y_0;$$

$$\Delta^2 y_1 = \Delta y_2 - \Delta y_1 = (y_3 - y_2) - (y_2 - y_1) = y_3 - 2y_2 + y_1$$

etc. Generally:

$$\Delta^2 y_i = y_{i+2} - 2y_{i+1} + y_i.$$

The same procedure can also be applied to differences of higher orders.

2. A constant number can be taken outside the difference symbol of any order and, conversely, can be brought inside this difference symbol.

Let:

$$y = c \cdot f(x).$$

where c is a constant number. On the basis of the definition of differences, we have:

$$\Delta y = c \cdot f(x + h) - c \cdot f(x) = c \cdot [f(x + h) - f(x)] = c \cdot \Delta f(x).$$

Thus,

$$\Delta [c \cdot f(x)] = c \cdot \Delta f(x).$$

By rewriting this relation from right to left, we obtain:

$$c\Delta f(x) = \Delta [cf(x)].$$

This is a mathematical formulation of the second part of the assertion under consideration.

3. For an entire function of the k^{th} degree:

$$y = A_0(x - x_i)^k + A_1(x - x_i)^{k-1} + \ldots + A_{k-1}(x - x_i) + A_k;$$

the k^{th} differences are identical.

Let us confine ourselves to the case of $k = 2$, so that:

$$y = A_0(x - x_i)^2 + A_1(x - x_i) + A_2.$$

Taking:

$$x_{i+1} = x_i + h;$$

$$x_{i+2} = x_i + 2h;$$

$$x_{i+3} = x_i + 3h.$$

we obtain from the formula for the entire function:

$$y_i = A_2; \qquad \Delta y_i = y_{i+1} - y_i = A_0h^2 + A_1h;$$

$$y_{i+1} = A_0h^2 + A_1h + A_2; \qquad \Delta y_{i+1} = y_{i+2} - y_{i+1} = 3A_0h^2 + A_1h;$$

$$y_{i+2} = 4A_0h^2 + 2A_1h + A_2; \qquad \Delta y_{i+2} = y_{i+3} - y_{i+2} = 5A_0h^2 + A_1h;$$

$$y_{i+3} = 9A_0h^2 + 3A_1h + A_2;$$

and finally:

$$\Delta^2 y_i = \Delta y_{i+1} - \Delta y_i = 2A_0h^2;$$

$$\Delta^2 y_{i+1} = \Delta y_{i+2} - \Delta y_{i+1} = 2A_0h^2.$$

The differences $\Delta^2 y_i$ and $\Delta^2 y_{i+1}$ are found to be identical (because A_0 and h are constant numbers), which is what we set out to prove.

4) Determination of Coefficients of an Entire Function.

Let us undertake the task of expressing the coefficients of an entire function in terms of differences of this function. To do this for the case of k = 2, we shall make use of the following relations:

$$y_i = A_2; \quad \Delta y_i = A_0 h^2 + A_1 h; \quad \Delta^2 y_i = 2A_0 h^2.$$

Substituting the value of A_0 from the last relation, namely:

$$A_0 = \frac{\Delta^2 y_i}{2h^2},$$

into the equation for Δy_1, we find:

$$\Delta y_i = \frac{1}{2} \Delta^2 y_i + A_1 h,$$

whence

$$A_1 = \frac{1}{2h} (2\Delta y_i - \Delta^2 y_i).$$

Thus, finally:

$$A_0 = \frac{\Delta^2 y_i}{2h^2}; \qquad A_1 = \frac{2\Delta y_i - \Delta^2 y_i}{2h^2}; \qquad A_2 = y_1.$$

The coefficients for an entire third degree function can be found in a similar manner:

$$A_0 = \frac{\Delta^3 y_i}{6h^3}; \qquad A_1 = \frac{\Delta^2 y_i - \Delta^3 y_i}{2h^2};$$

$$A_2 = \frac{6\Delta y_i - 3\Delta^2 y_i + 2\Delta^3 y_i}{6h}; \qquad A_3 = y_i.$$

These coefficients will be needed subsequently for deriving the formula of the interpolation function. It is of interest to note the fact that the coefficients of the entire function are here expressed in terms of differences with the same subscript i.

The same method may be used to determine these coefficients in terms of differences with different subscripts. Thus, for example, by taking t_i, t_{i-1}, t_{i-2}, t_{i-3} (instead of t_i, t_{i+1}, t_{i+2}, t_{i+3}), we shall find the following relations:

$$A_0 = \frac{\Delta^3 y_{i-3}}{6h^3}; \qquad A_1 = \frac{\Delta^2 y_{i-2} + \Delta^3 y_{i-3}}{2h^2};$$

$$A_2 = \frac{6\Delta y_{i-1} + 3\Delta^2 y_{i-2} + 2\Delta^3 y_{i-3}}{6h}; \qquad A_3 = y_i.$$

These relations, which the reader himself will be able to derive directly by using the same method, will find an application in the derivation of working formulas for the numerical integration of equations.

5) The Practical Value of Differences.

Differences are employed for the following purposes:

a) Determination of intermediate values of a function.

b) Computation of intermediate values of the argument.

c) Factual determination of derivatives of various orders of a function.

d) Determination of definite integrals.

e) Numerical integration of ordinary differential equations.

The universal character of ~~these~~ differences ~~used~~ in the operations enumerated above is reflected in the fact that it is perfectly immaterial whether the function has been stated in the form of a tabulation, a diagram, ~~or~~ a formula.

It has already been established that the k^{th} difference of an entire function of the k^{th} degree is constant. It is not difficult to show that, conversely, if the differences of the k^{th} order of a certain function are constant, the latter is an entire function of the k^{th} degree.

This proposition constitutes the cornerstone of the utilization of differences in all of the operations indicated above, for the following reason. If the formulation of the table of differences of a certain function shows that the k^{th} differences are almost constant, we have the right to replace our function by an entire function of the k^{th} degree and then subject the latter function to all the necessary operations.

It is this entire function which is designated as the interpolation function. A general expression for it must be derived. When $k = 3$, we have:

$$y = A_0(x - x_1)^3 + A_1(x - x_1)^2 + A_2(x - x_1) + A_3.$$

Let us introduce such a variable ξ that:

$$x = x_i + \xi h,$$

where h is the step. Then:

$$\xi = \frac{x - x_i}{h}.$$

The quantity ξ is called the coefficient of interpolation.

Using the coefficient of interpolation and the general relations for the coefficients A_0, A_1, A_2, A_3 of the entire function as derived in Subsection 4, we shall rewrite the formula for the interpolation function as follows:

$$y = \frac{\Delta^3 y_i}{6}\xi^3 + \frac{\Delta^2 y_i - \Delta^3 y_i}{2}\xi^2 + \frac{6\Delta y_i - 3\Delta^2 y_i + 2\Delta^3 y_i}{6}\xi + y_i,$$

or, finally, after regrouping the terms:

$$y = y_i + \xi \cdot \Delta y_i - \frac{1}{2}\xi(1 - \xi)\Delta^2 y_i + \frac{1}{6}\xi(1 - \xi)(2 - \xi)\Delta^3 y_i. \qquad (105)$$

The interpolation function is thus found to be expressed in terms of differences of the given function up to and including those of the third order, with all these differences having one and the same subscript.

In replacing any function (regardless of the manner in which it is stated) by this interpolation function, it is categorically imperative to direct attention, on the basis of the table of differences of the given function, to the character of the variation of these differences; it is necessary, and triply necessary, that the k^{th} differences be nearly constant.

Obviously this condition will be satisfied the better the smaller the absolute values of the differences of any order.

Rule. In order to reduce the differences of any order, the step must be reduced.

This rule results from the following relation:

$$\Delta^k y = y^{(k)} \cdot h^k + h^k \cdot \varepsilon = h^k(y^{(k)} + \varepsilon),$$

where $y^{(k)}$ is the k^{th}-order derivative of x, and ε is an infinitely small quantity of the highest order. The relation itself, which is derived at the proper stage of the differential computation, represents a generalization of the better-known relation between the increment of a function and its differential:

$$\Delta y = dy + h\varepsilon$$

or:

$$\Delta y = y'h + h\varepsilon = h(y' + \varepsilon).$$

Thus, as the step is halved, the first differences decrease approximately by a factor of two, the second differences decrease by a factor of four, and the third differences decrease by a factor of eight.

6) Determination of Intermediate Values of a Given Function.

To determine the intermediate values of a given function, regardless of how it is stated, it is necessary to formulate a table of differences of this function and, having made certain that the second differences are constant or are very small in comparison with the particular values of the function itself, replace it by an entire function of the second degree, i.e., by an interpolation function (105):

$$y = y_i + \xi \cdot \Delta y_i - \frac{1}{2} \xi(1 - \xi)\Delta^2 y_i. \qquad (106)$$

where y_i designates the tabular value of the given function corresponding to the next smaller value of the argument nearest to that value of x for which the intermediate value of the function is sought.

Knowing the intermediate value of x for which the intermediate value of the given function is sought, and having obtained from the table of differences of this function the nearest value of x_i, we can determine the value of the coefficient of interpolation ξ and the unknown value of y by means of formula (106).

Example 1. Given the following table of pressure impulses:

Table 4-c

$t \cdot 10^3$	20	21	22
I	71.0	80.6	92.1

where the impulses are expressed in $kg \cdot dm^{-2} \cdot sec$, determine the pressure impulse for $t = 0.020556$.

Solution. Let us formulate the table of differences:

Table 4-d.

$t \cdot 10^3$	20	21	22
I	71.0	80.6	92.1
ΔI	9.6	11.5	
$\Delta^2 I$	1.9		

Let us find the coefficient of interpolation:

$$\xi = \frac{t - t_i}{h} = \frac{0.020556 - 0.020}{0.001} = 0.556.$$

The desired intermediate value of the function is:

$$I = I_i + \xi \cdot \Delta I_i - \frac{1}{2}\xi(1 - \xi)\Delta^2 I_i = 71.0 + 0.556 \cdot 9.6 - \frac{1}{2}0.556(1 - 0.556)1.9,$$

since

$$\xi = 0.556; \quad I_i = 71.0; \quad \Delta I_i = 9.6 \text{ and } \Delta^2 I_i = 1.9.$$

We finally obtain:

$$I = 71.0 + 5.33 - 0.25 \approx 76 \text{ kg} \cdot \text{dm}^{-2} \cdot \text{sec}$$

7) Computation of Intermediate Values of the Argument.

Cases are sometimes encountered in practice of a contrary nature in which an intermediate value y of the function is given and it is required to find the coefficient of interpolation ξ and the intermediate value of the argument x. Such an operation is sometimes called an inverse interpolation.

Then, discarding the last term in formula (106), we obtain as a first approximation:

$$\xi_1 = \frac{y - y_i}{\Delta y_i}.$$

As a second approximation, we find for ξ from the same formula (106):

$$\xi = \frac{y - y_i}{\Delta y_i - \frac{1}{2}(1 - \xi)\Delta^2 y_i}.$$

Let us assume in the right-hand side of the above expression $\xi = \xi_1$, then in the second approximation:

$$\xi_2 = \frac{y - y_i}{\Delta y_i - \frac{1}{2}(1 - \xi_1)\Delta^2 y_i},$$

where ξ_1 is already known from the first approximation.

Sometimes the following expression is used as the third approximation:

$$\xi_3 = \frac{y - y_i}{\Delta y_i - \frac{1}{2}(1 - \xi_2)\Delta^2 y_i + \frac{1}{6}(1 - \xi_2)(2 - \xi_2)\Delta^3 y_i}.$$

This formula can be easily obtained from the general expression for the interpolation function (105) by placing ξ outside the parentheses, determining ξ, and substituting in the right-hand side $\xi = \xi_2$ known from the second approximation.

The desired value of x will be:

$$x = x_i + \xi_2 h \text{ or } x = x_i + \xi_3 h.$$

Example 2. We have the following table for the relative portion ψ of the charge:

Table 4-e.

$t \cdot 10^3$	20	21	22
ψ	0.034	0.042	0.054

It is desired to find the value of t corresponding to the inflow of gases $\psi = 0.038$.

Solution. Let us formulate the following table of differences:

Table 4-f.

$t \cdot 10^3$	20	21	22
ψ	0.034	0.042	0.054
$\Delta\psi$	0.008	0.012	-
$\Delta^2\psi$	0.004	-	-

We obtain as a first approximation:

$$\xi_1 = \frac{\psi - \psi_i}{\Delta\psi_i} = \frac{0.038 - 0.034}{0.008} = \frac{1}{2}.$$

We find as a second approximation:

$$\xi_2 = \frac{\psi - \psi_i}{\Delta\psi_i - \frac{1}{2}(1 - \xi_1)\Delta^2\psi_i} = \frac{0.038 - 0.034}{0.008 - \frac{1}{2}\cdot\frac{1}{2}0.004} = \frac{4}{7} = 0.556.$$

The desired value of the argument is:

$$t = t_i + \xi_2 h = 0.020 + 0.556(0.024 - 0.023) = 0.020556.$$

8) Numerical Differentiation of Functions.

For this operation it is necessary, first of all, to formulate a table of differences of the given function and to make absolutely certain that the differences of the third order are nearly constant. This fact makes it possible to replace the given function (regardless of the manner in which it is stated) by the interpolation function (105):

$$y = y_i + \xi \cdot \Delta y_i - \frac{1}{2}\xi(1 - \xi)\Delta^2 y_i + \frac{1}{6}\xi(1 - \xi)(2 - \xi)\Delta^3 y_i.$$

From this we find the desired derivative:

$$\frac{dy}{dx} = \frac{d\xi}{dx}\Delta y_i + \frac{1}{2}\left(2\xi\frac{d\xi}{dx} - \frac{d\xi}{dx}\right)\Delta^2 y_i + \frac{1}{6}\left(2\frac{d\xi}{dx} - 6\xi\frac{d\xi}{dx} + 3\xi^2\frac{d\xi}{dx}\right)\Delta^3 y_i,$$

keeping in mind that the differences y_i, Δy_i, $\Delta^2 y_i$, and $\Delta^3 y_i$ are certain constant numbers.

We place the derivative $\frac{d\xi}{dx}$ outside the parentheses:

$$\frac{dy}{dx} = \frac{d\xi}{dx}\left[\Delta y_i + \frac{1}{2}(2\xi - 1)\Delta^2 y_i + \frac{1}{6}(2 - 6\xi + 3\xi^2)\Delta^3 y_i\right];$$

But

$$\xi = \frac{x - x_i}{h};$$

$$\frac{d\xi}{dx} = \frac{1}{h}.$$

Therefore, finally:

$$\frac{dy}{dx} = \frac{1}{h}\left[\Delta y_i + \left(\xi - \frac{1}{2}\right)\Delta^2 y_i + \left(\frac{1}{2}\xi^2 - \xi + \frac{1}{3}\right)\Delta^3 y_i\right].$$

Such is the general working formula for numerical differentiation by means of differences of the given function. It permits finding the value of the derivative at any value of the independent variable x (i.e., at any ξ). This formula assumes its simplest form at $\xi = 0$:

$$\boxed{\frac{dy}{dx} = \frac{1}{h}\left[\Delta y_i - \frac{1}{2}\Delta^2 y_i + \frac{1}{3}\Delta^3 y_i\right].}$$

This relation permits determining the derivative only for those values of the argument which appear in the table of differences of the given function (i.e., for $\xi = 0$), by applying the subscript i successively to each column of this table.

If the step h is not a prime number (for example, $\frac{557}{738}$), the constant h may be introduced under the sign of the difference of any desired order. Therefore, upon introducing into the process the auxiliary function:

$$\phi = \frac{1}{h}y,$$

we obtain

$$\Delta\Phi_i = \frac{1}{h}\Delta y_i; \quad \Delta^2\Phi_i = \frac{1}{h}\Delta^2 y_i; \quad \Delta^3\Phi_i = \frac{1}{h}\Delta^3 y_i,$$

and the working formula for numerical differentiation acquires its final form:

$$\left(\frac{dy}{dx}\right)_i = \Delta\Phi_i - \frac{1}{2}\Delta^2\Phi_i + \frac{1}{3}\Delta^3\Phi_i .$$

Example 3. We have the following table for ψ, the relative portion of the charge:

Table 4-g.

$t \cdot 10^3$	21	21.5	22.0	22.5
$\psi \cdot 10^3$	42	47	54	64

It is desired to find the rate of gas formation $\frac{d\psi}{dt}$ at the instant $t = 0.0210$.

Solution. Let us formulate the following table of differences:

Table 4-h.

$t\cdot 10^3$	21	21.5	22.0	22.5
$\psi\cdot 10^3$	42	47	54	64
$\Delta\psi\cdot 10^3$	5	7	10	
$\Delta^2\psi\cdot 10^3$	2	3		
$\Delta^3\psi\cdot 10^3$	1			

Since the given values are $t_i\cdot 10^3 = 21$ and $\psi_i\cdot 10^3 = 42$, it follows that:

$$\Delta(\psi_i\cdot 10^3) = 5;\quad \Delta^2(\psi_i\cdot 10^3) = 2;\quad \Delta^3(\psi_i\cdot 10^3) = 1.$$

The first of the working formulas for the derivative $\frac{dy}{dx}$ will make it possible to determine the rate of gas formation ($h = 0.5\cdot 10^3$):

$$\frac{d(\psi\cdot 10^3)}{dt} = \frac{1}{h}\left[\Delta(\psi_i\cdot 10^3) - \frac{1}{2}\Delta^2(\psi_i\cdot 10^3) + \frac{1}{3}\Delta^3(\psi_i\cdot 10^3)\right],$$

from which:

$$\left(\frac{d\psi}{dt}\right)_{t=0.021} = \frac{5-1}{0.5} = 8\ \frac{1}{\sec}.$$

9) Computation of Definite Integrals.

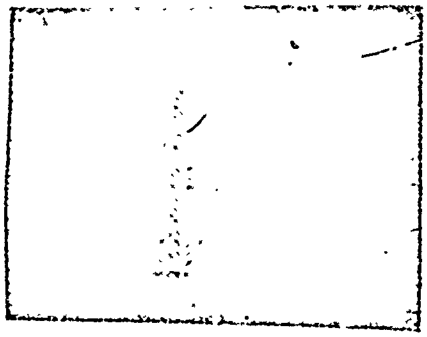

Fig. 152 - Determination of $\int_a^b ydx$ as a function of x from the curve $y = f(x)$.

If it is necessary to find any definite integral:

$$Y = \int_a^b ydx,$$

where the limits of integration a and b are finite, this operation is equivalent to computing the shaded area shown in fig. 152, which is limited at the top by a curve representing the function whose integral is to be found. To compute this area Y, the usual method is applied first: the interval of integration (b - a) must be divided into n equal portions, lines normal to the O-x axis are then erected at the points of division, and the unknown area is divided into elementary areas:

$$\Delta Y_0, \quad \Delta Y_1, \ldots, \quad \Delta Y_i, \ldots, \quad \Delta Y_{n-1}.$$

The problem is then reduced to the determination of these elementary areas. For this purpose it is sufficient to replace the portion of the curve $f(x_i)$ corresponding to the elementary area

ΔY_i by a cubic parabola, kee, 'g in mind that:

$$\Delta Y_i = \int_{x_i}^{x_{i+1}} y dx.$$

This replacement is accomplished by means of the interpolation function:

$$y = y_i + \xi \cdot \Delta y_i - \frac{1}{2}\xi(1 - \xi)\Delta^2 y_i + \frac{1}{6}\xi(1 - \xi)(2 - \xi)\Delta^3 y_i,$$

where

$$x = x_1 + \xi h,$$

so that:

$$dx = hd\xi; \quad \xi_i = 0; \quad \xi_{i+1} = 1.$$

Consequently, after replacing the variables, we have:

$$\Delta Y_i = h \int_0^1 \left[y_i + \xi \cdot \Delta y_i - \frac{1}{2}\xi(1 - \xi)\Delta^2 y_i + \frac{1}{6}\xi(1 - \xi)(2 - \xi)\Delta^3 y_i \right] d\xi.$$

We integrate the right-hand side of this relation keeping in mind the fact that y_i, Δy_i, $\Delta^2 y_i$ and $\Delta^3 y_i$, being particular values of the corresponding differences, are constant numbers:

$$\Delta Y_i = h \left| y_i\xi + \frac{1}{2}\xi^2 \cdot \Delta y_i - \frac{1}{2}\left(\frac{\xi^2}{2} - \frac{\xi^3}{3}\right)\Delta^2 y_i + \frac{1}{6}\left(\xi^2 - \xi^3 + \frac{\xi^4}{4}\right)\Delta^3 y_i \right|_0^1,$$

from which, after substituting the extreme values of ξ, we obtain:

$$\boxed{\Delta Y_i = h\left(y_i + \frac{1}{2}\Delta y_i - \frac{1}{12}\Delta^2 y_i + \frac{1}{24}\Delta^3 y_i\right).}$$

This is the working formula employed in finding a definite integral by the method of numerical integration.

To determine the unknown integral, it remains necessary to summate gradually and successively the individual elementary areas:

$$Y_1 = Y_0 + \Delta Y_0; \qquad Y_i = Y_{i-1} + \Delta Y_{i-1};$$

$$Y_2 = Y_1 + \Delta Y_1; \qquad \ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots \qquad Y_n = Y_{n-1} + \Delta Y_{n-1}.$$

<u>Example 4.</u> We have the following table for a function to be integrated:

Table 4-i.

$t \cdot 10^3$	8	12	16	20
p, $kg \cdot cm^{-2}$	26	32	48	88

It is desired to find the elementary area ΔI_i, the subscript i being applied to the first column.

<u>Solution.</u> Let us formulate the following table of differences:

Table 4-j.

$t \cdot 10^3$	8	12	16	20
p	26	32	48	88
Δp	6	16	40	-

Table 4-j (Cont'd.)

$\Delta^2 p$	10	24	-	-
$\Delta^3 p$	14	-	-	-

Since:

$$h = 4 \cdot 10^{-3}; \quad y_i = 26; \quad \Delta y_i = 6; \quad \Delta^2 y_i = 10; \quad \Delta^3 y_i = 14,$$

the elementary area under consideration will be:

$$\Delta Y_i = h\left(y_i + \frac{1}{2}\Delta y_i - \frac{1}{12}\Delta^2 y_i + \frac{1}{24}\Delta^3 y_i\right) =$$

$$= 4 \cdot 10^{-3}\left(26 + 3 - \frac{10}{12} + \frac{14}{24}\right) \approx 4 \cdot 10^{-3}(26 + 3 - 1 + 1) \approx$$

$$\approx 4 \cdot 10^{-3} \cdot 29 \approx 0.116 \text{ kg} \cdot \text{sec} \cdot \text{cm}^{-2} \approx 11.6 \text{ kg} \cdot \text{sec} \cdot \text{dm}^{-2}.$$

10) Determination of Pressure Impulse from a Pressure-Bomb Test.

As is already known from the preceding course in internal ballistics, the pressure impulse is expressed by the following relation:

$$I = \int_0^t p dt.$$

It is clear that in order to determine it from a bomb test yielding directly the (p, t) curve, it is necessary to compute a definite integral, for which the pressure of the powder gases p is the function being integrated and the time t is the independent variable. This can be accomplished of course by means of one of the usual methods of quadratures. However, it is usually necessary to have a curve for the pressure impulse as a function of the gas inflow ψ; in other words, in addition to the pressure impulse I_K at the end of the burning of the powder, it also becomes necessary to find a series of intermediate values for this quantity, which will correspond to intermediate values of the gas inflow. This problem is solved most simply by the method of numerical integration.

Example 5. Given the pressure curve (p, t) represented by Table 4, find the correlation (I, ψ) by means of the table correlating ψ with t.

Table 4-k - Experimental Table of ψ as a Function of t.

$t \cdot 10^3$	0	4	8	12	16	18	20	21	22	23	23.5	24	24.5
$\psi \cdot 10^3$	0	2	3	6	14	22	34	42	54	79	101	127	155
$t \cdot 10^3$	25	25.5	26.0	26.5	27.0	27.5	28.0	28.5	29.0	29.5	30.0	30.5	31.0
$\psi \cdot 10^3$	186	221	260	304	354	409	470	539	619	711	813	919	1000

Solution. We perform all computations on the working form in Table 5, without considering third differences, and designating:

$$\Sigma = y + \frac{1}{2}\Delta y - \frac{1}{12}\Delta^2 y = p + \frac{1}{2}\Delta p - \frac{1}{12}\Delta^2 p;$$

from which it follows that:

$$\Delta I_i = \Delta Y_i = h\left(y_i + \frac{1}{2}\Delta y_i - \frac{1}{12}\Delta^2 y_i\right) = h \cdot \Sigma.$$

											17	25	47	99	–	48	51	
4	$\Delta^2 p$	1	3	10	24	–	–	10	15	–	–	6	24	52	–	–	3	5	
5	p	21	23	26	32	–	–	48	63	–	–	88	105	128	–	–	175	223	2
6	$\frac{1}{2}\Delta p$	1	2	3	8	–	–	8	12	–	–	8	12	24	–	–	24	26	
7	$-\frac{1}{12}\Delta^2 p$	0	0	–1	–2	–	–	–1	–1	–	–	0	–2	–4	–	–	0	0	
8	Σ	22	25	28	38	–	–	55	74	–	–	96	115	148	–	–	199	249	3
9 10	$\Delta I = h\Sigma$ I	8.8 0	10.0 8.8	11.2 18.8	15.2 30.0	– 45.2	– –	11.0 45.2	14.8 56.2	– 71.0	– –	9.6 71.0	11.5 80.6	14.8 92.1	– 106.9	– –	10.0 106.9	12.4 116.9	1 1

Continued

1	$t \cdot 10^3$	24.5	25	25.5	26	26.5	27	27.5	28	28.5	29	29.5	30	30.5	31	30.5	30.75	
2 3	p Δp	330 64	394 72	466 80	546 90	636 103	739 118	857 133	990 147	1137 175	1312 204	1516 227	1743 240	1983 192	2175 –	1983 114	2097 78	2 –
4	$\Delta^2 p$	8	8	10	13	15	15	14	28	29	23	13	–18	–	–	–36	–	–
5	p	330	394	466	546	636	739	857	990	1137	1312	1516	1743	–	–	1983	2097	–
6	$\frac{1}{2}\Delta p$	32	36	40	45	52	59	66	74	88	102	114	120	–	–	57	39	–
7	$-\frac{1}{12}\Delta^2 p$	–1	–1	–1	–1	–1	–1	–1	–2	–2	–2	–1	4	–	–	3	(3)	–
8	Σ	361	429	505	590	687	797	922	1062	1223	1412	1629	1867	–	–	2043	2139	–
9 10	$\Delta I = h\Sigma$ I	18.0 144.3	21.4 162.3	25.2 183.7	29.5 208.9	34.4 238.4	39.8 272.8	46.1 312.6	53.1 358.7	61.2 411.8	70.6 473.0	81.4 543.6	93.4 625.0	– 718.4	– –	51.1 718.4	53.5 769.5	– 82

Table 5 – Computation of Pressure Impulse I.

1	$t \cdot 10^3$	0	4	8	12	16	20	16	18	20	22	20	21	22	23	24	23	23.5	24
2	p	21	23	26	32	48	88	48	63	88	128	88	105	128	175	274	175	223	27
3	Δp	2	3	6	16	40	–	15	25	40	–	17	23	47	99	–	48	51	5
4	$\Delta^2 p$	1	3	10	24	–	–	10	15	–	–	6	24	52	–	–	3	5	[illegible]
5	p	21	23	26	32	–	–	48	63	–	–	88	105	128	–	–	175	223	27
6	$\frac{1}{2}\Delta p$	1	2	3	8	–	–	8	12	–	–	8	12	24	–	–	24	26	2[illegible]
7	$-\frac{1}{12}\Delta^2 p$	0	0	–1	–2	–	–	–1	–1	–	–	0	–2	–4	–	–	0	0	–
8	Σ	22	25	28	38	–	–	55	74	–	–	96	115	148	–	–	199	249	30[illegible]
9	$\Delta I = h\Sigma$	8.8	10.0	11.2	15.2	–	–	11.0	14.8	–	–	9.6	11.5	14.8	–	–	10.0	12.4	15[illegible]
10	I	0	8.8	18.8	30.0	45.2	–	45.2	56.2	71.0	–	71.0	80.6	92.1	106.9	–	106.9	116.9	12[illegible]

Continued

1	$t \cdot 10^3$	24.5	25	25.5	26	26.5	27	27.5	28	28.5	29	29.5	30	30.5	~~31~~	30.5	30.75	31
2	p	330	394	466	546	636	739	857	990	1137	1312	1516	1743	1983	2175	1983	2097	217[illegible]
3	Δp	64	72	80	90	103	118	133	147	175	204	227	240	192	–	114	78	–
4	$\Delta^2 p$	8	8	10	13	15	15	14	28	29	23	13	–18	–	–	–36	–	–
5	p	330	394	466	546	636	739	857	990	1137	1312	1516	1743	–	–	1983	2097	–
6	$\frac{1}{2}\Delta p$	32	36	40	45	52	59	66	74	88	102	114	120	–	–	57	39	–
7	$-\frac{1}{12}\Delta^2 p$																	

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The computations in Table 5 are conducted in rows, proceeding from left to right in each row and downward from one row to the next, in the following manner.

1) The first row is filled with values of the argument, $t \cdot 10^3$.

2) The numbers for the pressure are taken from Table 4 (page 616).

3) From each number in the second row there is subtracted the preceding number in the same row, and the result is written under the left-hand number of this pair:

$$23 - 21 = 2; \; 26 - 23 = 3; \text{ etc.}$$

4) The numbers in the fourth row are obtained in the same manner as in the preceding row:

$$3 - 2 = 1; \; 6 - 3 = 3; \text{ etc.}$$

5) The numbers of the second row are repeated.

6) The numbers in the third row are halved.

7) The numbers in the fourth row are divided by 12 and written with the opposite sign.

8) The numbers in the three preceding rows are added.

9) The numbers in the eighth row are multiplied by the step h, the decimal point being correctly placed (cf. Example 4) to convert $kg \cdot sec \cdot cm^{-2}$ into $kg \cdot sec \cdot dm^{-2}$.

10) To obtain the next succeeding value of I_i, it is necessary to add to the preceding value of I_{i-1} the corresponding increment ΔI_{i-1}:

$$I_i = I_{i-1} + \Delta I_{i-1},$$

or, in other words, it is necessary to add the two numbers in the ninth and tenth rows of the preceding column:

$$0 + 8.8 = 8.8;$$

$$8.8 + 10.0 = 18.8, \text{etc.}$$

By taking the values for the gas inflow ψ and the pressure impulse I corresponding to the same instants t, it is easy to obtain the desired correlation between I and ψ (cf. Table 6).

Table 6 - Correlation between I and Ψ for Test.

I $kg \cdot sec \cdot dm^{-2}$	0	9	19	30	45	56	71	81	92	107	117	129	144	162
$\psi \cdot 10^3$	0	2	3	6	14	22	34	42	54	79	101	127	155	186
I $kg \cdot sec \cdot dm^{-2}$	184	209	238	273	313	359	412	473	544	625	718	770	823	
$\Psi \cdot 10^3$	221	260	304	354	409	470	539	619	711	813	919	967	1000	

Table 6 will be needed subsequently for the solution of the principal problem of internal ballistics (determination of the curves for the speed of the projectile and for the pressure of the powder gases as a function of the path of the projectile). This solution must be prepared by a discussion of the necessary theory, which will be undertaken in the next section.

2. NUMERICAL INTEGRATION OF DIFFERENTIAL EQUATIONS

11) Numerical Integration of First-Order Equation

We have such an equation in its general form:

$$F(y, y', x) = 0.$$

It is composed of the following elements: the independent variable x, the unknown function y, and its first derivative y'. It is necessary, first of all, to determine this derivative from the general equation:

$$y' = f(y, x).$$

It is this equation which is solved by the method of numerical integration, regardless of the form of the function which constitutes its right-hand side. In practice, what is usually found is not the total integral of the last equation (in which case, of course, numerical integration cannot be employed), but, rather, a partial integral, in consequence of which it is necessary to indicate the "initial conditions", i.e., the values of y_0 and x_0. In addition, the value of x_n at the end of the finite region of integration must also be known. Under these conditions the desired function will be:

$$y = \int_{x_0}^{x_n} y'dx$$

or

$$y = \int_{x_0}^{x_n} f(y, x)dx.$$

The problem is consequently reduced to the finding of a definite integral whose characteristic feature is such that the part of the function to be integrated is played by the derivative of the function.

It would seem that, as on the previous occasion involving the finding of a definite integral, it might be possible to perform computations by rows and to make use of a working formula containing differences of an auxiliary function with the same subscript i. In actual practice, however, this procedure cannot be adopted. As a matter of fact, the last formula shows that the function to be integrated includes the original function itself, and we thus obtain a "vicious circle;" in order to determine the particular value y_i of the function, we must know the particular value Φ_i, and in order to determine this particular value of the auxiliary function:

$$\Phi_i = f(y_i, x_i)h$$

we must have the value of y_i which is a constituent of the auxiliary function.

It is now clear that in performing numerical integration of the equations it is not possible to perform computations by rows, it being necessary instead to proceed gradually, step by step, performing the required computations downward along each column and to the right from one column to the next.

Let us further assume that the values Φ_{i-3}, Φ_{i-2}, Φ_{i-1} and Φ_i of the auxiliary function have already been found. With their aid it is possible to compute the following differences.

Table 6-a.

Φ	Φ_{i-3}	Φ_{i-2}	Φ_{i-1}	Φ_i
$\Delta\Phi$	$\Delta\Phi_{i-3}$	$\Delta\Phi_{i-2}$	$\Delta\Phi_{i-1}$	
$\Delta^2\Phi$	$\Delta^2\Phi_{i-3}$	$\Delta^2\Phi_{i-2}$		
$\Delta^3\Phi$	$\Delta^3\Phi_{i-3}$			

The above table shows that in order to determine Δy_i, it is not possible to make use of the following formula for determining definite integrals:

$$\Delta y_i = \Phi_i + \frac{1}{2}\Delta\Phi_i - \frac{1}{12}\Delta^2\Phi_i + \frac{1}{24}\Delta^3\Phi_i,$$

because the differences $\Delta\Phi_i$, $\Delta^2\Phi_i$ and $\Delta^3\Phi_i$ are not yet contained in the table. Consequently, it becomes necessary to derive an additional formula containing the differences:

$$\Delta\Phi_{i-1}, \quad \Delta^2\Phi_{i-2}, \quad \Delta^3\Phi_{i-3},$$

already contained in the table.

For this purpose, as on the previous occasion, we shall replace the function to be integrated, $y' = f(y, x)$, by the following interpolation function:

$$y' = A_0(x - x_i)^3 + A_1(x - x_i)^2 + A_2(x - x_i) + A_3,$$

and then:

$$\Delta y_i = \int_{x_i}^{x_{i+1}} f(y, x)dx = \int_{x_i}^{x_{i+1}} [A_0(x - x_i)^3 + A_1(x - x_i)^2 + A_2(x - x_i) + A_3]dx =$$

$$= \left| \frac{1}{4}A_0(x - x_i)^4 + \frac{1}{3}A_1(x - x_i)^3 + \frac{1}{2}A_2(x - x_i)^2 + A_3(x - x_i) \right|_{x_i}^{x_{i+1}},$$

whence

$$\Delta y_i = \frac{1}{4} A_0h^4 + \frac{1}{3} A_1h^3 + \frac{1}{2}A_2h^2 + A_3h,$$

because

$$x_{i+1} - x_i = h.$$

In contrast with the preceding case, we shall substitute here the following values for the coefficients:

$$A_0 = \frac{\Delta^3 y'_{i-3}}{6h^3}; \qquad A_1 = \frac{\Delta^2 y'_{i-2} + \Delta^3 y'_{i-3}}{2h^2};$$

$$A_2 = \frac{6\Delta y'_{i-1} + 3\Delta^2 y'_{i-1} + 2\Delta^3 y'_{i-3}}{6h}; \quad A_3 = y'_i,$$

which are expressed in terms of differences of the function y under the integral sign provided with the required subscripts.

Following obvious transformations, we obtain:

$$\boxed{\Delta y_i = h(y'_i + \frac{1}{2}\Delta y'_{i-1} + \frac{5}{12}\Delta^2 y'_{i-2} + \frac{3}{8}\Delta^3 y'_{i-3}).}$$

It is usually convenient to make use of the following auxiliary function:

$$\Phi = hy',$$

the derivative y' being computed in accordance with the given equation:

$$y' = f(x, y).$$

Then:

$$hy'_i = \Phi_i; \quad h\Delta y'_{i-1} = \Delta\Phi_{i-1}; \quad h\Delta^2 y'_{i-2} = \Delta^2\Phi_{i-2}; \quad h\Delta^3 y'_{i-3} = \Delta^3\Phi_{i-3},$$

and the working formula for the numerical integration of ordinary differential equations of the first order will assume the following form:

$$\boxed{\Delta y_i = \Phi_i + \frac{1}{2}\Delta\Phi_{i-1} + \frac{5}{12}\Delta^2\Phi_{i-2} + \frac{3}{8}\Delta^3\Phi_{i-3}.}$$

This formula now includes differences of the auxiliary function Φ provided with precisely those subscripts which are required in accordance with the table of difference presented above.

It will be seen from the same table that in order to determine Δy_i, it becomes necessary to take differences from different columns, which, of course, is inconvenient. For this reason it is desirable to modify the system of writing the differences in such a manner as to have in the same column differences with different subscripts, namely:

Table 6-b.

Φ	Φ_{i-3}	Φ_{i-2}	Φ_{i-1}	Φ_i
$\Delta\Phi$ $\Delta^2\Phi$ $\Delta^3\Phi$		$\Delta\Phi_{i-3}$	$\Delta\Phi_{i-2}$ $\Delta^2\Phi_{i-3}$	$\Delta\Phi_{i-1}$ $\Delta^2\Phi_{i-2}$ $\Delta^3\Phi_{i-3}$

In such a procedure each column will contain those differences which are necessary for the application of the new working formula for numerical integration.

Rule for writing new differences. From the right-hand number in a row there is subtracted the left-hand number in the same row, but the result is written under the right-hand number.

Finally, from the last table of differences there follows the most disagreeable characteristic feature of the numerical integration of equations: in order to determine Δy_1, it is necessary to know the differences:

$$\Delta\Phi_{i-1}, \quad \Delta^2\Phi_{i-2}, \quad \Delta^3\Phi_{i-3},$$

i.e., to have the following particular values of the function:

$$\Phi_{i-1}, \quad \Phi_{i-2}, \quad \Phi_{i-3}.$$

These will become known in the course of the process; but at the start of the integration only one particular value of this function, namely Φ_0, is known from the originally stated equation for y'.

In order to find

$$\Phi_{-1}, \quad \Phi_{-2}, \quad \Phi_{-3}$$

it is necessary to employ special supplementary methods, of which the most commonly used is the method of successive approximations. The essence of this method resides in the fact that, at the start of the computations, there is adopted a gradual and stepwise advance, each new approximation permitting one additional step, in which those differences that have already appeared earlier are utilized. There exist several variants of this method, one of which, the most exact one, can be best studied by the aid of a concrete example.

Example 6. Solve by the method of numerical integration the equation:

$$y' = x - y$$

over the interval 0; 0.5, with a step of h = 0.1, if y = 1 when x = 0.

Solution. All the computations are presented in Table 7.

Table 7 – Integration of Equation $y' = x - y$.

1	x	0	0.1	0	0.1	0.2	0	0.1	0.2	0.3	0	0.1	0.2
2	$-y$	−1.0000	−0.9000		−0.9100	−0.8385		−0.9096	−0.8373	−0.7815		0.9097	0.8375
3	y'	−1.0000	−0.8000		−0.8100	−0.6385		−0.8096	0.6373	−0.4815		−0.8097	0.6375
4	$\Phi = hy'$	−0.1000	−0.0800	−0.1000	−0.0810	−0.0638	−0.1000	−0.0810	−0.0637	−0.0482	−0.1000	−0.0810	−0.0638
5	$\Delta\Phi$	(200)	200	(208)	190	172	(206)	190	173	155	(−206)	190	172
6	$\Delta^2\Phi$			(−18)	(−18)	−18	(−15)	(−16)	−17	−18	(−15)	(−16)	−18
7	$\Delta^3\Phi$						(−1)	(−1)	(−1)	−1	(−1)	(−1)	(1)
8	Φ	−0.1000		−0.1000	−0.0810		−0.1000	−0.0810	−0.0637		−0.1000	−0.0810	−0.0638
9	$\frac{1}{2}\Delta\Phi$			100	95		104	95	86		103	95	86
10	$\frac{5}{12}\Delta^2\Phi$						−8	−8	−7		−6	−7	−8
11	$\frac{3}{8}\Delta^3\Phi$										0	0	0
12	$\Delta y = \Sigma$	−0.1000		−0.0900	−0.0715		−0.0904	−0.0723	−0.0558		−0.0903	−0.0722	−0.0560
13	y	1.0000	0.9000	1.0000	0.9100	0.8385	1.0000	0.9096	0.8373	−0.7815	1.0000	0.9097	0.8375
		First approximation		Second approximation			Third approximation				Normal computation		

Table 7 - Integration of Equation y' = x - y.

	0.1	0	0.1	0.2	0	0.1	0.2	0.3	0	0.1	0.2	0.3	0.4	0.5
00	-0.9000		-0.9100	-0.8385		-0.9096	-0.8373	-0.7815		0.9097	0.8375	-0.7815	-0.7400	
00	-0.8000		-0.8100	-0.6385		-0.8096	0.6373	-0.4815		-0.8097	0.6375	-0.4815	-0.3400	
00	-0.0800	-0.1000	-0.0810	-0.0638	-0.1000	-0.0810	-0.0637	-0.0482	-0.1000	-0.0810	-0.0638	-0.0482	-0.0340	
)	200	(208)	190	172	(206)	190	173	155	(-206)	190	172	156	142	
		(-18)	(-18)	-18	(-15)	(-16)	-17	-18	(-15)	(-16)	-18	-16	-14	
					(-1)	(-1)	(-1)	-1	(-1)	(-1)	(1)	1	0	
00		-0.1000	-0.0810		-0.1000	-0.0810	-0.0637		-0.1000	-0.0810	-0.0638	-0.0482	-0.0340	
		100	95		104	95	86		103	95	86	78	71	
					-8	-8	-7		-6	-7	-8	-7	-6	
									0	0	0	0	0	
00		-0.0900	-0.0715		-0.0904	-0.0723	-0.0558		-0.0903	-0.0722	-0.0560	-0.0411	-0.0275	
00	0.9000	1.0000	0.9100	0.8385	1.0000	0.9096	0.8373	-0.7815	1.0000	0.9097	0.8375	0.7815	0.7404	0.7129
	First roximation	Second approximation			Third approximation				Normal computations					

E

First Approximation

First Column

13) This is the leading row and is filled first; in the given case $y_0 = 1.0000$ from the conditions of the problem.

1) $x_0 = 0$ from the conditions of the problem.

2) $-y_0 = -1.0000$ in accordance with the thirteenth row.

3) $y_0' = -1$; in accordance with the given equation for y', we add the numbers in the first and second rows above.

4) $\Phi_0 = -0.1000$; the number in the third row is multiplied by the step $h = 0.1$, since:

$$\Phi_0 = hy_0'$$

The fifth, sixth and seventh rows are not filled in for lack of data. Then:

8) $\Phi_0 = -0.1000$; the number in the fourth row is repeated.

The ninth, tenth and eleventh rows are not filled in for lack of data. Thereupon:

12) $\Delta y_0 = -0.1000$; in this case it is necessary to repeat the number in the eighth row.

Second Column (of the First Approximation)

13) Leading row; $y_1 = 0.9000$; we add (algebraically) the numbers in the twelfth and thirteenth rows of the first column, since:

$$y_1 = y_0 + \Delta y_0;$$

1) $x_1 = 0.1$; we add the step h.

2) $-y_1 = -0.9000$ in accordance with the thirteenth row.

3) $y_1' = -0.8000$; in accordance with the given equation for y' we add the numbers in the first and second rows of this column (above).

4) $\Phi_1 = -0.0800$; the number in the third row is multiplied by the step $h = 0.1$.

Thus a number appears in the fifth row; $\Delta\Phi_0 = 200$, for which we subtract from the number (-0.0800) in the fourth row above the number (-1.0000) at the left in the same row, omitting the zeros at the left for greater convenience.

Wedge of First Approximation.

It consists merely of a single number $\Delta\Phi_{-1}$ and is surrounded by a heavy line. To obtain it, we postulate that

$$\Delta\Phi_{-1} = \Delta\Phi_0 = 200.$$

Second Approximation

First Column.

13) Leading row: $\Phi_0 = 1$; 1) $x_0 = 0$.

The second and third rows are not filled, as this is no longer necessary (cf. first column of first approximation). Then:

4) $\Phi_0 = -0.1000$, as in the same row of the first approximation.

Omitting the fifth, sixth, and seventh rows, we write:

8) $\Phi_0 = -0.1000$; the number in the fourth row is repeated.

9) $\frac{1}{2}\Delta\Phi_{-1} = 100$; we take the difference $\Delta\Phi_{-1}$ from the wedge of the first approximation.

The tenth and eleventh rows are not filled for lack of necessary data.

12) $\Delta y_0 = -0.0900$; we add the numbers in the eighth and ninth rows above.

The second column of this approximation is formed in an analogous manner, with the only difference that numbers appear in the first and second rows (cf. the number in the thirteenth row of this column) and in the fifth row, for which we subtract from the number (-0.0810) in the fourth row the number (-0.1000) in the same row, but on the left. The number in the leading row is obtained by adding the numbers in the twelfth and thirteenth rows of the preceding column.

The third column is distinguished from second in that a number (-17) appears in the sixth row, for which we subtract from the number 173 in the fifth row the number 190 at the left, and another number (-8) appears in the tenth row, for which the number in the sixth row is multiplied by $\frac{5}{12} = \frac{10}{24}$.

Filling in the Wedge of the Second Approximation.

In the sixth line we repeat twice the number (-18), noting that:

$$\Delta^2\Phi_{-2} = \Delta^2\Phi_{-1} = \Delta^2\Phi_0 = -18.$$

The number 208 in the fifth row of the first column of the second approximation will appear in accordance with the definition of the second differences:

$$\Delta^2\Phi_{-1} = \Delta\Phi_0 - \Delta\Phi_{-1},$$

from which:

$$\Delta\Phi_{-1} = \Delta\Phi_{-0} - \Delta^2\Phi_1 = 190 - (-18) = 208.$$

Third Approximation.

In the first column, after the thirteenth and first rows are filled, the places in the second, third, fifth, sixth, and seventh rows are left blank. Then:

8) The number in the fourth row is repeated.

9) $\frac{1}{2}\Delta\Phi_{-1} = 104$; the number 208 in the wedge of the second approximation is halved.

10) $\frac{5}{12}\Delta^2\Phi_{-2} = -8$; the number (-18) in the wedge of the second approximation is multiplied by $\frac{10}{24}$.

The eleventh row is omitted. Thereupon:

12) The numbers in the eighth, ninth, and tenth rows above are added:

$$-0.1000 + 0.0104 - 0.0008 = -0.0904.$$

In the second column, after the thirteenth, first, second, third, fourth and fifth rows are filled, the sixth and seventh rows are omitted. Then:

8) The number in the fourth row is repeated.

9) $\frac{1}{2}\Delta\Phi_0 = 95$; the number in the fifth row of this column is halved.

10) $\frac{5}{12}\Delta^2\Phi_{-2} = -8$; the number (-18) in the second column of the preceding wedge is multiplied by $\frac{10}{24}$.

The eleventh row is likewise omitted. Thereupon:

12) The numbers in the eighth, ninth, and tenth rows above are added.

In the third column, only the thirteenth, first, second, third, fourth, fifth and sixth rows are filled. There appears for the first time a number in the seventh row:

$$-18 - (-17) = -1.$$

<u>Filling in the Wedge of the Third Approximation.</u>

We make:

$$\Delta^3\Phi_{-3} = \Delta^3\Phi_{-2} = \Delta^3\Phi_{-1} = \Delta^3\Phi_0 = -1.$$

The number (-16) in the sixth row of the second column of the wedge will appear from the definition of the third differences:

$$\Delta^2\Phi_{-1} = \Delta^2\Phi_0 - \Delta^3\Phi_{-1} = -17 -(-1) = -16.$$

In an analogous manner, the blanks in the rows of the first column of the wedge will be filled as follows:

$$\Delta^2\Phi_{-2} = \Delta^2\Phi_{-1} - \Delta^3\Phi_{-2} = -16 - (-1) = -15;$$

$$\Delta^2\Phi_{-1} = \Delta\Phi_0 - \Delta^2\Phi_{-1} = 190- (-16) = 206.$$

For convenience of computation, the numbers of this wedge are transferred to their places in the columns of the subsequent normal computations. The approximations are considered to be completed because of the closeness of the results of the third and second approximations, respectively.

Normal Computations.

First Column.

13) As before, $y_0 = 1.0000$; 2) $x_0 = 0$.

The second and third rows need not be filled. The numbers in the fifth, sixth, and seventh rows are already in place. Thereupon:

8) $\Phi_0 = -0.1000$; the number in the fourth row is repeated.

9) $\frac{1}{2}\Delta\Phi_{-1} = 103$; the number 206 in the fifth row is halved.

10) $\frac{5}{12}\Delta^2\Phi_{-2} = -6$; the number (-15) in the sixth row is multiplied by $\frac{10}{24}$.

11) $\frac{3}{8}\Delta^2\Phi_{-3} = 0$; the number (-1) in the seventh row is multiplied by $\frac{3}{8}$.

12) $\Delta y_0 = -0.0903$; the numbers in the eighth, ninth, tenth, and eleventh rows are added.

The subsequent columns are filled in exactl, the same manner.

Attention is once again directed to the order of computation in each column of normal computations.

a) First of all, the leading thirteenth row is filled by adding the numbers in the twelfth and thirteenth rows of the preceding column, because

$$\Delta y_i = y_{i-1} + \Delta y_{i-1};$$

b) Thereupon, the spaces in each column are filled from top to bottom without omissions.

12) Use of Numerical Integration of the First-Order Equation with Argument v.

The method of numerical integration must be applied to the solution of the following equation:

$$\frac{dl}{dv} = \frac{1}{v_{np}^2} \frac{2v}{\psi - \frac{v^2}{v_{np}^2}} (l_\psi + l),$$

where:

$$l_\psi = l_\Delta - l_\alpha \psi.$$

In the physical law of powder burning the correlation between ψ and v is given by a table, whereas in the geometric law:

$$\psi = \psi_0 + \frac{\varkappa \sigma_0 \varphi m}{sI_k} v + \frac{\varkappa \lambda \varphi^2 m^2}{s^2 I_k^2} v^2.$$

Since it is also necessary as a rule to find the pressure curve, it is preferable to make use of the following equation:

$$\varphi m v \frac{dv}{dl} = ps,$$

whence

$$\frac{dl}{dv} = \frac{\varphi m}{s} \frac{v}{p},$$

where

$$p = \frac{f\omega}{s}\,\frac{\psi - \dfrac{v^2}{v_{\text{пр}}^2}}{l_\psi + l}.$$

It is not difficult to see that the auxiliary function Φ is found in the given case from the following relations:

$$l_\psi = l_\Delta - l_\alpha\psi; \quad p = \frac{f\omega}{s}\,\frac{\psi - \dfrac{v^2}{v_{\text{пр}}^2}}{l_\psi + l}; \qquad \Phi = \frac{\varphi mh}{s}\,\frac{v}{p}. \qquad (*)$$

As for the working formula of numerical integration, it has the usual form:

$$\Delta l_i = \Phi_i + \frac{1}{2}\Delta\Phi_{i-1} + \frac{5}{12}\Delta^2\Phi_{i-2} + \frac{3}{8}\Delta^3\Phi_{i-3}.$$

The purpose of the preliminary computations is to determine all the constants:

$$1)\ \frac{\omega}{s}; \quad 2)\ \frac{\varphi m}{s} = \frac{q\varphi}{gs}; \quad 3)\ v_{\text{пр}}^2 = 2\,\frac{f}{\theta}\,\frac{\omega}{s} : \frac{\varphi m}{s};$$

$$4)\ l_\Delta = \frac{\omega}{s}\left(\frac{1}{\Delta} - \frac{1}{\delta}\right); \quad 5)\ l_\alpha = \frac{\omega}{s}\left(\alpha - \frac{1}{\delta}\right);$$

$$6)\ \psi_0 = \frac{\dfrac{1}{\Delta} - \dfrac{1}{\delta}}{\dfrac{f}{p_0} + \alpha - \dfrac{1}{\delta}}; \quad 7)\ \frac{f\omega}{s}.$$

To determine the step of integration h, the speed of the projectile v_K at the end of powder burning must be known, and then:

$$h = \frac{v_K}{n},$$

where n - the number of sections - is taken in the range of 10-40, depending upon the density of loading (as this density increases, the number n must also be increased).

While determining l and p, the same working form may be used to find the time of motion of the projectile in accordance with the following relation:

$$\frac{dt}{dv} = \frac{\varphi m}{s}\frac{1}{p},$$

in the form of a definite integral.

Example 7. Determine the projectile velocity and gas pressure curves for a 76 mm gun, given the following conditions: $W_0 = 1.654$; $s = 0.4693$; $l_Д = 18.44$; $q = 6.5$; $p_0 = 30,000$; $f = 900,000$; $\alpha = 1$; $\delta = 1.6$; $\theta = 0.2$; $\omega = 0.930$; $\varphi = 1.05$; $g = 98.1$, using strip powder type SP (1 x 18 x 40 mm). It is assumed that the powder burns in conformity with the physical law derived from the pressure-bomb test presented in Table 5 (correlation between I and ψ).

To start with, we find:

$$\log\frac{\omega}{s} = 0.2970;\quad \log\frac{\varphi m}{s} = \bar{1}.1709;\quad \log v^2_{\Pi p} = \log\left(\frac{2g}{\varphi}\frac{f}{\theta}\frac{\omega}{q}\right) = 8.0803;$$

$$l_\Delta = l_0\left(1 - \frac{\Delta}{\delta}\right) = 2.284;\ \log l_a = \bar{1}.8710;\ \psi_0 = 0.038;\ \log\frac{f\omega}{s} = 6.2512.$$

From the value ψ_0 found we compute the value of t_0 by inverse interpolation with the aid of the data in Table 2 (cf. Example 2, Subsection 7)

$$t_0 = 0.020556 \text{ sec}$$

For this value of t_0 we determine the initial impulse I_0 by direct interpolation with the aid of Table 4 (cf. Example 1, Subsection 6):

$$I_0 = 76 \text{ kg}\cdot\text{sec}\cdot\text{dm}^{-2}$$

We find the velocity of the projectile at the end of burning of the powder, keeping in mind that $I_K = 823$ (cf. Table 5):

$$v_K = \frac{s}{\varphi m}(I_K - I_0) = 5040 \text{ dm}\cdot\text{sec}^{-1}$$

Bt choosing $n = 20$ as the number of sections, we obtain the step as being:

$$h = \frac{v_K}{20} = 252 \text{ dm}\cdot\text{sec}^{-1}$$

so that

$$\log\left(\frac{\varphi m}{s}h\right) = 1.5723; \quad \log\left(\frac{\varphi m}{s}\, h\cdot 10^5\right) = 6.5723.$$

On the basis of Table 5, we plot the (ψ, I) curve to the following scale: for ψ, 1 mm = 0.001; for I, 1 mm = 1 $\text{kg}\cdot\text{sec}\cdot\text{dm}^{-2}$.

We read on it the values of the gas inflow ψ corresponding to equal intervals

$$\frac{823 - 76}{20} = 37.35 \text{ kg·sec·dm}^{-2}$$

for the pressure impulse, or, what is the same thing, to equal intervals $h = 252$ dm·sec^{-1} for the velocity of the projectile.

These values of ψ are given subsequently in the working form used for performing the computations (in the fifth row).

1	v	0	252	0	252	5
2	$2 \log v$		4.8028			5.4
3	$-\log v^2_{np}$		9.9197			9.9
4	$\log \frac{v^2}{v^2_{np}}$		$\bar{4}.7225$			3.3
5	ψ	0.038	0.092			0.1
6	$-\frac{v^2}{v^2_{np}}$		-0.001			-0.0
7	$\psi - \frac{v^2}{v^2_{np}}$		0.091			0.1
8	$\log l_\alpha$	1.8710	1.8710			1.8
9	$\log \psi$	2.5793	2.9638			1.2
10	$\log l_\alpha \psi$	2.4503	2.8348			1.0
11	l_Δ	2.284	2.284			2.2
12	$-l_\alpha \psi$	-0.028	-0.068			-0.1
13	l_ψ	2.256	2.216		2.216	2.1
14	l	0.000	0.000		0.064	0.2
15	$l_\psi + l$	2.256	2.216		2.280	2.4
16	$\log \frac{f\omega}{s}$	6.2512	6.2512		6.2512	6.2
17	$\left(\log \psi - \frac{v^2}{v^2_{np}}\right)$	2.5793	2.9590		2.9590	1.2
18	$-\log(l_\psi + l)$	1.6466	1.6544		1.6421	1.6
19	$\log p$	4.4771	4.8646		4.8523	5.0
20	p	300	732		711	1
21	$\log\left(\frac{\gamma m}{s} h\right)$		1.5723		1.5723	1.5
22	$\log v$		2.4014		2.401[illegible]	2.7
23	$-\log p$		5.1354		5.1477	6.9
24	$\log \Phi$		1.1091		1.1214	1.1
25	Φ	0	0.129	0.000	0.132	0.1
26	$\Delta \Phi$	$\boxed{(129)}$	129		132	
27	$\Delta^2 \Phi$					-10
[illegible]	$\Delta^3 \Phi$					

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Form for Computations with Argument v.

252	0	252	504	756	1008	1260
4.8028			5.4048	5.7570	6.0068	6.2008
9.9197			9.9197	9.9197	9.9197	9.9197
4.7225			3.3245	3.6767	3.[illegible]265	2.1205
0.092			0.166	0.227	0.284	0.339
-0.001			-0.002	-0.005	-0.009	-0.013
0.091			0.164	0.222	0.275	0.326
1.8710			1.8710	1.8710	1.8710	1.8710
2.9638			1.2201	1.3560	1.4533	1.5302
2.8348			1.0911	1.2270	1.3243	1.4012
2.284			2.284	2.284	2.284	2.284
-0.068			-0.123	-0.169	0.211	0.252
2.216		2.216	2.161	2.115	2.073	2.032
0.000		0.064	0.262	0.385	0.573	0.789
2.216		2.280	2.423	2.500	2.646	2.821
6.2512		6.2512	6.2512	6.2512	6.2512	6.2512
2.9590		2.9590	1.2148	1.3464	1.4393	1.5132
1.6544		1.6421	1.6257	1.6021	1.5774	1.5496
4.8646		4.8523	5.0817	5.1997	5.2679	5.3150
732		711	1207	1586	1853	2065
1.5723		1.5723	1.5723	1.5723	1.5723	1.5723
2.4014		2.401[illegible]	2.7024	2.8785	3.0034	3.1004
5.1354		5.1477	6.9183	6.8003	6.7321	6.6850
1.1091		1.1214	1.1930	1.2511	1.3078	1.3577
0.129	0.000	0.132	0.156	0.178	0.203	0.228
129		132	24	22	25	25
			-109			

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18	$-\log(l_\psi + l)$	1.6466	1.6544		1.6421	1.6257
19	$\log p$	4.4771	4.8646		4.8523	5.0817
20	p	300	732		711	1207
21	$\log\left(\frac{\varphi m}{s} h\right)$		1.5723		1.5723	1.5723
22	$\log v$		2.4014		2.401	2.7024
23	$-\log p$		5.1354		5.1477	6.9183
24	$\log \Phi$		1.1091		1.1214	1.1930
25	Φ	0	0.129	0.000	0.132	0.156
26	$\Delta\Phi$	$\boxed{(129)}$	129		132	24
27	$\Delta^2\Phi$					-108
28	$\Delta^3\Phi$					
29	Φ	0		0.000	0.132	0.156
30	$\frac{1}{2}\Delta\Phi$			64	66	12
31	$\frac{5}{12}\Delta^2\Phi$					-45
32	$\frac{3}{8}\Delta^3\Phi$					
33	$\Delta l = \Sigma$	0		0.064	0.198	0.123
34	l	0	0	0.000	0.064	0.262
35	$\log \frac{\varphi m}{s} h \cdot 10^5$			6.5723	6.5723	6.5723
36	$-\log p$			5.5229	5.1477	6.9183
37	$\log F$			2.0952	1.7200	1.4906
38	F			124	52	31
39	ΔF			-72	-21	-7
40	$\Delta^2 F$			51	14	3
41	$\Delta^3 F$			-37		
42	F			124	52	31
43	$\frac{1}{2}\Delta F$			-36	-10	-4
44	$-\frac{1}{12}\Delta^2 F$			-4	-1	
45	$\frac{1}{24}\Delta^3 F$			-2		
46	$\Delta t \cdot 10^5 = \Sigma$			82	41	27
47	$t \cdot 10^5$			0	82	123

1.6544		1.6421	1.6257	1.6021	1.5774	1.5196
4.8646		4.8523	5.0817	5.1997	5.2679	5.3150
732		711	1207	1586	1853	2065
1.5723		1.5723	1.5723	1.5723	1.5723	1.5723
2.4014		2.401[illegible]	2.7024	2.8785	3.0034	3.1004
5.1354		5.1477	6.9183	6.8003	6.7321	6.6850
1.1091		1.1214	1.1930	1.2511	1.3078	1.3577
0.129	0.000	0.132	0.156	0.178	0.203	0.228
129		132	24	22	25	25
			-108	-2	3	0
	0.000	0.132	0.156	0.178	0.203	0.228
	64	66	12	11	12	12
			-45	-1	1	0
	0.064	0.198	0.123	0.188	0.216	0.240
0	0.000	0.064	0.262	0.385	0.573	0.789
	6.5723	6.5723	6.5723	6.5723	6.5723	6.5723
	5.5229	5.1477	6.9183	6.8003	6.7321	6.6850
	2.0952	1.7200	1.4906	1.3726	1.3044	1.2573
	124	52	31	24	20	18
	-72	-21	-7	-4	-2	-1
	51	14	3			
	-37					
	124	52	31	24	20	18
	-36	-10	-4	-2	-1	0
	-4	-1				
	-2					
	82	41	27	22	19	18
	0	82	123	150	172	191

19	log P		73[illegible]		711	12[illegible]
20	P		1.5723		1.5723	1.572[illegible]
21	$\log\left(\frac{c^m}{s} h\right)$		2.4014		2.401[illegible]	2.702
22	[illegible]					
23	log p		5.1354		5.1477	6.918
24	log Φ		1.1091		1.1214	1.193[illegible]
25	Φ	0	0.129	0.000	0.132	0.156
26	$\Delta\Phi$	(129)	129		132	2[illegible]
27	$\Delta^2\Phi$					-108
28	$\Delta^3\Phi$					
29	Φ	0		0.000	0.132	0.156
30	$\frac{1}{2}\Delta\Phi$			64	66	12
31	$\frac{5}{12}\Delta^2\Phi$					-45
32	$\frac{3}{8}\Delta^3\Phi$					
33	$\Delta l = \Sigma$	0		0.064	0.198	0.123
34	l	0	0	0.000	0.064	0.262
35	$\log \frac{\varphi m}{s} h \cdot 10^5$			6.5723	6.5723	6.5723
36	$-\log p$			5.5229	5.1477	6.9183
37	log F			2.0952	1.7200	1.4906
38	F			124	52	31
39	ΔF			-72	-21	-7
40	$\Delta^2 F$			51	14	3
41	$\Delta^3 F$			-37		
42	F			124	52	31
43	$\frac{1}{2}\Delta F$			-36	-10	-4
44	$-\frac{1}{12}\Delta^2 F$			-4	-1	
45	$\frac{1}{24}\Delta^3 F$			-2		
46	$\Delta t \cdot 10^5 = \Sigma$			82	41	27
47	$t \cdot 10^5$			0	82	123

732		711	1207	1586	1853	2065
1.5723		1.5723	1.5723	1.5723	1.5723	1.5723
2.4014		2.401	2.7024	2.8785	3.0034	3.1004
5.1354		5.1477	6.9183	6.8003	6.7321	6.6850
1.1091		1.1214	1.1930	1.2511	1.3078	1.3577
0.129	0.000	0.132	0.156	0.178	0.203	0.228
129		132	24	22	25	25
			-108	-2	3	0
	0.000	0.132	0.156	0.178	0.203	0.228
	64	66	12	11	12	12
			-45	-1	1	0
	0.064	0.198	0.123	0.188	0.216	0.240
0	0.000	0.064	0.262	0.385	0.573	0.789
	6.5723	6.5723	6.5723	6.5723	6.5723	6.5723
	5.5229	5.1477	6.9183	6.8003	6.7321	6.6850
	2.0952	1.7200	1.4906	1.3726	1.3044	1.2573
	124	52	31	24	20	18
	-72	-21	-7	-4	-2	-1
	51	14	3			
	-37					
	124	52	31	24	20	18
	-36	-10	-4	-2	-1	0
	-4	-1				
	-2					
	82	41	27	22	19	18
	0	82	123	150	172	191

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The working form can be broken down into three main sections. The upper section contains lines 1-24 and is reserved for computations necessary for determining the logarithm of the auxiliary function in accordance with the relations (*). Consequently, the pressure of the powder gases is also found at the same time. The values for the gas inflow in line 5 are first read from the (ψ_t I) curve in the case of the physical law of burning of the powder or computed in accordance with the corresponding formula in the case of the geometric law of burning.

The upper section is the most involved part of the work and makes the use of four-place logarithms obligatory.

In the middle section, which consists of lines 25-34, are performed computations necessary for the use of the working formula for the numerical integration of a first-order equation:

$$\Delta l_i = \Phi_i + \frac{1}{2}\Delta\Phi_{i-1} + \frac{5}{12}\Delta^2\Phi_{i-2} + \frac{3}{8}\Delta^3\Phi_{i-3},$$

and for finding the path of the projectile: $l_i = l_{i-1} + \Delta l_{i-1}$.

Finally, the lower section, consisting of lines 35-47, is reserved for finding the time of motion of the projectile in the bore in the form of a definite integral, using the following working formula:

$$\Delta t_i = F_i + \frac{1}{2}\Delta F_i - \frac{1}{12}\Delta^2 F_i + \frac{1}{24}\Delta^3 F_i.$$

This section is best filled after completion of the numerical integration of the equation for the path of the projectile:

$$\frac{dt}{dv} = \left(\frac{\varphi m}{g}\ h\right) \frac{y}{p},$$

i.e., after the two sections above it are filled.

It is useful to point out some of the characteristic features of the computations.

a) The path of the projectile is determined with an accuracy of 0.001 dm; the time t, within 0.00001 seconds.

b) To facilitate computations, only one approximation is made, and only one wedge is filled out (cf. line 26, first column of the working form).

c) At first, the differences of the auxiliary function up to and including the second order are introduced.

d) The third differences of this function are utilized only after p_m is passed.

e) The second difference of the function Φ appears only in the second column of the normal computations.

f) The rows which are not filled are: lines 2, 3, 4, 6, 7, 21, 22, 23, 24, 27, 28, 30, 31, 32, and 35-47 in the first column, lines 27-33 and 35-47 in the second column of the approximation, lines 2-24, 26, 27, 28, 31, and 32 of the first column of normal computations, and, finally, lines 2-12, 27, 28, 31, and 32 of the second column of normal computations.

g) Line 34 is the leading row and is filled first in each column.

For a more definite conception of the character of the work involved, when filling in the two upper sections of the working form it would be desirable to describe in detail the computations performed for one of the columns, for example, for the column of normal computations with v = 504.

34) The numbers in lines 33 and 34 of the preceding column are added:

$$l = 0.198 + 0.064 = 0.262.$$

since:

$$l_i = l_{i-1} + \Delta l_{i-1};$$

1) We add the integration step h:

$$v = 252 + h = 252 + 252 = 504;$$

2) The number in the preceding row, v = 504, gives:

$$2 \log v = 5.4048;$$

3) The complement of the logarithm of v_{np}^2 is taken:

$$-\log v_{np} = \overline{9}.9197;$$

4) The logarithms in lines 2 and 3 are added:

$$g\,\frac{v^2}{v_{np}^2} = 5.4048 + \overline{9}.9197 = \overline{3}.3245;$$

5) The value of ψ is taken from the (ψ, I) curve:

$$\psi = 0.166;$$

6) The logarithm in line 4 is used to find the number within 0.001:

$$\frac{v^2}{v^2_{np}} = 0.002;$$

7) The number in line 6 is subtracted from the number in line 5:

$$\psi - \frac{v^2}{v^2_{np}} = 0.166 - 0.002 = 0.164;$$

8) The logarithm of l_α has been found by preliminary computations:

$$\log l_\alpha = 1.8710;$$

9) The number in line 5 gives the logarithm:

$$\log \psi = \bar{1}.2201;$$

10) The logarithms in lines 8 and 9 above are added:

$$\log l_\alpha \psi = \bar{1}.8710 + \bar{1}.2201 = \bar{1}.0911;$$

11) The reduced length l_Δ has been found by preliminary computations:

$$l_\Delta = 2.284;$$

12) The logarithm in line 10 is used to find the number within 0.001:

$$l_\alpha \psi = 0.123;$$

13) The number in line 12 is subtracted from the number in line 11:

$$l_\psi - l_\Delta - l_\alpha \psi = 2.284 - 0.123 = 2.161;$$

14) The number in the leading row, line 34, is taken:

$$l = 0.262,$$

15) The numbers in lines 13 and 14 are added:

$$l_\psi + l = 2.161 + 0.262 = 2.423;$$

16) This logarithm has been obtained by preliminary computations:

$$\log \frac{f\omega}{s} = 6.2512;$$

17) The logarithm of the number in line 7 is found:

$$\log \left(\psi - \frac{v^2}{v^2_{np}}\right) = \bar{1}.2148;$$

18) The complement of the logarithm of the number in line 15 is taken:

$$-\log(l_\psi + l) = -\log 2.423 = \bar{1}.6257;$$

19) The logarithms in lines 16, 17, and 18 are added:

$$\log p = 6.2512 + \bar{1}.2148 + \bar{1}.6257 = 5.0817;$$

20) The logarithm in line 19 is used to find the pressure at $v = 504$:

$$p = 1207 \text{ kg}\cdot\text{cm}^{-2}$$

21) This logarithm has been obtained by preliminary computations:

$$\log\left(\frac{\varphi m}{s} h\right) = 1.5723;$$

22) From the number in line 1 we have:

$$\log v = 2.7024$$

23) The complement of the logarithm in line 19 is taken:

$$-\log p = -5.0817 = \bar{6}.9183$$

24) The logarithms in lines 21, 22, and 23 are added:

$$\log \Phi = 1.5723 + 2.7024 + \bar{6}.9183 = \bar{1}.1930.$$

We now proceed to the next, middle, section.

25) The logarithm in line 24 is used to obtain the number with an accuracy of 0.001:

$$\Phi_2 = 0.156;$$

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$$\Delta l_2 = \Phi_2 + \frac{1}{2}\Delta\Phi_1 + \frac{5}{12}\Delta^2\Phi_0 = 0.123.$$

It is now possible to take the next step, i.e., proceed to the next column with v = 756, starting the computations therein, as always, from the leading row, line 34.

At the muzzle, we have: $v_д = 583$ m/sec; $p_д = 625$ kg/cm^2.

13) Numerical Integration of the Second-Order Differential Equation.

In the general case, a differential equation of the second order contains the following components: the independent variable x, the function y itself, the first derivative y' of the function y with respect to x, and the second derivative y" of the same function with respect to x. Consequently, this equation can be represented in the following form:

$$F(x, y, y', y'') = 0;$$

where the symbol F represents an elementary function or a combination of such elementary functions.

Numerical integration makes it possible to solve this equation in any form, provided only that the given equation permits the determination of the second derivative as an explicit function of all the remaining variable quantities:

$$y'' = f(x, y, y').$$

In this numerical integration the increment of the derivative $\Delta y'$ is determined by the aid of the following formula:

$$\Delta y_i' = h\left(y_i'' + \frac{1}{2}\Delta y_{i-1}'' + \frac{5}{12}\Delta^2 y_{i-2}'' + \frac{3}{8}\Delta^3 y_{i-3}''\right)$$

or

$$\Delta y_i' = \Phi_i + \frac{1}{2}\Delta\Phi_{i-1} + \frac{5}{12}\Delta^2\Phi_{i-2} + \frac{3}{8}\Delta^3\Phi_{i-3},$$

where use is made of the following new auxiliary function:

$$\Phi = hy'',$$

expressed in terms of the second derivative of the desired function. It is not necessary to derive the fundamental formula for $\Delta y_i'$ by means of y" and differences of this second derivative, inasmuch as this formula is already obtained from the previously derived relation

$$\Delta y_i = h\left(y_i' + \frac{1}{2}\Delta y_{i-1}' + \frac{5}{12}\Delta^2 y_{i-2}' + \frac{3}{8}\Delta^3 y_{i-3}'\right)$$

by simply replacing y_i by y_i' and y_i' by y_i''.

On the other hand it is necessary to derive a working formula for the increment Δy of the function itself in terms of differences of the second derivative rather than of the first. Such a formula should simplify the work, because in computing Δy it will be possible to make use of the already available differences of the auxiliary function Φ, and it will not be necessary to find the differences of still another auxiliary function:

$$F = hy'$$

For this purpose, we shall utilize the equality

$$\Delta y_i = \int_{x_i}^{x_{i+1}} y'dx.$$

But in all cases

$$y' = y_i' + \Delta y' = y_i' + \int_{x_i}^{x} y''dx.$$

Let us replace here the derivative y" by the following interpolation function:

$$y'' = A_0(x - x_i)^3 + A_1(x - x_i)^2 + A_2(x - x_i) + A_3,$$

and then

$$y' = y_i' + \int_{x_i}^{x} [A_0(x - x_i)^3 + A_1(x - x_i)^2 + A_2(x - x_i) + A_3] dx =$$

$$= y_i' + \frac{1}{4} A_0(x - x_i)^4 + \frac{1}{3} A_1(x - x_i)^3 + \frac{1}{2} A_2(x - x_i)^2 + A_3(x - x_i),$$

so that, consequently

$$\Delta y_i = \int_{x_i}^{x_{i+1}} \Big[y_i' + \frac{1}{4} A_0(x - x_i)^4 + \frac{1}{3} A_1(x - x_i)^3 +$$

$$+ \frac{1}{2} A_2(x - x_i)^2 + A_3(x - x_i)\Big] dx = \Big| y_i'(x - x_i) + \frac{1}{20} A_0(x - x_i)^5 +$$

$$+ \frac{1}{12} A_1(x - x_i)^4 + \frac{1}{6} A_2(x - x_i)^3 + \frac{1}{2} A_3(x - x_i)^2 \Big|_{x_i}^{x_{i+1}},$$

whence

$$\Delta y_i = h \left[y_i' + \frac{1}{20} A_0 h^4 + \frac{1}{12} A_1 h^3 + \frac{1}{6} A_2 h^2 + \frac{1}{2} A_3 h \right],$$

because

$$x_{i+1} - x_i = h.$$

We shall substitute into the last expression the following values for the coefficients:

$$A_0 = \frac{\Delta^3 y''_{i-3}}{6h^3}; \qquad A_1 = \frac{\Delta^2 y''_{i-2} + \Delta^3 y''_{i-3}}{2h^2};$$

$$A_2 = \frac{6\Delta y''_{i-1} + 3\Delta^2 y''_{i-2} + 2\Delta^3 y''_{i-3}}{6h}; \quad A_3 = y''_i,$$

because the part of the function is here played by the derivative y", which has been replaced by the interpolation function.

After obvious transformations this substitution gives:

$$\Delta y_i = h\left[y'_i + h\left(\frac{1}{2}y''_i + \frac{1}{6}\Delta y''_{i-1} + \frac{1}{8}\Delta^2 y''_{i-2} + \frac{19}{180}\Delta^3 y''_{i-3}\right)\right]$$

or

$$\Delta y_i = h\left[y'_i + h\left(\frac{1}{2}y''_i + \frac{1}{6}\Delta y''_{i-1} + \frac{1}{8}\Delta^2 y''_{i-2} + \frac{1}{10}\Delta^3 y''_{i-3}\right)\right],$$

if, instead of the inconvenient number $\frac{19}{180}$ we take the closely approching it number:

$$\frac{18}{180} = \frac{1}{10}$$

It remains necessary to make use once again of the auxiliary function:

$$\Phi = hy'',$$

and then, finally, in the numerical integration of the second-order differential equation, we shall have to deal with working formulas of the following type:

$$\Delta y'_i = \Phi_i + \frac{1}{2}\Delta\Phi_{i-1} + \frac{5}{12}\Delta^2\Phi_{i-2} + \frac{3}{8}\Delta^3\Phi_{i-3};$$

$$\Delta y_i = h\left(y'_i + \frac{1}{2}\Phi_i + \frac{1}{6}\Delta\Phi_{i-1} + \frac{1}{8}\Phi_{i-2} + \frac{1}{10}\Delta^3\Phi_{i-3}\right).$$

To simplify the computations, it is better, however, to conduct the numerical integration with so small a step h as would enable us to do without third differences.

At the start of the numerical integration, as before, most frequent use is made of the method of successive approximations.

The number of rows in the lower part of the form will increase (in comparison with the number of rows in the lower part of the form for the integration of the first-order equation) by at least seven rows, because in addition to the ten rows:

$$\Phi, \Delta\Phi, \Delta^2\Phi, \Delta^3\Phi, \Phi, \frac{1}{2}\Delta\Phi, \frac{5}{12}\Delta^2\Phi, \frac{3}{8}\Delta^3\Phi, \Sigma, \quad \Delta y' = h\Sigma, \; y',$$

corresponding to the lower part of the form for the numerical integration of the first-order equation, there will also appear the following additional rows:

$$\frac{1}{2}\Phi, \; \frac{1}{6}\Delta\Phi, \; \frac{1}{8}\Delta^2\Phi, \; \frac{1}{10}\Delta^3\Phi, \Sigma_1, \quad \Delta y = h\Sigma_1, \; y.$$

In the method under consideration, the computations themselves are in no way different from similar computations for the solution of the first-order equation, so that the need for citing a special example is obviated.

14) Use of Numerical Integration of Second-Order Equations with Argument t.

In this process the leading part is played by the equation for the forward motion of the projectile:

$$\varphi m \frac{dv}{dt} = sp,$$

from which it follows that

$$l''_t = \frac{s}{\varphi m} p, \tag{107}$$

because

$$\frac{dv}{dt} = v'_t = \left(\frac{dl}{dt}\right)'_t = l''_t,$$

if the phenomenon of recoil is not considered in the explicit form.

Equation (107) is subject to numerical integration by means of the following relations:

$$p = \frac{f\omega}{s} \frac{\psi - \frac{v^2}{v^2_{np}}}{l_\psi + l} \quad \text{and} \quad l_\psi = l_\Delta - l_\alpha \psi,$$

which have already been employed earlier. The auxiliary function Φ is equal in this case to:

$$\Phi = h l''_t = \frac{sh}{\varphi m} p.$$

where the step h is already a certain time interval of the motion of the projectile in the bore. This step is chosen in advance.

The working formulas for the numerical integration will be:

$$\Delta v_i = \Phi_i + \frac{1}{2}\Delta\Phi_{i-1} + \frac{5}{12}\Delta^2\Phi_{i-2} + \frac{3}{8}\Delta^3\Phi_{i-3};$$

$$\Delta l_i = h\left(v_i + \frac{1}{2}\Phi_i + \frac{1}{6}\Delta\Phi_{i-1} + \frac{1}{8}\Delta^2\Phi_{i-2} + \frac{1}{10}\Delta^3\Phi_{i-3}\right),$$

because the function y to be determined represents in this case the relative path l of the projectile in the bore, and its derivative y' is the velocity of the projectile v.

The preliminary computations are reduced to the determination of constants:

1) $\log \frac{\omega}{s}$; 2) $\log \frac{\varphi m}{s}$; 3) $\log v^2_{\text{пр}}$; 4) l_Λ ; 5) $\log l_\alpha$; 6) ψ_0;

7) $\log \frac{f\omega}{s}$,

as in the case of numerical integration of a first-order equation.

The values of I_0, I_K and v_K are found in an analogous manner. The gas inflow ψ is read off the (I, ψ) curve, but, in contrast with the preceding case, it is necessary here to find for each point, during the process of integration itself:

$$I = I_0 + \frac{\varphi m}{s} v,$$

because the integration gives values of the velocity v which are separated from one another by unequal intervals. Once the value of the pressure impulse is had, we can obtain the required value of ψ from the I, ψ curve. Consequently, this curve cannot be utilized in advance, but it must be available in the course of the entire work of numerical integration.

In the case of the geometric law of burning, the I, ψ curve is replaced by the following relation:

$$\psi = \psi_0 + \frac{\varkappa \sigma_0 \varphi m}{s I_K} v + \frac{\varkappa \lambda \varphi^2 m^2}{s^2 I_K^2} v^2.$$

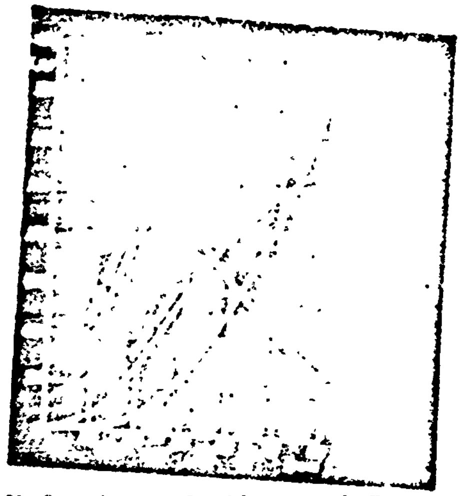

Fig. 153 - Velocity, Path, and Pressure Curves.

1) Time; 2) pressure curve; 3) velocity curve;
4) path curve.

Figure 153 contains curves for p, v, and l as functions of time t, obtained by numerical integration with respect to the argument t.

Computa'ion Fc

1	$t.10^5$	0	25	0	25	5C
2	$\log \frac{\varphi m}{s}$		1.1709		1.1709	1,17
3	$\log v$		1.7076		1.7404	2.07
4	$\log \frac{\varphi m}{s} v$		0.8785		0.9113	1.24
34	$\frac{1}{2}\Delta\Phi$			4	4	
35	$\frac{5}{12}\Delta^2\Phi$					
36	$\frac{3}{8}\Delta^3\Phi$					
37	$\Delta v - \Sigma$	51		55	64	
38	v	0	51	0	55	
39	$\frac{1}{2}\Phi$	26		26	30	
40	$\frac{1}{6}\Delta\Phi$			2	2	
41	$\frac{1}{8}\Delta^2\Phi$					
42	Σ_1	26		28	87	
43	$\Delta l - h\Sigma_1$	0.006		0.007	0.0022	
44	l	0.000	0.006	0.000	0.007	0.(

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Computa.on Form with Arg

1	$t.10^5$	0	25	0	25	50	0
2	$\log \frac{\varphi m}{s}$		1.1709		1.1709	1.1709	
3	$\log v$		1.7076		1.7404	2.0755	
4	$\log \frac{\varphi m}{s} v$		0.8785		0.9113	1.2464	
5	$\frac{\varphi m}{s} v$		8		8	18	
6	I_0		76		76	76	
7	I	76	84		84	94	
8	$2 \log v$		3.4152		3.4808	4.1510	
9	$-\log v^2_{np}$		9.9197		9.9197	9.9197	
10	$\log \frac{v^2}{v^2_{np}}$		5.3349		5.4005	4.0707	
11	ψ		0.045		0.045	0.057	
12	$-\frac{v^2}{v^2_{np}}$		0.000		0.000	0.000	
13	$\psi - \frac{v^2}{v^2_{np}}$	0.038	0.045		0.045	0.057	
14	$\log l_\alpha$	1.8710	1.8710		1.8710	1.8710	
15	$\log \psi$	2.5793	2.6532		2.6532	2.7559	
16	$\log l_\alpha \psi$	2.4503	2.5242		2.5242	2.6269	
17	l_Δ	2.284	2.284		2.284	2.284	
18	$-l_\alpha \psi$	-0.028	-0.033		-0.033	-0.042	
19	l_ψ	2.256	2.251		2.251	2.242	
20	l	0	0.006		0.007	0.029	
21	$l_\psi + l$	2.256	2.257		2.258	2.271	
22	$\log \frac{f\omega}{s}$	6.2512	6.2512		6.2512	6.2512	
23	$\log\left(\psi - \frac{v^2}{v^2_{np}}\right)$	2.5793	2.6532		2.6532	2.7559	
24	$-\log (l_\psi + l)$	1.6466	1.646[illegible]		1.6463	1.6438	
25	$\log p$	4.4771	4.5508		4.5509	4.6509	
26	$\log\left(\frac{sh}{\varphi m}\right)$	3.2270	3.2270		3.2370	3.2270	

Computation Form with Argument t.

5	0	25	50	0	25	50	75	100	125
709		1.1709	1.1709			1.1709	1.1709	1.1709	1.1709
076		1.7404	2.0755			2.0864	2.3160	2.5211	2.7152
785		0.9113	1.2464			1.2575	1.4869	1.6920	1.8861
8		8	18			18	31	49	77
6		76	76				76	76	76
4		84	94				107	125	153
52		3.4808	4.1510				4.6320	5.0422	5.4304
97		9.9197	9.9197				9.9197	9.9197	9.9197
49		5.4005	4.0707				4.5517	4.9619	3.3501
5		0.045	0.057				0.079	0.119	0.172
0		0.000	0.000				0.000	-0.001	-0.002
5		0.045	0.057				0.079	0.118	0.170
10		1.8710	1.8710				1.8710	1.8710	1.8710
32		2.6532	2.7559				2.8976	1.0755	1.2355
42		2.5242	2.6269				2.7686	2.9465	1.1065
4		2.284	2.284				2.284	2.284	2.284
3		-0.033	-0.042				-0.059	-0.088	-0.128
		2.251	2.242				2.225	2.195	2.156
		0.007	0.029				0.069	0.135	0.239
		2.258	2.271				2.294	2.330	2.395
		6.2512	6.2512				6.2512	6.[illegible]12	6.2512
2		2.6532	2.7559				2.8976	1.0719	1.2304
		1.6463	1.6438				1.6394	1.[illegible]326	1.6209
8		4.5509	4.6509				4.7882	4.9557	5.1025
0		3.2370	3.2270				3.2270	3.2270	3.2270

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	$\log \frac{\ }{v^2_{np}}$						
11	ψ		0.045		0.045	0.057	
12	$-\frac{v^2}{v^2_{np}}$		0.000		0.000	0.000	
13	$\psi - \frac{v^2}{v^2_{np}}$	0.038	0.045		0.045	0.057	
14	$\log l_\alpha$	1.8710	1.8710		1.8710	1.8710	
15	$\log \psi$	2.5793	2.6532		2.6532	2.7559	
16	$\log l_\alpha \psi$	2.4503	2.5242		2.5242	2.6269	
17	l_Δ	2.284	2.284		2.284	2.284	
18	$-l_\alpha \psi$	-0.028	-0.033		-0.033	-0.042	
19	l_ψ	2.256	2.251		2.251	2.242	
20	l	0	0.006		0.007	0.029	
21	$l_\psi + l$	2.256	2.257		2.258	2.271	
22	$\log \frac{f\omega}{s}$	6.2512	6.251[illegible]		6.2512	6.2512	
23	$\log\left(\psi - \frac{v^2}{v^2_{np}}\right)$	2.5793	2.6532		2.6532	2.7559	
24	$-\log(l_\psi + l)$	1.6466	1.646[illegible]		1.6463	1.6438	
25	$\log p$	4.4771	4.5508		4.5509	4.6509	
26	$\log\left(\frac{sh}{\varphi m}\right)$	3.2270	3.2270		3.2370	3.2270	
27	$\log \Phi$	1.7041	1.7778		1.7779	1.8779	
28	p kg·cm^{-2}	300	355	300	355	448	300
29	Φ	51	60	51	60	76	51
30	$\Delta\Phi$	$\boxed{(9)}$	9	(2)	9	16	(2)
31	$\Delta^2\Phi$			(7)	(7)	7	(7)
32	$\Delta^3\Phi$						
33	Φ	51		51	60		51
34	$\frac{1}{2}\Delta\Phi$			4	4		1
35	$\frac{5}{12}\Delta^2\Phi$						2
36	$\frac{3}{8}\Delta^3\Phi$	-					
37	$\Delta v - \Sigma$	51		55	64		54

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[illegible]		[illegible]	[illegible]				[illegible]	[illegible]	3.3[illegible]
		0.64[illegible]	0.[illegible]				[illegible]	[illegible]	[illegible]
		0.00	0.[illegible]				[illegible]	[illegible]	[illegible]
		0.64[illegible]	0.[illegible]				[illegible]	[illegible]	[illegible]
[illegible]		[illegible]	1.87[illegible]				[illegible]	[illegible]	1.8710
[illegible]		[illegible]	[illegible].75[illegible]				1.897[illegible]	[illegible]	1.[illegible]
[illegible]42		2.5242	2.626[illegible]				2.768[illegible]	2.[illegible]	1.10[illegible]
[illegible]		2.284	2.284				2.284	2.284	2.284
[illegible]		-0.035	-0.042				-0.059	-0.088	-0.128
[illegible]		2.25[illegible]	2.24[illegible]				2.225	2.[illegible]95	2.156
[illegible]		0.067	0.029				0.069	0.135	0.239
[illegible]7		2.258	2.27[illegible]				2.294	2.330	2.395
[illegible]		6.2512	6.2512				6.2512	6.[illegible]12	6.2512
[illegible]32		2.6532	2.7559				2.897[illegible]	[illegible]	[illegible]
[illegible]		1.6463	[illegible].643[illegible]				.6[illegible]	[illegible]	1.62[illegible]
[illegible]08		4.5509	4.6509				4.788[illegible]	4.9557	[illegible]
[illegible]70		3.2370	3.2270				3.2270	3.227[illegible]	3.227[illegible]
[illegible]78		1.7779	[illegible].8779				2.0150	2.1827	2.32[illegible]
[illegible]5	300	355	448	300	355	448	614	902	126[illegible]
[illegible]0	51	60	76	51	60	76	104	152	214
[illegible]9	(2)	9	16	(2)	9	16	28	48	62
	(7)	(7)	7	(7)	(7)	7	12	20	14
							5	8	-6
	51	60		51	60	76	104	152	214
	4	4		1	4	8	14	24	31
				2	2	3	5	8	6
							2	[illegible]	-2

678 e

21	$l_\psi + l$	2.256	2.257		2.258	2.271		
22	$\log \frac{f\omega}{s}$	6.2512	6.2512		6.2512	6.2512		
23	$\log\left(\psi - \frac{v^2}{v^2_{np}}\right)$	2.5793	2.6532		2.6532	2.7559		
24	$-\log(\psi + l)$	1.6466	1.646[illegible]		1.6463	1.6438		
25	$\log p$	4.4771	4.5508		4.5509	4.6509		
26	$\log\left(\frac{sh}{\varphi m}\right)$	3.2270	3.2270		3.2370	3.2270		
27	$\log \Phi$	1.7041	1.7778		1.7779	1.8779		
28	p kg·cm-2	300	355	300	355	448	300	
29	Φ	51	60	51	60	76	51	
30	$\Delta\Phi$	$\boxed{(9)}$	9	(2)	9	16	(2)	
31	$\Delta^2\Phi$			(7)	(7)	7	(7)	
32	$\Delta^3\Phi$							
33	Φ	51		51	60		51	
34	$\frac{1}{2}\Delta\Phi$			4	4		1	
35	$\frac{5}{12}\Delta^2\Phi$						2	
36	$\frac{3}{8}\Delta^3\Phi$							
37	$\Delta v = \Sigma$	51		55	64		54	
38	v	0	51	0	55	119	0	
39	$\frac{1}{2}\Phi$	26		26	30		26	
40	$\frac{1}{6}\Delta\Phi$			2	2		0	
41	$\frac{1}{8}\Delta^2\Phi$						1	
42	Σ_1	26		28	87		27	
43	$\Delta l = h\Sigma_1$	0.006		0.007	0.0022		0.007	(
44	l	0.000	0.006	0.000	0.007	0.029	0.000	(

						[illegible]	[illegible]	2.156
	0.067	0.029				0.069	0.135	0.239
	2.258	2.271				2.294	2.330	2.395
	6.2512	6.2512				6.2512	6.[illegible]512	6.2512
	2.6532	2.7559				2.8976	1.0719	1.2304
	1.6463	1.6438				1.6394	1.[illegible]26	1.6209
	4.5509	4.6509				4.7882	4.9557	5.1025
	3.2370	3.2270				3.2270	3.2270	3.2270
	1.7779	1.8779				2.0150	2.1827	2.3295
300	355	448	300	355	448	614	902	1266
51	60	76	51	60	76	104	152	214
(2)	9	16	(2)	9	16	28	48	62
(7)	(7)	7	(7)	(7)	7	12	20	14
						5	8	-6
51	60		51	60	76	104	152	214
4	4		1	4	8	14	24	31
			2	2	3	5	8	6
						2	3	-2
55	64		54	66	87	125	187	249
0	55	119	0	54	120	207	332	519
26	30		26	30	38	52	76	107
2	2		0	2	3	5	8	10
			1	1	1	2	2	2
28	87		27	87	162	266	418	638
0.007	0.0022		0.007	0.022	0.040	0.066	0.104	0.159
0.000	0.007	0.029	0.000	0.007	0.029	0.069	0.135	0.239

CHAPTER 2. SOLUTION BY EXPANSION IN TAYLOR'S SERIES

Integration of equations of internal ballistics describing the relations existing between the fundamental elements of a shot leads to rather complex integrals, which can be solved only by means of quadratures with any desired degree of accuracy (Professor Drozdov's solution). In this connection, in order to make integration possible, some of the variable parameters in the fundamental equations (θ, u_1, p_0, etc.) are usually assumed to be constant.

The method of numerical integration makes it possible to solve a system of differential equations not only without simplifying the functions entering same, but even by assigning variable values to those quantities which are usually assumed to be constant. This makes it possible to solve the problem on the basis of any desired hypotheses concerning the character of burning of the powder, the law of resistance to motion, the design of the bore, etc.

In solving the problem by the method of numerical integration, it is necessary to proceed successively from one value of the argument to another by the addition of its finite differences, starting at the very beginning. For this reason, for example, it is not possible to determine in advance the values of p_m or the values of the variables v, l and p corresponding to the end of burning of the powder, it being necessary, instead, to compute successively, point by point, the elements of the curves of pressure p, the path of the projectile l, its velocity v, etc. This constitutes the disadvantage of this method. Moreover, the method of numerical integration gives the relation between the individual variables only in the form of numerical tables, rather than in the form of analytical formulas.

In spite of these disadvantages, the method of numerical integration may serve as a means for indirect verification of the degree of accuracy obtained by the aid of the various approximate analytical methods available.

In this connection, when developing new theoretical problems, numerical integration may be utilized for determining the errors involved in the transition from exact equations and relations to others that are less exact but more convenient from the analytical point of view. Numerical integration may be employed with equal success both in the case of the geometric law of burning and in the case of the more complex physical law of burning; it may also be applied to barrels having a bore of variable cross section.

In the USSR the method of numerical integration was first applied to the solution of ballistic problems by V.M. Trofimov in 1918. This method has been developed in great detail by G.V. Oppokov, who employed the method of finite differences (1931-1938) discussed above.

In 1932 P.V. Melentyev proposed to apply the method of expansion in Taylor's series for the numerical solution of equations in ballistics, and this method, after being subjected to certain modifications, has been found to be sufficiently convenient.

Investigation of the fundamental relations expressing the conditions accompanying a shot shows that all the principal elements (l, v, ψ, and p) can be expressed in one form or another as functions of the path l and of its derivatives with respect to time up to the third order inclusive.

As a matter of fact, taking the fundamental system of equations expressing the relationship between the elements of a shot, we obtain the following.

1) The fundamental equation of pyrodynamics or the equation of transformation of energy:

$$ps(l_{\psi} + l) = f\omega\psi - \frac{\theta}{2}\varphi m v^2.$$

2) The equations expressing the law of burning of the powder:

$$\psi = \varkappa z(1 + \lambda z);$$

$$\frac{de}{dt} = u = u_1 p;$$

$$\frac{d\psi}{dt} = \frac{S_1}{\Lambda_1}\sigma u_1 p = \frac{\varkappa}{e_1}\sigma u_1 p.$$

3) The equation of motion:

$$ps = \varphi m \frac{dv}{dt} = \varphi m \frac{d^2 l}{dt^2} = \varphi m l''_t,$$

where

$$v = \frac{dl}{dt} = l'_t,$$

v being related to z by the following equation:

$$v = \frac{sI_K}{\varphi m}(z - z_0),$$

whence

$$z = z_0 + \frac{\varphi m}{sI_K} v.$$

All the variables entering into the fundamental system of equations can be expressed in terms of the path l and of its derivatives, since $v = l'_t$, $z = z_0 + \frac{\varphi m}{sI_K} l'_t$, ψ - being a function of z - will also be expressed as a function of l'_t, the pressure is proportional to l''_t, and the derivative $\frac{dp}{dt}$ is proportional to l'''_t. Consequently, if the time t of travel of the projectile through the bore is taken as the independent variable, and the path of the projectile l is taken as the function to be expanded in a series, it becomes possible to employ Taylor's series for finding the value of the path l_{n+1} and of its derivatives for the neighboring segment corresponding to the time $t_{n+1} = t_n + \Delta t = t_n + h$, provided that the values of the path l_n and of its derivatives for the preceding instant t_n are known. It is thus possible to find all the elements of burning of the powder and of the motion of the projectile during a shot, i.e., z, ψ, v, p, l, and t.

Let it be assumed that for a certain instant of time t_n the path l_n and its derivatives with respect to time l'_n, l''_n, l'''_n,... are known; if a sufficiently small increment of time $\Delta t = h$ is assumed and consideration is limited to derivatives up to and including the third order, then, in accordance with Taylor's formula, we shall have for $t_{n+1} = t_n + h$:

$$\ell_{n+1} = \ell_n + h\ell'_n + \frac{h^2}{2}\ell''_n + \frac{h^3}{2\cdot 3}\ell'''_n. \qquad (108)$$

Differentiating with respect to t and rejecting the terms containing ℓ_n^{IV}, i.e., considering that ℓ'''_n is constant over the given interval Δt and equals its mean value, we obtain:

$$\ell'_{n+1} = \ell'_n + h\ell''_n + \frac{h^2}{2}\ell'''_n;$$

$$\ell''_{n+1} = \ell''_n + h\ell'''_{n\,av.} = \ell''_n + h\frac{\ell'''_n + \ell'''_{n+1}}{2},$$

where $\frac{\ell'''_n + \ell'''_{n+1}}{2}$ is the mean value of the third derivative in the interval under consideration (Fig. 154). From the last equation we obtain the following value for ℓ'''_{n+1}: $\ell'''_{n+1} = \frac{2}{h}\ell''_{n+1} - \frac{2}{h}\ell''_n - \ell'''_n$.

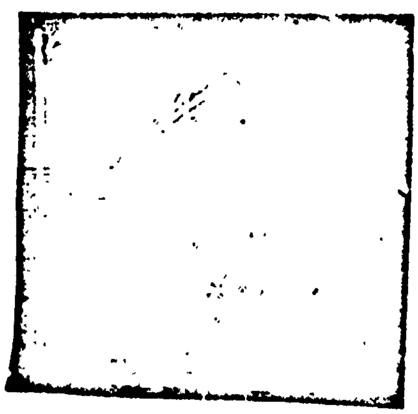

Fig. 154 - Diagram for ℓ''_t and ℓ'''_t.

As has been shown by P.V. Melentyev, it is more convenient to compute not the derivatives themselves, but, rather, the quantities proportional to them, namely: $h\ell'$, $h^2\ell''$ and $\frac{h^3}{2}\ell'''$. Therefore, by multiplying ℓ'_{n+1} by h and ℓ'''_{n+1} by $\frac{h^3}{2}$, we will obtain the following equations:

$$h l'_{n+1} = h l'_n + h^2 l''_n + \frac{h^3}{2} l'''_n; \qquad (109)$$

$$\frac{h^3}{2} l'''_{n+1} = h^2 l''_{n+1} - h^2 l''_n - \frac{h^3}{2} l'''_n. \qquad (110)$$

Comparing these two equations with the initial equation (108), it will be seen that l enters everywhere without a coefficient, l' enters with the coefficient h, l'' enters with the coefficient h^2, and l''' enters with the coefficient $\frac{h^3}{2}$. This considerably accelerates the subsequent computations.

The expression for the second derivative l''_t will be:

$$l''_t = \frac{s}{\varphi m} p$$

or, multiplying by h^2:

$$h^2 l''_t = h^2 \frac{s}{\varphi m} p. \qquad (111)$$

By combining the resulting values for the path l and its derivatives with the equations of the fundamental system, we obtain the totality of formulas necessary for the solution in the following form and sequence, which corresponds to the order of their application, the constants encountered being designated below as follows:

$$\frac{\varphi m}{s I_K} = k_1, \quad \frac{f\omega}{s} = k_2, \quad \frac{\Theta\varphi m}{2f\omega} = \frac{1}{v^2_{np}} = k_3, \quad \frac{s}{\varphi m} = k_4, \quad \frac{\omega}{s}\left(\alpha - \frac{1}{\delta}\right) = a.$$

System I

$$\begin{cases} h l'_{n+1} = h l'_n + h^2 l''_n + \frac{h^3}{2} l'''_n; & \text{(I)} \\ v_{n+1} = \frac{h l'_{n+1}}{h}; & \text{(II)} \\ z_{n+1} = z_0 + \frac{\varphi m}{s I_K} v_{n+1} = z_0 + k_1 v_{n+1}; & \text{(III)} \\ \psi_{n+1} = \varkappa z_{n+1} + \varkappa\lambda z^2_{n+1}; & \text{(IV)} \\ l_{n+1} = l_n + h l'_n + \frac{1}{2} h^2 l''_n + \frac{1}{3} \frac{h^3}{2} l'''_n; & \text{(V)} \\ p_{n+1} = \frac{f\omega}{s} \frac{\psi_{n+1} - \frac{v^2_{n+1}}{v^2_{np}}}{l_{\psi n+1} + l_{n+1}} = k_2 \frac{\psi_{n+1} - k_3 v^2_{n+1}}{l_\Delta - a\psi_{n+1} + l_{n+1}}; & \text{(VI)} \\ h^2 l''_{n+1} = h^2 \frac{s}{\varphi m} p_{n+1} = h^2 k_4 p_{n+1}; & \text{(VII)} \\ \frac{h^3}{2} l'''_{n+1} = h^2 l''_{n+1} - h^2 l''_n - \frac{h^3}{2} l'''_n. & \text{(VIII)} \end{cases}$$

The subscript (n+1) designates those values of the derivatives at the end of the given interval of time which, in the process of computation, are transferred from the column being computed into the corresponding rows of the right-hand neighboring column; however, the transferred values now bear the index n because they characterize the initial value of the given quantity in the next column.

In order to perform the computation by means of this totality of formulas, the values of l and of its derivatives at the start of the projectile's motion, i.e., at the instant $t = 0$, must be known. Since at the start of motion the path l and the speed v are equal to zero, we obtain:

$$(l)_0 = 0, \quad (l')_0 = (v)_0 = 0, \quad (l'')_0 = k_4 p_0, \; h^2(l'')_0 = h^2 k_4 p_0,$$

where p_0 is the forcing pressure usually specified beforehand. As regards the third derivative $(l''')_0$, we shall first find an expression for it at the present instant in the form of l'''.

To determine l''', we differentiate the equation $l''_t = k_4 p$ with respect to t:

$$l'''_t = k_4 p'_t.$$

But the quantity p'_t has already been derived:

$$p'_t = \frac{p}{l_\psi + l}\left[\frac{f\omega}{s}\,\frac{\varkappa\sigma}{I_K}\left(1 + \frac{p}{f\delta_1}\right) - v(1 + \theta)\right],$$

for the start of motion when $l = 0$, $v = 0$, $p = p_0$, and $\psi = \psi_0$; we will therefore obtain: $(p'_t)_0 = \frac{p_0}{l_{\psi_0}}\,\frac{f\omega}{s}\,\frac{\varkappa\sigma_0}{I_K}\left(1 + \frac{p_0}{f\delta_1}\right) = k_2\,\frac{\varkappa\sigma_0}{I_K}\left(1 + \frac{p_0}{f\delta_1}\right)\frac{p_0}{l_{\psi_0}}.$

The quantity $\left(1 + \frac{p_0}{f\delta_1}\right) = \frac{l_\Delta}{l_{\psi_0}}$, and this expression is therefore sometimes given in the following form:

$$(p_t')_0 = k_2 \frac{\varkappa \sigma_0}{I_K} \frac{l_\Delta}{l^2_{\psi_0}} p_0.$$

Consequently, the value of the third derivative for the initial instant is likewise known:

$$\frac{h^3}{2} l_0''' = \frac{h^3}{2} k_4 (p_t')_0 = \frac{h^3}{2} k_4 k_2 \frac{\varkappa \sigma_0}{I_K} \frac{l_\Delta}{l^2_{\psi_0}} p_0,$$

and it is possible to begin the successive solution of System (I) first for the first column corresponding to the interval of time $\Delta t = h$, then for the second column, etc., thus obtaining a successive series of values for l, v, z, ψ and p as functions of t.

The quantity $h = \Delta t$ must be so chosen as to obtain 10-15 columns for the period of burning of the powder, which will give a corresponding number of points for each of the quantities p, v, l, and ψ.

Since the time of burning is fundamentally determined by the thickness of the powder, the interval of time $h = \Delta t$ may be taken approximately according to the formula:

$$h \approx 0.001\, e_1,$$

where $e_1 = \frac{1}{2}$ the thickness of the powder in millimeters, h being rounded off to one or two significant figures (to 5 in the second significant figure). For example, if $2e_1 = 1.28$ mm:

$$h = 0.001\ (0.64) = 0.00064 = 0.0006 \text{ or } 0.00065$$

Since v_Δ and l_Δ are known at least approximately in advance, it is possible, after computing the average time of motion of the projectile $t_{av.} = \frac{2l_\Delta}{v_\Delta}$, to take for the value of the time step (increment) $\Delta t = h \approx \frac{t_{av.}}{15}$, rounded off to two significant figures.

<u>Sequence of Computation.</u> All the constants are computed first:

$$\varkappa, \lambda, \varkappa\lambda, \Lambda, \psi_0 = \frac{\frac{1}{\Delta} - \frac{1}{\delta}}{\frac{f}{p_0} + \alpha - \frac{1}{\delta}} = \frac{\frac{1}{\Delta} - \frac{1}{\delta}}{\frac{f}{p_0} + \frac{1}{\delta_1}}; \quad \sigma_0 = \sqrt{1 + 4\frac{\lambda}{\varkappa}\psi_0};$$

$$z_0 = \frac{2\psi_0}{\varkappa(\sigma_0 + 1)}; \quad I_K = \frac{e_1}{u_1}; \quad l_0 = \frac{W_0}{s}; \quad l_\Delta = \frac{1}{s}\left(W_0 - \frac{\omega}{\delta}\right) = l_0\left(1 - \frac{\Delta}{\delta}\right);$$

$$a = \frac{\omega}{s}\left(\alpha - \frac{1}{\delta}\right) = \frac{\omega}{s\delta_1}; \quad l_{\psi_0} = l_\Delta - a\psi_0; \quad \varphi m = \frac{\varphi q}{98.1}; \quad h \approx 0.001\ e_1;$$

$$k_1 = \frac{\varphi m}{sI_K}; \quad k_2 = \frac{f\omega}{s}; \quad k_3 = \frac{\theta\varphi m}{2f\omega} = \frac{1}{v^2_{np}} \text{ (small quantity)}; \quad k_4 = \frac{s}{\varphi m};$$

$$h^2 l_0'' = h^2 k_4 p_0; \quad \frac{h^3}{2} l_0''' = \frac{h^3}{2} k_4 k_2 \frac{\varkappa\sigma_0}{I_K} \frac{l_\Delta}{l^2_{\psi_0}} p_0 = \frac{h^3}{2} k_4 \frac{\varkappa\sigma_0}{I_K}\left(1 + \frac{p_0}{f\delta_1}\right)\frac{p_0^2}{\psi_0}.$$

The sequence of computation is not affected regardless of whether the computation is performed for a degressive or a progressive powder.

In computing the segment after the decomposition of the progressive powder, it is necessary to substitute for the usual formula the following previously derived formula:

$$\psi = \psi_s + \varkappa_z(z - 1)[1 + \lambda_2(z - 1)] = \psi_s + \varkappa_2(z - 1) + \varkappa_2\lambda_2(z - 1),$$

where z varies from 1 to $1 + \frac{\rho}{e_1}$, and $\varkappa_2$ and λ_2 are characteristics of the powder form after decomposition.

A form for conducting such computations is presented on pages 691-692.

Table 7-c - Computation Form for the Solution of Problems in Ballistics by Expansion in Taylor's Series.

Computation Formulas		Column No.	1	2	3
		Subscript n:	0	1	2
		Time $t_{n+1} = (n+1)h$	0.0008	0.0016	0.0024
$h = 0.0008$	1	$h\ell'_n$	0	1.1322	0.3856
$h\ell'_{n+1} = h\ell'_n + h^2\ell''_n + \frac{h^3}{2}\ell'''_n$	2	$h^2\ell''_n$	0.0958	0.1928	0.3546
$h\ell''_0 = 0$	3	$\frac{h^3}{2}\ell'''_n$	0.0364	0.0606	0.1012
$h^2\ell''_0 = 0.0958$ $\frac{h^3}{2}\ell'''_0 = 0.0364$	4	$h\ell'_{n+1}$	0.1322	0.3856	0.8414
$v_{n+1} = \frac{h\ell'_{n+1}}{h}$	5	$v_{n+1} = \frac{h\ell'_{n+1}}{h}$	165.3	482	
$z_{n+1} = z_0 + k_1 v_{n+1}$	6	$k_1 v_{n+1}$	0.0308	0.0899	
$k_1 = 0.0001864$	7	+ z_0	0.0297	0.0297	0.0297
	8	z_{n+1}	0.0605	0.1196	
$\psi_{n+1} = \varkappa z_{n+1} + \varkappa\lambda z^2_{n+1}$ $\varkappa = 1.06$	9	$\varkappa z_{n+1}$ +	0.0641	0.1268	
$\varkappa\lambda = -0.06$	10	$\varkappa\lambda z^2_{n+1}$	0.0002	0.0009	
	11	ψ_{n+1} −	0.0639	0.1259	
	12	$k_3 v^2_{n+1}$	0.0002	0.0016	
$k_3 = 0.087030$	13	$\psi_{n+1} - k_3 v^2_{n+1}$	0.0637	0.1243	
$k_2 = 2,850,000$	14	$k_2(\psi_{n+1} - k_3 v^2_{n+1}) =$ $= A_{n+1}$	181500	354300	

-c - Computation Form for the Solution of Problems in Internal Ballistics by Expansion in Taylor's Series.

	Subscript n	0	1	2	3
	Column No.	1	2	3	Remarks
	Time $t_{n+1} = (n+1)h$	0.0008	0.0016	0.0024	
1	$h\,l'_n$	0	1.1322	0.3856	From line 4 of preced ; column
2	$h^2\,l''_n$	0.0958	0.1928	0.3546	From line 25 of preceding column. Into line 26 of the given column.
3	$\frac{h^3}{2}\,l'''_n$	0.0364	0.0606	0.1012	From line 28 of preceding column. Into line 27 of the given column.
4	$h\,l'_{n+1}$	0.1322	0.3856	0.8414	Into line 1 of next column. Into line 16 of next column.
5	$v_{n+1} = \frac{h\,l'_{n+1}}{h}$	165.3	482		
6	$k_1 v_{n+1}$ +	0.0308	0.0899		
7	z_0	0.0297	0.0297	0.0297	In all columns.
8	z_{n+1}	0.0605	0.1196		
9	$\varkappa z_{n+1}$ +	0.0641	0.1268		
10	$\varkappa\lambda z^2_{n+1}$	0.0002	0.0009		
11	ψ_{n+1} −	0.0639	0.1259		
12	$k_3 v^2_{n+1}$	0.0002	0.0016		
13	$\psi_{n+1} - k_3 v^2_{n+1}$	0.0637	0.1243		
14	$k_2(\psi_{n+1} - k_3 v^2_{n+1}) = A_{n+1}$	181500	354300		

Table 7-c (Cont'd.)

Computation Formulas		Column No.	1	2	3
		n	0	1	2
		Time $t_{n+1} = (n+1)h$	0.0008	0.0016	0.0024
$p_{n+1} = k_2 \dfrac{\psi_{n+1} - k_3 v^2_{n+1}}{l_\Delta - a\psi_{n+1} + l_{n+1}}$	15	l_n	0	0.0600	0.3088
$l_{n+1} = l_n + h l'_n + \frac{1}{2}h^2 l''_n + \frac{1}{3}h^3 l'''_n$	16	$h l'_n$	0	0.1322	0.3856
$(l)_0 = 0$: $h l'_0 = 0$	17	$+\ \frac{1}{2} h^2 l''_n$	0.0479	0.0964	0.1773
	18	$\frac{1}{3} \frac{h^3}{2} l'''_n$	0.0121	0.0202	0.0337
	19	l_{n+1}	0.0600	0.3088	0.9054
	20	$+\ l_\Delta$	3.016	3.016	3.016
$a = \frac{\omega}{s}\left(\alpha - \frac{1}{\delta}\right) = 1.065$	21	$l_{n+1} + l_\Delta$	3.076	3.325	
	22	$-\ a\psi_{n+1}$	0.068	0.134	
	23	$B_{n+1} = l_{n+1} + l_\Delta - a\psi_{n+1}$	3.008	3.191	
	24	$p_{n+1} = \frac{A_{n+1}}{B_{n+1}}$ kg/cm^2	604	1100	
$h^2 l''_{n+1} = k_4 h^2 p_{n+1}$	25	$h^2 l''_{n+1}$	0.1928	0.3546	-
$k_4 h^2 = 0.0531192$	26	$-h^2 l''_n$	-0.0958	-0.1928	-0.3546
$\frac{h^3}{2} l'''_{n+1} = h^2 l''_{n+1} - h^2 l''_n - \frac{h^3}{2} l'''_n$	27	$-\ \frac{h^3}{2} l'''_n$	-0.0364	-0.0606	-0.1012
	28	$\frac{h^3}{2} l'''_{n+1}$	0.0606	0.1012	

	n	0	1	2	3
	Column No.	1	2	3	Remarks
	Time $t_{n+1} = (n+1)h$	0.0008	0.0016	0.0024	
15	l_n	0	0.0600	0.3088	From line 19 of preceding column
16	$h\,l'_n$	0	0.1322	0.3856	From line 4 of preceding column.
17	$+\ \frac{1}{2} h^2 l''_n$	0.0479	0.0964	0.1773	$\frac{1}{2}$ (of line 2 of the given column)
18	$\frac{1}{3} \frac{h^3}{2} l'''_n$	0.0121	0.0202	0.0337	$\frac{1}{3}$ (of line 3 of the given column)
19	l_{n+1}	0.0600	0.3088	0.9054	Into line 15 of next column
20	$+\ l_\Delta$	3.016	3.016	3.016	In all columns
21	$l_{n+1} + l_\Delta$	3.076	3.325		
22	$-\ a\psi_{n+1}$	0.068	0.134		
23	$B_{n+1} = l_{n+1} + l_\Delta - a\psi_{n+1}$	3.008	3.191		
24	$p_{n+1} = \frac{A_{n+1}}{B_{n+1}}$ kg/cm^2	604	1100		
25	$h^2 l''_{n+1}$	0.1928	0.3546	-	Into lines 2 and 26 of next column
26	$-h^2 l''_n$	-0.0958	-0.1928	-0.3546	From line 25 of preceding column
27	$-\frac{h^3}{2} l'''_n$	-0.0364	-0.0606	-0.1012	From line 28 of preceding column
28	$\frac{h^3}{2} l'''_{n+1}$	0.0606	0.1012		Into lines 3 and 27 of next column

Formulas for the second period:

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$+\frac{1}{2}h^2 l''_n + \frac{1}{3}h^3 l'''_n$ $(l)_0 = 0;\ h l'_0 = 0$	18	$\frac{1}{3}\ \frac{h^3}{2} l'''_n$	0.0121	0.0202	0.0337	1 c
	19	l_{n+1}	0.0600	0.3088	0.9054	I c
	20	l_Δ	3.016	3.016	3.016	I
$a = \frac{\omega}{s}\left(\alpha - \frac{1}{\delta}\right) = 1.065$	21	$l_{n+1} + l_\Delta$	3.076	3.325		
	22	$a\psi_{n+1}$	.068	0.134		
	23	$B_{n+1} = l_{n+1} + l_\Delta - a\psi_{n+1}$	3.008	3.191		
	24	$p_{n+1} = \frac{\Delta_{n+1}}{B_{n+1}} kg/cm^2$	604	1100		
$h^2 l''_{n+1} = k_4 h^2 p_{n+1}$	25	$h^2 l''_{n+1}$	0.1928	0.3546	-	I n
$k_4 h^2 = 0.0531192$	26	$-h^2 l''_n$	-0.0958	-0.1928	-0.3546	F c
$\frac{h^3}{2} l'''_{n+1} = h^2 l''_{n+1} - h^2 l''_n - \frac{h^3}{2} l'''_n$	27	$-\frac{h^3}{2} l'''_n$	-0.0364	-0.0606	-0.1012	F c
	28	$\frac{h^3}{2} l'''_{n+1}$	0.0606	0.1012		I n

Formulas for the second period:

$$v = v_{np}\sqrt{1 - \left(\frac{l_1 + l_K}{l_1 + l}\right)^{\theta}(1 - k_3 v_K^2)}; \quad p = p_K\left(\frac{l_1 + l_K}{l_1 + l}\right)^{1+\theta}$$

when $l = l_A$ $\quad v = v_A \quad p = p_A$.

p_K and l_K are determined from first period when $\psi = 1$.

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8	$\frac{1}{3}\ \frac{h^3}{2}\ l'''_n$	0.0121	0.0202	0.0337	$\frac{1}{3}$ (of line 3 of the given column)
9	l_{n+1}	0.0600	0.3088	0.9054	Into line 15 of next column
10	$+\ l_\Delta$	3.016	3.016	3.016	In all columns
11	$l_{n+1} + l_\Delta$	3.076	3.325		
12	$-\ a\psi_{n+1}$	0.068	0.134		
13	$B_{n+1} = l_{n+1} + l_\Delta - a\psi_{n+1}$	3.008	3.191		
14	$p_{n+1} = \frac{A_{n+1}}{B_{n+1}}$ kg/cm^2	604	1100		
15	$h^2 l''_{n+1}$	0.1928	0.3546	-	Into lines 2 and 26 of next column
16	$-h^2 l''_n$	-0.0958	-0.1928	-0.3546	From line 25 of preceding column
17	$-\ \frac{h^3}{2}\ l'''_n$	-0.0364	-0.0606	-0.1012	From line 28 of preceding column
18	$\frac{h^3}{2}\ l'''_{n+1}$	0.0606	0.1012		Into lines 3 and 27 of next column

Formulas for the second period:

$$\sqrt{1 - \left(\frac{l_1 + l_K}{l_1 + l}\right)^{\theta} (1 - k_3 v_K^2)}; \quad p = p_K\left(\frac{l_1 + l_K}{l_1 + l}\right)^{1+\theta}$$

when $l = l_A$ $\quad v = v_A \quad p = p_A$.

are determined from first period when $\psi = 1$.

The extreme left column contains the "computation formulas" and constants of System (I); in the next column to the right these formulas are broken down into individual operations, which are followed in the computations.

To start with, the first column (No. 1) corresponding to the time interval 0 to h is filled in first. In this column the subscript n relates to the start of the interval, and the subscript n+1 relates to its end; for this column n = 0 and n+1 = 1.

For the next (second) column, n = 1, n+1 = 2, etc. For the first column, computation of the constants gives us at n = 0 $l_n = 0$ (the path at the start of the motion), which we write on line 15; $hl'_n = 0$ (the velocity at the start of the motion) is written on lines 1 and 16; $h^2 l''_n = h^2 l''_0 = 0.0958$ is written on lines 2 and 26; $\frac{h^3}{2} l'''_0$ is written on lines 3 and 27. Line 17 is filled with $\frac{1}{2}(h^2 l''_0)$, and line 18 with $\frac{1}{3}(\frac{h^3}{2} l'''_0)$. Thus, all the quantities with the subscript n = 0 are inserted in the first column. We now subject them to the necessary operations. The sum of the first three rows gives the fourth $hl'_{n+1} = hl'_{0+1}$, which is immediately transferred to lines 1 and 16 of the neighboring column wherein, provided with the subscript n, it characterizes the value of this quantity at the start of the next interval; we then determine v_{n+1}, z_{n+1}, ψ_{n+1}, and A_{n+1}. By adding the four rows from line 15 through line 18, we obtain in line 19 l_{n+1} - the path of the projectile at the end of the given interval of time - and this quantity, provided with the subscript n, is transferred to line 15 of the neighboring column. After determining $p_{n+1} = \frac{A_{n+1}}{B_{n+1}}$ and multiplying it by $k_4 h^2$, we obtain $h^2 l''_{n+1}$, which we write on line 25 of the first column and on lines 26 and 2 of the next column, where this quantity acquires the subscript n, as does also $\frac{1}{2}(h^2 l''_{n+1})$, which is written in line 17 of the next column. After performing the operations indicated in the form with lines 25, 26, and 27, we obtain in line 28 of the first column $\frac{h^3}{2} l'''_{n+1}$, which we immediately transfer to lines 3 and 27 of the next column, while $\frac{1}{3}(\frac{h^3}{2} l'''_{n+1})$ is written in line 18 of the same column.

Thus, all operations with the quantities bearing the subscript n in the second column are already prepared, and the second column is then treated in the same manner as was the first.

Constants such as z_0 in line 7 and l_Δ in line 20 are inserted in the series of columns in advance.

By applying the same rules to the neighboring second column, we shall gradually, step by step, obtain values for v, z, ψ, l, and p as functions of $t = (n+1)h$, and this is continued to the end of burning or to the instant of emergence of the projectile from the bore, it being necessary to use $\psi = 1$ after the end of burning.

In performing the computations it is necessary to exercise extreme care not to commit any errors, because an error in one of the preceding columns will distort the results obtained in the succeeding columns.

It is best to follow up the computation of the data in each column by plotting them on graph paper as a function of t. In so doing an error in the given column will cause a deviation from the regular disposition of the points derived from the preceding columns, and such an error can be detected and corrected.

As a criterion of accuracy, it is also useful to plot on the diagram the third derivative (or $\frac{h^3}{2} l'''_{n+1}$ in the last row), which should first increase, then pass through a maximum, then become zero ($p'_t = 0$) at the instant p_m is attained, and thereupon acquire a negative value, fluctuating slightly in either direction.

The instant of time cut off on the diagram at $p'_t = 0$ or $\frac{h^3}{2} l''' = 0$ corresponds to the instant of maximum pressure, and all elements for it are best taken from the diagram

The time t_K corresponding to the end of burning of the powder is determined from the diagram on the basis of the ψ, t curve at $\psi = 1$; thereupon, the elements corresponding to the end of burning of the powder for this time are found by interpolation. If, without changing the segments $\Delta t = h$, the second period is computed as a direct continuation of the first, assuming $\psi = 1$ and $l_\psi = l_1$ throughout, the third derivative l'''_t usually begins to fluctuate, sometimes entering the region of positive values, which contradicts the physical nature of the process of pressure change.

For this reason, once the elements of the end of burning t_K, v_K, l_K, and p_K have been obtained from the computation of the first period, the procedure is continued by adopting the same step $\Delta t = h$ with t_K as the starting point by first computing the values of the path and of its derivatives for the start of the second period in accordance with the following formulas:

$$l_{(0)} = l_K; \quad hl'_{(0)} = hv_K; \quad h^2 l''_{(0)} = h^2 \frac{s}{\varphi m} p_K = h^2 k_4 p_K;$$

$$\frac{h^3}{2} l'''_{(0)} = -\frac{h^3}{2} \frac{s}{\varphi m} \frac{(1+\theta) v_K p_K}{(l_1 + l_K)} = -\frac{h^3}{2} k_4 (1+\theta) \frac{v_K p_K}{l_1 + l_K};$$

whereupon they are written in the initial column for computating the data of the second period and subjected to the same operations as in the first period, with the sole exceptions that $\psi = 1$ is assumed in line 11 and that lines 6-10 are omitted.

The computation is continued in this manner until $l_{n+1} \gg l_{A}$.

If $l_{n+1} = l_{A}$, the remaining elements v_{A} and p_{A} are obtained automatically in the same column for the subscript n+1; if $l_{n+1} > l_{A}$, the computations in this column must be carried as far as line 24, with lines 6-10 omitted, whereupon the value of l_{A} is used to obtain t_{A} by interpolation in the last column for the purpose of subsequently obtaining the elements p_{A} and v_{A}.

Instead of expanding in a series after obtaining the elements corresponding to the end of burning of the powder, it is possible to compute v_{A} and p_{A} by means of the usual second-period formulas, but without determining the time t.

The solution by expansion in a series is applicable to both the geometric and the physical law of burning of powder. In the latter case:

$$v = \frac{s}{\varphi m} \int_{\psi_0}^{\psi} pdt = \frac{s}{\varphi m}(I - I_0),$$

from which we have the following expression for I:

$$I = I_0 + \frac{\varphi m}{s} v,$$

and the correlation between ψ and z is replaced by the graphical dependence of I upon ψ, which is found from the bomb test.

The equation for $(p_t')_0$ is replaced by the following expression:

$$(p_t')_0 = k_2 \Gamma_0 \frac{I_\Delta}{I_{\psi_0}^2} p_0$$

and:

$$\frac{h^3}{2} I_0''' = \frac{h^3}{2} k_4 k_2 \Gamma_0 \frac{I_\Delta}{I_{\psi_0}^2} p_0,$$

where Γ_0 is the value of the experimental function Γ corresponding to the quantity $\psi = \psi_0$.

In the case of ballooning powders, Γ_0 is greater than in the case of the geometric law of burning, and therefore the values of both the first derivative p_t' and of p itself will increase more rapidly, and the maximum pressure will occur earlier. If the propellant force of the powder is the same, the maximum pressure will be greater in the case of the physical law of burning with ballooning than in the case of the geometric law of burning.