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T. C. T. Ting*

Abstract

In one dimensional wave propagation such as longitudinal waves in a rod, an elastic-plastic boundary may start at the end $x = 0$ of the rod depending on the stresses prescribed at $x = 0$. The initial slope of the elastic-plastic boundary at $x = 0$ can be determined easily if the time derivative σ_t of the stress σ on both sides of the elastic-plastic boundary are not zero. In this paper, the initial slope of the elastic-plastic boundary (or boundaries) is determined analytically when σ_t at $x = 0$ is continuous and vanishes at time $t = t_o$ while the second derivative σ_{tt} at t_o may or may not be continuous. It is seen that an elastic region can be generated near the end of the rod even though the stress state at the end is continuously plastic.

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*Associate Professor of Applied Mechanics, Department of Materials Engineering, University of Illinois at Chicago Circle, Chicago, Illinois 60680; visiting at Stanford University, Department of Applied Mechanics, Stanford, California 94305.

1. Introduction

The equation of motion for one dimensional wave propagation in the x-direction is

$$\sigma_x = \rho v_t \tag{1}$$

where σ is the stress, v the particle velocity and ρ is the density of the material. The subscripts x and t denote partial differentiation with respect to these variables. The continuity condition, together with the stress-strain relation, gives

$$\sigma_t = \rho c^2 v_x \tag{2}$$

where $c^2 = \frac{1}{\rho}\frac{d\sigma}{d\epsilon}$ is the wave speed. Equations (1) and (2) apply to both elastic and plastic regions. In elastic regions, the wave speed becomes $c_o^2 = E/\rho$ where E is the Young's modulus. We shall assume that $c^2 \leq c_o^2$.

In the $x - t$ plane, let λ denote the slope of an elastic-plastic boundary. λ is also identified as the speed of an unloading boundary (denoted by c_u) or a loading boundary (denoted by c_ℓ) depending on whether the material at the section changes from a plastic state to an elastic state or vice versa. In [1] and [2], it was shown that if σ_t on both sides of an elastic-plastic boundary are not zero, λ satisfies the relation

$$\frac{\sigma_t^p}{\sigma_t^e} = \frac{\frac{1}{c_o^2} - \frac{1}{\lambda^2}}{\frac{1}{c^2} - \frac{1}{\lambda^2}} \tag{3}$$

where the superscripts e and p denote values in the elastic and plastic regions respectively. Since $\sigma_t^p/\sigma_t^e \leq 0$ for an unloading while $\sigma_t^p/\sigma_t^e \geq 0$ for a loading, we have

$$\begin{gathered} c \leq c_u \leq c_o \\ c_o \leq c_\ell \quad \text{or} \quad c_\ell \leq c \ . \end{gathered} \tag{4}$$

If σ_t vanish on both sides of the elastic-plastic boundary while σ_{tt} are not zero on both sides of the boundary, λ satisfies the relation (see [3])

$$\frac{\sigma_{tt}^p}{\sigma_{tt}^e} = \frac{\frac{1}{c_o^2} - \frac{1}{\lambda^2}}{\frac{1}{c^2} - \frac{1}{\lambda^2}} \ . \tag{5}$$

Now, $\sigma_{tt}^p/\sigma_{tt}^e \geq 0$ for an unloading while $\sigma_{tt}^p/\sigma_{tt}^e \leq 0$ for a loading. Therefore, we have

$$\begin{gathered} c_o \leq c_u \quad \text{or} \quad c_u \leq c \\ c \leq c_\ell \leq c_o \ . \end{gathered} \tag{6}$$

Comparing Eqs. (6) with (4), it is seen that restrictions on the unloading wave speed c_u and the loading wave speed c_ℓ are interchanged. This was pointed out in [3].

For a rod which occupies $0 \leq x \leq L$, where L can be finite or infinite, suppose that we have obtained a solution of Eqs. (1) and (2) for $0 \leq x \leq L$, $t \leq t_o$ for certain initial and boundary conditions. For a positive quantity δ and a function $f(x, t)$, we define:

$$f^b = \lim_{\delta \to 0} f(0,\ t_o - \delta)$$

$$f^a = \lim_{\delta \to 0} f(0,\ t_o + \delta) \ . \qquad (7)$$

Thus σ_t^b and σ_t^a are the values of σ_t "before" and "after" $t = t_o$ at the end $x = 0$. If σ_t is not continuous at t_o, i.e. if $\sigma_t^a \neq \sigma_t^b$, the following three cases arise:

$$\begin{aligned} &(i) \quad \sigma_t^b < 0 \ , \quad \sigma_t^a \ \text{arbitrary} \\ &(ii) \quad \sigma_t^b \geq 0 \ , \quad \sigma_t^a \geq 0 \\ &(iii) \quad \sigma_t^b \geq 0 \ , \quad \sigma_t^a < 0 \ . \end{aligned} \qquad (8)$$

For case (i), there is no elastic-plastic boundary generated. The discontinuity in σ_t is simply propagated along $x = c_o(t - t_o)$. For cases (ii) and (iii), if there is an elastic-plastic boundary, the slope of this boundary is determined by Eq. (3). If there is no elastic-plastic boundary generated, the discontinuity in σ_t is simply propagated along $x = c_o(t - t_o)$ or $x = c(t - t_o)$ depending on whether the region near $x = 0$, $t = t_o$ is elastic or plastic.

The situation becomes complicated when $\sigma_t^a = \sigma_t^b = 0$ but σ_{tt}^a and σ_{tt}^b are not both zero. Again, we have three cases:

$$\begin{aligned} &(i) \quad \sigma_t^a = \sigma_t^b = 0 \ , \quad \sigma_{tt}^b > 0 \ , \quad \sigma_{tt}^a \ \text{arbitrary} \\ &(ii) \quad \sigma_t^a = \sigma_t^b = 0 \ , \quad \sigma_{tt}^b \leq 0 \ , \quad \sigma_{tt}^a \leq 0 \\ &(iii) \quad \sigma_t^a = \sigma_t^b = 0 \ , \quad \sigma_{tt}^b \leq 0 \ , \quad \sigma_{tt}^a > 0 \ . \end{aligned} \qquad (9)$$

For case (i), there is no elastic-plastic boundary generated.

For cases (ii) and (iii), if there is an elastic-plastic boundary, Eq. (5) alone is not sufficient to determine the slope of the elastic-plastic boundary. In some instances, Eq. (5) cannot be used and a modified version of Eq. (3) has to be employed. In this paper, we demonstrate how the initial slope of the elastic-plastic boundary or boundaries can be determined analytically by using Eq. (5) and the additional conditions which will be presented below and in the next section. We will restrict our attention to cases (ii) and (iii) for all combinations of σ_{tt}^{a} and σ_{tt}^{b}. An interesting result is obtained for case (iii) in which an unloading wave followed by a loading wave may be generated even though $\sigma(0, t)$ is non-decreasing at the boundary.

The validity of Eq. (5) requires that $\sigma_t = 0$ on both sides of the elastic-plastic boundary as well as along the elastic-plastic boundary. In the problem under consideration, $\sigma_t^a = \sigma_t^b = 0$ at $x = 0$, but σ_t may not be zero for $x \neq 0$ along the elastic-plastic boundary which starts from $x = 0$, $t = t_o$. Thus, we have to use Eq. (3) instead of Eq. (5) with some modification of the left hand side of Eq. (3). To obtain a modified version of Eq. (3), first consider $\sigma_t(x, t)$ in the neighborhood of $x = 0$, $t = t_o$. Since $\sigma_t(0, t_o) = 0$, by expanding $\sigma_t(x, t)$ into Taylor series, we have,

$$\sigma_t(x, t) = \sigma_{tx}(0, t_o)\, x + \sigma_{tt}(0, t_o)(t - t_o) + \ldots$$

Along $x = \lambda(t - t_o)$,

$$\sigma_t(x, t)\Big|_{x = \lambda(t - t_o)} = [\sigma_{tx}(0, t_o) + \frac{1}{\lambda}\sigma_{tt}(0, t_o)]\, x + 0(x^2) \qquad (10)$$

With Eq. (10), Eq. (3) becomes, by taking the limit $x \to 0$,

$$\frac{\sigma^p_{tx} + \frac{1}{\lambda}\sigma^p_{tt}}{\sigma^e_{tx} + \frac{1}{\lambda}\sigma^e_{tt}} = \frac{\frac{1}{c_o^2} - \frac{1}{\lambda^2}}{\frac{1}{c^2} - \frac{1}{\lambda^2}} \tag{11}$$

This is the modified version of Eq. (3). Equation (11) applies to an elastic-plastic boundary in which σ_t on both sides of the boundary are not zero except at the point concerned.

Since Eq. (11) introduces new unknowns σ_{tx}, we need additional conditions to determine them. One of these conditions is furnished by the consideration of the continuity of second derivatives along the boundary. Let d/dt denote the total derivative along any curve which starts at $x = 0$, $t = t_o$ with the slope $\frac{dx}{dt} = \alpha(t)$. Then,

$$\frac{d\sigma}{dt} = \sigma_x \alpha + \sigma_t \tag{12}$$

$$\frac{d^2\sigma}{dt^2} = \sigma_{xx}\alpha^2 + 2\sigma_{tx}\alpha + \sigma_{tt} + \sigma_x \alpha_t \tag{13}$$

If $[f]$ denote the discontinuity in the values of f on both sides of the curve, Eq. (12) gives

$$[\sigma_x]\,\alpha + [\sigma_t] = 0 \ . \tag{14}$$

In the present problem, $[\sigma_t] = 0$ at $x = 0$, $t = t_o$ and hence

$$[\sigma_x] = 0 \quad \text{at} \quad x = 0,\ t = t_o \ . \tag{15}$$

Moreover, if we eliminate v between Eqs. (1) and (2), we obtain, since $\sigma_t = 0$ at $x = 0$, $t = t_o$,

$$\sigma_{xx} = \frac{1}{c^2}\sigma_{tt} \quad \text{at} \quad x = 0,\ t = t_o . \tag{16}$$

With Eqs. (15) and (16), Eq. (13) yields

$$[(1 + \frac{\alpha^2}{c^2})\sigma_{tt}] + 2\alpha[\sigma_{tx}] = 0, \quad \text{at} \quad x = 0,\ t = t_o . \tag{17}$$

Equations (5), (11) and (17) will be used for determining the slope of an elastic-plastic boundary.

In this paper, we consider the situation in which the stress state in the region $0 \leq x < \delta$, $t_o - \delta < t \leq t_o$ is plastic. Since $\sigma_t(x, t_o) = \sigma^b_{tx} x + \ldots$, this implies that $\sigma^b_{tx} > 0$. The situation in which the region below $t = t_o$ is elastic can be analysed by essentially the same procedure. Therefore we will study the slope of the elastic-plastic boundaries for given $\sigma^b_{tt} < 0$, $\sigma^b_{tx} > 0$, and σ^a_{tt}.

2. <u>The Elastic-Plastic Boundaries</u>

We define β by the ratio

$$\beta = -\sigma^b_{tt}/\sigma^b_{tx} . \tag{18}$$

Since $\sigma^b_{tt} \leq 0$, $\sigma^b_{tx} > 0$, $\beta \geq 0$. In the neighborhood of $x = 0$, $t = t_o$,

$$\sigma_t(x, t) = \sigma^b_{tx} x + \sigma^b_{tt}(t - t_o) + \ldots .$$

Therefore $\sigma_t(x, t) = 0$ along the line $x = \beta(t - t_o)$. In the analysis of the elastic-plastic boundaries, the value of β plays an important role. For a given $\sigma^b_{tt} \leq 0$, $\beta \geq 0$, and σ^a_{tt}, there are six different situations depending on the values of σ^b_{tt}, β and σ^a_{tt}. These are discussed separately below.

A. Solutions for $\sigma_t^a = \sigma_t^b = 0, \quad \sigma_{tt}^b \leq 0 \ , \quad \sigma_{tt}^a \leq 0$.

There are three cases depending on the values of β. It is understood that σ_{tt}^b and σ_{tt}^a are not both zero.

A1. $c_o < \beta$ (Fig. 1)

For this case, the unloading wave speed c_u is simply equal to β. The second derivatives of the stress are discontinuous across c_u and c_o as shown by solid lines in Fig. 1. The letters P and E denote respectively the plastic regions and the elastic regions. To determine σ_{tx}^m, σ_{tt}^m and σ_{tx}^a, we use Eqs. (5) and (17) which can be written as

$$\frac{\sigma_{tt}^b}{\sigma_{tt}^m} = \frac{\frac{1}{c_o^2} - \frac{1}{c_u^2}}{\frac{1}{c^2} - \frac{1}{c_u^2}} \tag{19}$$

$$(1 + \frac{c_u^2}{c^2})\sigma_{tt}^b + 2c_u\sigma_{tx}^b = (1 + \frac{c_u^2}{c_o^2})\sigma_{tt}^m + 2c_u\sigma_{tx}^m \tag{20}$$

$$2\sigma_{tt}^m + 2c_o\sigma_{tx}^m = 2\sigma_{tt}^a + 2c_o\sigma_{tx}^a \ . \tag{21}$$

Equation (20) is obtained by applying Eq. (17) to the line $x = c_u(t - t_o)$ while Eq. (21) is obtained by applying Eq. (17) to the line $x = c_o(t - t_o)$.

When $\beta = c_u = c_o$, Eqs. (20) and (21) can be combined to give

$$(1 + \frac{c_o^2}{c^2})\sigma_{tt}^b + 2c_o\sigma_{tx}^b = 2\sigma_{tt}^a + 2c_o\sigma_{tx}^a$$

which yields σ_{tx}^a.

A2. $c^* \leq \beta \leq c_o$ (Fig. 2)

We will show that if c^* satisfies the relation

$$\frac{\sigma_{tt}^a}{\sigma_{tt}^b} = \frac{\frac{2}{c}\left(\frac{1}{c^*} - \frac{1}{c}\right)}{\left(\frac{1}{c^2} - \frac{1}{c_o^2}\right)} \qquad (22)$$

then the unloading wave speed c_u has the value $c < c_u < c_o$, and this elastic-plastic boundary is the only boundary across which the discontinuities in second derivatives may occur. In Fig. 2, the dotted lines c and c_o are for reference only. There are no discontinuities across these dotted lines. Notice that $c^* \leq c$. Also notice that c_u cannot be equal to β.

By applying Eqs. (11) and (17) to the unloading boundary, we obtain

$$\frac{\sigma_{tx}^b + \frac{1}{c_u}\sigma_{tt}^b}{\sigma_{tx}^a + \frac{1}{c_u}\sigma_{tt}^a} = \frac{\frac{1}{c_o^2} - \frac{1}{c_u^2}}{\frac{1}{c^2} - \frac{1}{c_u^2}} \qquad (23)$$

$$\left(1 + \frac{c_u^2}{c^2}\right)\sigma_{tt}^b + 2c_u\sigma_{tx}^b = \left(1 + \frac{c_u^2}{c_o^2}\right)\sigma_{tt}^a + 2c_u\sigma_{tx}^a \,. \qquad (24)$$

Elimination of σ_{tx}^a between Eqs. (23) and (24) yields, using Eq. (18)

$$\frac{\sigma_{tt}^a}{\sigma_{tt}^b} = 1 + \left(\frac{1}{c^2} - \frac{1}{c_o^2}\right)\frac{\left(\frac{1}{c_o^2} + \frac{2}{\beta c_u} - \frac{3}{c_u^2}\right)}{\left(\frac{1}{c_u^2} - \frac{1}{c_o^2}\right)^2} \,. \qquad (25)$$

Equation (25) provides a solution for c_u. We will show next that there exists only one c_u satisfying the condition $c \leq c_u \leq c_o$ for given $c^* \leq \beta \leq c_o$ and $0 \leq \sigma_{tt}^a/\sigma_{tt}^b \leq \infty$.

First, notice that for the unloading boundary to be valid, we must have $c_u > \beta$. With this, it can be shown that the right hand side of Eq. (25) is an increasing function of c_u for $\beta < c_u \leq c_o$. Next, we consider the cases $c \leq \beta \leq c_o$ and $\beta \leq c$ separately. When $c \leq \beta \leq c_o$, $\beta < c_u \leq c_o$, and the right hand side of Eq. (25) assumes a negative value when $c_u = \beta$ and becomes infinite when $c_u = c_o$. Hence there is a unique solution for c_u for given $c \leq \beta \leq c_o$ and $0 \leq \sigma_{tt}^a/\sigma_{tt}^b \leq \infty$. If $\beta \leq c$, we have $c \leq c_u \leq c_o$. The right hand side of Eq. (25) is still infinite when $c_u = c_o$ but assumes the value

$$\frac{\frac{2}{c}\left(\frac{1}{\beta} - \frac{1}{c}\right)}{\frac{1}{c^2} - \frac{1}{c_o^2}} \tag{26}$$

when $c_u = c$. In order that a unique solution exists, we must have

$$\frac{\sigma_{tt}^a}{\sigma_{tt}^b} \geq \frac{\frac{2}{c}\left(\frac{1}{\beta} - \frac{1}{c}\right)}{\frac{1}{c^2} - \frac{1}{c_o^2}} .$$

In view of the definition of c^* in Eq. (22), this implies that $c^* \leq \beta$. The proof is completed.

A particular solution in which $\sigma_{tt}(0, t)$ is continuous, (i.e. $\sigma_{tt}^a = \sigma_{tt}^b$), and the solution for $t \leq t_o$ is a simple wave solution was obtained by Biderman (see [4]). Biderman's solution can be recovered by setting $\sigma^a_{tt}/\sigma^b_{tt} = 1$, $\beta = c$ in Eq. (25). The result is

$$\frac{1}{c_o^2} + \frac{2}{cc_u} - \frac{3}{c_u^2} = 0$$

or

$$\frac{c_u}{c_o} = \sqrt{\frac{c_o^2}{c^2} + 3} - \frac{c_o}{c} \quad . \qquad (27)$$

A3. $0 \leq \beta \leq c^*$ (Fig. 3)

In this case the elastic-plastic boundary has the slope $0 < c_u \leq c$ and the discontinuities in derivatives occur across c_u and c as shown by solid lines in Fig. 3. Since $0 < c_u \leq c$ which satisfies Eq. (6), σ_t must be zero along $x = c_u(t - t_o)$. By Eq. (10), we have

$$\sigma^m_{tx} + \frac{1}{c_u} \sigma^m_{tt} = 0 \quad . \qquad (28)$$

Applying Eq. (5) to the unloading boundary, we have

$$\frac{\sigma^m_{tt}}{\sigma^a_{tt}} = \frac{\frac{1}{c_o^2} - \frac{1}{c_u^2}}{\frac{1}{c^2} - \frac{1}{c_u^2}} \quad . \qquad (29)$$

On the other hand, across the characteristics c we obtain, by Eq. (17)

$$\sigma^b_{tt} + c\,\sigma^b_{tx} = \sigma^m_{tt} + c\,\sigma^m_{tx} \quad . \qquad (30)$$

Equations (28) to (30) are sufficient to determine c_u, σ^m_{tx} and σ^m_{tt}. Elimination of σ^m_{tx} and σ^m_{tt} yields

$$\frac{\sigma_{tt}^{a}}{\sigma_{tt}^{b}} = \left(\frac{1}{\beta} - \frac{1}{c}\right)\frac{\left(\frac{1}{c_u} + \frac{1}{c}\right)}{\left(\frac{1}{c_u^2} - \frac{1}{c_o^2}\right)} . \qquad (31)$$

The right hand side of this equation is an increasing function of c_u and assumes values from zero to the one expressed in (26) as c_u varies from zero to c. Thus a unique solution is assured if $0 \leq \beta \leq c^*$.

B. <u>Solutions for $\sigma_t^a = \sigma_t^b = 0$, $\sigma_{tt}^b \leq 0$, $\sigma_{tt}^a > 0$</u>

When $\sigma_t^a = \sigma_t^b = 0$, $\sigma_{tt}^b \leq 0$ and $\sigma_{tt}^a > 0$, the stress at $x = 0$ is continuously loading. For prescribed $\sigma(0, t)$, $t \leq t_o$, the solution is uniquely determined up to the line $x = c(t - t_o)$. When $\beta > c$, $\sigma_t = 0$ along the line $x = \beta(t - t_o)$ which is below the line $x = c(t - t_o)$. Therefore unloading must take place even if $\sigma(0, t)$ is continuously increasing.

B1. <u>$c_o \leq \beta$ (Fig. 4)</u>

In this case the unloading wave c_u is simply equal to β and we have a loading wave $c_\ell \leq c$ as shown in Fig. 4. Discontinuities may occur across the line c_o as indicated by a solid line.

Applying Eq. (5) to the unloading boundary c_u and Eq. (11) to the loading boundary c_ℓ respectively, we have

$$\frac{\sigma_{tt}^{b}}{\sigma_{tt}^{m}} = \frac{\frac{1}{c_o^2} - \frac{1}{c_u^2}}{\frac{1}{c^2} - \frac{1}{c_u^2}} \qquad (32)$$

$$\frac{\sigma_{tx}^{a} + \frac{1}{c_\ell}\sigma_{tt}^{a}}{\sigma_{tx}^{n} + \frac{1}{c_\ell}\sigma_{tt}^{n}} = \frac{\frac{1}{c_o^2} - \frac{1}{c_\ell^2}}{\frac{1}{c^2} - \frac{1}{c_\ell^2}} \tag{33}$$

On the other hand, application of Eq. (17) to the lines c_u, c_o and c_ℓ yields

$$\left(1 + \frac{c_u^2}{c^2}\right)\sigma_{tt}^{b} + 2c_u\,\sigma_{tx}^{b} = \left(1 + \frac{c_u^2}{c_o^2}\right)\sigma_{tt}^{m} + 2c_u\,\sigma_{tx}^{m} \tag{34}$$

$$\sigma_{tt}^{m} + c_o\,\sigma_{tx}^{m} = \sigma_{tt}^{n} + c_o\,\sigma_{tx}^{n} \tag{35}$$

$$\left(1 + \frac{c_\ell^2}{c^2}\right)\sigma_{tt}^{a} + 2c_\ell\,\sigma_{tx}^{a} = \left(1 + \frac{c_\ell^2}{c_o^2}\right)\sigma_{tt}^{n} + 2c_\ell\,\sigma_{tx}^{n}\ . \tag{36}$$

Equations (32) to (36) have six unknowns and hence require another equation for a solution. This is supplied by the condition that the stress on the unloading boundary c_u must be identical to the stress on the loading boundary c for the same position x.

Referring to Fig. 4 this implies that

$$\int_{t_1}^{t_2} \sigma_t(x,\ t)\ dt + \int_{t_2}^{t_3} \sigma_t(x,\ t)\ dt = 0$$

or

$$\int_{t_1}^{t_2}\int_{t_1}^{\tau} \sigma_{tt}(x,\ t)\ dt\ d\tau + \int_{t_2}^{t_3}\left\{\sigma_t(x,\ t_2) + \int_{t_2}^{\tau}\sigma_{tt}(x,\ t)dt\right\}d\tau = 0$$

i.e.

$$\int_{t_1}^{t_2}(t_2-t)\sigma_{tt}(x,t)dt + \sigma_t(x,t_2)(t_3-t_2) + \int_{t_2}^{t_3}(t_3-t)\sigma_{tt}(x,t)dt = 0 \ .$$

Now, by the Mean Value Theorem, we obtain,

$$\frac{1}{2}(t_2-t_1)^2\,\sigma_{tt}(x,t_{12}) + \sigma_t(x,t_2)(t_3-t_2) + \frac{1}{2}(t_3-t_2)^2\,\sigma_{tt}(x,t_{23}) = 0 \quad (37)$$

where $t_1 \leq t_{12} \leq t_2$, $t_2 < t_{23} < t_3$. As $x \rightarrow 0$,

$$t_2 - t_1 \cong \left(\frac{1}{c_o} - \frac{1}{c_u}\right)x \ , \quad \sigma_{tt}(x,\ t_{12}) \cong \sigma^m_{tt}$$

$$t_3 - t_2 \cong \left(\frac{1}{c_\ell} - \frac{1}{c_o}\right)x \ , \quad \sigma_{tt}(x,\ t_{23}) \cong \sigma^n_{tt}$$

while

$$\sigma_t(x,\ t_2) \cong \left(\sigma^m_{tx} + \frac{1}{c_o}\sigma^m_{tt}\right)x$$

$$= \left(\sigma^n_{tx} + \frac{1}{c_o}\sigma^n_{tt}\right)x$$

by Eq. (10). Therefore, in the limit $x \rightarrow 0$, Eq. (37) yields

$$\frac{1}{2}\left(\frac{1}{c_o} - \frac{1}{c_u}\right)^2\sigma^m_{tt} + \left(\frac{1}{c_\ell} - \frac{1}{c_o}\right)\left(\sigma^m_{tx} + \frac{1}{c_o}\sigma^m_{tt}\right) + \frac{1}{2}\left(\frac{1}{c_\ell} - \frac{1}{c_o}\right)^2\sigma^n_{tt} = 0 \ . \quad (38)$$

Equation (38), together with Eqs. (32) to (36), give a complete description of the problem.

Equations (32) to (36) and Eq. (38) can be solved for c_ℓ. Omitting the lengthy process of eliminating the unknowns, we present the result below:

$$-\frac{\sigma^a_{tt}}{\sigma^b_{tt}} = \frac{\frac{1}{c^2} - \frac{1}{c_u^2}}{\frac{1}{c_\ell^2} - \frac{1}{c^2}} + \frac{\frac{2}{c_\ell}\left(\frac{1}{c_\ell} + \frac{1}{c_o}\right)\left(\frac{1}{c_\ell} - \frac{1}{c_u}\right)\left(\frac{1}{c^2} - \frac{1}{c_u^2}\right)}{\left(\frac{1}{c_o} + \frac{1}{c_u}\right)\left(\frac{1}{c_\ell^2} - \frac{1}{c^2}\right)^2} . \tag{39}$$

The left hand side of Eq. (39) can have any value from zero to infinite. The right hand side of Eq. (39) is an increasing function of c_ℓ and varies from zero to infinite as c_ℓ changes from zero to c. Therefore Eq. (39) provides a unique solution for c_ℓ for given $0 \leq -\sigma^a_{tt}/\sigma^b_{tt} \leq \infty$.

B2. <u>$c \leq \beta \leq c_o$ (Fig. 5)</u>

For this case we have two elastic-plastic boundaries: $c \leq \beta < c_u \leq c_o$ and $c_\ell \leq c$. Applying Eq. (11) to the unloading and the loading boundaries, we obtain

$$\frac{\sigma^b_{tx} + \frac{1}{c_u}\sigma^b_{tt}}{\sigma^m_{tx} + \frac{1}{c_u}\sigma^m_{tt}} = \frac{\frac{1}{c_o^2} - \frac{1}{c_u^2}}{\frac{1}{c^2} - \frac{1}{c_u^2}} \tag{40}$$

$$\frac{\sigma^a_{tx} + \frac{1}{c_\ell}\sigma^a_{tt}}{\sigma^m_{tx} + \frac{1}{c_\ell}\sigma^m_{tt}} = \frac{\frac{1}{c_o^2} - \frac{1}{c_\ell^2}}{\frac{1}{c^2} - \frac{1}{c_\ell^2}} . \tag{41}$$

Application of Eq. (17) to both boundaries yields

$$\left(1 + \frac{c_u^2}{c^2}\right)\sigma^b_{tt} + 2c_u\,\sigma^b_{tx} = \left(1 + \frac{c_u^2}{c_o^2}\right)\sigma^m_{tt} + 2c_u\,\sigma^m_{tx} \tag{42}$$

$$\left(1+\frac{c_\ell^2}{c^2}\right)\sigma_{tt}^a + 2c_\ell\,\sigma_{tx}^a = \left(1+\frac{c_\ell^2}{c_o^2}\right)\sigma_{tt}^m + 2c_\ell\,\sigma_{tx}^m\ . \tag{43}$$

Again, we need another equation which is supplied by the condition (see Fig. 5)

$$\int_{t_1}^{t_2} \sigma_t(x,\ t)\ dt = 0\ .$$

Following the procedure in deriving Eq. (38), the condition yields, as $x \to 0$,

$$\sigma_{tx}^m + \frac{1}{2}\,\sigma_{tt}^m\left(\frac{1}{c_\ell}+\frac{1}{c_u}\right) = 0\ . \tag{44}$$

Equations (40) to (44) give a complete description of the problem.

Elimination of σ_{tx}^m, σ_{tt}^m between Eqs. (40), (42) and (44) yields

$$\beta = \frac{\left(\frac{1}{c^2}-\frac{1}{c_u^2}\right)\left(\frac{1}{c_u^2}-\frac{1}{c_o^2}\right)+\frac{1}{c_u}\left(\frac{1}{c_\ell}-\frac{1}{c_u}\right)\left(\frac{1}{c^2}-\frac{1}{c_o^2}\right)}{\frac{1}{2}\left(\frac{1}{c^2}-\frac{1}{c_u^2}\right)\left(\frac{1}{c_u^2}-\frac{1}{c_o^2}\right)\left(\frac{1}{c_\ell}+\frac{1}{c_u}\right)+\frac{1}{c_u^2}\left(\frac{1}{c_\ell}-\frac{1}{c_u}\right)\left(\frac{1}{c^2}-\frac{1}{c_o^2}\right)} \tag{45}$$

while elimination of σ_{tx}^m, σ_{tt}^m, σ_{tx}^b, σ_{tx}^a between Eqs. (40) to (44) yields

$$-\frac{\sigma_{tt}^a}{\sigma_{tt}^b} = \left(\frac{\frac{1}{c^2}-\frac{1}{c_u^2}}{\frac{1}{c_\ell^2}-\frac{1}{c^2}}\right)^2\left\{\frac{\left(\frac{1}{c_\ell^2}-\frac{1}{c^2}\right)\left(\frac{1}{c_\ell^2}-\frac{1}{c_o^2}\right)+\frac{1}{c_\ell}\left(\frac{1}{c_\ell}-\frac{1}{c_u}\right)\left(\frac{1}{c^2}-\frac{1}{c_o^2}\right)}{\left(\frac{1}{c^2}-\frac{1}{c_u^2}\right)\left(\frac{1}{c_u^2}-\frac{1}{c_o^2}\right)+\frac{1}{c_u}\left(\frac{1}{c_\ell}-\frac{1}{c_u}\right)\left(\frac{1}{c^2}-\frac{1}{c_o^2}\right)}\right\}. \tag{46}$$

It can be shown that the right hand side of Eq. (45) is an increasing function of c_u and c_ℓ, for $c \le c_u \le c_o$ and $0 \le c_\ell \le c$. Since $\beta = c$ when $c_u = c$ and $\beta = c_o$ when $c_u = c_o$, Eq. (45) yields, for a given $c \leq \beta \leq c_o$, c_u as a single-valued function of c_ℓ for $0 \leq c_\ell \leq c$. Therefore, for each $0 \leq c_\ell \leq c$ and $c \leq \beta \leq c_o$, there exists only one $c \leq c_u \leq c_o$ which satisfies Eq. (45). In fact one can even show that $c_u > \beta$. Now, the right hand side of Eq. (46) varies from zero to infinite as c_ℓ assumes values from zero to c. Thus, by Eqs. (45) and (46), there exists a solution c_u and c_ℓ satisfying the relations $c \leq \beta < c_u \leq c$ and $0 \leq c_\ell \leq c$ for given $c \leq \beta \leq c_o$ and $0 \leq - \sigma^a_{tt}/\sigma^b_{tt} \leq \infty$.

Two limiting cases can be considered here. First, when $\beta = c_o$, Eq. (45) gives $c_u = c_o$ and Eq. (46) becomes

$$- \frac{\sigma^a_{tt}}{\sigma^b_{tt}} = \frac{\frac{1}{c^2} - \frac{1}{c_o^2}}{\frac{1}{c_\ell^2} - \frac{1}{c^2}} + \frac{\frac{1}{c_\ell}\left(\frac{1}{c^2} - \frac{1}{c_o^2}\right)\left(\frac{1}{c_\ell^2} - \frac{1}{c_o^2}\right)}{\frac{1}{c_o}\left(\frac{1}{c_\ell^2} - \frac{1}{c^2}\right)^2} . \qquad (47)$$

As can be verified easily, this limit can also be obtained by letting $c_u = c_o$ in Eq. (39). Thus, the solution is continuous from case B to B2. Next, when $\beta = c$, Eqs. (45) and (46) give $c_u = c_\ell = c$. This is a limiting case of A3 which is discussed below.

B3. $0 < \beta \leq c$

For this case, there is no elastic-plastic boundary generated. The discontinuities in second derivatives are simply propagated along $x = c(t - t_o)$.

3. Discussion

The initial slope of the elastic-plastic boundary at the end of the rod is determined analytically when the first derivative of the stress σ_t at the end of the rod is continuous and vanishes at time $t = t_o$

while the second derivatives σ_{tt} at t_o may or may not be continuous. Solutions are obtained for all possible combinations of σ_{tt} before and after $t = t_o$ except those cases in which no elastic-plastic boundaries are generated.

The determination of the complete shape of an elastic-plastic boundary for general initial-boundary value problems usually requires a numerical scheme. The procedure can be divided into two stages. In the first stage the slope of this boundary near the end of the rod must be found, and in the second stage the remainder of the boundary is determined. The problem considered here offers an analytical solution for the first stage.

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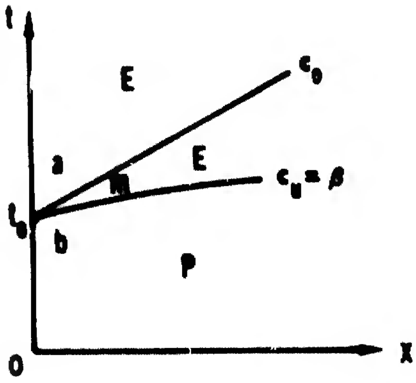

FIG. 1

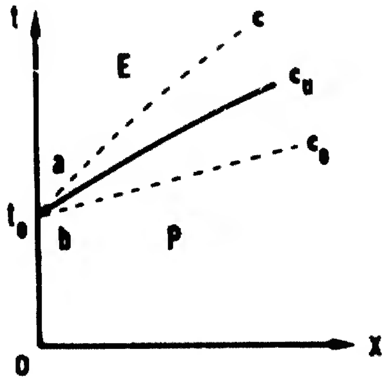

FIG. 2

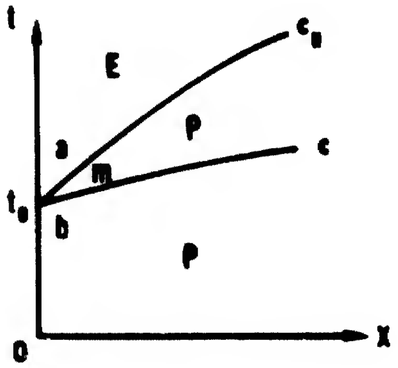

FIG. 3

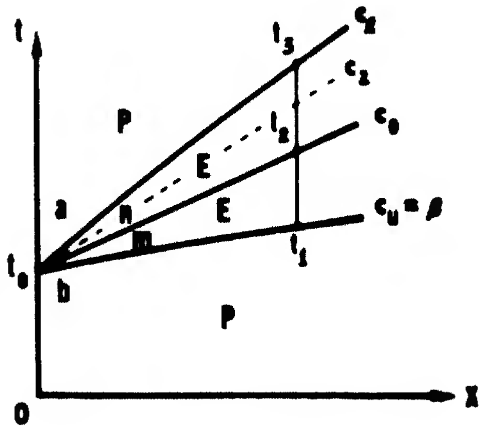

FIG. 4

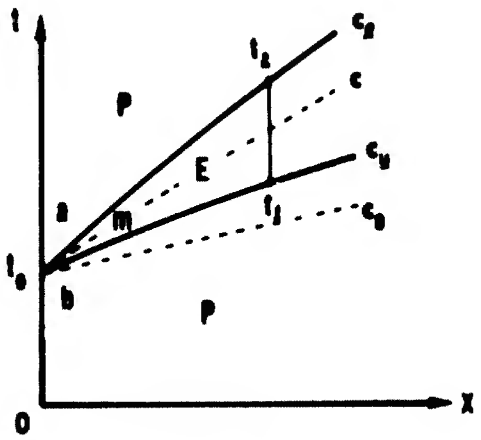

FIG. 5

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13. ABSTRACT

In one dimensional wave propagation such as longitudinal waves in a rod, an elastic-plastic boundary may start at the end $x = 0$ of the rod depending on the stresses prescribed at $x = 0$. The initial slope of the elastic-plastic boundary at $x = 0$ can be determined easily if the time derivative σ_t of the stress σ on both sides of the elastic-plastic boundary are not zero. In this paper, the initial slope of the elastic-plastic boundary (or boundaries) is determined analytically when σ_t at $x = 0$ is continuous and vanishes at time $t = t_o$ while the second derivative σ_{tt} at t_o may or may not be continuous. It is seen that an elastic region can be generated near the end of the rod even though the stress state at the end is continuously plastic.

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