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NUMERICAL SOLUTION OF A HARMONIC MIXED BOUNDARY VALUE PROBLEM BY LINEAR PROGRAMMING

J. R. Whiteman

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ABSTRACT

A method using the linear programming algorithm is given for obtaining a numerical solution with an error bound to a harmonic mixed boundary value problem in which the region of definition is a rectangle containing a slit. Harmonic functions which satisfy the boundary conditions on part of the boundary are matched using a Chebychev fit with the boundary conditions at a finite number of points on the remainder of the boundary so that the maximum error is minimized. The availability of a maximum principle makes this maximum error on the boundary a bound for the error throughout the region.

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1. Introduction

Let us consider the two dimensional harmonic mixed boundary value problem in which the function $u(x, y)$ satisfies the three equations

$$\Delta u \equiv \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{1}$$

throughout a simply connected region R which has boundary S,

$$u = g(x, y) \quad \text{on} \quad S_1 , \tag{2}$$

and

$$\frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad S_2 , \tag{3}$$

where $S_1 \cup S_2 = S$, and $\partial/\partial\nu$ is the derivative in the outward normal direction to the boundary. A much used method for obtaining a numerical solution to this type of problem is that of finite-differences. When the boundary S is sufficiently smooth, the finite-difference technique will produce reasonably accurate approximations which can be shown to converge to the exact solution as the mesh length $h \to 0$. However, when the region contains a re-entrant corner — e.g. as in Figure 1 at O where the interval angle $\phi > \pi$ — the convergence of the finite-difference solution to the exact solution of the problem will be so slow in the neighbourhood of O (this will here be called $\Gamma[O]$), that in $\Gamma[O]$ the approximation will be most inaccurate. Various methods have been suggested for improving the accuracy in $\Gamma[O]$ (see Motz [1], Woods[2], Whiteman [3] and [4].) However,

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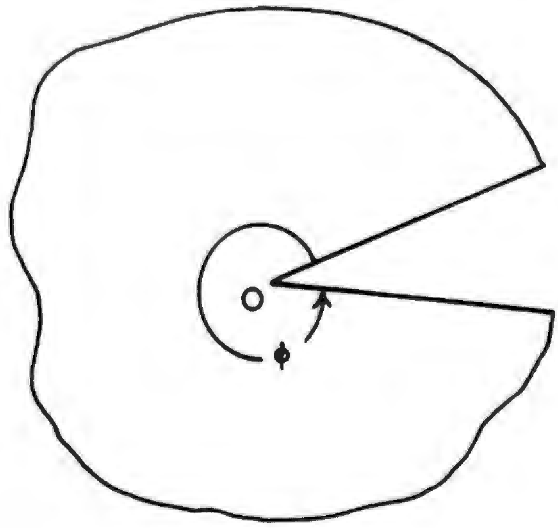

Figure 1

neither of the methods of Motz or Woods, nor those which use finite-differences in [3], have any form of error analysis, whilst the method in [4] is partly analytic and is thus restricted as to the shape of region in which it can be used.

Another technique for obtaining a numerical solution to this type of problem is that in which a function which is the analytic solution of the governing differential equation, and which also satisfies certain of the boundary conditions — e.g. on the arms of the re-entrant corner — is taken in the form of an infinite series involving unknown coefficients (see [3], Chapter 5). These coefficients will here be called b_i, $i = 1,2,3,\ldots$. The series is truncated after a number N terms, and the first N unknown coefficients are found by matching the truncated series with the given conditions at a number M points on the remainder of the boundary. There are thus M linear equations in the N unknowns b_i, $i = 1,2,\ldots,N$, and these are written here in matrix form as

$$A_{M\times N}\,\underline{b}_{N\times 1} = \underline{d}_{M\times 1} \qquad (4)$$

If $M = N$ the method is one of collocation, and, assuming that the matrix A is non-singular, the solution of (4) is unique. If $M > N$, then the calculated solution of (4), if it exists, is one of least squares. A disadvantage of the method is that, as the number of equations increases, the matrix A becomes ill-conditioned in the sense that small changes in the elements of A produce large changes in $\underline{b}$. Consequently, when the equations (4) are solved numerically, the rounding errors introduced, which can be regarded as equivalent to a perturbation of the matrix A, cause large errors in the calculated solution. It may be possible to improve the accuracy of this solution iteratively using double length arithmetic for some of the calculations (see e.g. Martin [5] or Moler [6].) Another possibility is to use double- or multiple-length arithmetic throughout the calculations. The success or failure of either technique depends on the size of the condition number of A, and this is found in practice often to be such that accurate solution of (4) is not possible. It is thus apparent that all the above methods are in some way unsatisfactory for solving the problem (1), (2), (3) when the boundary contains a re-entrant corner, and consequently yet another approach will be used in this paper. The technique here is to set up the equations (4) with $M > N$, but, instead of solving them using a least squares technique, we use linear programming methods to minimize the maximum error at the M points on the boundary. In problems where a maximum principle exists the maximum error on the boundary is a bound for the error throughout the region. A similar technique is used by Mangasarian in [8] to obtain a numerical solution to a biharmonic boundary value problem.

2. Maximum Principle

It is well known that a non-constant solution of (1) can attain neither a maximum nor a minimum at a point of R. Further, if u attains its maximum at a point $P \subset S$, where P lies on the boundary of a circle K in R, and if u is continuous in $R \cup P$, then, for non-constant u, $\frac{\partial u}{\partial \nu} > 0$ at P (see e.g. [7], Chapter 2). It follows for the problem (1), (2), (3) that, if S is sufficiently smooth, then tne maximum value of u must occur on S_1 .

Suppose that the harmonic function v is an approximation to the exact solution u of the problem defined by (1), (2) and (3), so that $v = u + \rho$ where ρ is the error in the approximation. Δ is a linear operator, so that $\Delta v = \Delta u + \Delta \rho$, and hence $\Delta \rho = 0$. If $|v-g| \leq \delta$ on S_1, where δ is some constant, then

$$|\rho| \leq \delta \text{ on } S_1 \ . \tag{5}$$

Similarly if $-\omega\delta < \frac{\partial v}{\partial \nu} < 0$ on S_2, where ω is some positive weighting function, then

$$-\omega\delta \leq \frac{\partial \rho}{\partial \nu} < 0 \text{ on } S_2 \ , \tag{6}$$

and hence since $\frac{\partial \rho}{\partial \nu}$ is always negative on S_2 the maximum value of ρ occurs on S_1 and is less than δ in absolute value.

3. Model Harmonic Mixed Boundary Value Problem

The techniques used in [1], [2], [3] and [4] for obtaining accurate approximations to the solutions of harmonic boundary value problems containing re-entrant corners have all been applied to a particular model problem. This problem, which is stated below, will also be used to illustrate the method of solution using linear programming.

The function u is harmonic in a rectangular region OBCDEFGHO, which has boundary containing a slit, so that at O the re-entrant angle $\phi = 2\pi$, Figure 2. The boundary conditions are $u = 1000$ on $\overline{BC}$, $u = 0$ on $\overline{GH}$ and $\frac{\partial u}{\partial \nu} = 0$ on the rest of the boundary including the slit. Since the function $u - 500$ is antisymmetrical about $\overline{BE}$, when u is put equal to 500 on $\overline{EO}$, the solution over the whole region can be obtained by considering only the top rectangle OBCDEO, Figure 3. The dimensions of the region are taken for convenience such that $\overline{EO} = \overline{OB} = \overline{ED} = \pi/2$.

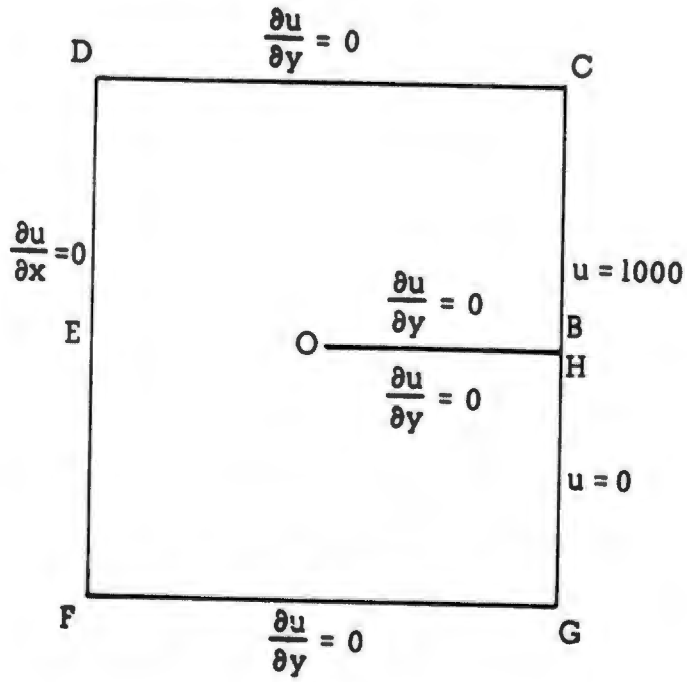

Figure 2

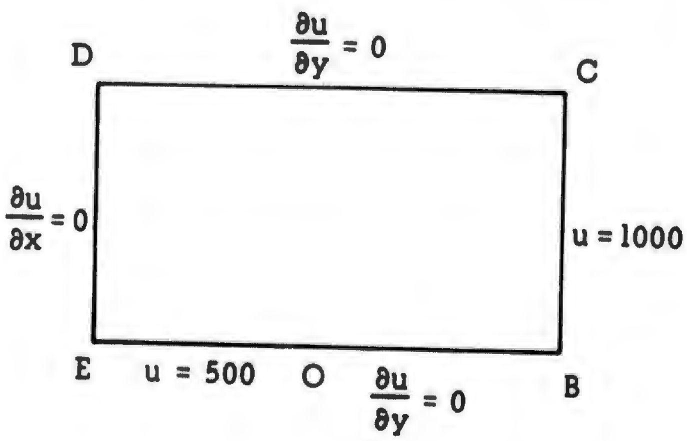

Figure 3

4. Numerical Solution

The solution of (1) which satisfies the boundary conditions on $\overline{EO}$ and $\overline{OB}$ is, in polar co-ordinates (r,θ) with O as origin and OB as zero angle,

$$u = 500 + \sum_{n=0}^{\infty} b_n r^{n+\frac{1}{2}} \cos(n+\tfrac{1}{2})\theta \quad ; \tag{7}$$

The b's again being unknown coefficients as in Section 1.
The series in (7) is truncated after N terms, and the resulting function U is matched with the requisite boundary conditions at the number M points on $\overline{BC}$, $\overline{CD}$ and $\overline{DE}$ using the linear programming technique with the weighting function ω in (6) taken equal to unity. One immediate question is that of how to choose

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the M and the N . The N must be large enough to make U an accurate approximation to u . The choice of M can be made by taking the value which is such that any larger value will not further increase the error bound δ; in this way the maximum error at the chosen points on BC is trapped. In order to check whether this has in fact been done, a fine mesh is put on $\overline{BC}$, and U is evaluated at these intermediate mesh points. Provided that the errors come within the calculated bound, it is reasonable to assume that the maximum error has indeed been trapped, and that the calculated value for δ is the upper bound.

The approximation U is calculated for the two cases $N = 8$ and $N = 15$, and the values on a grid with mesh length $\pi/7$ are shown in Figure 4. In each case it is found that the maximum has been trapped when the points of match are taken at intervals of $\pi/28$ on $\overline{BC}$, $\overline{CD}$ and $\overline{DE}$, so that $M = 53$. In order to ensure that $\frac{\partial u}{\partial \nu}$ is negative on $\overline{CD}$ and $\overline{DE}$ it is found necessary on these sides to make

$$-\delta < \frac{\partial u}{\partial \nu} < -\epsilon \quad .$$

In practice the value $\epsilon = 0.02$ is sufficient for this; again $\frac{\partial u}{\partial \nu}$ is evaluated at points on a fine mesh on $\overline{CD}$ and $\overline{DE}$ to check that it is negative. The results obtained by Whiteman [4], and those obtained using the standard finite-difference method with mesh length $h = \frac{\pi}{28}$, are also shown for comparison in Figure 4. It should be noted that the method in [4] produces an error bound, but this bound is dominated by a factor e^{-y}, so that, although it tends to zero rapidly as y increases, it has the value 15 when $y = 0$. As already stated there is no available error bound for the methods in [1] and [2]. The calculated values of the first five coefficients of the series in (7) are given in Figure 5 together with the values calculated by Motz [1].

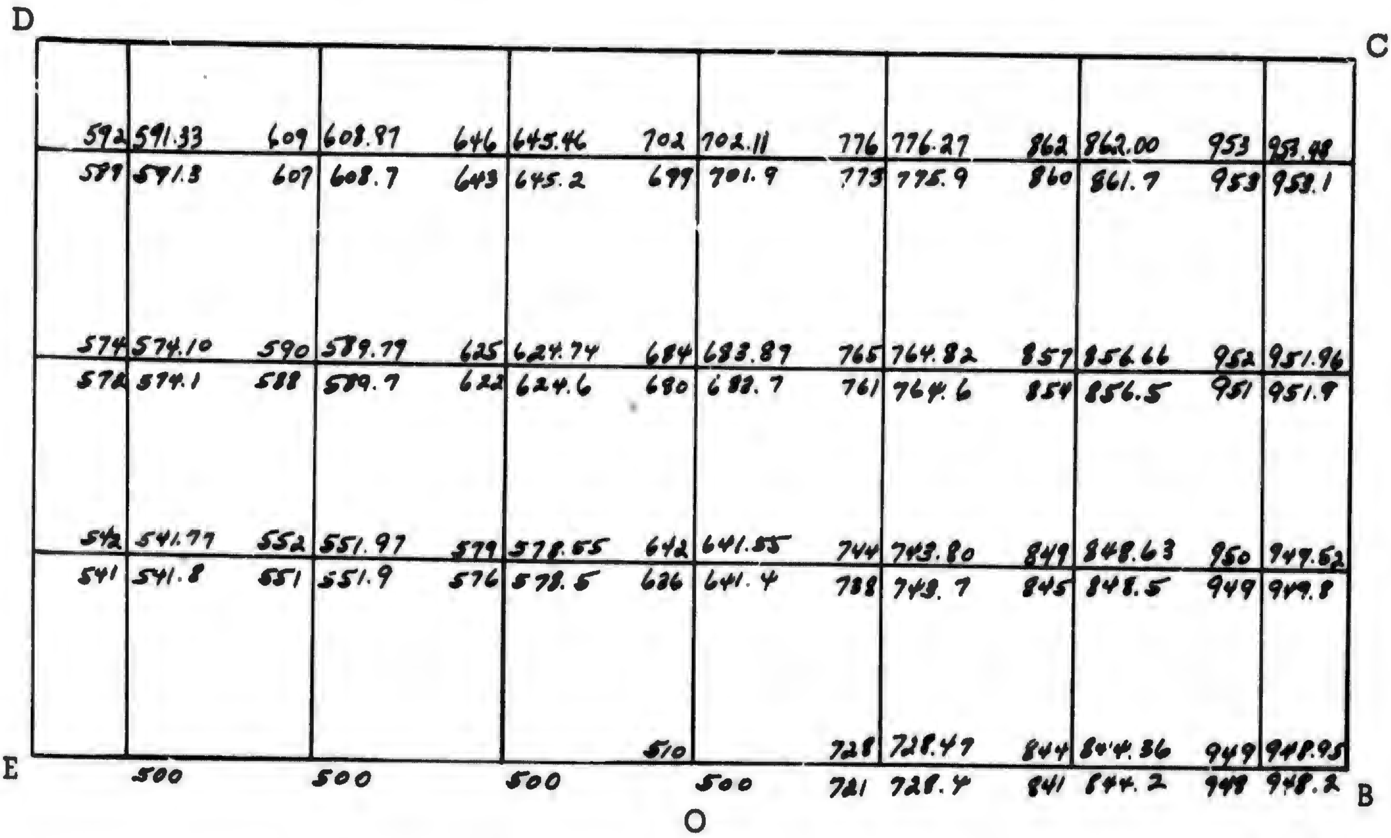

Figure 4

The figures at any mesh point P have the following significance

Solution due to Whiteman [4]	Solution with N = 15
Finite-difference solution with $h = \pi/28$	Solution with N = 8

	Coefficients for N = 8	Coefficients for N = 15	Coefficients are calculated by Motz
b_0	320.1	320.1	301.0
b_1	44.7	44.5	42.0
b_2	5.6	5.8	Not used by Motz
b_3	-1.7	-1.7	
b_4	0.2	0.2	

Figure 5

The bounds given by the linear programming method for the error on $\overline{BC}$ are $\delta = 0.6$ for the case $N = 8$ and $\delta = 0.06$ for the case $N = 15$. The times on the CDC 3600 for the computation of the solutions with these values of N are repectively 32 and 47 seconds. It should be noted that the points B, C, D and E do not lie on circles in R. However, it is felt that a slight rounding of the corners at these points is reasonable as the problem (1), (2), (3) is well posed in the Hadamard sense, and in this way the problem is such that the maximum principle of Section 2 can be applied. The bounds for the error at any point in R are thus 0.6 and 0.06 for the respective solutions.

5. Discussion

A numerical solution to the model boundary value problem has also been calculated using the linear programming technique with the function

$$u = 1000 - \sum_{n=0}^{\infty} B_n \cosh\{(n+\tfrac{1}{2})(y-\pi/2)\}\cos\{(n+\tfrac{1}{2})(x+\pi/2)\} . \qquad (8)$$

This function satisfies the differential equation (1) and the boundary conditions on $\overline{BC}$, $\overline{CD}$ and $\overline{DE}$, so that the matching is now done on $\overline{EO}$ and $\overline{OB}$. The function u in (8) satisfies boundary conditions which are remote from O, and the series is very slowly convergent in the neighbourhood of the singularity. In order to obtain a reasonable approximation it is therefore necessary to keep a large number of terms in the truncated series, and this in turn necessitates the use of a large number of matching points on $\overline{EO}$ and $\overline{OB}$. Even when this is done, the calculated solution is not very accurate, and the error bound is large ($\delta = 10$ when 30 terms of the series are taken and the matching is done at 56 points). It appears that in this case the roundoff error is having considerable effect, and this suggests that in solving boundary value problems involving singularities of this type it is desirable to use a series solution which is rapidly convergent; such a solution will exhibit the form of the singularity.

An important criterion by which the success of any method for solving boundary value problems may be judged is the ease with which it produces accurate solutions to problems in irregular shaped regions. One great advantage of the linear programming technique discussed here is that it is applicable to such problems, as the matching can be done on more or less any type of boundary for any type of boundary condition. There may of course be no maximum principle available for such a problem, but the technique will in general produce a numerical solution together with a bound for the error on the boundary.

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