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THE APPLICATION OF DYNAMIC RELAXATION TO THE FINITE ELEMENT METHOD OF STRUCTURAL ANALYSIS

by

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FOREWORD

This technical report was prepared by the Dynamical Systems Group under Project THEMIS at the University of Notre Dame, College of Engineering.

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ABSTRACT

Dynamic Relaxation, as developed by Day, Otter, Cassell et al, is a matrix iterative method for the solution of simultaneous linear equations. The application of the technique to problems of structural analysis under static stress conditions, as used by these investigators, has been almost exclusively associated with the finite difference formulation in space of both the equations of motion and the constitutive relationships.

In this report the authors discuss the constraints which the earlier investigators have placed on the method of Dynamic Relaxation by restricting themselves to the finite difference approach, particularly in relation to the static stress analysis of plates and shells having arbitrary geometric discontinuities and boundaries. To overcome this problem, an alternative approach using finite elements in space, in lieu of finite differences, is proposed and developed.

The mathematical basis for the application of Dynamic Relaxation to the solution of the equations of motion, and the constitutive equations is given for the finite element approach in space. The convergence of the method is analyzed by transforming the process into a standard eigenvalue problem for the error vector, and its dependance on the conditioning of the equations is demonstrated. Values of the parameters used in the iteration to optimize convergence are derived. The theory for the optimum asymptotic convergence of Dynamic Relaxation is shown to be applicable to a much wider range of problems than is the method of Successive Overrelaxation. For the special case of a tridiagonal form of the coefficient matrix, a comparison of the optimum convergence of Dynamic Relaxation is made with that of the other basic iterative methods, namely Point Jacobi, Gauss-Seidel, and Successive Overrelaxation. In this particular instance, the convergence of Successive Overrelaxation is shown to be better than that of Dynamic Relaxation.

A strategy for approaching the optimum convergence of Dynamic Relaxation is developed and demonstrated by the application to plane stress problems which include the static stress analysis of flat plates with circular, elliptical and filleted square holes.

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LIST OF SYMBOLS

b	parameter of generalized acceleration
C	constant of damping proportionality
E	Young's Modulus
h	time increment
K	Ch/ρ
N	conditioning number of $B - \lambda_{B\,MAX}/\lambda_{B\,MIN}$
q	amount by which the eigenvalues of B are shifted
u_i^k	displacement in the X direction at time k at i grid points from the origin
$\dot{u}$	velocity in the X direction
XX_i^k	normal stress in the X direction at time k at i grid points from the origin
$\Delta X, \Delta Y, \Delta Z$	mesh lengths in the X, Y and Z directions
α	$(2 - Ch/\rho)/(2 + Ch/\rho)$
β	over or under relaxation factor
β'	overrelaxation factor in Successive Overrelaxation for which the process has two equal real roots
$\bar{\beta}$	overrelaxation factor in Successive Overrelaxation for which the process has two equal real roots corresponding to $\lambda_{PJ\,MAX}$
λ_{PJ} or μ_i	eigenvalues of the Point Jacobi method.

λ_{EG}	eigenvalues of the Extrapolated Gauss method
λ_{GS}	eigenvalues of the Gauss-Seidel method
λ_{SOR}	eigenvalues of the Successive Overrelaxation method
λ_{DR}	eigenvalues of the Dynamic Relaxation method
λ_{B}	eigenvalues of $\mathbf{B}$
λ'_{B}	estimated minimum eigenvalue of $\mathbf{B}$
ρ	constant of mass proportionality
$\mathbf{A}$	coefficient matrix of a system of linear equations
$\mathbf{b}$	right hand side of a system of linear equations
$\mathbf{C}$	damping matrix
$\mathbf{D}$	diagonal matrix whose elements are the elements of the main diagonal of $\mathbf{A}$
$\boldsymbol{\epsilon}^k$	$\mathbf{X}^k - \mathbf{X}$ where $\mathbf{X}$ is the true solution vector
$\mathbf{K}$	stiffness matrix
$\mathbf{L}$	$\mathbf{D}^{-1}\mathbf{P}$
$\mathbf{M}$	mass matrix
$\mathbf{P}$	lower triangular matrix whose elements are the elements of $\mathbf{A}$ below the main diagonal
$\mathbf{Q}$	upper triangular matrix whose elements are the elements of $\mathbf{A}$ above the main diagonal
$\mathbf{R}$	nodal force vector
$\mathbf{r}$	nodal displacement vector

$\dot{r}$	nodal velocity vector
$\ddot{r}$	nodal acceleration vector
U	$D^{-1}Q$
X^k	unknown vector at iteration k

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I INTRODUCTION

1-1 Development and Applications of Dynamic Relaxation

The numerical iterative method for solving systems of linear equations known as Dynamic Relaxation was originally used by Otter and Day (1) in tidal flow calculations for the Thames estuary and the North Sea in the years 1958 to 1960. These were actual dynamical calculations. Since that time this technique has been applied to the static stress analysis of: (a) prestressed concrete pressure vessels by Otter (2); (b) planar frames and plates by Day (3); and (c) arch dams by Hobbs (4), Sefton (5), Kinsey (6), Otter, Cassell and Hobbs (7), and Cassell, Kinsey and Sefton (8).

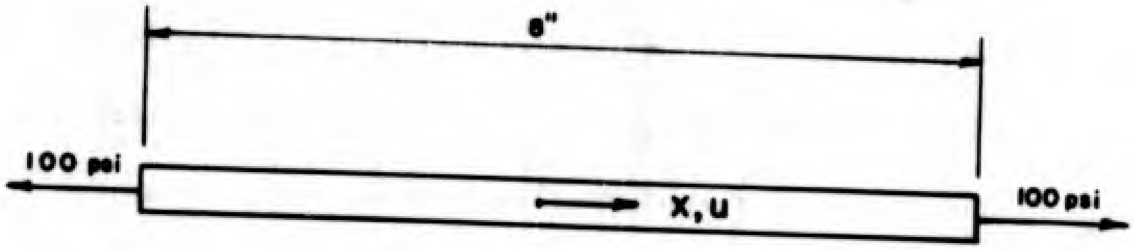
1-2 Description of the Method

As pointed out by Otter et al. (7), Dynamic Relaxation is what Frankel (9) refers to as the Second-Order Richardson Method. When applied in the field of structural mechanics the method consists essentially of solving the static stress equilibrium equations by integrating the damped wave equation until a steady state strain response due to a constant load function is achieved. The constitutive relationships are satisfied at each step of the integration, thus insuring compatibility. The wave equation is integrated using the standard central finite difference form in time and space until the velocity and acceleration terms reach acceptably small values thereby providing static equilibrium, while simultaneous satisfaction of the constitutive equations guarantees compatibility.

1-3 One Dimensional Example

A simple one dimensional numerical example taken from the paper by Otter et al (7) will serve to illustrate the technique.

The problem is the static stress analysis of the bar shown in Figure 1.1 subjected to a uniform tension of 100 psi using the finite difference formulation of Dynamic Relaxation in time and space. As in reference (7) let XX represent σ_{XX} the normal stress in the X direction.



BAR PROPERTIES:

$E = 3 \times 10^6$ psi

WEIGHT = g/12 lb./in.^{3}

Figure 1.1

The one dimensional form of the damped wave equation is given by (1.1)

$$\frac{\partial XX}{\partial X} = \rho \frac{\partial u}{\partial t^2} + \frac{K}{h} \rho \frac{\partial u}{\partial t} \tag{1.1}$$

where ρ is the mass density, u is the displacement in the X direction, K is the constant of proportionality between mass and damping and h is the time increment. It should be noted that the damping has been taken as proportional to the mass.

The constitutive relationship is

$$XX = E \frac{\partial u}{\partial X} \tag{1.2}$$

where E is Young's Modulus.

The formulation of the problem using one dimensional interlacing nets in space and taking advantage of symmetry is shown in Figure 1.2.

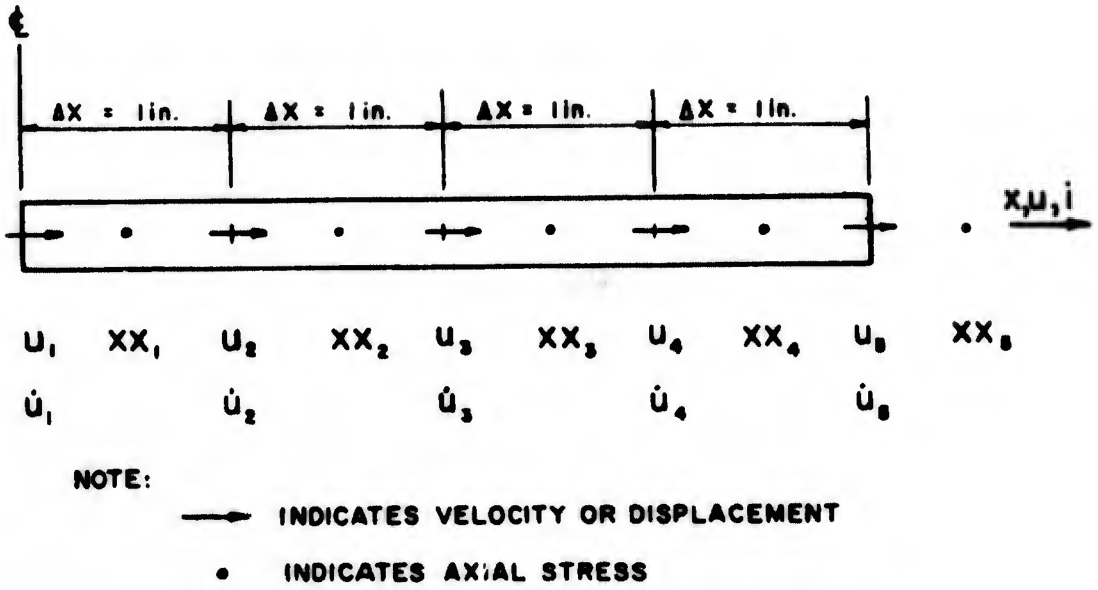


Figure 1.2

The equation of motion and constitutive equation in finite difference form in time and space then become

$$\frac{xx_i^k - xx_{i-1}^k}{\Delta x} = \frac{\rho}{h}(\dot{u}_i^{k+1} - \dot{u}_i^k) + \frac{K}{h}\rho\frac{(\dot{u}_i^{k+1} + \dot{u}_i^k)}{2} \qquad (1.3)$$

$$xx_i^{k+1} = \frac{E}{\Delta x}(u_{i+1}^{k+1} - u_i^{k+1}) \qquad (1.4)$$

where Δx is the mesh length in the x direction, and the dots denote time derivatives.

The relationship of displacement to velocity using interlacing nets in time is given by

$$u_i^{k+1} = u_i^k + \dot{u}_i^{k+1} h \qquad (1.5)$$

In equations (1.3) through (1.5) and hereafter the subscripts refer to space location and the superscripts to time stations.

Solving equation (1.3) for $\dot{u}_i^{k+1}$ yields the following

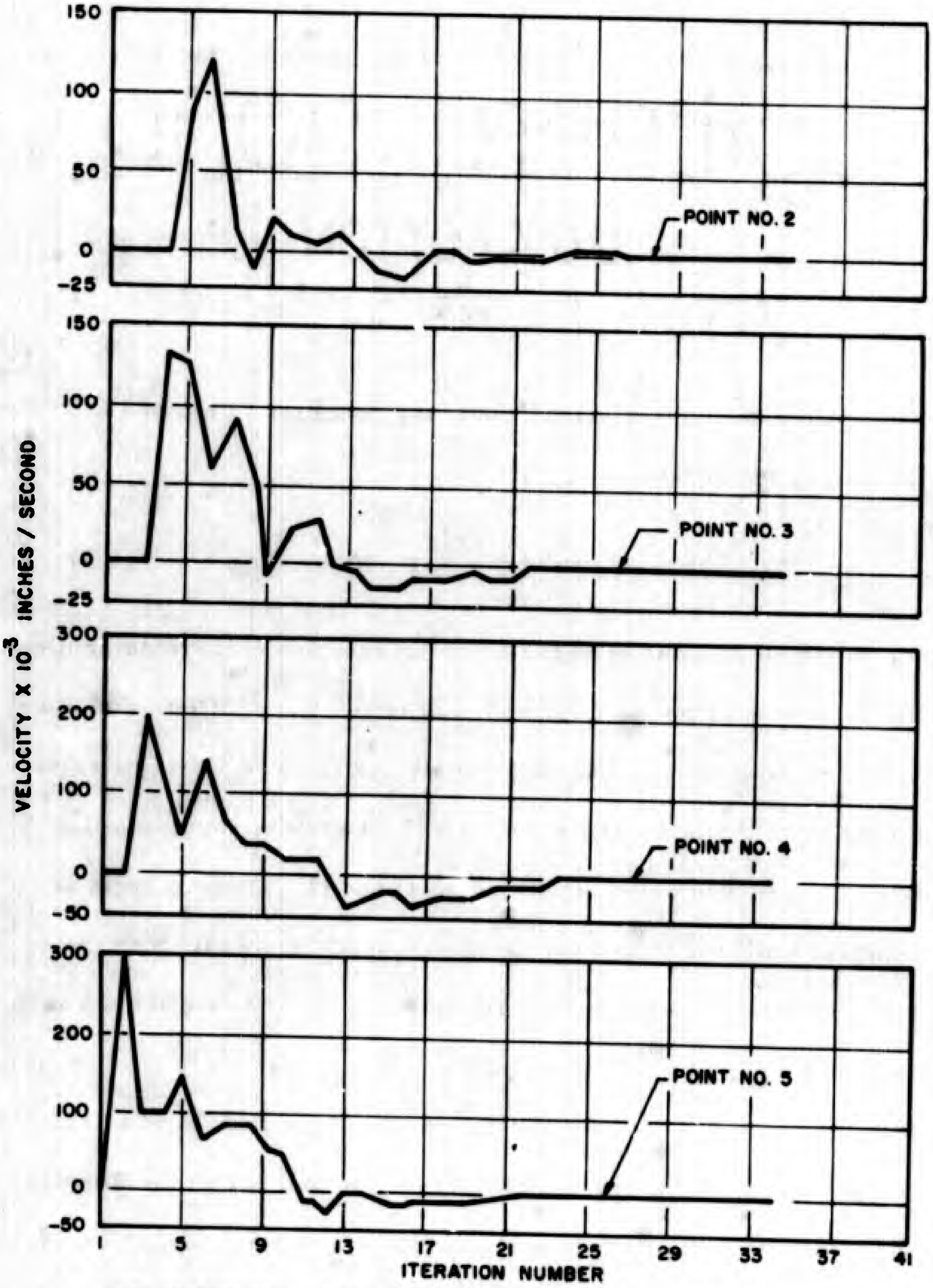
$$\dot{u}_i^{k+1} = \frac{1}{1+\frac{K}{2}} \left\{ \left(1-\frac{K}{2}\right) \dot{u}_i^k + \frac{h}{\rho \Delta x} \left[xx_i^k - xx_{i-1}^k \right] \right\} \qquad (1.6)$$

After setting the stress boundary condition at the end of the bar by the equation

$$\frac{xx_4^k + xx_5^k}{2} = 100 \qquad (1.7)$$

the iterative process is begun by using equation (1.6) to determine the velocities at points 2 through 5 at time $k+1$ (Note the other boundary condition is that u_1 equals zero). The $k+1$ displacements are calculated using equation (1.5) and the corresponding $k+1$ stresses using equations (1.4) and (1.7). This cycle is repeated until the velocities become acceptably small, then the right hand side of equation (1.3) tends to zero and equilibrium is satisfied.

Figures 1.3 to 1.5 show the values of the problem variables at the mesh points along the bar versus the number of the iteration for a solution with $K=0.4$ and $h=150$ microseconds. Figures 1.6 to 1.8 show another solution to the same problem with $K=0.71$



ONE DIMENSIONAL EXAMPLE WITH K = 0.40 AND h = 150μ SEC.

Fig. 1.3 - Velocity variation during Dynamic Relaxation solution

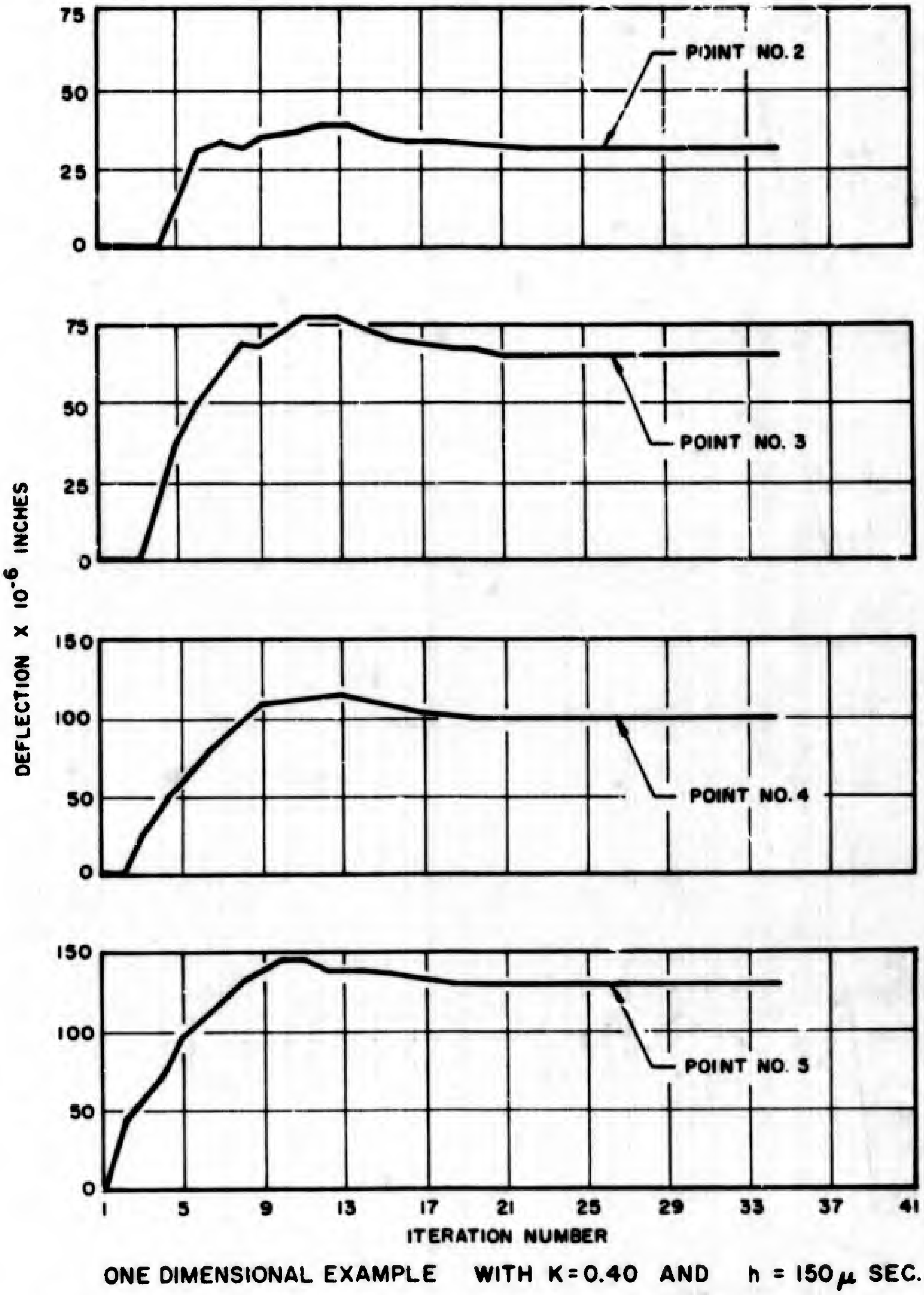


Fig. 1.4 - Deflection variation during Dynamic Relaxation solution

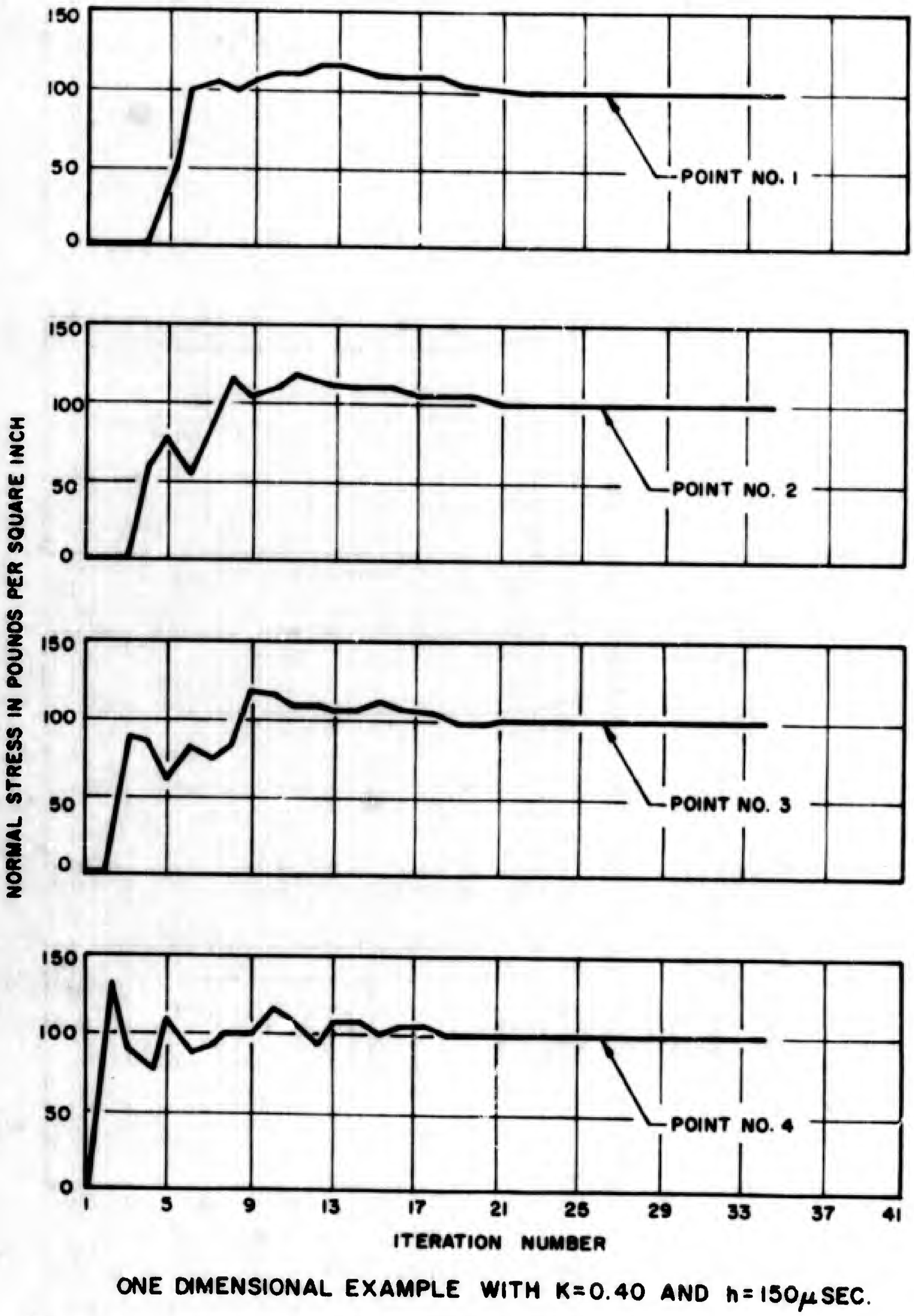
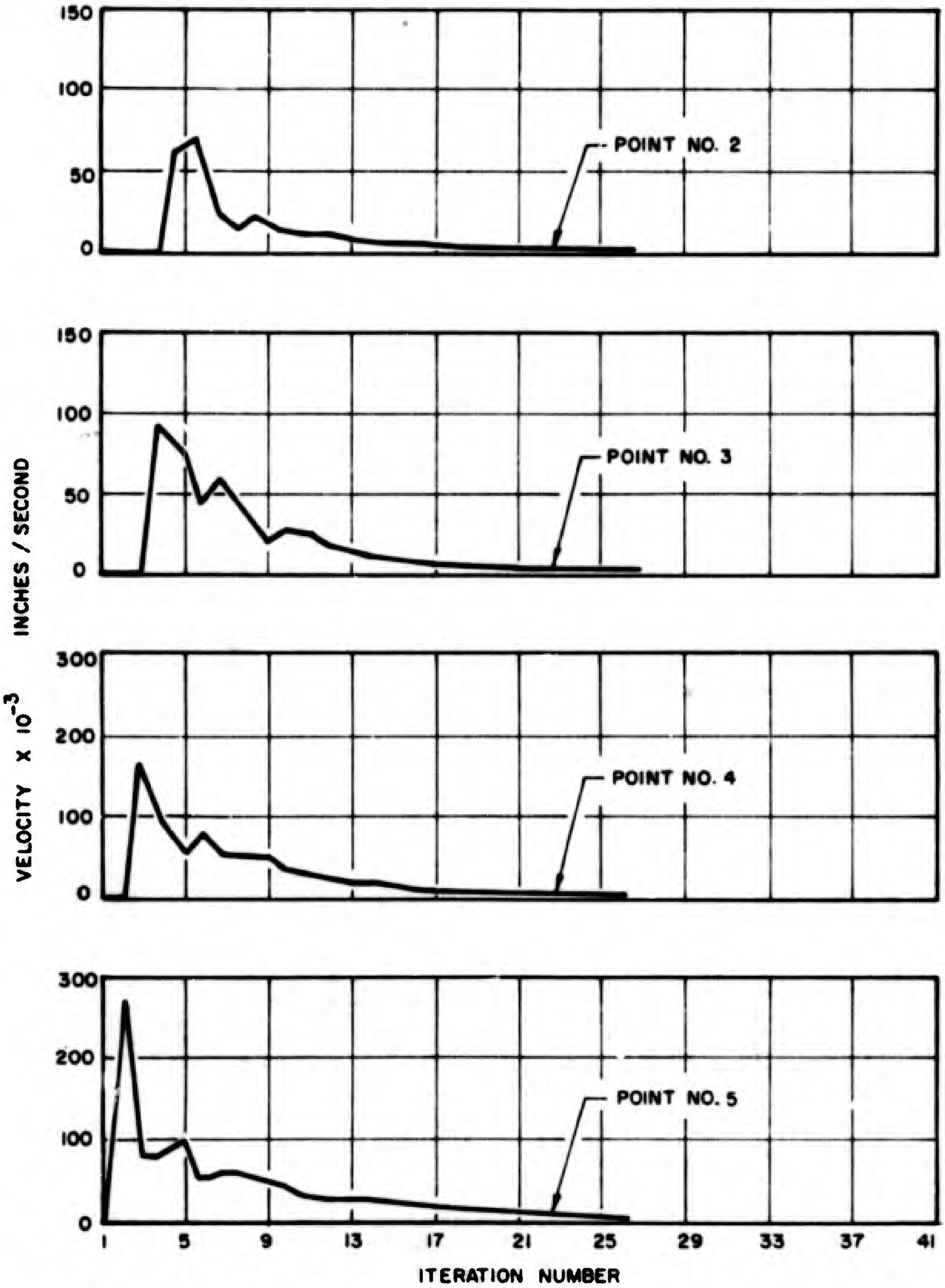
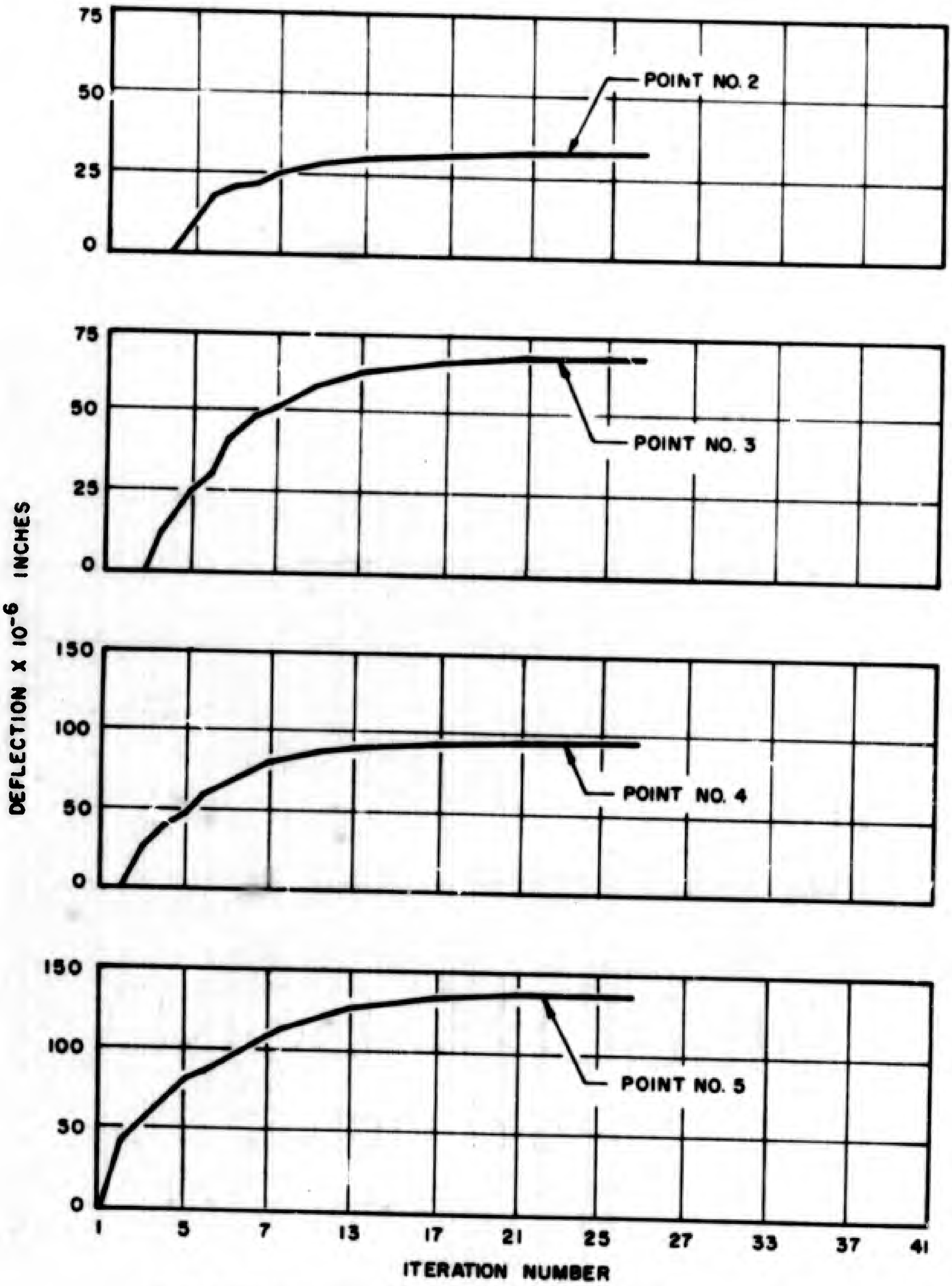


Fig. 1.5 - Normal stress variation during Dynamic Relaxation solution



ONE DIMENSIONAL EXAMPLE WITH K = 0.71 AND h = 150μ SEC.

Fig. 1.6 - Velocity variation during Dynamic Relaxation solution



ONE DIMENSIONAL EXAMPLE WITH K = 0.71 AND h = 150 μ SEC.

Fig. 1.7 - Deflection variation during Dynamic Relaxation solution

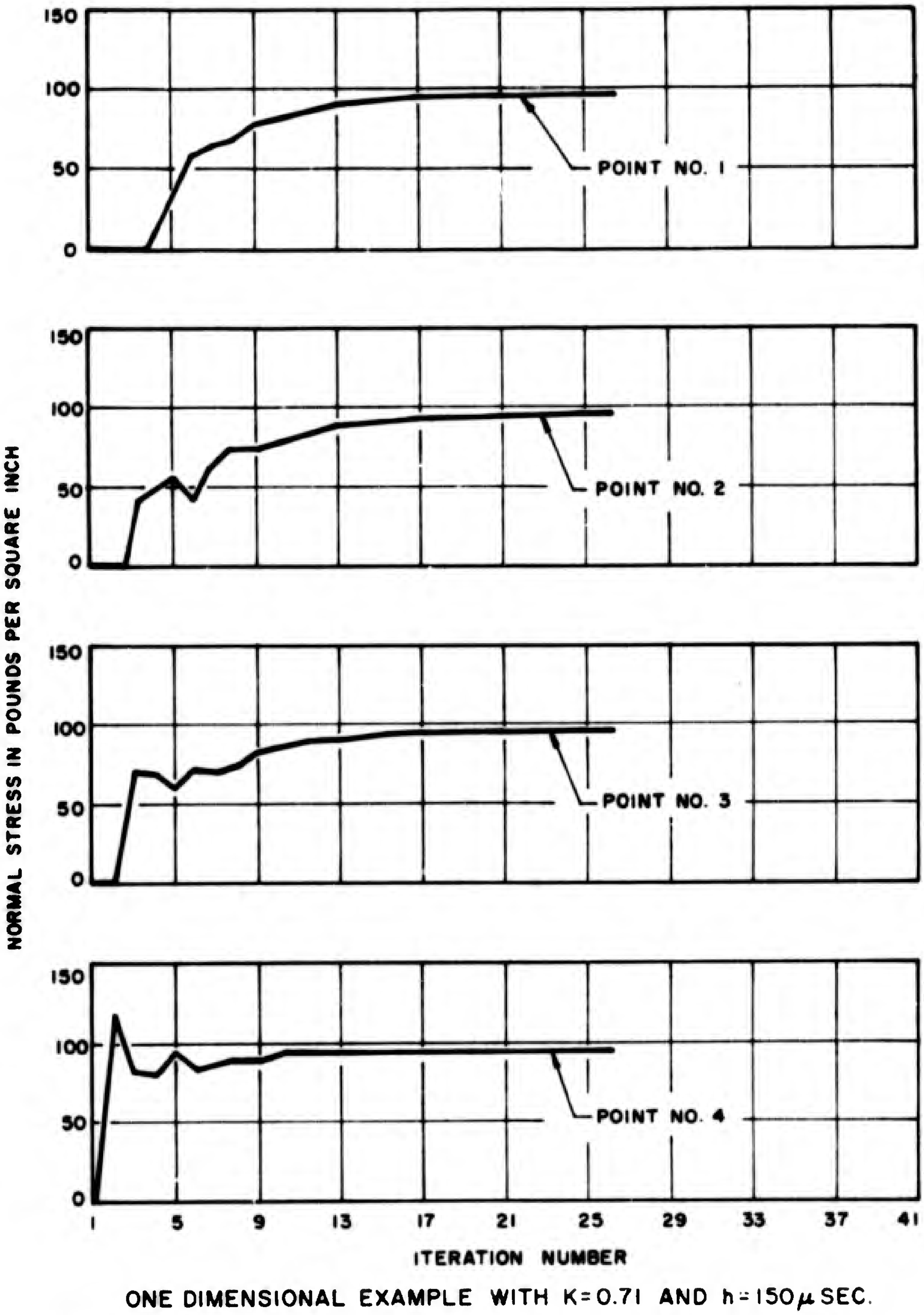


Fig. 1.8 - Normal stress variation during Dynamic Relaxation solution

and $h = 150$ microseconds.

It will be noted that the second solution converges in 26 iterations while the first takes 34. The problem of choosing K and h for optimal convergence is discussed by Day (3) who concludes, "The easiest way to ascertain these quantities (K and h) is by trial and error until suitable values are found. In practice this is fairly easy, especially when finding the time interval, for if the wrong interval is used the calculations are unstable."

Otter (2) proposes that h be chosen for the one dimensional problem so that

$$Ch / \Delta X < 1$$

where C is the wave velocity in the continuum. In (7) an extension of this criteria into three dimensions is given as

$$h \leq \frac{1}{C} \left\{ \left(\frac{1}{\Delta X}\right)^2 + \left(\frac{1}{\Delta Y}\right)^2 + \left(\frac{1}{\Delta Z}\right)^2 \right\}^{-\frac{1}{2}}$$

In regard to the optimum value of K , in reference (11) Otter using the physical analog as a guide in choosing the damping factor states, "the critical damping factor (K) is the same as twice the lowest undamped frequency of the system." Thus the determination of the optimum or critical K becomes a problem in determining the fundamental frequency of the system, which for complex systems Cassell and Otter, by extending the physical analogy, propose running the iteration with zero damping and from the deflection response determining the fundamental period.

1-4 Purpose of this Technical Report

The purpose of this study is to determine the feasibility of the application of Dynamic Relaxation to the static stress analysis of plates with geometrically arbitrary cut outs.

In all prior investigations using the method of Dynamic Relaxation, the method of finite differences had been employed almost exclusively in formulating the spatial relationships. This was pointed out by Otter _et al._ in their paper on Dynamic Relaxation and its subsequent discussion (10) (7), the single exception being the application of Dynamic Relaxation to planar frames by Day (3).

As a natural outgrowth of this earlier development, preliminary investigations were concentrated primarily upon a further extension of the method of Dynamic Relaxation and its usual coupling with the method of finite differences in the formulation of the equilibrium and constitutive relations. This effort was made in order to retain the advantage of the small computer storage requirements of this approach. The finite difference method can be thought of as being equivalent to a stiffness method where the stiffness matrix is composed of predominantly identical terms. These terms can be represented by a few simple equations thus eliminating the need for the explicit formulation of a stiffness matrix. Of course this advantage is most obvious when the technique is used for a problem defined in a single coordinate system where the continuum and its boundaries are easily defined without resorting to such devices as the use of cut meshes.

The application of the finite difference formulation of Dynamic Relaxation in space to the static stress analysis of a flat rectangular plate with a circular hole presented boundary definition difficulties that were not encountered if the hole was rectangular or square. In order to avoid cut meshes on the boundaries in the case of the plate with a circular hole, two coordinate systems were used, a polar coordinate system in the vicinity of the hole and a rectangular one around the outer boundaries of the plate. An artificial internal boundary was then created in an area where the stress gradient was low, between the two coordinate systems.

However, it became apparent for this particular type of problem that a continuance of the established close coupling of Dynamic Relaxation with the finite difference formulation in space quickly lost the computer storage advantage mentioned previously. In addition computer programs developed using this approach were complicated and limited in application to a special type of problem, that is one program for rectangular plates with circular holes, another program for elliptical holes, etc.

The use of cut meshes in conjunction with very irregular cut outs in flat plates presented the same difficulties as those encountered in the use of two different coordinate systems for regular cutouts.

As a consequence it was decided to employ the method of Dynamic Relaxation, coupled with finite elements in space, in lieu of finite differences in the formulation of the equilibrium and constitutive equations, since the finite element representation is far more amenable to conformance with structures of arbitrary geometry.

In Chapter II the convergence of the basic iterative methods, Point Jacobi (Gauss), Gauss- Seidel, and Successive Overrelaxation is examined by transforming the iterative process into an eigenvalue problem in terms of the error vector. This same approach is then used in Chapter III to examine the convergence of the matrix formulation of Dynamic Relaxation using finite elements in space. The theory for the optimum convergence of Dynamic Relaxation is developed and shown to be more general than that for Successive Overrelaxation. A comparison of the convergence of the basic iterative methods is made with that of Dynamic Relaxation for the special case of the tridiagonal form of the coefficient matrix. This was the only comparison possible since this is the only case where the theory for selecting the optimum overrelaxation factor in Successive Overrelaxation applies. In this particular case the convergence of Successive Overrelaxation was found to be better than that of Dynamic Relaxation.

Chapter IV contains examples of the application of the finite element form of Dynamic Relaxation to the static stress analysis of plates with cut outs and a proposed strategy for approaching the optimum convergence of the process for plane stress problems.

A summary of the results, conclusions and recommendations will be found in Chapter V.

II BASIC MATRIX ITERATIVE METHODS (12)

2-1 Introduction

As pointed out by Fox (13) matrix iterative methods for solving linear systems of equations can be advantageously used in at least two categories of problems. One class where the convergence is known *a priori* to be rapid and the other where the coefficient matrix is of large order and sparsely populated.

It is the latter class of problems that is of primary concern when the finite element method of structural analysis is employed. For example if the triangular finite element is being used in a plane stress problem with six elements connected to the typical node, the column and row of the stiffness matrix corresponding to the unit displacement of this node in the X or Y direction will contain fourteen non-zero elements out of possibly as many as three or four hundred elements.

Varga (14) lists the three basic iterative methods as the Point Jacobi (Gauss), Gauss-Seidel, and Successive Overrelaxation. Although these methods are known by several other names the terminology used by Varga will be adopted.

2-2 Matrix Formulation of the Basic Iterative Methods

The aim of all the iterative methods is to solve a set of linear equations in the form:

$$\begin{bmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \cdots & a_{nj} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_j \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_j \\ \vdots \\ b_n \end{bmatrix} \tag{2.1}$$

or using matrix notation (2. 1) may be expressed as

$$AX = b \tag{2.2}$$

where A is the coefficient matrix, X is the unknown vector and b represents the right hand side of the non-homgeneous linear equations in (2. 1).

Let

$$A = \left[P + D + Q \right] \tag{2.3}$$

where P is a lower triangular matrix whose elements are the elements of A below the main diagonal, D is a diagonal matrix whose elements are the elements of the main diagonal of A , and Q is an upper triangular matrix whose elements are the elements of A above the main diagonal.

$$D^{-1}AX = D^{-1}b$$

$$BX = C \tag{2.4}$$

where $B = D^{-1}A = [L + I + U]$

and $L = D^{-1}P \qquad U = D^{-1}Q$

This form of the linear equations to be solved will be used in the subsequent discussion.

In the Point Jacobi (Gauss) method the iteration is begun by assuming the unknown vector X to be equal to the diagonal of B. This vector is then used to compute a new estimate of X using the following recurrence equation

$$X^{k+1} = C - [L + U] X^k \qquad (2.5)$$

or

$$X^{k+1} - X^k = C - BX^k \qquad (2.6)$$

where $X^{k+1} - X^k$ is called the correction vector and $C - BX^k$ is the residual vector. As can be seen from (2.5) the elements of X^{k+1} are computed using all the elements of X^k.

The Gauss-Seidel method improves on the Point Jacobi (Gauss) method by utilizing the most recently computed components of X^{k+1} in computing subsequent components of this vector. The process can be written as

$$X^{k+1} = (I + L)^{-1} (C - UX^k) \qquad (2.7)$$

or

$$X^{k+1} - X^k = (I + L)^{-1} (C - BX^k) \qquad (2.8)$$

Relaxation factors can be introduced in either of these two methods. Using β as this factor it may be applied in the Point Jacobi iteration to give the Extrapolated Gauss method

$$X^{k+1} - X^{k} = \beta\left(C - BX^{k}\right) \tag{2.9}$$

or in the Gauss-Seidel iteration to give the method of Successive Overrelaxation

$$X^{k+1} - X^{k} = \beta\left(I + \beta L\right)^{-1}\left(C - BX^{k}\right) \tag{2.10}$$

The fact that the Point Jacobi and Extrapolated Gauss methods do not utilize the most recently computed components of X^{k+1} in computing subsequent components of this vector and that the Gauss-Seidel and Successive Overrelaxation methods do, has lead to the classification of the former two as simultaneous methods and the latter two as successive methods.

2-3 General Form of the Basic Iterative Methods

From the previous discussion it can be seen that a general form of the basic iterative methods is

$$X^{k+1} - X^{k} = H\left(C - BX^{k}\right) \tag{2.11}$$

Using equations (2.6), (2.8), (2.9) and (2.10) the H matrix for the various methods are:

Point Jacobi

$$H = I \tag{2.12}$$

Gauss-Seidel $H=(I+L)^{-1}$ (2.13)

Extrapolated Gauss $H=\beta I$ (2.14)

Successive Overrelaxation $H=\beta(I+\beta L)^{-1}$ (2.15)

2-4 Convergence of the Basic Iterative Methods

A very useful method of quantitatively examining the convergence of matrix iterative processes is to transform the iterative process into a standard eigenvalue problem for error vectors. This approach is developed here and used to examine the convergence of the three basic iterative methods. The same technique is used again in Chapter III for the analysis of the convergence of the finite element formulation of Dynamic Relaxation.

Equation (2.11) may alternatively be expressed in the following form

$$X^{k+1} = HC+(I-HB)X^{k}$$

$$X^{k+1} = HC+MX^{k} \qquad (2.16)$$

Subtracting the true solution from equation (2.16) gives the relationship between successive error vectors.

$$\epsilon^{k+1} = M\epsilon^{k} \qquad (2.17)$$

where $\epsilon^{k} = X^{k}-X$ and X is the true solution vector.

In terms of matrix and vector norms, an alternative method of examining convergence, a sufficient condition for convergence of the iterative process represented in (2.17) is that the matrix norm of $\mathbf{M}$ written $\|\mathbf{M}\|$ be less than one since

$$\|\mathbf{M}\,\boldsymbol{\epsilon}^k\| \leq \|\mathbf{M}\|\;\|\boldsymbol{\epsilon}^k\|$$

However, this approach to studying convergence of iterative methods was not adopted in this study. The interested reader will find details in Fox (13).

Convergence for purposes of this discussion will refer to the rate at which the error vector decays with each iterative step. Considering an error vector of such form that it is changed in magnitude only, by a factor λ during each iterative step.

$$\boldsymbol{\epsilon}^{k+1} = \lambda\boldsymbol{\epsilon}^k \qquad (2.18)$$

From equations (2.17) and (2.18)

$$\lambda\boldsymbol{\epsilon}^k = \mathbf{M}\boldsymbol{\epsilon}^k$$

or

$$[\lambda\mathbf{I}-\mathbf{M}]\boldsymbol{\epsilon} = 0 \qquad (2.19)$$

If $\mathbf{M}$ is $n \times n$ there are n eigenvalues λ_i and n associated eigenvectors $\mathbf{P}_i$.

In order to understand the effect of each iterative step, suppose $\boldsymbol{\epsilon}^0$ is an initial error vector. It may be represented in eigenvector components as

$$\epsilon^{\circ}_{(n\times1)} = \phi_{10} P_{1\,(n\times1)} + \text{-----} \phi_{i0} P_i + \text{-----} \phi_{n0} P_n$$

Applying the first iterative step gives

$$\epsilon^{1} = M\epsilon^{\circ} = \sum_{i=1}^{n} \phi_{i0} M P_i = \sum_{i=1}^{n} \lambda_i \phi_{i0} P_i$$

then for k iterative steps

$$\epsilon^{k} = \sum_{i=1}^{n} \lambda_i^{k} \phi_{i0} P_i$$

Thus it can be seen that the effect of iteration is to multiply the factors ϕ_{i0} by λ_i at each step. For convergence $\lambda^k \phi_{i0} \to 0$ as k increases provided all the eigenvalues of are less than one in modulus. In the complete process only the error mode with the largest modulus λ_i need be considered since this dictates the asymptotic rate of convergence.

For the Point Jacobi (Gauss) method from equations (2. 12) and (2. 16) it can be seen thar

$$M_{PJ} = (I - B) \tag{2. 20}$$

Using the form of equation (2. 18)

$$\left[(1 - \lambda_{PJ}) I - B \right] = 0 \tag{2. 21}$$

Let λ_B be the eigenvalues of B so

$$\left[\lambda_B I - B \right] \epsilon = 0 \tag{2.22}$$

From (2.21) and (2.22) the following relationship between the eigenvalues of M_{PJ} and B can be established

$$\lambda_{PJ} = I - \lambda_B$$

For convergence then $|\lambda_{PJ}| < 1$ which in turn requires that $0 < |\lambda_B| < 2$. If A is a positive definite matrix it can be shown that $0 < |\lambda_B|$ and the only requirement for convergence is that $|\lambda_B| < 2$.

The Extrapolated Gauss method provides a method of adjusting the eigenvalues of the Point Jacobi process for optimum convergence. This is accomplished by making the modulus of the extreme eigenvalues of the process equal.

$$M_{EG} = (I - \beta B) \tag{2.23}$$

Then in the form of the eigenvalue problem

$$\left[\left(\frac{1 - \lambda_{EG}}{\beta} \right) I - B \right] \epsilon = 0 \tag{2.24}$$

From (2.21) and (2.24)

$$\lambda_{EG} = 1 - \beta (1 - \lambda_{PJ})$$

The relaxation factor β that makes $|\lambda_{EG\,min}|$ equal to $|\lambda_{EG\,max}|$ can be expressed in terms of the extreme eigenvalues of B , when $\mathbf{B}$ is a positive definite matrix as

$$\beta = \frac{2}{\lambda_{B\,min} + \lambda_{B\,max}} \tag{2.25}$$

then

$$|\lambda_{EG\,max}| = \frac{\lambda_{B\,max} - \lambda_{B\,min}}{\lambda_{B\,max} + \lambda_{B\,min}} \tag{2.26}$$

Defining the conditioning number as

$$N = \frac{\lambda_{max}}{\lambda_{min}}$$

the modulus of the largest eigenvalue for the Extrapolated Gauss method or its optimum asymptotic convergence can be written in terms of the conditioning number of $\mathbf{B}$ as

$$|\lambda_{EG\,max}| = \frac{N-1}{N+1}$$

In the case of the Gauss-Seidel method

$$\mathbf{M}_{GS} = \left[\mathbf{I} - (\mathbf{I} + \mathbf{L})^{-1}\,\mathbf{B}\right]$$

or

$$\mathbf{M}_{GS} = (\mathbf{I} + \mathbf{L})^{-1}\mathbf{U} \tag{2.27}$$

If $\mathbf{A}$ is a positive definite matrix it can be shown that although the eigenvalues of $\mathbf{M}_{GS}$ may be complex, their moduli are always less than one, thus the Gauss-Seidel iteration will converge.

In the following analysis of the asymptotic convergence of the Successive Overrelaxation method, which is due to David Young, the development and results are limited to tridiagonal forms of the coefficient matrix $\mathbf{A}$.

For Successive Overrelaxation

$$\mathbf{M}_{SOR} = (\mathbf{I}+\beta\mathbf{L})^{-1}\left[(1-\beta)\,\mathbf{I}-\beta\mathbf{U}\right] \qquad (2.28)$$

and as before transforming the process into the standard eigen-value form yields

$$\left[\lambda_{SOR}\,\beta\mathbf{L}+(\lambda_{SOR}-1+\beta)\,\mathbf{I}+\beta\mathbf{U}\right]\boldsymbol{\epsilon}=\mathbf{0} \qquad (2.29)$$

The characteristic equation resulting from (2.29) can be transformed into one similar in form to the one given by equation (2.21) for the Point Jacobi method as follows. Premultiply $\left[\lambda_{SOR}\beta\mathbf{L}+\left(\lambda_{SOR}-1+\beta\right)\mathbf{I}+\beta\mathbf{U}\right]$ by a diagonal matrix, the general term of which is $\lambda_{SOR}^{-(n-1)/2}$ and post multiply by another diagonal matrix with a diagonal term equal to $\lambda_{SOR}^{(n-1)/2}$, where n is the number of the diagonal term. This transformation does not alter the value of the determinant of the original matrix.

The transformed matrix gives a characteristic equation which can be written

$$\lambda_{SOR}^{\frac{n}{2}}\,\beta^{n}\,F\left(\frac{\lambda_{SOR}-1+\beta}{\lambda_{SOR}^{\frac{1}{2}}\,\beta}\right)$$

as compared to the characteristic equation of the Point Jacobi method which is simply

$$F(\lambda_{PJ})$$

where in both of these equations $F(\ )$ represents a function of the quantity in brackets.

Since for the Point Jacobi method when applied to a system of equations with a coefficient matrix in tridiagonal form the eigenvalues of the method are either zero or occur in equal (in magnitude) and opposite (in sign) pairs the characteristic equation for this method can be written as

$$\lambda_{PJ}^{n-2P} \prod_{i=1}^{P} \left(\lambda_{PJ}^{2} - \mu_i^2 \right) = 0 \tag{2.30}$$

where P equals the number of zero roots and λ_{PJ} equals $\pm \mu_i$.

In a similar manner the characteristic equation for the Successive Overrelaxation method can be written

$$\left(\lambda_{SOR} - 1 + \beta\right)^{n-2P} \prod_{i=1}^{P} \left[\left(\lambda_{SOR} - 1 + \beta \right)^2 - \lambda_{SOR} \beta^2 \mu_i^2 \right] = 0 \tag{2.31}$$

A comparison of (2.30) and (2.31) shows that the value of λ_{SOR} that corresponds to λ_{PJ} equal to zero is

$$\lambda_{SOR} = 1 - \beta \tag{2.32}$$

and the one for non-zero values is

$$\left(\lambda_{SOR} - 1 + \beta\right)^2 = \lambda_{SOR}\,\beta^2\,\mu_i^2 \qquad (2.33)$$

It is worth noting at this point that if $\beta = 1$, Successive Overrelaxation is merely the Gauss-Seidel iteration and from (2.32) and (2.33)

$$\lambda_{GS} = 0 \quad \text{or} \quad \lambda_{GS} = \mu_i^2 = \lambda_{PJ}^2$$

which indicates that the reduction in the error vector accomplished by one cycle of the Gauss-Seidel method requires two cycles of the Point Jacobi method.

Now let $\lambda_{SOR} = \alpha_{SOR}^2$ in equation (2.33) and solve for α_{SOR}

$$\alpha_{SOR} = \frac{\beta\,\mu_i \pm \sqrt{\beta^2\mu_i^2 - 4(\beta - 1)}}{2}$$

and

$$\lambda_{SOR} = \left[\frac{\beta\,\mu_i \pm \sqrt{\beta^2\mu_i^2 - 4(\beta - 1)}}{2}\right]^2 \qquad (2.34)$$

Upon examining (2.34) it can be seen that for $0 < \beta < 1$, λ_{SOR} has two real values. For $\beta > 1$, λ_{SOR} has two real values only if $\beta^2\mu_i^2 > 4(\beta-1)$ and two equal real values if $\beta^2\mu_i^2 = 4(\beta-1)$. If $\beta^2\mu_i^2 < 4(\beta-1)$, λ_{SOR} is complex with a modulus that is independent of μ_i .

A plot of the roots of equation (2.34) versus β is shown in Figure 2.1.

If β' is the value of β for which λ_{SOR} has two equal roots, then it can be shown that for a value of β' between one and two

$$\beta' = \frac{2}{1+\sqrt{1-\mu_i^2}} \tag{2.35}$$

and from (2.34)

$$\lambda_{SOR} = \frac{\beta'^2 \mu_i^2}{4} = \beta' - 1$$

If

$$\bar{\beta} = \frac{2}{1+\sqrt{1-\lambda_{PJ\,max}^2}} \tag{2.36}$$

it can be shown that all eigenvalues of the Successive Overrelaxation process have the same modulus. Theoretically all will be complex except for the eigenvalues associated with $\lambda_{PJ\,max}$ which will be real. Thus for optimum convergence of the Successive Overrelaxation method when applied to tridiagonal matrices, the overrelaxation factor should be chosen equal to $\bar{\beta}$ as given in equation (2.36). The optimum convergence rate is then given by

$$|\lambda_{SOR}| = \bar{\beta} - 1 = \frac{2}{1+\sqrt{1+\lambda_{PJ\,max}^2}} - 1$$

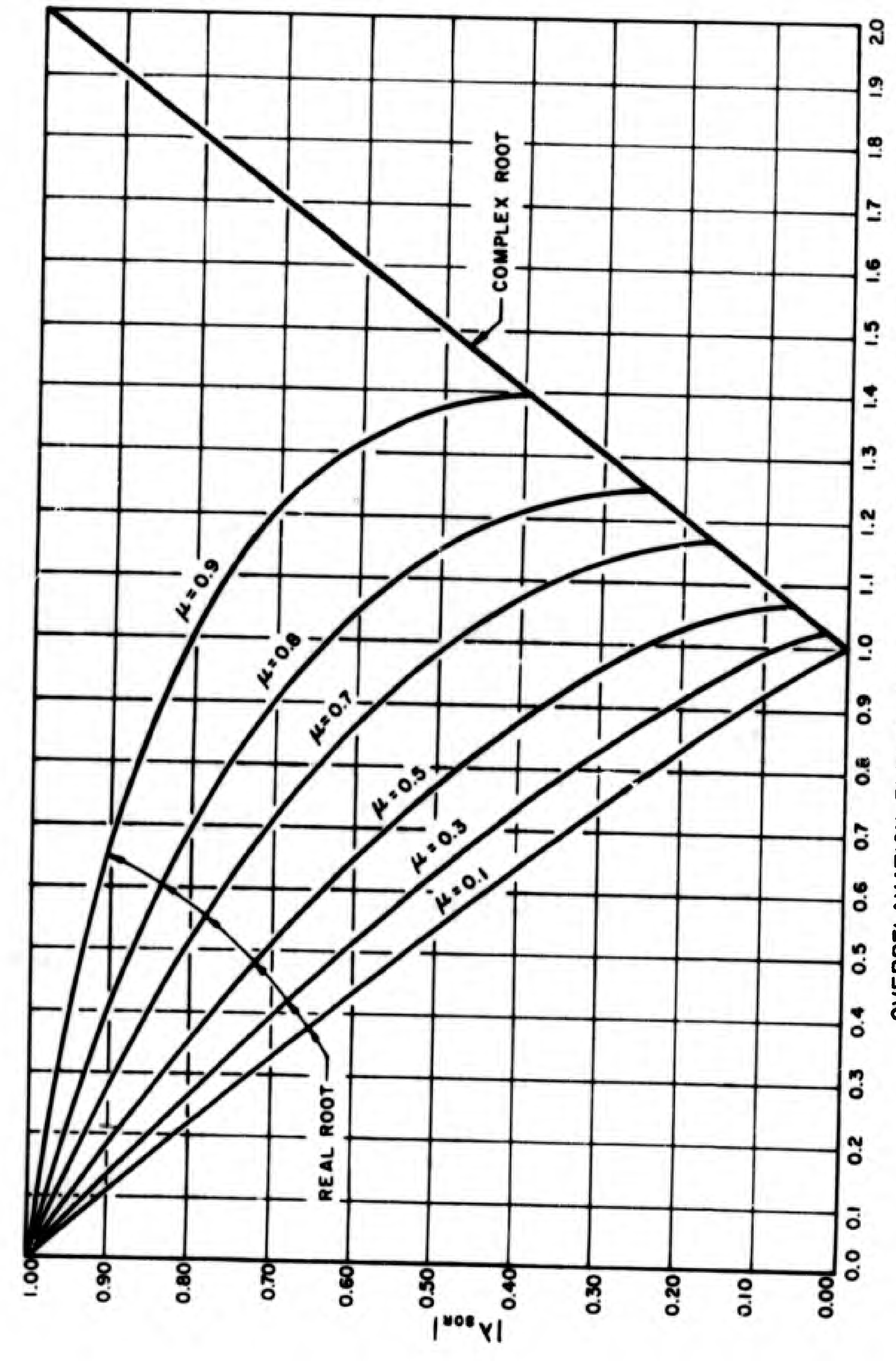


Fig. 2.1 - Relation between asymptotic convergence and overrelaxation factor in Successive Overrelaxation

$$\left|\lambda_{SOR}\right| = \frac{1-\sqrt{1-\lambda_{PJ\,max}^2}}{1+\sqrt{1-\lambda_{PJ\,max}^2}} \tag{2.37}$$

or alternatively

$$\left|\lambda_{SOR}\right| = \frac{1-\sqrt{1-\lambda_{GS}}}{1+\sqrt{1-\lambda_{GS}}} \tag{2.38}$$

These expressions, (2.37) and (2.38), for the asymptotic convergence of Successive Overrelaxation can also be expressed in terms of the conditioning number of $\mathbf{B}$

$$\left|\lambda_{SOR}\right| = \left(\frac{\sqrt{N}-1}{\sqrt{N}+1}\right)^2$$

2-5 A Comparison of the Convergence of the Basic Iterative Methods

This comparison of the asymptotic convergence of the basic iterative methods is valid only when they are applied to a system of equations where the coefficient matrix $\mathbf{A}$ is tridiagonal in form. Summarizing from 2-4

For Point Jacobi

$$\left|\lambda_{PJ}\right| = \frac{N-1}{N+1}$$

For Gauss-Seidel

$$\left|\lambda_{GS}\right| = \left(\frac{N-1}{N+1}\right)^2$$

For Successive Overrelaxation

$$\left|\lambda_{SOR}\right| = \left(\frac{\sqrt{N}-1}{\sqrt{N}+1}\right)^2$$

III MATHEMATICAL FORMULATION OF DYNAMIC RELAXATION USING FINITE ELEMENTS

3-1 Introduction

As was discussed in Chapter I the purpose of Dynamic Relaxation is to solve a system of linear equations. Although the method is perfectly general and can be applied to any system of linear equations in which the coefficient matrix is positive definite, the following discussion will be concerned with the solution of the system of linear equations that result when the stiffness method of structural analysis is employed. This viewpoint will help in developing the method and indicate the reason for Day's choice of the name Dynamic Relaxation.

In the stiffness method the system of equations for which a solution is sought is given in matrix notation by

$$\mathbf{K}\,\mathbf{r} = \mathbf{R} \qquad (3.1)$$

In a structural sense equation (3.1) represents an idealized system of connected finite elements with stiffness properties contained in $\mathbf{K}$, whose nodal connections have displaced some distance $\mathbf{r}$ and hold a system of external nodal point loads $\mathbf{R}$ in static equilibrium. The solution sought is

$$\mathbf{r} = \mathbf{K}^{-1}\mathbf{R} \qquad (3.2)$$

In order to achieve this solution by the method of Dynamic Relaxation, equation (3.1) is transformed into an equation of motion by introducing imaginary point masses at the nodes and viscous damping forces.

$$\mathbf{M}\ddot{\mathbf{r}} + \mathbf{C}\dot{\mathbf{r}} + \mathbf{K}\mathbf{r} = \mathbf{R} \tag{3.3}$$

where $\mathbf{M}$ and $\mathbf{C}$ are the mass and damping matrices respectively. This equation is then integrated for displacement response under the step load $\mathbf{R}$. The integration is continued until the system achieves a steady state response, for which the steady state displacement is the solution sought.

As was previously pointed out this formulation uses the finite element approach in space to form the equations of motion and the constitutive relationships, in contrast to the finite difference formulation in space used by Otter, Cassell, et al. However in both approaches the standard central finite difference method has been employed to integrate the equations of motion. The possible use of other methods will be discussed later.

Since equation (3.3) does not have to represent the true dynamic behavior of the structural system, but is merely a means to an end, the end being the steady state response, the selection of the parameters involving mass, damping and length of time is in principle arbitrary. The following development will show the effect these parameters have on the integration procedure and how they may be selected to achieve the required steady state response in as few iterative steps as possible, thus optimizing the technique.

3-2 Formulation of the Iterative Process

Equation (3.3) may be rewritten using central finite difference in time as

$$\mathbf{M}\left[\frac{\dot{\mathbf{r}}^{k+1}-\dot{\mathbf{r}}^{k}}{h}\right]+\mathbf{C}\left[\frac{\dot{\mathbf{r}}^{k+1}+\dot{\mathbf{r}}^{k}}{2}\right]+\mathbf{K}\mathbf{r}^{k}=\mathbf{R} \qquad (3.4)$$

where superscripts are used to indicate time stations. Equation (3.4) is solved for $\dot{\mathbf{r}}^{k+1}$ giving

$$\dot{\mathbf{r}}^{k+1}=\left[\frac{1}{h}\mathbf{M}+\frac{1}{2}\mathbf{C}\right]^{-1}\left[\frac{1}{h}\mathbf{M}-\frac{1}{2}\mathbf{C}\right]\dot{\mathbf{r}}^{k}+\left[\frac{1}{h}\mathbf{M}+\frac{1}{2}\mathbf{C}\right]^{-1}\left[\mathbf{R}-\mathbf{K}\mathbf{r}^{k}\right] \qquad (3.5)$$

In order to keep the iterative step as simple as possible $\mathbf{M}$ and $\mathbf{C}$ have been chosen to be diagonal matrices. This choice avoids the need to perform an inversion in the iterative step which could be self defeating, since the object of the method is to obtain the inverse of $\mathbf{K}$ without using an elimination method.

Both the mass and damping were assumed to be proportional to the main diagonal terms of $\mathbf{K}$, that is if $\mathbf{D}$ is a diagonal matrix of the main diagonal terms of $\mathbf{K}$

$$\mathbf{M} = \rho \mathbf{D} \quad \text{and} \quad \mathbf{C} = c\mathbf{D}$$

This assumption, as will be seen in the subsequent development leads to making the Dynamic Relaxation iteration process dependent on the ratio of the extreme eigenvalues of a matrix $\mathbf{D}^{-1}\mathbf{K}$. Assuming the mass and damping proportional to unit matrices has the effect of making the process depend on the extreme eigenvalues of $\mathbf{K}$. This as was pointed out in the numerical example in Chapter I is what Otter et al. did. In this investigation it has been found that the former approach, that is making mass and damping proportional to $\mathbf{D}$, has for the same set of equations, given slightly better convergergence than the latter choice.

Using mass and damping proportional to $\mathbf{D}$ equation (3.5) then becomes

$$\dot{\mathbf{r}}^{k+1} = \left[\frac{2 - \frac{Ch}{\rho}}{2 + \frac{Ch}{\rho}}\right]\dot{\mathbf{r}}^{k} + \left[\frac{\frac{2h}{\rho}}{2 + \frac{Ch}{\rho}}\right]\left[\mathbf{D}^{-1}\mathbf{R} - \mathbf{B}\mathbf{r}^{k}\right] \quad (3.6)$$

where $\mathbf{B} = \mathbf{D}^{-1}\mathbf{K}$

Using the standard central finite difference form for the relationship between displacements and velocities the equation for $\mathbf{r}^{k+1}$ becomes

$$\mathbf{r}^{k+1} = \mathbf{r}^{k} + \alpha h\dot{\mathbf{r}}^{k} + \gamma\left[\mathbf{D}^{-1}\mathbf{R} - \mathbf{B}\mathbf{r}^{k}\right] \quad (3.7)$$

where $$\alpha = \frac{2 - \frac{Ch}{\rho}}{2 + \frac{Ch}{\rho}} \qquad \gamma = \frac{\frac{2h^2}{\rho}}{2 + \frac{Ch}{\rho}}$$

If equation (3.7) is to represent the true solution then

$$\dot{r} = 0 \qquad \text{and} \qquad K r = R$$

Equation (3.7) represents the iterative process for the finite element formulation in space of Dynamic Relaxation. It is carried out until both $\dot{r}$ and the quantities $(D^{-1} R - B r)$ reach acceptably small values.

3-3 Convergence of Dynamic Relaxation

The same method that was developed and employed in Chapter II to examine the convergence of the basic iterative methods will be used here to investigate the asymptotic convergence of the finite element formulation of Dynamic Relaxation, that is the transformation of the process into a standard eigenvalue problem for error vectors.

Rewriting equation (3.7) in the following form

$$r^{k+1} - r^k = \alpha r^k - \alpha r^{k-1} + \gamma \left[D^{-1} R - B r^k \right]$$

and subtracting the true solution from it gives

$$\epsilon^{k+1} - \epsilon^k = \alpha \epsilon^k - \alpha \epsilon^{k-1} - \gamma B \epsilon^k \qquad (3.8)$$

where as in Chapter II

$$\epsilon^k = X^k - X$$

and $\mathbf{X}$ is the true solution vector.

Letting $\beta = \alpha + 1$ and rearranging (3. 8) becomes

$$\epsilon^{K+1} = \left[\beta \, I - \gamma B\right] \epsilon^{K} - \alpha \, \epsilon^{K-1} \tag{3.9}$$

From equation (2. 17)

$$\epsilon^{K+1} = \lambda_{DR} \epsilon^{K} = \lambda_{DR}^{2} \epsilon^{K-1}$$

Substituting these relations into (3. 9) the eigenvalue problem in terms of the error vector can be stated as

$$\left[\left(\frac{\lambda_{DR}^{2} - \lambda_{DR}\beta + \alpha}{\lambda_{DR}\gamma}\right) I + B\right] \epsilon = 0 \tag{3.10}$$

Use is now made of equation (2. 22) in Chapter II which is

$$\left[\lambda_{B} \, I - B\right] \epsilon = 0$$

in order to establish a relationship between λ_{DR} and λ_{B} .

From equations (3. 10) and (2. 22) comes the following relation

$$\lambda_{DR}^{2} - \left(\beta - \gamma \lambda_{B}\right) \lambda_{DR} + \alpha = 0 \tag{3.11}$$

Using the following general form of a quadratic equation

$$\lambda^{2} - 2a\lambda + a^{2} + b^{2} = \left[\lambda - (a + ib)\right]\left[\lambda - (a - ib)\right] \tag{3.12}$$

it can be seen from (3. 11) that

$$-2a = -(\beta - \gamma \lambda_B) \quad \text{or} \quad a = \left(\frac{\beta - \gamma \lambda_B}{2}\right)$$

$$a^2 + b^2 = \alpha$$

$$\text{and } b^2 = \alpha - \left(\frac{\beta - \gamma \lambda_B}{2}\right)^2 \text{ with } b = \sqrt{\alpha - \left(\frac{\beta - \gamma \lambda_B}{2}\right)^2}$$

Therefore the roots of equation (3. 11) will be:

(1) complex conjugate if $\left(\frac{\beta - \gamma \lambda_B}{2}\right)^2 < \alpha$

(2) two equal real roots if $\left(\frac{\beta - \gamma \lambda_B}{2}\right)^2 = \alpha$

(3) two unequal real roots if $\left(\frac{\beta - \gamma \lambda_B}{2}\right)^2 > \alpha$

With complex conjugate roots the modulus of λ_{DR} is given by

$$|\lambda_{DR}| = \sqrt{\alpha} = \sqrt{\frac{2 - \frac{Ch}{\rho}}{2 + \frac{Ch}{\rho}}} \qquad (3.13)$$

and is independent of λ_B .

The roots will be real and equal if

$$\alpha = \left(\frac{\beta - \gamma \lambda_B}{2}\right)^2$$

which after substitution and simplification gives the following relationship

$$\frac{C^2h^2}{\rho^2} = \frac{\lambda_B h^2}{\rho}\left(4 - \frac{\lambda_B h^2}{\rho}\right) \qquad (3.14)$$

For two unequal real roots the modulus of the larger is given by

$$|\lambda_{DR}| = \left| \left(\frac{\beta - \gamma\lambda_B}{2}\right) + \sqrt{\left(\frac{\beta - \gamma\lambda_B}{2}\right)^2 - \alpha} \right|$$

which reduces to

$$|\lambda_{DR}| = \left| \frac{1}{2 + \frac{Ch}{\rho}} \left[\left| 2 - \frac{\lambda_B h^2}{\rho} \right| + \sqrt{\frac{\lambda_B^2 h^4}{\rho^2} - \frac{4\lambda_B h^2}{\rho} + \frac{C^2 h^2}{\rho^2}} \right] \right| \qquad (3.15)$$

Figure 3.1 is a plot of the largest modulus of λ_{DR} versus $\frac{Ch}{\rho}$ for constant values of λ_B .

If the value of Ch/ρ for which the roots of equation (3.11) are real and equal is considered to be analogous to critical damping in terms of the physical problem represented by equation (3.3), a very important difference between these two quantities is apparent from the plot of the largest modulus of λ_{DR} versus $\lambda_B h^2/\rho$ shown in Figure 3.2. In the physical problem critical damping is associated with the fundamental mode or frequency, as can be readily seen from Figure 3.2 in Dynamic Relaxation when dealing with the eigenvalues of $\mathbf{B}$ both the lowest and highest eigenvalues of $\mathbf{B}$

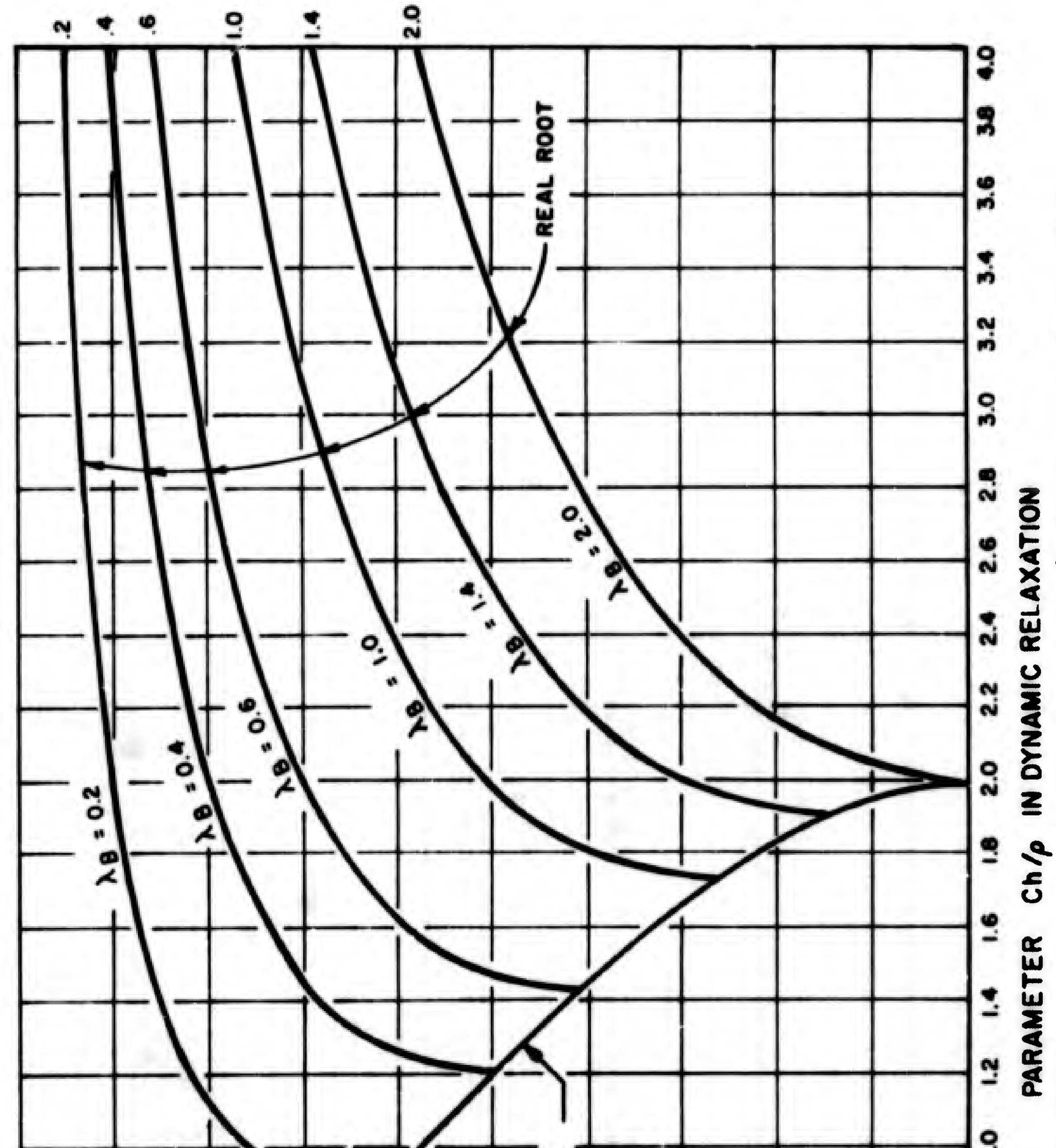


Fig. 3.1 - Relation between asymptotic convergence and Ch/ρ in Dynamic Relaxation

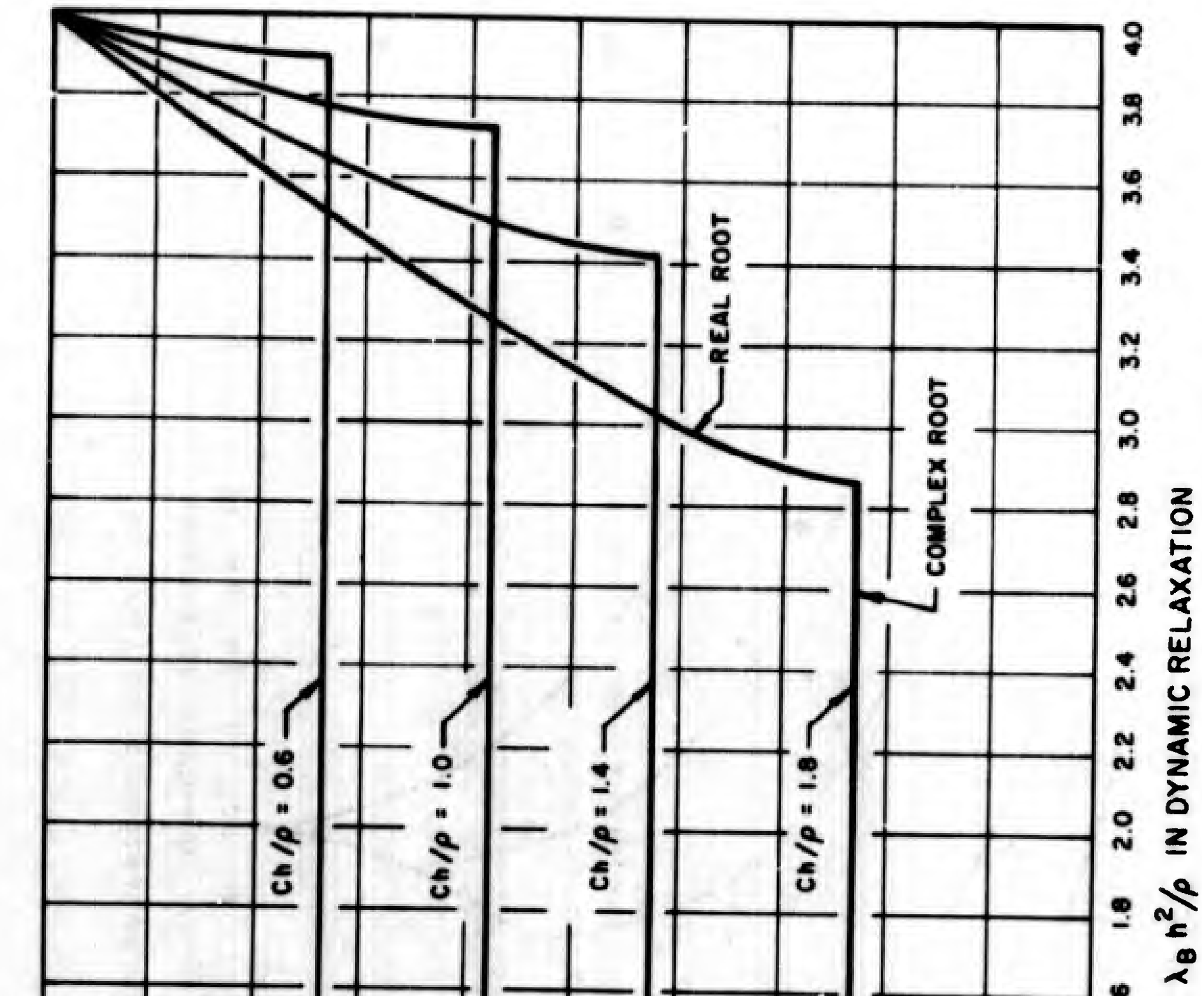


Fig. 3.2 - Relation between asymptotic convergence and $\lambda_B h^2/\rho$ in Dynamic Relaxation

have associated with them critical damping. In pursuing the physical analogy Otter et al. overlooked this point.

In addition Figure 3.2 makes clear the strategy to follow in order to optimize the asymptotic convergence of the Dynamic Relaxation iteration. A simple numerical example will illustrate this optimal strategy.

An analysis of $\mathbf{B}$ gives the extreme eigenvalues as $\lambda_{B\min} = 0.466$ and $\lambda_{B\max} = 2.800$. The parameter h^2/ρ is now chosen in order to make the two parameters $\lambda_{B\min} h^2/\rho$ and $\lambda_{B\max} h^2/\rho$ symmetrical about the ordinate where $\lambda_B h^2/\rho$ equals 2.0 . This value of h^2/ρ is approximately 1.22 . Next the value of Ch/ρ is chosen in order to make the roots associated with $\lambda_{B\min}$ and $\lambda_{B\max}$ real and equal. This is the value of Ch/ρ referred to previously as being analogous to critical damping. In this example it means choosing a value of Ch/ρ of about 1.4 which would then make all the roots equal in modulus and for this example equal to about 0.42 . This strategy then guarantees that all the error modes are decaying at exactly the same rate, of course it also presumes the ability to evaluate the extreme eigenvalues of $\mathbf{B}$.

A more formal statement of the optimal strategy in Dynamic

Relaxation is as follows.

The parameter h^2/ρ should be selected so that

$$\left(\frac{h^2}{\rho}\right)_{opt.} = \frac{4.0}{\lambda_{B\,min} + \lambda_{B\,max}} \tag{3.16}$$

Then from (3.14)

$$\left(\frac{Ch}{\rho}\right)_{opt.} = \frac{4\sqrt{\lambda_{B\,min}\ \lambda_{B\,max}}}{\lambda_{B\,min} + \lambda_{B\,max}} \tag{3.17}$$

Chapter IV contains examples and a discussion of the application of this strategy for optimizing the convergence of Dynamic Relaxation when applied to the finite element analysis of plane stress problems.

The optimum convergence achievable in the Dynamic Relaxation iteration can be expressed in terms of the conditioning number of $\mathbf{B}$ which is

$$N = \frac{\lambda_{B\,max}}{\lambda_{B\,min}}$$

Substituting the optimum value of Ch/ρ given by (3.17) into equation (3.13) gives for the optimum value of the asymptotic convergence of Dynamic Relaxation

$$\left|\lambda_{DR}\right|_{opt.} = \left|\frac{\sqrt{N}-1}{\sqrt{N}+1}\right| \tag{3.18}$$

It is now informative to summarize this result and the corresponding results for the other iterative methods from Chapter II when applied to tridiagonal matrices.

3-4 Comparison of the Convergence of Dynamic Relaxation with the Other Basic Iterative Methods

In making this comparison it must again be emphasized, as was pointed out in Chapter I, that the theory just developed for the application of Dynamic Relaxation is not limited to systems of equations with a tridiagonal form of the coefficient matrix as is the case for the theory of selecting the optimum overrelaxation factor in Successive Overrelaxation. In the finite element method of structural analysis the generality of the Dynamic Relaxation technique is a real strength since sometimes it is impossible to formulate the problem so that the stiffness matrix is tridiagonal in form.

Summarizing the convergence analyses of Chapters II and III gives the following convergence rates for the various methods when applied to a system of equations with a coefficient matrix of tridiagonal form.

Simultaneous Methods	Successive Methods
Point Jacobi	Gauss-Seidel
$\left|\lambda_{PJ}\right|_{opt.} = \frac{N-1}{N+1}$	$\left|\lambda_{GS}\right|_{opt.} = \left(\frac{N-1}{N+1}\right)^2$
Dynamic Relaxation	Successive Overrelaxation
$\left|\lambda_{DR}\right|_{opt.} = \frac{\sqrt{N}-1}{\sqrt{N}+1}$	$\left|\lambda_{SOR}\right| = \left(\frac{\sqrt{N}-1}{\sqrt{N}+1}\right)^2$

It can be seen from this comparison that reduction in the error vector accomplished in one cycle of the Successive Over-relaxation method takes two cycles in the Dynamic Relaxation method, the same relationship that exists between Gauss-Seidel and Point Jacobi.

A plot of the optimum convergence rates for the various methods versus the conditioning number of $\mathbf{B}$ is shown in Figure 3.3. The number of iterations to reduce the original error by a factor of 10^{-4} versus the conditioning number is shown in Figure 3.4.

3-5 Other Techniques for Integrating the Equation of Motion

Day (3) and Otter et al. (7) refer to stability problems with the Dynamic Relaxation method. Their proposals for coping with this aspect of the method were given in Chapter I. From the development in this Chapter it is quite clear how an instability

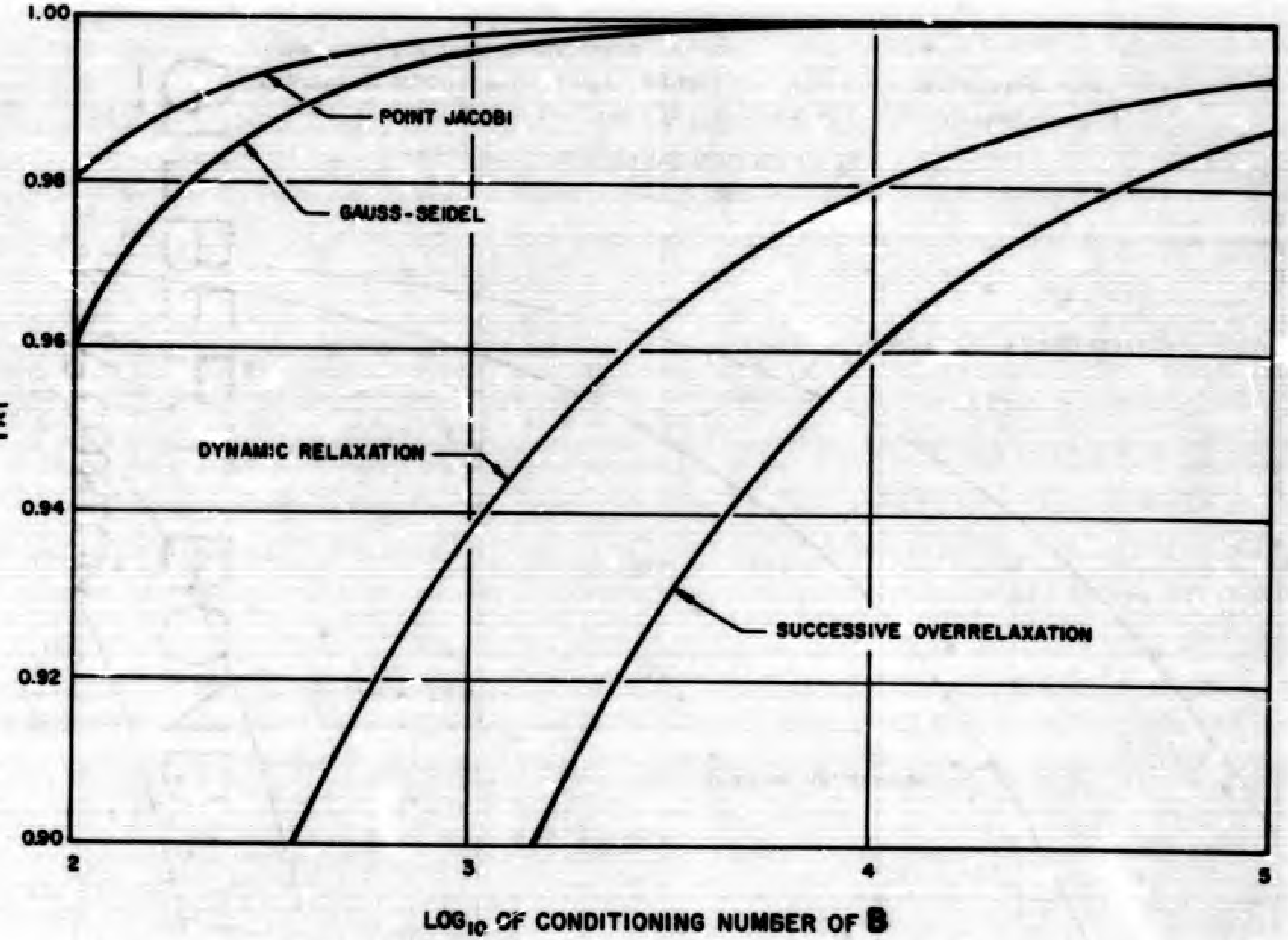


Fig. 3.3 - Comparison of optimum asymptotic convergence rates of Basic Iterative Methods with that of Dynamic Relaxation for tridiagonal form of coefficient matrix

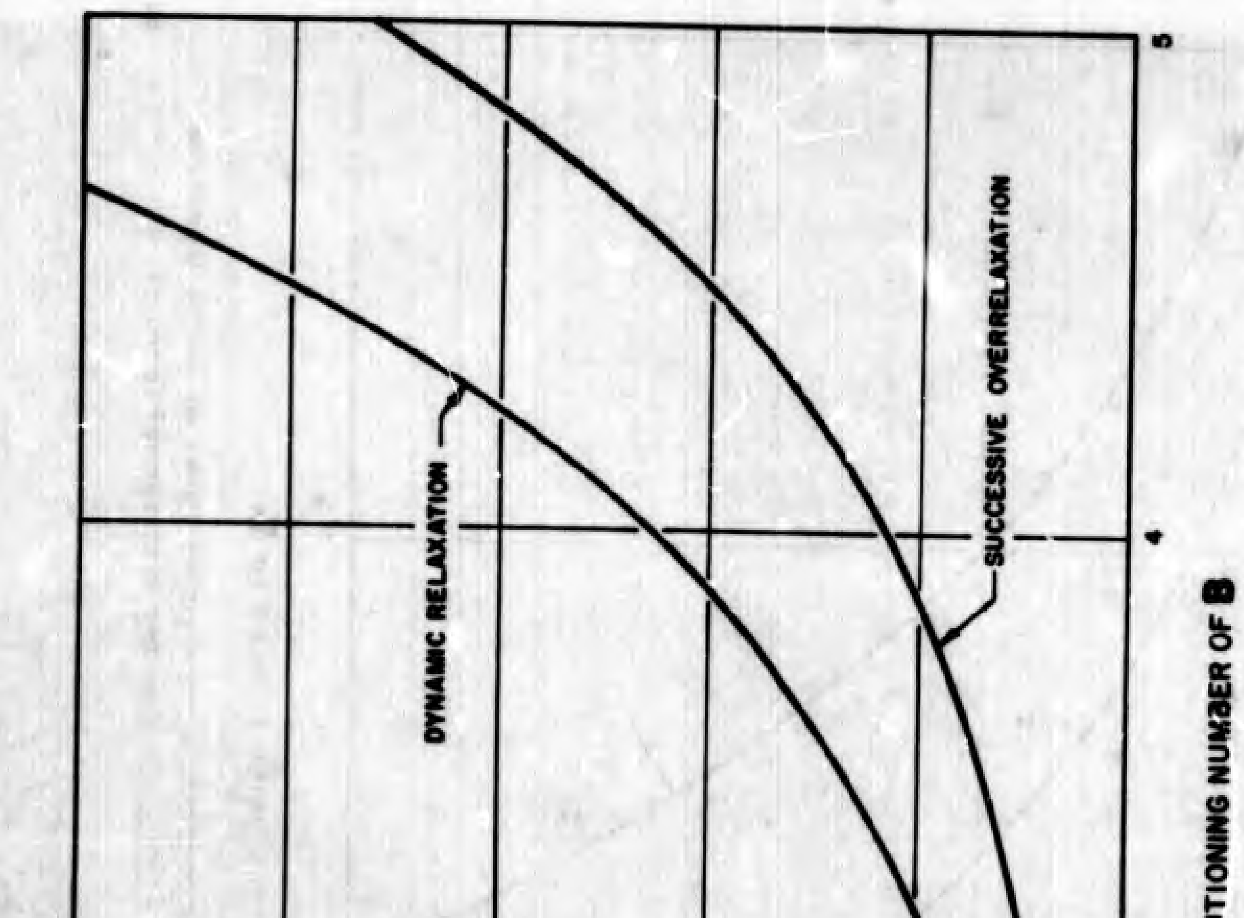


Fig. 3.4 - Comparison of cycles required for a given error reduction by Basic Iterative Methods with those required by Dynamic Relaxation for tridiagonal form of coefficient matrix

might be introduced into the Dynamic Relaxation iteration. Referring to Figure 3.2 consider what happens to the process when the parameter h^2/ρ is chosen so that $\lambda_{B max} h^2/\rho > 4.0$. This means that the associated $|\lambda_{DR}|$ is greater than one. Equation (2.17) then shows that the magnitude of at least one error mode is being increased each cycle. The number of cycles required for a manifestation of an instability in the process will depend on the modulus of the λ_{DR} associated with $\lambda_{B max}$, how many other modes have moduli greater than one and the initial magnitudes of those error modes.

Klein and Sylvester (15) use a method for integrating the equation of motion developed by Chan, Cox and Benfield (16) which guarantees stability regardless of the interval of integration. In view of the previous discussion the method proposed seems attractive.

Applying the recurrence relationship given by Chan et al. to equation (3.3) results in the following equation

$$D r^{k+1} = P r^{k} - F r^{k-1} + h^2 R \tag{3.19}$$

where $$D = M + \frac{h}{2} C + b h^2 K$$

$$P = 2M - (1-2b) h^2 K$$

$$F = M - \frac{h}{2} C + b h^2 K$$

and b is a parameter of generalized acceleration which for infinite stability should equal 1/4.

Unfortunately this method with b equal to other than zero requires the inversion of K in order to evaluate the vector r^{k+1}. This is the same problem encountered if M and C are both taken to be proportional to K. With b equal to zero (3. 19) reduces to (3. 7). Nevertheless it is informative to analyze the iterative process given by equation (3. 19).

Subtracting the true solution from equation (3. 19) gives the eigenvalue problem in terms of the error vector for the proposed technique.

$$\left[\lambda_{DR}^2 D - \lambda_{DR} P + R\right]\epsilon = 0 \tag{3.20}$$

After substitution and simplification (3. 20) becomes

$$\left[\frac{\lambda_{DR}^2 - \beta\lambda_{DR} + \alpha}{\gamma/4\,(\lambda_{DR}+1)^2}\right] I + B = 0 \tag{3.21}$$

Once again using equation (2. 22), which is

$$\left[\lambda_B I - B\right]\epsilon = 0$$

in order to establish a relationship between λ_{DR} and λ_B gives the following equation

$$\lambda_{DR}^2 - \beta\lambda_{DR} + \alpha = \frac{-\gamma\lambda_B}{4}\left(\lambda_{DR}+1\right)^2$$

which may be rewritten as

$$\lambda_{DR}^2 + \left(\frac{2\gamma\lambda_B - 4\beta}{\gamma\lambda_B + 4}\right)\lambda_{DR} + \left(\frac{\gamma\lambda_B + 4\alpha}{\gamma\lambda_B + 4}\right) = 0 \qquad (3.22)$$

Comparison with the quadratic form given by (3.12) shows that

$$a = \frac{2\beta - \gamma\lambda_B}{4 + \gamma\lambda_B} \quad \text{and} \quad b = \sqrt{\frac{\gamma\lambda_B + 4\alpha}{\gamma\lambda_B + 4} - \left(\frac{2\beta - \gamma\lambda_B}{4 + \gamma\lambda_B}\right)^2}$$

The roots of equation (3.22) will be:

(1) complex conjugate if $\dfrac{\gamma\lambda_B + 4\alpha}{\gamma\lambda_B + 4} > \left(\dfrac{2\beta - \gamma\lambda_B}{4 + \gamma\lambda_B}\right)^2$

(2) two equal real roots if $\dfrac{\gamma\lambda_B + 4\alpha}{\gamma\lambda_B + 4} = \left(\dfrac{2\beta - \gamma\lambda_B}{4 + \gamma\lambda_B}\right)^2$

(3) two unequal real roots if $\dfrac{\gamma\lambda_B + 4\alpha}{\gamma\lambda_B + 4} < \left(\dfrac{2\beta - \gamma\lambda_B}{4 + \gamma\lambda_B}\right)^2$

With complex conjugate roots the modulus of λ_{DR} is

$$|\lambda_{DR}| = \sqrt{\frac{2 - \frac{Ch}{\rho} + \frac{\lambda_B h^2}{2\rho}}{2 + \frac{Ch}{\rho} + \frac{\lambda_B h^2}{2\rho}}} \qquad (3.23)$$

The roots will be real and equal if

$$\frac{\gamma\lambda_B + 4\alpha}{\gamma\lambda_B + 4} = \left(\frac{2\beta - \gamma\lambda_B}{4 + \gamma\lambda_B}\right)^2$$

which after the appropriate substitutions and simplification becomes

$$\frac{C^2h^2}{\rho^2} = \frac{4\lambda_B h^2}{\rho} \tag{3.24}$$

If the roots are real and unequal the modulus of the larger is given by

$$|\lambda_{DR}| = \left|\frac{2 - \frac{\lambda_B h^2}{2\rho}}{2 + \frac{Ch}{\rho} + \frac{\lambda_B h^2}{2\rho}}\right| + \sqrt{\frac{\frac{C^2h^2}{\rho^2} - 4\frac{\lambda_B h^2}{\rho}}{\left(2 + \frac{Ch}{\rho}\right)^2 + \frac{\lambda_B h^2}{\rho}\left(2 + \frac{Ch}{\rho} + \frac{\lambda_B h^2}{4\rho}\right)}} \tag{3.25}$$

It is interesting to compare equations (3.23), (3.24) and (3.25) with (3.13), (3.14), and (3.15) respectively.

Figure 3.5 is a plot of the largest modulus of λ_{DR} versus $\lambda_B h^2/\rho$ for constant values of Ch/ρ. From Figure 3.5 or equations (3.23) and (3.25) and the fact that B is a positive definite matrix it can be seen that this method of integration of the equation of motion does indeed provide infinite stability for the iterative procedure regardless of the value of the parameter h^2/ρ selected. Of course this analysis does not take into account noise which might become a problem when the modulus of approaches one.

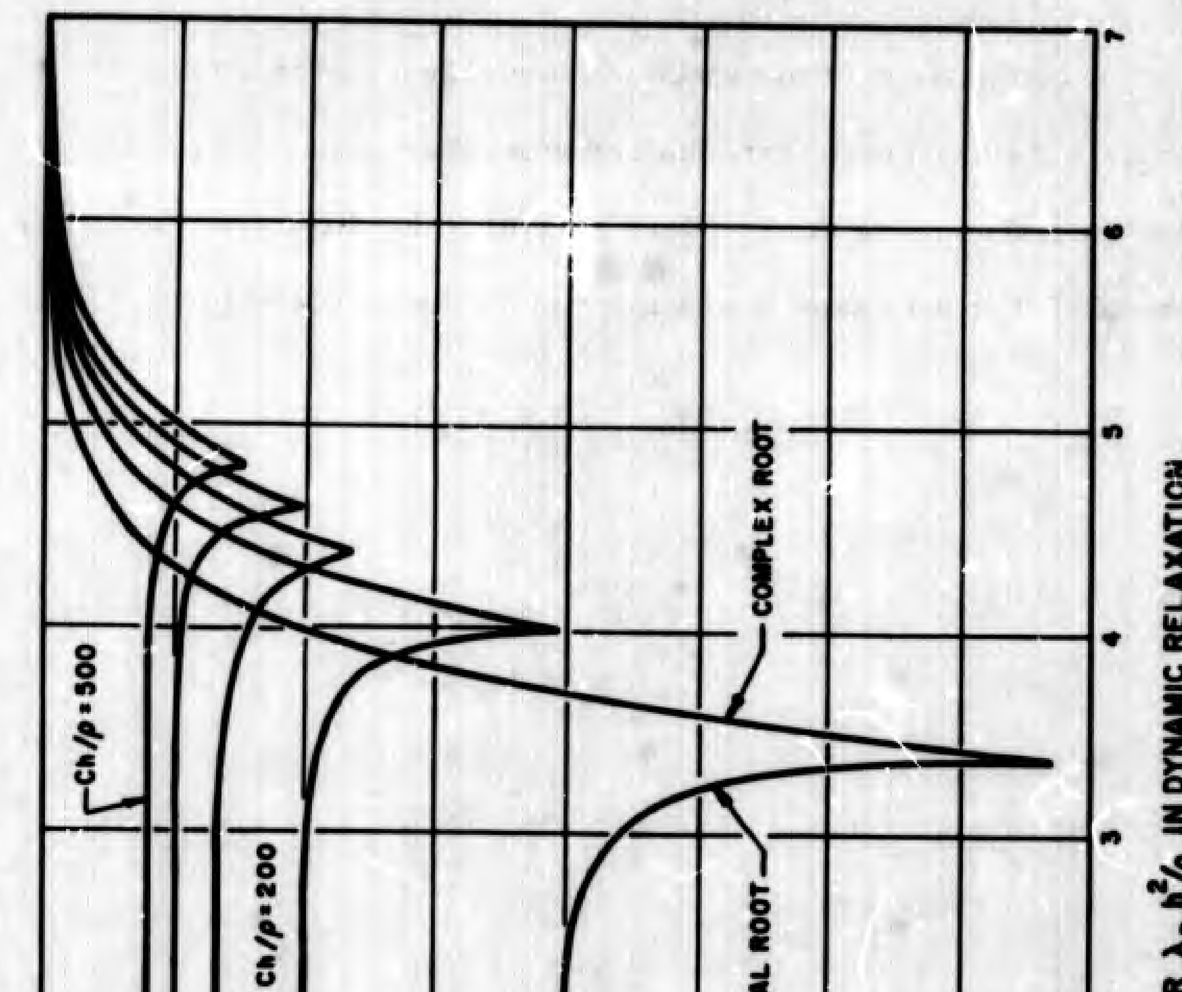


Fig. 3.5 - Relation between asymptotic convergence and $\lambda_B h^2/\rho$ in infinitely stable form of Dynamic Relaxation

A comparison of the optimum convergence of this form of Dynamic Relaxation to that obtained when the equation of motion was integrated using the standard central finite difference technique showed that in all cases it was inferior to the standard form.

IV APPLICATION OF DYNAMIC RELAXATION TO PLANE STRESS PROBLEMS USING FINITE ELEMENTS

4-1 Introduction

As developed in Chapter III the optimization of the Dynamic Relaxation iterative process for solving linear equations depends on the maximum and minimum eigenvalues of $\mathbf{B}$. If these are known precisely then it is no problem to choose the parameters h^2/ρ and Ch/ρ for an optimal solution, which from Chapter III are given by

$$\left(\frac{h^2}{\rho}\right)_{opt.} = \frac{4}{\lambda_{B\,min} + \lambda_{B\,max}} \qquad (4.1)$$

and

$$\left(\frac{Ch}{\rho}\right)_{opt.} = \frac{4\sqrt{\lambda_{B\,min}\,\lambda_{B\,max}}}{\lambda_{B\,min} + \lambda_{B\,max}} \qquad (4.2)$$

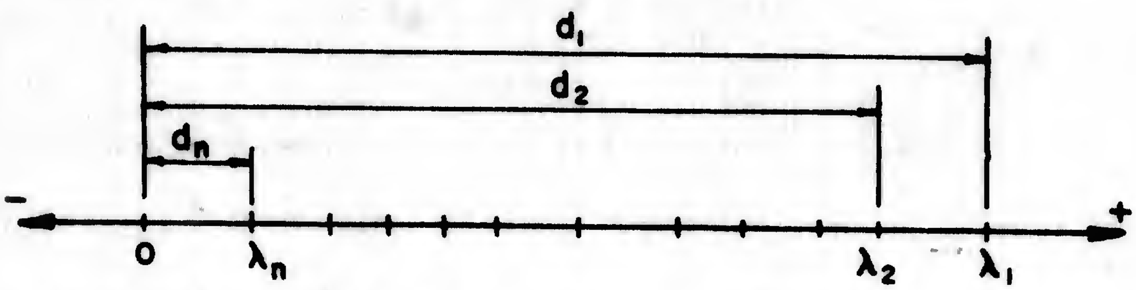
The following discussion is devoted to developing a strategy for achieving a near optimal solution by estimating $\lambda_{B\,min}$ and $\lambda_{B\,max}$ before the iteration process is begun and then, if neccessary, adjusting the parameters as the process continues.

4-2 Estimation of the Maximum Eigenvalue of $\mathbf{B}$

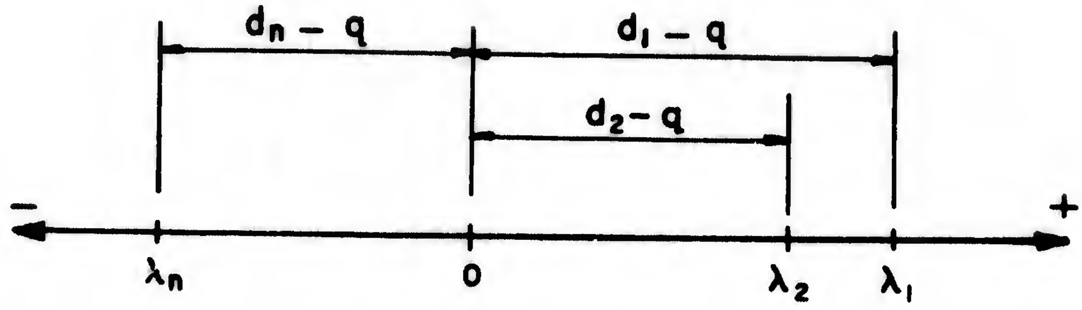
The method adopted for estimating the maximum eigenvalue of $\mathbf{B}$ is the well known technique of direct iteration described by Fox (13). The details of this technique are also given by Allen (17) who refers to this approach as the Intensification Method.

The use of this approach provided a simple way of obtaining a reliable estimate of the maximum eigenvalue of $\mathbf{B}$. Since the convergence of the iteration to the dominant eigenvalue depends on the ratio of the sub-dominant eigenvalue to the dominant one a shift in the eigenvalues of $\mathbf{B}$ can decrease this ratio and thus accelerate the convergence to the dominant eigenvalue. This procedure can be illustrated schematically as follows.

If λ_1, λ_2 --------- λ_n are the eigenvalues of $\mathbf{B}$ (a positive definite matrix) arranged in order of decreasing magnitude they can be shown in a one dimensional plot.



This range of eigenvalues can be shifted a distance q , so they are now represented in the following plot.



The shift has decreased the ratio λ_2 / λ_1, thus accelerating the convergence of the iteration to the dominant eigenvalue. Of course care must be exercised to insure that the magnitude of the shift does not make λ_n the dominant eigenvalue. In most plane stress problems the magnitude of the smallest eigenvalue of $\mathbf{B}$ can be estimated closely enough to avoid this difficulty.

The shifting of the eigenvalues of $\mathbf{B}$ can be stated in a more formal way as follows, if

$$\mathbf{B}\mathbf{X} = \lambda \mathbf{X}$$

then
$$(\mathbf{B} - q\,\mathbf{I})\mathbf{X} = (\lambda - q)\mathbf{X}$$

and $\lambda - q$ is an eigenvalue of $\mathbf{B} - q\,\mathbf{I}$. The magnitude of q is chosen to decrease the ratio of the sub-dominant eigenvalue to the dominant one, the dominant eigenvalue of the new matrix is determined by the direct iteration procedure. This eigenvalue plus the magnitude of the shift then gives the dominant eigenvalue of $\mathbf{B}$.

Employing the shift decreased the number of iterations required to determine the maximum eigenvalue of $\mathbf{B}$ by 10 per cent at worst to 50 per cent at best.

The Rayleigh Quotient was used to produce a better final estimate of λ, that is

$$\lambda = \frac{X_{k+1}^T X_k}{X_k^T X_k}.$$

This approach gave estimates which were just a few percent less than the maximum eigenvalue. The use of Aitken's formula described by Fox (13) showed no consistency in either appreciably improving or accelerating the convergence to the maximum eigenvalue in the class of problems which were studied, namely plane stress problems using triangular finite elements.

It is of interest to note that in this particular class of finite element problem the maximum eigenvalue of $\mathbf{B}$ was found to be rather consistently in the neighborhood of 2.8.

4-3 Estimation of the Minimum Eigenvalue of $\mathbf{B}$

Initially the feasibility of estimating the minimum eigenvalue of $\mathbf{B}$ from the analysis of the deflection response of the system with zero damping was examined. As noted in Chapter I this is the approach suggested by Cassell and Otter. With the damping set equal to zero the eigenvalues of the iterative method occur in complex conjugate pairs. The parameters used in developing the Dynamic Relaxation method in Chapter III then become

$$a = \frac{2 - \frac{Ch}{\rho}}{2 + \frac{Ch}{\rho}} = 1$$

$$\beta = a + 1 = 2$$

$$\gamma = \frac{\frac{2h^2}{\rho}}{2 + \frac{Ch}{\rho}} = \frac{h^2}{\rho}$$

From equation (3. 13) the modulus is then equal to one. Using equation (3. 12) the eigenvalues are

$$\lambda_{DR} = \left(1 - \frac{\lambda_B h^2}{\rho}\right) \pm i\sqrt{1 - \left(1 - \frac{\lambda_B h^2}{2\rho}\right)^2} \tag{4. 3}$$

or $\lambda_{DR} = \cos\theta \pm i \sin\theta$

Setting $\cos\theta = \cos\omega_{min} h = 1 - \frac{\lambda_{B\,min} h^2}{2\rho}$ (4. 4)

the problem then becomes one of evaluating the fundamental frequency of the system, from which the minimum eigenvalue of can be determined using equation (4. 4).

In order to evaluate the fundamental frequency, the system was run with zero damping until a periodic deflection response was observed. No simple periodic form was observed. The use of the first cycle of the response, which was definitely non-periodic, lead to estimates of the minimum eigenvalue of **B** which were well over twice the magnitude of the actual value. When the iterative process was run long enough, sometimes for as many as three or four hundred cycles, to develop a periodic response, a Fourier analysis had to be employed in order to obtain an estimate of the minimum eigenvalue. The harmonic analysis using the Fourier Coefficient method did, however, give very good estimates of the minimum eigenvalue of **B** in the problems where it was employed.

This technique was abandoned because of the uncertainty involved in knowing _a priori_ the required running time with zero damping in order to develop a periodic deflection response. As mentioned previously in some of the cases where it was employed the number of iterations needed to develop a periodic response exceeded the number of iterations required for the optimum solution by a factor of two or more.

The simplest and most direct way of estimating the minimum eigenvalue proved to be, once again, the direct iteration method, the same one used to estimate the maximum eigenvalue. In this case, however, it was applied after the eigenvalues of **B** had been shifted enough to make the minimum eigenvalue of **B** the dominant eigenvalue of the new matrix generated from **B** . In terms of the discussion of this method in Section 4-2, the magnitude of the shift, q , found to be most efficient for estimating the minimum eigenvalue of B in the least number of cycles was 55 per cent of the maximum eigenvalue of B .

This technique gave estimates which were approximately twice the actual minimum eigenvalue of B .

4-4 Estimation of the Minimum Eigenvalue of B using the Dynamic Relaxation Iteration

After establishing $\lambda_{B\,max}$, if the iteration parameters h^2/ρ and Ch/ρ are so chosen that the λ_{DR} associated with $\lambda_{B\,min}$ is real and given by equation (3. 15) then this value of is the dominant eigenvalue of the iteration process and can be estimated as follows.

Equation (2. 17) which is

$$\epsilon^{k+1} = M\epsilon^{k}$$

may be rewritten as

$$x^{k+1} - x = M(x^{k} - x) \tag{4. 5}$$

or similarly as

$$x^{k} - x = M(x^{k-1} - x) \tag{4. 6}$$

Subtracting (4. 6) from (4. 5) gives

$$x^{k+1} - x = M(x^{k} - x^{k-1}) \tag{4. 7}$$

This is the same eigenvalue problem as that given by equation (2.18) except that it is in terms of successive correction vectors instead of error vectors. A series of approximations to the dominant λ_{DR} can be obtained by calculating

$$\lambda_{DR} = \frac{|X^{k+1} - X^{k}|}{|X^{k} - X^{k-1}|} \tag{4.8}$$

When this quantity has converged to a relatively constant value it can be used in the following equation developed in Chapter III to estimate $\lambda_{B\,min}$.

$$-\lambda_{B} = \frac{\lambda_{DR}^{2} - \lambda_{DR}\beta + \alpha}{\lambda_{DR}\gamma}$$

This estimate of $\lambda_{B\,min}$ can then be used with equations (4.1) and (4.2) to select the optimum iteration parameters.

The estimates of the minimum eigenvalue of $\mathbf{B}$ using this approach were no better and were achieved in about the same number of cycles as those obtained using the technique described in Section 4-3.

4-5 Solution Procedure Using Dynamic Relaxation

Since the method used for estimating the eigenvalues of $\mathbf{B}$ produced much more exact estimates of the maximum eigenvalue than of the minimum one the following procedure for approaching the optimum solution in terms of the number of iterative steps required was evolved.

The eigenvalues of $\mathbf{B}$ are estimated as described previously. Because the estimate of the maximum eigenvalue is slightly low, it is increased three per cent to make certain that the estimate is greater than the actual value. Referring to Figure 4.1 the reason for this adjustment in the maximum eigenvalue of $\mathbf{B}$ is clear. For optimum convergence, as pointed out in Chapter III, it is desirable to select h^2/ρ so that the parameter $\lambda_{B\,max} h^2/\rho$ makes the associated roots of the iterative process real and equal. From Figure 4.1 it can be seen that when estimating $\lambda_{B\,max}$ to approach the optimum, it is desirable from the viewpoint of convergence and stability of the process that $\lambda_{B\,max}$ be slightly overestimated. This insures that the actual maximum eigenvalue times h^2/ρ will in fact be less than 4.0 for stability, and that the associated root, although now complex. will be very close in magnitude to the optimum of two equal real roots.

For reasons of convergence only, the estimate of the minimum eigenvalue is adjusted to insure that it is smaller than the actual value. Again referring to Figure 4.1 it can be seen that from the point of view of convergence that this adjustment makes the associated root of the iterative procedure complex but close in magnitude to the optimum of two equal real roots.

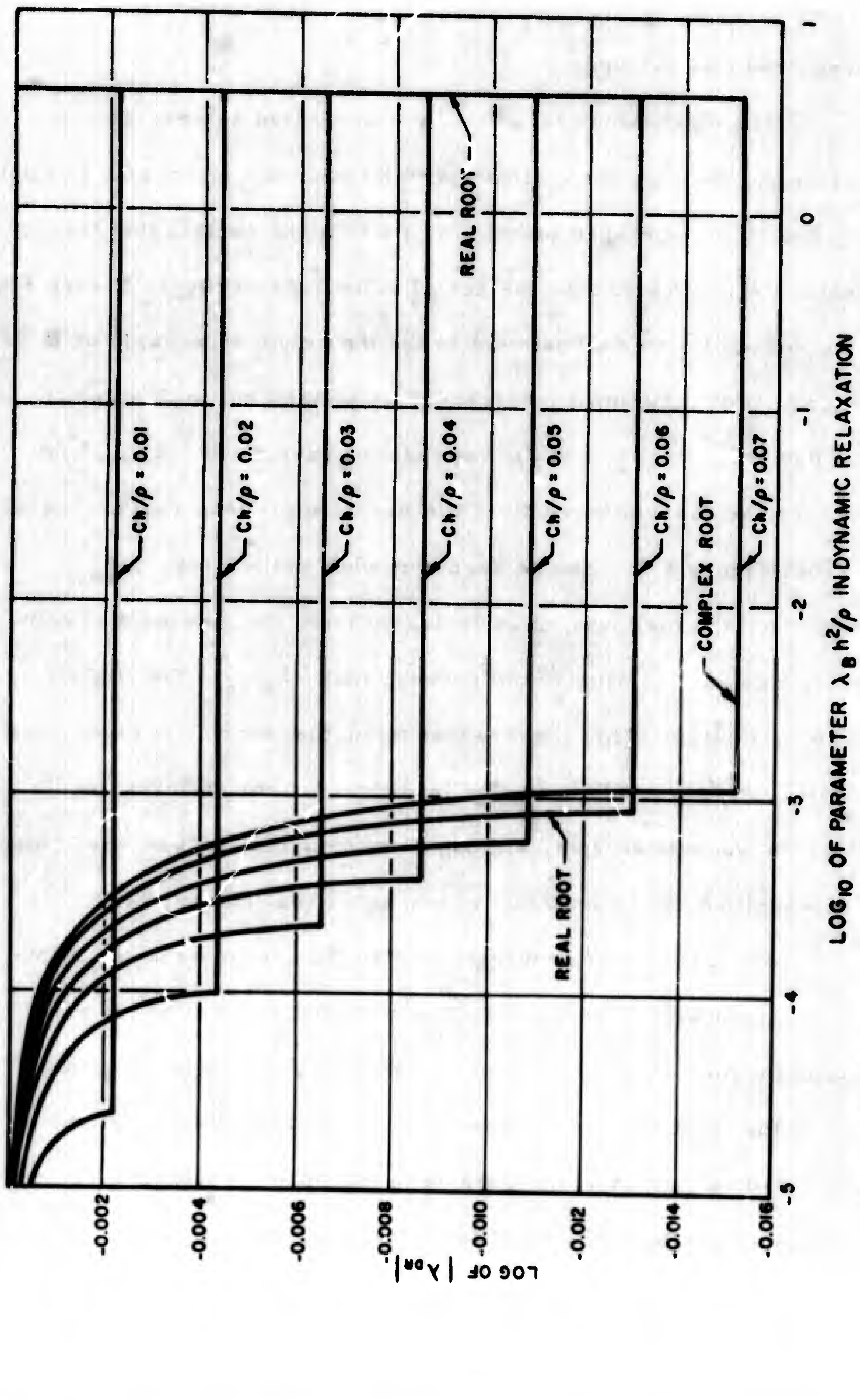


Fig. 4.1 - Relation between asymptotic convergence and $\lambda_B h^2/\rho$ in Dynamic Relaxation

Since B is a positive definite matrix there is no stability consideration in adjusting the estimate of the minimum eigenvalue, that is, no matter how small h^2/ρ is, the root associated with $\lambda_{B\,min} h^2/\rho$ will never be greater than one in modulus.

Using these adjusted extreme eigenvalue estimates the optimum parameters h^2/ρ and Ch/ρ are computed using equations (4.1) and (4.2). This places the procedure in its starting position as shown schematically in Figure 4.2.

The iteration process is now begun and as it progresses simultaneous adjustments are made in the parameters h^2/ρ and Ch/ρ. These adjustments are made so that the process approaches the final position shown in Figure 4.2 where $\lambda_{B\,min} h^2/\rho$ is so positioned that the associated $|\lambda_{DR}|$ has just become real and equal.

4-6 Adjustment of Dynamic Relaxation During the Iteration Process

The need for adjustment of the Dynamic Relaxation process during the solution depends on the detection that the process is indeed at some position between the starting and final positions as shown in Figure 4.2. One way of accomplishing this is to evaluate the modulus of the complex λ_{DR}. If it agrees with the value predicted by equation (3.13), that is

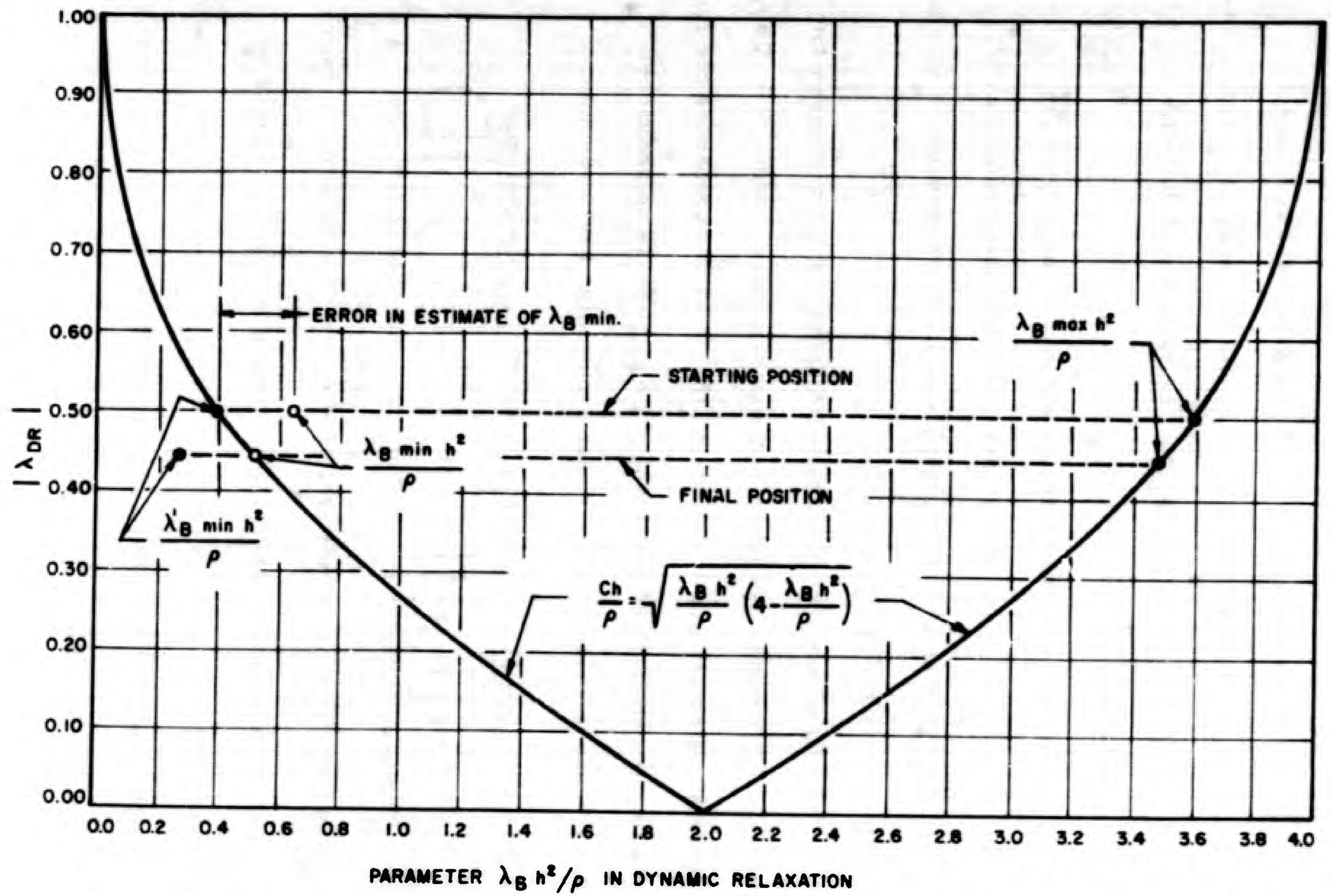


Fig. 4.2 - Schematic representation of strategy for approaching optimal convergence of Dynamic Relaxation

$$|\lambda_{DR}| = \sqrt{\frac{2 - \frac{Ch}{\rho}}{2 + \frac{Ch}{\rho}}}$$

it is possible that the parameters h^2/ρ and Ch/ρ can be adjusted to improve convergence. This cycle of computing the modulus of λ_{DR} and then adjusting the parameters can then be repeated until a position close to or just beyond the final position in Figure 4.2 is achieved.

A method for evaluating the modulus of the complex λ_{DR} was developed using a recursive form of relationship among three successive error vectors. Since the error vector is not available during the iterative procedure the correction vector was used. This effort met with no success, possibly because none of the eigenvalues were dominant.

In a search for some alternative technique, the behavior of several solution processes was studied in order to ascertain whether or not there was some other consistent indicator that could be used to detect the fact that the parameters h^2/ρ and Ch/ρ could be adjusted to improve convergence during the iteration process.

For the plane stress problems studied there appears to be what may prove to be a reasonably dependable indicator. This technique for detecting the fact that an improvement in convergence is possible is best illustrated by an example.

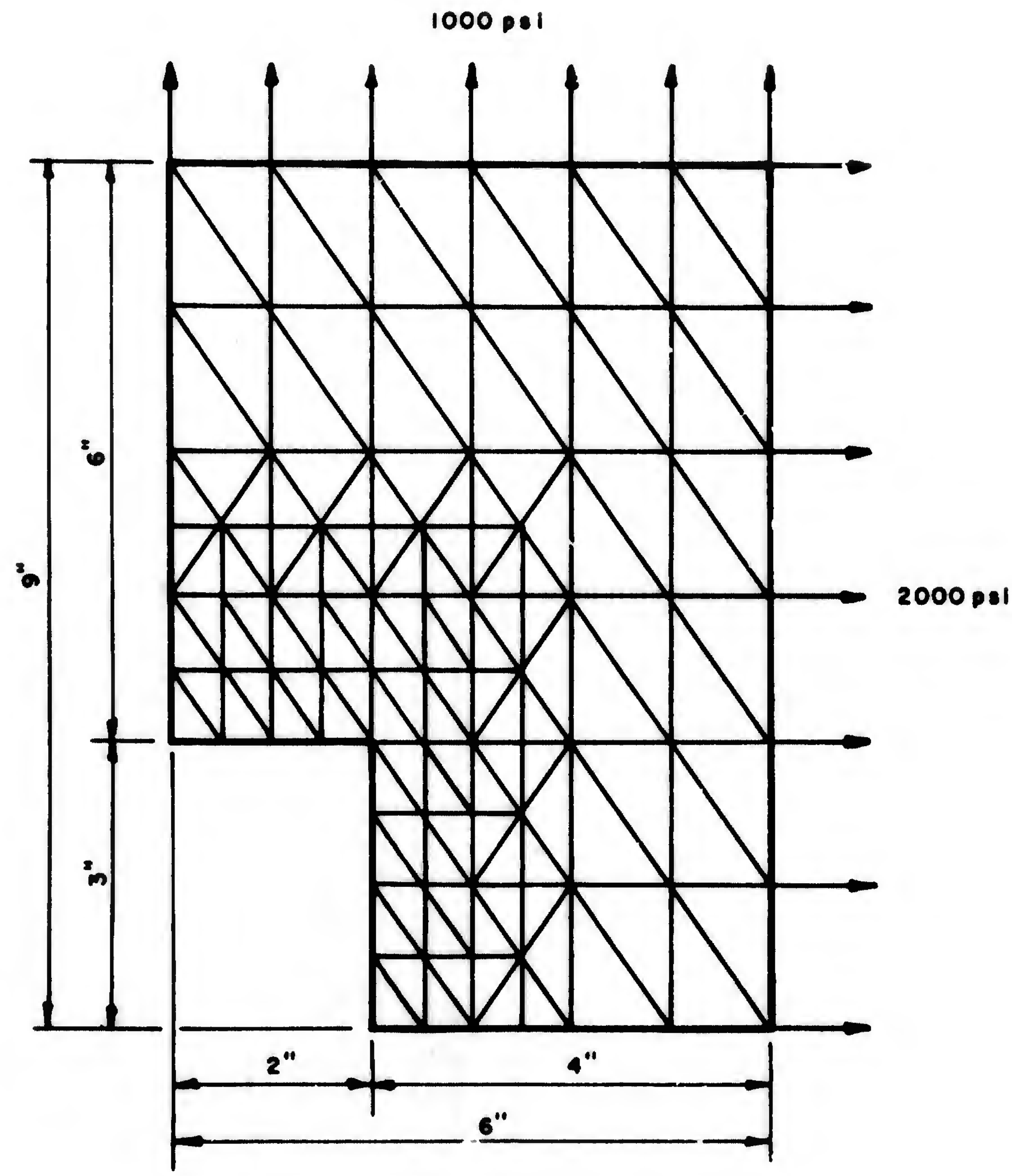
A plate with a square hole shown in Figure 4.3 was analyzed under the loading shown. Plots of certain vector component norms during the iterative process were made for the solution process using three different estimates of the minimum eigenvalue of $\mathbf{B}$, or essentially, three different estimates of the conditioning number. These estimates of the conditioning number were: (1) 800, where the $|\lambda_{DR}|$ associated with $\lambda_{B\,min} h^2/\rho$ was real and considerably larger than all of the other values; (2) 1050, where $|\lambda_{DR}|$ had just become real with a slightly larger modulus than the others, and; (3) 1500, where all the λ_{DR}'s were complex and equal in modulus. The norms plotted were

$$\sum_{i=1}^{n} |r_x|+|r_y| \qquad \text{in Figure 4.4}$$

$$\sum_{i=1}^{n} |r_x| \qquad \text{in Figure 4.5}$$

$$\sum_{i=1}^{n} |r_y| \qquad \text{in Figure 4.6}$$

The characteristic of the optimum solution (estimated conditioning number of 1050) which proved to be typical of all the



QUARTER PLATE

FINITE ELEMENT MODEL
NUMBER OF NODES = 81
NUMBER OF ELEMENTS = 128

Figure 4.3

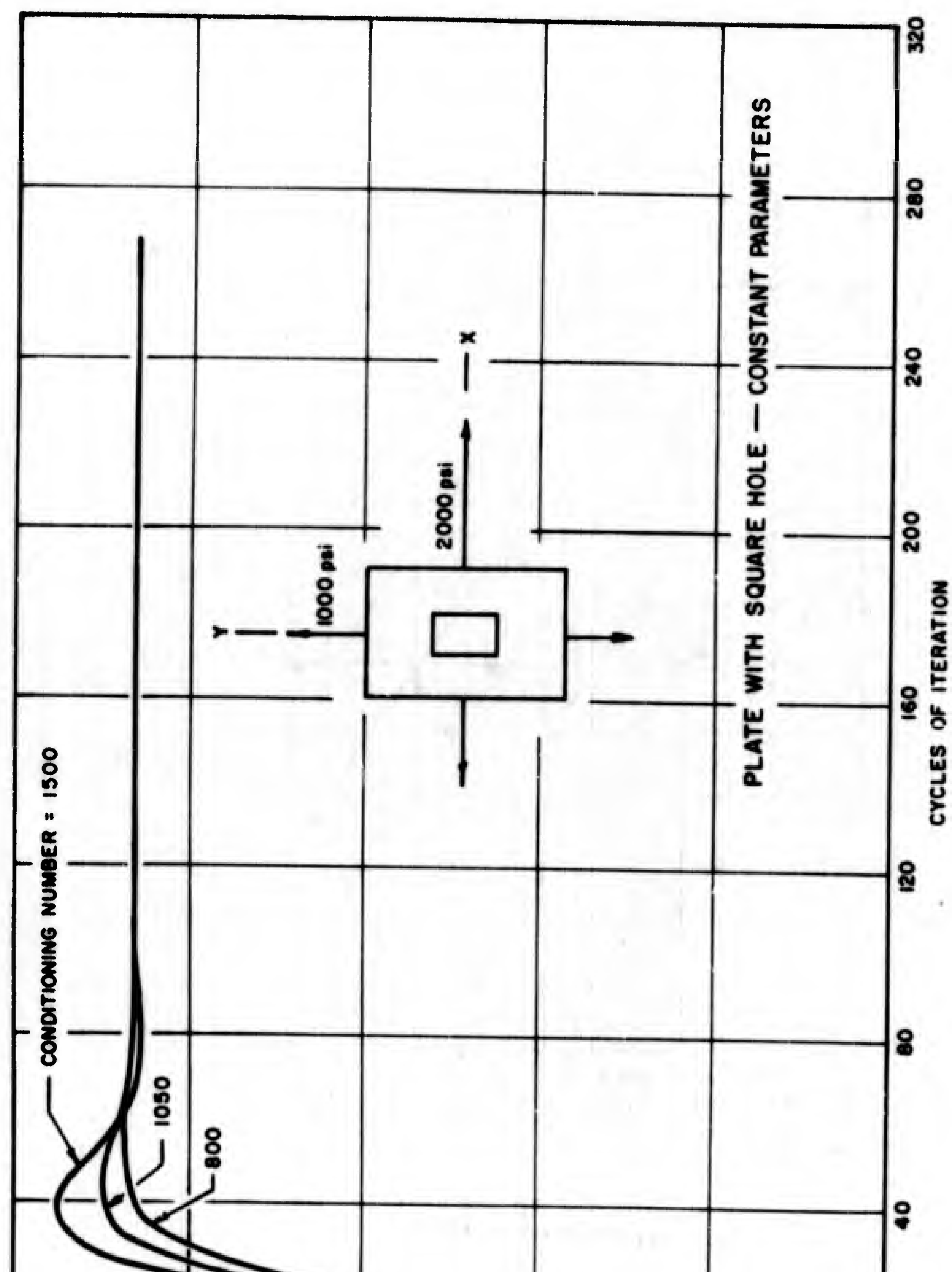


Fig. 4.4 - Total deflection norm variation during Dynamic Relaxation solution with constant parameters

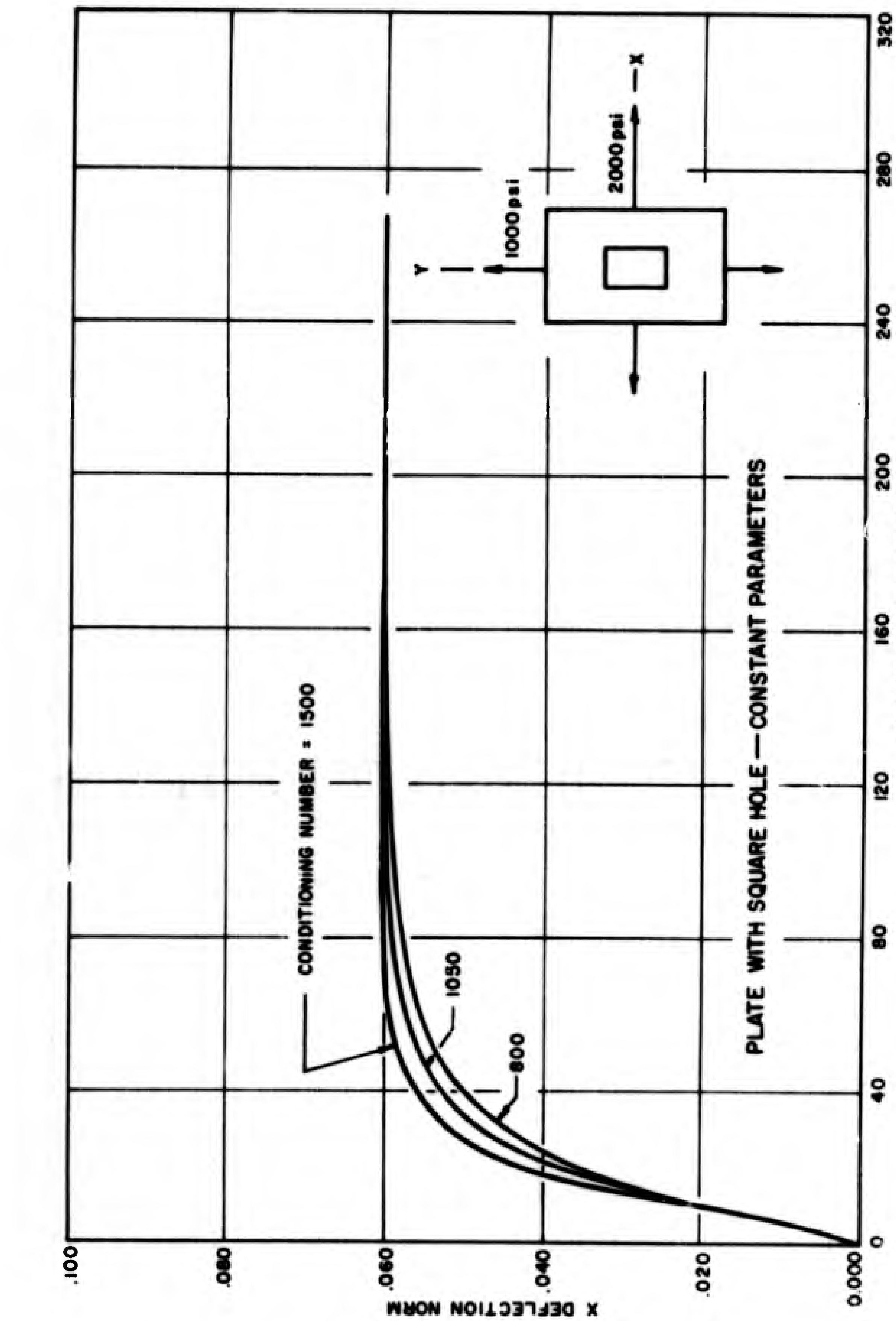


Fig. 4.5 - X deflection norm variation during Dynamic Relaxation solution with constant parameters

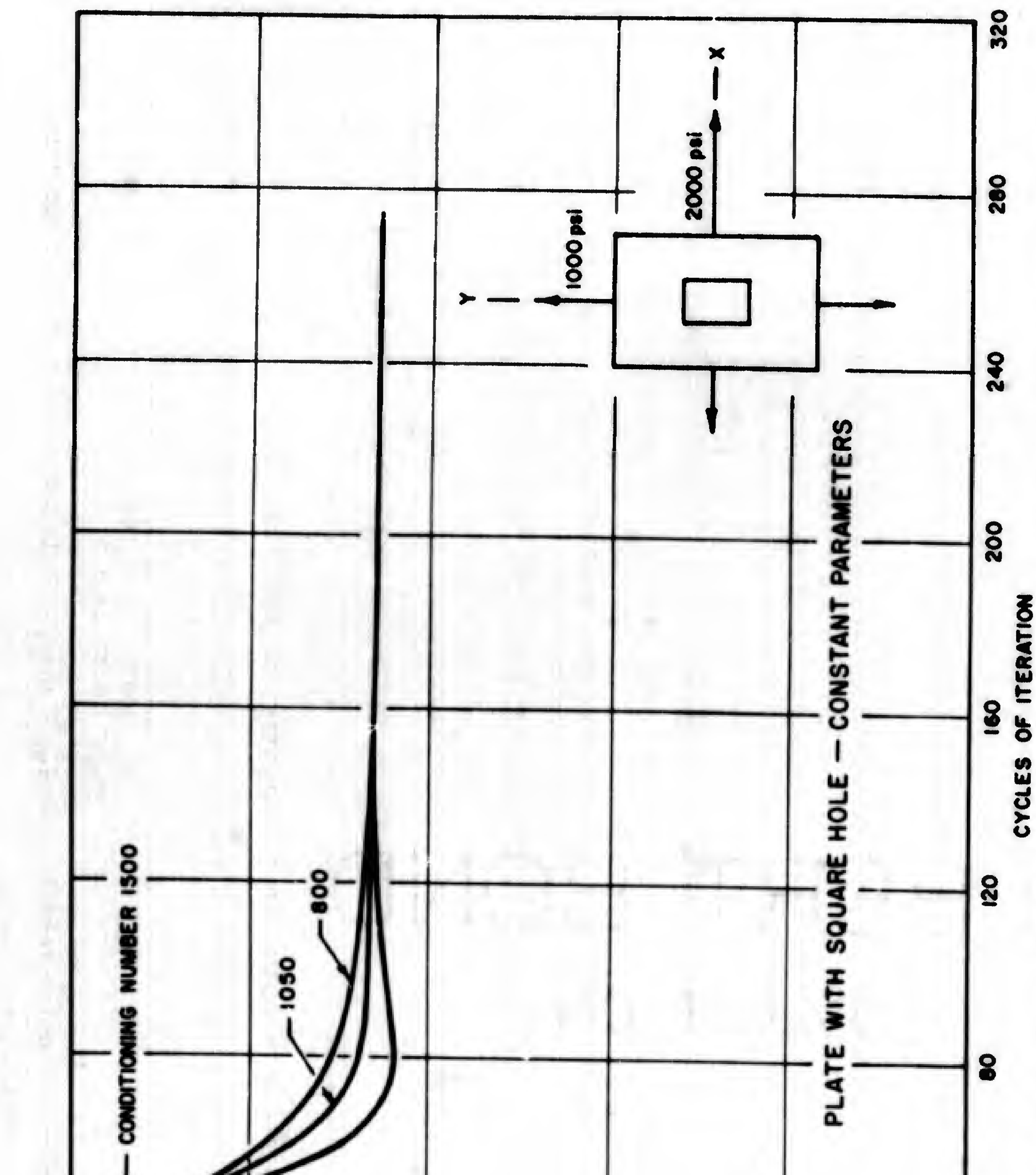


Fig. 4.6 - Y deflection norm variation during Dynamic Relaxation solution with constant parameters

problems analyzed in this study is that the larger of the two displacement component norms, in this case $\sum_{i=1}^{n} |r_x|$ behaved very much like the deflection response of a critically damped, one degree of freedom, spring-mass-damper system. From Figure 4.5 it can be seen that for a conditioning number of 800, the norm behavior was similar to an overdamped response and for a conditioning number of 1500, that of an underdamped response. Of course it is quite clear that this is a rather pragmatic approach, but it is not without some rationale, based on the mathematical development of Dynamic Relaxation in Chapter III.

If this approach is used, then, after the iterative process is begun, if the larger of the two norms exhibits the underdamped characteristic, an adjustment in the parameters is made. This kind of behavior can quite readily be detected by checking the slope of the deflection vector norm for changes during the iteration.

The amount of adjustment to be made in the parameters h^2/ρ and Ch/ρ once the need for it has been determined, is largely dictated by how accurate the estimate of $\lambda_{B\,min}$ is considered to be. In this study the adjustment was made in the following manner.

Let $K_1 = \frac{-M}{\log_{10}\lambda_1}$ and $K_2 = \frac{-M}{\log_{10}\lambda_2}$

where K_1 and K_2 are the estimated number of cycles to reduce the initial error by 10^{-M} using the corresponding λ_1 or λ_2, that is

$$\epsilon_k = \lambda^k \epsilon_0$$

or

$$\epsilon_k = 10^{-M} \epsilon_0$$

$$\lambda^k = 10^{-M}$$

$$K = \frac{-M}{\log_{10} K}$$

Then letting $I = K_1 - K_2$ the following relationship can be derived.

$$\log_{10}\lambda_2 = \frac{M}{1 + (M/\log_{10}\lambda_1)} \qquad (4.9)$$

where λ_1 is the current value of $|\lambda_{DR}|$

λ_2 is the new value of $|\lambda_{DR}|$

I is the estimated number of iterations saved if the change is made.

M is a measure of the reduction of the original error in P iterations given by

$$\epsilon_P = 10^{-M} \epsilon_0$$

A new λ_{DR} was computed from the current one using equation (4.9) every time the change in slope of the displacement vector norm indicated underdamped behavior.

The values of **I** and **M** used in the following numerical examples were 10 and 5 respectively.

4-7 Application to Plane Stress Problems

The approach previously described was programmed and applied to three plane stress problems of flat plates with cutouts of a circular, elliptical, and filleted square shape. The finite element employed was triangular with a stiffness matrix based on assumed uniform strain given in reference (18). The **T** and **S** matrices used to define **K** are given in Appendix A. The finite element models of the flat plates employed are also shown in Appendix A.

Figures 4.7, 4.8 and 4.9 give the number of iterations required for this approach compared to solutions where the parameters were held constant throughout the process. Figures 4.10 through 4.15 illustrate the behavior of the corresponding displacement component norms during the solution.

For purposes of comparison, the edge stresses around the cutout obtained by this finite element formulation were compared to those of an infinite flat plate. Houghton and Rothwell (19) developed complex stress functions for stresses around cutouts in infinite flat plates using Muskhelishvili's method of conformal

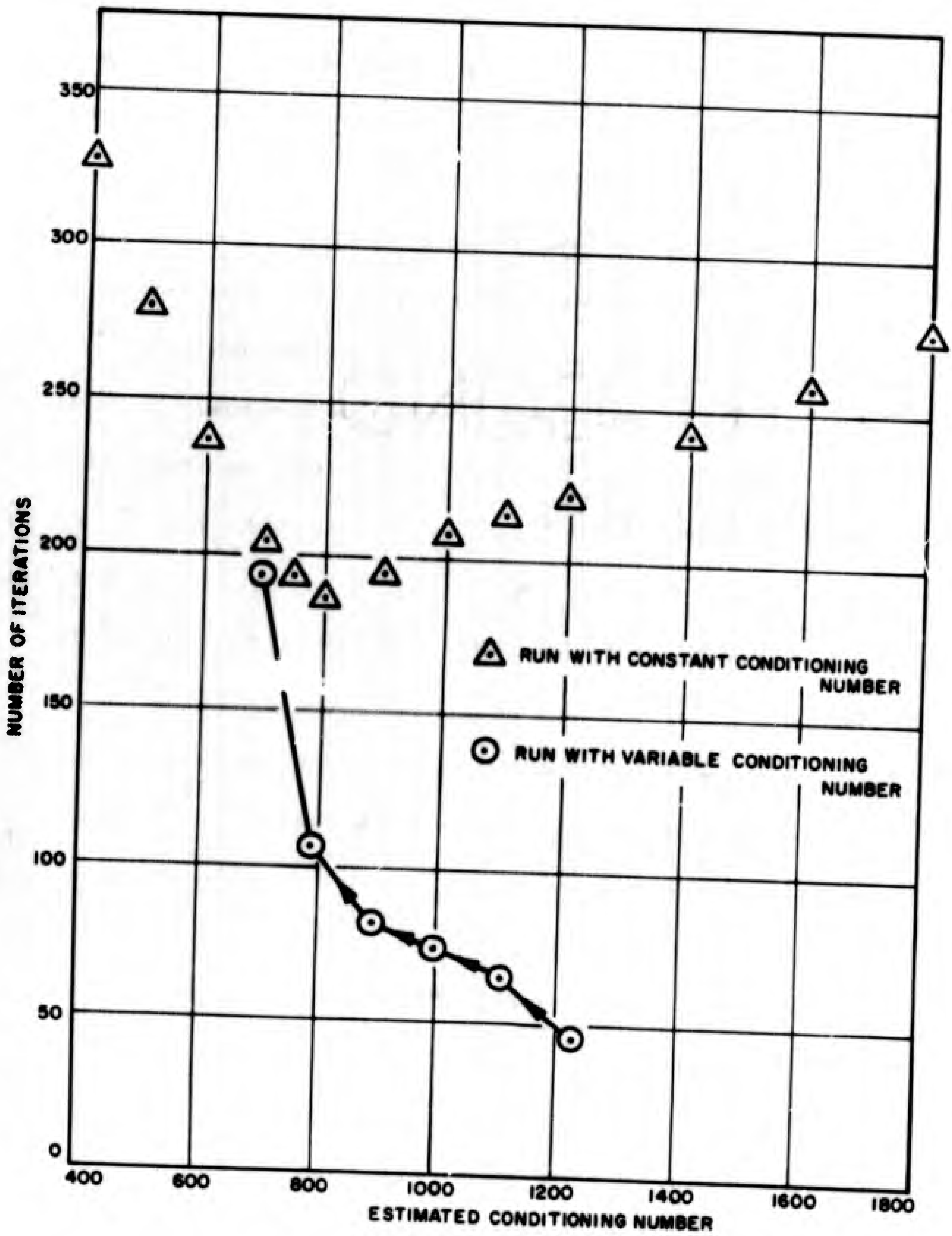


PLATE WITH CIRCULAR HOLE

Fig. 4.7 - Comparison of constant parameter solutions with variable parameter solution in Dynamic Relaxation

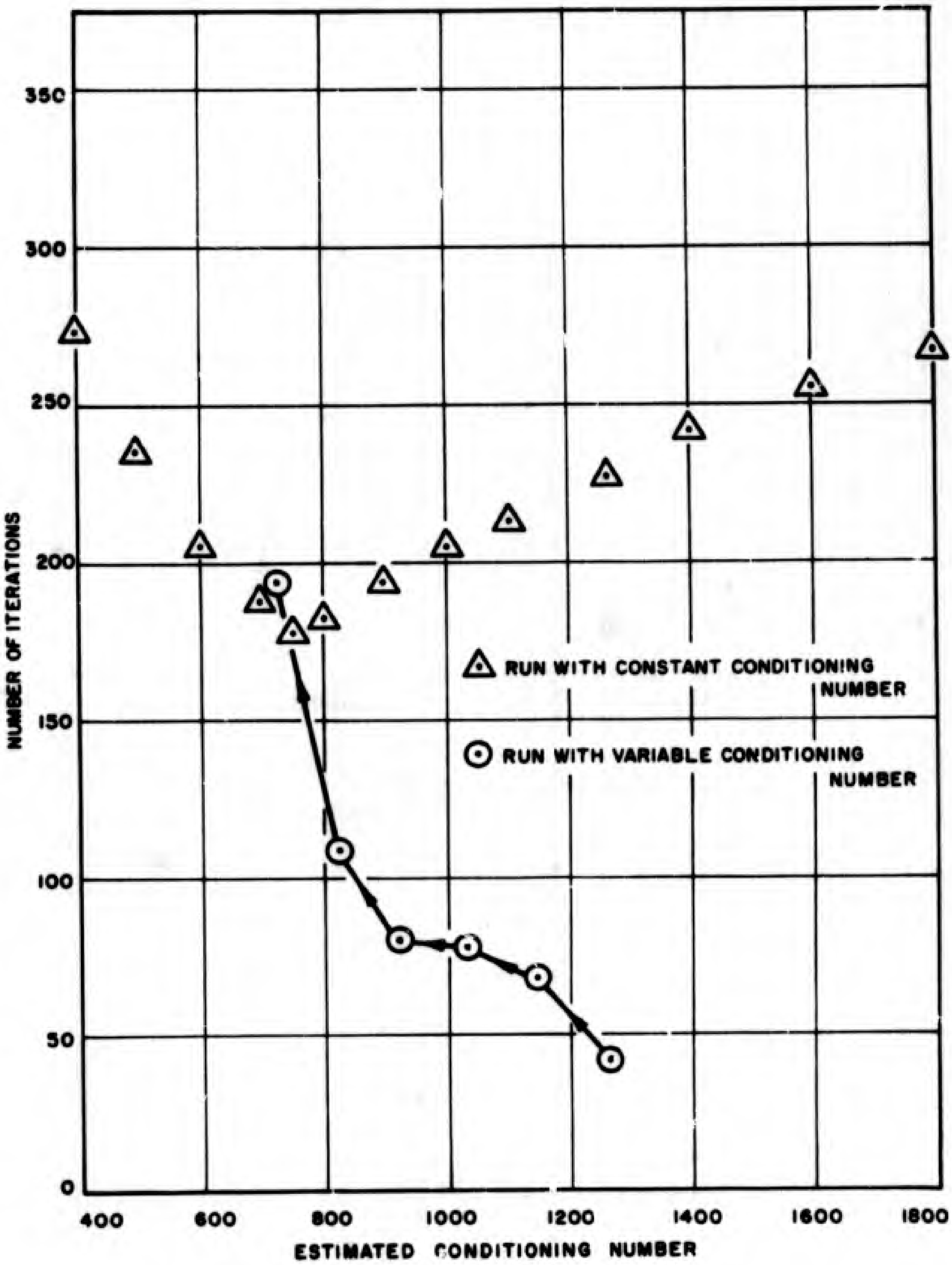


PLATE WITH ELLIPTICAL HOLE

Fig. 4.8 - Comparison of constant parameter solutions with variable parameter solution in Dynamic Relaxation

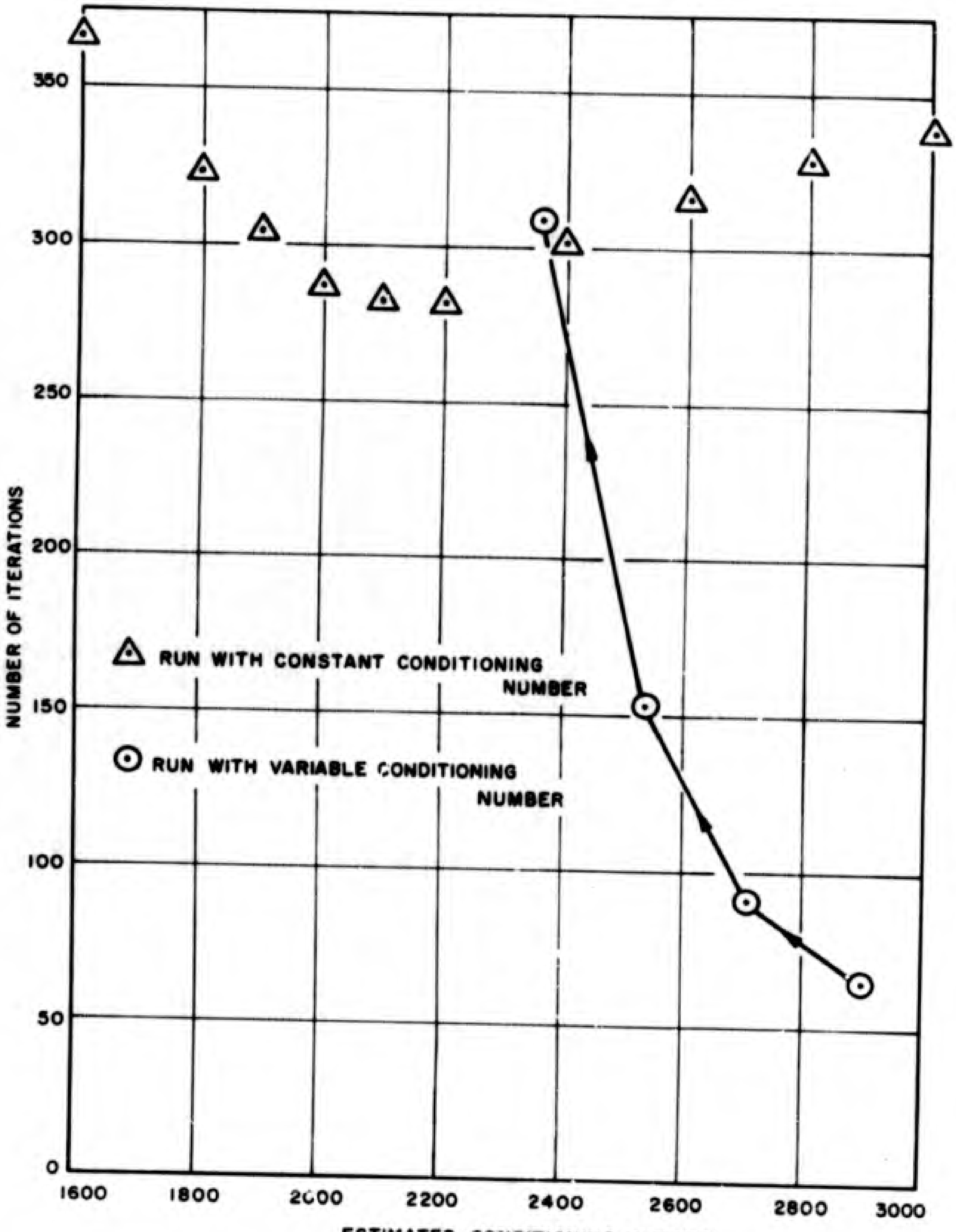


PLATE WITH SQUARE HOLE

Fig. 4.9 - Comparison of constant parameter solutions with variable parameter solution in Dynamic Relaxation

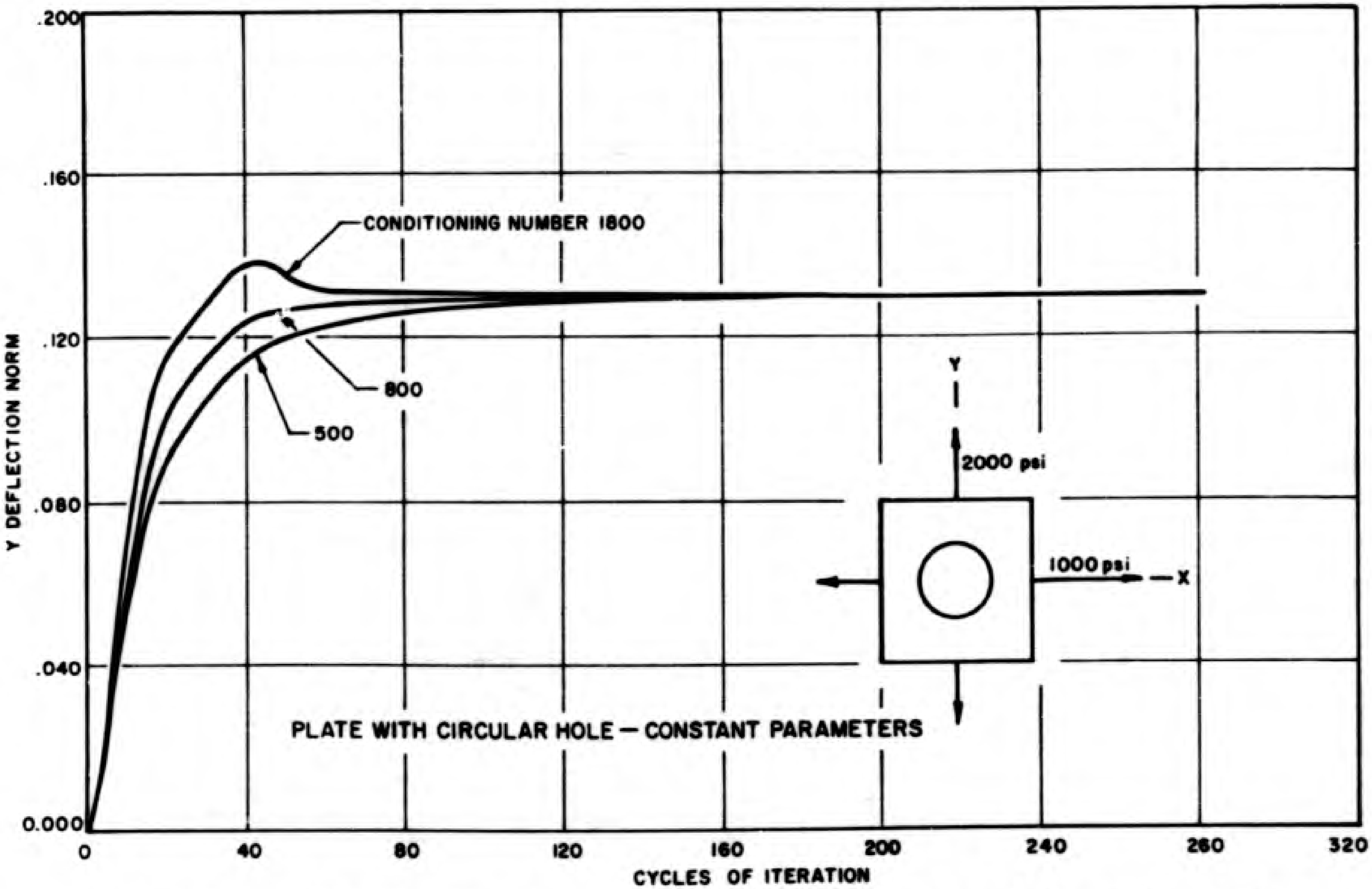


Fig. 4.10 - Y deflection norm variation during Dynamic Relaxation solution with constant parameters

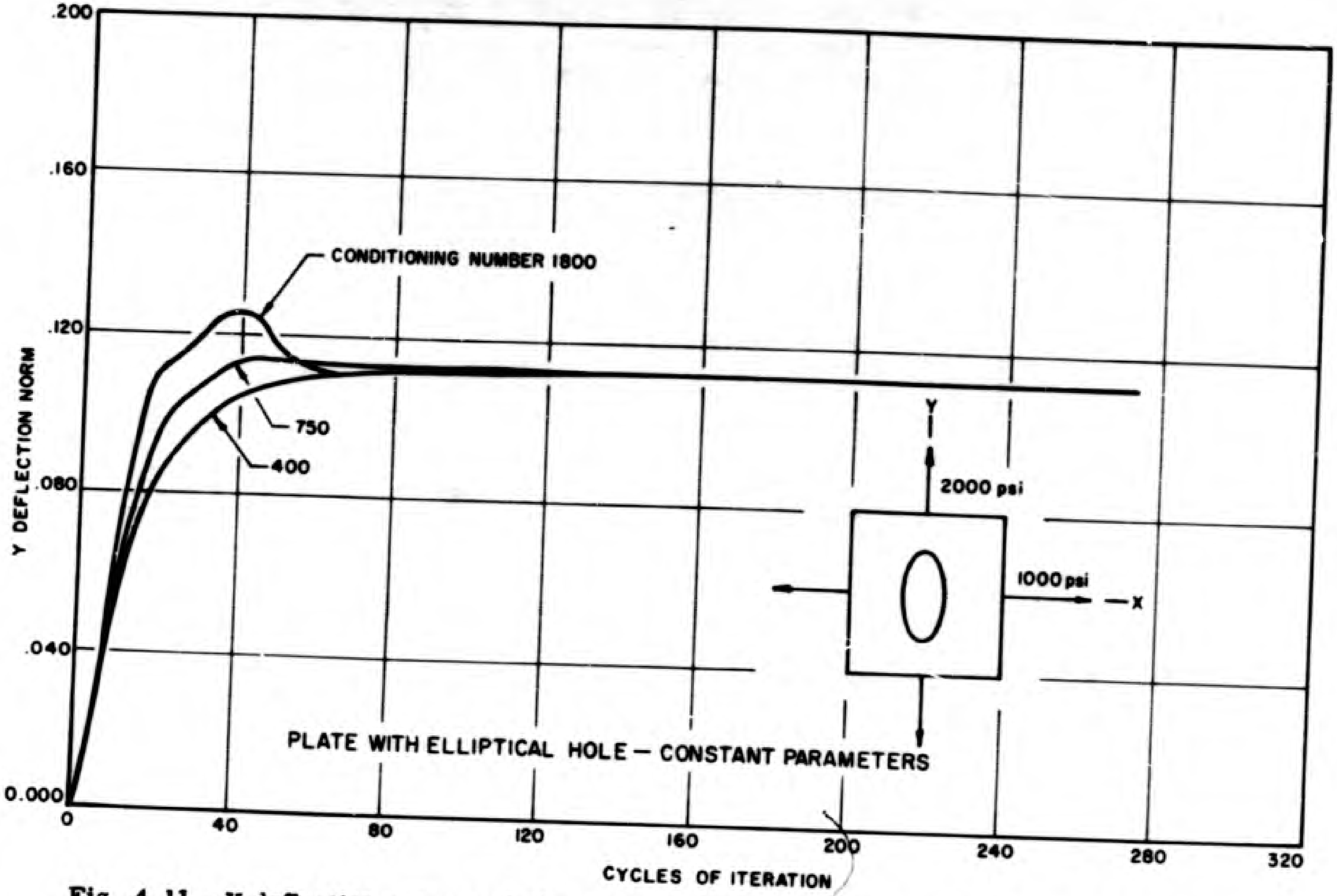


Fig. 4.11 - Y deflection norm variation during Dynamic Relaxation solution with constant parameters

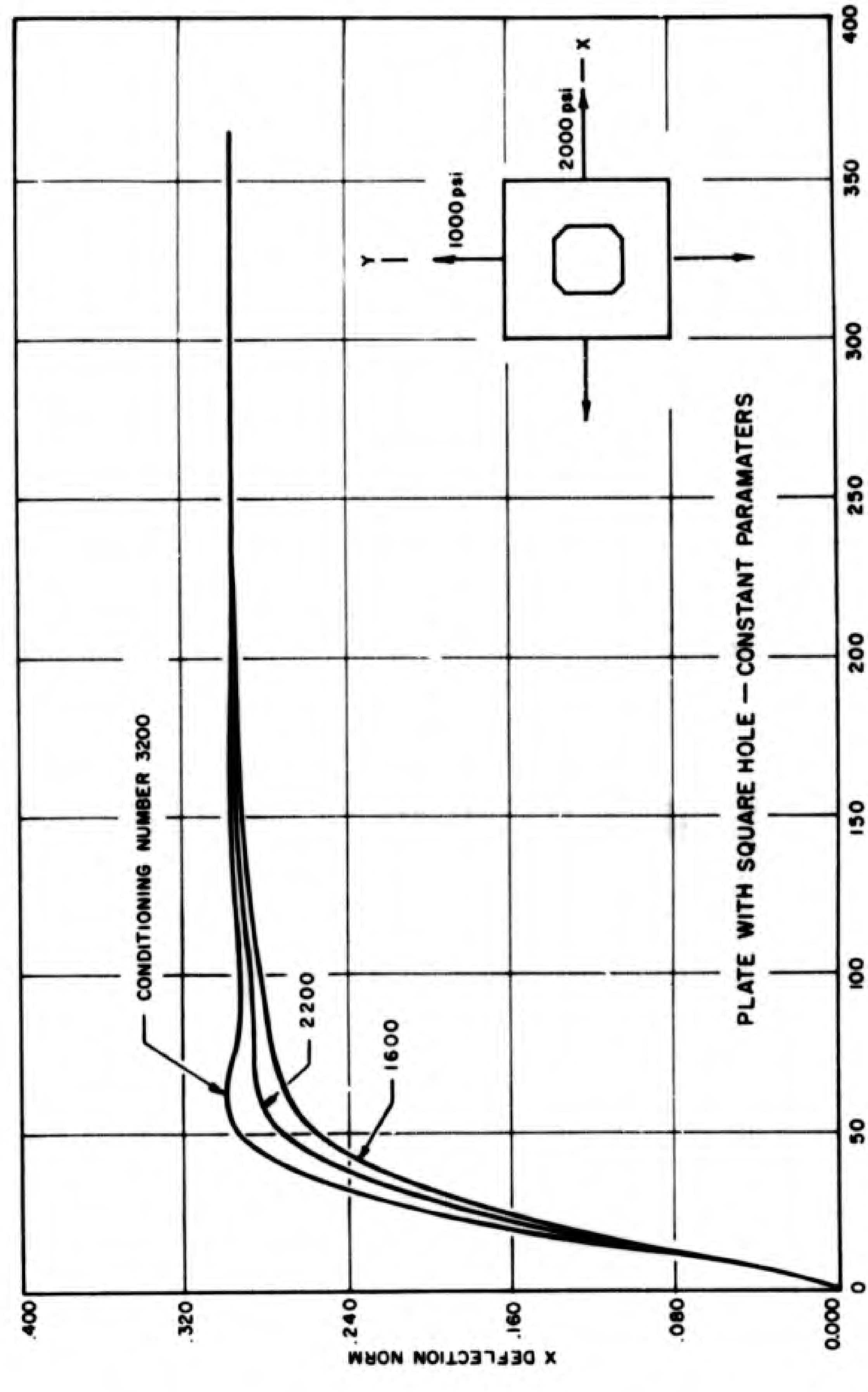


Fig. 4.12 - X deflection norm variation during Dynamic Relaxation solution with constant parameters

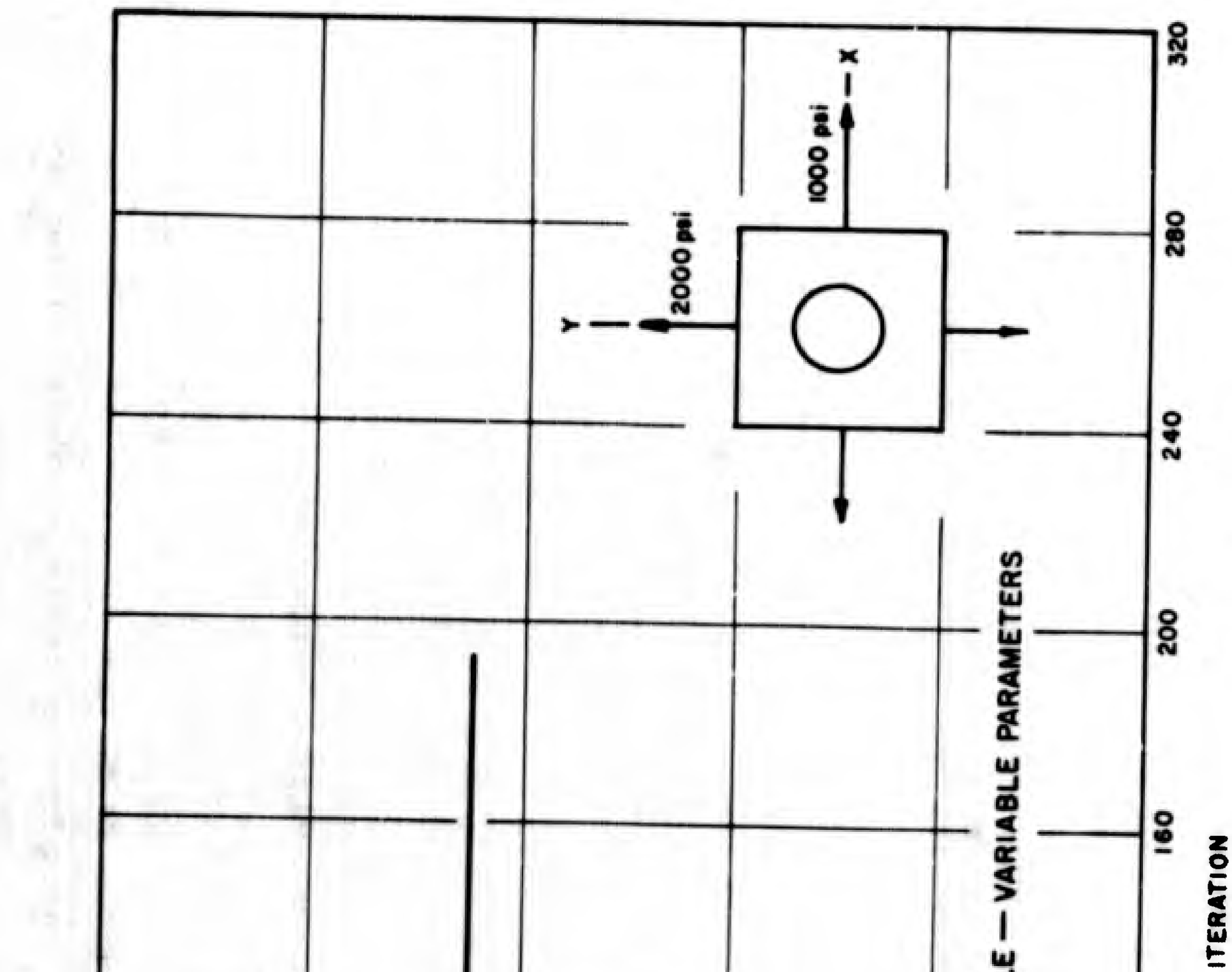


Fig. 4.13 - Y deflection norm variation during Dynamic Relaxation solution with variable parameters

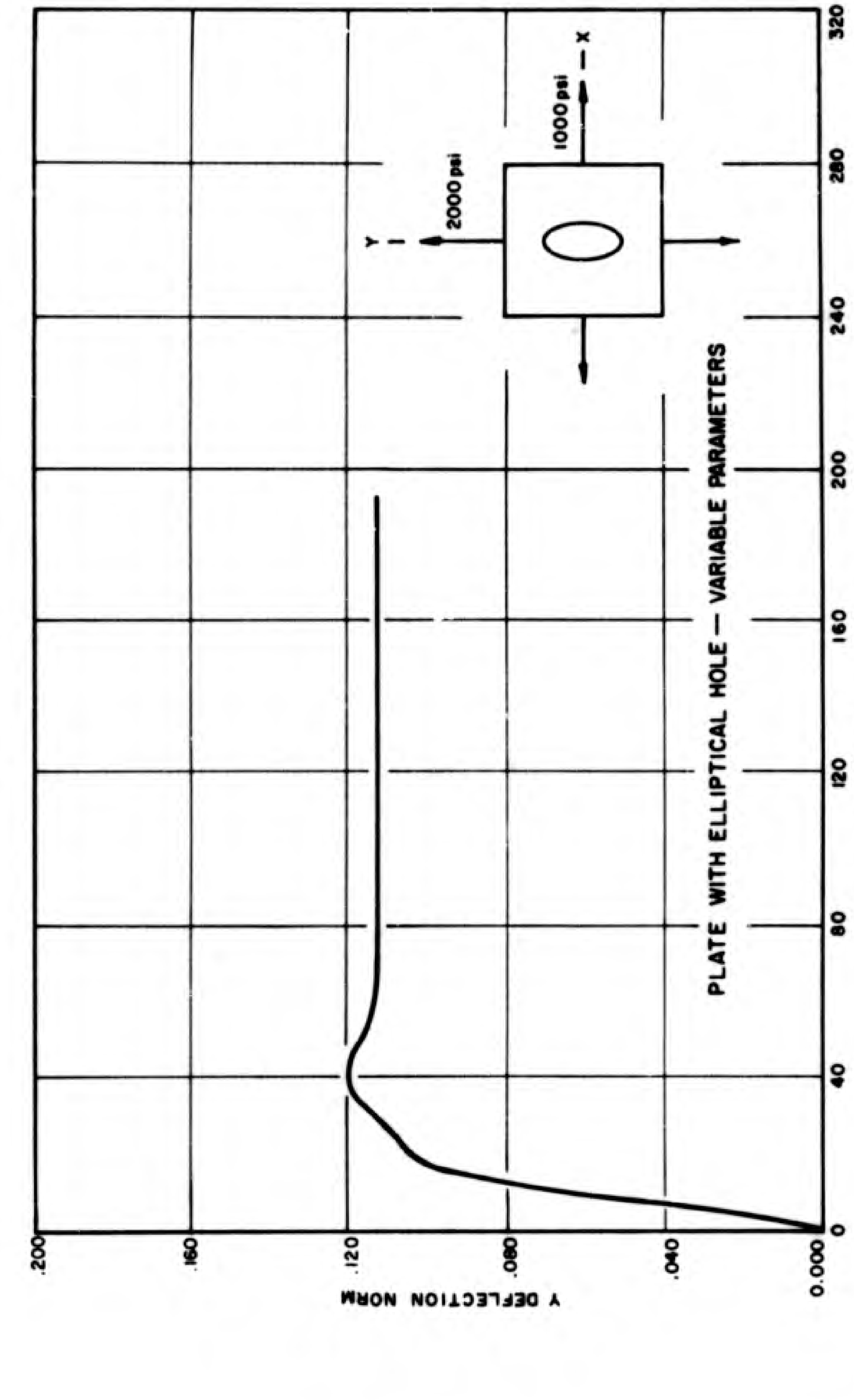


Fig. 4.14 - Y deflection norm variation during Dynamic Relaxation solution with variable parameters

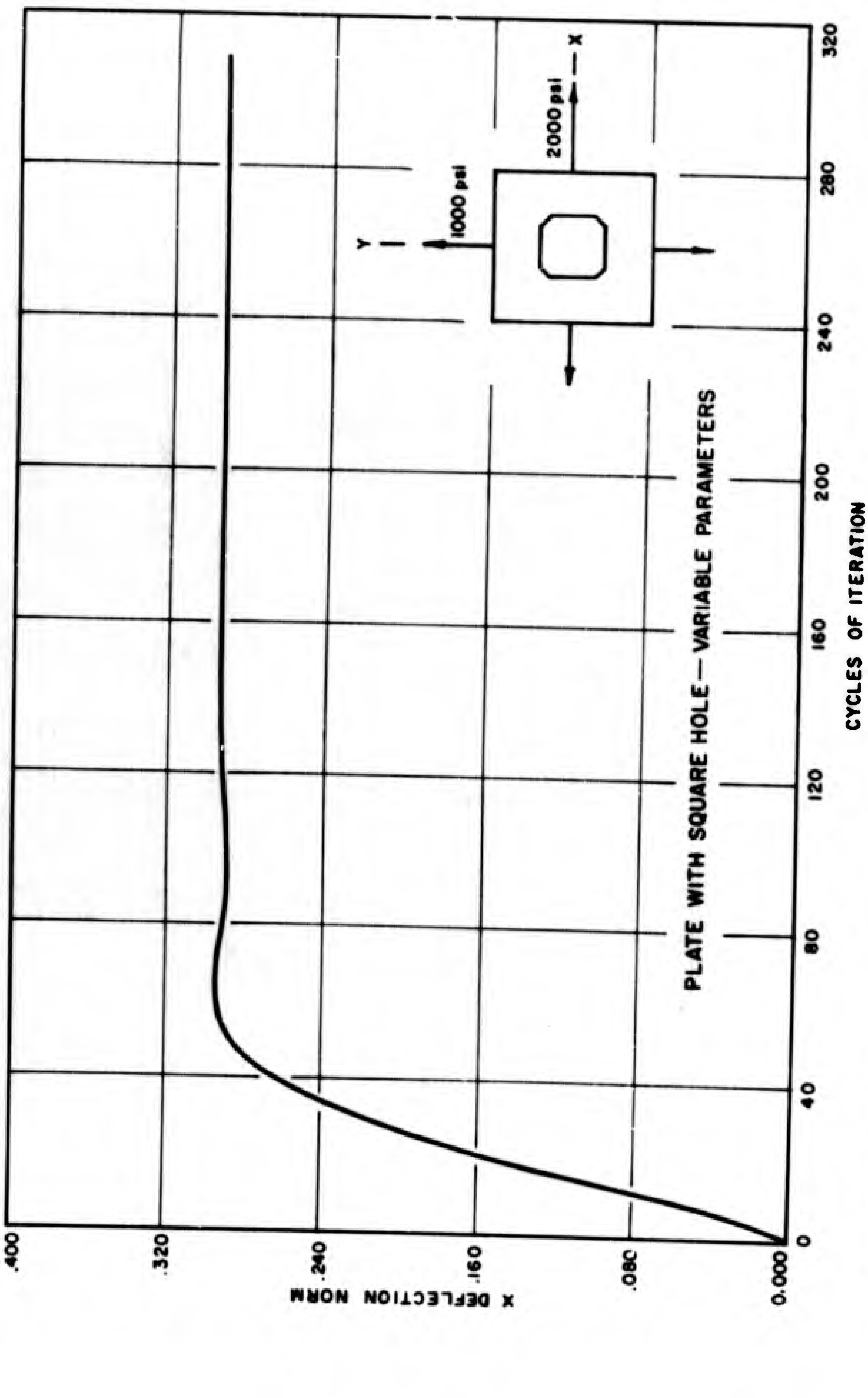


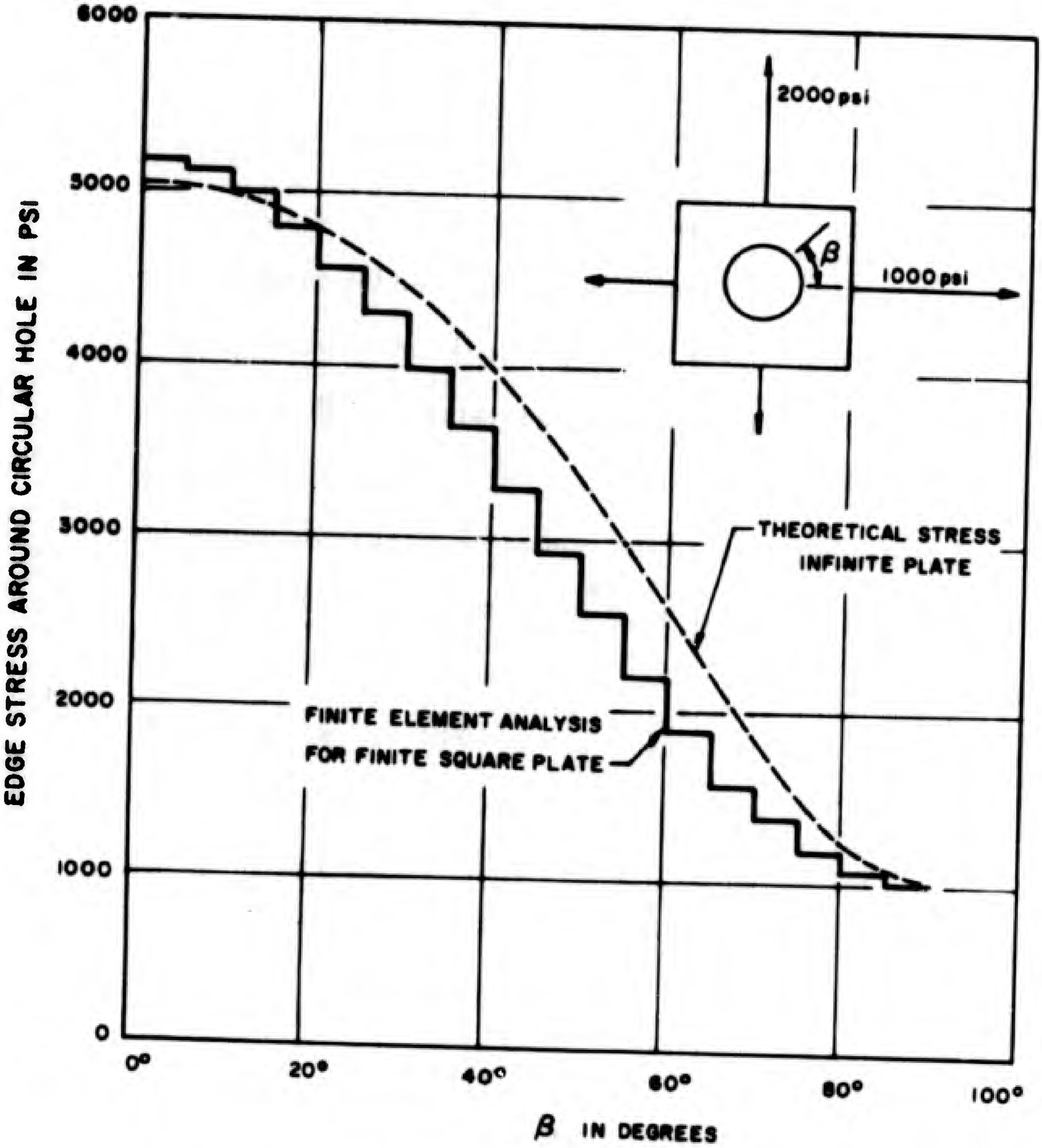
Fig. 4.15 - X deflection norm variation during Dynamic Relaxation solution with variable parameters

transformation. The edge stresses around the cutout in an infinite flat plate were computed using these stress functions, which are given in Appendix B, and compared to those that resulted from the finite element analysis. The results of this comparison are shown in Figures 4.16, 4.17 and 4.18.

4-8 Effect of the Amount of Adjustment on the Iterative Process

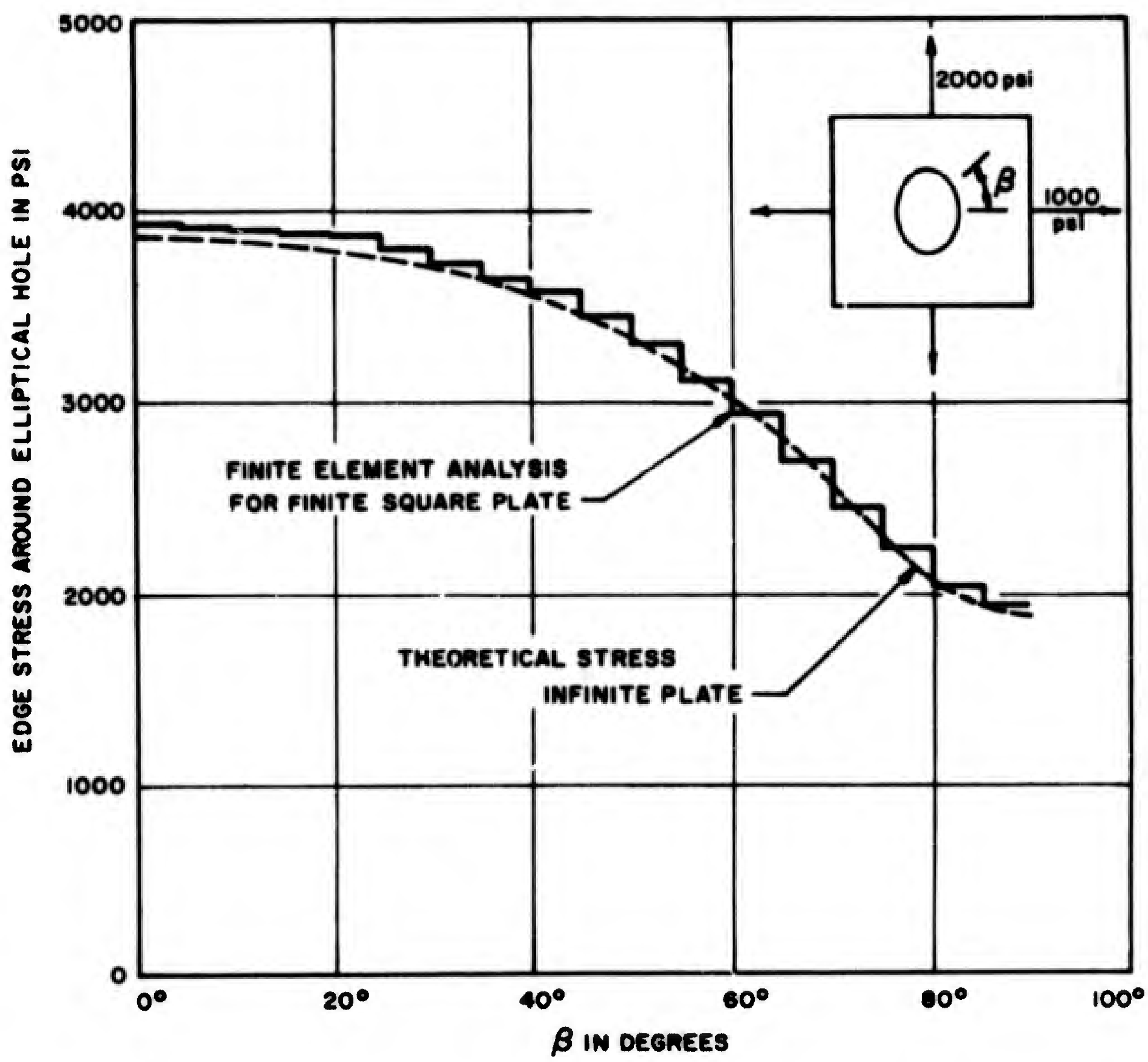
The strategy for approaching the optimum convergence of Dynamic Relaxation was examined for the effect the magnitude of the adjustment, the value of I in equation (4.9), has on the process. The minimum eigenvalue of $\mathbf{B}$ was deliberately underestimated by a factor of approximately 2.5. The strategy previously described was then applied using various values of I in equation (4.9). Figure 4.19 shows the results of this investigation. It is clear that making large adjustments may take the process beyond the optimum and as a result hurt the convergence.

It is possible to detect the fact that an adjustment away from the optimum has been made. When such an adjustment in parameters has been made using this strategy, it forces the root associated with $\lambda_{B\,min}\, h^2/\rho$ to become real and dominant in the iteration process. This fact can be detected by using equation (4.8). When an indication that the adjustment in parameters has created a dominant root then the parameters can be readjusted toward the optimum.



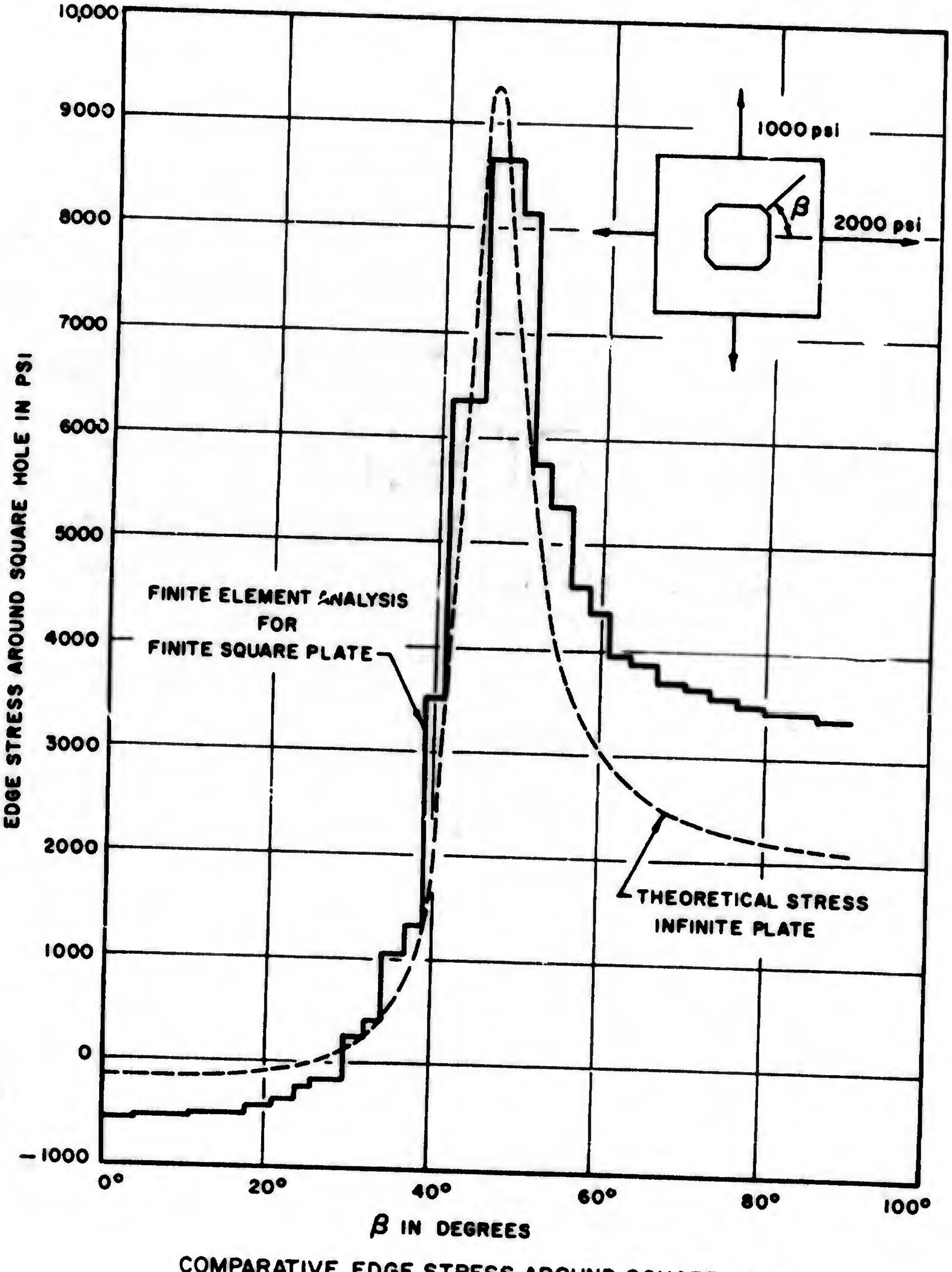
COMPARATIVE EDGE STRESS AROUND A CIRCULAR HOLE

Figure 4.16



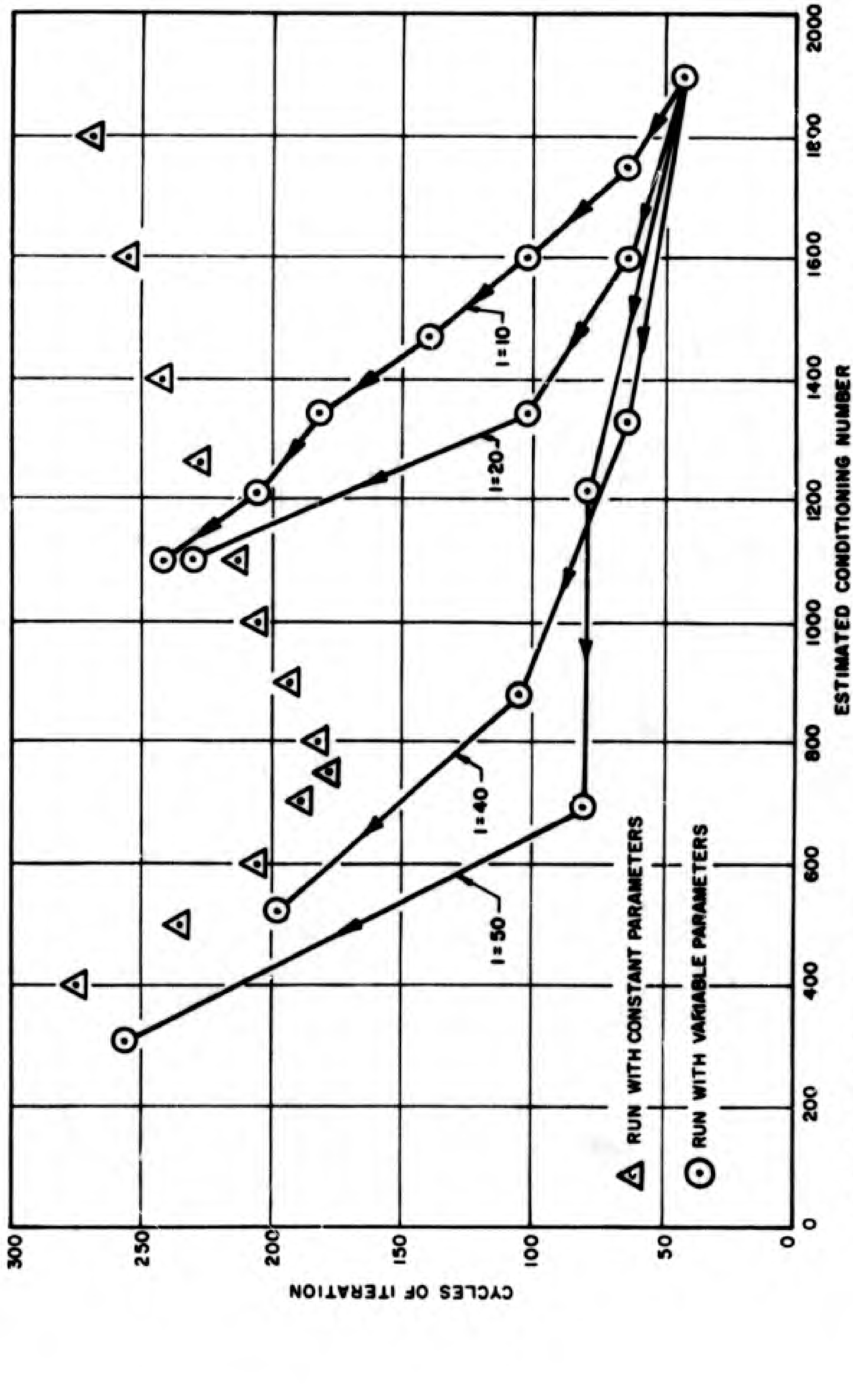
COMPARATIVE EDGE STRESS AROUND ELLIPTICAL HOLE

Figure 4.17



COMPARATIVE EDGE STRESS AROUND SQUARE HOLE

Figure 4.18



EFFECT OF THE MAGNITUDE OF THE ADJUSTMENT OF PARAMETERS — PLATE WITH ELLIPTICAL HOLE

Fig. 4.19 - Comparison of constant parameter solutions with variable parameter solutions in Dynamic Relaxation

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V CONCLUSIONS

5-1 Results of Investigation

The theory for the application of Dynamic Relaxation to the solution of linear equations which have been expressed in finite element form in space has been developed. The dependence of the choice of the parameters for optimum convergence on the conditioning of these equations has been demonstrated. The very significant result of this part of the investigation is the fact that the theory for the optimum convergence of the Dynamic Relaxation method is clearly defined for all forms of a positive definite coefficient matrix. This makes the method considerably more general than Successive Overrelaxation, for which the theory of optimum convergence applies only when the coefficient matrix is tridiagonal in form. This feature of Dynamic Relaxation makes the technique particularly advantageous when employing the finite element method of structural analysis.

The asymptotic convergence of the basic iterative methods and Dynamic Relaxation has been examined by transforming each of these iterative methods into a standard eigenvalue problem for the error vector. The convergence rates of these methods were compared for the special case of the tridiagonal form of the coefficient matrix. As a result of this comparison it was found in this particular case, that the convergence of Successive Over-relaxation is superior to that of Dynamic Relaxation. However, the comparison showed that the convergence of Dynamic Relaxation is superior to that of both the Point Jacobi and Gauss-Seidel methods.

A technique has been developed whereby the optimum convergence condition for Dynamic Relaxation is approached by starting the iterative process with parameters based on estimates of the maximum and minimum eigenvalues of $\mathbf{B}$ and then adjusting these parameters as the iteration progresses. This technique has been applied with success to the finite element analysis of flat plates with circular, elliptical and filleted square cutouts. It is worth noting here that these particular problems could not have been solved by either the Point Jacobi or Extrapolated Gauss methods since in all cases the maximum eigenvalue of $\mathbf{B}$ was greater than two.

5-2 Conclusions

The theory for the application of Dynamic Relaxation to the optimum solution of equations based on a finite element representation in space is more clearly defined for a broader class of problems than would be possible by Successive Overrelaxation. It is in the generality of this theory that the real strength of the technique lies. It is true that, in order to apply this method optimally to the solution of a system of linear equations, the maximum and minimum eigenvalues of **B** must be determined or at least estimated. If the current literature on the topic of matrix error analysis is indicative of future trends in the field, however, this should be considered as an advantage of Dynamic Relaxation, rather than a deterrent to its use. To cite just one example which emphasizes this point, the following conclusions are quoted from "Matrix Error Analysis for Engineers", by Rosanoff and Ginsburg (20).

"1. The conditioning numbers (defined by them also as $\lambda_{max} / \lambda_{min}$) are adequate tools for matrix error analysis.

2. The measurement of conditioning is usually inexpensive and should be routine for most calculations.

3. Since the significance of ill conditioning is the absence of information, the future of matrix error analysis lies with avoidance of ill conditioning. Pre-conditioning and careful inversion techniques can be of only limited help."

Thus the determination of the conditioning of the equations to be solved should be a necessity, regardless of the technique employed to solve them. Of course, Dynamic Relaxation uses this very same information in order to select optimum parameters, so the method fits neatly in with the increasingly important field of matrix error analysis, providing not only a solution to a broad class of problems, but in the process it also provides information regarding the validity of the answers obtained.

5-3 Future Areas of Investigation

As a result of this study of Dynamic Relaxation, the following areas are recommended for future investigation:

1. A search for more precise methods of evaluating the extreme eigenvalues of **B** , particularly the minimum one. This would reduce the number of adjustments in parameters required during the iteration process, or possibly eliminate the need for adjustment altogether. Obviously it is important that these methods be simple and not require more effort than the solution process itself.

2. The effect of making the **D** matrix a super diagonal matrix of two by two sub-matrices representing the direct stiffness of the nodes should be studied. Of course to have any impact this approach would have to improve the conditioning of **B** . This could be similar to the improvement in convergence that was noted in using **B** instead of **K** in forming the re-

currence equation for Dynamic Relaxation. In effect, the suggested procedure would amount to a form of block relaxation.

3. A study should be made to ascertain if there are any problems peculiar to the application of Dynamic Relaxation to systems of equations with conditioning numbers of 10000 and greater. It is possible that new strategies for achieving optimum convergence would have to be adopted in this category of problem.

4. Finally, although there is no mathematical theory to substantiate it, the extension of Dynamic Relaxation to the solution of systems of geometrically non-linear finite element equations should be investigated. It is possible that Dynamic Relaxation may be used to solve these equations as well as, or better than, any other existing technique.

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APPENDIX A

Turner et al. (18) develop the stiffness matrix of a plane triangle based on the assumption of constant strain in the following manner.

The nodal displacements are expressed in terms of the strains by integrating the expressions for constant strain. Next a relationship between stress and nodal displacements is developed in the following form

$$\sigma = S\delta$$

where σ is the stress vector and δ is the nodal displacement vector.

An equilibrium relationship between the nodal forces and element stresses in them developed.

$$F = T\sigma$$

where F is the nodal force vector and σ is the stress vector.

Using these relationships the following equation is formed

$$F = TS\delta$$

therefore

$$K = TS$$

where

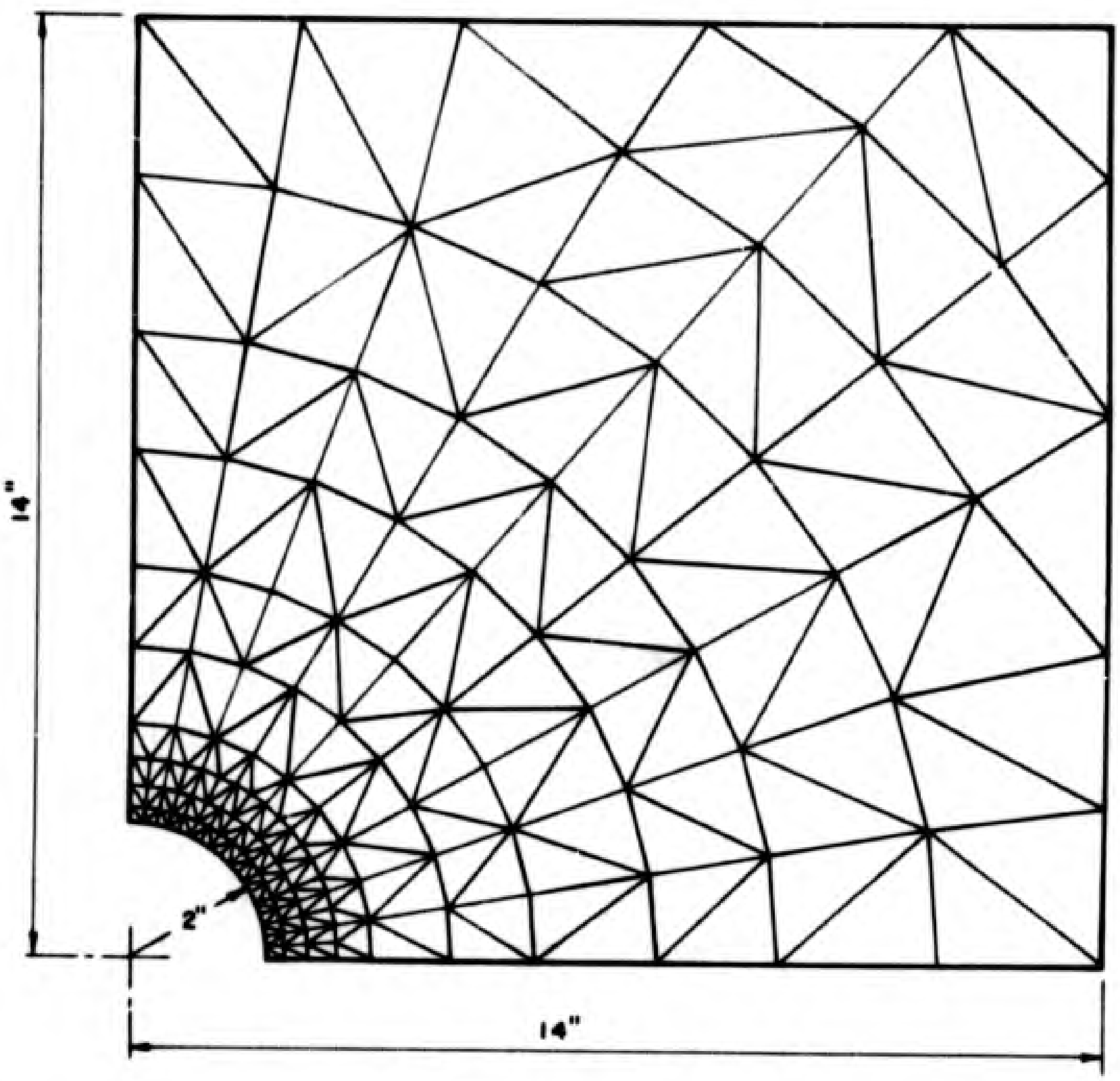
$$\mathbf{T} = \frac{t}{2} \begin{bmatrix} Y_{23} & 0 & X_{32} \\ 0 & X_{32} & Y_{23} \\ Y_{31} & 0 & X_{13} \\ 0 & X_{13} & Y_{31} \\ Y_{12} & 0 & X_{21} \\ 0 & X_{21} & Y_{12} \end{bmatrix}$$

$$\mathbf{S} = \frac{E}{(1-\nu^2)(X_{21}Y_{31}-X_{31}Y_{21})} \begin{bmatrix} -Y_{32} & \nu X_{32} & Y_{31} & -\nu X_{31} & -Y_{21} & \nu X_{21} \\ -\nu Y_{32} & X_{32} & \nu Y_{31} & -X_{31} & -\nu Y_{21} & X_{21} \\ \lambda_1 X_{32} & -\lambda_1 Y_{32} & -\lambda_1 X_{31} & \lambda_1 Y_{31} & \lambda_1 X_{21} & -\lambda_1 Y_{21} \end{bmatrix}$$

in which $X_{ij} = X_i - X_j$ and $Y_{ij} = Y_i - Y_j$

$$\lambda_1 = \frac{(1-\nu)}{2}$$

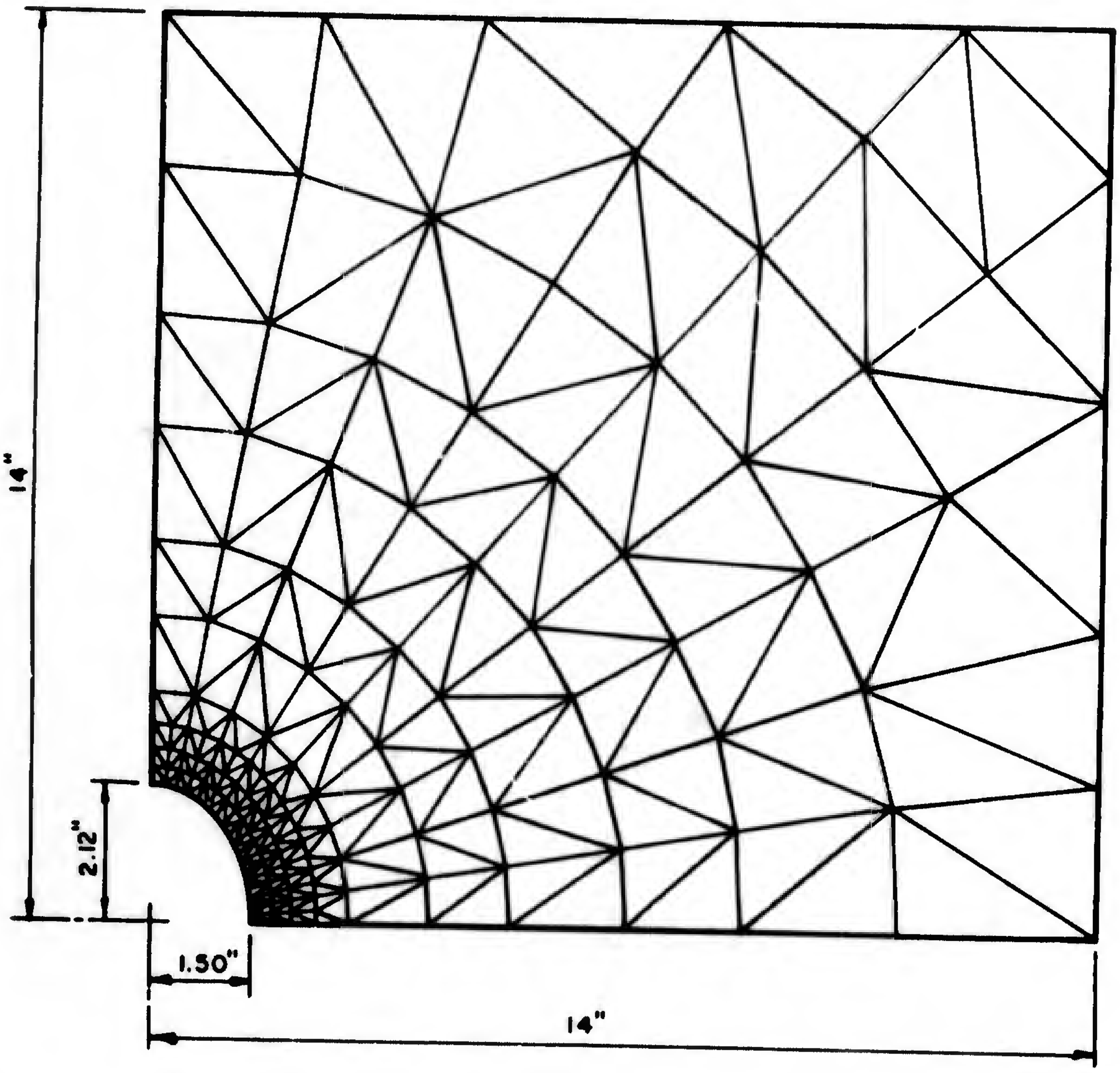
FINITE ELEMENT MODEL
OF
FLAT PLATE WITH A CIRCULAR HOLE



NUMBER OF NODES = 153
NUMBER OF ELEMENTS = 256

Figure A. 1

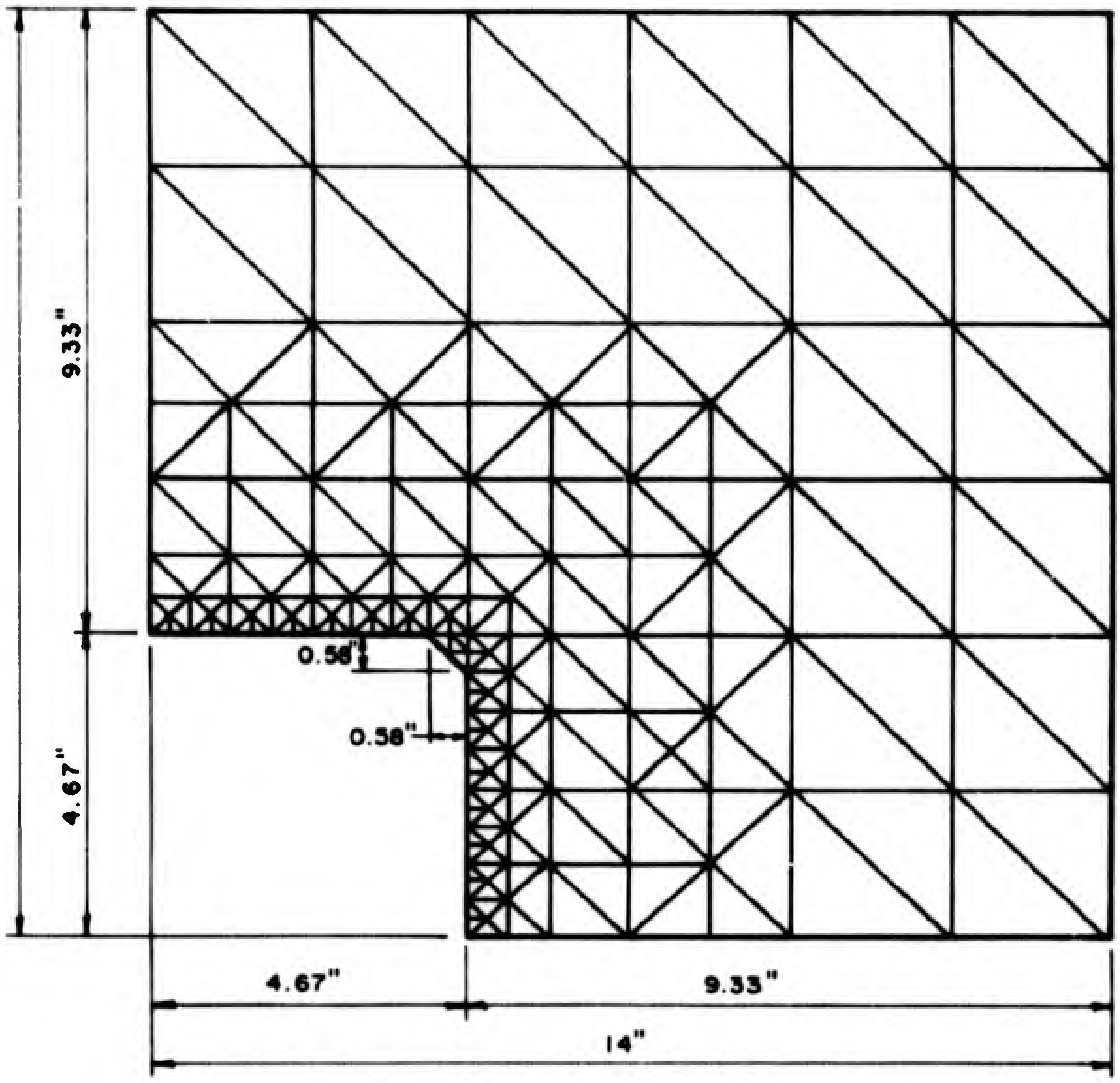
FINITE ELEMENT MODEL
OF
FLAT PLATE WITH ELLIPTICAL HOLE



NUMBER OF NODES = 153
NUMBER OF ELEMENTS = 256

Figure A. 2

FINITE ELEMENT MODEL
OF
FLAT PLATE WITH A FILLETED SQUARE
CUTOUT



NUMBER OF NODES = 142

NUMBER OF ELEMENTS = 226

Figure A. 3

APPENDIX B

The stresses in the region of cut-outs in an infinite plane sheet have been obtained by the method of conformal transformation which has been developed by Muskhelishvili (21) and used extensively by Savin (22), (23). This method can be used for the solution of an unreinforced cut-out of any shape provided that the transformation function can be expressed as a simple polynomial function.

The following theory of the method has been taken from (19).

The method and notation of Muskhelishvili have been followed closely here.

The stress components in an elastic plane stress system may be expressed in terms of a single real stress function (the Airy stress function)

$$U(\rho, \theta),$$

satisfying, in the absence of body forces, the biharmonic equation

$$\nabla^4 U = 0$$

where

$$\nabla^2 = \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \cdot \frac{\partial}{\partial \rho} + \frac{1}{\rho^2} \cdot \frac{\partial^2}{\partial \theta^2}$$

The stress components in the plane, referred to a polar coordinate system (ρ, θ) are given by

$$\sigma_\rho = \frac{1}{\rho} \cdot \frac{\partial U}{\partial \rho} + \frac{1}{\rho^2} \cdot \frac{\partial^2 U}{\partial \theta^2}$$

$$\sigma_\theta = \frac{\partial^2 U}{\partial \rho^2}$$

and

$$\tau_{\rho\theta} = - \frac{\partial}{\partial \rho} \left(\frac{1}{\rho} \cdot \frac{\partial U}{\partial \theta} \right) .$$

Using the complex variable

$$Z = \rho e^{i\theta}$$

the biharmonic equation becomes

$$\frac{\partial^4 U}{\partial Z^2 \partial \bar{Z}^2} = 0 ,$$

and the real stress function may be expressed in complex form, in terms of the two complex functions, $\phi(Z)$ and $\chi(Z)$,

by $2U = \bar{Z}\phi(Z) + Z\overline{\phi(Z)} + \chi(Z) + \overline{\chi(Z)}$

where a bar denotes the complex conjugate.

If $\phi(Z)$ and $\chi(Z)$ are holomorphic functions, every expression of this form represents a biharmonic function.

The stress components in the polar co-ordinate system are then given by

$$\sigma_\rho + \sigma_\theta = 2\left[\Phi(Z) + \overline{\Phi(Z)}\right]$$

and

$$\sigma_\theta - \sigma_\rho + 2i\tau_{\rho\theta} = 2\left[\bar{Z}\,\Phi' + \Psi(Z)\right] e^{2i\theta}$$

where $\psi(Z) = \chi'(Z)$,

$$\Phi(Z) = \phi'(Z),$$

$$\psi(Z) = \psi'(Z),$$

and the prime denotes differentiation with respect to Z .

The region Σ in the ζ-plane may be transformed into the region S in the z-plane by a conformal transformation function

$$Z = \omega(\zeta),$$

where $\omega(\zeta)$ is a single valued analytic function.

Transformation functions of the form

$$\omega(\zeta) = R\left[\zeta + \frac{c_1}{\zeta} + \frac{c_2}{\zeta^2} + \cdots\cdots\right]$$

have been obtained, which transform an infinite plane sheet containing an elliptic or approximate square cut-out in the z-plane into the region outside the unit circle

$$\sigma = e^{i\theta}$$

in the ζ-plane. Similar transformation functions may be obtained for cut-outs of other shapes.

If the functions previously written

$$\phi(Z) \quad \psi(Z) \quad \Phi(Z) \quad \Psi(Z)$$

are now denoted by

$$\phi_*(Z) \quad \psi_*(Z) \quad \Phi_*(Z) \quad \Psi_*(Z)$$

then in the new notation

$$\phi(\zeta) = \phi_*(z) = \phi_*\left[\omega(\zeta)\right],$$

$$\psi(\zeta) = \psi_*(z) = \psi_*\left[\omega(\zeta)\right],$$

and

$$\Phi(\zeta) = \frac{\phi'(\zeta)}{\omega'(\zeta)}$$

$$\Psi(\zeta) = \frac{\psi'(\zeta)}{\omega'(\zeta)}$$

In the transformed plane the stress components, denoted by σ_r, σ_θ and $\tau_{r\theta}$ in the curvilinear co-ordinate system corresponding to the polar co-ordinate system in the ζ-plane, are given by

$$\sigma_\rho + \sigma_\theta = 2\left[\Phi(\zeta) + \overline{\Phi(\zeta)}\right] = 4\,\mathrm{Re}\cdot\left[\Phi(\zeta)\right],$$

and

$$\sigma_\theta - \sigma_\rho + 2i\tau_{\rho\theta} = \frac{2\zeta^2}{\rho^2\overline{\omega'(\zeta)}}\left[\overline{\omega(\zeta)}\cdot\Phi'(\zeta) + \omega'(\zeta)\Psi(\zeta)\right],$$

where Re denotes the real part.

At the edge of an unreinforced cut-out

$$\sigma_\rho = \tau_{\rho\theta} = 0,$$

so that to find the stress at the edge of the cut-out the function is not required, and the problem is reduced to that of finding the function $\Phi(\zeta)$ satisfying the boundary conditions of the problem.

The expression for the edge stress then becomes

$$\sigma_\theta = 4\,\mathrm{Re.}\left[\Phi(\zeta)\right]$$

The boundary condition on the edge of the cut-out (since there are no external forces applied to the edge of the cut-out) may be written

$$\phi(\sigma) + \frac{\omega(\sigma)}{\omega'(\sigma)} \cdot \overline{\phi'(\sigma)} + \overline{\psi(\sigma)} = 0 ,$$

where $\sigma = e^{i\theta}$ on the unit circle.

The complex stress functions $\phi(\zeta)$ and $\psi(\zeta)$ may be expressed $\phi(\zeta) = \phi_o(\zeta) + \phi_1(\zeta)$

and $\psi(\zeta) = \psi_o(\zeta) + \psi_1(\zeta)$,

where the functions $\phi_1(\zeta)$ and $\psi_1(\zeta)$ represent the state of stress at infinity, and $\phi_o(\zeta)$, $\psi_o(\zeta)$ are the perturbation functions due to the presence of the cut-out.

Hence $\phi_1(\zeta)$ and $\psi_1(\zeta)$ are given by

$$\psi_1(\zeta) = R\,\Gamma_1\,\zeta$$

and

$$\psi_1(\zeta) = R\,\Gamma_2\,\zeta$$

where

$$\Gamma_1 = \frac{1}{4}(N_1 - N_2) ,$$

$$\Gamma_2 = -\frac{1}{2}(N_1 - N_2)\,e^{-2i\alpha} ,$$

and N_1, N_2 are the values of the principal stresses at infinity, α is the angle made by the direction of N_1 with the axis $\theta = 0$, and R is the scale factor in the transformation function.

The functions $\phi_0(\zeta)$ and $\psi_0(\zeta)$, which are homomorphic outside the unit circle including the point at infinity, may be expanded into series of the form

$$\phi_0(\zeta) = \frac{a_1}{\zeta} + \frac{a_2}{\zeta^2} + \frac{a_3}{\zeta^3} + \text{--------} = \sum_1^\infty a_n \zeta^{-n}$$

and

$$\psi_0(\zeta) = \frac{b_1}{\zeta} + \frac{b_2}{\zeta^2} + \frac{b_3}{\zeta^3} + \text{---------} = \sum_1^\infty b_n \zeta^{-n}$$

On the unit circle

$$\zeta = \sigma = e^{i\theta},$$

and the term

$$\frac{\omega(\sigma)}{\omega'(\sigma)}$$

may be expanded into a series of ascending powers of σ. By substituting this, together with the series expansions for $\phi(\zeta)$ and $\psi(\zeta)$, into the boundary condition on the edge of the cut-out the coefficients a_n may be determined by equating coefficients of powers of σ in the boundary condition.

Since the function $\phi(\zeta)$ alone is sufficient to obtain the stress at the edge of the cut-out, it is not necessary to determine the coefficients b_n.

Using this technique Houghton and Rothwell (19) develop the stress distribution σ_θ at the edge of elliptical and square cut-outs. They are given in the appendix of (19) as follows.

For the elliptical cutout of any eccentricity in a plane sheet subjected to 2:1 bi-axial tension

$$\sigma_\theta = \frac{3 - 2\cos 2\theta - 3m^2 + 2m}{1 - 2m\cos 2\theta + m^2}$$

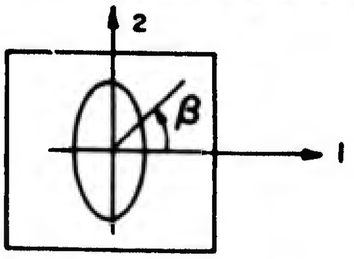
where $$m = \frac{1-K}{1+K}$$

and $$K = \frac{\text{minor axis}}{\text{major axis}}$$

It should be noted that the angle θ in the expression for the edge stress refers to the unit circle, and the corresponding angle β on the ellipse is given by

$$\tan\beta = K\tan\theta$$

where β is shown on the following sketch



For the circular cutout, it may be regarded as a special case of an elliptical cutout with $m = 0$, and in this case the edge stress becomes

$$\sigma_\theta = 3 - 2\cos 2\theta$$

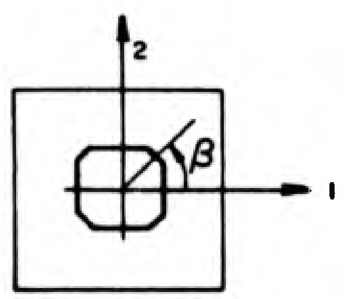
For a square cutout having a corner radius of 0.06 times the hole dimension in a plane sheet subjected to 2:1 bi-axial tension.

$$\sigma_\theta = \frac{63 - 72 \cos 2\theta}{35 + 28 \cos 4\theta}$$

Again the angle θ refers to the unit circle, and the corresponding angle β on the square is given by

$$\tan \beta = \frac{\sin \theta + \frac{1}{6} \sin 3\theta}{\sin \theta + \frac{1}{6} \cos 3\theta}$$

wher β is shown on the following sketch.



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13. ABSTRACT

The report is a study in depth of the method of Dynamic Relaxation, a matrix iterative method for the solution of simultaneous linear equations, and used principally in problems of structural analysis under static stress conditions.

Whereas earlier investigators have restricted themselves almost exclusively to the finite difference formulation in space of both the equations of motion and the constitutive relationships, the present study formulates an alternative approach using finite elements in space.

The mathematical basis for the finite element approach in space is developed, and followed by an analysis of convergence based upon a transformation into a standard eigenvalue problem for the error vector, intimately associated with the conditioning of the equations. Optimum convergence is studied and comparisons made with other iterative methods.

Dynamic Relaxation is demonstrated by applying it to plane stress problems of statically loaded plates having discontinuities in the form of circular, elliptical and filleted square holes.

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Eigenvalue						
Error Vector						
Successive Overrelaxation						
Point Jacobi						
Gauss-Seidel						
Optimum convergence						