NEW CONDITIONS FOR CENTRAL LIMIT THEOREMS

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1. Introduction

A general formulation of the central limit problem for sums of independent random variables is the following (see Loève [1], p. 291). Let

$$S_n = \sum_k X_{nk}$$

where $k=1,\ldots,k_n$, $k_n \to \infty$ as $n \to \infty$, and for each n $\{X_{nk}\}$ are independent random variables with probability distribution functions F_{nk} and $EX_{nk} = 0$. Let $\{F_n\}$ be the distribution functions of $\{S_n\}$ and let $\Phi(x)$ be the distribution function of a Normal random variable with zero-mean and variance σ^2. Under these conditions it is possible to show the following.

<u>Theorem 1.1: Let</u> $\max_k \mathrm{Var}\, X_{nk} \to 0$ <u>and</u> $\sum_k \mathrm{Var}\, X_{nk} \to \sigma^2 < \infty$ <u>where</u> σ^2 <u>is a positive constant. The sums</u> S_n <u>are asymptotically Normal</u> (i.e., $F_n(x) \to \Phi(x)$) <u>if and only if for every</u> $\epsilon > 0$

$$g_n(\epsilon) = \sum_k \int_{|x|\geq\epsilon} x^2 dF_{nk} \to 0 . \tag{1.1}$$

Except in special cases, the application of condition 1.1 is difficult because of the integrals involved. By assuming the existence of

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fourth-order moments, we are able to prove new necessary and sufficient conditions for both Normal and Poisson convergence which involve only moments. The proof of the theorem makes use of a characterization of the Normal distribution among infinitely divisible (I.D.) laws which was perhaps first recognized by Borges [3] and later independently by the author [4].

2. Normal Convergence

Theorem 2.1: Let $E|S_n|^{(4+\delta)}$ *be uniformly bounded for some* $\delta > 0$. *Let* $\max_k \mathrm{Var}\, X_{nk} \to 0$, $\sum_k \mathrm{Var}\, X_{nk} \to \sigma^2 < \infty$ *where* σ^2 *is a positive constant. Then* S_n *is asymptotically Normal if and only if*

$$ES_n^4 - 3\{ES_n^2\}^2 \to 0. \tag{2.1}$$

Proof: The asymptotic normality of S_n implies condition 2.1 by the moment convergence theorem (see Loeve [1], p. 184) and the fact that for a zero-mean Normal random variable S, $ES^4 - 3\{ES^2\}^2 = 0$.

To prove the converse it is sufficient to show that every convergent subsequence $\{F_{n'}\}$ of $\{F_n\}$ converges to $\Phi(x)$ (see Feller [2], p. 261).

Let F be the limit of $F_{n'}$, then F is an infinitely divisible law with characteristic function f(u) such that (see Loève [1], p. 293),

$$\log f(u) = \int (e^{iux} - 1 - iux) \frac{1}{x^2}\, dK(x) \tag{2.2}$$

where K(x) is monotone increasing and of bounded variation, $K(-\infty) = 0$, and $K(\infty) = \sigma^2 < \infty$. The integrand is defined by continuity at the origin. It is known that F(x) is a Normal law if and only if K(x) increases only at x = 0.

Since $E|S_n|^{(4+\delta)}$ are uniformly bounded, all moments of S_n, of order 4 or less converge to those of F. Since $ES_n^4 - 3\{ES_n^2\}^2 \to 0$,

$$0 = \int x^4 dF(x) - 3 \int x^2 dF(x)^{\;2}$$

$$= \frac{\partial^4}{\partial u^4} \log f(u)\Big|_{u=0}$$

$$= \int x^2 dK(x). \tag{2.3}$$

This last equation is obtained by differentiating the RHS of Eq. (2.2) under the integral sign. This is justified in the following way. That

$$-\frac{d^2}{du^2} \log f(u) = \int e^{iux} dK(x) \tag{2.4}$$

is shown by Loeve [1], p. 293. Thus the L.H.S. of Eq. (2.4) is a characteristic function. Since this characteristic function is twice differentiable, its second derivative is given by (Loève [1], p. 200)

$$\frac{d^4}{du^4} \log f(u) = \int x^2 e^{iux} dK(x)$$

and Eq. (2.3) follows. Thus K(x) increases only at x = 0.

Finally, since $\sum_k \mathrm{Var}\, X_{nk} \to \sigma^2$ we have shown that $F_{n'} \to \Phi(x)$ and the proof is complete.

Condition 2.1 becomes even simpler when we note that

$$ES_n^4 - 3\left\{ES_n^2\right\}^2 = \sum_k \left[EX_{nk}^4 - 3\left\{EX_{nk}^2\right\}^2\right] .$$

The L.H.S. is usually called the fourth cumulant of S_n. This identity says that the fourth cumulant of a sum of independent random variables equals the sum of the fourth cumulants.

If the distributions F_n are known to be infinitely divisible (I.D.), then moments higher than 4 are not required.

<u>Theorem 2.2: If</u> F_n <u>are</u> I.D., <u>then</u> $F_n(x) \to \Phi(x)$ <u>if and only if</u> $ES_n^2 \to \sigma^2$ <u>and</u>

$$ES_n^4 - 3\left\{ES_n^2\right\}^2 \to 0.$$

<u>Proof</u>: The characteristic functions $f_n(u)$ are given by

$$\log f_n(u) = \int (e^{iux} - 1 - iux) \frac{1}{x^2} dK_n(x).$$

For any $\epsilon > 0$

$$\int_{|x|>\epsilon} dK_n(x) \le \int \frac{x^2}{\epsilon^2} dK_n(x) \to 0$$

as $n \to \infty$. Thus $K_n(x)$ converges to a step function at the origin of size σ^2.

The converse is obtained from the moment convergence theorem.

3. Poisson Convergence

The key feature of the results above is that the fourth cumulant (i.e., $ES_n^4 - 3\{ES_n^2\}^2$) corresponds to $\int x^2 dK(x)$ and is a good test for a jump of K(x) at the origin. However, this method can be used to test for jumps at other points also. For example, the equation

$$\int (x-1)^2 dK(x) = 0$$

implies that K(x) can only have a jump at x = 1. This leads to the following results. First we note that

$$\int (x-1)^2 dK_n(x) = ES_n^4 - 3\left\{ES_n^2\right\}^2 - 2ES_n^3 + ES_n^2$$

$$= \sum_k \left[EX_{nk}^4 - 3\left\{EX_{nk}^2\right\}^2 - 2EX_{nk}^3 + EX_{nk}^2\right] .$$

<u>Theorem 3.1: Let</u> $E|S_n|^{(4+\delta)}$ <u>be uniformly bounded for some</u> $\delta > 0$. <u>Then</u> S_n <u>is asymptotically Poisson</u> (i.e., $\log f_n(u) \to [\sigma^2(e^{iu}-1)-iu\sigma^2]$) <u>if and only if</u>

$$ES_n^4 - 3\left\{ES_n^2\right\}^2 - 2ES_n^3 + ES_n \to 0.$$

In a similar way we obtain:

<u>Theorem 3.2: If</u> F_n <u>are I.D., then</u> F_n <u>is asymptotically Poisson if and only if</u> $ES_n^2 \to \sigma^2$ <u>and</u>

$$ES_n^4 - 3\left\{ES_n^2\right\}^2 - 2ES_n^3 + ES_n \to 0.$$

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